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QUANTUM MOTION ON THE THREE-DIMENSIONAL SPHERE: THE ELLIPSO-CYLINDRICAL BASES

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ABSTRACT

We study the free quantum motion on the three-dimensional sphere $S^{(3)}$ in ellipso-cylindrical coordinates, where we distinguish between prolate elliptic and oblate elliptic coordinates. The oblate and prolate elliptic wave-functions are constructed in terms of the hyper-spherical and cylindrical wave-functions. A perturbation theory for the elliptic wave-functions in powers of the elliptic parameter is established. Finally, the interbasis expansions are used to set up path integral identities for the two ellipso-cylindrical coordinate systems on $S^{(3)}$.

1 Introduction.

The investigation of separation of variables in the Helmholtz, respectively Schrödinger equation, has a long history, and the case of spaces of non-vanishing constant curvature has been thoroughly discussed by Olevskiĭ [?]. This includes the two- and three-dimensional sphere with two and six orthogonal coordinate systems, and the two- and three-dimensional hyperboloid with nine and 34 orthogonal coordinate systems, respectively. The corresponding cases for the flat spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ with four and eleven orthogonal coordinate systems can be found in [?]. However, many of these coordinate systems are not well known, and the usual studies of physical systems, potential problems, the incorporation of magnetic fields, monopole problems, perturbation theory, or scattering theory, are studied in the more familiar systems, e.g., in cartesian, spherical or parabolic coordinates, depending which coordinate system is best suited to match the symmetry properties of the problem under consideration.

The choice of a coordinate system emphasizes which observables are considered to be the most appropriate for a particular investigation. Actually, the number of coordinate systems which can be found in an homogeneous space equals the number of sets of functionally independent observables in this space. For instance, in a problem with spherical symmetry, the total angular momentum operator $\mathbf{L}^2$, and the L_z operator are sufficient for a comprehensive understanding of classical and quantum motion in $\mathbb{R}^3$ with spherical symmetry and a preferred axis.

The incorporation of interaction, i.e., potentials or magnetic fields, generally reduces the number of coordinate systems in which the Schrödinger equation is separable. If it turns out that the problem is separable in more that one coordinate system, it follows that there are additional observables exceeding the numbers of degrees of freedom. If more than one such coordinate system exist, there will be different representations of the solutions of the Schrödinger equation, and to each of them corresponds a set of commuting observables, which may be simultaneously assigned definite eigenvalues. These systems are called super-integrable, c.f. [?]–[?]. The most important super-integrable systems are the oscillator and the Kepler-Coulomb problem, which are said to be maximally super-integrable, and have in three dimensions five functionally independent observables. A comprehensive discussion of super-integrable systems in two and three dimensions in spaces of constant curvature can be found in [?] for the Euclidean space, in [?] for the two- and three-dimensional sphere, and in [?, ?] for the two- and three-dimensional hyperboloid.

However, the investigation of separation of variables goes beyond the search of super-integrable systems, and the corresponding observables. Perturbating a super-integrable system often leads to a problem which may be separable only in one coordinate system, and it may even not be exactly solvable. Then, the unperturbed system may serve as a starting point for a perturbative investigation, and the energy-levels of a perturbed system may be classified according to the dynamical symmetry group of the unperturbed system. Because the algebra of the angular momentum operator $\mathbf{L}$ and the Pauli-Lenz-Runge operator $\mathbf{K}$ (suitable rescaled) closes for constant energy, the dynamical group $\mathrm{O}(4)$ describes the discrete spectrum, and the Lorentz group $\mathrm{O}(3,1)$ the continuous spectrum [?]. For instance, the oblate elliptic system is of interest because it enables one to set up a complete classification of the energy-levels of the quadratic Zeeman effect (c.f. Herrick [?], Lakshmann and Hasegawa [?], Solov'ev [?], and Brown and Solov'ev [?]).

The spherical, the sphero-conical and a rotated version of the prolate elliptic coordinate system [?] separate the Kepler-Coulomb problem on $S^{(3)}$, and the ellipsoidal system

separates the Neumann model [?, ?].

A classification of the coordinate systems on $S^{(3)}$, including an enumeration of the corresponding observables has been already performed in Refs. [?, ?]. The simpler case of the two-dimensional sphere is due to [?, ?]. Whereas on the two-dimensional sphere, the solution of the Schrödinger equation in spherical coordinates in terms of the spherical functions $Y_l^m(\vartheta, \varphi)$, and in elliptic coordinates in terms of Lamé polynomials [?, ?] is sufficiently understood, including the corresponding interbasis expansions, not very much is known in the three-dimensional case.

In order to solve the Schrödinger equation in different coordinate systems there are two approaches. The first consists of the separation of the Schrödinger equation and a direct solution of the emerging differential equations in terms of powers of the variables, say. In [?] a first attempt was undertaken to characterize and classify the free quantum motion on $S^{(3)}$, and the authors determined the ellipso-cylindrical wave-functions by making an Ansatz in terms of powers of Jacobi elliptic functions, and called the emerging polynomial solutions *associated Lamé polynomials*.

The second is based on the knowledge of the complete set of observables. The first investigation of this kind was done by Coulson and Joseph [?] for the basis of the hydrogen atom in spheroidal coordinates by constructing an expansion in terms of the spherical basis. This method was also used by Patera and Winternitz [?] for the sphero-conical wave-functions on $S^{(2)}$. Mardoyan et al. [?] used it for constructing the spheroidal basis in terms of the parabolic basis for the Coulomb-Kepler problem, and Kibler et al. [?] analysed by this method a generalized Coulomb-Kepler problem.

The second approach is superior to the first because it does not only allows one to calculate the wave-functions, at least recursively, but also determines at the same time the coefficients of the interbasis expansion with respect to another basis. After obtaining recurrence relations for the interbasis coefficients, they can serve as a starting point for an algebraic perturbation description [?] of the construction of the wave-functions.

Having this in mind, the present paper is devoted to the study of the two ellipso-cylindrical bases on the three-dimensional sphere $S^{(3)}$.

The further content of our paper is as follows. In the next section we discuss the two ellipso-cylindrical coordinate systems on $S^{(3)}$, and define the relevant observables. In the third section, we construct the ellipso-cylindrical bases in terms of the hyper-spherical and cylindrical bases. In the fourth section we consider a perturbation theory for the ellipso-cylindrical bases. The cases of the elliptic parameter $0 < |a| \ll 1$ and $a \gg 1$ are discussed separately. In the fifth section the interbasis expansions are used to establish path integral identities in the ellipso-cylindrical coordinate systems. A first attempt to such a formulation was tried in [?], however only on a heuristic level. In the sixth section we give a summary and discussion of our results, and in the two appendices some results are compiled concerning the matrix elements of operators with respect to the cylindrical and the hyper-spherical bases.

2 The Schrödinger Equation and the Integrals of Motion.

The Schrödinger-, respectively the Helmholtz equation, for the free motion on the three-dimensional unit sphere $S^{(3)}$

$$s_1^2 + s_2^2 + s_3^2 + s_4^2 = 1 \tag{2.1}$$

embedded in four-dimensional Euclidean space has the form

$$-\frac{\hbar^2}{2M}\Delta\Psi = \frac{\hbar^2}{2M}J(J+2)\Psi \ , \qquad J = 0, 1, 2, \ldots . \tag{2.2}$$

Δ is the Laplace operator which is given by

$$\Delta = -\frac{1}{\hbar^2}(\mathbf{L}^2 + \mathbf{M}^2) \ . \tag{2.3}$$

L_i and M_i, $i = 1, 2, 3$, are the six generators of the group O(4) which have the following form

$$L_i = \frac{\hbar}{\mathrm{i}}\epsilon_{ijk}s_i\frac{\partial}{\partial s_k} \ , \qquad M_i = \frac{\hbar}{\mathrm{i}}\left(s_i\frac{\partial}{\partial s_4} - s_4\frac{\partial}{\partial s_i}\right) \ , \qquad i = 1, 2, 3 \ . \tag{2.4}$$

They satisfy the commutation relations

$$[L_i, L_j] = \mathrm{i}\hbar\epsilon_{ijk}L_j \ , \qquad [M_i, M_j] = \mathrm{i}\hbar\epsilon_{ijk}L_j \ , \qquad [L_i, M_j] = \mathrm{i}\hbar\epsilon_{ijk}M_j \ . \tag{2.5}$$

Equation (**??**) is separable in six coordinate systems which we list in table **??**, together with their corresponding characteristic observables.

2.1 The Ellipso-Cylindrical Coordinate Systems.

On the three-dimensional sphere there are two ellipso-cylindrical coordinate systems, the *oblate elliptic system*, also called elliptic-cylindrical 1 system, and the *prolate elliptic system*, also called elliptic-cylindrical 2 system.

Oblate elliptic coordinates. Let us start with the oblate elliptic system. In terms of the Jacobi elliptic functions cn, sn and dn [**?**, p.910] it has the form [**?**]

$$\left.\begin{aligned} s_1 &= \mathrm{sn}(\mu,k)\mathrm{dn}(\nu,k')\cos\varphi \ , \\ s_2 &= \mathrm{sn}(\mu,k)\mathrm{dn}(\nu,k')\sin\varphi \ , \\ s_3 &= \mathrm{cn}(\mu,k)\mathrm{cn}(\nu,k') \ , \\ s_4 &= \mathrm{dn}(\mu,k)\mathrm{sn}(\nu,k') \ , \end{aligned}\right\} \tag{2.6}$$

where $0 \le \mu \le 2K$, $-K' \le \nu \le K'$, $0 \le \varphi < 2\pi$. k is the modulus of the elliptic functions, with $k'^2 = 1 - k^2$. K and K' are the complete elliptic integrals. In the following, if not explicitly stated otherwise, we usually omit in the Jacobi elliptic functions the explicit dependence on the modulus k. The coordinates (**??**) are an one-parametric coordinate system depending on k. In the limit $k \to 0$, respectively $k \to 1$, simpler coordinate systems emerge.

Table 1: Coordinate Systems on the Three-Dimensional Sphere

Coordinate System	Coordinates	Observables I_1, I_2
I. Cylindrical $\gamma \in [0, \pi/2]$ $\varphi_{1,2} \in [0, 2\pi)$	$s_1 = \cos\gamma \cos\varphi_1$ $s_2 = \cos\gamma \sin\varphi_1$ $s_3 = \sin\gamma \sin\varphi_2$ $s_4 = \sin\gamma \cos\varphi_2$	$I_1 = M_3^2$ $I_2 = L_3^2$
II. Sphero-Elliptic $\chi \in [0, \pi]$ $\alpha \in [-K, K]$ $\beta \in [-2K', 2K']$	$s_1 = \sin\chi \mathrm{sn}(\alpha, k)\mathrm{dn}(\beta, k')$ $s_2 = \sin\chi \mathrm{cn}(\alpha, k)\mathrm{cn}(\beta, k')$ $s_3 = \sin\chi \mathrm{dn}(\alpha, k)\mathrm{sn}(\beta, k')$ $s_4 = \cos\chi$	$I_1 = \mathbf{L}^2$ $I_2 = L_1^2 + k'^2 L_2^2$
III. Spherical $\chi \in [0, \pi]$ $\vartheta \in [0, \pi]$ $\varphi \in [0, 2\pi)$	$s_1 = \sin\chi \sin\vartheta \cos\varphi$ $s_2 = \sin\chi \sin\vartheta \sin\varphi$ $s_3 = \sin\chi \cos\vartheta$ $s_4 = \cos\chi$	$I_1 = \mathbf{L}^2$ $I_2 = L_3^2$
IV. Oblate Elliptic $\mu \in [0, 2K]$ $\nu \in [-K', K']$ $\varphi \in [0, 2\pi)$	$s_1 = \mathrm{sn}(\mu, k)\mathrm{dn}(\nu, k') \cos\varphi$ $s_2 = \mathrm{sn}(\mu, k)\mathrm{dn}(\nu, k') \sin\varphi$ $s_3 = \mathrm{cn}(\mu, k)\mathrm{cn}(\nu, k')$ $s_4 = \mathrm{dn}(\mu, k)\mathrm{sn}(\nu, k')$	$I_1 = (1-k^2)\mathbf{L}^2 - k^2 M_3^2$ $I_2 = L_3^2$
V. Prolate Elliptic $\mu \in [-K, K]$ $\nu \in [-K', K']$ $\varphi \in [0, 2\pi)$	$s_1 = \mathrm{cn}(\mu, k)\mathrm{cn}(\nu, k') \cos\varphi$ $s_2 = \mathrm{cn}(\mu, k)\mathrm{cn}(\nu, k') \sin\varphi$ $s_3 = \mathrm{sn}(\mu, k)\mathrm{dn}(\nu, k')$ $s_4 = \mathrm{dn}(\mu, k)\mathrm{sn}(\nu, k')$	$I_1 = \mathbf{L}^2 + k^2 M_3^2$ $I_2 = L_3^2$
VI. Ellipsoidal $a_1 < \varrho_1 < a_2 < \varrho_3$ $< a_3 < \varrho_4 < a_4$ $a_{1,2,3,4} \in \mathbb{R}$	$s_1^2 = \frac{(\varrho_1 - a_1)(\varrho_2 - a_1)(\varrho_3 - a_1)}{(a_4 - a_1)(b_3 - a_1)(a_2 - a_1)}$ $s_2^2 = \frac{(\varrho_1 - a_2)(\varrho_2 - a_2)(\varrho_3 - a_2)}{(a_4 - a_2)(a_3 - a_2)(a_1 - a_2)}$ $s_3^2 = \frac{(\varrho_1 - a_3)(\varrho_2 - a_3)(\varrho_3 - a_3)}{(a_1 - a_3)(a_2 - a_3)(a_4 - a_3)}$ $s_4^2 = \frac{(\varrho_1 - a_4)(\varrho_2 - a_4)(\varrho_3 - a_4)}{(a_1 - a_4)(a_2 - a_4)(a_3 - a_4)}$	$I_1 = (a_1 + a_4)L_1^2 + (a_2 + a_4)L_2^2 + (a_3 + a_4)L_3^2$ $+(a_2 + a_3)M_1^2 + (a_1 + a_3)M_2^2 + (a_1 + a_2)M_3^2$ $I_2 = a_1 a_4 L_1^2 + a_2 a_4 L_2^2 + a_3 a_4 L_3^2$ $+a_2 a_3 M_1^2 + a_1 a_3 M_2^2 + a_1 a_2 M_3^2$

In the limiting case $k \to 0$, i.e., $k' \to 1$ we obtain

$$\begin{aligned} &\mathrm{cn}(\mu, k) \to \cos\vartheta \ , && 0 \le \vartheta \le \pi \ , \\ &\mathrm{sn}(\nu, k') \to \tan\nu \equiv \cos\chi \ , && 0 \le \chi \le \pi \ . \end{aligned} \tag{2.7}$$

Therefore the oblate elliptic coordinate system yields in this limit spherical coordinates:

$$\left.\begin{aligned} s_1 &= \sin\chi \sin\vartheta \cos\varphi \ , \\ s_2 &= \sin\chi \sin\vartheta \sin\varphi \ , \\ s_3 &= \sin\chi \cos\vartheta \ , \\ s_4 &= \cos\chi \ . \end{aligned}\right\} \tag{2.8}$$

Alternatively we can consider $k \to 1$, i.e. $k' \to 0$, and we obtain for the Jacobi elliptic functions

$$\begin{aligned} &\mathrm{sn}(\mu, k) \to \tan\mu \equiv \sin\gamma \ , && 0 \le \gamma \le \pi \ , \\ &\mathrm{cn}(\nu, k') \to \sin\varphi_2 \ , && 0 \le \varphi_2 \le \pi \ . \end{aligned} \tag{2.9}$$

Hence oblate elliptic coordinates yield cylindrical coordinates on $S^{(3)}$ ($\varphi \equiv \varphi_1$)

$$\left.\begin{aligned} s_1 &= \sin\gamma\cos\varphi_1 \ , \\ s_2 &= \sin\gamma\sin\varphi_1 \ , \\ s_3 &= \cos\gamma\sin\varphi_2 \ , \\ s_4 &= \cos\gamma\cos\varphi_2 \ . \end{aligned}\right\} \tag{2.10}$$

Finally, the Laplace operator in oblate elliptic coordinates is given by

$$\Delta = \frac{1}{k^2\mathrm{cn}^2\mu + k'^2\mathrm{cn}^2\nu}\left(\frac{\partial^2}{\partial\mu^2} + \frac{\mathrm{cn}\mu\mathrm{dn}\mu}{\mathrm{sn}\mu}\frac{\partial}{\partial\mu} + \frac{\partial^2}{\partial\nu^2} - k'^2\frac{\mathrm{sn}\nu\mathrm{cn}\nu}{\mathrm{dn}\nu}\frac{\partial}{\partial\nu}\right) + \frac{1}{\mathrm{sn}^2\mu\mathrm{dn}^2\nu}\frac{\partial^2}{\partial\varphi^2} \ . \tag{2.11}$$

Prolate Elliptic Coordinates. Next, we consider prolate elliptic coordinates which have in terms of the Jacobi elliptic functions the form

$$\left.\begin{aligned} s_1 &= \mathrm{cn}(\mu,k)\mathrm{cn}(\nu,k')\cos\varphi \ , \\ s_2 &= \mathrm{cn}(\mu,k)\mathrm{cn}(\nu,k')\sin\varphi \ , \\ s_3 &= \mathrm{sn}(\mu,k)\mathrm{dn}(\nu,k') \ , \\ s_4 &= \mathrm{dn}(\mu,k)\mathrm{sn}(\nu,k') \ . \end{aligned}\right\} \tag{2.12}$$

Here are $-K \leq \mu \leq K$, $-K' \leq \nu \leq K'$, $0 \leq \varphi \leq 2\pi$. In comparison to the oblate elliptic coordinate system, the prolate elliptic system has for the two limiting cases $k \to 0$ and $k \to 1$, respectively $k' \to 1$ and $k' \to 0$, the same limiting coordinate system, namely the spherical system. In the limit $k \to 0$ we find the coordinates (**??**). For the limit $k \to 1$ we find

$$\begin{aligned} &\mathrm{sn}(\mu,k) \to \tan\mu \equiv \cos\chi \ , \quad 0 \leq \chi \leq \pi \ , \\ &\mathrm{sn}(\nu,k') \to \cos\vartheta \ , \qquad\quad 0 \leq \vartheta \leq \pi \ , \end{aligned} \tag{2.13}$$

and therefore we obtain the coordinate system representation

$$\left.\begin{aligned} s_1 &= \sin\chi\sin\vartheta\cos\varphi \ , \\ s_2 &= \sin\chi\sin\vartheta\sin\varphi \ , \\ s_3 &= \cos\chi \ , \\ s_4 &= \sin\chi\cos\vartheta \ , \end{aligned}\right\} \tag{2.14}$$

where in comparison to (**??**) the coordinates s_3 and s_4 are interchanged.

We finally note that both elliptic coordinate systems can be transformed into each other by the substitution

$$k \to \frac{\mathrm{i}k}{k'} \ , \qquad k' \to \frac{1}{k'} \ , \qquad \mu \to K - \mu \ . \tag{2.15}$$

The Laplacian in prolate elliptic coordinates has the form

$$\Delta = \frac{1}{k^2\mathrm{cn}^2\mu + k'^2\mathrm{cn}^2\nu}\left(\frac{\partial^2}{\partial\mu^2} - \frac{\mathrm{sn}\mu\mathrm{dn}\mu}{\mathrm{cn}\mu}\frac{\partial}{\partial\mu} + \frac{\partial^2}{\partial\nu^2} - \frac{\mathrm{sn}\nu\mathrm{dn}\nu}{\mathrm{cn}\nu}\frac{\partial}{\partial\nu}\right) + \frac{1}{\mathrm{cn}^2\mu\mathrm{cn}^2\nu}\frac{\partial^2}{\partial\varphi^2} \ . \tag{2.16}$$

2.2 The Observables.

We consider the Schrödinger equation of the free motion on $S^{(3)}$ in oblate spheroidal coordinates (??). We make the following Ansatz for the wave-functions

$$\Psi(\mu,\nu,\varphi;k) = \psi_1(\mu;k)\psi_2(\nu;k)\frac{\mathrm{e}^{\mathrm{i}m\varphi}}{\sqrt{2\pi}} \ , \qquad m \in \mathbb{Z} \ , \tag{2.17}$$

and obtain the two coupled ordinary differential equations

$$\frac{\mathrm{d}^2\psi_1}{\mathrm{d}\mu^2} + \frac{\mathrm{cn}\mu\mathrm{dn}\mu}{\mathrm{sn}\mu}\frac{\mathrm{d}\psi_1}{\mathrm{d}\mu} - \left[k^2J(J+2)\mathrm{sn}^2\mu + \frac{m^2}{\mathrm{sn}^2\mu}\right]\psi_1 \ = \ -\lambda_q(k)\psi_1 \ , \tag{2.18}$$

$$\frac{\mathrm{d}^2\psi_2}{\mathrm{d}\nu^2} - k'^2\frac{\mathrm{sn}\nu\mathrm{cn}\nu}{\mathrm{dn}\nu}\frac{\mathrm{d}\psi_2}{\mathrm{d}\nu} + \left[J(J+2)\mathrm{dn}^2\nu + k^2\frac{m^2}{\mathrm{dn}^2\nu}\right]\psi_2 \ = \ +\lambda_q(k)\psi_2 \ . \tag{2.19}$$

The quantum number q labels the eigenvalues of the observables of the oblate elliptic system. In order to determine the operator Λ corresponding to the eigenvalue λ we eliminate the quantum number J from the equations (??, ??). This yields

$$\begin{aligned}\Lambda \ = \ \frac{1}{k^2\mathrm{sn}^2\mu - \mathrm{dn}^2\nu}\Bigg(&\mathrm{dn}^2\nu\frac{\partial^2}{\partial\mu^2} + k^2\mathrm{sn}^2\mu\frac{\partial^2}{\partial\nu^2} + \frac{\mathrm{cn}\mu\mathrm{dn}\mu\mathrm{dn}^2\nu}{\mathrm{sn}\mu}\frac{\partial}{\partial\mu} - k^2k'^2\frac{\mathrm{sn}\nu\mathrm{cn}\nu\mathrm{sn}^2\mu}{\mathrm{dn}\nu}\frac{\partial}{\partial\nu}\Bigg) \\ &- \frac{k^2\mathrm{sn}^2\mu + \mathrm{dn}^2\nu}{\mathrm{sn}^2\mu\mathrm{dn}^2\nu}\frac{\partial^2}{\partial\varphi^2} \ .\end{aligned} \tag{2.20}$$

If we reinsert the coordinate $\mathbf{s}$ in (??) it follows that the operator Λ can be represented as

$$\hbar^2\Lambda = (1-k^2)\mathbf{L}^2 + k^2(L_3^2 - M_3^2) + k^2\mathcal{L}_1 \ . \tag{2.21}$$

Here is $\mathcal{L}_1 = -\hbar^2\Delta$. We can introduce another operator $\mathcal{L}_2$ by defining

$$\begin{aligned}\hbar^2\mathcal{L}_2^{oblate} \ &= \ \frac{1}{1-k^2}\left[\hbar^2\Lambda - k^2\mathcal{L}_1 + k^2L_3^2\right] \\ &= \ \mathbf{L}^2 - dM_3^2 \ , \qquad d = \frac{k^2}{1-k^2} \in [0,\infty) \ .\end{aligned} \tag{2.22}$$

After the transformation (??) we obtain for the prolate coordinate system the corresponding operator

$$\hbar^2\mathcal{L}_2^{prolate} = \mathbf{L}^2 + k^2M_3^2 \ . \tag{2.23}$$

Both operators (??, ??) can be cast into one operator yielding

$$\hbar^2\mathcal{L}_2 = \mathbf{L}^2 - aM_3^2 \ , \tag{2.24}$$

where now $a \in [-1,\infty)$. For a positive we have oblate elliptic coordinates, for $a \in [-1,0]$ prolate elliptic coordinates. Therefore the operator $\mathcal{L}_2 = \mathcal{L}_2(a)$ is a continuous function of the parameter a. It is possible to analytically continue the dependence on a to the entire real line by means of the mapping $k^2 \to 1/k^2$ which maps the domain $[0,\infty) \to (-\infty,-1]$.

The Laplace operator $\mathcal{L}_1 = -\hbar^2\Delta$, the elliptic operator $\mathcal{L}_2$, $\mathcal{L}_3 = -\hbar^2\partial^2/\partial\varphi^2$ and the parity operators Π_3 : $\Pi_3(s_3) = -s_3$, Π_4 : $\Pi_4(s_4) = -s_4$, form a complete set of commuting operators and fix the elliptic basis uniquely.

The limiting cases of $a \to \infty$, $a \to 0$, and $a \to -1$ give us the corresponding observables for the cylindrical and hyper-spherical systems, i.e.,

$$\left.\begin{array}{rcl} \lim\limits_{a\to 0} \mathcal{L}_2 & = & \mathbf{L}^2 \ , \\ \lim\limits_{a\to -1} \mathcal{L}_2 & = & \mathbf{L}^2 + M_3^2 \ , \\ \lim\limits_{a\to \infty} \dfrac{\mathcal{L}_2}{a} & = & -M_3^2 \ . \end{array}\right\} \tag{2.25}$$

3 The Ellipso-Cylindrical Bases.

3.1 Expansion of the Elliptic Basis with Respect to the Cylindrical Basis.

Let us construct the elliptic basis in terms of the cylindrical basis according to

$$\Psi_{Jqm}(\mu,\nu,\varphi;a) = \sum_{m_2=-(J-|m|)}^{J-|m|} T_{Jqm_2}(a)\Psi_{Jmm_2}(\vartheta,\varphi_1,\varphi_2) \ , \tag{3.1}$$

with the wave-functions $\Psi_{Jmm_2}(\vartheta,\varphi_1,\varphi_2)$ as defined in (**??**), where m_2 is even or odd depending on the parity of $J-|m|$, and we use $m \equiv m_1$ in the cylindrical basis. Inserting the expansion (**??**) into the operator equation

$$\mathcal{L}_2\Psi_{Jqm} = \lambda_q\hbar^2\Psi_{Jqm} \ , \tag{3.2}$$

taking into account the orthogonality of the cylindrical wave-functions Ψ_{Jmm_2}, and using the formulæ (**??**) and (**??**), we obtain

$$(\lambda_q + am_2^2)T_{Jqm_2}(a) = \frac{1}{\hbar^2}\sum_{m_2'=-(J-|m|)}^{J-|m|} T_{Jqm_2'}(a)(\mathbf{L}^2)_{m_2m_2'} \ , \tag{3.3}$$

where

$$(\mathbf{L}^2)_{m_2m_2'} = \int d\mathbf{\Omega}\Psi^*_{Jmm_2}(\gamma,\varphi_1,\varphi_2)\mathbf{L}^2\Psi_{Jmm_2'}(\gamma,\varphi_1,\varphi_2) \ , \tag{3.4}$$

and $d\mathbf{\Omega}$ denotes the invariant volume element on $S^{(3)}$. Using, e.g.,

$$\mathbf{L}^2 = \frac{1}{2}(L_+L_- + L_-L_+) + L_3^2 \ , \tag{3.5}$$

and the formulæ in Appendix A, we have the following expression of the matrix elements

$$\frac{1}{\hbar^2}(\mathbf{L}^2)_{m_2m_2'} = -A_{m_2-2}\delta_{m_2,m_2'-2} + B_{m_2}\delta_{m_2m_2'} - A_{m_2}\delta_{m_2,m_2'+2} \ , \tag{3.6}$$

with the coefficients A_{m_2} and B_{m_2} given by

$$A_{m_2} = \frac{1}{4}\sqrt{(J-m_2-m)(J-m_2+m)(J+m_2+m+2)(J+m_2-m+2)} \ , \tag{3.7}$$

$$B_{m_2} = \frac{1}{2}[J(J+2) + (m^2-m_2^2)] \ . \tag{3.8}$$

The coefficients A_{m_2} and B_{m_2} satisfy the recurrence relations

$$A_{-m_2} = A_{m_2-2} \ , \qquad B_{-m_2} = B_{m_2} \ . \tag{3.9}$$

Inserting (**??**) into (**??**) we obtain a three-term recurrence relation for the coefficients $T_{qm_2} \equiv T_{Jqm_2}(a)$:

$$A_{m_2} T_{q,m_2+2} + \Big(am_2^2 + \lambda_q - B_{m_2}\Big) T_{qm_2} + A_{m_2-2} T_{q,m_2-2} = 0 \ . \tag{3.10}$$

These recurrence relations can be rewritten into the form of two sets of homogeneous linear equations, one for each parity class, depending on whether the quantum number m_2 is even or odd, respectively. For $J - |m| = 2k$ we have on the one hand

$$\left.\begin{aligned}
0 &= (B_{2k} - 4ak^2 - \lambda_q) T_{q,2k} - A_{2k-2} T_{q,2k-2} \ , \\
0 &= A_{2k-2} T_{q,2k} - \Big(B_{2k-2} - 4a(k-1)^2 - \lambda_q\Big) T_{q,2k-2} + A_{2k-4} T_{q,2k-4} \ , \\
&\dots \\
0 &= A_0 T_0 - (B_0 - \lambda_q) T_0 + A_{-2} T_{q,-2} \ , \\
&\dots\dots\dots \\
0 &= A_{-2k} T_{q,-2k+2} - (B_{-2k} - 4ak^2 - \lambda_q) T_{q,-2k} \ ;
\end{aligned}\right\} \tag{3.11}$$

for $J - |m| = 2k + 1$ we have on the other

$$\left.\begin{aligned}
0 &= \Big(B_{2k+1} - (2k+1)^2 a - \lambda_q\Big) T_{q,2k+1} - A_{2k-1} T_{q,2k-1} \ , \\
0 &= A_{2k-1} T_{q,2k+1} - \Big(B_{2k-1} - (2k-1)^2 a - \lambda_q\Big) T_{q,2k-1} + A_{2k-3} T_{q,2k-3} \ , \\
&\dots \\
0 &= A_1 T_3 - (B_1 - a - \lambda_q) T_1 + A_{-1} T_{q,-1} \ , \\
0 &= A_{-1} T_{q1} - (B_{-1} - a - \lambda_q) T_{q,-1} + A_{-3} T_{q,-3} \ , \\
&\dots \\
0 &= A_{-2k-1} T_{q,-2k+1} - \Big(B_{-2k-2} - (2k-1)^2 a - \lambda_q\Big) T_{q,-2k-1} \ .
\end{aligned}\right\} \tag{3.12}$$

The homogeneous systems (**??**, **??**) satisfy the symmetry conditions

$$T_{q,-k} = p T_{qk} \ , \qquad p = \pm 1 \ . \tag{3.13}$$

We can determine the classes $(r,p) = (\pm,\pm)$, where $p = \pm 1$ and $r = (-1)^{|m_2|} = (-1)^{J-|m|}$, and we see that the recurrence relations for the T_{qm_2} can be put into four classes:

i) $J - |m| = 2k, m_2 = 2n$:

$$\begin{aligned}
&\sqrt{(k-n)(k+n+1)(J-k-n)(J-k+n+1)}\,(1 \pm \delta_{n0}) T^{(+,\pm)}_{q,2n+2} \\
&\qquad + \Big\{\lambda_q + 4an^2 - \tfrac{1}{2}[J(J+2) + (J-2k-2n)(J-2k+2n)]\Big\} T^{(+,\pm)}_{q,2n} \\
&\qquad + \sqrt{(k+n)(k-n+1)(J-k+n)(J-k-n+1)}\, T^{(+,\pm)}_{q,2n-2} = 0 \ ,
\end{aligned} \tag{3.14}$$

ii) $J - |m| = 2k + 1, m_2 = 2n + 1$:

$$\begin{aligned}
&\sqrt{(k-n)(k+n+2)(J-k-n-1)(J-k+n+1)}\, T^{(-,\pm)}_{q,2n+3} \\
&\qquad + \Big\{\lambda_q + a(2n+1)^2 - [J(J+2) + (2k+1)J + 2(k-n)(k-n-1)]\Big\} T^{(-,\pm)}_{q,2n+1} \\
&\qquad + \sqrt{(k-n+1)(k+n+1)(J-k-n)(J-k+n)}\, T^{(-,\pm)}_{q,2n-1} \\
&\qquad \pm (J-k)(k+1)\delta_{n0} T^{(-,\pm)}_{q,1} = 0 \ .
\end{aligned} \tag{3.15}$$

The homogeneous equations corresponding to the three-term recurrence relations (**??**, **??**) have non-trivial solutions if their determinants are equal to zero. These conditions lead to

four algebraic equations and give us the eigenvalue $\lambda_q \equiv \lambda^{(r,p)}_{Jqm}(a)$ of the elliptic observable Λ. For each class we have for the quantum number q

$$J-|m|=2k: \qquad \left\{ \begin{array}{lll} q=0,2,\ldots,2k; & k+1 & \text{states of class } (+,+)\ , \\ q=1,3,\ldots,2k-1; & k & \text{states of class } (+,-)\ , \end{array} \right. \tag{3.16}$$

$$J-|m|=2k+1: \qquad \left\{ \begin{array}{lll} q=0,2,\ldots,2k; & k+1 & \text{states of class } (-,+)\ , \\ q=1,3,\ldots,2k+1; & k+1 & \text{states of class } (-,-)\ . \end{array} \right. \tag{3.17}$$

The quantum number q labels the elliptic observable and counts according to the oscillation theorem the number of zeros of the "angular" elliptic wave-function $\psi_1(\mu,k)$, and the multiplicity of the degeneracy for fixed J is $(J+1)^2$.

Therefore, we can write the interbasis expansions of the elliptic wave-functions $\Psi^{(r,p)}_{Jqm}$ in the following form

$$\Psi^{(+,+)}_{Jqm}(\mu,\nu,\varphi;a) = \sum_{n=0,1}^{(J-|m|)/2} (1-\tfrac{1}{2}\delta_{n0}) T^{(+,+)}_{q,2n} \Psi^{(+,+)}_{J,m,2n}(\gamma,\varphi_1,\varphi_2)\ , \tag{3.18}$$

$$\Psi^{(+,-)}_{Jqm}(\mu,\nu,\varphi;a) = \sum_{n=1,2}^{(J-|m|)/2} T^{(+,-)}_{q,2n} \Psi^{(+,-)}_{J,m,2n}(\gamma,\varphi_1,\varphi_2)\ , \tag{3.19}$$

$$\Psi^{(-,+)}_{Jqm}(\mu,\nu,\varphi;a) = \sum_{n=0,1}^{(J-|m|-1)/2} T^{(-,+)}_{q,2n+1} \Psi^{(-,+)}_{J,m,2n+1}(\gamma,\varphi_1,\varphi_2)\ , \tag{3.20}$$

$$\Psi^{(-,-)}_{Jqm}(\mu,\nu,\varphi;a) = \sum_{n=0,1}^{(J-|m|-1)/2} T^{(-,-)}_{q,2n+1} \Psi^{(-,-)}_{J,m,2n+1}(\gamma,\varphi_1,\varphi_2)\ , \tag{3.21}$$

where

$$\Psi^{(+,\pm)}_{J,m,2n}(\gamma,\varphi_1,\varphi_2) = \frac{1}{\sqrt{2}}\Big[\Psi_{J,m,2n}(\gamma,\varphi_1,\varphi_2) \pm \Psi_{J,m,-2n}(\gamma,\varphi_1,\varphi_2)\Big]\ , \tag{3.22}$$

$$\Psi^{(-,\pm)}_{J,m,2n+1}(\gamma,\varphi_1,\varphi_2) = \frac{1}{\sqrt{2}}\Big[\Psi_{J,m,2n+1}(\gamma,\varphi_1,\varphi_2) \pm \Psi_{J,m,-2n-1}(\gamma,\varphi_1,\varphi_2)\Big]\ . \tag{3.23}$$

The ellipso-cylindrical wave-functions are subject to the orthogonality relation

$$\int d\boldsymbol{\Omega}\, \Psi^{(r,p)*}_{Jqm}(\mu,\nu,\varphi;a) \Psi^{(r,p)}_{J'q'm'}(\mu,\nu,\varphi;a) = \delta_{JJ'}\delta_{qq'}\delta_{mm'}\delta_{rr'}\delta_{pp'}\ , \tag{3.24}$$

and the following orthonormalization conditions hold

$$\left.\begin{array}{l} \displaystyle\sum_{n=0,1}^{(J-|m|)/2} \Big(1-\tfrac{1}{2}\delta_{n0}\Big) T^{(+,+)*}_{q,2n} T^{(+,+)}_{q',2n} = \sum_{n=1,2}^{(J-|m|)/2} T^{(+,-)*}_{q,2n} T^{(+,-)}_{q',2n} \\ = \displaystyle\sum_{n=0,1}^{(J-|m|-1)/2} T^{(-,+)*}_{q,2n} T^{(-,+)}_{q',2n} = \sum_{n=0,1}^{(J-|m|-1)/2} T^{(-,-)*}_{q,2n} T^{(-,-)}_{q',2n} = \delta_{q,q'}\ . \end{array}\right\} \tag{3.25}$$

The expansions (??–??) can be inverted due to their unitarity. Using for instance the orthogonality conditions (??) one shows that

$$\Big(T^{(r,p)}_{qm_2}\Big)^{-1} = T^{(r,p)*}_{qm_2}\ . \tag{3.26}$$

Hence we can write the inverse expansion of the cylindrical basis over the elliptic basis in the following form

$$\Psi^{(r,p)}_{Jmm_2}(\vartheta,\varphi_1,\varphi_2) = \sum_q T^{(r,p)*}_{qm_2}(a) \Psi^{(r,p)}_{Jqm}(\mu,\nu,\varphi;a)\ , \tag{3.27}$$

Table 2: Coefficients for Interbasis Expansion (??).

Coefficients for $(r,p)=(+,+)$

J	m	n	$T^{(+,+)}_{q,2n}$	$\lambda_q^{(+,+)}$
0	0	0	$1/\sqrt{2}$	$\lambda_q^{(+,+)}=0$
1	±1	0	1	$\lambda_q^{(+,+)}=2$
2	0	0	$\sqrt{\dfrac{\lambda_q^{(+,+)}+4a-2}{\lambda_q^{(+,+)}+2a-3}}$	$[2-4a-\lambda_q^{(+,+)}][4-\lambda_q^{(+,+)}]=8$
2	0	1	$-\sqrt{\dfrac{\lambda_q^{(+,+)}-4}{2(\lambda_q^{(+,+)}+2a-3)}}$	$[2-4a-\lambda_q^{(+,+)}][4-\lambda_q^{(+,+)}]=8$
2	±2	0	$1/\sqrt{2}$	$\lambda_q^{(+,+)}=6$
3	±1	0	$\sqrt{\dfrac{\lambda_q^{(+,+)}+4a-6}{\lambda_q^{(+,+)}+2a-7}}$	$[6-4a-\lambda_q^{(+,+)}][8-\lambda_q^{(+,+)}]=24$
3	±1	1	$-\sqrt{\dfrac{\lambda_q^{(+,+)}-8}{2(\lambda_q^{(+,+)}+2a-7)}}$	$[6-4a-\lambda_q^{(+,+)}][8-\lambda_q^{(+,+)}]=24$
3	±3	0	$1/\sqrt{2}$	$\lambda_q^{(+,+)}=12$

Coefficients for $(r,p)=(+,-)$

J	m	n	$T^{(+,-)}_{q,2n}$	$\lambda_q^{(+,-)}$
2	0	1	1	$\lambda_q^{(+,-)}=2-4a$
3	±1	1	1	$\lambda_q^{(+,-)}=6-4a$

Coefficients for $(r,p)=(-,+)$

J	m	n	$T^{(-,+)}_{q,2n+1}$	$\lambda_q^{(-,+)}$
1	0	0	1	$\lambda_q^{(-,+)}=-a$
2	±1	0	1	$\lambda_q^{(-,+)}=2-a$
3	0	0	$\sqrt{\dfrac{\lambda_q^{(-,+)}+9a-3}{2(\lambda_q^{(-,+)}+5a-3)}}$	$[3-9a-\lambda_q^{(-,+)}][3-a-\lambda_q^{(-,+)}]=9$
3	0	1	$-\sqrt{\dfrac{\lambda_q^{(-,+)}+a-3}{2(\lambda_q^{(-,+)}+5a-3)}}$	$[3-9a-\lambda_q^{(-,+)}][3-a-\lambda_q^{(-,+)}]=9$
3	±2	0	1	$\lambda_q^{(-,+)}=6-a$

Coefficients for $(r,p)=(-,-)$

J	m	n	$T^{(-,-)}_{q,2n+1}$	$\lambda_q^{(-,-)}$
1	0	0	1	$\lambda_q^{(-,-)}=2-a$
2	±1	0	1	$\lambda_q^{(-,-)}=6-a$
3	0	0	$\sqrt{\dfrac{\lambda_q^{(-,-)}+9a-3}{2(\lambda_q^{(-,-)}+5a-7)}}$	$[3-9a-\lambda_q^{(-,-)}][11-a-\lambda_q^{(-,-)}]=9$
3	0	1	$-\sqrt{\dfrac{\lambda_q^{(-,-)}+a-11}{2(\lambda_q^{(-,-)}+5a-7)}}$	$[3-9a-\lambda_q^{(-,-)}][11-a-\lambda_q^{(-,-)}]=9$
3	±2	0	1	$\lambda_q^{(-,-)}=12-a$

where q is given by (??, ??) and it follows by taking into account the proper parity classes

$$\left.\begin{aligned}&\sum_{q=0,2}^{J-|m|}\left(1-\tfrac{1}{2}\delta_{n0}\right)T^{(+,+)*}_{q,2n}T^{(+,+)}_{q,2n'}=\sum_{q=1,3}^{J-|m|-1}T^{(+,-)*}_{q,2n}T^{(+,-)}_{q,2n'}\\&=\sum_{q=0,2}^{J-|m|}T^{(-,+)*}_{q,2n+1}T^{(-,+)}_{q,2n'+1}=\sum_{q=1,3}^{J-|m|-1}T^{(-,-)*}_{q,2n+1}T^{(-,-)}_{q,2n'+1}=\delta_{n,n'}\ .\end{aligned}\right\}\tag{3.28}$$

Some results for the coefficients $T^{(r,p)}_{q,2n}$ and $\lambda^{(r,p)}_q$ are listed in table 2.

3.2 Expansion of the Elliptic Basis with Respect to the Hyper-Spherical Basis.

Let us consider the interbasis expansion of the elliptic basis with respect to the hyperspherical basis

$$\Psi_{Jqm}(\mu,\nu,\varphi;a)=\sum_{l=|m|}^{J}N_{Jql}(a)\Psi_{Jlm}(\chi,\vartheta,\varphi)\ ,\tag{3.29}$$

with the wave-functions $\Psi_{Jlm}(\chi,\vartheta,\varphi)$ as defined in (??). Performing the same manipulations as in the previous subsection we obtain

$$\frac{1}{a}[-\lambda_q+l(l+1)]N_{ql}(a)=\frac{1}{\hbar^2}\sum_{l'=|m|}^{J}N_{Jql'}(a)(M^2_3)_{ll'}\ ,\tag{3.30}$$

where

$$(M^2_3)_{ll'}=\int d\mathbf{\Omega}\Psi^*_{Jlm}(\chi,\vartheta,\varphi)M^2_3\Psi_{Jl'm}(\chi,\vartheta,\varphi)\ .\tag{3.31}$$

Using the formulæ of Appendix B we have

$$\frac{1}{\hbar^2}(M^2_3)_{ll'}=-C_{l-2}\delta_{l,l'-2}+E_l\delta_{ll'}-C_l\delta_{l,l'+2}\ ,\tag{3.32}$$

with the coefficients C_l and E_l given by

$$\begin{aligned}C_l=&\sqrt{\frac{(J-l-1)(J-l)(J+l+2)(J+l+3)}{(2l+1)(2l+3)^2(2l+5)}}\\&\times\sqrt{(l-|m|+1)(l-|m|+2)(l+|m|+1)(l+|m|+2)}\ ,\end{aligned}\tag{3.33}$$

$$\begin{aligned}E_l=&\frac{(J-l+1)(J+l+1)(l-|m|)(l+|m|)}{(2l-1)(2l+1)}\\&+\frac{(J-l)(J+l+2)(l-|m|+1)(l+|m|+1)}{(2l+1)(2l+3)}\ .\end{aligned}\tag{3.34}$$

Inserting (??) in (??) we obtain a three-term recurrence relation for the interbasis coefficients $N_{ql}\equiv N_{Jql}(a)$:

$$C_lN_{q,l+2}+\left\{\frac{1}{a}[l(l+1)-\lambda_q]-E_l\right\}N_{ql}+C_{l-2}N_{q,l-2}=0\ .\tag{3.35}$$

¿From the three-term recurrence relation (**??**) we see that the index l increases by $\Delta l = 2$. This means that we get four classes of coefficients which can again be classified according to their parity. The first class emerges if the index l starts with $l = |m|$, the second for $l = |m| + 1$. Their respective parity is given by $p = (-1)^{l-|m|}$. The third and fourth class are obtained if l is between $|m|$ and J, respectively $|m| + 1$ and $J - 1$. Their respective parity is given by $r = (-1)^{J-|m|}$. Therefore the interbasis expansion (**??**) splits into four contributions with parity classes $p = (-1)^{l-|m|}$ and $r = (-1)^{J-|m|}$, and they have the form

$$\Psi^{(+,+)}_{Jqm}(\mu,\nu,\varphi;a) = \sum_{l=|m|,|m|+2}^{J} N^{(+,+)}_{ql}\Psi_{Jlm}(\chi,\vartheta,\varphi) \ , \tag{3.36}$$

$$\Psi^{(+,-)}_{Jqm}(\mu,\nu,\varphi;a) = \sum_{l=|m|+1,|m|+3}^{J-1} N^{(+,-)}_{ql}\Psi_{Jlm}(\chi,\vartheta,\varphi) \ , \tag{3.37}$$

$$\Psi^{(-,+)}_{Jqm}(\mu,\nu,\varphi;a) = \sum_{l=|m|,|m|+2}^{J-1} N^{(-,+)}_{ql}\Psi_{Jlm}(\chi,\vartheta,\varphi) \ , \tag{3.38}$$

$$\Psi^{(-,-)}_{Jqm}(\mu,\nu,\varphi;a) = \sum_{l=|m|+1,|m|+3}^{J} N^{(-,-)}_{ql}\Psi_{Jlm}(\chi,\vartheta,\varphi) \ . \tag{3.39}$$

The interbasis coefficients N_{ql} satisfy the following four sets of homogeneous equations, and we have listed in table 3 some simple examples for small values of J, l, m.

1. $\underline{(r,p) = (+,+)}$: $J - |m|$ even, $l - |m|$ even:

$$\left.\begin{aligned}
0 &= \left[\frac{|m|(|m|+1)-\lambda_q}{a} - E_{|m|}\right] N_{q,|m|} + C_{|m|}N_{q,|m|+2} \ , \\
0 &= C_{|m|}N_{q,|m|} \\
&\quad + \left[\frac{(|m|+2)(|m|+3)-\lambda_q}{a} - E_{|m|+2}\right] N_{q,|m|+2} + C_{|m|+2}N_{q,|m|+4} \ , \\
&\cdots \\
0 &= \left[\frac{J(J+1)-\lambda_q}{a} - E_J\right] N_{qJ} + C_{J-2}N_{q,J-2} \ .
\end{aligned}\right\} \tag{3.40}$$

2. $\underline{(r,p) = (+,-)}$: $J - |m|$ even, $l - |m|$ odd:

$$\left.\begin{aligned}
0 &= \left[\frac{(|m|+1)(|m|+2)-\lambda_q}{a} - E_{|m|+1}\right] N_{q,|m|+1} + C_{|m|+1}N_{q,|m|+3} \ , \\
0 &= C_{|m|+1}N_{q,|m|+1} \\
&\quad + \left[\frac{(|m|+3)(|m|+4)-\lambda_q}{a} - E_{|m|+5}\right] N_{q,|m|+3} + C_{|m|+3}N_{q,|m|+5} \ , \\
&\cdots \\
0 &= C_{J-3}N_{q,-3} + \left[\frac{(J-1)J-\lambda_q}{a} - E_{J-1}\right] N_{q,-1} \ .
\end{aligned}\right\} \tag{3.41}$$

Table 3: Coefficients for Interbasis Expansion (??).

Coefficients for $(r,p)=(+,+)$

J	m	l	$N_{ql}^{(+,+)}$	$\lambda_q^{(+,+)}$
0	0	0	1	$\lambda_q^{(+,+)}=0$
1	±1	1	1	$\lambda_q^{(+,+)}=2$
2	0	0	$\sqrt{\frac{\lambda_q^{(+,+)}+4a/3-6}{2(\lambda_q^{(+,+)}+2a-3)}}$	$[6-\frac{4}{3}a-\lambda_q^{(+,+)}][\frac{8}{3}a+\lambda_q^{(+,+)}]=-\frac{32}{9}a^2$
2	0	2	$\sqrt{\frac{\lambda_q^{(+,+)}+8a/3}{2(\lambda_q^{(+,+)}+2a-3)}}$	$[6-\frac{4}{3}a-\lambda_q^{(+,+)}][\frac{8}{3}a+\lambda_q^{(+,+)}]=-\frac{32}{9}a^2$
2	±2	2	1	$\lambda_q^{(+,+)}=6$
3	±1	1	$\sqrt{\frac{\lambda_q^{(+,+)}+8a/5-12}{2(\lambda_q^{(+,+)}+2a-7)}}$	$[2-\frac{12}{5}a-\lambda_q^{(+,+)}][12-\frac{8}{5}a-\lambda_q^{(+,+)}]=\frac{96}{25}a^2$
3	±1	3	$\sqrt{\frac{\lambda_q^{(+,+)}+12a/5-2}{2(\lambda_q^{(+,+)}+2a-7)}}$	$[2-\frac{12}{5}a-\lambda_q^{(+,+)}][12-\frac{8}{5}a-\lambda_q^{(+,+)}]=\frac{96}{25}a^2$
3	±3	3	1	$\lambda_q^{(+,+)}=12$

Coefficients for $(r,p)=(+,-)$

J	m	l	$N_{ql}^{(+,-)}$	$\lambda_q^{(+,-)}$
2	0	1	1	$\lambda_q^{(+,-)}=2-4a$
3	±1	2	1	$\lambda_q^{(+,-)}=6-4a$

Coefficients for $(r,p)=(-,+)$

J	m	l	$N_{ql}^{(-,+)}$	$\lambda_q^{(-,+)}$
1	0	0	1	$\lambda_q^{(-,+)}=-a$
2	±1	1	1	$\lambda_q^{(-,+)}=2-a$
3	0	0	$\sqrt{\frac{\lambda_q^{(-,+)}+5a}{2(\lambda_q^{(-,+)}+5a-3)}}$	$[\lambda_q^{(-,+)}+5a][-6+5a+\lambda_q^{(-,+)}]=16a^2$
3	0	2	$\sqrt{\frac{\lambda_q^{(-,+)}+5a-6}{2(\lambda_q^{(-,+)}+5a-3)}}$	$[\lambda_q^{(-,+)}+5a][-6+5a+\lambda_q^{(-,+)}]=16a^2$
3	±2	2	1	$\lambda_q^{(-,+)}=6-a$

Coefficients for $(r,p)=(-,-)$

J	m	l	$N_{ql}^{(-,-)}$	$\lambda_q^{(-,-)}$
1	0	1	1	$\lambda_q^{(-,-)}=2-a$
2	±1	2	1	$\lambda_q^{(-,-)}=6-a$
3	0	1	$\sqrt{\frac{\lambda_q^{(-,-)}+41a/5-2}{2(\lambda_q^{(-,-)}+5a-7)}}$	$[\lambda_q^{(-,-)}-2+\frac{41}{5}a][\lambda_q^{(-,-)}-12+\frac{9}{5}a]=\frac{144}{25}a^2$
3	0	3	$\sqrt{\frac{\lambda_q^{(-,-)}+9a/5-12}{2(\lambda_q^{(-,-)}+5a-7)}}$	$[\lambda_q^{(-,-)}+\frac{41}{5}a-2][\lambda_q^{(-,-)}+\frac{9}{5}a-12]=\frac{144}{25}a^2$
3	±2	3	1	$\lambda_q^{(-,-)}=12-a$

3. $\underline{(r,p)=(-,+)}$: $J-|m|$ odd, $l-|m|$ even:

$$\left.\begin{aligned}
0 &= \left[\frac{|m|(|m|+1)-\lambda_q}{a}-E_{|m|}\right]N_{q,|m|}+C_{|m|}N_{q,|m|+2} \quad , \\
0 &= C_{|m|}N_{q,|m|} \\
&\quad +\left[\frac{(|m|+2)(|m|+3)-\lambda_q}{a}-E_{|m|+2}\right]N_{q,|m|+2}+C_{|m|+2}N_{q,|m|+4} \quad , \\
&\cdots \\
0 &= C_{J-3}N_{q,-3}+\left[\frac{J(J-1)-\lambda_q}{a}-E_{J-1}\right]N_{q,-1} \quad .
\end{aligned}\right\} \tag{3.42}$$

4. $\underline{(r,p)=(-,-)}$: $J-|m|$ odd, $l-|m|$ odd:

$$\left.\begin{aligned}
0 &= \left[\frac{(|m|+1)(|m|+2)-\lambda_q}{a}-E_{|m|+1}\right]N_{q,|m|+1}+C_{|m|+1}N_{q,|m|+3} \quad , \\
0 &= C_{|m|+1}N_{q,|m|+1} \\
&\quad +\left[\frac{(|m|+3)(|m|+4)-\lambda_q}{a}-E_{|m|+3}\right]N_{q,|m|+3}+C_{|m|+3}N_{q,|m|+5} \quad , \\
&\cdots \\
0 &= C_{J-2}N_{q,-2}+\left[\frac{J(J+1)-\lambda_q}{a}-E_{J}\right]N_{qJ} \quad .
\end{aligned}\right\} \tag{3.43}$$

The following normalization conditions hold

$$1=\sum_{l=|m|,|m|+2}^{J}\left|N_{ql}^{(+,+)}\right|^2=\sum_{l=|m|,|m|+2}^{J}\left|N_{ql}^{(-,+)}\right|^2=\sum_{l=|m|+1}^{J}\left|N_{ql}^{(+,-)}\right|^2=\sum_{l=|m|+1}^{J}\left|N_{ql}^{(-,-)}\right|^2 \quad , \tag{3.44}$$

and the ellipso-cylindrical wave-functions are again subject to the orthogonality relation (??). Similarly as in (??) we have a unitary condition

$$\left(N_{ql}^{(r,p)}\right)^{-1}=N_{ql}^{(r,p)\,*} \quad . \tag{3.45}$$

If we take into account the parity of the hyper-spherical wave-functions according to $\Psi_{Jlm}\equiv\Psi_{Jlm}^{(r,p)}$, we can obtain the inverse interbasis expansion of the hyper-spherical basis in terms of the elliptic basis by means of

$$\Psi_{Jlm}^{(r,p)}(\chi,\vartheta,\varphi)=\sum_q N_{ql}^{(r,p)\,*}\Psi_{Jqm}^{(r,p)}(\mu,\nu,\varphi;a) \quad , \tag{3.46}$$

with q as in (??, ??).

3.3 Symmetry Properties of the Wave-Functions.

The interbasis expansions (??–??) or (??–??) together with their corresponding three-term recurrence relations completely determine the oblate and prolate elliptic bases. Now, we consider the symmetry properties of the elliptic bases under reflection. For the cylindrical and hyper-spherical bases we have

1. Reflection Π_3: $s_3\to -s_3$

$$\begin{aligned}
\Pi_3\Psi_{Jm_1m_2}^{(r,p)}(\gamma,\varphi_1,\varphi_2) &= p\Psi_{Jm_1m_2}^{(r,p)}(\gamma,\varphi_1,\varphi_2) \quad , \\
\Pi_3\Psi_{Jlm}^{(r,p)}(\chi,\vartheta,\varphi) &= p\Psi_{Jlm}^{(r,p)}(\chi,\vartheta,\varphi) \quad .
\end{aligned} \tag{3.47}$$

2. Reflection Π_4: $s_4 \to -s_4$

$$\begin{aligned} \Pi_4 \Psi^{(r,p)}_{J m_1 m_2}(\gamma, \varphi_1, \varphi_2) &= r \Psi^{(r,p)}_{J m_1 m_2}(\gamma, \varphi_1, \varphi_2) \ , \\ \Pi_4 \Psi^{(r,p)}_{J l m}(\chi, \vartheta, \varphi) &= r \Psi^{(r,p)}_{J l m}(\chi, \vartheta, \varphi) \ . \end{aligned} \tag{3.48}$$

Applying the reflection operators to the interbasis expansions (??–??) or (??–??) we obtain the complete set of operators defining the elliptic bases which are given by

$$\left.\begin{aligned} \mathcal{L}_1 \Psi^{(r,p)}_{J q m}(\mu, \nu, \varphi; a) &= \hbar^2 J(J+2) \Psi^{(r,p)}_{J q m}(\mu, \nu, \varphi; a) \ , \\ \mathcal{L}_2 \Psi^{(r,p)}_{J q m}(\mu, \nu, \varphi; a) &= \hbar^2 \lambda_q(a) \Psi^{(r,p)}_{J q m}(\mu, \nu, \varphi; a) \ , \\ \Pi_3 \Psi^{(r,p)}_{J q m}(\mu, \nu, \varphi; a) &= p \Psi^{(r,p)}_{J q m}(\mu, \nu, \varphi; a) \ , \\ \Pi_4 \Psi^{(r,p)}_{J q m}(\mu, \nu, \varphi; a) &= r \Psi^{(r,p)}_{J q m}(\mu, \nu, \varphi; a) \ . \end{aligned}\right\} \tag{3.49}$$

3.4 Limiting Cases.

As we have seen in the previous section, the two ellipso-cylindrical systems are the most general one-parametric coordinate systems on $S^{(3)}$ and contain as limiting cases the cylindrical and the hyper-spherical coordinate systems.

According to this, the elliptic bases in the limiting cases $a \to \infty$ and $a \to 0$ degenerate to the hyperspherical and cylindrical bases. Therefore we must determine the quantum numbers l_0 and $\widetilde{m}_2$ which characterize the limiting hyperspherical and cylindrical wave-functions, for q fixed. Using the fact that the number q is independent of the parameter a and the expression for the wave-functions (??) and (??,??), we obtain

$$l_0 = q + |m| \ , \qquad \widetilde{m}_2 \equiv \widetilde{m}_2^{(r,p)} \ , \tag{3.50}$$

where

$$\left.\begin{aligned} \widetilde{m}_2^{(+,+)} &= J - |m| - q &&= 2k - q \ , \\ \widetilde{m}_2^{(+,-)} &= J - |m| - q + 1 &&= 2k - q + 1 \ , \\ \widetilde{m}_2^{(-,+)} &= J - |m| - q - 1 &&= 2k - q \ , \\ \widetilde{m}_2^{(-,-)} &= J - |m| - q &&= 2k - q + 1 \ . \end{aligned}\right\} \tag{3.51}$$

We analyse the two limits separately.

3.4.1 The Hyper-Spherical Limit.

The limit $a \to 0$ can be analysed by means of the recurrence relation (??). Ignoring all terms which depend explicitly on a we get

$$\lim_{a \to 0} \lambda_q(a) = l_0(l_0+1) = (q + |m|)(q + |m| + 1) \ , \tag{3.52}$$

$$\lim_{a \to 0} N_{ql}(a) = \delta_{l l_0} \ , \tag{3.53}$$

and in particular for the wave-functions

$$\lim_{a \to 0} \Psi_{J q m}(\mu, \nu, \varphi; a) = \Psi_{J l_0 m}(\chi, \vartheta, \varphi) \ . \tag{3.54}$$

Let us consider the expansion of the elliptic bases in terms of cylindrical basis, as $a \to 0$. By means of the quantum numbers $\{J, m, l, m_2\}$ we define the new quantum numbers

$$\mu_1 = \frac{|m| + m_2}{2} \ , \qquad \mu_2 = \frac{|m| - m_2}{2} \ , \qquad J = 2j_1 \ , \qquad l = j \ , \tag{3.55}$$

and the three-term recurrence relation (??) for the $T_{qm} \equiv T_{\mu_1\mu_2}$ takes on the form

$$\begin{aligned}
&\sqrt{(j_1-\mu_1)(j_1+\mu_2)(j_1+\mu_1+1)(j_1-\mu_2+1)}\,T_{\mu_1+1,\mu_2-1} \\
&\quad +\Big[j(j+1)-2j_1(j_1+1)-2\mu_1\mu_2\Big]T_{\mu_1\mu_2} \\
&\quad +\sqrt{(j_1+\mu_1)(j_1-\mu_2)(j_1+\mu_2+1)(j_1-\mu_1+1)}\,T_{\mu_1-1,\mu_2+1}=0 \quad , \qquad (3.56)
\end{aligned}$$

which coincide up to a phase factor with the three-term recurrence relation for the Clebsch-Gordan coefficients $< j_1 j \mu_1\mu_2 | j, \mu_1+\mu_2 >$ [?], and we find

$$\lim_{a\to 0} T_{Jqm}(a) = (-1)^{(\mu_1-\mu_2)/2} < j_1 j \mu_1\mu_2 | j, \mu_1+\mu_2 > \quad . \qquad (3.57)$$

Therefore the expansion (??) yields

$$\Psi_{j_1,j,\mu_1+\mu_2}(\chi,\vartheta,\varphi) = \sum_{\mu_1,\mu_2} (-1)^{(\mu_1-\mu_2)/2} < j_1 j \mu_1\mu_2 | j, \mu_1+\mu_2 > \Psi_{j_1,\mu_1,\mu_2}(\gamma,\varphi_1,\varphi_2) \quad , \qquad (3.58)$$

which is the well-known expansion of the hyper-spherical basis in terms of the cylindrical basis [?].

3.4.2 The Cylindrical Limit.

For the limit $a \to \infty$ we find from the recurrence relations (??, ??)

$$\left.\begin{aligned}
\lim_{a\to\infty} \frac{\lambda_q^{(r,p)}(a)}{a} &= -[\widetilde{m}_2^{(r,p)}]^2 \quad , \\
\lim_{a\to\infty} T_{qm_2}^{(r,p)}(a) &= \delta_{\widetilde{m}_2^{(r,p)},m_2} \quad ,
\end{aligned}\right\} \qquad (3.59)$$

and therefore we get

$$\lim_{a\to\infty} \Psi_{Jqm}^{(r,p)}(\mu,\nu,\varphi;a) = \Psi_{J,\widetilde{m}_2^{(r,p)},m_2}^{(r,p)}(\gamma,\varphi_1,\varphi_2) \quad . \qquad (3.60)$$

We consider the interbasis expansion (??). Using the quantum numbers (??) we can rewrite the three-term recurrence relation (??) as follows ($N_j \equiv N_{ql}$)

$$\begin{aligned}
&\sqrt{\frac{(2j_1-j-1)(2j_1-j)(2j_1+j+2)(2j_1+j+3)}{(2j+1)(2j+3)^2(2j+5)}} \\
&\quad \times\sqrt{(j-\mu_1-\mu_2+1)(j-\mu_1-\mu_2+2)(j+\mu_1+\mu_2+1)(j+\mu_1+\mu_2+2)}\,N_{j+2} \\
&\qquad +\left\{(\mu_1-\mu_2)^2-\left[\frac{\Big((2j+1)^2-j_1^2\Big)\Big(j^2-(\mu_1+\mu_2)^2\Big)}{4j^2-1}\right.\right. \\
&\qquad\qquad \left.\left.+\frac{\Big((2j-1)^2-(j_1+1)^2\Big)\Big((j+1)^2-(\mu_1+\mu_2)^2\Big)}{(2j+1)(2j+3)}\right]\right\} N_j \\
&+\sqrt{\frac{(2j_1-j+1)(2j_1-j+2)(2j_1+j)(2j_1+j+1)}{(2j-1)(2j+1)^2(2j+3)}} \\
&\quad \times\sqrt{(j-\mu_1-\mu_2-1)(j-\mu_1-\mu_2)(j+\mu_1+\mu_2-1)(j+\mu_1+\mu_2)}\,N_{j-2}=0 \quad . \qquad (3.61)
\end{aligned}$$

This three-term recurrence relation coincides with the corresponding three-term recurrence relation for the Clebsch-Gordan coefficients which can be obtained from formula 8.6.5(27) in [?]. Therefore we get in the limit $a \to \infty$

$$\lim_{a\to\infty} N_{Jql}(a) = (-1)^{j/2} < j_1, j, \mu_1, \mu_2 | j, \mu_1 + \mu_2 > \ , \tag{3.62}$$

and the expansion (??) yields

$$\Psi_{j_1\mu_1\mu_2}(\gamma, \varphi_1, \varphi_2) = \sum_j (-1)^{j/2} < j_1, j, \mu_1, \mu_2 | j, \mu_1 + \mu_2 > \Psi_{j_1, j, \mu_2+\mu_2}(\chi, \vartheta, \varphi) \ , \tag{3.63}$$

which is the inverse expansion of (??).

4 Perturbation Theory.

The determination of the ellipso-cylindrical wave-functions $\Psi^{(r,p)}_{Jqm}$ by using (??,??) or (??), respectively, and solving them recursively is related with the solution of higher order algebraic equations. In general, such algebraic equations cannot be solved analytically. Nevertheless, the spherical and the cylindrical bases can be considered as a zero-approximation for a perturbation expansion. We do this in the following, first for $0 < |a| \ll 1$, second for $a \gg 1$.

4.1 The Case $0 < |a| \ll 1$.

According to (??) we can label the coefficients N_{ql} and λ_q with the index l_0 instead with q. Let us set $N_{ql} = \tilde{N}_{ll_0}, \lambda_q = \tilde{\lambda}_{l_0}$.

For the calculation of the higher order corrections we consider the perturbation expansion

$$\tilde{N}_{ll_0}(a) = \delta_{ll_0} + \sum_{s=1}^{\infty} N^{(s)}_{ll_0} a^s \ , \tag{4.1}$$

$$\tilde{\lambda}_{l_0}(a) = l_0(l_0+1) + \sum_{s=1}^{\infty} \lambda^{(s)}_{ll_0} a^s \ , \tag{4.2}$$

which fixes l_0 as the angular momentum number corresponding to the $\mathbf{L}^2$ operator in the limit $a = 0$. Inserting these expansions into the three-term recurrence relations (??) and comparing coefficients with the same power of a, we arrive after some algebraic manipulations at the following equations for $N^{(s)}_{ll_0}$ and $\lambda^{(s)}_{ll_0}$:

$$\begin{aligned}(l - l_0)(l + l_0 + 1) N^{(s)}_{ll_0} + C_l N^{(s-1)}_{l+2,l_0} \\ + C_{l-2} N^{(s-1)}_{l-2,l_0} - E_l N^{(s-1)}_{ll_0} - \sum_{t=0}^{s-1} \lambda^{(s-1)}_{l_0} N^{(t)}_{ll_0} = 0 \ .\end{aligned} \tag{4.3}$$

According to standard perturbation theory, e.g. Landau and Lifshitz [?], we use the initial conditions

$$N^{(0)}_{ll_0} = \delta_{ll_0} \ , \qquad N^{(s)}_{ll} = \delta_{s0} \ , \tag{4.4}$$

and we can express $\lambda^{(s)}_{l_0}$ for $s > 1$ in terms of the coefficients $N^{(s-1)}_{l_0 l_0}, N^{(s-1)}_{l_0, l_0 \pm 2}$ by means of

$$\lambda^{(s)}_{l_0} = C_{l_0} N^{(s-1)}_{l_0+2,l_0} - E_{l_0} N^{(s-1)}_{l_0 l_0} + C_{l_0-2} N^{(s-1)}_{l_2-2,l_0} \ . \tag{4.5}$$

Explicitly for $s=1$ and $s=2$ we have

$$\left.\begin{aligned}\lambda_{l_0}^{(1)} &= -E_{l_0}\ , \\ \lambda_{l_0}^{(2)} &= C_{l_0}N_{l_0+2,l_0}^{(1)} + C_{l_0-2}N_{l_0-2,l_0}^{(1)}\ ,\end{aligned}\right\} \tag{4.6}$$

and from (**??**) we obtain for $s=1$

$$N_{ll_0}^{(1)} = \frac{C_{l_0-2}}{2(2l_0-1)}\delta_{l,l_0-2} - \frac{C_{l_0}}{2(2l_0+3)}\delta_{l,l_0+2}\ , \tag{4.7}$$

which determines the second approximation for $\tilde{N}_{ll_0}(a)$ and $\tilde{\lambda}_{l_0}(a)$. Explicitly, we obtain for $\lambda_q(a)$ in second order perturbation theory

$$\lambda_q(a) = (q+|m|)(q+|m|+1) - E_{q+|m|}\cdot a - \frac{1}{2}\left[\frac{C_{q+|m|}^2}{2q+2|m|+3} - \frac{C_{q+|m|+2}^2}{2q+2|m|-1}\right]a^2 + O(a^3)\ . \tag{4.8}$$

Substituting (**??**) into (**??**) and expanding (**??**) we obtain for the elliptic basis in first order perturbation theory with respect to the hyper-spherical basis

$$\begin{aligned}\Psi_{Jqm}(\mu,\nu,\varphi;a) &= \Psi_{J,q+|m|,m}(\chi,\vartheta,\varphi) - \frac{a}{2}\left[\frac{C_{q+|m|}}{2q+2|m|+3}\Psi_{J,q+|m|+2,m}(\chi,\vartheta,\varphi)\right. \\ &\quad \left. - \frac{C_{q+|m|-2}}{2q+2|m|-1}\Psi_{J,q+|m|-2,m}(\chi,\vartheta,\varphi)\right] + O(a^2)\ .\end{aligned} \tag{4.9}$$

Finally, the case of the prolate elliptic basis is obtained if we replace in all relevant formulæ $a \to -a$.

4.2 The Case $a \gg 1$.

For the case of $a \gg 1$ we consider the power expansions

$$T_{qm_2}^{(r,p)}(a) = \delta_{\widetilde{m}_2^{(r,p)},m_2} + \sum_{s=1}^{\infty} T_{qm_2}^{(r,p),(s)}a^{-s}\ , \tag{4.10}$$

$$\frac{\lambda_q^{(r,p)}(a)}{a} = -[\widetilde{m}_2^{(r,p)}]^2 + \sum_{s=1}^{\infty}\lambda_q^{(r,p),(s)}a^{-s}\ , \tag{4.11}$$

where $\widetilde{m}_2^{(r,p)}$ as in (**??**). Substituting (**??**) and (**??**) into the recurrence relation (**??**, **??**) we obtain by the same reasoning as for $|a|$ small in second order perturbation theory for the quantities $\lambda_q^{(r,p)}(a)$

$$\begin{aligned}\lambda_q^{(+,+)}(a) &\simeq -a(2k-q)^2 + \tfrac{1}{2}\big[J(J+2) + (J-q)(J-4k+q)\big] \\ &\quad + \frac{1}{a}\left\{\frac{q(2k+\frac{q}{2}+1)(J-2k+\frac{q}{2})(J-\frac{q}{2}+1)}{8(2k-q+1)}\Big(1+\delta_{k,\frac{q}{2}}\Big)\right. \\ &\quad \left. - \frac{(q+2)(2k-\frac{q}{2})(J-\frac{q}{2})(J-2k+\frac{q}{2}+1)}{8(2k-q-1)}\Big(1+\delta_{k,\frac{q+1}{2}}\Big)\right\}\ , \\ \lambda_q^{(+,-)}(a) &\simeq -a(2k-q+1)^2 + \tfrac{1}{2}\Big[J(J+2) + (J-q+1)(J-4k+q-1)\Big]\end{aligned} \tag{4.12}$$

$$+\frac{1}{a}\left\{\frac{(q-2)(2k-\frac{q}{2}+\frac{3}{2})(J-2k+\frac{q}{2}-\frac{1}{2})(J-\frac{q}{2}+\frac{3}{2})}{8(2k-q+2)}\Big(1-\delta_{k,\frac{q-1}{2}}\Big)\right.$$
$$\left.-\frac{(q+1)(2k-\frac{q}{2}+\frac{1}{2})(J-2k+\frac{q}{2}+\frac{1}{2})(J-\frac{q}{2}+\frac{1}{2})}{8(2k-q)}\Big(1-\delta_{k,\frac{q+1}{2}}\Big)\right\}\quad , \tag{4.13}$$

$$\lambda_q^{(-,+)}(a)\simeq -a(2k-q+1)^2+J(J+2)+(2k+1)J+\tfrac{1}{2}q(q-2)+(J-k)(k+1)\delta_{k,\frac{q}{2}}$$
$$+\frac{1}{a}\left\{\frac{q(2k-\frac{q}{2}+2)(J-2k+\frac{q}{2}-1)(J-\frac{q}{2})}{8(2k-q+2)}\right.$$
$$\left.-\frac{(q+1)(2k-\frac{q}{2}+\frac{1}{2})(J-2k+\frac{q}{2})(J-\frac{q}{2})}{8(2k-q)}\Big(1-\delta_{k,\frac{q}{2}}\Big)\right\}\quad , \tag{4.14}$$

$$\lambda_q^{(-,-)}(a)\simeq -a(2k-q+2)^2+J(J+2)+(2k+1)J+\tfrac{1}{2}(q^2-1)-(J-k)(k+1)\delta_{k,\frac{q-1}{2}}$$
$$+\frac{1}{a}\left\{\frac{(q-1)(2k-\frac{q}{2}+\frac{5}{2})(J-2k+\frac{q}{2}-\frac{3}{2})(J-\frac{q}{2}+\frac{3}{2})}{8(2k-q+3)}\right.$$
$$\left.-\frac{(q+1)(2k-\frac{q}{2}+\frac{3}{2})(J-2k+\frac{q}{2}-\frac{1}{2})(J-\frac{q}{2}+\frac{1}{2})}{8(2k-q+1)}\Big(1-\delta_{k,\frac{q-1}{2}}\Big)\right\}\quad . \tag{4.15}$$

Therefore we obtain in first order for the elliptic wave-function $\Psi^{(r,p)}_{Jqm}(\mu,\nu,\varphi;a)$

$$\Psi^{(+,+)}_{Jqm}(\mu,\nu,\varphi;a)\simeq \Psi^{(+,+)}_{J,2k-q,m}(\gamma,\varphi_1,\varphi_2)$$
$$-\frac{1}{4a}\left\{\frac{\sqrt{q(2k+\frac{q}{2}+1)(J-2k+\frac{q}{2})(J-\frac{q}{2}+1)}}{\sqrt{2}\,(2k-q+1)}\Big(1+\delta_{k,\frac{q}{2}+1}\Big)\Psi^{(+,+)}_{J,2k-q+2,m}(\gamma,\varphi_1,\varphi_2)\right.$$
$$\left.-\frac{\sqrt{(q+2)(2k-\frac{q}{2})(J-\frac{q}{2})(J-2k+\frac{q}{2}+1)}}{\sqrt{2}\,(2k-q-1)}\,\Psi^{(+,+)}_{J,2k-q-2,m}(\gamma,\varphi_1,\varphi_2)\right\}\quad , \tag{4.16}$$

$$\Psi^{(+,-)}_{Jqm}(\mu,\nu,\varphi;a)\simeq \Psi^{(+,-)}_{J,2k-q+1,m}(\gamma,\varphi_1,\varphi_2)$$
$$-\frac{1}{4a}\left\{\frac{\sqrt{(q-1)(2k-\frac{q}{2}+\frac{3}{2})(J-2k+\frac{q}{2}-\frac{1}{2})(J-\frac{q}{2}+\frac{3}{2})}}{\sqrt{2}\,(2k-q+2)}\right.$$
$$\times\Big(1+\delta_{k,\frac{q+1}{2}}\Big)\Psi^{(+,-)}_{J,2k-q+3,m}(\gamma,\varphi_1,\varphi_2)$$
$$\left.-\frac{\sqrt{(q+1)(2k-\frac{q}{2}+\frac{1}{2})(J-\frac{q}{2}+\frac{1}{2})(J-2k+\frac{q}{2}+\frac{1}{2})}}{\sqrt{2}\,(2k-q)}\,\Psi^{(+,-)}_{J,2k-q-1,m}(\gamma,\varphi_1,\varphi_2)\right\}\quad , \tag{4.17}$$

$$\Psi^{(-,+)}_{Jqm}(\mu,\nu,\varphi;a)\simeq \Psi^{(-,+)}_{J,2k-q,m}(\gamma,\varphi_1,\varphi_2)$$
$$-\frac{1}{4a}\left\{\frac{\sqrt{q(2k-\frac{q}{2}+2)(J-2k+\frac{q}{2}-1)(J-\frac{q}{2})}}{\sqrt{2}\,(2k-q+2)}\,\Psi^{(-,+)}_{J,2k-q+2,m}(\gamma,\varphi_1,\varphi_2)\right.$$
$$\left.-\frac{\sqrt{(q+1)(2k-\frac{q}{2}+1)(J-\frac{q}{2})(J-2k+\frac{q}{2})}}{\sqrt{2}\,(2k-q)}\Big(1-\delta_{k,\frac{q}{2}}\Big)\Psi^{(-,+)}_{J,2k-q-2,m}(\gamma,\varphi_1,\varphi_2)\right\}\quad , \tag{4.18}$$

$$\Psi^{(-,-)}_{Jqm}(\mu,\nu,\varphi;a) \simeq \Psi^{(-,-)}_{J,2k-q+1,m}(\gamma,\varphi_1,\varphi_2)$$
$$-\frac{1}{4a}\left\{\frac{\sqrt{(q-1)(2k-\frac{q}{2}+\frac{5}{2})(J-2k+\frac{q}{2}-\frac{3}{2})(J-\frac{q}{2}+\frac{3}{2})}}{\sqrt{2}\,(2k-q+3)}\,\Psi^{(-,-)}_{J,2k-q+3,m}(\gamma,\varphi_1,\varphi_2)\right.$$
$$-\frac{\sqrt{(q+1)(2k-\frac{q}{2}+\frac{3}{2})(J-\frac{q}{2}+\frac{1}{2})(J-2k+\frac{q}{2}-\frac{1}{2})}}{\sqrt{2}\,(2k-q+1)}$$
$$\left.\times\left(1-\delta_{k,\frac{q-1}{2}}\right)\Psi^{(-,-)}_{J,2k-q-1,m}(\gamma,\varphi_1,\varphi_2)\right\}\quad . \tag{4.19}$$

Higher order perturbation corrections can be analogously calculated.

5 Two Path Integral Representations in Ellipso-Cylindrical Coordinates on $S^{(3)}$.

Equation (**??**) describes the expansion of the elliptic basis in terms of the cylindrical basis, where in addition the parity classes according to (**??**–**??**) must be taken into account properly. Vice versa we have for the expansion of the cylindrical basis in terms of the elliptic basis the expansion (**??**) which must be interpreted according to the parity classes of the expansion coefficients T_{qm_2}. However, we use the short-hand notation of (**??**) in order to make the corresponding formulæ not to lengthy.

The path integral representation of the propagator in cylindrical coordinates is given by [**?**, **?**] [note $\chi \in (0,\pi), \vartheta \in (0,\pi)$]

$$\begin{aligned}
&K(\gamma'',\gamma',\varphi_1'',\varphi_1',\varphi_2'',\varphi_2';T)\\
&=\int\limits_{\gamma(t')=\gamma'}^{\gamma(t'')=\gamma''}\mathcal{D}\gamma(t)\sin\gamma\cos\gamma\int\limits_{\varphi_1(t')=\varphi_1'}^{\varphi_1(t'')=\varphi_1''}\mathcal{D}\varphi_1(t)\int\limits_{\varphi_2(t')=\varphi_2'}^{\varphi_2(t'')=\varphi_2''}\mathcal{D}\varphi_2(t)\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\left(\dot\gamma^2+\cos^2\gamma\dot\varphi_1^2+\sin^2\gamma\dot\varphi_2^2\right)+\frac{\hbar^2}{8M}\left(4+\frac{1}{\cos^2\gamma}+\frac{1}{\sin^2\gamma}\right)\right]dt\right\}\\
&:=\lim_{N\to\infty}\left(\frac{M}{2\pi\mathrm{i}\epsilon\hbar}\right)^{3N/2}\prod_{j=1}^{N-1}\int_0^{\pi/2}\sin\gamma_j\cos\gamma_j d\gamma_j\int_0^{2\pi}d\varphi_j^{(1)}\int_0^{2\pi}d\varphi_j^{(2)}\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left[\frac{M}{2\epsilon}\left((\Delta\gamma_j)^2+\widehat{\cos^2}\,\gamma_j(\Delta\varphi_j^{(1)})^2+\widehat{\sin^2}\,\gamma_j(\Delta\varphi_j^{(2)})^2\right)\right.\right.\\
&\qquad\left.\left.+\epsilon\frac{\hbar^2}{8M}\left(4+\frac{1}{\cos^2\gamma_j\sin^2\gamma_j}\right)\right]\right\}\\
&=\sum_{Jmm_2}\exp\left[-\frac{\mathrm{i}\hbar T}{2M}J(J+2)\right]\Psi^*_{Jmm_2}(\gamma',\varphi_1',\varphi_2')\Psi_{Jmm_2}(\gamma'',\varphi_1'',\varphi_2'')\quad .
\end{aligned}\tag{5.1}$$

Here we have formulated the path integral in the canonical way according to, e.g. Refs. [**?**]–[**?**] by means of a "product lattice" [**?**] which is used in the following. We have used the abbreviations $\varphi_0^{(1)}=\varphi_1', \varphi_N^{(1)}=\varphi_1''$, etc., $\epsilon=(t''-t')/N\equiv T/N$, $\Delta\mathbf{q}_j=\mathbf{q}_j-\mathbf{q}_{j-1}$, $\mathbf{q}_j=\mathbf{q}(t'+j\epsilon)$ $(t_j=t'+\epsilon j, j=0,\ldots,N)$, $\widehat{f^2(\mathbf{q}_j)}=f(\mathbf{q}_{j-1})f(\mathbf{q}_j)$ for some function $f(\mathbf{q})$ of coordinates $\mathbf{q}$, and we interpret the limit $N\to\infty$ as equivalent to $\epsilon\to 0$, T fixed.

The path integral representation of the propagator in the hyper-spherical basis is given by [?, ?] [note $\gamma \in (0,\pi)$]

$$
\begin{aligned}
&K(\chi'',\chi',\vartheta'',\vartheta',\varphi'',\varphi';T) \\
&= \int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''} \mathcal{D}\chi(t)\sin^2\chi \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t)\sin\vartheta \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t) \\
&\quad \times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\Big(\dot\chi^2+\sin^2\chi\,\dot\vartheta^2+\sin^2\chi\sin^2\vartheta\dot\varphi^2\Big)\right.\right. \\
&\qquad\qquad \left.\left.+\frac{\hbar^2}{8M}\left(4+\frac{1}{\sin^2\chi}\left(1+\frac{1}{\sin^2\vartheta}\right)\right)\right]dt\right\} \\
&= \sum_{Jlm}\exp\left[-\frac{\mathrm{i}\hbar T}{2M}J(J+2)\right]\Psi^*_{Jlm}(\chi',\vartheta',\varphi')\Psi_{Jlm}(\chi'',\vartheta'',\varphi'') \ . \qquad (5.2)
\end{aligned}
$$

In [?] we have derived the path integral representation on the sphere $S^{(D-1)}$ which has the form

$$
K(\mathbf{\Omega}'',\mathbf{\Omega}';T) = \int\limits_{\mathbf{\Omega}(t')=\mathbf{\Omega}'}^{\mathbf{\Omega}(t'')=\mathbf{\Omega}''} \mathcal{D}\mathbf{\Omega}(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\dot{\mathbf{\Omega}}^2+\hbar^2\frac{(D-1)(D-3)}{8M}\right]dt\right\} \ . \qquad (5.3)
$$

Here $\mathbf{\Omega}$ is a unit vector on the D-dimensional sphere. The quantity $\dot{\mathbf{\Omega}}^2$ is defined in its lattice representation by means of $\dot{\mathbf{\Omega}}^2 \to (\mathbf{\Omega}_j-\mathbf{\Omega}_{j-1})^2/\epsilon$, which in turn can be restated yielding

$$
\frac{1}{\epsilon}(\mathbf{\Omega}_j-\mathbf{\Omega}_{j-1})^2 = \frac{2}{\epsilon}(1-\cos\psi_{j,j-1}) \ , \qquad (5.4)
$$

and $\psi_{j,j-1}$ is the angle between two vectors $\mathbf{\Omega}_j$ and $\mathbf{\Omega}_{j-1}$ on $S^{(D-1)}$. The addition theorem for $\cos\psi_{1,2}$ has the form

$$
\begin{aligned}
\cos\psi_{1,2} &= \cos\vartheta_1^{(1)}\cos\vartheta_2^{(1)} + \sum_{m=1}^{D-2}\cos\vartheta_1^{(m+1)}\cos\vartheta_2^{(m+1)}\prod_{n=1}^{m}\sin\vartheta_1^{(n)}\sin\vartheta_2^{(n)} + \prod_{n=1}^{D-1}\sin\vartheta_1^{(n)}\sin\vartheta_2^{(n)} \\
&= \cos\chi_1\cos\chi_2+\sin\chi_1\sin\chi_2\Big[\cos\vartheta_1\cos\vartheta_2+\sin\vartheta_1\sin\vartheta_2\cos(\varphi_2-\varphi_1)\Big] \ , \qquad (5.5)
\end{aligned}
$$

where in the last line we have inserted the spherical coordinates on $S^{(3)}$. In order to evaluate the path integral, we need the expansion theorem [?, p.980]

$$
\mathrm{e}^{z\cos\psi} = \left(\frac{z}{2}\right)^{-\nu}\Gamma(\nu)\sum_{J=0}^{\infty}(J+\nu)I_{J+\nu}(z)C_J^{\nu}(\cos\psi) \qquad (5.6)
$$

for $\nu=(D-2)/2$. The $C_n^{\nu}(x)$ are Gegenbauer polynomials [?, p.1029] and the $I_\mu(z)$ are modified Bessel functions [?, p.958]. By means of the addition theorem for the hyper-spherical harmonics on the $S^{(D-1)}$-sphere [?]

$$
\sum_{\mu=1}^{M} S_l^{\mu}(\mathbf{\Omega}')S_l^{\mu}(\mathbf{\Omega}'') = \frac{1}{\Omega(D)}\frac{2J+D-2}{D-2}C_J^{\frac{D-2}{2}}(\cos\psi^{',''}) \ , \qquad (5.7)
$$

where $\Omega(D) = 2\pi^{D/2}/\Gamma(D/2)$, $M = (2J+D-2)(J+D-3)!/J!(D-2)!$, we obtain the following expansion

$$\mathrm{e}^{z(\mathbf{\Omega}'\cdot\mathbf{\Omega}')} = \mathrm{e}^{z\cos\psi^{'}{}^{''}} = 2\pi\left(\frac{2\pi}{z}\right)^{\frac{d-2}{2}}\sum_{J=0}^{\infty}\sum_{\mu=1}^{M} S_J^{\mu}(\mathbf{\Omega}')S_J^{\mu}(\mathbf{\Omega}'')I_{J+\frac{D-2}{2}}(z)\ . \tag{5.8}$$

We consider in the short time kernel of the path integral (**??**) the quantity $\mathrm{e}^{z_j\cos\psi_{j-1,j}}$ with $z_j = M/\mathrm{i}\epsilon\hbar$. Using the expansion (**??**) and the interbasis expansion (**??**) we get

$$\begin{aligned}
&\mathrm{e}^{z_j\cos\psi_{j-1,j}} = 2\pi\left(\frac{2\pi}{z_j}\right)^2\sum_{Jlm}\Psi^*_{Jlm}(\chi_{j-1},\vartheta_{j-1},\varphi_{j-1})\Psi_{Jlm}(\chi_j,\vartheta_j,\varphi_j)I_{J+2}(z_j)\\
&= 2\pi\left(\frac{2\pi}{z_j}\right)^2\sum_{Jlm}\sum_{qq'}N^{-1}_{ql}{}^{*}N^{-1}_{q'l}\Psi^*_{Jqm}(\mu_{j-1},\nu_{j-1},\varphi_{j-1};a)\Psi_{Jq'm}(\mu_j,\nu_j,\varphi_j;a)I_{J+2}(z_j)\\
&\simeq \left(\frac{2\pi\mathrm{i}\epsilon}{M}\right)^{3/2}\sum_{Jqm}\Psi^*_{Jqm}(\mu_{j-1},\nu_{j-1},\varphi_{j-1};a)\Psi_{Jq'm}(\mu_j,\nu_j,\varphi_j;a)\\
&\qquad\times\exp\left[-\frac{\mathrm{i}}{\hbar}\frac{M}{\epsilon}-\mathrm{i}\epsilon\hbar\frac{(J+2)^2-1/4}{2M}\right]\ .
\end{aligned}\tag{5.9}$$

In the last line we have applied the asymptotic expansion $I_\nu(z)\simeq(2\pi z)^{-1/2}\mathrm{e}^{z-(\nu^2-1/4)/2z}$, $|z|\to\infty$ according to [**?**], where in order that $|\arg z|<\pi/2$ a Wick rotation must be performed [**?**]. The path integration of the propagator (**??**) starting from the hyper-spherical basis is thus reduced to a multiple integration over the orthogonal bases $\Psi_{Jqm}(\mu,\nu,\varphi;a)$ yielding

$$\begin{aligned}
&K(\chi'',\chi',\vartheta'',\vartheta',\varphi'',\varphi';T)\\
&=\sum_{Jlm}\exp\left[-\frac{\mathrm{i}\hbar T}{2M}J(J+2)\right]\Psi^*_{Jlm}(\chi',\vartheta',\varphi')\Psi_{Jlm}(\chi'',\vartheta'',\varphi'')\\
&=\sum_{J}\exp\left[-\frac{\mathrm{i}\hbar T}{2M}J(J+2)\right]\sum_{qm}\Psi^*_{Jqm}(\mu',\nu',\varphi';a)\Psi_{Jqm}(\mu'',\nu'',\varphi'';a)
\end{aligned}\tag{5.10}$$

Oblate Elliptic Coordinates, $a\geq 0$:

$$\begin{aligned}
&\equiv\int_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathcal{D}\mu(t)\int_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathcal{D}\nu(t)(k^2\mathrm{cn}^2\mu+k'^2\mathrm{cn}^2\nu)\mathrm{sn}\mu\mathrm{dn}\nu\int_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\Big((k^2\mathrm{cn}^2\mu+k'^2\mathrm{cn}^2\nu)(\dot\mu^2+\dot\nu^2)+\mathrm{sn}^2\mu\mathrm{dn}^2\nu\dot\varphi^2\Big)\right.\right.\\
&\qquad\left.\left.+\frac{\hbar^2}{8M}\frac{1}{k^2\mathrm{cn}^2\mu+k'^2\mathrm{cn}^2\nu}\Big(\frac{\mathrm{cn}^2\nu\mathrm{dn}^2\nu}{\mathrm{sn}^2\nu}+k^4\frac{\mathrm{sn}^2\mu\mathrm{cn}^2\mu}{\mathrm{dn}^2\mu}\Big)\right]dt+\frac{\mathrm{i}\hbar T}{2M}\right\}\ ,
\end{aligned}\tag{5.11}$$

Prolate Elliptic Coordinates, $a\in[-1,0]$:

$$\begin{aligned}
&\equiv\int_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathcal{D}\mu(t)\int_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathcal{D}\nu(t)(k^2\mathrm{cn}^2\mu+k'^2\mathrm{cn}^2\nu)\mathrm{cn}\mu\mathrm{cn}\nu\int_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\Big((k^2\mathrm{cn}^2\mu+k'^2\mathrm{cn}^2\nu)(\dot\mu^2+\dot\nu^2)+\mathrm{cn}^2\mu\mathrm{cn}^2\nu\dot\varphi^2\Big)\right.\right.\\
&\qquad\left.\left.+\frac{\hbar^2}{8M}\frac{1}{k^2\mathrm{cn}^2\mu+k'^2\mathrm{cn}^2\nu}\Big(\frac{\mathrm{sn}^2\nu\mathrm{dn}^2\nu}{\mathrm{cn}^2\nu}+k^4\frac{\mathrm{sn}^2\mu\mathrm{dn}^2\mu}{\mathrm{cn}^2\mu}\Big)\right]dt+\frac{\mathrm{i}\hbar T}{2M}\right\}\ .
\end{aligned}\tag{5.12}$$

In addition, we have included a direct implementation of the interbasis expansion (??) in the propagator. Of course, a similar relation holds for the case, where the interbasis expansion with the coefficients T_{qm_2} is used.

6 Discussion and Summary.

In this paper we have discussed the solution of the Schrödinger equation on the three-dimensional sphere $S^{(3)}$ in terms of the two one-parametric ellipso-cylindrical coordinate systems, i.e, the oblate elliptic and the prolate elliptic coordinate systems. The elliptic wave-functions $\Psi_{Jqm}(\mu,\nu,\varphi;a)$ have been constructed in terms of the cylindrical and the hyper-spherical wave-functions, respectively. The coefficients of the interbasis expansions have been found to satisfy three-term recurrence relations.

The two interbasis expansions (??) and (??) have allowed us the determination of the elliptic wave-functions in any desired order. In analogy to the corresponding wave-functions on the two-dimensional sphere $S^{(2)}$, where *Lamé polynomials* are involved, we may call, following [?] the elliptic wave-functions *associated Lamé polynomials*. The improvement of our solution in comparison to [?] lies in the fact that we have not just defined a power series expansion in an unambiguous way. Our approach provides at the same time the necessary coefficients of the interbasis expansions which allows us on the one hand to determine the elliptic wave-functions and the corresponding observables to any desired accuracy, and on the other enables us to switch from one basis to another, appropriate for our purposes.

We have also developed a perturbation theory for the elliptic wave-functions Ψ_{Jqm} for the two limiting cases of the elliptic parameter a, i.e, for the cases $0 < |a| \ll 1$ and $a \gg 1$, respectively. Formulæ (??, ??) represent first order perturbation theory for the elliptic wave-functions. For higher order contributions, the recurrence relations (??, ??) or (??–??) can be used in perturbation theory by implementing them in a symbolic computer program, like REDUCE, MAPLE, or MATHEMATICA.

The elliptic observable is contained in the approximate "third observable" which describes the diamagnetic Kepler problem (the quadratic Zeeman effect) [?, ?, ?, ?], which is not separable. The "third observable" can be introduced to have the form [?, ?] $\hat{\Lambda} = 4\mathbf{A}^2 - 5A_z$ ($\mathbf{A} = \mathbf{M}/(J+1)^2$ is the Pauli-Runge-Lenz vector). Then, "$\hat{\Lambda}$ commutes with the operator $\hat{\rho}^2$ $[= (J+1)^2\{5J(J+2) - 4 + L_z^2 - 4\mathcal{L}_2\}/2$, $\rho^2 = x^2 + y^2$, $d = -5/4]$ in an n-layer, since the operator $\hat{\Lambda}$ is an exact integral of motion for the hydrogen atom in the absence of a magnetic field.... The eigenfunctions Ψ_{nqm} are the correct wave functions in the zeroth approximation for the hydrogen atom in a weak magnetic field" [?], and the elliptic observable can be used in the weak-field limit in an algebraic perturbation investigation according to (??).

Using the results for the interbasis coefficients T_{qm_2} and N_{ql} we have derived by a simple change of basis path integral identities for the propagator on $S^{(3)}$ in oblate elliptic and prolate elliptic coordinates. In the two identities (??) and (??) we could furthermore factorize the wave-functions Ψ_{Jqm} according to (??), and separate off the φ-path integral. This would yield path integral identities for potentials according to

$$V_1(\mu,\nu) = \frac{\hbar^2}{2M}\frac{m^2-1/4}{\mathrm{sn}^2\mu\mathrm{dn}^2\nu}\ , \qquad V_2(\mu,\nu) = \frac{\hbar^2}{2M}\frac{m^2-1/4}{\mathrm{cn}^2\mu\mathrm{cn}^2\nu}\ , \qquad m \in \mathbb{Z}\ , \tag{6.1}$$

and m^2 can be analytically continued to any real number [?, ?]. However, this is obvious, and we omit the explicit formulæ, c.f. [?] for the two spheroidal cases in $\mathbb{R}^3$.

It is legitimate to ask, whether the two path integral identities (??) and (??) are of use from a practicable point of view. Here one must keep in mind that all known path integral relations and identities require more or less the knowledge of expansion formulæ, where some kind of higher transcendental functions are involved. The calculation of the radial harmonic oscillator requires the expansion of plane waves into a product of circular waves, i.e. spherical harmonics, and Bessel functions [?], the calculation of the path integral for the (modified) Pöschl-Teller potential requires the knowledge of expansion theorems on the SU(2) and SU(1,2) group manifolds [?]. In the cases of path integration on homogeneous spaces in terms of (one-) parametric coordinate systems, i.e., elliptic and spheroidal coordinates in Euclidean space [?], where Matthieu and spheroidal wave-functions are involved, or elliptic (conical) [?] and ellipso-cylindrical coordinates on spheres, where Lamé and associated Lamé polynomials are involved, one has to accept the fact that these unusual wave-functions are but another kind of higher transcendental special functions. Unfamiliarity does not imply unreasonability.

We have not explicitly included in our discussion the dependence on the radius R of the three-dimensional sphere. In the flat space limit the following identities must emerge (up to normalization, $\delta = pd/2$, $R \to \infty, k \to 0$, where $Rk \equiv d$ fixed)

$$\begin{aligned} \psi_1(\mu;k) &\to \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,\mathrm{ps}_l^m(\vartheta;\delta^2) \ , \\ \psi_2(\nu;k) &\to \sqrt{\frac{2}{\pi}}\,pS_l^{m\,(1)}(\cosh\xi;\delta) \ . \end{aligned} \tag{6.2}$$

ps_l^n and $S_l^{n\,(1)}$ are prolate spheroidal wave-functions in terms of spheroidal coordinates ξ, ϑ in $\mathbb{R}^3$ [?]. This would require an expansion of the quantity ($\mathbf{x}, \mathbf{p} \in \mathbb{R}^4$, where for $\mathbf{x}$ appropriate prolate elliptic and for $\mathbf{p}$ spherical coordinates are used)

$$\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}} = \exp\left[\mathrm{i}pR(\sin\chi\sin\vartheta\mathrm{sn}\mu\mathrm{dn}\nu\cos(\varphi-\psi) + \sin\chi\cos\vartheta\mathrm{cn}\mu\mathrm{cn}\nu + \cos\chi\mathrm{dn}\mu\mathrm{sn}\nu)\right] \tag{6.3}$$

in terms of the ellipso-cylindrical bases Ψ_{Jqm}, as $R \to \infty$. However, such considerations go beyond the scope of the present paper and will be discussed elsewhere.

It is obvious that our methods lead to further investigations: 1) the determination of the interbasis expansion for ellipsoidal coordinates on the three-dimensional sphere, 2) the determination of the interbasis expansion for the analogous one-parametric coordinate systems on the three-dimensional hyperboloid. This also would include a detailed investigation of the limiting cases, as $R \to \infty$. These issues will be discussed in future investigations.

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A Matrix Elements in the Cylindrical Basis.

The wave-functions in the cylindrical coordinates have the following form

$$\Psi_{Jm_1m_2}(\gamma,\varphi_1,\varphi_2) = \frac{\mathrm{e}^{\mathrm{i}m_1\varphi_1}}{\sqrt{2\pi}} C^J_{m_1m_2}(\gamma) \frac{\mathrm{e}^{\mathrm{i}m_2\varphi_2}}{\sqrt{2\pi}} \quad , \tag{A.1}$$

and are normalized according to

$$\int_0^{\pi/2} \sin\gamma\cos\gamma d\gamma \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \Psi^*_{Jm_1m_2}(\gamma,\varphi_1,\varphi_2)\Psi_{J'm_1'm_2'}(\gamma,\varphi_1,\varphi_2) = \delta_{JJ'}\delta_{m_1m_1'}\delta_{m_2m_2'} \, . \tag{A.2}$$

The cylindrical wave-function $C^J_{m_1m_2}(\gamma)$ is given by

$$\begin{aligned} C^J_{m_1m_2}(\gamma) &= \sqrt{\frac{2(J+1)\left(\frac{J-|m_1|-|m_2|}{2}\right)!\left(\frac{J+|m_1|+|m_2|}{2}\right)!}{\left(\frac{J-|m_1|+|m_2|}{2}\right)!\left(\frac{J+|m_1|-|m_2|}{2}\right)!}} \\ &\qquad \times(\sin\gamma)^{|m_1|}(\cos\gamma)^{|m_2|}P^{(|m_1|,|m_2|)}_{(J-|m_1|-|m_2|)/2}(\cos 2\gamma) \qquad (A.3) \\ &= \sqrt{2(J+2)}\, d^{J/2}_{(m_2+m_1)/2,(m_1-m_2)/2}(\cos 2\gamma) \quad . \qquad (A.4) \end{aligned}$$

The $P^{(\alpha,\beta)}_n(x)$ are Jacobi polynomials [**?**, p.1035], and the $d^n_{\kappa,\lambda}(x)$ are Wigner functions [**?**]. We have for the quantum numbers $m_{1,2} \in \mathbb{Z}$, where for fixed J they vary according to $-J \le m_1+m_2 \le J$, and $(J-|m_1|-|m_2|)$ is even. The operators L_1, L_2, L_3 are explicitly given by

$$\left.\begin{aligned} L_1 &= -\frac{\hbar}{\mathrm{i}}\Big(\sin\varphi_1\sin\varphi_2\frac{\partial}{\partial\gamma} - \tan\gamma\sin\varphi_1\cos\varphi_2\frac{\partial}{\partial\varphi_2} + \cot\gamma\cos\varphi_1\sin\varphi_2\frac{\partial}{\partial\varphi_1}\Big) \quad , \\ L_2 &= \frac{\hbar}{\mathrm{i}}\Big(\cos\varphi_1\sin\varphi_2\frac{\partial}{\partial\gamma} - \tan\gamma\cos\varphi_1\cos\varphi_2\frac{\partial}{\partial\varphi_2} - \cot\gamma\sin\varphi_1\sin\varphi_2\frac{\partial}{\partial\varphi_1}\Big) \quad , \\ L_3 &= \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\varphi_1} \quad . \end{aligned}\right\} \tag{A.5}$$

The functions (**??**) are eigenfunctions of the set of the three operators $\{\mathcal{L}_2, M_3^2, L_3^2\}$:

$$\left.\begin{aligned} \mathcal{L}_1\Psi_{Jm_1m_2} &= \hbar^2 J(J+2)\Psi_{Jm_1m_2} \quad , \\ L_3^2\Psi_{Jm_1m_2} &= \hbar^2 m_1^2\Psi_{Jm_1m_2} \quad , \\ M_3^2\Psi_{Jm_1m_2} &= \hbar^2 m_2^2\Psi_{Jm_1m_2} \quad . \end{aligned}\right\} \tag{A.6}$$

The matrix-elements of the operators $L_\pm$ have the following form ($L_\pm = L_1 \pm \mathrm{i}L_2$)

$$\begin{aligned} < Jm_1'm_2'|L_+|Jm_1m_2 > &= -\frac{\hbar}{2\mathrm{i}}\sqrt{(J-m_1-m_2)(J+m_1+m_2+2)}\,\delta_{m_1',m_1+1}\delta_{m_2',m_2+1} \\ &\quad +\frac{\hbar}{2\mathrm{i}}\sqrt{(J-m_1+m_2)(J+m_1-m_2+2)}\,\delta_{m_1',m_1+1}\delta_{m_2',m_2-1} \quad , \qquad (A.7) \\ < Jm_1'm_2'|L_-|Jm_1m_2 > &= -\frac{\hbar}{2\mathrm{i}}\sqrt{(J+m_1-m_2)(J-m_1+m_2+2)}\,\delta_{m_1',m_1-1}\delta_{m_2',m_2+1} \\ &\quad +\frac{\hbar}{2\mathrm{i}}\sqrt{(J+m_1+m_2)(J-m_1-m_2+2)}\,\delta_{m_1',m_1-1}\delta_{m_2',m_2-1} \quad . \qquad (A.8) \end{aligned}$$

B Matrix Elements in the Hyper-Spherical Basis.

The wave-functions in spherical coordinates have the following form

$$\Psi_{Jlm}(\chi,\vartheta,\varphi) = S_{Jl}(\chi)Y_l^m(\vartheta,\varphi) \ , \tag{B.1}$$

where the $Y_l^m(\vartheta,\varphi)$ are the usual spherical harmonics, and the $S_{Jl}(\chi)$ are given by Gegenbauer polynomials, i.e.,

$$S_{Jl}(\chi) = 2^l l! \sqrt{\frac{2(J+1)(J-l)!}{\pi(J+l+1)!}} (\sin\chi)^l C_{J-l}^{l+1}(\cos\chi) \ . \tag{B.2}$$

They are normalized according to

$$\int_0^\pi \sin^2\chi d\chi \int_0^\pi \sin\vartheta d\vartheta \int_0^{2\pi} d\varphi \, \Psi^*_{Jlm}(\chi,\vartheta,\varphi)\Psi_{J'l'm'}(\chi,\vartheta,\varphi) = \delta_{JJ'}\delta_{ll'}\delta_{mm'} \ . \tag{B.3}$$

The functions (**??**) are eigenfunctions of the set of the three operators $\{\mathcal{L}_2, \mathbf{L}^2, L_3^2\}$:

$$\left.\begin{aligned} \mathcal{L}_1\Psi_{Jlm} &= \hbar^2 J(J+2)\Psi_{Jlm} \ , \\ \mathbf{L}^2\Psi_{Jlm} &= \hbar^2 l(l+1)\Psi_{Jlm} \ , \\ L_3^2\Psi_{Jlm} &= \hbar^2 m^2\Psi_{Jlm} \ . \end{aligned}\right\} \tag{B.4}$$

The operators $M_{1,2,3}$ in spherical coordinates have the form

$$\left.\begin{aligned} M_1 &= \frac{\hbar}{\mathrm{i}}\Big(\sin\vartheta\cos\varphi\frac{\partial}{\partial\chi} + \cot\chi\cos\vartheta\cos\varphi\frac{\partial}{\partial\vartheta} - \cot\chi\frac{\sin\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\Big) \ , \\ M_2 &= \frac{\hbar}{\mathrm{i}}\Big(\sin\vartheta\sin\varphi\frac{\partial}{\partial\chi} + \cot\chi\cos\vartheta\sin\varphi\frac{\partial}{\partial\vartheta} + \cot\chi\frac{\cos\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\Big) \ , \\ M_3 &= \frac{\hbar}{\mathrm{i}}\Big(\cos\vartheta\frac{\partial}{\partial\chi} - \cot\chi\sin\vartheta\frac{\partial}{\partial\vartheta}\Big) \ , \\ M_\pm &= M_1 \pm \mathrm{i}M_2 = \frac{\hbar}{\mathrm{i}}\mathrm{e}^{\pm\mathrm{i}\varphi}\Big(\sin\vartheta\frac{\partial}{\partial\chi} + \cot\chi\cos\vartheta\frac{\partial}{\partial\vartheta} \pm \mathrm{i}\frac{\cot\chi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\Big) \ . \end{aligned}\right\} \tag{B.5}$$

The corresponding matrix elements are given by

$$\begin{aligned} < Jl'm'|M_+|Jlm > = & \frac{\hbar}{\mathrm{i}}\sqrt{\frac{(l+m+1)(l+m+2)(J+l+2)(J-l)}{(2l+1)(2l+3)}}\,\delta_{l',l+1}\delta_{m',m+1} \\ & + \frac{\hbar}{\mathrm{i}}\sqrt{\frac{(l-m-1)(l-m)(J+l+1)(J-l+1)}{(2l+1)(2l-1)}}\,\delta_{l',l-1}\delta_{m',m+1} \ , \end{aligned} \tag{B.6}$$

$$\begin{aligned} < Jl'm'|M_-|Jlm > = & \frac{\hbar}{\mathrm{i}}\sqrt{\frac{(l-m+1)(l-m+2)(J+l+2)(J-l)}{(2l+1)(2l+3)}}\,\delta_{l',l+1}\delta_{m',m-1} \\ & - \frac{\hbar}{\mathrm{i}}\sqrt{\frac{(l+m+1)(l+m)(J+l+1)(J-l+1)}{(2l+1)(2l-1)}}\,\delta_{l',l-1}\delta_{m',m-1} \ , \end{aligned} \tag{B.7}$$

$$\begin{aligned} < Jl'm|M_3|Jlm > = & -\frac{\hbar}{\mathrm{i}}\sqrt{\frac{(l-m+1)(l+m+1)(J+l+2)(J-l)}{(2l+1)(2l+3)}}\,\delta_{l',l+1} \\ & + \frac{\hbar}{\mathrm{i}}\sqrt{\frac{(l-m)(l+m)(J+l+1)(J-l+1)}{(2l+1)(2l-1)}}\,\delta_{l',l-1} \ . \end{aligned} \tag{B.8}$$

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