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PATH INTEGRAL SOLUTION FOR NATANZON POTENTIALS

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ABSTRACT

We study Natanzon potentials in the path formalism. They generalize the known Gaussian, Besselian and Legendrian path integral representations in a systematic way.

1 Introduction.

The invention of the path integral by Feynman [?] has been proven fruitful in many branches in mathematics and theoretical physics. It has ever been a challenge to many to calculate path integrals exactly. However, the only path integral which could be calculated exactly for a long time was the path integral for the harmonic oscillator and closely related problems, i.e., physical systems which could be described by a quadratic Lagrangian.

The first major step towards the calculation of more complicated problems was the evaluation of the path integral for the hydrogen atom by Duru and Kleinert [?]. Their achievement could be accomplished by the invention of the space-time transformation technique, also called "Duru-Kleinert-transformation", "local-time-substitution", etc. This technique enables one to transform a complicated problem into a simpler one: in the case of the hydrogen atom the simpler system is just the harmonic oscillator.

Therefore it was possible to evaluate now many problems from ordinary textbook quantum mechanics by means of path integrals by using just a few Basic Path Integrals, the Gaussian Path Integral based on the quadratic Lagrangian [?], the Besselian Path Integral based on the radial harmonic oscillator [?], and the Legendrian Path Integral based on the Pöschl-Teller and modified Pöschl-Teller potentials [?, ?]. Actually, all Basic Path Integrals can be considered as a special path integral on a group manifold and the particular form of its solution in terms of the spectral expansion for a potential, say, can be seen as a matrix element in a coordinate space representation of the corresponding dynamical group, c.f. [?, ?, ?] and references therein.

However, almost all potential problems considered sofar allowed only the incorporation of two free parameters, for instance, in the Coulomb problem they are the Coulomb coupling constant and the angular momentum number. This is also the case in the so-called "conditionally solvable" potentials [?] which allow the incorporation of anharmonic potential forces. It is known, though, that there is a wider class of potentials, known as "Natanzon potentials" [?] which generalize the usual potentials in a systematic way. They have six free parameters, their explicit form may even not be known analytically, but the corresponding quantum mechanical problem is exactly solvable, at least implicitly. It is the purpose of this contribution to summarize some results concerning the path integral solutions of the two types of Natanzon potentials [?], the hypergeometric and confluent types, along with some further modifications in their structure which give rise to "conditionally solvable Natanzon potentials" [?]. These classes of potentials contain all former path integral solutions for particular potentials as special cases.

2 The Natanzon Potentials.

Due to lack of space our presentation can only be sketchy. For the formulation of the path integral we follow along the lines of Refs. [?] and [?, ?], and references therein. In the evaluations of the path integrals for all the Natanzon potentials we must perform the transformation $r \mapsto z$ together with the time-substitution $dt = ds/z'^2$ such that the new pseudo-time s'' can be introduced via the constraint $\int_0^{s''} ds/z'^2 = T = t'' - t'$, and with $z \to h$ for the confluent cases. This space-time transformation causes the emerging Schwarz derivative to cancel with the $\hbar^2$-term in each potential. In the case of the confluent Natanzon potentials, a Besselian Path Integral emerges, whereas in the case of the hypergeometric Natanzon potential an additional space-time transformation with $z = \tanh^2 x, x > 0$ must be performed.

2.1 Hypergeometric Natanzon Potential.

Let us start with the case of the hypergeometric Natanzon potential which is defined by

$$V(r) = \frac{\hbar^2}{2M}\frac{fz(z-1)+h_0(1-z)+h_1 z}{R(z)} + \frac{\hbar^2}{8M}\left(3\left(\frac{z''}{z'}\right)^2 - 2\frac{z'''}{z'}\right) \ , \tag{1}$$

where $R(z) = a_0 z^2 + b_0 z + c_0$, and $z = z(r)$ is implicitly defined by the differential equation $z' = 2z(1-z)/\sqrt{R(z)}$. The variable z varies in the interval $z \in (0,1)$. We get via the path integral solution of the modified Pöschl-Teeler potential for the corresponding path integral representation

$$\begin{aligned} K(r'',r';T) &= \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2}\dot{r}^2 - V(r)\right]dt\right\} \\ &= \int_{\mathbb{R}} \frac{dE}{2\pi\mathrm{i}} \mathrm{e}^{-\mathrm{i}ET/\hbar} G(r'',r';E) \ , \end{aligned} \tag{2}$$

with the Green's function $G(E)$ of the hypergeometric Natanzon potential given by

$$\begin{aligned} G(r'',r';E) = &\Big(R(r')R(r'')\Big)^{1/4}\frac{M}{\hbar^2}\frac{\Gamma(m_1-L_\nu)\Gamma(L_\nu+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)} \\ &\times\Big((1-z(r'))(1-z(r''))\Big)^{(m_1-m_2)/2}\Big(z(r')z(r'')\Big)^{(m_1+m_2)/2} \\ &\times {}_2F_1\Big(-L_\nu+m_1, L_\nu+m_1+1; m_1-m_2+1; 1-z_<^2\Big) \\ &\times {}_2F_1\Big(-L_\nu+m_1, L_\nu+m_1+1; m_1+m_2+1; z_>^2\Big) \ . \end{aligned} \tag{3}$$

${}_2F_1(a,b;c;z)$ is the hypergeometric function. C.f. [**?**, **?**, **?**] and references therein for the proper derivation of the Green's function for the Pöschl-Teller potential. Here denote $m_{1,2} = \frac{1}{2}(\eta \pm \sqrt{-2ME'}/\hbar)$, $L_\nu = \frac{1}{2}(\nu-1)$, and I have abbreviated $\eta^2 = h_0 + 1 - 2Mc_0E/\hbar^2$, $\nu^2 = f + 1 - 2Ma_0E/\hbar^2$, and $E' = (a_0+b_0+c_0)E - \hbar^2(h_1+1)/2M$. $z_{<,>}$ is the smaller/larger of z', z''. Note that the numbers η and ν are square roots and the specific sign they take may vary in different examples. The poles of the Green's function determine the bound state energy eigenvalues E_n ($n \in \mathbb{N}_0$)

$$\sqrt{h_0+1-\frac{2Mc_0E_n}{\hbar^2}} - \sqrt{f+1-\frac{2Ma_0E_n}{\hbar^2}} - \sqrt{h_1+1-\frac{2M}{\hbar^2}(a_0+b_0+c_0)E_n} = 2n+1 \ . \tag{4}$$

Equation (**??**) is an equation of fourth degree in E_n.

2.2 Confluent Natanzon Potentials.

The class of the confluent Natanzon potentials can be obtained by the substitution [**?**] $a_0 = \sigma_2/\tau^2$, $b_0 = \sigma_1/\tau$, $f = g_2/\tau^2$, $h_1 - h_0 - f = g_1/\tau$, $z = h/\tau$, and taking into account the limit $\tau \to 0$. This yields

$$V(r) = \frac{\hbar^2}{2M}\frac{g_2h^2+g_1h+\eta}{R(r)} + \frac{\hbar^2}{8M}\left(3\left(\frac{h''}{h'}\right)^2 - 2\frac{h'''}{h'}\right) \ , \tag{5}$$

where $R(r) = \sigma_2 h^2 + \sigma_1 h + c_0$, and $h = h(r)$ is implicitly defined by the differential equation $h'/2h = 1/\sqrt{R(r)}$. The variable r and the function $h = h(r)$ are assumed to be positive. Via the path integral for the radial harmonic oscillator we get the following path integral representation

$$K(r'',r';T) = \int_{\mathbb{R}} \frac{dE}{2\pi\mathrm{i}} \mathrm{e}^{-\mathrm{i}ET/\hbar} G(r'',r';E) \ , \tag{6}$$

Table 1: Natanzon Potentials

$R(z)$	$V(z), \quad \Delta V(z) = \frac{\hbar^2}{2M}\left(\left(\frac{z''}{z'}\right)^2 - \frac{z'''}{z'}\right)$	Range
$R = \sigma_2 h^2 + \sigma_1 h + c_0$	$\frac{\hbar^2}{2M}\frac{g_2 h^2 + g_1 h + \eta}{R(h)} + \Delta V(h)$	$h > 0$
$R = c_0$ $h = \mathrm{e}^{2x/\sqrt{c_0}}$	$\frac{\hbar^2}{2Mc_0}\left(g_2 \mathrm{e}^{4x/\sqrt{c_0}} + g_1 \mathrm{e}^{2x/\sqrt{c_0}} + \eta + 1\right)$	$x \in \mathbb{R}$
$R = \sigma_1 h$ $h = r^2/\sigma_1$	$\frac{\hbar^2}{2M}\left(\frac{g_2}{\sigma_1^2} r^2 + \frac{g_1}{\sigma_1} + \frac{\eta + 3/4}{r^2}\right)$	$r > 0$
$R = \sigma_2 h^2$ $h = 2r/\sqrt{\sigma_2}$	$\frac{\hbar^2}{2M}\left(\frac{g_2}{\sigma_2} + \frac{g_1/\sqrt{\sigma_2}}{r} + \frac{\eta}{4r^2}\right)$	$r > 0$
$R = a_0 z^2 + b_0 z + c_0$	$\frac{\hbar^2}{2M}\frac{fz(z-1) + h_0(1-z) + h_1}{R(h)} + \Delta V(h)$	$z \in (0,1)$
$R = 1$ $z = \frac{1}{2}(1 + \tanh x)$	$\frac{\hbar^2}{2M}\left(h_1 + \frac{h_0}{2} + 1 - \frac{h_0}{2}\tanh x - \frac{f/4}{\cosh^2 x}\right)$	$x \in \mathbb{R}$
$R = z$ $z = \tanh^2 r$	$\frac{\hbar^2}{2M}\left(h_1 + 1 + \frac{f - 3/4}{\cosh^2 r} + \frac{h_0 + h_1 + 3/4}{\sinh^2 r}\right)$	$r > 0$
$R = 1 - z$ $z = 1/\cosh^2 r$	$\frac{\hbar^2}{2M}\left(h_0 + h_1 + 1 - \frac{f + 3/4}{\cosh^2 r} + \frac{h_1 + 3/4}{\sinh^2 r}\right)$	$r > 0$
$R = z^2$ $z = 1 - \mathrm{e}^{-2r}$	$\frac{\hbar^2}{2M}\left(\frac{h_1}{4} + f + 1 + \frac{h_0 + h_1}{4\sinh^2 r} + \left(\frac{h_1}{2} - f\right)\coth r\right)$	$r > 0$
$R = 4z(1-z)$ $z = \frac{1}{2}(1 - \cos\varphi)$	$\frac{\hbar^2}{8M}\left(\frac{h_0 + h_1 + 3/4}{\sin^2(\varphi/2)} + \frac{h_1 + 3/4}{\cos^2(\varphi/2)} - (f+1)\right)$	$\varphi \in (0, \pi)$

with the Green's function $G(E)$ of the confluent Natanzon potential given by

$$G(r'', r'; E) = \left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')}\right)^{1/2} \frac{1}{\hbar}\sqrt{-\frac{M}{2E'}} \times \frac{\Gamma(\frac{1}{2} + \lambda - \kappa)}{\Gamma(2\lambda + 1)} W_{\kappa,\lambda}\left(\sqrt{-8ME'}\frac{h_>}{\hbar}\right) M_{\kappa,\lambda}\left(\sqrt{-8ME'}\frac{h_<}{\hbar}\right) \quad . \tag{7}$$

$W_{\kappa,\lambda}(z)$ and $M_{\kappa,\lambda(z)}$ are Whittaker functions. Here denote $\lambda^2 = \frac{1}{4}(\eta + 1 - 2Mc_0E/\hbar^2$, $\kappa = (E\sigma_1 - \hbar^2 g_1/2M/4\hbar)\sqrt{-M/2E'}$, and $E' = (\sigma_2 E - \hbar^2 g_2/2M)/4$. From the poles in the Green's function the bound state energy eigenvalues E_n can be determined by the equation ($n \in \mathbb{N}_0$)

$$\frac{E_n\sigma_1 - \hbar^2 g_1/2M}{2\hbar}\sqrt{\frac{M}{\hbar^2 g_2/M - 2\sigma_2 E_n}} - \sqrt{\eta + 1 - \frac{2Mc_0E_n}{\hbar^2}} = 2n + 1 \quad , \tag{8}$$

which is again an equation of fourth degree in E_n. Table 1 gives some examples for special cases of Natanzon potentials. In the confluent cases we must replace $z \to h$ in ΔV.

2.3 Conditionally Solvable Natanzon Potentials.

2.3.1 A generalized Coulomb potential.

As a first example of a conditionally solvable Natanzon potential I consider a modification of (**??**) by changing the power-behaviour of h in the g_1-term into a g_1h^3, in the σ_1-term into σ_1h^4, and we set $c_0 = \eta = 0$. Thus we obtain

$$V(r) = \frac{\hbar^2}{2M}\frac{g_1h^3 + g_2h^2}{R(r)} + \Delta V(h(r)) \ , \tag{9}$$

where $R(r) = \sigma_1h^4 + \sigma_2h^2$, $h'(r)/2h(r) = 1/\sqrt{R(r)}$. The variable r and the function $h = h(r)$ are assumed to be positive. We obtain via the path integral for a shifted harmonic oscillator the path integral representation [**?**]

$$K(r'', r'; T) = \int_{\mathbb{R}} \frac{dE}{2\pi\mathrm{i}} \mathrm{e}^{-\mathrm{i}ET/\hbar} G(r'', r'; E) \ , \tag{10}$$

where the Green's function $G(E)$ is given by ($\omega = -\sigma_1 E/2M$)

$$\begin{aligned} G(r'', r'; E) = & \left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')}\right)^{1/2} \sqrt{\frac{M}{\pi\hbar^3\omega}}\,\Gamma(-\nu) \\ & \times\Bigg\{ D_\nu\left[\sqrt{\frac{2M\omega}{\hbar}}\left(h_>(r) - \frac{\hbar^2 g_1}{4M\sigma_1 E}\right)\right] D_\nu\left[-\sqrt{\frac{2M\omega}{\hbar}}\left(h_<(r) - \frac{\hbar^2 g_1}{4M\sigma_1 E}\right)\right] \\ & - D_\nu\left[\sqrt{\frac{2M\omega}{\hbar}}\left(h(r') - \frac{\hbar^2 g_1}{4M\sigma_1 E}\right)\right] D_\nu\left[\sqrt{\frac{2M\omega}{\hbar}}\left(h(r'') - \frac{\hbar^2 g_1}{4M\sigma_1 E}\right)\right] \\ & \times D_\nu\left(\sqrt{\frac{2M\omega}{\hbar}}\,\frac{\hbar^2 g_1}{4M\sigma_1 E}\right) \Bigg/ D_\nu\left(-\sqrt{\frac{2M\omega}{\hbar}}\,\frac{\hbar^2 g_1}{4M\sigma_1 E}\right)\Bigg\} \ . \end{aligned} \tag{11}$$

Table 2: Confluent Conditionally Solvable Natanzon Potentials

$R(h)$	$V(h), \quad \Delta V(h) = \frac{\hbar^2}{2M}\left(\left(\frac{h''}{h'}\right)^2 - \frac{h'''}{h'}\right)$	Range
$R = \sigma_1h^4 + \sigma_2h^2$	$\frac{\hbar^2}{2M}\frac{g_1h^3 + g_2h^2}{R(h)} + \Delta V(h)$	$h > 0$
$R = 16h^4, h = \sqrt{r}$	$\frac{\hbar^2}{32M}\left(\frac{g_1}{\sqrt{r}} + \frac{g_2}{r} - \frac{3}{r^2}\right)$	$r > 0$
$R = \sigma_1h^3 + \sigma_2h^2$	$\frac{\hbar^2}{2M}\frac{g_1h^4 + g_2h^2}{R(h)} + \Delta V(h)$	$h > 0$
$R = 9h^3, h = r^{2/3}$	$\frac{\hbar^2}{18M}\left(g_1r^{2/3} + g_2r^{-2/3} - \frac{5}{4r^2}\right)$	$r > 0$

Note that a proper incorporation of the boundary conditions must be taken into account [**?**]. Here I have abbreviated

$$\nu = -\frac{1}{2} + \frac{1}{4\omega\hbar}\left(\sigma_2 E - \frac{\hbar^2 g_2}{2M} - \frac{\hbar^4 g_1^2}{16M^2\sigma_1 E}\right) \ . \tag{12}$$

This determines the energy-spectrum by zeros of the parabolic cylinder function, i.e.,

$$D_{\nu_n}\left(-\sqrt{\frac{2M\omega_n}{\hbar}}\,\frac{\hbar^2 g_1}{4M\sigma_1 E_n}\right)=0\ . \tag{13}$$

I have indicated by $\nu_n=\nu(E_n)$ and $\omega_n=\omega(E_n)$ the explicit dependence on E_n. In table 2 another example can be found.

2.3.2 A modified Rosen-Morse potential.

For another class of conditionally solvable potentials I want to investigate, I consider the hypergeometric Natanzon potential with the following modifications

$$V(r)=\frac{\hbar^2}{2M}\frac{\frac{3}{4}z(1-z)+h_0(1-z)+h_1 z^{1/2}}{R(z)}+\Delta V(z(r))\ , \tag{14}$$

where $R(z)=b_0z+c_0$, and $z=z(r)$ is implicitly defined by the differential equation $z'=2z(1-z)/\sqrt{R(z)}$. The variable z varies in the interval $z\in(0,1)$. Therefore we get via the path integral solution for the Rosen-Morse potential for the corresponding path integral

$$K(r'',r';T)=\int_{\mathbb{R}}\frac{dE}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}ET/\hbar}G(r'',r';E)\ , \tag{15}$$

with the Green's function $G(E)$ of the conditionally solvable hypergeometric Natanzon potential given by

$$\begin{aligned}G(r'',r';E)=&\frac{M}{\hbar^2}\left(\frac{R(r')R(r'')}{z(r')z(r'')}\right)^{1/4}\frac{\Gamma(m_1-L_E)\Gamma(L_E+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\\
&\times\left(\frac{2\sqrt{z(r')}}{1+\sqrt{z(r')}}\cdot\frac{2\sqrt{z(r'')}}{1+\sqrt{z(r'')}}\right)^{\frac{1}{2}(m_1+m_2+1)}\left(\frac{1-\sqrt{z(r')}}{1+\sqrt{z(r')}}\cdot\frac{1-\sqrt{z(r'')}}{1+\sqrt{z(r'')}}\right)^{\frac{1}{2}(m_1-m_2)}\\
&\times{}_2F_1\left(-L_E+m_1,L_E+m_1+1;m_1-m_2+1;\frac{1-\sqrt{z_>(r)}}{1+\sqrt{z_>(r)}}\right)\\
&\times{}_2F_1\left(-L_E+m_1,L_E+m_1+1;m_1+m_2+1;\frac{2\sqrt{z_<(r)}}{1+\sqrt{z_<(r)}}\right)\ .\end{aligned} \tag{16}$$

Here are

$$L_E=\frac{1}{2}\left(\sqrt{1-h_1-2M(b_0+c_0)E/\hbar^2}\ -1\right)\ , \tag{17}$$

$$m_{1,2}=\sqrt{h_0+1-2Mc_0E/\hbar^2}\pm\frac{1}{2}\sqrt{h_1+1-2M(b_0+c_0)E/\hbar^2}\ . \tag{18}$$

Furthermore I have abbreviated $\eta^2=h_0+1-2Mc_0E/\hbar^2$. Note that the number η is a square root and the specific sign it takes may vary in different examples. From the poles of the Green's function, we derive the quantization condition for the bound state wave-functions ($n\in\mathbb{N}$)

$$\begin{aligned}&\sqrt{h_0+1-2Mc_0E_n/\hbar^2}+(n+\tfrac{1}{2})\\
&\quad=\frac{1}{2}\left(\sqrt{1+h_1-2M(b_0+c_0)E_n/\hbar^2}-\sqrt{1-h_1-2M(b_0+c_0)E_n/\hbar^2}\right)\ .\end{aligned} \tag{19}$$

This equation is again an equation of fourth degree in E_n. In table 3 I have listed further possibilities of hypergeometric conditionally solvable Natanzon potentials. The cases $R=z$, $R=1-z$ and $R=4z(1-z)$ lead to already known potentials, the others to genuine new structures.

With our successful path integral solutions we have therefore accomplished new features in a Table of Path Integrals [**?**] which add to the endeavour "How to Solve Path Integrals in Quantum Mechanics" [**?**].

Table 3: Hypergeometric Conditionally Solvable Natanzon Potentials

$R(z)$	$V(z), \quad \Delta V = \frac{\hbar^2}{2M}\left(\left(\frac{z''}{z'}\right)^2 - \frac{z'''}{z'}\right)$	Range
$R = b_0 z + c_0$	$\frac{\hbar^2}{2M}\frac{3z(1-z)/4 + h_0(1-z) + h_1 z^{1/2}}{R(z)} + \Delta V(z)$	$z \in (0,1)$
$R = 1$ $z = \frac{1}{2}(1+\tanh x)$	$\frac{\hbar^2}{2M}\left(h_0 + 1 - \frac{h_0 - 3/4}{1+\mathrm{e}^{-2x}} + \frac{h_1}{\sqrt{1+\mathrm{e}^{-2x}}} - \frac{3}{4(1+\mathrm{e}^{-2x})^2}\right)$	$x \in \mathbb{R}$
$R = z$ $z = \tanh^2 r$	$\frac{\hbar^2}{2M}\left(\frac{h_0 + 3/4}{\sinh^2 r} + h_1 \coth r + 1\right)$	$r > 0$
$R = 1 - z$ $z = 1/\cosh^2 r$	$\frac{\hbar^2}{2M}\left(h_0 + 1 + \frac{3}{4\sinh^2 r} + h_1 \frac{\cosh r}{\sinh^2 r}\right)$	$r > 0$
$R = b_0 z + c_0$	$\frac{\hbar^2}{2M}\frac{3z(1-z)/4 + h_0(1-z) + h_1(1-z)^{1/2}}{R(z)} + \Delta V(z)$	$z \in (0,1)$
$R = 1$ $z = \frac{1}{2}(1+\tanh x)$	$\frac{\hbar^2}{2M}\left(h_0 + 1 - \frac{h_0 - 3/4}{1+\mathrm{e}^{-2x}} + \frac{h_1 \mathrm{e}^{-x}}{\sqrt{1+\mathrm{e}^{-2x}}} - \frac{3}{4(1+\mathrm{e}^{-2x})^2}\right)$	$x \in \mathbb{R}$
$R = z$ $z = \tanh^2 r$	$\frac{\hbar^2}{2M}\left(\frac{h_0 + 3/4}{\sinh^2 r} + h_1 \frac{\cosh r}{\sinh^2 r}\right)$	$r > 0$
$R = 1 - z$ $z = 1/\cosh^2 r$	$\frac{\hbar^2}{2M}\left(h_0 + 1 + \frac{3}{4\sinh^2 r} + h_1 \coth r\right)$	$r > 0$
$R = a_0 z^2 + b_0 z$	$\frac{\hbar^2}{2M}\frac{fz(z-1) - 3(1-z)/4 + h_1 z^{3/2}}{R(z)} + \Delta V(z)$	$z \in (0,1)$
$R = z$ $z = \tanh^2 r$	$\frac{\hbar^2}{2M}\left(h_1 \tanh r - \frac{f + 3/4}{\cosh^2 r} + 1\right)$	$r > 0$
$R = z^2$ $z = 1 - \mathrm{e}^{-2r}$	$\frac{\hbar^2}{2M}\left(f + 1 - \frac{f + 3/4}{1-\mathrm{e}^{-2r}} + \frac{h_1}{\sqrt{1-\mathrm{e}^{-2r}}} - \frac{3}{4(1-\mathrm{e}^{-2r})^2}\right)$	$r > 0$
$R = 4z(1-z)$ $z = \frac{1}{2}(1-\cos\varphi)$	$\frac{\hbar^2}{8M}\left[h_1 \frac{\sin(\varphi/2)}{\cos^2(\varphi/2)} - \left(f + \frac{1}{4}\right) + \frac{3}{4}\frac{1}{\cos^2(\varphi/2)}\right]$	$\varphi \in (0,\pi)$
$R = a_0 z^2 + b_0 z$	$\frac{\hbar^2}{2M}\frac{fz(z-1) - 3(1-z)/4 + h_1 z^{3/2}\sqrt{1-z}}{R(z)} + \Delta V(z)$	$z \in (0,1)$
$R = z$ $z = \tanh^2 r$	$\frac{\hbar^2}{2M}\left(h_1 \frac{\sinh r}{\cosh^2 r} - \frac{f + 3/4}{\cosh^2 r} + 1\right)$	$r > 0$
$R = z^2$ $z = 1 - \mathrm{e}^{-2r}$	$\frac{\hbar^2}{2M}\left(f + 1 - \frac{f + 3/4}{1-\mathrm{e}^{-2r}} + \frac{h_1 \mathrm{e}^{-r}}{\sqrt{1-\mathrm{e}^{-2r}}} - \frac{3}{4(1-\mathrm{e}^{-2r})^2}\right)$	$r > 0$
$R = 4z(1-z)$ $z = \frac{1}{2}(1-\cos\varphi)$	$\frac{\hbar^2}{8M}\left[h_1 \tan\frac{\varphi}{2} - \left(f + \frac{1}{4}\right) + \frac{3}{4}\frac{1}{\cos^2(\varphi/2)}\right]$	$\varphi \in (0,\pi)$

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