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A matrix S for all simple current extensions

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Abstract

A formula is presented for the modular transformation matrix S for any simple current extension of the chiral algebra of a conformal field theory. This provides in particular an algorithm for resolving arbitrary simple current fixed points, in such a way that the matrix S we obtain is unitary and symmetric and furnishes a modular group representation. The formalism works in principle for any conformal field theory. A crucial ingredient is a set of matrices S^J_{ab}, where J is a simple current and a and b are fixed points of J. We expect that these input matrices realize the modular group for the torus one-point functions of the simple currents. In the case of WZW-models these matrices can be identified with the S-matrices of the orbit Lie algebras that were introduced recently in [1]. As a special case of our conjecture we obtain the modular matrix S for WZW-theories based on group manifolds that are not simply connected, as well as for most coset models.

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1. Introduction

One of the more important unsolved problems in conformal field theory is that of classifying and understanding all modular invariant partition functions. Besides the diagonal modular invariant, one can often construct other modular invariant partition functions for conformal field theories. In spite of some recent progress, even in the most extensively studied case of WZW-models based on simple Lie algebras, the classification of these invariants is still incomplete. Moreover, even for the known non-diagonal invariants, a satisfactory interpretation as a full-fledged conformal field theory is available in only a few cases.

In this paper, we will be interested in modular invariants that suggest an extension of the chiral algebra, *i.e.* invariants of the general form[§]

$$\sum_i N_i \, |\sum_\ell m_{i,\ell} \mathcal{X}_\ell|^2 \ . \tag{1.1}$$

Here $\mathcal{X}_\ell$ is a character of the original theory (which we will call the unextended theory, even though its chiral algebra will in general itself be an extension of the Virasoro algebra), $m_{i,\ell}$ a non-negative integer and N_i a positive integer. The identity character of the unextended theory appears exactly once (by convention for $i = \ell = 0$, with $m_{0,0} = N_0 = 1$).

Such a partition function suggests an interpretation in terms of an extended algebra, with each term representing the contribution of an irreducible representation of that algebra. The fields which we would like to interpret as the generators of an extended chiral algebra can then be read off the term containing the identity. The existence and uniqueness of such an extended algebra is however by no means guaranteed. Indeed, several examples are known of partition functions of the form (1.1) that do not correspond to any conformal field theory (see *e.g.* [2,3]). We are not aware of examples of modular invariant combinations of characters of rational conformal field theories that can be interpreted in more than one way in terms of an extended chiral algebra, but this possibility cannot be ruled out either.

Having found a modular invariant partition function, the next logical step is to attempt to derive the modular transformation matrix S of the characters of the putative new theory. If such a matrix can indeed be written down, a further important consistency check is the computation of the fusion coefficients using Verlinde's formula [4]. If no inconsistency appears, one can try to compute operator product coefficients and correlation functions. In principle, any of these steps may fail or produce a non-unique answer.

§ In particular we do not study 'heterotic' invariants or fusion rule automorphisms, since our interest is in defining the matrix S for the chiral half of a theory.

Apart from a few trivial theories, essentially the only case where the whole programme can be carried through is the extension of WZW-models by currents of spin 1. These invariants can be interpreted as conformal embeddings, and hence the extended theory is again a WZW-model.

In this paper we will focus on another case that can be expected to be manageable, namely, for arbitrary rational conformal field theories, the so-called *simple current invariants* ([5,6], for a review see [7]). These invariants have been completely classified for any conformal field theory [8,9]. Since the construction of the partition function can be formulated in terms of orbifold methods, it is reasonable to expect a conformal field theory to exist. Therefore in particular there should exist a unitary and symmetric matrix S with all the usual properties. Unfortunately, orbifold methods do not seem to be of much help in actually determining this matrix. Such a computation has been carried out so far only for the $\mathbf{Z}_2$-orbifolds of the $c = 1$ models [10] and a few other simple examples. Therefore we will follow a different route. Here we will only consider the first step in the programme of describing the (putative) theory which corresponds to a given simple current modular invariant, namely the determination of S.

Our current knowledge indicates that for WZW-models based on simple Lie algebras nearly all off-diagonal invariants are simple current invariants. The remaining solutions, which are appropriately referred to as 'exceptional invariants', are rare (although there are a few infinite series) and unfortunately beyond the scope of this paper. For semi-simple algebras far less is known, but certainly the number of simple current invariants increases dramatically [8]. For most of these invariants the modular matrix S, one of the most basic quantities of a conformal field theory, could not be computed up to now. Although the most important application of our results appears to be in WZW-models, and also in coset theories (see below), we emphasize that simple current constructions are not *a priori* restricted to WZW-models. For this reason we will set up the formalism in its most general form, and focus on WZW-models only at the end.

For simple current invariants there are a few convenient simplifications in (1.1); for example the coefficients $m_{i,\ell}$ are either 0 or 1, and the vectors $\vec{m}_i$ are all orthogonal. The problem we address in this paper occurs whenever one of the multiplicities N_i is larger than 1. This situation occurs if one (or more) of the simple currents in the extension has a fixed point, *i.e.* if it maps a primary field to itself. If there are no fixed points, one can compute the matrix S simply by looking at the modular transformation properties of the characters. However, if $N_i > 1$ for some value of i, this may imply that the new theory has more than one character corresponding to the i^{th} term (the multiplicity will in fact be determined in this paper). In that case all characters in the ith term of the sum (1.1) are identical as functions of the modular parameter τ and possible Cartan angles of the unextended theory, and one cannot disentangle their transformation under $\tau \mapsto -\frac{1}{\tau}$.

Fixed points occur very often in simple current invariants. A simple and well-known example is the D-invariant of su(2) level 4, which has the form $|\mathcal{X}_0 + \mathcal{X}_4|^2 + 2\,|\mathcal{X}_2|^2$. There are two representations with character $\mathcal{X}_2$. The known modular transformations

of su(2) level 4 do not tell us how they transform into each other. Hence we cannot deduce the matrix S directly from that of su(2) level 4. If we assume that a new theory with an extended chiral algebra exists, we know more about S: it must be unitary and symmetric and form, together with the known matrix T, a representation of the modular group, hence satisfy $S^4 = \mathbb{1}$ and $(ST)^3 = S^2$. In the example the most general form of S that is symmetric and agrees with the known transformations of the $su(2)_4$ characters is

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2}+\epsilon & -\frac{1}{2}-\epsilon \\ 1 & -\frac{1}{2}-\epsilon & -\frac{1}{2}+\epsilon \end{pmatrix} ,$$

where ϵ is an unknown parameter. Imposing unitarity fixes ϵ up to a sign. Finally imposing $(ST)^3 = S^2$ fixes ϵ completely (and one obtains the matrix S of su(3) level 1). It is this solution that we wish to generalize to arbitrary conformal field theories with simple currents.

In the general case one can proceed as follows [2]. First one computes the naive matrix S associated with the partition function (1.1) by 'orbit-averaging' the matrix S of the original theory, and by resolving the ith row and column of S into (at most) N_i distinct rows and columns. To make the new matrix unitary, correction terms are needed for the entries between fixed point representations. These corrections can be described in terms of a matrix S^J that acts only on the fixed points (in fact there is such a matrix for every current J in the extension, hence the upper index). It can then be shown that the resolved matrix* $\tilde{S}$ is unitary and symmetric and satisfies $(\tilde{S}T)^3 = \tilde{S}^2$ if S^J has all those properties on the fixed points. Since T is known and unambiguous, this information can be used in some cases to get plausible *ansätze* for S^J.

The problem with this method is that one has to identify the T-eigenvalues of the degenerate representations with a known spectrum. Surprisingly, in many WZW-models these T-eigenvalues can be recognized as those of another WZW-theory (up to an overall phase). In [7] this was achieved for all simple current invariants of WZW-models based on simple, simply laced Lie algebras, as well as for a few other cases. However, the fixed point spectrum obtained for B_n and C_{2n} theories did not correspond (with a few exceptions) to that of a WZW-model or any other known conformal field theory. In addition, the application of this procedure to more complicated combinations of simple currents, with fixed points of all possible types, has never been formulated.

The main results reported here are:

- A conjecture is presented for the matrix $\tilde{S}$ for any simple current invariant of any conformal field theory for which the relevant matrices S^J are known. One important problem to be addressed is precisely how many irreducible representations of the extended algebra one gets if $N_i > 1$. We will present a conjecture for this case

$\star$ To prevent confusion between the matrices for the unextended and the extended theories, we denote the former as S and the latter as $\tilde{S}$.

as well; perhaps surprisingly, the answer is not always N_i. This means in particular that not even the spectrum of certain extended theories was known before.

- A matrix S^J is presented for any simple current of any WZW-model. This requires the extension of the results of [7] to all simple algebras. The construction of S^J was essentially already achieved in [1]. It was found that the 'missing' cases correspond to spectra of twisted affine Kac-Moody algebras. The matrices S^J for the missing cases have been obtained earlier from rank-level duality [11], but now for the first time they can be treated on an equal footing for all WZW-models: they can be identified with the modular matrices S of the 'orbit Lie algebras' that are associated to the Dynkin diagram automorphisms induced by the currents J. Although the fixed point resolution matrices S^J can in principle be extracted from [1] or [11], we believe it is worthwhile to present the result in a more accessible way.

The term 'conjecture' is used in the first item because the conditions we solve are necessary, but not sufficient. An important condition that in the general case is not easy to impose is that the new matrix $\tilde{S}$ must yield sensible fusion rule coefficients when substituted in Verlinde's formula. (Note that a rigorous proof of the conjecture would require in particular an explicit construction of the extended chiral algebra, as well as the proof that it gives rise to a reasonable conformal field theory.)

However, there are several reasons why we believe our solution is the correct one, namely:

— The solution is *mathematically natural* in the sense that a very simple closed formula can be given that applies to all cases.

— It has been checked by explicit computation to give non-negative integer fusion coefficients for all types of simple and many semi-simple algebras (of course, such checks have been done only for a limited range of ranks and levels).

— It can be *derived* rigorously as the matrix $\tilde{S}$ that describes the transformation of the characters of diagonal coset models.

Our results also allow the computation of the matrix $\tilde{S}$ for most coset models. The modular properties of coset models G/H can be described in terms of a formal tensor product of the G-theory with the complement of the H-theory (the complement of a conformal field theory has by definition a complex conjugate representation of the modular group). One gets a matrix $S_G \otimes S_H^*$ that acts on the branching functions. In many cases some of the branching functions vanish, while others have to be identified with each other, and correspond to a single primary field in the theory. This is known as *field identification*. Field identification can be formulated – as far as modular transformation properties are concerned – in terms of a simple current extension of this tensor product, except in a few rare cases (the so-called 'maverick' cosets [12]). Hence the computation of the matrix $\tilde{S}$ of coset models is technically identical to the computation for a suitably chosen integer spin simple current invariant so that our conjecture regarding fixed point resolution for $\tilde{S}$ covers this case as well.

However, there is an essential difference in the interpretation and computation of the fixed point characters. In an integer spin modular invariant each of the representations originating from a fixed point has the same character with respect to the chiral algebra of the unextended theory, namely the one appearing within the absolute value symbol in (1.1). On the other hand in coset models the latter character is to be interpreted as the *sum* of N characters that may be (and in general are) distinct as functions of τ. Hence the degeneracy is lifted, and we can determine $\tilde{S}$ directly from the transformation of the characters. All of this is useful only if one is able to compute the coset characters, which for $N > 1$ are not equal to the branching functions. The differences between the branching functions and the coset characters are called *character modifications*.

We have accomplished this for the diagonal coset models $G \times G/G$, by realizing field identification on the entire Hilbert space, and identifying the various eigenspaces of field identification on the fixed points [13]. Having done this, we can prove that for diagonal coset models the character modifications are equal to branching functions of twining characters. Twining characters have been defined in [1] and will be briefly described in section 6. For our present purpose all we need is the fact that in [1] the modular transformations of these characters were obtained. This allowed us to derive the modular transformations of the characters of diagonal coset models. The formula for $\tilde{S}$ we conjecture here is a generalization of the one in [13]. The formula is not identical, since in the general case a complication arises that does not occur for diagonal coset models.

While in the case of coset conformal field theories and for integer spin simple current invariants of WZW-theories the associated orbit Lie algebras provide natural candidates for the matrices S^J that implement fixed point resolution, it is not clear whether analogous data are available for arbitrary rational conformal field theories. However, we expect that the matrices that describe the transformation of the one-point functions of the simple currents on the torus will do the job. Note that it follows on quite general reasons that these one-point functions have good modular transformation properties [14] and are non-zero only for fixed points. The identification of the matrices S^J with the S-matrices for torus one-point functions implies in particular the conjecture that in the case of WZW-models the modular transformation properties of these one-point functions are described by the orbit Lie algebras. Apart from being conceptually elegant, this has the practical advantage that an explicit closed formula for the matrices S^J can be given, namely the Kac-Peterson [15] formula of the orbit Lie algebra.

The organization of this paper is as follows. In the next section we formulate the conditions we impose on the solution. They consist of six conditions that are beyond question, plus two additional ones that should be considered as working hypotheses. In section 3 we discuss what can be deduced about $\tilde{S}$ using only the six unquestionable conditions.

In section 4 we perform a Fourier transformation on the labels of the resolved fixed points. If one imposes the two additional conditions, this suggests an *ansatz* for the matrix $\tilde{S}$ in the general case, and leads naturally to a definition of the quantities S^J. This

ansatz is an additional assumption, and for this reason we do not claim to have found the most general solution satisfying all conditions. The characterization of the primary fields of the extended theory and the formula for S, given by equation (5.1), are the main results of this paper. It can be shown to satisfy all conditions given certain properties of the matrices S^J. This proof is independent of the heuristic arguments leading us to (5.1), and is briefly summarized in section 5. Readers who are only interested in the result may therefore in fact skip sections 3 and 4.

In section 6 we briefly review the concepts of twining characters and orbit Lie algebras and apply our formalism to WZW-models. Realizing that the WZW-model based on $G = \tilde{G}/\mathcal{Z}$, where $\tilde{G}$ is the universal covering Lie group of G and $\mathcal{Z}$ a subgroup of the center of $\tilde{G}$, is described by the corresponding simple current invariant, this leads in particular to a conjecture for the S-matrix of WZW-models based on non-simply connected compact Lie groups (for the precise definition of these models see [16]).

2. Conditions

With respect to the fusion product, the set of simple currents of a conformal field theory forms a finite abelian group, known as the center $\mathcal{C}$ of the theory. To any subgroup $\mathcal{G} \subseteq \mathcal{C}$ of mutually local integral spin simple currents one can associate a modular invariant partition function in which the chiral algebra is extended by this set of currents. (An explicit expression for the partition function will be given in (2.5).) Our goal is to write down for any such extended theory a pair of matrices $\tilde{S}$ and $\tilde{T}$, which must satisfy the following requirements:

$$\begin{aligned}
&[\mathrm{I}] && \tilde{S} \text{ and } \tilde{T} \text{ act 'correctly' on the characters.} \\
&[\mathrm{II}] && \tilde{S} \text{ is symmetric.} \\
&[\mathrm{III}] && \tilde{S} \text{ is unitary.} \\
&[\mathrm{IV}] && \tilde{S}^2 = \tilde{C}. \\
&[\mathrm{V}] && \tilde{S} \text{ and } \tilde{T} \text{ satisfy } (\tilde{S}\tilde{T})^3 = \tilde{S}^2.
\end{aligned} \tag{2.1}$$

Here $\tilde{C}$ is a matrix with entries 0 or 1 satisfying $\tilde{C}^2 = 1$, *i.e.* a permutation of order 2, which furthermore acts trivially on the identity. The characters of the theory are linear combinations of characters of the unextended theory. This gives us some information about their modular transformations in terms of the matrices S and T of the unextended theory. The meaning of the first condition is that the matrices $\tilde{S}$ and $\tilde{T}$ must reproduce this knowledge. This is the only condition that relates $\tilde{S}$ to the matrix S of the unextended theory. The matrix $\tilde{T}$ follows in a straightforward way from T using condition [I], and therefore we do not specify any explicit conditions for it. As usual, it is a unitary diagonal matrix.

Although we do not impose a general integrality condition on the fusion rules derived from $\tilde{S}$, we make an exception for certain simple current fusion rules, because they are of

special importance to us, and *can* be analyzed. Suppose we are given a simple current J in the unextended theory that is 'local' (i.e., has zero monodromy charge) with respect to all currents in $\mathcal{G}$, so that its orbit is an allowed field in the extended theory, i.e. J is not 'projected out' by the extension. We claim that this orbit gives rise to a simple current in the extended theory.[⋆] Note that neither the identity primary field nor the simple current J are fixed by the currents in $\mathcal{G}$. It is then easy to see that the S-matrix of the extended theory satisfies $\tilde{S}_{0,J} = \tilde{S}_{0,0}$. It follows that – if indeed $\tilde{S}$ leads to correct fusion rules – the primary field J in the extended theory has quantum dimension 1 and hence again is a simple current. Therefore we require

$$\text{[VI]} \qquad \tilde{\mathcal{N}}_{J,b}{}^{c} = \delta_{Jb,c}. \tag{2.2}$$

Here $\tilde{\mathcal{N}}$ are the fusion coefficients obtained from $\tilde{S}$ via the Verlinde [4] formula, and $Jb \equiv J \times b$ is another primary field in the extended theory, obtained as the fusion product of J and b. The fusion coefficients are finite since $\tilde{S}_{0,n} \neq 0$ after fixed point resolution.

Conditions [I] – [VI] are clearly necessary. It will be helpful to impose two additional working hypotheses, namely

$$\begin{aligned} &\text{[VII]} && \text{Consistency of successive extensions.} \\ &\text{[VIII]} && \text{Fixed Point Homogeneity.} \end{aligned} \tag{2.3}$$

Condition [VII] applies when there are several distinct paths to the final result. This is the case if there exist several distinct chains of subgroups of the form

$$\mathcal{G} \equiv \mathcal{H}_0 \supset \mathcal{H}_1 \supset \ldots \supset \mathcal{H}_n \equiv \{\mathbf{1}\}\,, \tag{2.4}$$

which is possible if the order of $\mathcal{G}$ is not prime. If we have a general formula that can deal with any extension, in particular it will give a result for each such chain, when the extensions are performed successively. Condition [VII] states that the answer should not depend on the specific chain chosen. Condition [VIII] means that in the final result the resolved fixed points coming from the same primary field a are indistinguishable in as far as their modular transformations and fusion rules are concerned. This condition has to be handled with some care; while for coset theories it does hold for $\tilde{S}$ and the fusion rules (in all known cases), it does *not* apply to the characters, for which one has to include so-called character modifications.

⋆ Note, however, that fixed point resolution might introduce additional simple currents that are not related to simple currents of the unextended theory.

An integer spin simple current modular invariant has the general form

$$Z = \sum_{\substack{\text{orbits } a \\ Q=0}} |\mathcal{S}_a| \cdot |\sum_{J \in \mathcal{G}/\mathcal{S}_a} \mathcal{X}_{Ja}|^2 . \tag{2.5}$$

Here $\mathcal{G}$ is a subgroup of the center whose elements have integer spin; the first sum is over all $\mathcal{G}$-orbits of primary fields in the theory with zero monodromy charge Q. The monodromy charges of primary fields a with respect to the simple current J are defined as the fractional parts

$$Q_J(a) = h(a) + h(J) - h(Ja) \bmod \mathbf{Z}$$

of combinations of conformal weights. The S-matrix elements of fields on the same simple current orbit are related by

$$S_{La,b} = \mathrm{e}^{2\pi \mathrm{i} Q_L(b)} S_{a,b} \, . \tag{2.6}$$

The group $\mathcal{S}_a$ appearing in (2.5), the *stabilizer* of a, is the subgroup of $\mathcal{G}$ that acts trivially on the orbit a; $\mathcal{X}_a$ is the character of the field a, and $\mathcal{X}_{Ja}$ is the character of the representation obtained when the simple current J acts on the orbit a. Since the center $\mathcal{C}$ is abelian, all elements of an orbit have the same stabilizer $\mathcal{S}_a$. The representations in the orbit a are called *fixed points* with respect to the currents in the stabilizer. Below we will also use the notation

$$\mathcal{G}_a := \mathcal{G}/\mathcal{S}_a \tag{2.7}$$

for the factor group of currents that acts non-trivially on a.

Implicit in the foregoing discussion is a choice of a representative within each orbit. In the following $a, b, c, \ldots$ always refer to a definite choice of orbit representatives. The primary fields in the unextended theory are then obtained as Ja with $J \in \mathcal{G}_a$. Quantities like S, T and the fusion coefficients $\mathcal{N}$ in the extended theory will be distinguished by a tilde.

On general grounds one expects [17] that it should be possible to rewrite the invariant (2.5) as a standard diagonal one, *i.e.* as

$$Z = \sum_{\alpha} |\tilde{\mathcal{X}}_\alpha|^2 \, . \tag{2.8}$$

The matrix $\tilde{S}$ acts on the new characters $\tilde{\mathcal{X}}_\alpha$. The relation between (2.8) and (2.5) is straightforward if there are no fixed points, *i.e.* if $|\mathcal{S}_a| = 1$ for all a. If $|\mathcal{S}_a| = 2$ or 3 there is only one possible interpretation, namely that the orbit a corresponds to precisely $|\mathcal{S}_a|$ representations in the extended theory. The characters of those representations are

identical with respect to the unextended algebra. If $|\mathcal{S}_a| \geq 4$ there are as many interpretations as there are ways of writing $|\mathcal{S}_a|$ as a sum of squares. Each such square can be absorbed in the definition of an extended character $\tilde{\mathcal{X}}_\alpha$ rather than being interpreted as a multiplicity.

In general there is thus ample room for ambiguities. First of all, even the number of primary fields is not evident. Furthermore, for each possible choice of the spectrum (which fixes the matrix $\tilde{T}$) there may exist more than one matrix $\tilde{S}$ that satisfies (2.1) and (2.2).

3. Fixed point resolution: Generalities

We will now examine the consequences of the first six conditions. The other two will be discussed later.

3.1. Condition [I]

Each character $\tilde{\mathcal{X}}_\alpha$ of the extended theory is in any case a sum of characters of the original theory, which belong to a definite orbit a. There may be more than one character of the extended theory that belongs to the same $\mathcal{G}$-orbit, so we need an extra label. Let us write $|\mathcal{S}_a|$ as a sum of squares,

$$|\mathcal{S}_a| = \sum_i (m_{a,i})^2 \ , \tag{3.1}$$

where i labels the different primaries into which a gets resolved (if we also impose condition [VIII], then $m_{a,i}$ has to be independent of i). Corresponding to this definition we have

$$\tilde{\mathcal{X}}_{a,i} = m_{a,i} \sum_{J \in \mathcal{G}_a} \mathcal{X}_{Ja} \tag{3.2}$$

so that $\sum_i |\tilde{\mathcal{X}}_{a,i}|^2 = |\mathcal{S}_a| \cdot |\sum_{J\in\mathcal{G}_a} \mathcal{X}_{Ja}|^2$. Clearly $\tilde{T}_{(a,i),(a,i)} = T_{a,a}$ independent of i. For $\tilde{S}$ we find

$$\tilde{\mathcal{X}}_{a,i}(-\frac{1}{\tau}) = m_{a,i} \sum_{J\in\mathcal{G}_a} \sum_b \sum_{K\in\mathcal{G}_b} S_{Ja,Kb}\, \mathcal{X}_{Kb}(\tau)\,.$$

Here and in the rest of this paper we write only the dependence on τ, but there might be additional variables (for example Cartan angles in affine Lie algebra characters). This may in fact be necessary to resolve ambiguities in the unextended theory. The last two sums form together a sum over all fields in the theory, but because of the sum on J only those fields Kb contribute that have zero monodromy charge with respect to all currents

in $\mathcal{G}$. We denote this as $Q_{\mathcal{G}} = 0$. For fields Kb with $Q_{\mathcal{G}}(Kb) = 0$ the matrix element $S_{Ja,Kb}$ is in fact independent of J, so we get

$$\tilde{\mathcal{X}}_{a,i}(-\frac{1}{\tau}) = m_{a,i}\,|\mathcal{G}_a| \sum_b \sum_{K\in\mathcal{G}_b} S_{a,b}\,\mathcal{X}_{Kb}(\tau)\ ,$$

where we have also used that as a consequence of $Q_{\mathcal{G}}(a) = 0$ we have $S_{a,Kb} = S_{a,b}$. Now we are faced with the problem that in general there is more than one character associated with the orbit whose representative is b. Hence we may write

$$\sum_{K\in\mathcal{G}_b} \mathcal{X}_{Kb}(\tau) = \frac{1}{N(\eta,b)} \sum_j \eta_{b,j}\tilde{\mathcal{X}}_{b,j}\ ,$$

where, in order to satisfy (3.2),

$$N(\eta,b) = \sum_j \eta_{b,j} m_{b,j}\ ,$$

and $\eta_{b,j}$ is a set of coefficients that is present for any b that splits into more than a single representation. We find thus the following formula for $\tilde{S}$:

$$\tilde{S}_{(a,i),(b,j)} = m_{a,i}\,|\mathcal{G}_a|\,S_{a,b}\,\frac{1}{N(\eta,b)}\,\eta_{b,j} + \Delta_{(a,i),(b,j)}\ .$$

Here $\Delta_{(a,i),(b,j)}$ is a possible extra term whose presence cannot be inferred from the previous arguments, because of possible degeneracies in the set of characters. (We are assuming here that the set of (generalized) characters of the unextended theory is linearly independent, and we will in any case only consider degeneracies that were introduced by the fixed point resolution.) These degeneracies allow for an additional term, provided it satisfies

$$\sum_j \Delta_{(a,i),(b,j)}\,m_{b,j} = 0\ . \tag{3.3}$$

3.2. Condition [II]

Now we impose the condition that $\tilde{S}$ must be symmetric. Multiplying this condition with $m_{a,i}$ and summing over i we get

$$|\mathcal{G}|\,\frac{1}{N(\eta,b)}\,\eta_{b,j}\,S_{a,b}+\sum_i m_{a,i}\,\Delta_{(a,i),(b,j)}=m_{b,j}\,|\mathcal{G}_b|\,S_{a,b}\,. \tag{3.4}$$

Now since S is unitary, for any b there exists an a such that $S_{a,b}\neq 0$. Let us fix b and pick one such a. Then (3.4) can be solved for $\eta_{b,j}$:

$$\frac{1}{N(\eta,b)}\,\eta_{b,j}=\frac{|\mathcal{G}_b|}{|\mathcal{G}|}\,m_{b,j}-\frac{\sum_i m_{a,i}\,\Delta_{(a,i),(b,j)}}{|\mathcal{G}_b|\,S_{a,b}}\,. \tag{3.5}$$

Note that the dependence of the last term on a should cancel. Substituting (3.5) into the formula for $\tilde{S}$ we get

$$\tilde{S}_{(a,i),(b,j)}=m_{a,i}m_{b,j}\,\frac{|\mathcal{G}_a|\,|\mathcal{G}_b|}{|\mathcal{G}|}S_{a,b}+\Gamma_{(a,i),(b,j)}\,, \tag{3.6}$$

where the last term is equal to Δ plus the contribution from the second term in (3.5),

$$\Gamma_{(a,i),(b,j)}=\Delta_{(a,i),(b,j)}-m_{a,i}\,\frac{|\mathcal{G}_a|}{|\mathcal{G}_b|}\,\sum_k m_{a,k}\Delta_{(a,k),(b,j)}\,.$$

Note that Γ satisfies a sum rule analogous to (3.3). Furthermore symmetry of $\tilde{S}$ implies that Γ must be symmetric.

3.3. Condition [III]

The remaining conditions involve a product $\mathcal{P}$ of two matrices, either $\tilde{S}^2$, $\tilde{S}\tilde{S}^\dagger$ or $(\tilde{S}\tilde{T})\tilde{S}$. Note that T is constant for fixed a or b. As a consequence, when we write such a product symbolically as $\mathcal{P}=\mathcal{P}_{S,S}+\mathcal{P}_{S,\Gamma}+\mathcal{P}_{\Gamma,S}+\mathcal{P}_{\Gamma,\Gamma}$, then the cross-terms $\mathcal{P}_{S,\Gamma}$ and $\mathcal{P}_{\Gamma,S}$ between the two terms in (3.6) always cancel due to condition (3.3). For $\mathcal{P}=\tilde{S}\tilde{S}^\dagger$, the term $\mathcal{P}_{S,S}$ reads $\mathcal{P}_{S,S}=\sum_{b,j} m_{a,i}m_{b,j}^2m_{c,k}\,S_{a,b}\,S^\dagger_{b,c}\,|\mathcal{G}_a||\mathcal{G}_b|^2|\mathcal{G}_c|\,/\,|\mathcal{G}|^2$. The sum over j can be done using (3.1):

$$\mathcal{P}_{S,S}=\sum_b m_{a,i}m_{c,k}\,\frac{|\mathcal{G}_a|\,|\mathcal{G}_b|\,|\mathcal{G}_c|}{|\mathcal{G}|}\,S_{a,b}S^\dagger_{b,c}=\sum_b\sum_{J,K} m_{a,i}m_{c,k}\,\frac{|\mathcal{G}_c|}{|\mathcal{G}|}\,S_{Ja,Kb}\,S^\dagger_{Kb,c}\,.$$

Here we have traded the factor $|\mathcal{G}_a||\mathcal{G}_b|$ for a sum over the orbits of a and b. Each term in these orbits gives the same contribution. The sum on b is over all orbit representatives

of $Q_{\mathcal{G}} = 0$ orbits. It can be extended to a sum over *all* orbits because the contributions of the $Q_{\mathcal{G}} \neq 0$ orbits cancel among each other owing to the sum on J. Together with the sum on K we now have a sum over all primary fields in the unextended theory, and we can use unitarity (respectively $S^2 = C$, or $STS = T^{-1}ST^{-1}$) in the unextended theory. Requiring unitarity of $\tilde{S}$ leads then to the condition

$$\sum_{b,j} \Gamma_{(a,i),(b,j)} \Gamma^{\dagger}_{(b,j),(c,k)} = \delta_{ac} \left(\delta_{ik} - \frac{m_{a,i}\, m_{a,k}}{|\mathcal{S}_a|} \right) .$$

Note that the right hand side is a projection operator,

$$P^a_{ik} \equiv \delta_{ik} - \frac{m_{a,i}\, m_{a,k}}{|\mathcal{S}_a|} .$$

A special case of this result was already obtained in [2], but there all multiplicities $m_{a,i}$ were assumed to be equal to 1.

3.4. Condition [IV]

The computation for $\tilde{S}^2$ yields in a similar way the relation

$$\sum_{b,j} \Gamma_{(a,i),(b,j)} \Gamma_{(b,j),(c,k)} = \tilde{C}_{(a,i),(c,k)} - \frac{m_{a,i}\, m_{c,k}}{|\mathcal{S}_c|} \sum_J C_{Ja,c} ,$$

where $\tilde{C}_{(a,i),(c,k)}$ is the charge conjugation matrix of the extended theory and $C_{a,c}$ that of the unextended one. The sum in the second term can only contribute if a and c are representatives of conjugate orbits, and in that case it contributes 1, and otherwise 0. We may thus introduce a matrix $\hat{C}_{a,c}$ on orbit representatives which is 1 if a is conjugate to some field on the $\mathcal{G}$-orbit of c, and 0 otherwise. Then we get

$$\sum_{b,j} \Gamma_{(a,i),(b,j)} \Gamma_{(b,j),(c,k)} = \tilde{C}_{(a,i),(c,k)} - \frac{m_{a,i}\, m_{c,k}}{|\mathcal{S}_c|}\, \hat{C}_{ac} .$$

Using the sum rule analogous to (3.3) that is valid for the matrix Γ, we conclude that $\sum_k \tilde{C}_{(a,i),(c,k)}\, m_{c,k} = m_{a,i}\, \hat{C}_{a,c}$. This implies that $\tilde{C}_{(a,i),(c,k)}$ can only be non-zero between orbits a and c with $\hat{C}_{a,c} = 1$. Furthermore, conjugate fields must have the same value of m. It follows that the set of numbers m_i must be identical on conjugate orbits. Hence

we may write

$$\tilde{C}_{(a,i),(c,k)} = \hat{C}_{a,c}\, C^c_{i,k}\,, \tag{3.7}$$

where $C^c_{i,k}$ is a conjugation matrix that is introduced by the fixed point resolution. Because $\tilde{C}$ and $\hat{C}$ are symmetric, the matrices C^c must satisfy

$$C^c = (C^{c^*})^{\mathrm{T}}$$

if c^* is the conjugate of c. The final result is therefore

$$\sum_{b,j} \Gamma_{(a,i),(b,j)} \Gamma_{(b,j),(c,k)} = \hat{C}_{a,c} \sum_l C^c_{i,l}\, P^c_{l,k}\,.$$

3.5. Condition [V]

Condition [V] is most conveniently dealt with in the equivalent form $\tilde{S}\tilde{T}\tilde{S} = \tilde{T}^{-1}\tilde{S}\tilde{T}^{-1}$. For $\tilde{S}\tilde{T}\tilde{S}$ we find

$$\sum_{b,j} \Gamma_{(a,i),(b,j)}\, \tilde{T}_{b,b}\, \Gamma_{(b,j),(c,k)} = (\tilde{T}^{-1}\tilde{S}\tilde{T}^{-1})_{(a,i),(c,k)} - \sum_J m_{a,i} m_{c,k}\, \frac{|\mathcal{G}_c|}{|\mathcal{G}|}\, T^{-1}_{Ja,Ja}\, S_{Ja,c}\, T^{-1}_{c,c}\,.$$

The sum on J just yields a factor $|\mathcal{G}_a|$, and then the last term cancels the first contribution from $\tilde{S}$. The result is

$$\sum_{b,j} \Gamma_{(a,i),(b,j)}\, \tilde{T}_{b,b}\, \Gamma_{(b,j),(c,k)} = (\tilde{T}^{-1}\Gamma\tilde{T}^{-1})_{(a,i),(c,k)}\,.$$

3.6. Some remarks on fusion rules

Although it seems to be quite difficult to examine the fusion rules in general, we can discuss the case that one of the three fields is *not* a fixed point (if even fewer fields are fixed points, the discussion is completely straightforward). Note that the formula $\sum_i m_{a,i}\Gamma_{(a,i),(b,j)} = 0$ implies that $\Gamma = 0$ if a field is 'resolved' into only one primary field of the extended theory. In particular, there are no correction matrices Γ for fields that are not fixed points.

We obtain the following formula for the fusion coefficients:

$$\tilde{\mathcal{N}}_{a,(b,j)}{}^{(c,k)} = \frac{|\mathcal{G}_b|\,|\mathcal{G}_c|}{|\mathcal{G}|^2}\, m_{b,j} m_{c,k} \sum_{J\in\mathcal{G}} \mathcal{N}_{a,Jb}{}^{c} + \sum_{d,l} \frac{S_{a,d}}{S_{0,d}}\, \Gamma_{(b,j),(d,l)} \Gamma^*_{(c,k),(d,l)}\,.$$

In principle the requirement that the coefficient should be a positive integer imposes restrictions on Γ, but these conditions do not look particularly useful. This is even more true for the fusion of three resolved fixed points. A few things can be learned, though. Multiplying with $m_{c,k}$ and summing over k we get

$$\sum_k \tilde{\mathcal{N}}_{a,(b,j)}{}^{(c,k)} m_{c,k} = m_{b,j} \sum_{J\in\mathcal{G}_b} \mathcal{N}_{a,Jb}{}^{c}\,.$$

This has a few implications. If $a \times b$ contains terms in the orbit of c, then $\tilde{\mathcal{N}}_{a,(b,j)}{}^{(c,k)}$ cannot be zero for all k. Furthermore, if $a \times b$ does not yield any contribution in the c orbit, then $\tilde{\mathcal{N}}_{a,(b,j)}{}^{(c,k)} = 0$. Thus the new fusion rules must respect the orbit-orbit maps of the original fusion rules, although the distribution of fields may be non-trivial.

We can in fact say more. If $n = \sum_{J\in\mathcal{G}_b} \mathcal{N}_{a,Jb}{}^{c} = 1$ and $|\mathcal{G}_b| = |\mathcal{G}_c|$, then it can be shown that the vector $\vec{m}_c$ is a permutation of $\vec{m}_b$. Since this is not a surprising result, we omit the details of the proof. The condition that $n = 1$ and $|\mathcal{G}_b| = |\mathcal{G}_c|$ is in particular satisfied if a is a simple current, but in general this is by no means the only possibility. If there is any non-fixed point field a that maps orbit b to c with multiplicity 1, then orbit b and c must have the same decomposition vector $\vec{m}$ (if $|\mathcal{G}_b| = |\mathcal{G}_c|$). The result indicates that fixed points which have the same stabilizer should also possess the same decomposition $\vec{m}$; it is difficult to imagine how a different decomposition could still provide a solution to all constraints.

3.7. Condition [VI]

Suppose there is a simple current L in the theory which is local with respect to all currents by which we extend the algebra. Then according to the remarks before equation (2.2) L will again be a simple current in the extended theory. (Note that, as before, L stands for a definite representative of the $\mathcal{G}$-orbit of the additional currents.) Hence L must act as a simple current on the resolved fixed points. Thus if in the unextended theory

$$L \times a = b\,,$$

then as seen above we have in the extended theory

$$L \times (a,i) = (b,j)$$

for some j. Hence we have both

$$Q_L(a) = h(a) + h(L) - h(b) \bmod 1 \tag{3.8}$$

and

$$\tilde{Q}_L((a,i)) = h((a,i)) + h(L) - h((b,j)) \bmod 1\,.$$

Since the respective conformal weights are the same up to integers, we see that on all fields $Q_L = \tilde{Q}_L$. Note that L was assumed to be local with respect to $\mathcal{G}$, so that $h(L)$ is a constant (modulo integers) on the $\mathcal{G}$-orbit of L, and hence the notation makes sense.

To relate the matrix element $\Gamma_{(a,i),(c,k)}$ to $\Gamma_{(b,j),(c,k)}$, we recall the relation (2.6) that a simple current L imposes on the S-matrix elements. Combining this formula with the analogous relation

$$\tilde{S}_{L(a,i),(b,j)} = \mathrm{e}^{2\pi\mathrm{i}Q_L(b)}\,\tilde{S}_{(a,i),(b,j)} \tag{3.9}$$

for $\tilde{S}$, we obtain an analogous relation for Γ:

$$\Gamma_{L(a,i),(c,k)} = \mathrm{e}^{2\pi\mathrm{i}Q_L(c)}\,\Gamma_{(a,i),(c,k)}\,. \tag{3.10}$$

4. Fourier decomposition

Suppose we consider an arbitrary fixed point resolution, where a fixed point a is split into M_a primary fields. From now on we will impose the homogeneity condition [VIII], and therefore in particular we will only consider the case that a is split into M_a primary fields with identical multiplicity factors $m_{a,i} = m_a$. Then

$$|\mathcal{S}_a| = (m_a)^2 M_a\,. \tag{4.1}$$

Suppose by some as yet unspecified procedure we obtain a matrix $\tilde{S}_{(a,i),(b,j)}$ satisfying all the requirements listed in section 2.

4.1. Group characters

For each fixed point choose an abelian discrete group $\mathcal{M}_a$ with as many characters as there are resolved fields, *i.e.* $|\mathcal{M}_a| = M_a$. Later we will identify this group as a subgroup of the stabilizer, but for the moment there is no need to be specific. An important role

will be played by the group characters Ψ_i^a, $i = 1, 2, \ldots, M_a$, of $\mathcal{M}_a$. The characters are a complete set of complex functions on the group satisfying

$$\begin{aligned} \Psi_i^a(g)\Psi_i^a(h) &= \Psi_i^a(gh)\,, \\ \Psi_i^a(g^{-1}) &= \Psi_i^a(g)^*\,, \\ \Psi_i^a(\mathbf{1}) &= 1\,, \end{aligned} \tag{4.2}$$

for all $g, h \in \mathcal{M}_a$ ($\mathbf{1}$ denotes the unit element of $\mathcal{M}_a$). For these characters the orthogonality and completeness relations

$$\sum_i \Psi_i^a(g)\Psi_i^a(h)^* = M_a\,\delta_{gh}\,, \qquad \sum_g \Psi_i^a(g)\Psi_j^a(g)^* = M_a\,\delta_{ij} \tag{4.3}$$

hold. For cyclic groups $\mathbf{Z}_N$ we will label the elements by integers $0 \le g < N$; the characters read

$$\Psi_\ell^a(g) = \mathrm{e}^{2\pi\mathrm{i}\,\ell g/N} \qquad \text{for} \quad 0 \le \ell < N\,.$$

The groups $\mathcal{M}_a$ are chosen isomorphic on conjugate $\mathcal{G}$-orbits, as well as on $\mathcal{G}$-orbits connected by any additional simple currents. This is possible since we have seen that the decomposition vector $\vec{m}$ is preserved by charge conjugation and simple current maps.

We define the Fourier components of $\tilde{S}$ with respect to the groups $\mathcal{M}_a$ as

$$S_{a,b}^{g,h} := \frac{1}{\sqrt{M_a M_b}} \sum_i \sum_j \Psi_i^a(g)^*\, \Psi_j^b(h)\, \tilde{S}_{(a,i),(b,j)}\,.$$

The inverse of this transformation is

$$\tilde{S}_{(a,i),(b,j)} = \frac{1}{\sqrt{M_a M_b}} \sum_g \sum_h \Psi_i^a(g)\, \Psi_j^b(h)^*\, S_{a,b}^{g,h}\,. \tag{4.4}$$

Now we will examine the implication of conditions [I] – [V] in terms of the Fourier components.

4.2. Condition [I]

Because of this condition some of the elements $S_{a,b}^{g,h}$ are already known. According to the general expression (3.6) for fixed point resolution, we have

$$\tilde{S}_{(a,i),(b,j)} = m_a m_b \, \frac{|\mathcal{G}|}{|\mathcal{S}_a|\,|\mathcal{S}_b|} \, S_{a,b} + \Gamma_{(a,i),(b,j)} \, . \tag{4.5}$$

Using (4.5), we can compute $S^{g,h}$ for $g = \mathbf{1}$ (or $h = \mathbf{1}$). The characters obey $\Psi_i^a(\mathbf{1}) = 1$ for all i. Using also

$$\sum_i m_{a,i} \, \Gamma_{(a,i),(b,j)} = 0$$

and the fact that the multiplicities are by assumption independent of i, it follows that in this case Γ does not contribute. The only contribution is thus

$$S_{a,b}^{\mathbf{1},h} = \frac{1}{\sqrt{M_a M_b}} \sum_i \sum_j \Psi_j^b(h) \, \frac{|\mathcal{G}|\, m_a m_b}{|\mathcal{S}_a|\,|\mathcal{S}_b|} \, S_{a,b} \, .$$

Because of the orthogonality relation of the characters, this vanishes unless $h = \mathbf{1}$. For $h = \mathbf{1}$ the sums over i and j yield $M_a M_b$, and the result is

$$S_{a,b}^{\mathbf{1},h} = S_{a,b}^{h,\mathbf{1}} = \frac{|\mathcal{G}|\, m_a m_b \sqrt{M_a M_b}}{|\mathcal{S}_a|\,|\mathcal{S}_b|} \, \delta_{h,\mathbf{1}} \, S_{a,b} = \frac{|\mathcal{G}|}{\sqrt{|\mathcal{S}_a|\,|\mathcal{S}_b|}} \, \delta_{h,\mathbf{1}} \, S_{a,b} \, . \tag{4.6}$$

4.3. Condition [II]

Symmetry of $\tilde{S}_{(a,i),(b,j)}$ implies

$$S_{b,a}^{h^{-1},g^{-1}} = S_{a,b}^{g,h} \, .$$

4.4. Condition [III]

Unitarity of $\tilde{S}$ can be shown to be equivalent to

$$\sum_{h,b} S_{a,b}^{g,h} (S_{b,c}^{h^{-1},f})^* = \delta_{ac} \, \delta_{g,f^{-1}} \, . \tag{4.7}$$

4.5. Condition [IV]

Consider now the product $\tilde{S}^2$. Using (3.7) we find

$$\sum_{h,b} S^{g,h}_{a,b}\, S^{h,f}_{b,c} = \frac{1}{\sqrt{M_a M_c}} \sum_i \sum_l \Psi^a_i(g)^*\, \Psi^c_l(f)\, \hat{C}_{a,c}\, C^c_{i,l}\,.$$

This vanishes unless $c = a^*$. If $c = a^*$, then $M_a = M_c$, and we may write the result as

$$\sum_{h,b} S^{g,h}_{a,b}\, S^{h,f}_{b,c} = \hat{C}_{a,c}\, \frac{1}{M_c} \sum_{i,l} \Psi^c_i(g)^*\, \Psi^c_l(f)\, C^c_{i,l}\,.$$

If $C^c_{i,l} = \delta_{il}$ the result is $\hat{C}_{a,c}\delta_{gf}$. Otherwise $C^c_{i,l}$ defines a permutation of the labels i of the characters. We may define

$$C^c_{g,f} := \frac{1}{M_c} \sum_{i,l} \Psi^c_i(g)^*\, C^c_{i,l}\, \Psi^c_l(f)\,,$$

so that the result is

$$\sum_{h,b} S^{g,h}_{a,b}\, S^{h,f}_{b,c} = \hat{C}_{a,c}\, C^c_{g,f}\,.$$

The matrix $C^c_{g,f}$ is unitary, but in general it is not a permutation even though the Fourier transform $C^c_{i,l}$ is.

4.6. Condition [V]

A completely analogous computation can be done for the relation $(\tilde{S}\tilde{T})^3 = \tilde{C}$ of the modular group. The result is

$$\sum_{h,b} S^{g,h}_{a,b} T_{b,b} S^{h,f}_{b,c} = T^{-1}_{a,a}\, S^{g,f}_{a,c}\, T^{-1}_{c,c}\,.$$

4.7. Condition [VI]

If the extended theory has a surviving simple current L we have

$$\tilde{S}_{L(a,i),(b,j)} = \mathrm{e}^{2\pi\mathrm{i}Q_L(b)}\,\tilde{S}_{(a,i),(b,j)}\,. \tag{4.8}$$

The action of L on the resolved fixed point moves the orbit representative a to La, and the label i to Li. Here Li denotes some other label of the resolved fixed points of the field La. Expanding the left and right hand side of (4.8) into Fourier modes, we get $\mathrm{e}^{2\pi\mathrm{i}Q_L(b)}\,S^{g,h}_{a,b} = \frac{1}{M_a}\sum_{i,f}\Psi^a_i(g)^*\,\Psi^a_{Li}(f)\,S^{f,h}_{La,b}$. Analogously as we did above for charge conjugation, we define a matrix

$$\mathcal{F}^a_{g,f}(L) := \frac{1}{M_a}\sum_i \Psi^a_i(g)^*\,\Psi^a_{Li}(f)\,,$$

so that we can write the result as

$$\mathrm{e}^{2\pi\mathrm{i}Q_L(b)}\,S^{g,h}_{a,b} = \sum_f \mathcal{F}^a_{g,f}(L)\,S^{f,h}_{La,b}\,. \tag{4.9}$$

4.8. Condition [VII]: Successive extensions

Condition [VII] has several consequences. We will first compare an extension in two steps with a complete extension, *i.e.* in the notation of (2.4) we compare

$$\mathcal{G} \supset \mathcal{H} \supset \{\mathbf{1}\} \quad \text{with} \quad \mathcal{G} \supset \{\mathbf{1}\}\,.$$

For simplicity we consider only the case where $p \equiv |\mathcal{G}|/|\mathcal{H}|$ is prime, which by recursion includes all other cases anyway.

Performing the extension by $\mathcal{H}$, we obtain a modular matrix $\tilde{S}_{(a,i),(b,j)}$, which can be described by Fourier components $S^{g,h}_{a,b}$. By assumption the extended theory has an integral spin simple current L of prime order p. When we further extend by this current L, the stabilizer of any field a remains either unchanged or is enlarged from $\mathcal{S}^{\mathcal{H}}_a$ to $\mathcal{S}^{\mathcal{G}}_a \supset \mathcal{S}^{\mathcal{H}}_a$. If it remains unchanged, then for $L \in \mathcal{G} \setminus \mathcal{H}$ we have $L(a,i) = (b,j)$ with $a \neq b$, and hence L has in any case no fixed points; this situation requires no further discussion. On the other hand, if the stabilizer is enlarged, then $|\mathcal{S}^{\mathcal{G}}_a| = p \cdot |\mathcal{S}^{\mathcal{H}}_a|$. Now two cases have to be distinguished:

- Case A: $L(a,i) = (a,i)\,.$
- Case B: $L(a,i) = (a,Li)$ with $Li \neq i\,.$

In case A a fixed point resolution is necessary for the primary field labelled by (a,i), whereas in case B a field identification takes place. Now condition [VIII] implies immediately that any fixed point of a simple current of prime order must be resolved into p fields with multiplicity $m_\alpha = 1$; hence in case A the field (a,i) is resolved into p new primary fields (a,i,α), $\alpha = 1,2,\ldots,p$. The field identification in case B combines p primaries (a,i) (with fixed a, but distinct values of i) into a single primary field of the $\mathcal{G}$-extended theory. In other words, the $M_a^{\mathcal{H}}$ fields (a,i) with given a are combined into $M_a^{\mathcal{H}}/p$ new fields (a,i_ℓ), where i_ℓ ($\ell = 1,2,\ldots,M_a^{\mathcal{H}}/p$) denotes some definite choice among the labels i, reducing the label set by a factor p. It follows that if all extensions were as in case A, the multiplicities m would always be equal to 1, and the total number of fields would be equal to $|\mathcal{S}_a|$, the order of the stabilizer of a. In contrast, case B amounts to a reduction of the number of primary fields by a factor p^2, which is accompanied by an enlargement of the multiplicity m by a factor of p due to the sum over the L-orbit. (Note that L must generate an orbit of order p on the resolved $\mathcal{H}$-fixed points, and under condition [VIII] this is only possible if $|\mathcal{H}|$ contains a factor p.) As a consequence, the number of primaries into which a fixed point a of the extension by $\mathcal{G}$ is resolved can in general be any integer $|\mathcal{S}_a|/N^2$ with $N \in \mathbf{Z}$. Inspecting successive extensions allows us to decide which of these possibilities is realized.

Let us now first consider case A. For the Fourier basis we may take, without any loss of generality, a subgroup $\mathcal{M}_a^{\mathcal{H}} \subset \mathcal{M}_a^{\mathcal{G}}$ with $|\mathcal{M}_a^{\mathcal{H}}| = |\mathcal{M}_a^{\mathcal{G}}|/p$. Then any $g \in \mathcal{M}_a^{\mathcal{G}}$ can be written in the factorized form $g = hw^\ell$, where $h \in \mathcal{M}_a^{\mathcal{H}}$, and where w is a coset representative of a non-trivial element of the coset $\mathcal{M}_a^{\mathcal{G}}/\mathcal{M}_a^{\mathcal{H}}$ (we take the identity $\mathbf{1} \in \mathcal{M}_a^{\mathcal{G}}$ as the representative for the trivial element of $\mathcal{M}_a^{\mathcal{G}}/\mathcal{M}_a^{\mathcal{H}}$); thus $w^\ell \notin \mathcal{M}_a^{\mathcal{H}}$ for $0 < \ell < p$.

The $\mathcal{M}_a^{\mathcal{G}}$-characters then act as

$$\Psi^a_{(i,\alpha)}(hw^\ell) = \Psi^a_{(i,\alpha)}(h)\Psi^a_{(i,\alpha)}(w^\ell) = \Psi^a_i(h)\Psi^a_{(i,\alpha)}(w^\ell)\,.$$

Here in the last equality we have used the fact that for $h \in \mathcal{M}_a^{\mathcal{H}}$ the characters satisfy $\Psi_{(i,\alpha)}(h) = \Psi_i(h)$, where the latter are characters of $\mathcal{M}_a^{\mathcal{H}}$.

We can now write down formulas for the first extension, the second extension, and the full extension. The non-fixed orbits can be taken into account by restricting the label α to a single value. To make the notation unambiguous, resolved matrices will now be distinguished by a superscript $\mathcal{G}$ and $\mathcal{H}$ (instead of a tilde), and other quantities are labelled in the same way. For the first extension we have

$$S^{\mathcal{H}}_{(a,i),(b,j)} = \frac{1}{\sqrt{M_a^{\mathcal{H}} M_b^{\mathcal{H}}}} \sum_{g\in\mathcal{M}_a^{\mathcal{H}}} \sum_{h\in\mathcal{M}_b^{\mathcal{H}}} S^{g,h\,;\mathcal{H}}_{a,b}\, \Psi^a_i(g)\, \Psi^b_j(h)^*\,, \tag{4.10}$$

and for the second extension

$$S^{\mathcal{G}}_{(a,i,\alpha),(b,j,\beta)} = \sqrt{\frac{M_a^{\mathcal{H}} M_b^{\mathcal{H}}}{M_a^{\mathcal{G}} M_b^{\mathcal{G}}}} \sum_{0\le n<p} \sum_{0\le m<p} S^{n,m}_{(a,i),(b,j)}\, \Psi^a_\alpha(n)\, \Psi^b_\beta(m)^*\,. \tag{4.11}$$

Note that the second extension requires a $\mathbf{Z}_p$-Fourier transform even if $\mathcal{M}^{\mathcal{G}}$ is not the direct product of $\mathcal{M}^{\mathcal{H}}$ by $\mathbf{Z}_p$.

For the full extension we find

$$S^{\mathcal{G}}_{(a,i,\alpha),(b,j,\beta)} = \frac{1}{\sqrt{M^{\mathcal{G}}_a M^{\mathcal{G}}_b}} \sum_{g\in\mathcal{M}^{\mathcal{G}}_a} \sum_{h\in\mathcal{M}^{\mathcal{G}}_b} S^{g,h\,;\mathcal{G}}_{a,b}\, \Psi^a_{(i,\alpha)}(g)\, \Psi^b_{(j,\beta)}(h)^* \,. \tag{4.12}$$

Furthermore we have the relations (*cf.* (4.6))

$$S^{\mathbf{1},\mathbf{1}\,;\mathcal{H}}_{a,b} = \frac{|\mathcal{H}|}{\sqrt{|\mathcal{S}^{\mathcal{H}}_a|\,|\mathcal{S}^{\mathcal{H}}_b|}}\, S_{a,b} \tag{4.13}$$

and

$$S^{\mathbf{1},\mathbf{1}\,;\mathcal{G}}_{a,b} = \frac{|\mathcal{G}|}{\sqrt{|\mathcal{S}^{\mathcal{G}}_a|\,|\mathcal{S}^{\mathcal{G}}_b|}}\, S_{a,b}\,, \qquad S^{\mathbf{1},\mathbf{1}}_{(a,i),(b,j)} = \frac{|\mathcal{G}|}{|\mathcal{H}|}\sqrt{\frac{|\mathcal{S}^{\mathcal{H}}_a|\,|\mathcal{S}^{\mathcal{H}}_b|}{|\mathcal{S}^{\mathcal{G}}_a|\,|\mathcal{S}^{\mathcal{G}}_b|}}\, S^{\mathcal{H}}_{(a,i),(b,j)}\,. \tag{4.14}$$

Using these identities it is not difficult to show that

$$S^{g,h\,;\mathcal{G}}_{a,b} = \frac{|\mathcal{G}|}{|\mathcal{H}|}\sqrt{\frac{|\mathcal{S}^{\mathcal{H}}_a|\,|\mathcal{S}^{\mathcal{H}}_b|}{|\mathcal{S}^{\mathcal{G}}_a|\,|\mathcal{S}^{\mathcal{G}}_b|}}\, S^{g,h\,;\mathcal{H}}_{a,b} \tag{4.15}$$

when $g \in \mathcal{M}^{\mathcal{H}}_a$ and $h \in \mathcal{M}^{\mathcal{H}}_b$. Furthermore, if $g \in \mathcal{M}^{\mathcal{H}}_a$ but $h \notin \mathcal{M}^{\mathcal{H}}_b$ (or vice versa), then $S^{g,h\,;\mathcal{H}}_{a,b}$ must vanish.

In case B we have $\mathcal{M}^{\mathcal{H}}_a \supset \mathcal{M}^{\mathcal{G}}_a$, and we can work with group characters $\Psi^a_{(i)}$ acting on $a \in \mathcal{M}^{\mathcal{H}}_a$, where the labelling is such that the subset $\Psi^a_{(i_p)}$ forms a set of characters of $\mathcal{M}^{\mathcal{G}}_a$. Again we will include the limiting case $\mathcal{M}^{\mathcal{G}} = \mathcal{M}^{\mathcal{H}}$, to allow also for fields that are not fixed points. Of course, the formula for the first extension remains the same as in case A. For the full extension we have

$$S^{\mathcal{G}}_{(a,i_p),(b,j_p)} = \frac{1}{\sqrt{M^{\mathcal{G}}_a M^{\mathcal{G}}_b}} \sum_{g\in\mathcal{M}^{\mathcal{G}}_a} \sum_{h\in\mathcal{M}^{\mathcal{G}}_b} S^{g,h\,;\mathcal{G}}_{a,b}\, \Psi^a_{i_p}(g)\, \Psi^b_{j_p}(h)^* \,, \tag{4.16}$$

and for the second extension

$$S^{\mathcal{G}}_{(a,i_p),(b,j_p)} = \frac{|\mathcal{G}|}{|\mathcal{H}|}\,\frac{|\mathcal{S}^{\mathcal{H}}_a||\mathcal{S}^{\mathcal{H}}_b|}{|\mathcal{S}^{\mathcal{G}}_a||\mathcal{S}^{\mathcal{G}}_b|}\, S^{\mathcal{H}}_{(a,i_p),(b,j_p)}\,. \tag{4.17}$$

This is just (3.6) with $m_i = 1$ and $\Gamma = 0$, since in the second step there are no fixed points to resolve. Note that all matrix elements of $S^{\mathcal{H}}_{(a,i_p),(b,j_p)}$ are the same on the L-orbits, so that the answer does not depend on which element we select for i_p.

Combining this information, we obtain again the result (4.15), except that this time it completely determines $S^{g,h\,;\mathcal{G}}$, i.e. is valid for all $g \in \mathcal{M}_a^{\mathcal{H}}$ and all $h \in \mathcal{M}_b^{\mathcal{H}}$.

In principle it might happen that cases A and B occur simultaneously for a given extension. Then there are matrix elements of $S^{\mathcal{G}}$ between fields a and b with $\mathcal{M}_a^{\mathcal{G}} \subset \mathcal{M}_a^{\mathcal{H}}$ and $\mathcal{M}_b^{\mathcal{G}} \supset \mathcal{M}_b^{\mathcal{H}}$. The analysis of this mixed case is essentially the same as in the previous case, and the result is once again (4.15), with the restriction that $S^{g,h} = 0$ whenever $h \notin \mathcal{M}_b^{\mathcal{H}}$.

All these cases are summarized by the formula

$$S^{\mathcal{G}}_{(a,I),(b,I')} = \frac{1}{\sqrt{M_a^{\mathcal{G}}\, M_b^{\mathcal{G}}}} \sum_{g\in\mathcal{M}_a^{\mathcal{G}}} \sum_{h\in\mathcal{M}_b^{\mathcal{G}}} S_{a,b}^{g,h\,;\mathcal{G}}\, \Psi_I^a(g)\, \Psi_{I'}^b(h)^* \tag{4.18}$$

for the full extension, where I now stands for either the combination of labels (i,α) or the single label i_p, or just the label i or α, and analogously for I'. Furthermore we always have the relation (4.15). In case A this only gives us part of $S^{g,h\,;\mathcal{G}}$ whereas for case B it gives us all of $S^{g,h\,;\mathcal{G}}$. Note that even though (4.18) is universal, the factor $1/\sqrt{M_a^{\mathcal{G}}}$ for case B is by a factor p larger than in case A.

4.9. Condition [VII]: Commutativity of extensions

In the previous section we have analyzed the consequences of condition [VII] by comparing two successive extensions to the full extension. Another aspect of condition [VII] is that two successive extensions should commute whenever each of them can be performed as the first extension. To check this commutativity, we have to compare the embedding chains

$$\mathcal{G} = \mathcal{H}_1 \times \mathcal{H}_2 \supset \mathcal{H}_1 \supset \{\mathbf{1}\} \qquad \text{and} \qquad \mathcal{G} = \mathcal{H}_1 \times \mathcal{H}_2 \supset \mathcal{H}_2 \supset \{\mathbf{1}\}\,.$$

Consider two fields a and b with stabilizer $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$, all with implicit labels a respectively b. For simplicity we will assume that case A applies in all cases, so that $|\mathcal{M}| = |\mathcal{S}|$. Without loss of generality we may then choose for the group $\mathcal{M}_i$ the stabilizer $\mathcal{S}_i$. Requiring that each of the two embedding chains yields the same answer leads to the condition

$$S^{g,h\,;\mathcal{G}} = \frac{|\mathcal{G}|}{|\mathcal{H}_i|} \sqrt{\frac{|\mathcal{S}_i^{\mathcal{H}}||\mathcal{S}_i^{\mathcal{H}}|}{|\mathcal{S}^{\mathcal{G}}||\mathcal{S}^{\mathcal{G}}|}}\, S^{g,h\,;\mathcal{H}_i} \tag{4.19}$$

when either g or h are restricted to the subgroup $\mathcal{M}_i$. In particular $S^{g,h\,;\mathcal{G}}$ vanishes when g and h are from different factors of $\mathcal{G}$. This holds equally well for any other decomposition of $\mathcal{G}$.

4.10. The matrices S^J

At this point it is worth stressing that so far there was no need to specify the discrete abelian group $\mathcal{M}_a$ (except for the restriction that the groups associated to successive embeddings are contained in each other). Rather, choosing a particular group $\mathcal{M}_a$ is merely a matter of convenience. However, while for any fixed point resolution the Fourier transformation (4.4) can be performed for any arbitrary choice of $\mathcal{M}_a$, this manipulation is not likely to lead to useful results unless a clever choice of $\mathcal{M}_a$ is made.

Now the results of the previous sections inspire us to make the following *ansatz* for $S_{a,b}^{g,h}$. First of all, we identify the elements g and h of the groups $\mathcal{M}_a$ and $\mathcal{M}_b$ with elements J and K of the stabilizer $\mathcal{S}_a$ or $\mathcal{S}_b$, respectively. This is obviously possible if all multiplicities $m_{a,i}$ are equal to 1, since in that case $M_a = |\mathcal{S}_a|$. Otherwise these numbers differ by a square (because of condition [VIII]), and one can always find a subgroup of $\mathcal{S}_a$ that has the right size. This subgroup may not be unique, but we will soon make a canonical choice. Now we make the ansatz

$$S_{a,b}^{J,K;\mathcal{G}} := \frac{|\mathcal{G}|}{\sqrt{|\mathcal{S}_a|\,|\mathcal{S}_b|}}\,\delta_{JK}\,S_{a,b}^{J}\,. \tag{4.20}$$

This defines a matrix $S_{a,b}^J$ for each current, which is however independent of the extension one considers. The precise definition of S^J can already be obtained by considering the minimal extension for which it appears, namely the extension by J itself (or more precisely, the discrete group $\mathcal{H}_J$ it generates). In that case (4.20) reads

$$S_{a,b}^{J,K;\mathcal{H}_J} = \delta_{J,K}\,S_{a,b}^{J}\,,$$

since $\mathcal{G} = \mathcal{H}_J$, and the identification of J with an element of $\mathcal{S}_a$ and one of $\mathcal{S}_b$ means that $S_{ab}^J = 0$ unless $Ja = a$ and $Jb = b$. Thus $J \in \mathcal{S}_a \cap \mathcal{S}_b$, which implies that $\mathcal{H}_J = \mathcal{S}_a = \mathcal{S}_b$.*

Note that for $\mathcal{G} = (\mathbf{Z}_2)^n$ this structure follows completely from the foregoing discussion. For each single $\mathcal{H}_J = \mathbf{Z}_2$ extension by a current J we have

$$S^{0,J} = 0 = S^{J,0} \tag{4.21}$$

$\star$ In principle it could happen that $J \in \mathcal{S}$, but $J \notin \mathcal{M}$. Since the Fourier transforms are defined using $\mathcal{M}$, this would imply that $S_{a,b}^J$ is then not defined for all primary fields a. It can be shown (see the appendix) that this situation – which can only be due to successive resolutions according to case B above – can never arise within a single cyclic group. (Also, one can argue that these matrix elements are anyway never needed in further extensions.) Hence if we extend the algebra only by J (and its powers), then all fixed points are fully resolved into fields that can be described in terms of $\mathcal{S}_a$.

and

$$S_{a,b}^{0,0\,;\mathcal{H}_J} = \frac{|\mathcal{H}_J|}{\sqrt{|\mathcal{S}_a^{\mathcal{H}_J}|\,|\mathcal{S}_b^{\mathcal{H}_J}|}}\,S_{a,b}\,, \qquad S_{a,b}^{J,J\,;\mathcal{H}_J} = \frac{|\mathcal{H}_J|}{\sqrt{|\mathcal{S}_a^{\mathcal{H}_J}|\,|\mathcal{S}_b^{\mathcal{H}_J}|}}\,S_{a,b}^J\ (=S_{a,b}^J)\,. \tag{4.22}$$

The last of these equalities is the definition of S^J, while the others follow from (4.6) and tell us that $S_{a,b}^{\mathbf{1}} = S_{a,b}$. Using (4.19), this implies immediately that (4.20) holds for any further extension. Therefore the non-trivial assumption in (4.20) is that the matrices $S^{J,K}$ vanish for distinct currents J and K even if they belong to the same cyclic subgroup of $\mathcal{G}$ and have the same order.

Now we substitute the ansatz (4.20) in conditions [II]–[V], derived for the most general form of the resolution procedure. We consider first the defining matrices obtained for the minimal extension. We find:

Condition [II]:

$$S_{a,b}^J = S_{b,a}^{J^{-1}}\,. \tag{4.23}$$

Condition [III]:

$$\sum_b S_{a,b}^J\,(S_{b,c}^{J^{-1}})^* = \delta_{bc}\,.$$

When combined with (4.23), this implies that S^J is unitary.

Condition [IV]:

We obtain $\sum_b S_{a,b}^J S_{b,c}^J \delta_{JK} = \hat{C}_{a,c} C_{J,K}^c$, where $C_{J,K}^c$ is a unitary matrix. The condition [IV] says that it must also be diagonal, so that its diagonal matrix elements must be phases. Hence we get

$$\sum_b S_{a,b}^J S_{b,c}^J = \hat{C}_{a,c}\,\eta_c^J$$

with some phases η_c^J.

The fact that $C_{J,K}^c$ is diagonal implies that $\frac{1}{M_a}\sum_{i,j}\Psi_i^a(J)C_{i,j}^c(J)\Psi_j^a(K)^* = \eta_c^J\delta_{JK}$. Multiplying with $\Psi_k^a(K)$ and summing over K then gives

$$\sum_i \Psi_i^a(J)\,C_{i,k}^c = \eta_c^{\,J}\,\Psi_k^a(J)\,.$$

In the sum on the left hand side only a single term survives. Define k^c by

$$C_{j,k}^c = \delta_{j,k^c}\,.$$

Then we get

$$\Psi^a_{k^c}(J) = \eta_c^{\,J}\,\Psi^a_k(J)\,.$$

This allows us to write η_c^J (considered as a function of J) as a ratio of two group characters. It then follows that it enjoys the group property

$$\eta_c^{J_1}\eta_c^{J_2} = \eta_c^{J_1 J_2}\,.$$

In particular $(\eta_c^J)^N = 1$ if J has order N. This implies in particular that $\eta_c^J = -1$ is allowed only for currents of even order that are not themselves a square of other currents.

The property $C^c_{i,j} = C^{c^*}_{j,i}$ implies that $\eta_c^J = (\eta_{c^*}^J)^*$.

Condition [V]:

$$\sum_b S^J_{a,b}T_{b,b}S^J_{b,c}\delta_{JK} = T^{-1}_{a,a}S^J_{a,b}T^{-1}_{b,b}\ .$$

We thus find that S^J must form a unitary representation of the modular group on the fixed points of J. This means that $S^J(S^J)^\dagger = \mathbb{1}$, $(S^JT)^3 = (S^J)^2$ and, due to $\eta_c^J = (\eta_{c^*}^J)^*$, $(S^J)^4 = \mathbb{1}$; note, however, that it is *not* required that S^J must be symmetric, nor does $(S^J)^2$ have to be a permutation.

Condition [VI]:

From the general result (4.9) we get directly

$$\mathrm{e}^{2\pi\mathrm{i}Q_L(b)}\,S^J_{a,b}\,\delta_{JK} = \sum_M \mathcal{F}^a_{J,M}(L)\,S^M_{La,b}\,\delta_{MK}\,.$$

This implies that the matrix $\mathcal{F}_{J,M}$ must be diagonal. Unitarity of S^J then requires that the elements of $\mathcal{F}$ should be phases. Hence

$$\frac{1}{M_a}\sum_i \Psi^a_i(J)\,\Psi^a_{Li}(M)^* = F(a,L,J)\,\delta_{JM} \tag{4.24}$$

with certain phases $F(a,J,K)$. The relation for S^J that we get is

$$S^J_{La,b} = F(a,L,J)\,\mathrm{e}^{2\pi\mathrm{i}Q_L(b)}\,S^J_{a,b}\,. \tag{4.25}$$

Multiplying (4.24) with $\Psi^a_k(M)$ and summing over M we then get $F(a,L,J)^*\Psi^a_k(J) =$

$\Psi^a_{L^{-1}k}(J)$, *i.e.*

$$F(a,L,J)^* = \Psi^a_{L^{-1}k}(J) \,/\, \Psi^a_k(J)\,.$$

Since this is a ratio of group characters, the phase satisfies the group property

$$F(a,L,J_1)\,F(a,L,J_2) = F(a,L,J_1J_2)\,. \tag{4.26}$$

The quantities $F(a,K,J)$ decide whether the current K acts non-trivially on the labels of the resolved fixed points of J. This action is trivial if and only if $F(a,K,J)=1$, but the value of F is relevant only if a is fixed by K. Indeed, in the appendix we show that the value of F on other fields can be modified by conjugating S^J by a diagonal unitary matrix. This changes both F and η, and shows that only the value of these parameters on fixed points and self-conjugate fields, respectively, is relevant information. In the appendix these transformations are used to choose a convenient 'gauge' such that $\eta=1$ on fields that are not self-conjugate and that both η and F are constant on simple current orbits, and such that F satisfies the group property also with respect to its second argument.

Furthermore, as is shown in the appendix, with this choice the phases F for integral spin currents J and K are equal to

$$F(a,K^pJ^q,K^nJ^m) = F(a,K,J)^{pm-nq}\,, \tag{4.27}$$

so that on each $\mathbf{Z}_N \times \mathbf{Z}_M$ subgroup of the center, F can be completely expressed in terms of a single phase.

We are now in a position to determine precisely under which conditions case B of section 4.8 applies. Clearly this situation occurs whenever two currents J and K from different orbits both fix a field a, with $F(a,K,J)\neq 1$. The relation (4.27) guarantees that the latter property is symmetric in J and K, so that it does not matter whether we first extend by J and then by K or the other way round. Inspired by these observations we define for each field the *untwisted stabilizer* $\mathcal{U}_a$ as

$$\mathcal{U}_a := \{J\in\mathcal{S}_a \,|\, F(a,K,J)=1 \text{ for all } \ K\in\mathcal{S}_a\}\,. \tag{4.28}$$

Because of the group property (4.26) this is a subgroup of $\mathcal{S}_a$. Because the defining relation is symmetric in J and K, $|\mathcal{S}_a/\mathcal{U}_a|$ is always a square. The discussion leading to (4.18) shows that the multiplicity factor(s) m_a should be chosen equal to the integer $\sqrt{|\mathcal{S}_a/\mathcal{U}_a|}$, and the number of fields should be reduced by the square of this factor, so that it is precisely $|\mathcal{U}_a|$. Consequently the untwisted subgroup $\mathcal{U}_a$ rather than the full stabilizer $\mathcal{S}_a$ is the natural group to use for the Fourier decomposition.

A few other observations which show that the untwisted stabilizer is a rather natural concept are the following. First, as follows from the result (5.6) below, not only $J \in \mathcal{S}_b$, but even $J \in \mathcal{U}_b$ is a necessary (though not sufficient) condition for $S^J_{a,b}$ to be non-vanishing. Second, as we will show in the appendix, the condition $F(a,K,J) = 1$ is symmetric in J and K, and the same group $\mathcal{U}_a$ is obtained if one replaces the F in (4.28) by $\bar{F}$ (which is the analogue of F for the action of the current on the second index of S^J). Finally, the groups $\mathcal{U}_a$ are invariant under the transformations (5.2) which respect the conditions on the matrices S^J.

We have now gathered all the ingredients for the formula for the matrix $\tilde{S}$. What is still missing is a proof that the matrices $S^{J,K}$ defined in (4.20) satisfy all the relevant requirements also for extensions that are not minimal. Since these matrices are expressed in terms of the matrices S^J for the minimal extension, this should now follow from the conditions on S^J. Before examining this, we summarize all these conditions and write down an explicit the formula for $\tilde{S}$.

5. The formula for $\tilde{S}$

5.1. The properties of the matrices S^J

The observations in the previous section can be summarized as follows. We are given a conformal field theory, with a set of mutually local integer spin simple currents forming a discrete group $\mathcal{G}$. To resolve the fixed points in the extension by $\mathcal{G}$ we need at least the following data.

For every simple current J we are given a matrix S^J that satisfies

- $\{1\}$ $S^J_{a,b} = 0$ if $Ja \neq a$ or $Jb \neq b$.

From now on S^J refers only to the restriction to the fixed points. Thus S^J is a non-trivial matrix only if J has fixed points, which can happen only if J has integer or half-integer spin.

- $\{2\}$ S^J is unitary.
- $\{3\}$ S^J satisfies $(S^J T^J)^3 = (S^J)^2$.

Here T^J is the T-matrix of the unextended theory, restricted to the fixed points of J.

We require the following simple current transformation rules:

- $\{4\}$ $S^J_{Ka,b} = F(a,K,J)\, \mathrm{e}^{2\pi \mathrm{i} Q_K(b)}\, S^J_{a,b}$,

  $S^J_{b,Ka} = \bar{F}(a,K,J)\, \mathrm{e}^{2\pi \mathrm{i} Q_K(b)}\, S^J_{b,a}$.

Here K is any other simple current that is local with respect to J, and $Q_K(b)$ is the monodromy charge of b with respect to K, defined for the unextended theory the matrix $S \equiv S^{\mathbf{1}}$ as in (3.8). This is not merely a definition of F and $\bar{F}$, but implies that they do not depend on b. In addition we require

- $\{4a\}$ $F(a,K,J_1)\,F(a,K,J_2) = F(a,K,J_1J_2)$

for all currents in $\mathcal{S}_a$. Owing to the group property $\{4a\}$ we have $F(a,K,\mathbf{1}) = 1$ (this also follows directly from condition $\{4\}$ because for $J = \mathbf{1}$, $\{4\}$ is the usual simple current relation (2.6) for the matrix $S^{\mathbf{1}} \equiv S$).

Using the functions $F(a,K,J)$ we define for each primary field a the untwisted stabilizer $\mathcal{U}_a$ as in (4.28).

The charge conjugation conditions can be stated in terms of a matrix η defined by

$$(S^J)^2 = \eta^J C^J \ ,$$

where C^J is the charge conjugation C of the unextended theory restricted to the fixed points of J. Note that if a is a fixed point of J, then so is its conjugate (denoted a^* in the following), so that the restriction makes sense. The matrices η must satisfy the following conditions:

- $\{5a\}$ η^J is diagonal.
- $\{5b\}$ $\eta^{J_1}_{a,a}\eta^{J_2}_{a,a} = \eta^{J_1J_2}_{a,a}$ for all $J_i \in \mathcal{U}_a$.
- $\{5c\}$ $\eta^J C^J = C^J(\eta^J)^\dagger$.

Thus the numbers $\eta^J_{a,a}$ are phases, conjugate fields have complex conjugate η-values, and on self-conjugate fields η can take only the values ± 1. Note that condition $\{5c\}$ is equivalent to $(S^J)^4 = \mathbb{1}$. Condition $\{5b\}$ is required only on the untwisted stabilizer $\mathcal{U}_a$ of a, not on the full stabilizer. Note also that $\eta^{\mathbf{1}}_{a,a} = 1$.

Furthermore we demand that S^J should satisfy

- $\{6\}$ $S^J_{a,b} = S^{J^{-1}}_{b,a}$.

Above we have postulated a lot of structure that should be present in any rational conformal field theory. One may wonder whether this structure is something completely new or whether it is already available in the usual data that are associated to a rational conformal field theory, e.g. those which appear in the polynomial equations [14]. Indeed, we conjecture that the matrix S^J coincides with the matrix that describes the modular transformation properties of the one-point function on the torus with insertion of the simple current $J(z)$. A proof of this conjecture is however beyond the scope of this paper. In section 6 we will show that for the important special case of WZW-models natural solutions to all these conditions can be written down explicitly.

5.2. The main formula

We work here with the group characters of the untwisted stabilizer, which have the usual properties, see section 4.1.

The primary fields of the extended theory can be described as follows. Each fixed point a of the unextended theory is resolved into $|\mathcal{U}_a|$ distinct fields, which are labelled by the group characters of the untwisted stabilizer $\mathcal{U}_a$. Then the following is the formula for the modular matrix $\tilde{S}$:

$$\tilde{S}_{(a,i),(b,j)} = \frac{|\mathcal{G}|}{\sqrt{|\mathcal{U}_a|\,|\mathcal{S}_a|\,|\mathcal{U}_b|\,|\mathcal{S}_b|}} \sum_{J\in\mathcal{G}} \Psi_i^a(J)\, S_{a,b}^J\, \Psi_j^b(J)^* \,. \tag{5.1}$$

Here the summation is formally over all $\mathcal{G}$, but in fact the only contributing terms are those with $J \in \mathcal{U}_a \cap \mathcal{U}_b$. In particular, if a primary field a is not a fixed point of any current, then $\mathcal{U}_a = \{\mathbf{1}\}$, and only $S^{\mathbf{1}}$ (the modular matrix S of the unextended theory) contributes.

The formula (5.1) follows directly from the Fourier decomposition (4.4) with $\mathcal{M}_a = \mathcal{U}_a$ and the diagonality assumption (4.20), which in its turn is strongly suggested by the arguments in sections 4.8 and 4.9.

5.3. Phase rotations

As mentioned in the previous section, all conditions on S^J are respected by the 'gauge' transformation

$$S^J \mapsto D^J S^J (D^J)^\dagger \,, \tag{5.2}$$

where D^J is a diagonal unitary matrix which, in order to preserve $\{6\}$, satisfies

$$D^J = (D^{J^{-1}})^* \,. \tag{5.3}$$

A sufficient condition for preserving the group properties of η and F, $\{5b\}$ and $\{4a\}$ is

$$D^{J_1} D^{J_2} = D^{J_1 J_2} \,. \tag{5.4}$$

However, F and η change only by ratios of the matrix elements of D^J, and therefore these latter conditions are necessary only for those ratios. These ratios are between conjugate fields or fields on the same simple current orbits. There are thus many phase rotations that are not restricted by (5.4).

This implies that in any case for the description of fixed point resolution we will have to deal with ambiguities. To see the effect of these phases, let us absorb them into the group characters by defining

$$\tilde{\Psi}_i^a(J) := D_{a,a}^J \, \Psi_i^a(J)$$

for each a. The new functions (which are generically not group characters any more) can be expanded with respect to the old ones,

$$\tilde{\Psi}_i^a(J) = \sum_j d_{ij}^a \Psi_j^a(J) \,.$$

To compute the coefficients d_{ij}^a we multiply both sides with $\Psi_k^a(J)^*$ and sum over J:

$$d_{ik}^a = \frac{1}{|\mathcal{U}_a|} \sum_J D_{a,a}^J \, \Psi_i^a(J) \, \Psi_k^a(J)^* \,.$$

Because of (5.3) and the inversion property of the characters, the numbers d_{ik} are real. Furthermore d is unitary:

$$\begin{aligned} \sum_k d_{ik}^a (d_{lk}^a)^* &= \frac{1}{|\mathcal{U}_a|^2} \sum_k \sum_{K,J} D_{a,a}^J \Psi_i^a(J) \Psi_k^a(J)^* (D_{a,a}^K)^* \Psi_l^a(K)^* \Psi_k^a(K) \\ &= \frac{1}{|\mathcal{U}_a|} \sum_J D_{a,a}^J \Psi_i^a(J) \, (D_{a,a}^J)^* \Psi_l^a(J)^* = \delta_{il} \,. \end{aligned}$$

Therefore it is orthogonal.

The new matrix $\tilde{S}$ is related to the old one as

$$\tilde{S}_{[\mathrm{new}]} = d \, \tilde{S}_{[\mathrm{old}]} \, d^\dagger \,,$$

where d is a block-diagonal matrix with blocks acting on each fixed point space. Since d is unitary, it preserves unitarity of $\tilde{S}$. Since it is orthogonal, it preserves symmetry of $\tilde{S}$, as well as $(\tilde{S}\tilde{T})^3 = \tilde{S}^2$ and $\tilde{S}^4 = \mathbb{1}$.

The transformation (5.2) thus generates a rotation among the primary fields into which a fixed point is resolved that respects all modular group representation properties. These transformations change the fusion rules unless the same transformation is applied to all fields simultaneously, in a sense that we will formulate more precisely in section 5.5. Since d is a generic orthogonal matrix, not necessarily a permutation, in general the transformation does not even respect integrality of the fusion coefficients. However, because of conditions $\{5b\}$ and $\{4a\}$ which must be respected by D^J, it will leave two important aspects of the fusion rules unaffected, namely the charge conjugation matrix and simple current fusion rules. Although both can change, they will in any basis have the correct form.

The existence of this freedom implies that given a set of matrices S^J satisfying all 6 conditions, one may discover that the matrix $\tilde{S}$ does not yield correct fusion rules. This does not necessarily imply that the fixed point resolution procedure has failed, but merely that one may have to choose a different basis using the phase rotations. In the application to WZW-models these considerations will not play a role, because there is a canonical basis for S^J which – by inspection – yields correct fusion rules.

Given a basis choice that yields proper fusion coefficients, there are still further phase rotations one can make that do not affect the integrality of the fusion rule coefficients. Namely, it is easy to show that if the matrices D^J satisfy (5.4), then the matrix d^a_{ij} is a permutation. If it depends on a it changes the fusion rules, but does not affect their integrality.

5.4. Proofs

Most of the properties of $\tilde{S}$ were already proved in the previous section. In particular the presence of the terms $S^{\mathbf{1}}$ guarantees that condition [I] is satisfied, and $\{6\}$ in combination with the group property of the characters under inversion imply that $\tilde{S}$ is symmetric.

The most crucial and non-trivial check is unitarity. This amounts to proving that the matrices $S^{J,K}$ of (4.20) satisfy (4.7). This can easily be done, but instead we prefer to give here a direct proof. Define

$$N(a,b) := \frac{|\mathcal{G}|}{\sqrt{|\mathcal{U}_a|\,|\mathcal{S}_a|\,|\mathcal{U}_b|\,|\mathcal{S}_b|}}$$

It is straightforward to show that

$$\sum_{b,j} \tilde{S}_{(a,i),(b,j)} \tilde{S}^*_{(b,j),(c,k)} = \sum_{\substack{b \\ Q=0;\ \mathrm{R}}} N(a,b)N(b,c)\,|\mathcal{U}_b| \sum_{J\in\mathcal{U}_a\cap\mathcal{U}_b\cap\mathcal{U}_c} \Psi^a_i(J)\,\Psi^c_k(J)^*\,S^J_{a,b}\,(S^J_{c,b})^* \,.$$

The symbol $\sum_{Q=0;\ \mathrm{R}}$ indicates a sum over one representative out of each $\mathcal{G}$-orbit with $Q_{\mathcal{G}}(b)=0$.

Now we sum over a full orbit of b rather than just a representative. Each term in such an orbit gives the same contribution because, due to $\{4\}$,

$$S^J_{a,Kb}(S^J_{c,Kb})^* = \mathrm{e}^{2\pi\mathrm{i}(Q_K(a)-Q_K(c))} F(b,K,J)F(b,K,J)^*\,S^J_{a,b}(S^J_{c,b})^* \qquad (5.5)$$

and $Q_{\mathcal{G}}(a)=Q_{\mathcal{G}}(c)=0$. Hence we can extend the sum over the full orbit, and divide by the orbit length $|\mathcal{G}|/|\mathcal{S}_b|$. After this operation we get a factor $N(a,b)N(b,c)|\mathcal{U}_b||\mathcal{S}_b|/|\mathcal{G}| = N(a,c)$, and in particular (except for the range of the J-summation) all dependence on b disappears. We wish to sum also over all b with $Q_{\mathcal{G}}(b)\neq 0$, because only in that case we

can use the unitarity relation for S^J. To do so,we observe that $S^J_{a,b} = F(a,K,J)^*\, S^J_{Ka,b}$ if $Q_K(b) = 0$. Hence we may replace $S^J_{a,b}$ by $\frac{1}{|\mathcal{G}|}\sum_{K\in\mathcal{G}} F(a,K,J)^* S^J_{Ka,b}$. Suppose we now consider a field b with non-zero monodromy charge. Then we have

$$\frac{1}{|\mathcal{G}|}\sum_{K\in\mathcal{G}} F(a,K,J)^*\, S^J_{Ka,b} = \frac{1}{|\mathcal{G}|}\sum_{K\in\mathcal{G}} \mathrm{e}^{2\pi\mathrm{i}Q_K(b)} S^J_{a,b} = 0\,.$$

Clearly we are therefore allowed to extend the sum over b to all fields. Now there is still one obstacle preventing us from performing this summation over b, namely the restriction of J to $\mathcal{U}_b$. Note, however, that $S^J_{Ka,b}$ vanishes if $J \notin \mathcal{U}_b$. This is clearly true for $J \notin \mathcal{S}_b$ since then by $\{1\}$ we have $S^J_{a,b} = S^J_{b,c} = 0$. Consider now $J \in \mathcal{S}_b$, but $J \notin \mathcal{U}_b$. Then there exists a current $L \in \mathcal{S}_b$ so that $F(b,L,J) \neq 1$. Thus the fact that $b = Lb$ and $Q_L(Ka) = 0$ imply that

$$S^J_{a,b} = S^J_{a,Lb} = F(b,L,J)\,\mathrm{e}^{2\pi\mathrm{i}Q_L(a)} S^J_{a,b} = F(b,L,J)\, S^J_{a,b}\,, \tag{5.6}$$

and hence $S^J_{a,b}$ has to vanish. Hence we may extend the sum over J to all of $\mathcal{G}$, but due to the restriction of the sum to $\mathcal{U}_a \cap \mathcal{U}_b \cap \mathcal{U}_c$ we are finally left with a sum over $\mathcal{U}_a \cap \mathcal{U}_c$. Now we can finally sum over b, using unitarity of S^J, to obtain

$$\sum_{b,j} \tilde{S}_{(a,i),(b,j)} \tilde{S}^*_{(b,j),(c,k)} = \frac{1}{\sqrt{|\mathcal{U}_a|\,|\mathcal{U}_c|\,|\mathcal{S}_a|\,|\mathcal{S}_c|}} \sum_{K\in\mathcal{G}} \sum_{J\in\mathcal{U}_a\cap\mathcal{U}_c} F(a,K,J)^*\, \Psi^a_i(J)\, \Psi^c_k(J)^*\, \delta_{Ka,c}\,.$$

The sum over K vanishes unless a and c are on the same K-orbit. But then they are identical, since we work with a definite set of representatives. Hence there is a contribution from every K that fixes a. In the sum over J all factors $F(a,K,J)$ are equal to 1 by the definition of $\mathcal{U}_a$. Note that the value of $F(a,K,J)$ for primary fields a that are not fixed points of K is irrelevant. This is as expected since those values can be modified without affecting unitarity. The sum over K yields a factor $|\mathcal{S}_a|\delta_{ac}$. Then there is only a contribution if $a = c$ and hence $\mathcal{S}_a = \mathcal{S}_c$ and $\mathcal{U}_a = \mathcal{U}_c$. Finally, using orthogonality of the group characters we get as the final result $\delta_{ac}\delta_{ik}$ as required.

The proofs of the relations $\tilde{S}^2 = \tilde{C}$ and $\tilde{S}\tilde{T}\tilde{S} = \tilde{T}^{-1}\tilde{S}\tilde{T}^{-1}$ work in essentially the same way, except that the cancellation of the factors F for the intermediate state works differently here: one now gets a factor $F(b,K,J)\bar{F}(b,K,J)$ which, however, is again unity because it equals $F(b,K,J)F(b,K,J^{-1}) = 1$.

For the charge conjugation matrix we find

$$\tilde{C}_{(a,i),(c,k)} = \hat{C}_{a,c}\, C^c_{i,k}\,,$$

where we defined

$$\hat{C}_{a,c} := \frac{1}{|\mathcal{S}_a|}\sum_{K\in\mathcal{G}} C_{Ka,c}$$

and

$$C^c_{i,k} := \frac{1}{|\mathcal{U}_c|} \sum_{J \in \mathcal{U}_c} \Psi^c_i(J)\, \eta^J_{c^*}\, \Psi^c_k(J)^* \,. \tag{5.7}$$

The matrix $\tilde{C}_{(a,i),(c,k)}$ is automatically symmetric since $\tilde{S}$ is symmetric. Hence

$$\hat{C}_{a,c}\, C^c_{i,k} = \hat{C}_{a,c}\, C^a_{k,i} \,.$$

This implies that

$$C^c = (C^{c^*})^T \,,$$

a property that can easily be checked explicitly. The matrix $\tilde{C}_{(a,i),(c,k)}$ is also automatically unitary, because $\tilde{S}$ is. We would like it to satisfy $\mathbb{1} = \tilde{C}^2 = \tilde{C}\tilde{C}^T$, and hence to be orthogonal. This will be the case if and only if it is real, and hence if and only if C^c is real. To verify this, replace the sum on J by a sum on J^{-1} and use (4.2). The result is

$$\begin{aligned}(C^c_{i,k})^* &= \frac{1}{|\mathcal{U}_c|} \sum_{J \in \mathcal{U}_c} \Psi^c_i(J)^*\, (\eta^J_{c^*})^*\, \Psi^c_k(J) = \frac{1}{|\mathcal{U}_c|} \sum_{J \in \mathcal{U}_c} \Psi^c_i(J^{-1})\, \eta^{J^{-1}}_{c^*}\, \Psi^c_k(J^{-1})^* \\ &= \frac{1}{|\mathcal{U}_c|} \sum_{J \in \mathcal{U}_c} \Psi^c_i(J)\, \eta^J_{c^*,c^*}\, \Psi^c_k(J)^* = C^c_{i,k} \,,\end{aligned}$$

so that indeed $\tilde{S}^4 = \tilde{C}^2 = \mathbb{1}$.

The last condition that C^c should satisfy is that it should be a permutation. Hence we need

$$\frac{1}{|\mathcal{U}_c|} \sum_{J \in \mathcal{U}_c} \Psi^c_i(J)\, \eta^J_{c^*}\, \Psi^c_k(J)^* = \delta_{k,\pi(i)}$$

for some permutation π. Multiplying both sides with $\Psi^c_k(K)$ and summing over k yields

$$\Psi^c_i(K)\, \eta^K_{c^*} = \Psi^c_{\pi(i)}(K) \tag{5.8}$$

This equation implies that, as a function of K, $\eta^K_{c^*}$ is a ratio of characters of $\mathcal{U}_c$, which shows that the group property of the η's (assumption $\{5b\}$) is rather natural in this context. The converse is also true. If η^J satisfies $\{5b\}$, then so does the left hand side of (5.8). Hence the left hand side is itself a $\mathcal{U}_c$-character, and must be equal to some character Ψ^c_k for some k. This is true for all i and defines a permutation $\pi(i)$. In other words, the η^J are the input which determines how the conjugation acts on the resolved fixed points in the extended theory.

We can also check the fusion rules of additional simple currents L that are not part of the extension, and are not projected out. We assume that for such a current condition $\{4\}$ holds, since it can in principle be used in a further extension.*

Now recall that on general grounds we have $\tilde S_{L(a,i),(b,j)} = \mathrm{e}^{2\pi\mathrm{i}Q_L(b)}\tilde S_{(a,i),(b,j)}$ (see (3.9)). Here the notation $L(a,i)$ stands for

$$L(a,i) \equiv (La, Li)\,,$$

with Li an allowed label for the fields into which La is resolved; thus L acts in the usual way, *i.e.* by the fusion product, on the first label, and as a permutation on the second label. Substituting this into the formula for $\tilde S$ we obtain

$$\begin{aligned}\tilde S_{(La,Li),(b,j)} &= \frac{|\mathcal{G}|}{\sqrt{|\mathcal{U}_a|\,|\mathcal{S}_a|\,|\mathcal{U}_b|\,|\mathcal{S}_b|}} \sum_{J\in\mathcal{G}} \Psi^a_{Li}(J)\, S^J_{La,b}\, \Psi^b_j(J)^* \\ &= \frac{|\mathcal{G}|}{\sqrt{|\mathcal{U}_a|\,|\mathcal{S}_a|\,|\mathcal{U}_b|\,|\mathcal{S}_b|}} \sum_{J\in\mathcal{G}} \Psi^a_{Li}(J)\, F(a,L,J)\, \mathrm{e}^{2\pi\mathrm{i}Q_L(b)} S^J_{a,b}\, \Psi^b_j(J)^*\,.\end{aligned}$$

Thus we clearly need

$$\Psi^a_{Li}(J)\, F(a,L,J) = \Psi^a_i(J) \quad \text{for all } J\,.$$

In other words, for given F there must exist a permutation L of the labels i such that this relation holds. By the same arguments as used above for charge conjugation, this is true if and only if F satisfies the group property $\{4a\}$ in J. In other words, the numbers $F(\cdot,L,\cdot)$ are the input which determines how a simple current L acts on the resolved fixed points in the extended theory.

This shows explicitly that the matrix $\tilde S$ satisfies conditions [I] – [VI]. That it also satisfies condition [VII] essentially follows already from its construction. We will not give a formal proof here, because condition [VII] (as well as [VIII]) is of lesser importance. If it were not satisfied, this would affect the uniqueness of the solution, but we do not have a proof that our solution is unique anyway.

$\star$ If L has fractional spin one may tensor the theory with another theory that has a complementary simple current, so that the total spin is integral.

5.5. Fusion rule automorphisms

Next we show that one can identify a symmetry which acts on the spectrum of the extended theory and is isomorphic to the discrete abelian group $\mathcal{G}$ of simple currents by which the chiral algebra was extended. This symmetry permutes the fields into which a fixed point is resolved. (Thus its role is analogous to the orbifold group in an (abelian) orbifold construction, which permutes the various twist sectors of the orbifold.) This group does not act freely in general, but it acts transitively on the fields into which a specific fixed point is resolved and therefore reflects the 'fixed point homogeneity', giving a concrete meaning to condition [VIII]. The symmetry we consider here is a special case of the phase rotations of section 5.3; they are now restricted to act in the same way on all fields, and are required to satisfy the group property (5.4). This ensures that they induce permutations.

Our argument goes as follows. We denote by $\mathcal{G}^*$ the group of characters of the identification group $\mathcal{G}$, and analogously for the subgroups $\mathcal{U}_b \subset \mathcal{G}$. Consider now some fixed point a and take an arbitrary $\mathcal{G}$-character $\phi \in \mathcal{G}^*$. Then to any character $\Psi_i^a \in \mathcal{U}_a^*$, $\Psi_i^a : \mathcal{U}_a \to \mathbf{C}$, of the untwisted stabilizer $\mathcal{U}_a$ we associate another element

$$ {}^{\phi}\Psi_i^a : \quad \mathcal{U}_a \to \mathbf{C} $$

of $\mathcal{U}_a^*$, which is defined by

$$ {}^{\phi}\Psi_i^a(J) := \phi(J)\, \Psi_i^a(J) \tag{5.9} $$

for all $J \in \mathcal{U}_a$. Combining the group properties of ϕ with respect to $\mathcal{U}_a$ (which are inherited from its group properties with respect to $\mathcal{G}$) and the basic properties of the $\mathcal{U}_a$-characters Ψ_i^a, it follows that the functions ${}^{\phi}\Psi_i^a$ are again characters of $\mathcal{U}_a$. As a consequence, (5.9) defines in fact an *action* of $\mathcal{G}^*$ on $\mathcal{U}_a$ and hence permutes the $\mathcal{U}_a$-characters. (This is completely analogous to the discussion of charge conjugation and additional simple currents in the previous section.) Furthermore, as an abstract group $\mathcal{G}^*$ is isomorphic to $\mathcal{G}$. Since the primary fields into which a fixed point a is resolved can be labelled by the elements of the character group $\mathcal{U}_a^*$, we can therefore describe the situation also as a permutation

$$ \Psi_i^a \;\mapsto\; {}^{\phi}\Psi_i^a = \Psi^a_{\pi_\phi(i)} \tag{5.10} $$

of these fields which is induced by the identification group $\mathcal{G}$.

We claim that when the permutation (5.10) is performed simultaneously (with one and the same $\mathcal{G}$-character ϕ) on all fixed points, then it is in fact an automorphism of the fusion rules, i.e. we have

$$ \tilde{\mathcal{N}}_{(a,\pi_\phi(i)),(b,\pi_\phi(j))}^{\quad\ (c,\pi_\phi(k))} = \tilde{\mathcal{N}}_{(a,i),(b,j)}^{\quad\ (c,k)} \,. \tag{5.11} $$

Note that the identity is never a fixed point, *i.e.* $\mathcal{G}^*$ acts trivially on it, so that the permutation π_ϕ definitely leaves the identity primary field of the extended theory fixed.

The relation (5.11) can be proven as follows. We insert the ansatz (5.1) for the S-matrix into the Verlinde formula and perform the resulting summation over $\mathcal{U}_d^*$. This leads to

$$\tilde{\mathcal{N}}_{(a,i),(b,j)}{}^{(c,k)} = \sum_d \frac{1}{S_{0\,d}} \sum_{\substack{J\in\mathcal{U}_a\cap\mathcal{U}_d,\\ K\in\mathcal{U}_b\cap\mathcal{U}_d}} S^J_{a,d} S^K_{b,d} (S^{JK}_{c,d})^* \, \Psi^a_i(J)\Psi^b_j(K)\Psi^c_k(JK)^* \,. \qquad (5.12)$$

The equality (5.11) then immediately follows with the help of the group properties of the $\mathcal{G}$-character ϕ.

It can happen that the symmetry we just described is not the only symmetry that acts on the resolved fixed points. As an example, let us look at the integer spin simple current invariant of su(3) at level 3. It has a single fixed point of order three. It is easy to check that any permutation of the three primary fields into which this fixed point is resolved is a symmetry. Since this modular invariant describes the conformal embedding of su(3) at level 3 into so(8) at level 1, this $\mathcal{S}_3$-symmetry is nothing else but the usual triality of so(8). More generally, the symmetry is also enhanced whenever only one non-trivial type of stabilizer with identical untwisted stabilizer is present. Then any permutation performed on all resolved fixed points simultaneously is a symmetry of the theory.

6. WZW-models

In this section we will introduce the mathematical tools which are needed to apply our results to WZW-models. In particular, we will show how to obtain the matrices S^J in the case of WZW-models. For WZW-models the action of a simple current corresponds to a certain symmetry of the Dynkin diagram, and therefore we will have to introduce some structures which one can associate to any symmetry of a Dynkin diagram, namely the so-called twining characters and orbit Lie algebras. For a more detailed explanation, and for some generalizations beyond the results needed in this paper, we refer to [1] and [18].

6.1. Twining characters

Twining characters are defined in the following way. Consider the Dynkin diagram of a symmetrizable Kac-Moody algebra. While the results of [1] are much more general, for applications in conformal field theory we can restrict ourselves to untwisted affine Lie algebras. An automorphism of such a diagram acts in a canonical way on the generators corresponding to the simple roots and on the associated Cartan subalgebra generators. By imposing the automorphism properties, the action can be extended to all root generators, and with some additional work also to the derivation, *i.e.*, in physics terms, the zero-mode Virasoro generator, and even to the whole Virasoro algebra. In this way we can associate to any Dynkin diagram automorphism an algebra automorphism

[1]. Furthermore the action on the Cartan subalgebra induces a natural action on the dual space, *i.e.* the weight space, and in particular on the highest weights. Integrable highest weights are either mapped to other integrable highest weights, or are fixed by the automorphism.

The action of the automorphism ω on the Kac-Moody algebra and on the highest weights also induce an action of ω on representation spaces, *i.e.* ω can be implemented by an operator $\mathcal{T}_\omega$ which has a well defined action on each state in a representation space (note that this action is not simply determined by the weight of the state). This allows to define a new kind of character, the *twining character*, which is obtained from the usual characters by inserting $\mathcal{T}_\omega$ into the trace:

$$\mathcal{X}^{(\omega)}(\tau) = \mathrm{Tr}\ \mathcal{T}_\omega\, \mathrm{e}^{2\pi\mathrm{i}\tau L_0}\ ,$$

where the trace is over all states in the irreducible representation space. Note that $\mathcal{X}^{(id)}$ is just the ordinary character. Also, strictly speaking the definition only makes sense for highest weight representations with a highest weight that stays fixed under the automorphism, but if the highest weight is not fixed we can simply set $\mathcal{X}^{(\omega)} = 0$ for $\omega \neq id$.

6.2. Orbit Lie algebras

Given a diagram automorphism we define a new 'folded' Dynkin diagram which is essentially obtained by identifying nodes that lie on the same orbit under the automorphism and by assigning one node of the new diagram to each orbit. The only thing we then still need is a prescription for the number of lines between the nodes of the folded diagram. In terms of the Cartan matrix $\breve{A}$ of the folded diagram, this prescription is the following:

$$\breve{A}^{i,j} := s_i \sum_{l=0}^{N_i-1} A^{\omega^l i,j}\ .$$

Here A is the Cartan matrix of the original Lie algebra, and the labels i and j on the left hand side are (arbitrarily chosen) orbit representatives. N_i denotes the length of the orbit of node i, and ω is the symmetry of the Dynkin diagram. Finally, the integer s_i is defined as

$$s_i := 1 - \sum_{l=1}^{N_i} A^{\omega^l i,i}\ .$$

One can show that if A is the Cartan matrix of an affine Lie algebra, then so is $\breve{A}$, except for the order N automorphism of Lie algebras of untwisted affine type A_{N-1}. In that case there is only a single orbit, and $\breve{A}$ is formally the 1×1 matrix 0. In all other cases one has $s_i = 1$ or $s_i = 2$ for all i, and one obtains again an affine Lie algebra which

is called the *orbit Lie algebra* $\breve{G}$ associated to the original algebra and its automorphism ω. It can be a twisted or an untwisted affine Lie algebra. The main result of [1] is now that the twining characters are equal to the ordinary characters of the orbit Lie algebra (this holds for the full characters, not just for the Virasoro-specialized ones). Moreover, in the exceptional case of the order N automorphisms of affine A_{N-1}, the twining characters can also be computed (they are just a power of $e^{2\pi i\tau}$).

The relation between integrable highest weights of fixed points of G and integrable highest weights of $\breve{G}$ is determined straightforwardly from the folding prescription of the Dynkin diagram. Dynkin labels a_i of G that are identified by the diagram automorphism get mapped to a single Dynkin label a on the corresponding node of the Dynkin diagram of $\breve{G}$. This prescription can be extended to include also the Cartan angles into the twining characters.

6.3. Diagram automorphisms corresponding to Simple Currents

For our purposes we need these results for the special case of diagram automorphisms related to simple currents. Simple currents act as a permutation on the highest weight representations; the same is true for diagram automorphisms. The correspondence is not one-to-one: some diagram automorphisms, such as charge conjugation, do not correspond to simple currents.[⋆] We will only need simple currents for which there does exist such an automorphism. For these currents one finds the following orbit Lie algebras:

algebra G	level	current	spin	orbit Lie algebra $\breve{G}$	level	shift δ
$A_n^{(1)}$	$\frac{p(n+1)}{\ell}$	J^ℓ	$\frac{1}{2}p(n+1-\ell)$	$A_{\ell-1}^{(1)}$	p	$\frac{[(n+1)^2-\ell^2]}{24\ell}p$
$B_{n+1}^{(1)}$	p	J	$\frac{1}{2}p$	$\tilde{B}_n^{(2)}$	p	$\frac{2n+3}{24}p$
$C_{2n}^{(1)}$	p	J	$\frac{1}{2}pn$	$\tilde{B}_n^{(2)}$	p	$\frac{5n}{24}p$
$C_{2n+1}^{(1)}$	$2p$	J	$\frac{1}{2}(2n+1)p$	$C_n^{(1)}$	p	$\frac{2n+1}{8}p$
$D_n^{(1)}$	$2p$	J_v	p	$C_{n-2}^{(1)}$	p	$\frac{p}{4}$
$D_{2n}^{(1)}$	$2p$	J_s	$\frac{1}{2}pn$	$B_n^{(1)}$	p	$\frac{np}{8}$
$D_{2n+1}^{(1)}$	$4p$	J_s	$\frac{1}{2}(2n+1)p$	$C_{n-1}^{(1)}$	p	$\frac{2n+3}{8}p$
$E_6^{(1)}$	$3p$	J	$2p$	$G_2^{(1)}$	p	$\frac{2}{3}p$
$E_7^{(1)}$	$2p$	J	$\frac{3}{2}p$	$F_4^{(1)}$	p	$\frac{3}{8}p$

(Here in the first line ℓ is a divisor of $n+1$. The orbit Lie algebra for a power of a current J that has the same order as J is the same as that of J.) This table is identical

⋆ On the other hand, there is also an exceptional simple current, namely for E_8 at level 2, for which there is no corresponding diagram automorphism.

to the one that has already appeared in [7], except for the second and third lines. The algebra $\tilde{B}_n^{(2)}$ is a twisted affine Lie algebra (we use the notation of [19]), and appears precisely in those cases for which no 'fixed point theory' could be identified in [7]. The third column specifies the group $\mathcal{G}$ of simple currents by means of that simple current which generates it. For algebras and simple currents that are not listed in the table, there are no fixed points. The table in [7] contained one extra entry for the full $\mathbf{Z}_2 \times \mathbf{Z}_2$ center of $D_{2n}^{(1)}$. We did not include it here, because first of all it can be obtained by folding the Dynkin diagram of one of the orbit Lie algebras once more, and secondly because in our formalism we need only fixed point resolution matrices corresponding to cyclic subgroups of the center, even if the full extension is non-cyclic. The last column of the table describes the shift δ in the modular anomaly $h - \frac{c}{24}$; the entry in the table is $(h - \frac{c}{24})|_G - (h - \frac{c}{24})|_{\check{G}}$. This shift is universal in the sense that it only depends on the level of the algebra $\check{G}$, but not on the specific primary field.

Although in the last column we have given the full shift (except for $\tilde{B}$, see below), in the following we need only its fractional part. On general grounds (see e.g. [7]), this should be a 12th root of unity for all fixed points of integer spin simple currents (and hence an odd power of a 24th root of unity for fixed points of half-integer spin currents), and indeed it is.

6.4. The matrices S^J for WZW-models.

The orbit Lie algebras appearing in the table all have the property that their characters (which are equal to the twining characters of the parent Lie algebra) span a unitary representation of the modular group (which, incidentally, is no longer true for the other twisted affine Lie algebras which do not arise in our construction). This provides us in a natural way with a matrix S^J for every simple current.

Using the general results of [7] for fixed point theories, we can also describe how the shift δ enters in the precise definition of the fixed point resolution matrix S^J. Namely, let $\check{S}^J$ be the matrix S of the orbit Lie algebra corresponding to the folding generated by J; then the matrix S^J that will appear in the fixed point resolution is given by

$$S^J = \mathrm{e}^{-6\pi\mathrm{i}\delta}\,\check{S}^J\,. \tag{6.1}$$

The shift δ ensures that S^J satisfies condition {3} of section 5, *i.e.* together with the matrix T of the unextended theory restricted to the fixed points it forms a representation of the modular group.

Since the twisted algebras $\tilde{B}_n^{(2)}$ do not correspond to conformal field theories, the exact definition of the shift cannot be made directly in terms of differences of conformal weights, but obviously all we need is that condition {3} is satisfied. The shift δ is chosen in such a way that this is the case if we use in (6.1) for $\check{S}^J$ the Kac-Peterson [15] formula

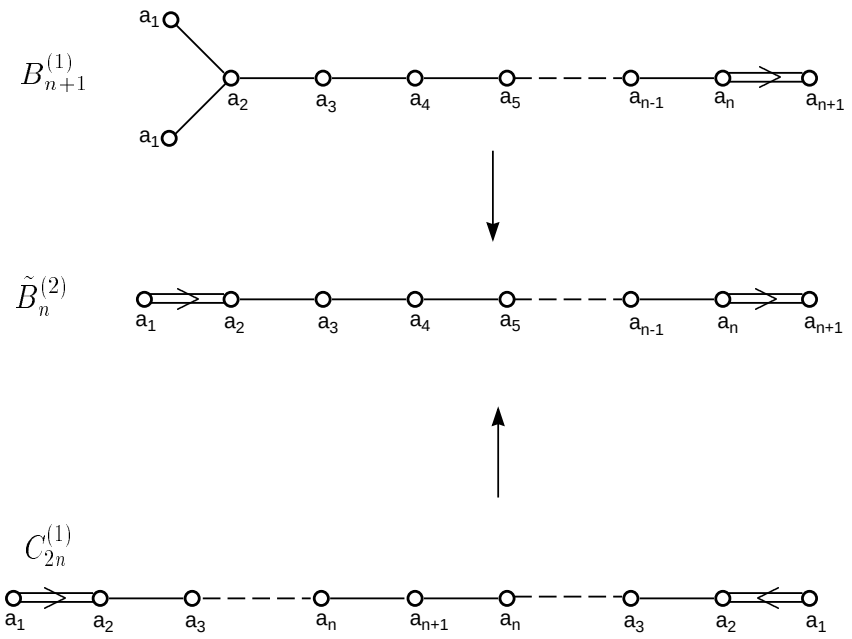
for the S-matrix of $\tilde{B}_n^{(2)}$ (or $A_{2n}^{(2)}$ in the notation of [20]).† Since this formula is an essential ingredient in (5.1), for completeness we include it here:

$$S_{\Lambda,\Lambda'} = \mathrm{i}^{|\Delta_+|} \left| \frac{M^*}{(k+g)M} \right|^{-\frac{1}{2}} \sum_{w \in W} \epsilon(w) \exp\left[-2\pi \mathrm{i} \, \frac{(\Lambda+\rho, w(\Lambda'+\rho))}{k+g} \right] .$$

This formula holds for all algebras in columns 1 or 5 of the table. To use it one has to select first a definite simple Lie algebra, whose Dynkin diagram is obtained by removing one node from the Dynkin diagram of the affine algebra. For the untwisted Kac-Moody algebras $X^{(1)}$ this is the usual horizontal subalgebra X. For $\tilde{B}_n^{(2)}$ one chooses the unique C_n-type sub-diagram of the Dynkin diagram to define the horizontal subalgebra. The level k appearing in the formula is the one appearing in columns 2 or 6 of the table, and g is the dual Coxeter number, whose value is $2n+1$ for $\tilde{B}_n^{(2)}$. Furthermore ρ is the Weyl vector of the simple Lie algebra, and Λ and Λ' are the restrictions of highest weights of the affine algebra to its horizontal subalgebra, obtained by ignoring the Dynkin label at the additional node. For fixed level these restrictions label the distinct representations uniquely. Finally, $|\Delta^+|$ is the number of positive roots of the horizontal algebra, the sum is over the Weyl group of the horizontal algebra weighted by the determinant $\epsilon(w)$, and the normalization factor involves the lattice M of the translation subgroup of the Weyl group of the affine algebra and its dual M^*. For untwisted affine Lie algebras $X^{(1)}$, M is the coroot lattice of X, while for $\tilde{B}_n^{(2)}$ it is the *root* lattice of B_n.

To use the matrices S^J defined by (6.1), one must also know the precise relation between the highest weights of the fixed points and those of the representations of the orbit Lie algebra. This relation follows in a straightforward way from the folding; it was already represented diagrammatically in [7], except for the cases involving $\tilde{B}_n^{(2)}$, which were interpreted differently in [7]. The prescription for those cases is displayed in the following diagram:

† One may also compare the formula for the conformal anomaly given in [20] to the conformal anomalies of the fixed points. This does not yield precisely the shift δ. The difference appears to be due to the fact that in this case one of the additional arguments (Cartan angles) of the twining characters is shifted by a constant with respect to the $A_{2n}^{(2)}$ (orbit Lie algebra) characters as defined in [20]. Since modular transformations act not only τ, but also on the other arguments, this results in an extra phase in the transformation of the twining characters.



Let us now examine the remaining conditions. Since we want to do this for arbitrary WZW-models, not just those based on simple Lie algebras, we will first discuss how the problem factorizes in the semi-simple case. Consider thus a tensor product of N simple WZW-models. We denote their primary fields as $(a_1, a_2, \ldots, a_N)$. Consider an integer spin simple current $(J_1, J_2, \ldots, J_N)$. It can only have fixed points if each of its components J_i does. Each such component can only have a fixed point if it has integer or half-integer spin. Note that the conditions $\{1\}$ – $\{6\}$ were only formulated for integer spin simple currents J (while the current K appearing in $F(a, K, J)$ can however have any spin). Hence we will need to generalize them to half-integer spin currents. This generalization has to be such that for any integral spin combination the conditions $\{1\}$ – $\{6\}$ are automatically satisfied. Note that the need for such a generalization is not limited to WZW-models, since the possibility of tensoring exists for any conformal field theory.

It is clear that conditions $\{1\}$ – $\{6\}$ are satisfied for all extensions of any tensor product if and only if they are satisfied for each factor in the tensor product, provided that we also allow for currents J of half-integer spin.

The matrix S^J acts only on the fixed points of J, and therefore condition $\{1\}$ is satisfied. Furthermore S^J is unitary, so $\{2\}$ is satisfied as well. Moreover, the matrices S^J are symmetric, and they are identical for any two currents J that belong to the same cyclic subgroup and have the same order, so that $\{6\}$ is satisfied as well. Condition $\{3\}$ is also satisfied, as pointed out after (6.1) above.

Unless all fixed points are self-conjugate, the folded Dynkin diagram inherits a non-trivial charge conjugation symmetry from the unfolded one. The matrix S^J satisfies $(S^J)^2 = \eta^J C^J$, where $C^J = \breve{C}$ is the charge conjugation matrix derived from the folded diagram. This is thus the unfolded charge conjugation matrix restricted to the fixed

points. Hence the phase η is just the square of the phase appearing in (6.1):

$$\eta^J = \mathrm{e}^{-12\pi\mathrm{i}\delta(J)} \,. \tag{6.2}$$

For integer spin currents (even those built out of half-integral spin currents) $\delta(J)$ is a 12th root of unity, and hence η can only be $+1$ or -1. Moreover, it is independent of the fixed points, so that $\{5a\}$ and $\{5c\}$ are manifestly satisfied. For half-integral spin currents there is no reason why $\{5c\}$ should be satisfied; indeed in this case we have $\eta = \pm\mathrm{i}$, and hence $\{5c\}$ does not hold.

If the fixed points lie on non-trivial orbits of further simple currents, this will still be the case after folding [13]. Thus the orbit Lie algebra inherits simple current symmetries from the parent algebra. By the standard simple current relations for the S-matrix (see (2.6)), these simple current symmetries are of the form

$$S^J_{Ka,b} = \mathrm{e}^{2\pi\mathrm{i}Q^J_K(b)} S^J_{a,b} \,,$$

where

$$Q^J_K(b) = h^J(a) + h^J(K^J) - h^J(b) \,. \tag{6.3}$$

Here the superscript J refers to quantities defined directly in the orbit Lie algebra. The conformal weights $h^J(a)$ and $h^J(b)$ differ by a constant (independent of a and b) from the corresponding quantities in the parent algebra; this constant (*i.e.* the shift δ) cancels in (6.3). (On the other hand, the primary field K^J is a simple current in the orbit Lie algebra. In particular, K^J and K are not related the way fixed points are related to fields of the orbit Lie algebra, and hence the difference $h^J(K^J) - h(K)$ has nothing to do with the shift δ.) The relation (6.3) can thus be rewritten as $Q^J_K(b) = Q_K(b) + h^J(K^J) - h(K)$, so that

$$F(a,K,J) = \mathrm{e}^{2\pi\mathrm{i}\,[h^J(K^J) - h(K)]} \,. \tag{6.4}$$

Actually the definition (6.3) only makes sense if we can interpret the orbit Lie algebra as a conformal field theory. This is possible in all cases, except for the $\tilde{B}$-type algebras, but fortunately the Dynkin diagrams of those algebras do not possess any symmetries. Once we have a conformal field theory interpretation, $h^J(K^J)$ is simply the conformal spin of the simple current corresponding to the symmetry induced by K.

We conclude that the phase factor $F(a,K,J)$ is determined by $h^J(K^J) - h(K)$, and is independent of a. Furthermore, since all matrices are symmetric, we have $F = \bar{F}$, and then (A.14) implies that F is real. If K is equal to J, then K is mapped to the identity current of the orbit Lie algebra, which maps every field to itself. In conformal field theories this current is just the identity primary field, but even for the $\tilde{B}$-cases we

should interpret $h^J(K^J)$ as zero for $K = J$ (so that (6.3) holds with $Q_J^J \equiv 0$ and $a = b$). Hence

$$F(J,J) \equiv F(a,J,J) = (-1)^{h(J)}$$

$$F(J,J) \equiv F(a,J,J) = (-1)^{h(J)} \tag{6.5}$$

(see the appendix for a different derivation).

The last requirements to be checked are the group properties of η and F. Here we have to pay special attention to currents of half-integral spin. We note the following.

— The shifts δ are an even multiple of $\frac{1}{24}$ for integral spin simple currents and an odd multiple of $\frac{1}{24}$ for half-integral spin currents [7]. Therefore $\eta = \pm 1$ for integral spin currents, while $\eta = \pm \mathrm{i}$ for half-integral spin currents.

— According to (A.13), (6.5), for half-integral spin currents we have $F(J,J) = -1$, while for integral spin currents this quantity is 1.

Because of the first point the group property $\{5b\}$ of η cannot be satisfied as it stands for half-integer spin currents of order 2. The second property implies that integral spin currents built out of overlapping combinations of half-integer spin currents (like $K = (J_1, J_2, \mathbf{1})$ and $J = (\mathbf{1}, J_2, J_3)$) will have $F(K,J) = -1$.

A check of the properties of F and η has to be performed only at the lowest level at which a current J has fixed points. This is because J then has fixed points precisely at each multiple of that level, and all relevant quantities (the spins of simple currents and the shifts δ) are linear in the level. Hence F and η scale with powers of the level. By explicit computation it can be checked that $\{4a\}$ is satisfied for all WZW-models with a cyclic center. In that case the phase $F(J,K)$ is given by (A.1) for all currents J and K that have fixed points, and this expression implies $\{4a\}$.

The only simple WZW-theories with a non-cyclic center are those based on D_{2n}. By inspection we see that $F(J_s,J_v) = F(J_c,J_v) = F(J_s,J_c) = -1$ for D_{4m} at levels $4k+2$, and that $F(J_s,J_v) = F(J_c,J_v) = -1, F(J_s,J_c) = 1$ for D_{4m+2} level $4k+2$ (consequently at levels that are a multiple of 4 all these quantities are 1). The values for equal arguments are directly determined, through (A.1), by the spins of the currents. It is then straighforward to check that also in this case $\{4a\}$ is indeed satisfied.

As remarked above, the factors η^J cannot satisfy $\{5b\}$ for arbitrary currents J. Note however that $\{5b\}$ only needs to hold on the untwisted stabilizer and only for integer spin currents. We find that in general instead of $\{5b\}$ in WZW-models the relation

$$\eta^{J_1}\eta^{J_2} = F(J_1,J_2)\,\eta^{J_1 J_2} \tag{6.6}$$

holds. This implies immediately that indeed $\{5b\}$ does hold, because the factor $F(J,K)$ is equal to 1 for any two currents in the untwisted stabilizer by the definition of the latter. This is in fact a non-trivial check on the applicability of our formalism to WZW-models.

Although establishing (6.6) empirically is sufficient for our purpose, a deeper insight would be welcome. Note that *a priori* the quantities F and η are not in any obvious way related. The relation (6.6) as well as the fact that $F(a,K,J)$ is independent of a suggest that F plays a more fundamental role than we described so far. We comment briefly on this is in the conclusions.

Finally we give two examples where the untwisted stabilizer is smaller than the stabilizer. First consider the following modular invariant of D_4 level 2:

$$|\mathcal{X}_{\mathbf{1}} + \mathcal{X}_v + \mathcal{X}_s + \mathcal{X}_c|^2 + 4\,|\mathcal{X}_{28}|^2\,.$$

Here the subscripts denote, respectively, the identity field, the three simple currents J_v, J_s and J_c, and the unique primary field with ground state dimension 28 that is fixed by all currents. Because the currents have spin 1, this is a conformal embedding, and it can be interpreted in terms of E_7 level 1, provided we take the factor 4 inside the square. This provides the correct ground state dimension 56 for the E_7-representation, and it yields the correct total number of primaries, namely 2. The stabilizer of the (28) is $\mathcal{S}_{28} = \{\mathbf{1}, J_v, J_s, J_c\}$, but the untwisted stabilizer is just the identity, $\mathcal{U}_{28} = \{\mathbf{1}\}$.

An example with reduction of the stabilizer due to spin-1/2 currents is also easy to find. Consider the product of three B_n WZW-theories at level 1. We may extend the algebra by the currents $(J,J,\mathbf{1})$, $(J,\mathbf{1},J)$ and $(\mathbf{1},J,J)$. This extension has again an interpretation as a conformal embedding, namely as $\mathrm{so}(d_1)\oplus\mathrm{so}(d_2)\oplus\mathrm{so}(d_3) \subset \mathrm{so}(d_1+d_2+d_3)$, with all d_i odd. The spinor of each $\mathrm{so}(d_i)$ is fixed by the simple current J, and hence the product of the three spinors has a stabilizer of order 4. But $\mathrm{so}(d_1+d_2+d_3)$ has only one spinor representation, so that once again we see that the factor 4 must be absorbed into the character. In this way one gets the correct normalization $2^{(d_1+d_2+d_3-1)/2}$ of the ground state.

We have applied our formula to many other cases, in particular with the aim of checking the fusion rules. In all cases the fusion coefficients were positive integers, but a proof of this property is still lacking. A computer program that implements our formalism and calculates the (conjectured) fusion coefficients and the matrix $\tilde{S}$ for any simple current extended WZW-model is available from the second author.

7. Conclusions

In this paper we have presented strong evidence for the formula (5.1) which describes the modular S-matrix for extensions of the chiral algebra by integer spin simple currents. The basic ingredients leading to this formula are the following. For any simple current J in the extension we postulate the existence of a representation of the modular group defined on the fixed points only, leading to the matrices S^J. These matrices determine a subgroup of the stabilizer of each fixed point, the so-called untwisted stabilizer. A formalism could then be developed which expresses the matrix $\tilde{S}$ in terms of the matrices S^J and the group characters of the untwisted stabilizer groups $\mathcal{U}_a$.

Some important open problems remain. We have not shown that our solution for $\tilde{S}$ always leads to non-negative integral fusion coefficients, not to mention even the existence of a conformal field theory for each extension. We have also not shown that our solution is unique. The arguments in section 4 suggest strongly that it is the only solution to the conditions [I] – [VIII], given the matrices S^J, but uniqueness of the latter is by no means proved. Nevertheless, the existence of natural candidates (the modular matrices for torus one-point functions in general, and S-matrices of orbit Lie algebras for WZW-models) suggests that our solution is unique in this respect.

In some situations fixed point resolutions which are still more general than the ones considered here can work, which however violate the postulate of fixed point homogeneity. In this case equation (4.1) is replaced by a more general decomposition of $|\mathcal{S}_a|$ into squares. An example we have checked is the simple current invariant of A_4 level 5. If one splits the fixed point using $m_1 = 2$, $m_2 = 1$ instead of the homogeneous splitting $m_i = 1$, $i = 1, 2, \ldots, 5$, and uses the same matrix S^J, then one arrives at a matrix $\tilde{S}$ that satisfies all conditions (except homogeneity). Furthermore both resolutions yield correct fusion rules for a different number of primary fields. Of course this does not mean that both yield a conformal field theory.

It is tempting to speculate and interpret our results directly in terms of the underlying extension of the chiral algebra $\mathcal{A}$. The simple currents used in the extension form a group $\mathcal{G}$. One would like to construct the extended chiral algebra $\tilde{\mathcal{A}}$ in terms of this group $\mathcal{G}$ and the original chiral algebra $\mathcal{A}$. More precisely, instead of the group $\mathcal{G}$ one should work with its group algebra $\mathcal{A}_{\mathcal{G}}$. The fact that we have to distinguish between the ordinary stabilizers and the untwisted stabilizers shows, however, that this idea is somewhat too naive. Interestingly enough, there is a generalization of the notion of a group algebra, the so-called twisted group algebras. The twisted group algebra $\tilde{A}_{\mathcal{G}}$ associated to $\mathcal{G}$ is still associative, as required for a chiral algebra, but no longer commutative, even though $\mathcal{G}$ is. The representation theory of $\tilde{A}_{\mathcal{G}}$ precisely fits with what we expect for $\tilde{\mathcal{A}}$ from our results. In particular, the irreducible representations of $\tilde{A}_{\mathcal{G}}$ are not one-dimensional any more; they are labelled by the characters of a subgroup $\mathcal{U}$ of $\mathcal{G}$ and have dimension $\sqrt{|\mathcal{G}|/|\mathcal{U}|}$; this should explain the prescription (4.1) for decomposing the stabilizer groups.

Unfortunately, such an explicit description of $\tilde{\mathcal{A}}$ is still lacking. A first step in this direction would be an explicit construction of the chiral vertex operators corresponding

to simple currents. In the case of WZW-theories, this should in particular lead to a proof of our expectation that the matrices S^J describe the modular transformation properties of the one-point functions of simple currents on the torus. Such a construction would also justify our assumption on the existence of the matrices S^J for arbitrary conformal field theories. Note that in the case of WZW-theories one can identify the matrices S^J with the S-matrices of the relevant orbit Lie algebras, and therefore all ingredients for the S-matrix formula are known.

Let us also mention possible applications of our formula beyond conformal field theory. When combined with Verlinde's formula, the Kac-Peterson formula for the S-matrix allows to compute the rank of certain vector bundles over the moduli space of semi-stable bundles with structure group G, where G is a simply connected compact Lie group. Since integer spin simple current invariants can be interpreted in terms of WZW-theories on non-simply connected compact Lie groups [16][21], it is reasonable to expect that our formula, combined again with the Verlinde formula, can be used to compute analogous quantities for the case when the structure group G is not simply connected.

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APPENDIX

Properties of F and η

There are several restrictions on F and η due to conditions $\{1\}$ – $\{6\}$,* and in addition there is a residual transformation that respects all conditions. Here we will use the latter to restrict the values of F and η.

First of all, because of the unitarity condition $\{2\}$ we have

$$\begin{aligned} 1 &= \sum_b S^J_{Ka,b}(S^J_{b,Ka})^\dagger \\ &= \sum_b F(a,K,J)F(a,K,J)^* S^J_{a,b}(S^J_{a,b})^* \ = F(a,K,J)F(a,K,J)^*, \end{aligned}$$

so that F is a phase. Note that for this conclusion it is essential that the numbers F do not depend on b. Furthermore we have manifestly $F(a,\mathbf{1},J) = F(a,K,\mathbf{1}) = 1$.

Consider now first the values of $F(a,K,J)$ when a is a fixed point of K. Then condition $\{4\}$ implies that either $F(a,K,J) = \mathrm{e}^{-2\pi\mathrm{i}Q_K(b)}$ or $S^J_{a,b} = 0$. Since $S^J_{a,b}$ cannot vanish for all b because of unitarity, this implies that

$$F(a,K,J) = \mathrm{e}^{2\pi\mathrm{i}\,q(a,K,J)}\,,$$

where $q(a,K,J)$ is an allowed charge with respect to the current K, *i.e.* an Mth root of unity, where M is the order of K in $\mathcal{G}$.

If $K = J$ (or a power of J) we have $Q_J(b) = 0 = Q_K(b)$, because b is a fixed point of the (integer spin) simple current J. It follows that $F(a,J^n,J) = 1$. Hence by the group property in the third argument we have $F(a,J^n,J^m) = 1$ for all n and m. Thus F is trivial within cyclic groups generated by integer spin currents.

The matrices S^J can also be defined for half-integer spin currents, the only currents of fractional spin that can possess fixed points. In that case the fixed points of J have charge $\frac{1}{2}$ with respect to J, and the foregoing result generalizes to

$$F(a,J^n,J^m) = (-1)^{nm\,h(J)}\,, \qquad \text{(A.1)}$$

where $h(J)$ is the spin of J. Any half-integer spin simple current is local with respect to itself. Therefore $h(J^m) = mh(J)$ mod 1. Relation (A.1) thus implies that for currents J and K from the same cyclic orbit, $F(a,J,K)$ is equal to -1 if both J and K have half-integer spin, and equal to 1 otherwise.

$\star$ Curly brackets refer to the conditions stated in section 5.

Now we compute the value of $F(a,K,J)$ on the $\mathcal{G}$-orbit through a, still for the case $Ka=a$. Suppose that L is a current in $\mathcal{G}$. Then

$$S^J_{KLa,b} = F(La,K,J)\,\mathrm{e}^{2\pi\mathrm{i}Q_K(b)}\,S^J_{La,b} = F(La,K,J)F(a,L,J)\,\mathrm{e}^{2\pi\mathrm{i}[Q_K(b)+Q_L(b)]}S^J_{a,b}\,. \tag{A.2}$$

Using commutativity of the currents we have also

$$S^J_{LKa,b} = F(Ka,L,J)\,\mathrm{e}^{2\pi\mathrm{i}Q_L(b)}\,S^J_{Ka,b} = F(Ka,L,J)F(a,K,J)\,\mathrm{e}^{2\pi\mathrm{i}[Q_K(b)+Q_L(b)]}S^J_{a,b}\,. \tag{A.3}$$

If K fixes a, then when equating (A.2) and (A.3) the factor $F(a,L,J)$ cancels out, and (choosing b such that $S^J_{a,b}\neq 0$, which is again possible because S^J is unitary) we find

$$F(La,K,J) = F(a,K,J)\,, \tag{A.4}$$

so that $F(a,K,J)$ is constant on all simple current orbits of a. A third way of writing $S^J_{KLa,b}$ is as follows:

$$S^J_{LKa,b} = F(a,LK,J)\,\mathrm{e}^{2\pi\mathrm{i}[Q_K(b)+Q_L(b)]}\,S^J_{a,b}\,;$$

comparing with (A.2) or (A.3) we see that

$$F(a,KL,J) = F(a,K,J)F(a,L,J) \tag{A.5}$$

if a is a fixed point of K as well as L.

Next we recall from section 5 that conditions $\{1\}$ – $\{6\}$ remain valid if we transform S^J as in (5.2), provided that the numbers D^J satisfy (5.3) and condition (5.4), or at least a sufficient subset of the latter to preserve the group properties of η^J and $F(a,K,J)$. This transformation changes both F and η, except the value of $F(a,K,J)$ if $Ka=a$ and the value of η on self-conjugate fields. The phase factor $F(a,K,J)$ changes to $D^J_a(D^J_{Ka})^*F(a,K,J)$, where D_a denotes the diagonal matrix element $D_{a,a}$. We will use the analogous notation also for η. On the K-orbit of a we then have

$$F'(K^\ell a,K,J) = D^J_{K^\ell a}(D^J_{K^{\ell+1}a})^*F(K^\ell a,K,J)\,. \tag{A.6}$$

The matrices D^J can be used to make all F' equal to a constant X on the K-orbit of K (this works for all currents K simultaneously, as can *e.g.* be checked by considering one generator for each cyclic subgroup). This can be seen as follows. Suppose N

is the smallest positive integer for which $K^N a = a$. Multiplying the equations for $\ell = 0, 1, \ldots, N-1$ we then find

$$X^N = \prod_{\ell=0}^{N-1} F(K^\ell a, K, J)\,. \tag{A.7}$$

The right hand side can be computed as

$$\begin{aligned} S^J_{K^N a,b} &= F(K^{N-1}a, K, J)\, \mathrm{e}^{2\pi \mathrm{i} Q_K(b)} S^J_{K^{N-1}a,b} \\ &= F(K^{N-1}a, K, J)\, \mathrm{e}^{2\pi \mathrm{i} Q_K(b)} F(K^{N-2}a, K, J)\, \mathrm{e}^{2\pi \mathrm{i} Q_K(b)} S^J_{K^{N-2}a,b} \\ &= \ldots = \mathrm{e}^{2\pi \mathrm{i} N Q_K(b)} \prod_{\ell=0}^{N-1} F(K^\ell a, K, J) S^J_{a,b}\,. \end{aligned} \tag{A.8}$$

On the other hand we have

$$S^J_{K^N a,b} = \mathrm{e}^{2\pi \mathrm{i} N Q_K(b)} F(a, K^N, J)\, S^J_{a,b}\,. \tag{A.9}$$

Comparing (A.7), (A.8) and (A.9) we see that X is an Nth root of $F(a, K^N, J)$. The latter quantity has been discussed already above, since K^N fixes a. For any valid choice of X the equations (A.6) then allow us to express the coefficients D^J_{Ka} recursively in terms of D^J_a. Since only the ratios enter, the latter remains a free parameter. It is natural to choose the values of X in such a way that the group property (A.5) of the F coefficients is respected for all primaries a, not just for fixed points of K. This can be achieved (although not uniquely) by choosing the phase $q(a, K, J)$ in the range $0 \le q < 1$ and defining

$$F(a, K^\ell, J) = \exp[2\pi \mathrm{i} \frac{\ell}{N} q(a, K^N, J)]\,.$$

Note that if $K^M = \mathbf{1}$, then $(K^N)^{M/N} = \mathbf{1}$, and then q is an integer multiple of $\frac{N}{M}$. This implies that $F(a, K^M, J) = 1$, as is clearly required. This definition can be directly generalized to arbitrary currents $K \in \mathcal{G}$, namely as

$$F(a, \prod_i K_i^{\ell_i}, J) = \exp[2\pi \mathrm{i} \sum_i \frac{\ell_i}{N_i} q(a, K_i^{N_i}, J)]\,, \tag{A.10}$$

which for given J manifestly has the required group property in the second argument. On the other hand, the group property $\{4a\}$ in the third argument is not necessarily respected for all J. This is because in solving the equations (A.6) we did not pay attention yet to (5.4), which does not allow us an arbitrary choice of the Nth root of unity for each J. This problem can be solved by choosing (A.10) on a set of generators J_i of $\mathcal{S}_a$, and extending it over $\mathcal{S}_a$ with the help of the group property.

Since $F(a,K,J) = 1$ if K and J belong to the same cyclic group, let us assume now that they are generators of two different cyclic groups $\mathbf{Z}_N$ and $\mathbf{Z}_M$, respectively. Then $F(a,K,J) = \Phi_a$, where Φ_a is a phase factor, with $(\Phi_a)^{\gcd(N,M)} = 1$ ($\gcd(N,M)$ denotes the greatest common divisor of N and M). If J and K have integral spin, then using the group property in the last argument and $F(a,K,K^n) = 1$ we find that $F(a,K,K^nJ^m) = (\Phi_a)^m$. Now we may use the group property in the second argument as well as $F(a,(K^nJ^m)^\ell, K^nJ^m) = 1$ to derive $F(a,K^t(K^nJ^m)^\ell, K^nJ^m) = (\Phi_a)^{tm}$. The power of Φ_a can be written as $tm = pm - nq$ with $p := t + n\ell$ and $q := m\ell$. The final result is thus

$$F(a,K^pJ^q,K^nJ^m) = (\Phi_a)^{pm-nq}\,. \tag{A.11}$$

This implies in particular the relation

$$F(a,J_1,J_2) = F(a,J_2,J_1)^* \tag{A.12}$$

for arbitrary currents J_1 and J_2.

For mutually local* currents K and J of arbitrary integer or half-integer spin the foregoing result generalizes to

$$F(a,K^pJ^q,K^nJ^m) = (\Phi_a)^{pm-nq}(-1)^{pnh(K)+qmh(J)}\,, \tag{A.13}$$

which again implies (A.12).

This concludes the discussion of F. Having fixed F, we also know $\bar F$ using {4} and {6}:

$$\bar F(a,K,J) = F(a,K,J^{-1}) = F(a,K,J)^*\,. \tag{A.14}$$

Now consider the phases η. On conjugate representations a and $c = a^*$ we have $(Ka)^* = K^{-1}c$, so that

$$\begin{aligned} C^J_{a,c} = C^J_{Ka,K^{-1}c} &= (\eta^J_{Ka,Ka})^* \sum_b S^J_{Ka,b} S^J_{b,K^{-1}c} \\ &= F(a,K,J)\bar F(c,K^{-1},J)(\eta^J_{Ka})^* \eta^J_a\, C^J_{a,c}\,. \end{aligned} \tag{A.15}$$

If a is a fixed point of K, then the factors η^J cancel, and we find

$$F(a,K,J) = \bar F(a^*,K^{-1},J)^* = F(a^*,K^{-1},J) = F(a^*,K,J)^*\,. \tag{A.16}$$

Because of the definition (A.10) of F for any a and the way this definition was extended over $\mathcal{S}_a$, this relation holds automatically for all values of a if it holds for fixed points of

$\star$ Note that mutually non-local currents cannot have simultaneous fixed points, since the action of one current changes the charge with respect to the other. The definition of F is only relevant for two currents that fix the same field.

K. Then the factors F in (A.15) cancel for all a, and hence the factors η must cancel as well. Thus we get

$$\eta^J_{Ka} = \eta^J_a \quad \text{for all} \quad K \in \mathcal{G}\,.$$

Under the transformation (5.2) η_a is changed to $\eta_a D^J_a (D^J_{a^*})^*$, and η_{a^*} is transformed by the complex conjugate of this factor. If a and a^* lie on different orbits, this allows us to transform both η_a and η^*_a to the value 1. Since we have already made the value of η constant on $\mathcal{G}$-orbits, we thus need just one parameter per conjugate pair to fix η_a. This parameter is precisely available, because one parameter D^J_a per orbit remained free. On the other hand, after fixing η, there is still one free parameter per pair of conjugate orbits. The group property $\{5b\}$ for η is manifestly satisfied if we set η^J to 1 for all J.

If a and a^* lie on the same orbit, we have $\eta_a = \eta_{a^*} = (\eta_a)^*$ so that $\eta_a = \pm 1$. We do not have any free parameters to change this value, since the ratio D_a/D_{a^*} is already fixed. Note that even if an orbit is self-conjugate, it may nevertheless happen that it does not contain any self-conjugate field.† Therefore this conclusion does not follow directly from $\{5c\}$. Note that the group property for η, $\{5b\}$, is satisfied because the group property of D^J was respected.

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