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Nonperturbative expansion in QCD and e^+e^- annihilation into hadrons at low energies

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Abstract

The process of e^+e^- annihilation into hadrons is analyzed at low energies by using a new nonperturbative approach in QCD based on a small expansion parameter. For this aim, we apply the momentum renormalization scheme and the smearing method of Poggio, Quinn and Weinberg to compare our theoretical prediction with experimental data. We derive values of the light pole quark masses, construct a nonperturbative β-function and discuss its connection with the static quark-antiquark potential.

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1 Introduction

The idea of constructing the so-called floating or variational series in quantum theories was suggested and applied to the anharmonic oscillator case in [1]. Within this approach, a certain variational principle is combined with the possibility of calculating corrections to the main contribution which gives a possibility to answer the question about the validity of the principal contribution and the region of applicability of the results obtained. At present, this idea finds many applications in developing various approaches, which somehow make it possible to go beyond perturbation theory. Among them, there are the Gaussian effective potential method [2–4], the optimized δ-expansion [5–7] and the variational perturbation theory [8–12]. In this paper, we will apply the recently obtained nonperturbative QCD expansion based on a new small expansion parameter. Within this method, a quantity under consideration can be approximated by a series, which is different from the perturbative expansion and which can be used to go beyond the weak-coupling regime. This allows one to deal with considerably lower energies than in the case of perturbation theory. In particular, in this paper we will consider a momentum renormalization scheme and parametrize our result in terms of so-called pole quark masses. In the case of perturbation theory, a corresponding renormalization prescription of such type was considered in [13–16]. In the framework of this scheme, the effective coupling constant depends on the quark masses which provides a natural way to include into consideration the threshold effects without any additional matching procedure. The nonperturbative expansion in the case of the massless MS–like renormalization scheme has been earlier considered in [17,18], and some applications of this method were given in [19–22]. Within this approach, a given quantity is represented in the form of a power series with a new expansion parameter a associated with the initial coupling constant $\lambda = g^2/(4\pi)^2 = \alpha_s/(4\pi)$ by the following equation

$$\lambda = \frac{1}{C}\,\frac{a^2}{(1-a)^3}\,. \tag{1.1}$$

It is clear that for all values of the coupling constant $\lambda \geq 0$ the expansion parameter a obeys the inequality $0 < a \leq 1$. The positive parameter C plays the role of a variational parameter, which is associated with the use of the variational or floating series. The original quantity, which is approximated by this expansion, does not depend on the auxiliary parameters C; however, any finite approximation depends on it on account of the truncation of the series. Here we will fix this parameter using some further information, which comes from the potential approach to meson spectroscopy.

The outline of this paper is as follows. For selfconsistency, we first explain the main ideas of the nonperturbative expansion in quantum chromodynamics in section 2 and demonstrate how a small expansion parameter appears. In section 3 the pole masses of light quarks are calculated. In section 4 we consider the renormalization procedure in the momentum renormalization scheme and discuss a connection between the static quark-antiquark potential and the behaviour of the nonperturbative β-function at large values of the coupling constant. The process of e^+e^- annihilation into hadrons at low energies is analyzed in the last section. The results are summarized briefly in the conclusion.

2 Nonperturbative expansion in QCD: general method and results

In this section we will briefly summarize the main ideas and results of the nonperturbative expansion method which has been proposed in [17,18]. This approach to QCD is based on the concepts of variational perturbation theory (VPT) [8–12]. According to [17], an expansion, different from a perturbative series, can be constructed in QCD on the basis of the Gaussian functional integrals (like in perturbation theory). For small values of the coupling constant g, the Nth-partial sum of the new series reproduces the Nth order of standard perturbation theory to within $O(g^{N+1})$. In the nonperturbative region, when the running coupling constant of QCD becomes large and perturbation theory does not work, the new arising expansion parameter remains small and the method remains still valid.

Let us write the QCD action functional in the form

$$S(A,q,\varphi) = S_2(A) + S_2(q) + S_2(\varphi) + g\,S_3(A,q,\varphi) + g^2\,S_4(A)\,, \qquad (2.1)$$

where $S_2(A)$, $S_2(q)$ и $S_2(\varphi)$ are free action functionals of the gluon, quark, and ghost fields, respectively; the term $S_2(A)$ also contains a term fixing the covariant α_G-gauge. The term $S_3(A,q,\varphi)$ describes the Yukawa interaction of gluons, gluons with quarks , and gluons with ghosts

$$S_3(A,q,\varphi) = S_3(A) + S_3(A,q) + S_3(A,\varphi)\,, \qquad (2.2)$$

and the term $S_4(A)$ in (2.1) generates the four–gluon vertex. We will transform the latter term by introducing auxiliary fields $\chi^a_{\mu\nu}$ [17]. Upon the χ-transformation, the diagrams of the Green functions will consist only of diagrams of the Yukawa type. In addition to the usual three–line vertices of QCD, vertices of the type $A\chi A$ will appear. Thus, a certain Green function of QCD can be represented in the following functional integral form

$$\begin{aligned} G(\cdots) &= \int D\chi\, D_{\rm QCD}\,(\cdots) \\ &\times \exp\Big\{ i\Big[S(A,\chi) + S_2(q) + S_2(\varphi) + S_2(\chi) + g\,S_3(A,q,\varphi)\Big]\Big\}\,, \end{aligned} \qquad (2.3)$$

where

$$S(A,\chi) = \frac{1}{2}\int dxdy\; A^a_\mu(x)\Big[D^{-1}(x,y|\chi)\Big]^{ab}_{\mu\nu} A^b_\nu(y) \qquad (2.4)$$

with the gluon propagator $D(x,y|\chi)$ in the χ-field

$$\Big[D^{-1}(x,y|\chi)\Big]^{ab}_{\mu\nu} = \big[(-g_{\mu\nu}\,\partial^2 + \partial_\mu\partial_\nu)\,\delta^{ab} + g\sqrt{2}\,f^{abc}\,\chi^c_{\mu\nu} + \text{gauge terms}\big]\,\delta(x-y)\,, \qquad (2.5)$$

and the term $(\cdots)$ is a set of ν gluon, quark and ghost fields. The integration measure $D_{\rm QCD}$ in (2.3) defines standard integrations over gluon, quark, and ghost fields.

Following the ideas of the VPT method, we introduce auxiliary parameters ζ and ξ and rewrite the action in (2.3) in the form

$$S(A,q,\varphi,\chi) = S'_0(A,q,\varphi,\chi) + S'_I(A,q,\varphi,\chi)\,. \qquad (2.6)$$

where

$$S_0'(A,q,\varphi,\chi) \;=\; \zeta^{-1}\left[S(A,\chi)+S_2(q)+S_2(\varphi)\right]+\xi^{-1}\,S_2(\chi)\,, \tag{2.7}$$

$$S_I'(A,q,\varphi,\chi) = g\,S_3(A,q,\varphi)-(\zeta^{-1}-1)\left[\,S(A,\chi)+S_2(q)+S_2(\varphi)\right]-(\xi^{-1}-1)\,S_2(\chi)\,. \tag{2.8}$$

The exact value of the quantity under consideration, for instance, the Green function, does not depend on the parameters ζ and ξ. However, the approximation of that quantity by a finite number of terms of the VPT series, that results from the expansion in powers of the action $S_I'(A,q,\varphi,\chi)$, does depend on those parameters. We can employ the freedom in the choice of the parameters ζ and ξ for our aim to find a new small parameter of the expansion.

It is further convenient to rewrite $S_0'(A,q,\varphi,\chi)$ in (2.7) by replacing ζ^{-1} to $[1+\kappa\,(\zeta^{-1}-1)]$ and ξ^{-1} to $[1+\kappa\,(\xi^{-1}-1)]$ and putting $\kappa=1$ at the end of calculations. In this case, any power of the expression $(\zeta^{-1}-1)\left[\,S(A,\chi)+S_2(q)+S_2(\varphi)\right]+(\xi^{-1}-1)\,S_2(\chi)$, appearing upon expanding the r.h.s. of the Green function (2.3) in powers of (2.8), can be obtained by differentiating with respect to the parameter κ as many times as required. Then, in the integrand, the prefactor of the exponential will contain only the powers of the action $g\,S_3(A,q,\varphi)$ that generate the QCD Yukawa diagrams with modified propagators defined by appropriate quadratic forms in the new free action S_0'. The VPT series for the Green function is given by

$$\begin{aligned} G(\cdots) \;&=\; \sum_n\sum_{k=0}^{n}\frac{1}{(n-k)\,!}\Big(-\frac{\partial}{\partial\kappa}\Big)^{n-k}\frac{i^k}{k!} \\ &\times\;\int D\chi\,D_{\rm QCD}(\cdots)\left[g\,S_3(A,q,\varphi\right]^k\exp\left[\,i\,S_0'(A,q,\varphi,\chi)\,\right] \end{aligned} \tag{2.9}$$

with the above replacement in $S_0'(A,q,\varphi,\chi)$. Further, it is convenient to rescale the fields

$$(A,q,\varphi)\Rightarrow\frac{(A,q,\varphi)}{\sqrt{1\,+\,\kappa\,(\zeta^{-1}-1)}}\quad,\quad\chi\Rightarrow\frac{\chi}{\sqrt{1\,+\,\kappa\,(\xi^{-1}-1)}}\,. \tag{2.10}$$

As a result, the propagators acquire the standard form and only the vertices get modified. Finally, integrating over the field χ, we obtain for the Green function

$$\begin{aligned} G(\cdots) \;&=\; \sum_n\sum_{k=0}^{n}\frac{1}{(n\,-\,k)\,!}\Big(-\frac{\partial}{\partial\kappa}\Big)^{n\,-\,k}\frac{i^k}{k!}\,\frac{1}{\left[\,1\,+\,\kappa\,(\zeta^{-1}-1)\,\right]^{\nu/2}} \\ &\times\;\int D_{\rm QCD}\,(\cdots)\left[g_3\,S_3(A,q,\varphi)\right]^k\exp\Big\{\,i\left[\,S_0(A,q,\varphi)\,+\,g_4^2\,S_4(A)\right]\Big\}\,. \end{aligned} \tag{2.11}$$

Here $S_0(A,q,\varphi)$ does no longer contain the term describing the field χ and represents a usual functional of the QCD free action, whereas g_3 and g_4 in the Yukawa and four-gluon vertices are defined as follows:

$$g_3\;=\;\frac{g}{\left[\,1\,+\,\kappa\,(\zeta^{-1}-1\,)\right]^{3/2}}\,,\qquad g_4\;=\;\frac{g}{\left[\,1\,+\,\kappa\,(\xi^{-1}-1)\right]^{1/2}}\,. \tag{2.12}$$

As the analysis of the structure of the VPT series shows [17], we will succeed in constructing the small expansion parameter if we put $\xi=\zeta^3$ and if the parameter ζ is

connected with the coupling constant by the equation

$$\lambda = \frac{g^2}{(4\pi)^2} = \frac{1}{C}\frac{a^2}{(1-a)^3}, \quad a = 1-\zeta, \tag{2.13}$$

where C is a positive constant. As follows from (2.13), at any values of the coupling constant g, the new expansion parameter a obeys the inequality $0 \leq a < 1$.

Let us next consider the renormalization group β-function of the expansion parameter a,

$$\beta_a(a) = \mu^2 \frac{\partial a}{\partial \mu^2} = \frac{2\beta_0}{C}\frac{1}{f'(a)}, \tag{2.14}$$

where $\beta_0 = 11 - 2/3\,N_f$ is the first coefficient of the standard β-function in perturbative expansion, and N_f is the number of active quarks. By finding the renormalization constants in the massless renormalization scheme with an accuracy $O(a^3)$, for the function $f(a)$ we get

$$f(a) = \frac{2}{a^2} - \frac{6}{a} - 48\ln a - \frac{18}{11}\frac{1}{1-a} + \frac{624}{121}\ln(1-a) + \frac{5184}{121}\ln(1+\frac{9}{2}a). \tag{2.15}$$

In Fig. 1, we plot the function $-\beta_a(a)$ for $N_f = 3$ and $C = 4.1$. This choice of the value of C taken from [18] corresponds to the assumption that the renormalization group β_λ-function at large enough values of the coupling constant λ behaves as $\beta_\lambda \simeq -\lambda$. Such a behaviour corresponds to the singular infrared behaviour of the running coupling constant $\lambda(Q^2) \sim Q^{-2}$ and leads to the linear growth of the static quark-antiquark potential (see also discussion in section 4). As it should be, the β_a-function has an infrared fixed point at $a = 1$.

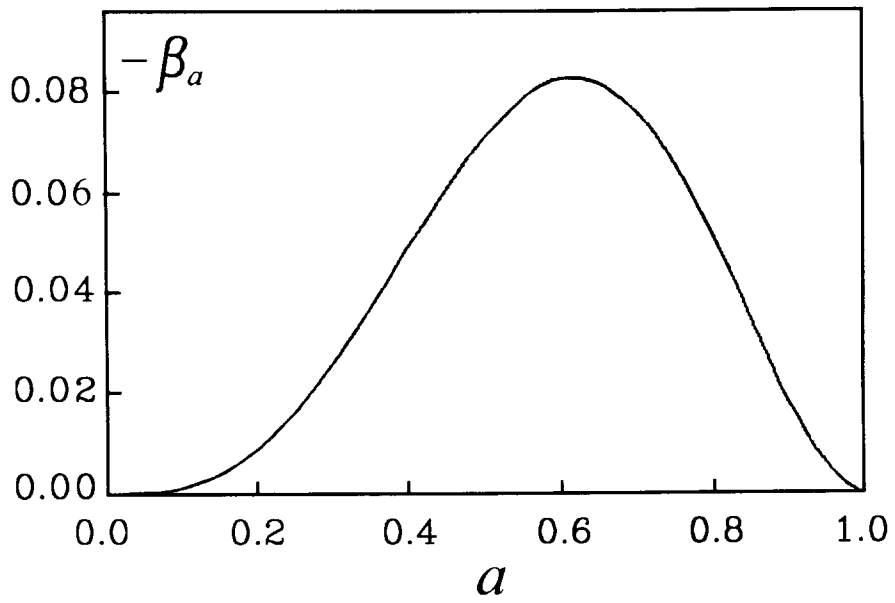

Figure 1: The function $-\beta_a(a)$ defined by Eq. (2.14) versus a.

By solving the renormalization group equation (2.14) we find the momentum dependence of the running expansion parameter a as a solution of the following transcendental equation

$$\ln\frac{Q^2}{Q_0^2} = \frac{C}{2\beta_0}\left[f(a) - f(a_0)\right]. \tag{2.16}$$

For any values of Q^2, this equation has a unique solution $a(Q^2)$ in the interval between 0 and 1.

In Fig. 2, we demonstrate the behaviour of the running expansion parameter $a(Q^2)$. To define the normalization point a_0 we have used as input the value of $\alpha_s(Q_0^2) = 0.35$ at the τ lepton mass scale: $Q_0 = M_\tau = 1.777$ GeV.

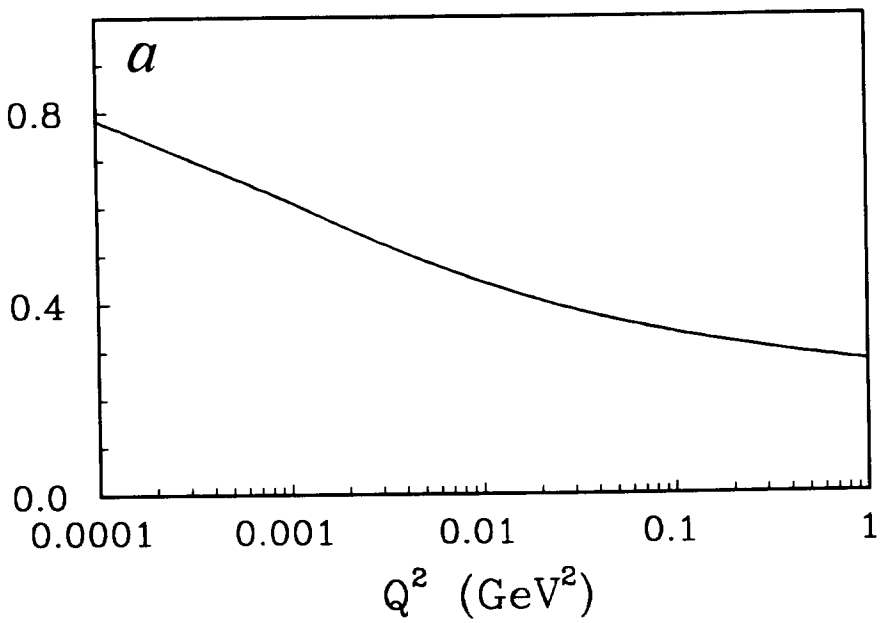

Figure 2: The running expansion parameter $a(Q^2)$ defined by Eq. (2.16) versus Q^2.

The curve, in Fig. 2, shows that the running expansion parameter $a(Q^2)$ remains small $(0.34 \div 0.44)$ for sufficiently small $Q^2 \simeq 0.01 \div 0.1$. Actually, the real expansion parameter is associated with a^2/C which, of course, is smaller than a. This is important for the applicability of our following estimates of the light pole quark masses.

3 Pole quark masses

Let us write the inverse quark propagator in the following form

$$S_f^{-1}(\not{p}, \mu^2) = \not{p} - m_f(\mu^2) - \Sigma_f(\not{p}, \mu^2), \tag{3.1}$$

with $m_f(\mu^2)$ being the renormalized quark mass of flavour f (the index f will be omitted in the following). For the renormalized self-energy $\Sigma(\not{p}, \mu)$, we will use the representation

$$\Sigma(\not{p}, \mu^2) = m(\mu^2)\,\Sigma_1(p^2, \mu^2) + (\not{p} - m(\mu^2))\,\Sigma_2(p^2, \mu^2). \tag{3.2}$$

Following [23], we define the renormalization point and renormalization scheme invariant mass as follows

$$m_{\rm eff}(p^2) = m(\mu^2)\left[1 + \frac{\Sigma_1(p^2, \mu^2)}{1 - \Sigma_2(p^2, \mu^2)}\right] = \overline{m}(|p^2|)\left[1 + \frac{\Sigma_1(p^2, |p^2|)}{1 - \Sigma_2(p^2, |p^2|)}\right], \tag{3.3}$$

where $\overline{m}(|p^2|)$ is the running mass. The pole quark mass, M, is then defined by the location of the quark propagator pole and obeys the following equation

$$M = m_{\rm eff}(M^2)\,. \tag{3.4}$$

Notice that the pole mass, M, as also $m_{\rm eff}$, is renormalization point and renormalization scheme invariant. Thus, we can use, for example, the MS–scheme to derive the values of the pole quark masses.

By using the nonperturbative approach let us next compute the pole quark masses. Note, that in the framework of standard perturbation theory it is possible to carry out such calculations for heavy quarks only [23] because in the case of the light quarks the running coupling constant becomes too large to apply the perturbative expansion. However, within the nonperturbative expansion this task may be considered for the light quarks as well since the expansion parameter a remains small, and we do not find ourselves outside the region of applicability of the approach.

Let us first consider the anomalous mass dimension γ_m defined by

$$\gamma_m = -\lambda \frac{\partial b_1}{\partial \lambda}\,, \tag{3.5}$$

where $b_1(\lambda)$ is given from the mass renormalization constant Z_m ($m_0 = Z_m\, m$) in dimensional regularization with $D = 4 - 2\epsilon$,

$$Z_m = 1 + \frac{b_1}{\epsilon} + \frac{b_2}{\epsilon^2} \cdots .$$

In an approximation to $O(a^5)$ we find

$$b_1 = -\left\{\gamma_0 \frac{a^2}{C} + 3\,\gamma_0 \frac{a^3}{C} + \left[6\,C\,\gamma_0 + \frac{\gamma_1}{2}\right] \frac{a^4}{C^2} + \left[10\,C\,\gamma_0 + 3\,\gamma_1\right] \frac{a^5}{C^2}\right\}\,, \tag{3.6}$$

where

$$\gamma_0 = 4\,, \qquad \gamma_1 = 4\left(\frac{101}{6} - \frac{5}{9} N_f\right) \tag{3.7}$$

with N_f active flavours.

The running quark mass as a solution of the renormalization group equation with boundary condition $\overline{m}(Q^2 = \mu^2) = m(\mu^2)$ can then be found in the form

$$\overline{m}(Q^2) = m(\mu^2) \exp\left[-\int_{\lambda(\mu^2)}^{\lambda(Q^2)} d\lambda' \frac{\gamma_m(\lambda')}{\beta(\lambda')}\right]\,. \tag{3.8}$$

In difference from the previous section, to obtain the nonperturbative β-function we now use the renormalization constants including terms $O(a^5)$ [18], and get

$$\beta = \mu^2 \frac{\partial \lambda}{\partial \mu^2} = -\beta_0 \frac{a^4}{C^2} \frac{\phi(a)}{(1 - a)^2 (1 + a/2)}\,, \tag{3.9}$$

where

$$\phi(a) = 1 + \frac{9}{2} a + 2(6 + a)a^2 + 5(5 + 3B)a^3 \tag{3.10}$$

and

$$B = \frac{\beta_1}{2C\beta_0}, \qquad \beta_0 = 11 - \frac{2}{3}N_f \qquad \beta_1 = 102 - \frac{38}{3}N_f . \tag{3.11}$$

The modified value of the parameter $C = 21.5$ has been found in [18] using an information coming from the meson spectroscopy.

Substituting (3.9) into (3.8) and taking into account (3.5) and (3.6) we find the running mass in the following form

$$\overline{m}(Q^2) = m(\mu^2) \exp\left\{ \frac{2\gamma_0}{\beta_0} \frac{1}{5(5+3B)} \left[L(a(Q^2)) - L(a(\mu^2)) \right] \right\}, \tag{3.12}$$

where

$$L(a) = x_1 S(a, a_1) + x_2 S(a, a_2) + x_3 S(a, a_3) \tag{3.13}$$

and

$$\begin{aligned} S(a,b) = & - \left(\frac{51}{2}A + \frac{255}{4} \right) \frac{\ln(1-a)}{1-b} \\ & + \left[\left(2b + 16b^2 + \frac{15}{2}b^3 \right) A + \frac{1}{b} + 5 + \frac{57}{4}b + 31b^2 + \frac{25}{2}b^3 \right] \frac{\ln(a-b)}{1-b} \\ & - \left(\frac{47}{2}a + \frac{15}{4}a^2 + \frac{15}{2}ab \right) A - \frac{87}{2}a - \frac{25}{4}a^2 - \frac{25}{2}ab - \frac{\ln a}{b} \end{aligned} \tag{3.14}$$

with

$$A = \frac{\gamma_1}{2C\gamma_0} . \tag{3.15}$$

The values of a_i in Eq. (3.13) are the solutions of the equation $\phi(a_i) = 0$, and

$$x_1 = \frac{1}{(a_1 - a_2)(a_1 - a_3)}, \quad x_2 = \frac{1}{(a_2 - a_1)(a_2 - a_3)}, \quad x_3 = \frac{1}{(a_3 - a_1)(a_3 - a_2)} . \tag{3.16}$$

For the pole mass, M, we obtain the following equation

$$M = \overline{m}(M^2) \left[1 + 4\frac{a^2(M^2)}{C} \left(1 + 3a(M^2) \right) \left(\frac{4}{3} + \ln 4\pi - \gamma \right) \right] . \tag{3.17}$$

For small values of the coupling constant Eq. (3.17) reproduces the corresponding equation for the pole quark mass in the two-loop approximation of perturbation theory [23].

By using the expression (3.9) for the β-function we can find the running expansion parameter $a(M^2)$ as a solution of the equation

$$\ln \frac{M^2}{{\mu_0}^2} = \frac{C}{2\beta_0} \left[\tilde{f}(a) - \tilde{f}(a_0) \right] , \tag{3.18}$$

where

$$\tilde{f}(a) = \frac{1}{5(5+3B)} \left[x_1 J(a, a_1) + x_2 J(a, a_2) + x_3 J(a, a_3) \right] \tag{3.19}$$

with

$$\begin{aligned} J(a,b) = & - \frac{2}{a^2 b} - \frac{4}{a b^2} - \frac{12}{a b} - \frac{9}{(1-a)(1-b)} + \frac{4 + 12b + 21b^2}{b^3} \ln a \\ & + \frac{30 - 21b}{(1-b)^2} \ln(1-a) - \frac{(2+b)^2}{b^3 (1-b)^2} \ln(a-b) . \end{aligned} \tag{3.20}$$

and x_i are defined by Eq. (3.16).

To compute the values of the pole quark masses we will use as input the following values of the running masses at 1 GeV [24]

$$\overline{m}_u(1\mathrm{GeV}) = 5.6 \pm 1.1\mathrm{MeV}, \quad \overline{m}_d(1\mathrm{GeV}) = 9.9 \pm 1.1\mathrm{MeV}, \quad \overline{m}_s(1\mathrm{GeV}) = 199 \pm 33\mathrm{MeV}. \tag{3.21}$$

From Eq. (3.17) we then find

$$M_u = 63 \pm 6\,\mathrm{MeV} \quad M_d = 77 \pm 5\,\mathrm{MeV} \quad M_s = 379 \pm 43\,\mathrm{MeV}\,. \tag{3.22}$$

The above values of the pole quark masses differ from the standard constituent quark masses, but are close to the values of the so-called effective quark masses which have been found from an analysis of e^+e^- experimental data [25]. For heavy quarks the difference between the values of the pole quark masses which are computed perturbatively and in the framework of this approach is practically negligible and we will take $M_c = 1.5$ GeV, $M_b = 4.5$ GeV and $M_t = 174$ GeV.

One could give a basic argument against these considerations using pole quark masses by assuming that the full quark propagator should not have a pole due to quark confinement. Leaving this fundamental question aside, we shall see that pole masses nevertheless provide a suitable parametrization in the following considerations.

4 Momentum renormalization scheme and nonperturbative β-function

In the previous section we have calculated the values of pole quark masses which are renormalization point and renormalization scheme invariant, by using the MS procedure and as input the values of the running masses at 1 GeV. On the other hand, for an analysis of e^+e^- annihilation it is suitable to use a momentum renormalization scheme with pole quark masses in order to simulate threshold effects. In this section we will consider the renormalization procedure in the momentum scheme and establish a connection between the nonperturbative β-function and the static quark-antiquark potential.

In the first order of our approximation, the renormalization constant $Z_\lambda(\mu', \mu)$, which describes the modification of the coupling constant $\lambda(\mu)$ when changing the scale parameter from μ to μ', has the following form

$$Z_\lambda(\mu', \mu) = 1 + \lambda_{\mathrm{eff}} \left[J(\mu'^2) - J(\mu^2) \right]. \tag{4.1}$$

Here

$$\lambda_{\mathrm{eff}} = \frac{1}{C} a^2 (1 + 3a) \tag{4.2}$$

and

$$J(\mu'^2) - J(\mu^2) = 11 \ln \frac{\mu'^2}{\mu^2} - \frac{2}{3} \sum_f \left[I\left(\frac{M_f^2}{\mu'^2}\right) - I\left(\frac{M_f^2}{\mu^2}\right) \right], \tag{4.3}$$

where the function $I(M^2/\mu^2)$ is the well known one-loop integral

$$I\left(\frac{M^2}{\mu^2}\right) = 6 \int_0^1 dx\, x(1-x) \ln \left[1 + \frac{\mu^2}{M^2} x(1-x) \right]. \tag{4.4}$$

or

$$I(x) = -\frac{5}{3} + 4\,x + (1 - 2x)\sqrt{1 + 4x}\,\ln\frac{\sqrt{1 + 4x} + 1}{\sqrt{1 + 4x} - 1}\,. \tag{4.5}$$

Using the definition of the β-function

$$-\frac{\beta(\lambda)}{\lambda} = \left.\frac{\partial \ln Z_\lambda(\mu', \mu)}{\partial \ln \mu'^2}\right|_{\mu'=\mu} \tag{4.6}$$

we find

$$-\frac{\beta(\lambda)}{\lambda} = \lambda_{\rm eff}(a)\left[11 - \frac{2}{3}\sum_f F_1\left(\frac{M_f^2}{\mu^2}\right)\right], \tag{4.7}$$

where

$$F_1(x) = 1 - 6\,x + \frac{12\,x^2}{\sqrt{1 + 4x}}\ln\frac{\sqrt{1 + 4x} + 1}{\sqrt{1 + 4x} - 1}\,. \tag{4.8}$$

The renormalization scale dependence of the running expansion parameter $a = a(\mu^2)$ is now defined by the following equation

$$C\,[U(a) - U(a_0)] = 11\,\ln\frac{\mu^2}{\mu_0^2} - \frac{2}{3}\sum_f\left[I\left(\frac{M_f^2}{\mu^2}\right) - I\left(\frac{M_f^2}{{\mu_0}^2}\right)\right], \tag{4.9}$$

where μ_0 is some normalization point, $a_0 = a(\mu_0^2)$ and the function $U(a)$ has the following form

$$U(a) = \frac{1}{a^2} - \frac{3}{a} - 12\,\ln a + \frac{3}{4}\ln(1 - a) + \frac{45}{4}\ln(1 + 3a)\,. \tag{4.10}$$

Notice that the function $U(a)$ is monotonous and changes its values from plus to minus infinity when the parameter a changes in the interval $0 < a < 1$. Thus, the equation (4.9) has a unique solution for the running expansion parameter $a(\mu^2)$.

As seen from Eq. (4.7) at $\mu^2 = Q^2 < M_f^2$, the contribution of quarks is suppressed in the β-function. Thus, for sufficiently small Q^2 we can relate our result for the β-function with the static potential of QCD without dynamical quarks.

The non-relativistic static quark-antiquark potential used in meson spectroscopy is associated with the running coupling constant $\lambda(Q^2)$ as follows

$$V(r) \sim \int d\mathbf{Q}\,\exp(\,i\mathbf{Q}\cdot\mathbf{r}\,)\,\frac{\lambda(Q^2)}{Q^2}\,, \tag{4.11}$$

In order to implement the idea of quark confinement, one usually assumes that the running coupling constant has at small Q^2 an infrared singularity of the form

$$\lambda(Q^2) \sim \frac{1}{Q^2} \tag{4.12}$$

leading at long distances to a linearly increasing confinement potential

$$V(r) \sim r\,. \tag{4.13}$$

The infrared behaviour (4.12) of the running coupling constant then implies the following behaviour of the β-function at large values of the coupling constant

$$-\frac{\beta(\lambda)}{\lambda} = -\frac{1}{\lambda}\,\mu^2\,\frac{d\,\lambda(\mu^2)}{d\,\mu^2} \simeq 1\,. \tag{4.14}$$

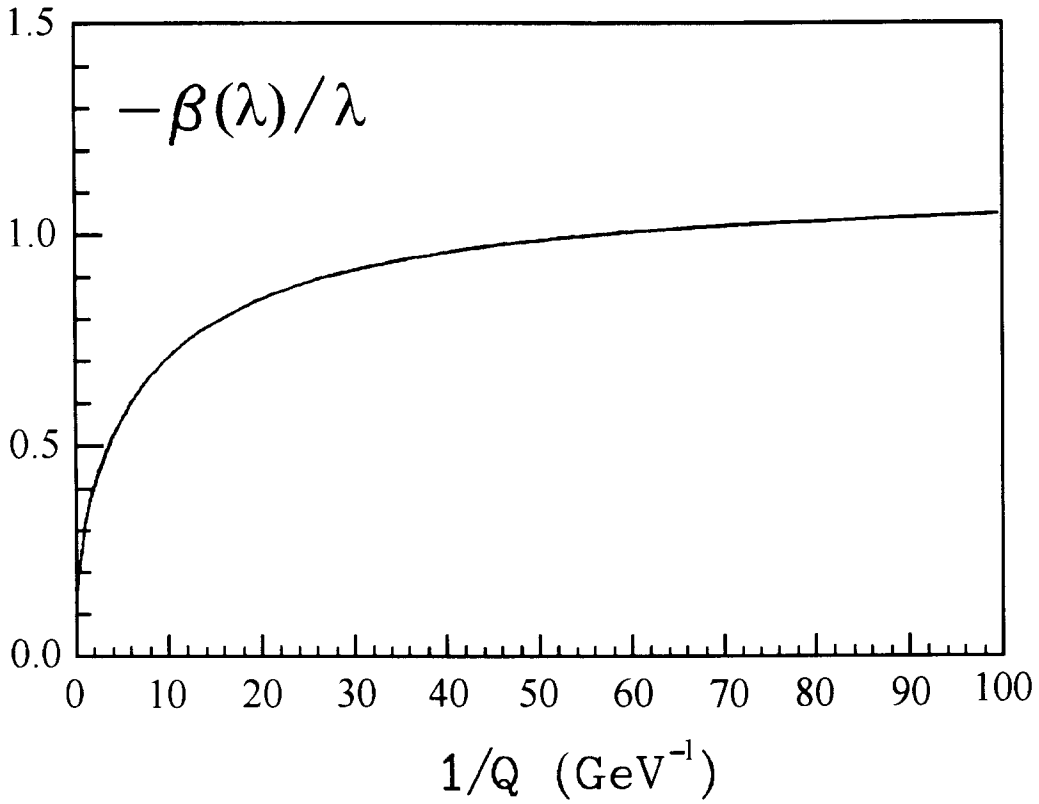

Figure 3: The function $-\beta(\lambda)/\lambda$ versus $1/Q$ for $C = 39$.

From Eq. (4.7) we find that in the limit of large values of the coupling constant ($\mu \to 0$, $a \to 1$)

$$-\frac{\beta(\lambda)}{\lambda} \simeq \frac{44}{C_{\text{asympt}}}. \tag{4.15}$$

This relation allows one to fix the parameter C within this approach: $C_{\text{asympt}} \simeq 44$. However, in an appropriate region of the momentum, we will choose a value of C slightly less than the value of C_{asympt}. Fig. 3 shows the function (4.7) for $C = 39$ calculated with pole quark masses defined in the previous section, as a function of $1/Q$. Thus, we have fixed all our parameters except a_0 which will be defined by considering the process of e^+e^- annihilation into hadrons.

5 e^+e^- annihilation at low energies

In this section we will apply the results for the pole quark masses and the running expansion parameter $a(Q^2)$ obtained in the previous sections to the process of e^+e^- annihilation into hadrons at low energies. A direct comparison of the theoretical prediction with experimental data in the region of resonances is impossible due to the large contributions coming from higher order terms. However, following [26], we can avoid this problem of threshold singularities by introducing so-called smearing quantities. For the $R(s)$-ratio

the corresponding smearing function can be defined as follows

$$R_\Delta(s) = \frac{1}{2\,\mathrm{i}} \left[\Pi(s + \mathrm{i}\Delta) - \Pi(s - \mathrm{i}\Delta) \right] . \tag{5.1}$$

The smeared quantity $R_\Delta(s)$ then allows one to compare the experimental data with the theoretical predictions obtained in the framework of QCD.

Seemingly, one could try to calculate the function $R_\Delta(s)$ by employing perturbation theory for the correlation function $\Pi(q^2)$ in the Euclidean region $q^2 < 0$ using the perturbative running coupling constant $\alpha_s(Q^2 = -q^2)$, and then performing an analytic continuation to $q^2 = s \pm \mathrm{i}\Delta$. Let us recall that the correlation function has perfectly well defined analytic properties. Namely, $\Pi(q^2)$ is an analytic function in the complex q^2-plane with a cut along the positive part of the real axis. Notice that the parametrization of $\Pi(q^2)$ by the perturbative running coupling constant breaks these analytic properties (see discussion in the massless case in [20,22]). This immediately becomes also clear if one rewrites (5.1) by using the dispersion relation for the hadronic vacuum polarization $\Pi(q^2)$

$$R_\Delta(s) = \frac{\Delta}{\pi} \int_0^\infty ds' \, \frac{R(s')}{(s' - s)^2 + \Delta^2} , \tag{5.2}$$

where $R(s)$ is the ratio of e^+e^- annihilation into hadrons. The perturbative approximation of $R(s)$ by $\alpha_s(s)$ indeed leads to a singularity of the integrand on the integration contour. Applying a special optimization method to the third-order of perturbative approximation of $R(s)$, it has been shown in [27] how to calculate the smearing quantity $R_\Delta(s)$ and to compare the theoretical prediction with experimental data. Let us now for the same purpose apply the nonperturbative approach in the first order of our approximation. We will consider both the smeared quantity (5.1) and the following function

$$W_\Delta(q^2) = \frac{d\,R_\Delta(q^2)}{d\,q^2} = -\frac{1}{2\,\mathrm{i}} \left[\frac{D(q^2 + \mathrm{i}\Delta)}{q^2 + \mathrm{i}\Delta} - \frac{D(q^2 - \mathrm{i}\Delta)}{q^2 - \mathrm{i}\Delta} \right] . \tag{5.3}$$

which is more sensitive to the values of the pole quark masses.

Here, the $D(q^2)$–function can be written as follows

$$\begin{aligned} D(q^2) &= -\,q^2 \frac{d}{d\,q^2} \Pi(q^2) \\ &= \frac{3}{\pi} \sum_f Q_f^2 \left[F_1\left(\frac{M_f^2}{-q^2}\right) + 4\,\lambda_{\mathrm{eff}}(q^2)\, F_2\left(\frac{M_f^2}{-q^2}\right) \right] . \end{aligned} \tag{5.4}$$

For the function F_2 we will use the result obtained on the basis of the Schwinger approximation of the imaginary part of the vacuum polarization [28]

$$F_2\left(\frac{M^2}{-q^2}\right) = \frac{4\pi}{3} \frac{4\,M^2}{-q^2} \int_0^1 d\,v \, \frac{v^2\,(3 - v^2)}{\left[4\,M^2/(-q^2) + 1 - v^2\right]^2} f(v) \tag{5.5}$$

with

$$f(v) = \frac{\pi}{2\,v} - \frac{3 + v}{4} \left(\frac{\pi}{2} - \frac{3}{4\,\pi} \right) . \tag{5.6}$$

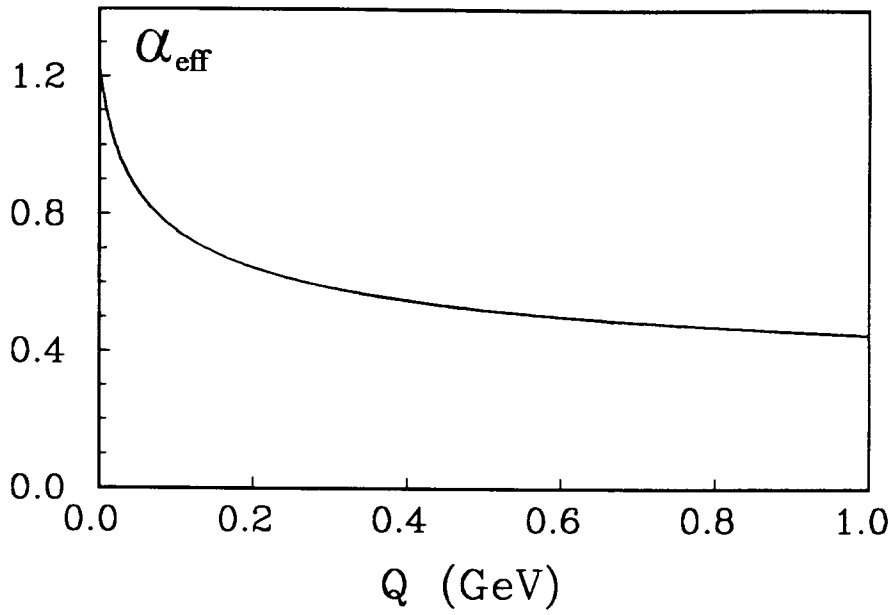

Figure 4: The running coupling constant $\alpha_{\text{eff}}(Q)$ versus Q at $C = 39$.

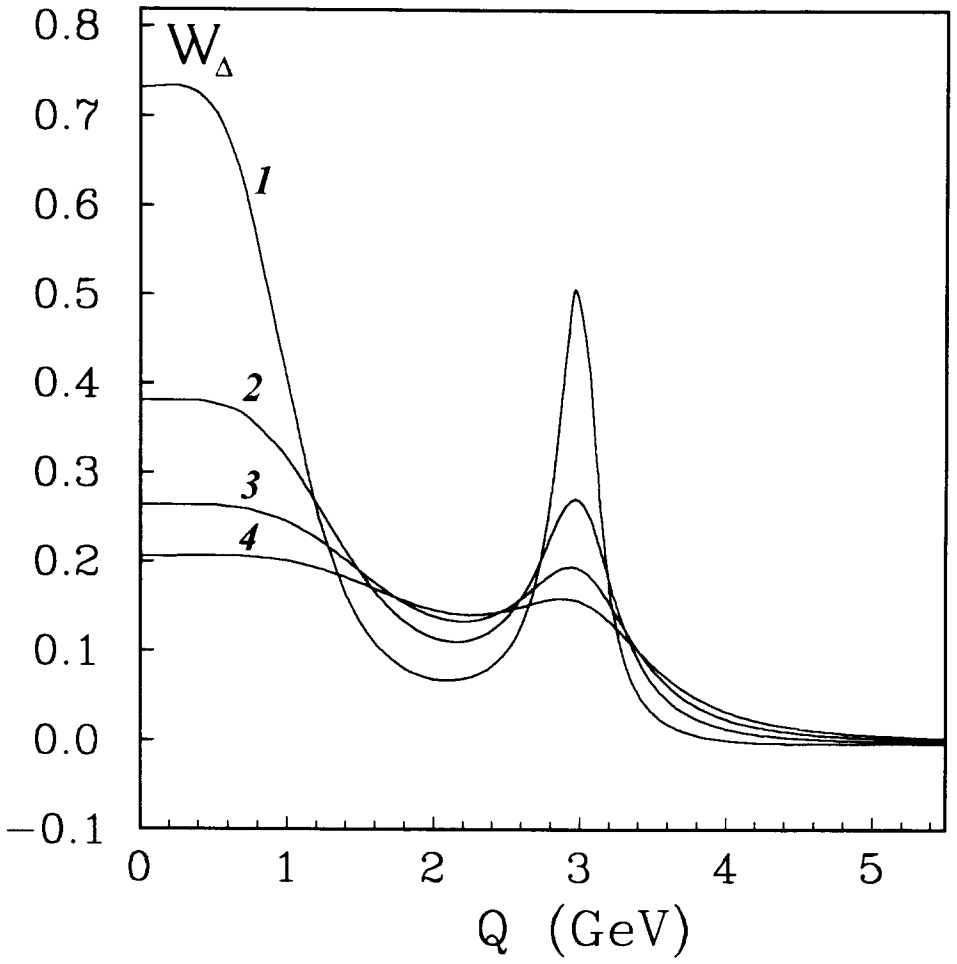

Figure 5: The function W_Δ as function of $Q = \sqrt{s}$ for Δ =1, 2, 3, and 4 GeV^2.

The effective coupling constant λ_{eff} in Eq. (5.4) is defined by Eq. (4.2) with the running expansion parameter $a(-q^2)$, which can be found as a solution of Eq. (4.9). Some details of the calculation of the W_Δ–function can be found in Appendix A.

We should note that in contrast to perturbation theory, where the running coupling constant has an infrared singularity contradicting the dispersion relation for the D–function, the effective coupling constant λ_{eff} is a finite function in the infrared region with freezing behaviour at small Q^2. The discussion of the behaviour and analytic properties of the effective coupling constant in the massless case have been considered in [20,22].

To define the parameter a_0 we will normalize our results using experimental data at sufficiently large momentum $Q_0 = 6$ GeV together with the values of the pole quark masses found in the section 3. In Fig. 4, we have plotted the effective coupling constant $\alpha_{\text{eff}} = 4\pi\lambda_{\text{eff}}$ at small Q. Indeed, the effective coupling constant α_s is seen to be frozen, and, at very small Q, $\alpha_{\text{eff}} \simeq 1.2$. The idea that the QCD coupling constant can become frozen at low energies has been discussed within many approaches (see the discussion in [27,29,30]). The following estimate of the fit-invariant integral

$$\int_0^1 dQ \frac{\alpha_s(Q^2)}{\pi} \simeq 0.2 \text{ GeV} \tag{5.7}$$

is the experimental result associated with a freezing of the coupling constant [27]. Inserting our expression α_{eff} into the integral (5.7) we obtain 0.182 GeV.

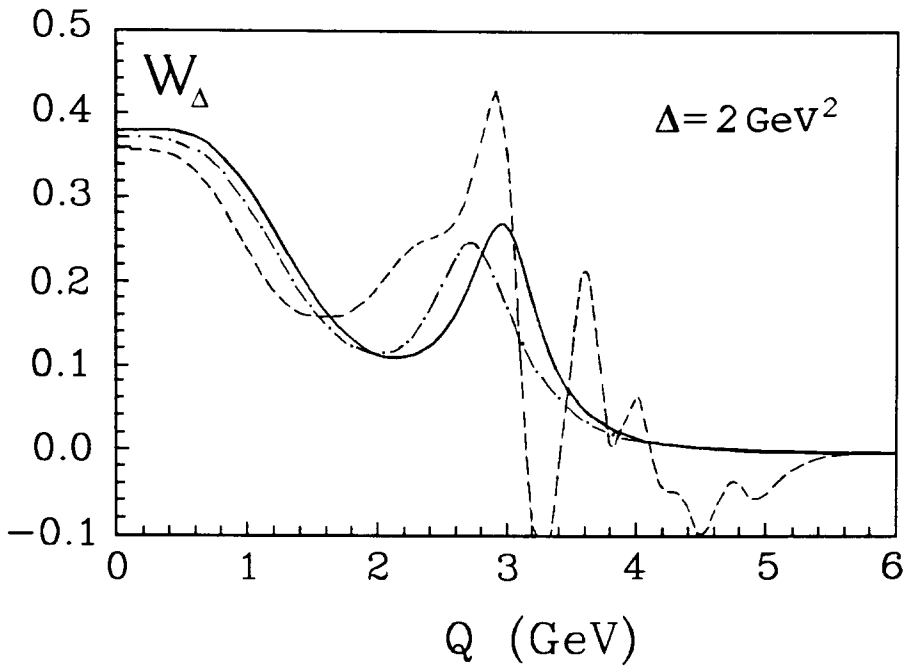

Figure 6: The function W_Δ versus $Q = \sqrt{s}$ for $\Delta = 2$ GeV^2. The solid curve is our result, the dashed line represents the smeared experimental data (taken from [27]), and the dot-dashed curve has been obtained in [27] by applying the optimization procedure to the third-order calculation of $R_{e^+e^-}$.

In Fig. 5 we show our result for the function $W_\Delta(s)$ at Δ =1, 2, 3, and 4 GeV^2 (curves with numbers 1, 2, 3, and 4, respectively). The two peaks in each of the curves are associated with the regions of the ρ-meson resonance and the charmonium.

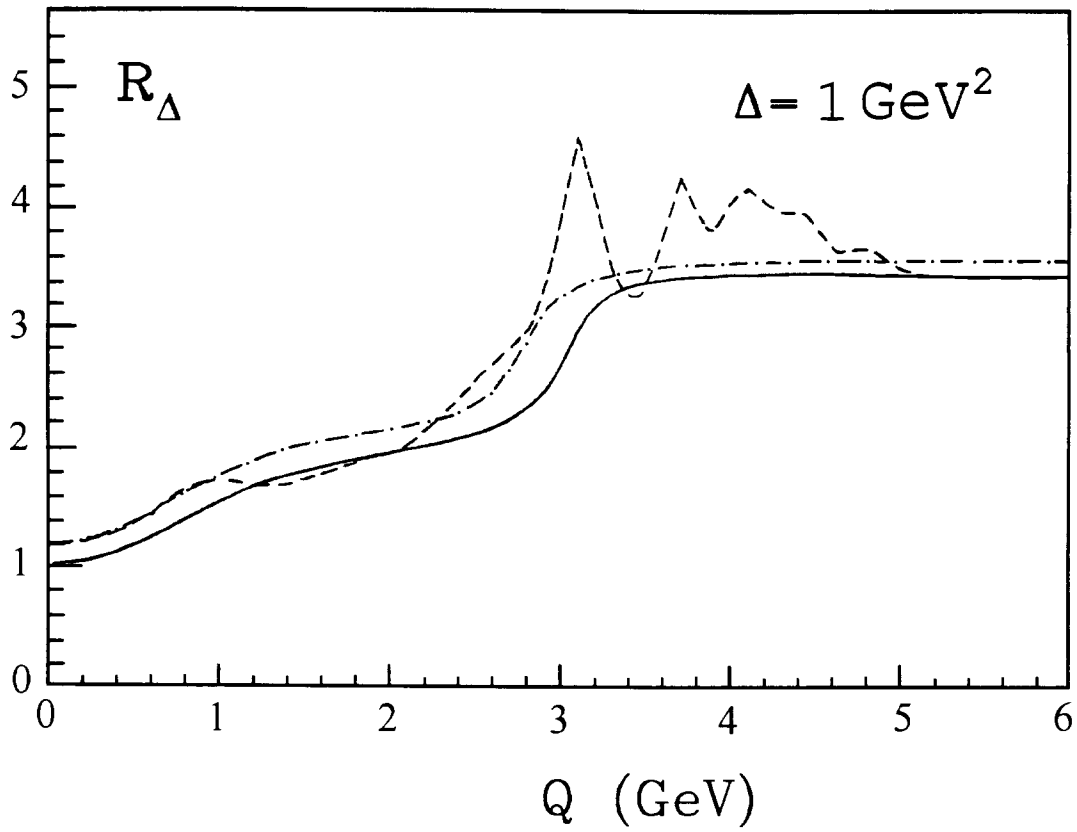

Figure 7: The function R_Δ versus $Q = \sqrt{s}$ for $\Delta = 1$ GeV^2. All lines are defined in the same way as in Fig. 6.

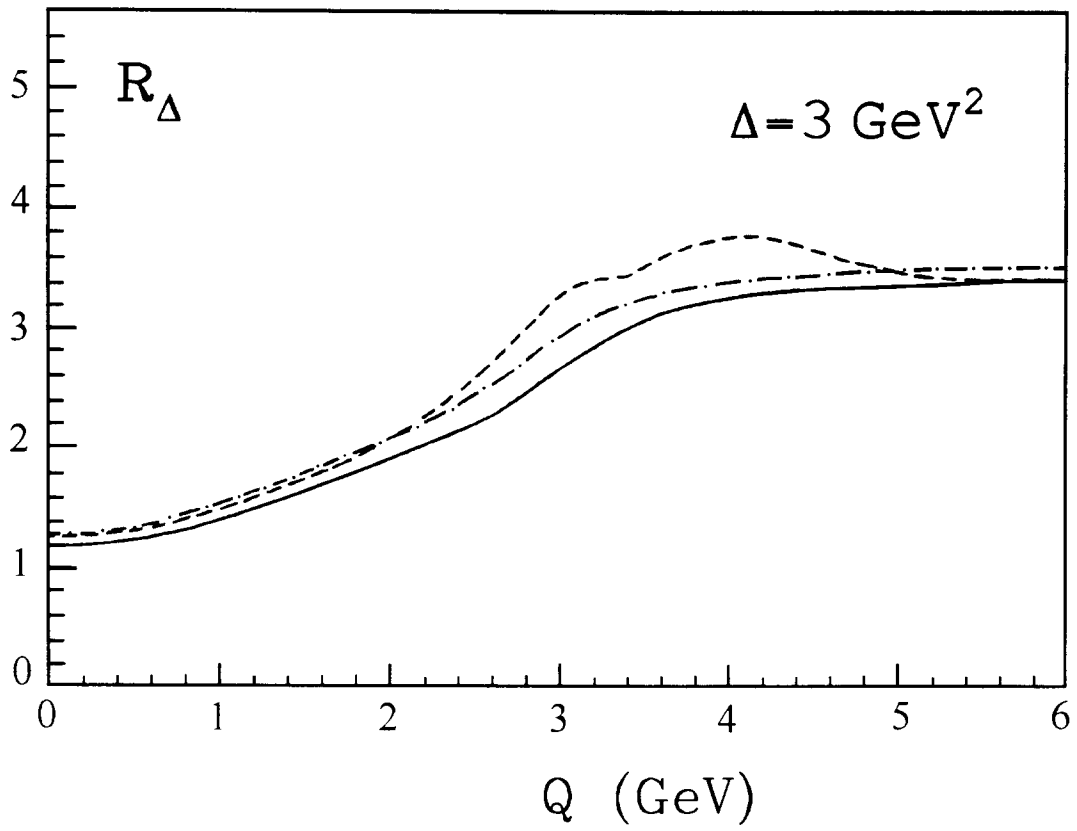

Figure 8: The function R_Δ versus $Q = \sqrt{s}$ for $\Delta = 3$ GeV^2. All lines are defined in the same way as in Fig. 6.

In Fig. 6, we compare the function $W_\Delta(s)$ for $\Delta = 2$ GeV^2 (solid line) with the smeared experimental data (dashed line) from [27]. It should be emphasized that our curve obtained in the first non-trivial order of $\lambda_{\rm eff}$ is close to the theoretical predictions obtained in [27] on the basis of optimization of the third-order QCD perturbative corrections to $R_{e^+e^-}$ (see dot-dashed curve in Fig. 6). As we can see from Fig. 6, the value of $\Delta = 2$ GeV^2 is not sufficiently large to smooth the region of charm resonances. When the value of Δ grows up to 4 GeV^2, the experimental curve approaches the theoretical prediction shown in Fig. 5 (curve 4).

In Fig. 7 and Fig. 8, we plot the function R_Δ versus $Q = \sqrt{s}$ for $\Delta = 1$ GeV^2 and $\Delta = 3$ GeV^2. The solid curve represents our theoretical prediction, the dashed curve is the smeared experimental data from [27], and the dot-dashed curve is the optimized third-order perturbative prediction [27]. As we can see from Fig. 7 and Fig. 8, the solid and dot-dashed curves are similar to each other. The gap between these lines in the region of small momenta is mainly due to the difference in values of light quark masses. This difference becomes smaller if one suppresses the contribution of light quarks by increasing Δ.

Note that for small $\Delta \simeq 1 \div 2$ GeV^2, experimental data for the smeared quantities R_Δ and W_Δ have a resonance structure near the charmonium family. By increasing Δ up to $3 \div 4$ GeV^2 we smooth this region and the agreement between the theoretical predictions (solid and dot-dashed curves in Figs. 7, 8) and experimental (dashed curves) data becomes improved.

6 Summary and conclusion

In this paper, we have considered the approach to quantum chromodynamics based on variational perturbation theory and its application to the process of e^+e^- annihilation into hadrons at low energies. In the framework of this method, the original action functional is first rewritten by using some variational term with auxiliary parameters of the variational type. A quantity under consideration does not depend on these parameters, but its finite approximation will do due to the truncation of the series. This fact allows one to use the freedom in the choice of the parameters to construct a new small expansion parameter in QCD. An important virtue of this approach is the fact that for sufficiently small α_s it reproduces the standard perturbative expansion. Therefore, in principle, all the high-energy physics is maintained by the method. In going to lower energies where standard perturbation theory ceases to be valid ($\alpha_s \sim 1$), our expansion parameter (really, $a^2(Q^2)/C$) remains small and we do not go beyond the range of applicability of the method.

Using our nonperturbative method with a new small expansion parameter, we have calculated the values of pole quark masses taking the values of the running masses at 1 GeV as input. Next, we have investigated the momentum renormalization scheme with the pole quark masses. Within this scheme the problem of the number of active quarks does not appear. The effective coupling constant now depends on all quark masses and does not require any additional definition of the dependence on the number of active flavours.

On the basis of these results we have then described the process of e^+e^- annihilation into hadrons at low energies using the smearing method to smooth the resonance structure. In particular, it was shown that for a sufficiently large Δ the theoretical predictions obtained in the first order of our approximation agree with experimental data up to lowest energies. Note that the full quantities R_Δ and W_Δ to which all quarks contribute are not very sensitive to the values of light quark masses. It will be further interesting to consider the corresponding quantities without the contribution of heavy quarks. In this case, there is a possibility to decrease the value of Δ up to the ρ meson scale and to investigate the values of the light pole quark masses in detail. We plan also to apply this approach to the inclusive τ decay where the heavy quark contribution does not play an essential role and sensitivity to the values of light pole quark masses will become noticeable.

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Appendix A

In this appendix we collect some formulae used in the text.

The function W_Δ can be written as follows

$$W_\Delta(s) = \frac{1}{s^2 + \Delta^2} \left[\Delta\, D_1(s,\Delta) - s\, D_2(s,\Delta) \right], \tag{A.1}$$

with

$$\begin{aligned} D_1(s,\Delta) &= \frac{3}{\pi} \sum_f Q_f^2 \left[A_1(s,M_f) + 4\left(\lambda_1(s) A_2(s,M_f) - \lambda_1(s) B_2(s,M_f) \right) \right], \\ D_2(s,\Delta) &= \frac{3}{\pi} \sum_f Q_f^2 \left[B_1(s,M_f) + 4\left(\lambda_1(s) B_2(s,M_f) + \lambda_1(s) A_2(s,M_f) \right) \right], \end{aligned} \tag{A.2}$$

where

$$\lambda_1(s) = \mathrm{Re}\, \lambda_{\mathrm{eff}}\left(a(s)\right), \qquad \lambda_2(s) = \mathrm{Im}\, \lambda_{\mathrm{eff}}\left(a(s)\right). \tag{A.3}$$

The complex function $a(s) = a_1 + \mathrm{i}a_2$ obeys the following system of equations:

$$\begin{aligned} C\, \mathrm{Re}\left[U(a_1 + \mathrm{i}a_2) - U(a_0) \right] &= 11 \ln \frac{\sqrt{s^2 + \Delta^2}}{Q_0^2} - \frac{2}{3} \sum_f \left[A_0(s,M_f) - I\left(\frac{M_f^2}{Q_0^2} \right) \right], \\ C\, \mathrm{Im}\, U(a_1 + \mathrm{i}a_2) &= -11 \left(\frac{\pi}{2} + \mathrm{atan} \frac{s}{\Delta} \right) - \frac{2}{3} \sum_f B_0(s,M_f). \end{aligned} \tag{A.4}$$

Finally, the functions $A_i(s,M)$ and $B_i(s,M)$ are defined by

$$A_0(s,M) = \int_0^1 du \, \frac{u^2(3-u^2)}{(1-a-u^2)^2 + b^2}(1-a-u^2), \tag{A.5}$$

$$B_0(s,M) = -b \int_0^1 du \, \frac{u^2(3-u^2)}{(1-a-u^2)^2 + b^2}, \tag{A.6}$$

$$A_1(s,M) = \frac{3}{2} \int_0^1 du \, \frac{(1-u^2)^2}{(1-a-u^2)^2 + b^2}(1-a-u^2), \tag{A.7}$$

$$B_1(s,M) = -\frac{3}{2} b \int_0^1 du \, \frac{(1-u^2)^2}{(1-a-u^2)^2 + b^2}, \tag{A.8}$$

$$A_2(s,M) = \frac{4\pi}{3} \int_0^1 du \, \frac{u^2(3-u^2)}{\left[(1-a-u^2)^2 + b^2\right]^2} f(u) \cdot \left\{ -a \left[(1-a-u^2)^2 - b^2\right] + 2b^2(1-a-u^2) \right\}, \tag{A.9}$$

$$B_2(s,M) = \frac{4\pi}{3} \int_0^1 du \, \frac{u^2(3-u^2)}{\left[(1-a-u^2)^2 + b^2\right]^2} f(u) \cdot \left\{ b \left[(1-a-u^2)^2 - b^2\right] + 2ab(1-a-u^2) \right\}. \tag{A.10}$$

Here

$$a = \frac{4M^2 s}{s^2 + \Delta^2}, \qquad b = \frac{4M^2 \Delta}{s^2 + \Delta^2} \tag{A.11}$$

and the function $f(u)$ is defined by Eq. (5.6) in the text.

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