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Simulating the massive Schwinger model with chiral defect fermions

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Abstract

Some time ago Kaplan proposed a new model for the description of chiral fermions on the lattice by adding an extra dimension for the fermions. A variant of this proposal was introduced by Shamir and can be used to describe vector-like theories in even dimensions. We used this model for the simulation of the massive Schwinger model at different gauge couplings. The prediction that the fermion mass gets only multiplicative renormalization was tested and verified.

1 Introduction

Some years ago Kaplan suggested a new method simulating chiral fermions on the lattice [1]. He started with a vector-like odd-dimensional theory and a mass term that depends on one space coordinate and has the form of a domain wall. He found a zero-mode bound to this domain wall. From the point of view of the even-dimensional domain wall, this zero-mode is a chiral fermion. The zero-mode exists up to a critical momentum, so that the chirality is a low energy effect. Anomalies in gauge currents arise from the flow of charge into the extra dimension. In the case of a finite extra dimension which has the topology of a circle and periodic boundary conditions an additional anti-domain wall with a chiral fermion of opposite handness is generated.

A similar but technicaly simpler approach was made by Shamir [2]. Instead of an infinite extra dimension and a mass that is a function of this dimension, he used a semi-infinite coordinate with free boundary conditions and a constant odd-dimensional mass. In the case of a finite lattice this corresponds to cutting the links at both domains in the original model and dropping the side with a negative mass. Therefore, in numerical simulations one has to take only half as many points to obtain similar results. The zero-states with different chirality are located at the boundaries. A connection of these two boundaries by a link of strength m will form a Dirac fermion from the two Weyl fermions whose mass is proportional to m, if the exponentially small overlap between the surface states can be neglected. This model can be

used to describe vector-like theories in even dimensions. Unlike the case of chiral gauge theories [3] in the vector-like model there is no problem to couple $(2n+1)$-dimensional fermions to a gauge field in $2n$ dimensions. It was shown that perturbative corrections to the quark mass are proportional to m, so that it undergoes only multiplicative renormalization to all orders [2]. Therefore, there is no fine tuning problem that appears for Wilson fermions. This model has in the massless case a very mild breaking of the axial symmetries. Moreover, the correct chiral limit can be proven [4].

We examined the case of $(2+1)$-dimensional fermions coupled to a two-dimensional $U(1)$ gauge field, i.e. the massive Schwinger model [5, 6]. To test the prediction that the fermion mass gets only multiplicative renormalization we studied the behaviour of the pseudoscalar isotriplet ('pion') mass if m tends to zero.

2 Free boundary fermions

We start our discussion with the free fermion case in $2+1$ dimensions and a lattice of size $\Omega = L^2 \cdot L_{\rm s}$. The action is given by

$$S_{\rm F} \;=\; \sum_x \left\{ \sum_{s=1}^{L_s} \overline{\Psi}_{x,s} \left[\partial\!\!\!/ + M + \frac{r}{2}\Delta \right] \Psi_{x,s} + m \left[\overline{\Psi}_{x,1} P_{\rm L} \Psi_{x,L_s} + \overline{\Psi}_{x,L_s} P_{\rm R} \Psi_{x,1} \right] \right\} , \qquad (1)$$

where $\partial\!\!\!/ = \sum_{\mu=1}^{3} \sigma_\mu \partial_\mu$ and $\Delta = \sum_{\mu=1}^{3} \Delta_\mu$. σ_μ are the Pauli matrices, r denotes the Wilson parameter and $P_{\rm R,L} = (1 \pm \sigma_3)/2$ are the projection operators. A lattice point is given by (x, s), where s labels the extra coordinate. In the following pages you will note that the lattice spacing is taken to be $a = 1$. The lattice derivation ∂_μ and Laplacian Δ_μ for $\mu = 1, 2$ are defined as usual and

$$\partial_3 \Psi_{x,s} \equiv \frac{1}{2} \begin{cases} \Psi_{x,2} & s = 1 \\ \Psi_{x,s+1} - \Psi_{x,s-1} & 2 \le s \le L_{\rm s} - 1 \\ -\Psi_{x,L_s-1} & s = L_{\rm s} \end{cases} , \qquad (2)$$

$$\Delta_3 \Psi_{x,s} \equiv \begin{cases} \Psi_{x,2} - 2\Psi_{x,1} & s = 1 \\ \Psi_{x,s+1} - 2\Psi_{x,s} + \Psi_{x,s-1} & 2 \le s \le L_{\rm s} - 1 \\ -2\Psi_{x,L_s} + \Psi_{x,L_s-1} & s = L_{\rm s} \end{cases} , \qquad (3)$$

i.e. the case $m = 0$ corresponds to open boundaries in the third direction. For $r = 1$ and $m = 1$ the model corresponds to the topology of a circle and periodic boundary condition in s, for $r = 1$ and $m = -1$ it describes the antiperiodic case. Thus, in these cases the (2+1)-dimensional boundary fermion model is equivalent to a three-dimensional model with Wilson fermions and the corresponding boundary conditions.

The action (1) leeds to the Hamiltonian

$$H = \sigma_2 \left[{\rm i}\, \sigma_1 \overline{k}_1 + \sigma_3 \partial_3 + M + \frac{r}{2} \left(-\hat{k}_1^2 + \Delta_3 \right) + m \left(\delta_{s,1} \delta_{s',L_s} P_{\rm L} + \delta_{s,L_s} \delta_{s',1} P_{\rm R} \right) \right] , \qquad (4)$$

$$\overline{k} = \sin(k) , \qquad \hat{k} = 2 \sin(k/2) .$$

Its eigenvalues occur in pairs corresponding to particle and anti-particle and were already examined for $m = 0$ [7, 8]. In this case the chiral zero-modes are confined to the two boundaries.

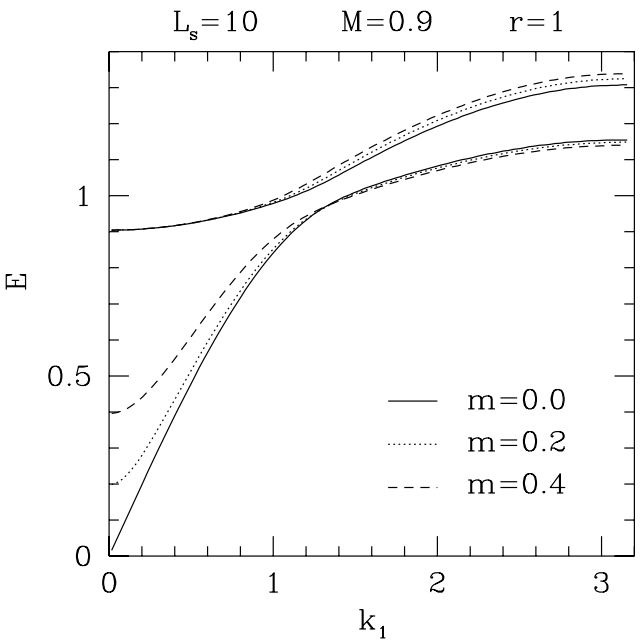

Figure 1: Part of the energy spectrum for a lattice extension of $L_\mathrm{s} = 10$ and a $(2+1)$-dimensional mass $M = 0.9$. The three different types of lines characterize the different values of the mass parameter m.

The wave functions have the form of plane waves in the two-dimensional space and decay exponentially with $(1-M)$ in s. The chiral zero-modes exist for momenta k_1 below some critical momentum k_c. For $0 < M/r < 2$ there is one zero-mode at each boundary. For $m \neq 0$, an infinite lattice in x_1 direction and a finite third direction one can calculate the lowest eigenvalues in the limit where m^2 and ${k_1}^2$ are small, i.e. m^2, ${k_1}^2 \ll M^2$. If one choose L_s large enough, so that the exponentially small mixing of the two chiral modes can be neglected ($(1-M)^{L_\mathrm{s}} \ll m$), the result is

$$ {E_0}^2 = {k_1}^2 + m^2 M^2 (2-M)^2 \ . \tag{5} $$

Therefore, the mass of the light Dirac fermion is

$$ \overline{m} = mM(2-M) \ . \tag{6} $$

In Fig. 1 we sketch the dependence of the energy spectrum for the two lightest states on the mass. One can observe that the lowest eigenvalue has indeed the predicted form.

3 Interacting theory

The interacting theory is defined by coupling the fermions to a two-dimensional gauge field, i.e. $U_{(x,s)\mu} = U_{x\mu}$ for $\mu = 1, 2$ and $U_{(x,s)3} = 1$. This leads for $r = 1$ to the following gauged fermionic part of the action

$$ S_\mathrm{F} \equiv \left(\overline{\Psi}, Q[U]\Psi\right) $$

$$= \sum_x \sum_{s=1}^{L_s} \left\{ \frac{1}{2} \sum_{\mu=1}^{2} \left[\overline{\Psi}_{x,s} \left(1+\sigma_\mu\right) U_{x\mu} \Psi_{x+\hat{\mu},s} + \overline{\Psi}_{x+\hat{\mu},s} \left(1-\sigma_\mu\right) U^*_{x\mu} \Psi_{x,s} \right] + \left(M-3\right) \overline{\Psi}_{x,s} \Psi_{x,s} \right\}$$
$$+ \sum_x \left\{ \sum_{s=1}^{L_s-1} \left[\overline{\Psi}_{x,s} P_{\rm R} \Psi_{x,s+1} + \overline{\Psi}_{x,s+1} P_{\rm L} \Psi_{x,s} \right] + m \left[\overline{\Psi}_{x,L_s} P_{\rm R} \Psi_{x,1} + \overline{\Psi}_{x,1} P_{\rm L} \Psi_{x,L_s} \right] \right\} , \qquad (7)$$

where $U_{x\mu}$ denotes the U(1) gauge variable. Apart from the unconventional sign of the mass term, the first term is the usual two-dimensional action. The preferred boundary conditions are periodic in space direction and antiperiodic in time direction. The gauge part is given by

$$S_{\rm G} = \beta \sum_P \left(1 - {\rm Re}\, U_P\right) \ , \qquad (8)$$

where U_P is the product of four U's around a plaquette and $\beta = 1/g^2$. The whole action is given by the sum of the two parts:

$$S = S_{\rm F} + S_{\rm G} \ . \qquad (9)$$

The case $m = 0$ should yield to the massless Schwinger model by taking first the limit $L_{\rm s} \to \infty$ and then $\beta \to \infty$.

The definitions of the vector and axial currents as well as their divergence equations can be found in [9, 4]. Here we just give a definition of the axial transformation (for even $L_{\rm s}$):

$$\delta^i_A \, \psi_{x,s} = {\rm i}\, q(s)\, \sigma_i \, \psi_{x,s}, \qquad (10)$$
$$\delta^i_A \, \overline{\psi}_{x,s} = -{\rm i}\, q(s)\, \overline{\psi}_{x,s}\, \sigma_i, \qquad (11)$$

where

$$q(s) \equiv \begin{cases} 1 & 1 \le s \le L_{\rm s}/2 \\ -1 & L_{\rm s}/2 < s \le L_{\rm s} \end{cases} \ . \qquad (12)$$

The non-invariance of the action under this transformation results from the coupling between the layers $s = L_{\rm s}/2$ and $s = L_{\rm s}/2 + 1$, respectively, $s = 1$ and $s = L_{\rm s}$ for $m \neq 0$.

For the numerical simulations we used the Hybrid Monte Carlo algorithm with pseudo-fermions [10]. This requires a positive definite determinant and therefore the flavour duplication of the fermion spectrum. The fermion matrix for the first species ist $Q[U]$, for the second $Q[U]^\dagger$. If we define the reflection $\mathcal{R}$ by

$$\mathcal{R} \Psi_{x,s} \equiv \Psi_{x,L_s+1-s} \ , \qquad (13)$$

the fermion matrix satisfies

$$\sigma_3 \mathcal{R}\, Q[U]\, \sigma_3 \mathcal{R} = Q[U]^\dagger \ . \qquad (14)$$

This yields to

$$\det Q[U] = \det Q[U]^\dagger \ . \qquad (15)$$

Thus, the simulated model describes the two flavour massive Schwinger model.

The topological charge is given by

$$Q_{\rm top} = \frac{1}{2\pi} \sum_x F_x \ , \qquad Q_{\rm top} \in \mathbb{Z} \ , \qquad (16)$$

$$\exp\left({\rm i}\, F_x\right) = U_{x,1}\, U_{x+\hat{1},2}\, U^*_{x+\hat{2},1}\, U^*_{x,2} \ , \qquad F_x \in [-\pi, \pi) \ . \qquad (17)$$

The different topological sectors are separated by a potential barrier of height 2β, so that the tunneling is suppressed. We restricted the simulations to the topological trivial case ($Q_{\rm top} = 0$)[1],

[1] Contributions of the non-trivial sectors were examined in [11]

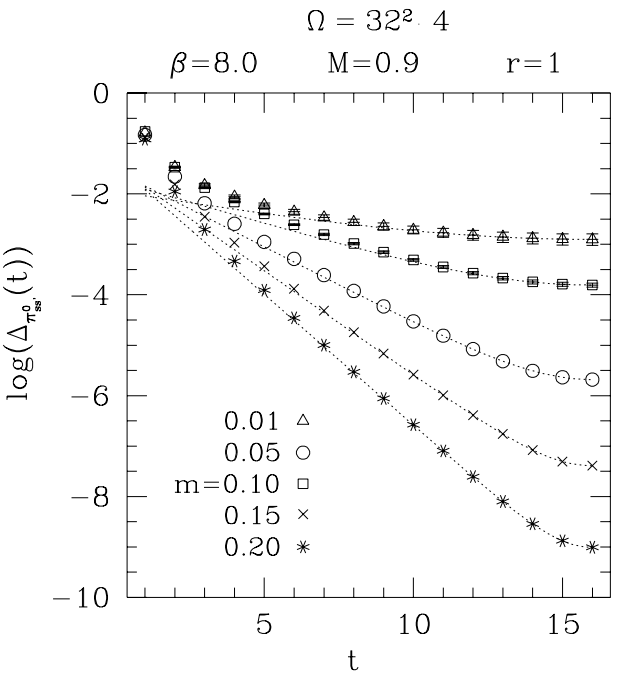

Figure 2: Correlation function for the pion for $(s, s') = (1, L_{\rm s})$. The error bars for $m \geq 0.1$ were omitted because, they are too small for a visualization (smaller than 3% of the measured values). The curves shown are the best fit in the interval $13 \leq t \leq 16$.

i.e. we start with $U_{x\mu} = 1$ and controlled that the topological charge doesn't change. The lightest particles in this model are a pseudoscalar isotriplet with mass M_{1^-} satisfying

$$\frac{M_{1^-}}{g} \underset{\frac{\overline{m}}{g} \to 0}{\longrightarrow} c \left(\frac{\overline{m}}{g} \right)^{\frac{2}{3}} , \tag{18}$$

$$c = 6 \sqrt{\frac{2}{\pi}} \left(\frac{e^{\gamma}}{2\pi} \right)^{\frac{2}{3}} \approx 2.07 . \tag{19}$$

This is the analogue to the pion in QCD and becomes massless as the fermion mass $\overline{m}$ goes to zero. The matrix inversions were performed by the conjugate gradient algorithm [12].

It can be shown that in the limit $L_{\rm s} \to \infty$ corrections to the fermion mass $\overline{m}$ are proportional m, so that the fermion mass in the boundary model gets only multiplicative renormalization. The reason is that in weak coupling perturbation theory, corrections to the inverse fermion propagator are exponentially suppressed with $|s - s'|$. Therefore, the zero-modes can't mix, except for an exponentially small factor, and a mass term can't develop. Thus, the chiral limit is attained if m goes to zero. We used the massive Schwinger model with two degenerate fermion species to test non-pertubatively the prediction that the fermion mass tends to zero for $m \to 0$.

The simulations were performed at $M = 0.9$, so that at tree level one has $\overline{m} = 0.99\, m$. For the gauge coupling the two values $\beta = 0.25$ and $\beta = 8.0$ were examined. In the last case the renormalization of M is small. It follows that the exponentially decay in the additional direction is fast and the exponentially contribution to the mass are even for small values of $L_{\rm s}$

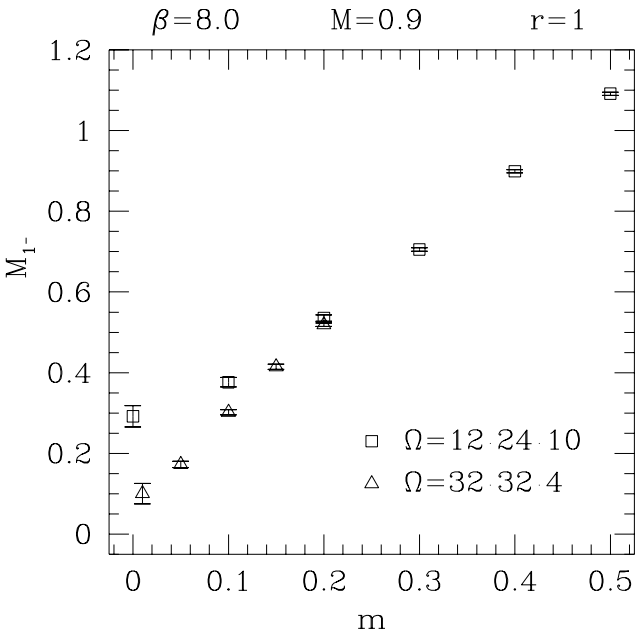

Figure 3: Pion mass as a function of m at $\beta = 8.0$.

strongly suppressed. Therefore, only small extensions of L_s are needed. As a check we used lattices of size $\Omega = L_\mathrm{x} \cdot L_\mathrm{t} \cdot L_\mathrm{s} = 12 \cdot 24 \cdot 10$ and $\Omega = 32^2 \cdot 4$. For the mass parameter m we have chosen the values $m = 0.0, 0.1, \ldots, 0.5$ in the first case and $m = 0.01, 0.05, 0.10, 0.15, 0.20$ in the second case. At $\beta = 0.25$ we examined $m = 0.0, 0.1, 0.2, 0.3$ for a $12 \cdot 24 \cdot 10$ lattice.

The pion mass M_{1^-} is calculated from the correlation function

$$\Delta_{\pi^0_{s\,s'}}(t) = \left\langle \pi^0_{ss'}(0)\pi^0_{ss'}(t) \right\rangle \ , \tag{20}$$

$$\pi^0_{ss'}(t) \equiv \sum_\mathrm{x} \overline{\psi}_{(\mathrm{x},\mathrm{t}),s}\sigma_3\tau_3\psi_{(\mathrm{x},\mathrm{t}),s'} \ , \tag{21}$$

where $(I, I_3)^P = (1, 0)^-$ and $x = (\mathrm{x}, \mathrm{t})$. For $(s, s') = (1, L_\mathrm{s})$ and $(s, s') = (L_\mathrm{s}, 1)$ this yields to

$$\Delta_{\pi^0_{1,L_s}}(t) = \Delta_{\pi^0_{L_s,1}}(t) \quad \sim \quad \sum_\mathrm{x} \left[\left| Q[U]^{-1}_{(0,0),(\mathrm{x},\mathrm{t});1,L_\mathrm{s}} \right|^2 + \left| Q[U]^{-1}_{(0,0),(\mathrm{x},\mathrm{t});L_\mathrm{s},1} \right|^2 \right] \ . \tag{22}$$

For every lattice size we performed 10000 sweeps. Every 100th sweep we calculated $Q[U]^{-1}$, the correlation function and the pion mass separately on every configuration. The statistical error was estimated by binning. The autocorrelation times for the pion mass were of order $\mathcal{O}(100)$ sweeps. The maximum value of approximate 500 sweeps was obtained for $\beta = 8.0$, $\Omega = 32^2 \cdot 4$ and $m = 0.01$.

In fig. 2 the different correlation functions for the $32^2 \cdot 4$ lattice and $(s, s') = (1, L_\mathrm{s})$ are plotted. From this we extracted the pion mass by a least square fit by a single cosh. The values are shown in fig. 3 together with the masses of the other lattice. One can see that the values of the two masses for $m = 0.20$ coincide within the statistical errors. For small values of m finite size effects appear. An estimate for the value of the critical mass by polynomial interpolation gives $|m_\mathrm{c}| \leq 0.01$, i.e. no deviation of a multiplicative renormalization for the mass is observable. Since we are in the weak coupling region renormalization effects are small

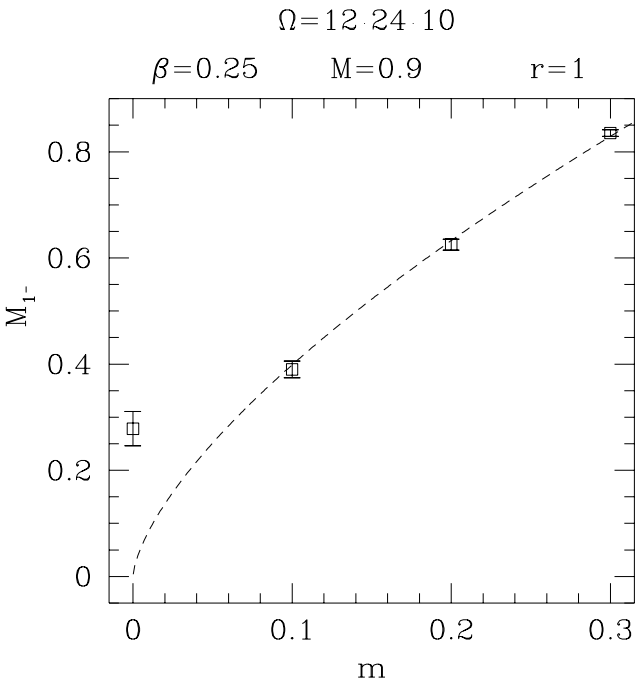

Figure 4: Pion mass as a function of m at $\beta = 0.25$. The dashed line indicates an ansatz proportional $m^{2/3}$.

anyway. The renormalization of the fermion mass can be obtained by comparing the data with the results of other simulations and perturbation theory. The result is $\overline{m} = 0.90(5)\, m$.

Renormalization effects at $\beta = 0.25$ are larger. Therefore the simulations at this point are a better check for the prediction that the fermion mass gets only multiplicative renormalization. As before there is no hint for a nonzero critical mass (fig. 4). Moreover m/g is small enough for a comparison with perturbation theory. The results are in agreement with theory and lead to $\overline{m} = 0.60(2)\, m$.

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