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Some topics in beam dynamics of storage rings

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Abstract

In the following report we want to review some beam dynamics problems in accelerator physics. Theoretical tools and methods are introduced and discussed, and it is shown how these concepts can be applied to the study of various problems in storage rings. The first part treats Hamiltonian systems (proton accelerators) whereas the second part is concerned with explicitly stochastic systems (e.g. electron storage rings).

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1 Introduction

As synchrotron radiation sources and as colliders, storage rings have become an important tool in physical research. Colliders are devices which allow two beams of ultrarelativistic charged particles circulating in opposite directions to be accumulated, stored and collided. (see Figure 1) These colliding

Figure 1: storage ring

beam facilities reaching energies from several hundreds of MeV up to the TeV range (see LHC project at CERN [1]) allow the investigation of the innermost structure of matter, and they have become an indispensable tool of high energy physics. The technical components of such an accelerator are magnets, a toroidal vacuum chamber and radio-frequency (rf) structures such as cavities and transmitters. Usually the stored beams consist of bunches, each of them containing 10^{10} to 10^{11} particles. The size of these bunches ranges from a tenth of a millimeter to a few centimeters. Some details of the electron-proton storage ring HERA at DESY in Hamburg are listed below:

.	$electron-ring$	$proton-ring$
$circumference$	$6.336km$	$6.336km$
$energy$	$26/30GeV$	$820GeV$
$No.\ of\ colliding\ bunches$	174	174
$particles/bunch$	3×10^{10}	4.5×10^{10}
$bunch-width\ at\ interaction\ point$	$0.24mm$	$0.22mm$
$bunch-height\ at\ interaction\ point$	$0.06mm$	$0.07mm$
$bunch-length\ at\ maximum\ energy$	$8mm$	$53mm$

An accelerator constitutes a complex many-body system - namely an ensemble of 10^{10} to 10^{13} charged ultrarelativistic ($v \approx c$) particles subject to external electromagnetic fields, radiation fields and various other influences such as restgas scattering, space charge effects and effects due to the beam induced electromagnetic fields in the surrounding metallic structures (vacuum tubes, cavities etc.), the so-called wakefields. Two key issues in the performance of colliders are:

- the storage and confinement of particles over many hours corresponding to 10^8 to 10^{10} revolutions in the toroidal vacuum chamber and
- a high reaction rate when the particle beams collide at the interaction points where the experimental detectors are located.

In order to achieve a high number of revolutions inside a tube with a diameter of a few centimeters one has to control and understand the effects which cause degradation of the beam lifetime or which cause the particles to hit the wall of the vacuum chamber and thus be lost. Various loss mechanisms are:

- intrabeam scattering
- reaction rate losses during colliding beam operation (wanted)
- restgas scattering and imperfect vacuum
- nonlinear dynamics effects
- effects due to noise

The reaction rate for a process of cross section, σ, obtained in a collider can be written as

$$R = L \cdot \sigma \tag{1}$$

where L is the luminosity, a parameter characterizing the colliding beam system. Generally, the luminosity is calculated by integrating over all possible collisions between the particles in both beams. In the case of head-on collisions of equal current bunched beams of opposite charge with Gaussian charge distribution the luminosity is given by

$$L = \frac{N \cdot N \cdot f}{4\pi\sigma_x \cdot \sigma_z \cdot B} \tag{2}$$

where N is the total number of particles per beam, f is the revolution frequency, B the number of bunches per beam and σ_x, σ_z the standard deviations in the transverse directions. A degradation or limitation of the luminosity can be due to a blow up of the transverse beam sizes (σ_x, σ_z or of the beam emittance, which measures the particle distribution in phase space) or due to a limitation of the number of particles, N, which can be stored in the accelerator (instabilities). Figure 2, for example, shows the blow up of the proton beam in HERA after several hours of luminosity operation [2], and Figure 3 shows how the beam-beam interaction has influenced the beam

Figure 2: beam blow up of HERA proton beam

dynamics in the electron-positron storage ring PETRA [3].

Concerning the theoretical methods and tools to describe and analyze the beam dynamics in storage rings there are two approaches:

- collective theory
- single particle theory.

Figure 3: Examples of beam blow-up at PETRA due to beam-beam interaction

Collective phenomena such as instabilities are treated within the framework of the Vlasov-equation [4], [5] and will not be studied in this review. A schematic description of the Vlasov theory is shown in Figure 4, and a recent theoretical review can be found for, example, in [6].

In this review we will restrict ourselves to the classical single particle dynamics, i.e. we study the equations of motion of a single charged ultrarelativistic particle under the influence of external electromagnetic fields and

radiation effects. In general these equations are nonlinear [7] [8] [9] [10] [11]. The main nonlinearities are due to the beam-beam interaction, where a test particle in one beam experiences strong nonlinear electromagnetic forces of the counter-rotating beam. Other sources are nonlinear cavity fields or transverse multipole fields. These multipole fields are either introduced artificially, e.g. by sextupoles which compensate the natural chromaticity or they occur naturally as deviations from linear fields due to errors. These field errors play an important role in superconducting magnets [12].

Figure 4: Vlasov scheme

Because of these nonlinearities a storage ring acts as a nonlinear device schematically sketched in Figure 5. a_{in} is some initial amplitude (position,

momentum given by the injection conditions) and a_{fin} is the amplitude after N (10^8 - 10^{10}) revolutions in the ring. In accelerator physics one often tries to define different zones according to a_{in}. For small amplitudes up to a certain boundary a_{lin} - the linear aperture - the storage ring behaves more or less like a linear element (at least for the time scales of interest, i.e. 10 - 20 hours storage time). For larger amplitudes the behaviour becomes more and more nonlinear and eventually at a_{dyn} - the dynamic aperture - the particle motion becomes unbounded. One problem of the single particle theory of accelerators is to make quantitative predictions of these different zones, or stated in a different way, to calculate quantities such as the linear aperture a_{lin} or the dynamic aperture a_{dyn}. Furthermore one wants to know how these quantities depend on various machine parameters and the type of the nonlinearity. A better and - from a practical point of view - more relevant question is: what is the lifetime of the particle, or what is the probability for the particle to hit the wall of the vacuum chamber after it has been injected into a certain volume of phase space.

Mathematically the single particle dynamics is formulated in terms of ordinary differential equations (dynamical system) of the type

$$\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}, t, ...\alpha_i(t)...) \tag{3}$$

describing the time evolution [1] of the six-dimensional phase space vector

[1] Instead of time t one often uses another independent variable in accelerator physics, namely the arclength s along a certain design trajectory (see Appendix A). However, since there is a one-to-one correspondence between these quantities, in the following we will always feel free to switch between both notations and use them synonymously. t is mainly used in a general context, whereas s is used in special accelerator applications.

Figure 5: storage ring as a nonlinear device

$\vec{x}(t) = (x_1....x_6)^T$ with $\vec{f} = (f_1....f_6)^T$ some vector function of $\vec{x}$. The quantities $\alpha_i(t)$ which may depend on time (deterministically or randomly) represent a set of control parameters.

As we will show later, the classical proton dynamics (heavy particle with negligible radiation effects) is described by a Hamiltonian. On the other hand the dynamics of light particles like the electron, where radiative effects play an important role, is governed by a stochastically and dissipatively perturbed Hamiltonian. In order to solve the dynamical equations (3), various numerical and analytical tools have been developed, some of which will be described in greater detail in the following.

This survey is organized as follows: In the first part we will consider storage rings where radiation phenomena can be neglected, i.e. accelerators for protons or heavy ions up to the TeV range. In HERA, for example, the radiation losses of a proton are about 6 eV at 820 GeV whereas an electron loses more than 83 MeV at 30 GeV. This large difference is due to the fact that the radiation losses are inversely proportional to the fourth power of the rest mass m_0. At the same energy the ratio of the energy losses of an electron and a proton is given by $\frac{P_{electron}}{P_{proton}} = (\frac{m_{0p}}{m_{0e}})^4 = 1.13 \cdot 10^{13}$. Thus proton storage rings can be modelled mathematically by nonlinear (in general nonintegrable) Hamiltonians. Nonintegrable means that the corresponding nonlinear equations of motion cannot be solved analytically. As we will see later the phase space dynamics of these systems shows a very rich and complicated structure. The questions we want to answer in the first part are:

- what does the Hamiltonian for the particle dynamics look like?
- what is in principle possible in these systems? (qualitative theory)
- which analytical (i.e. perturbative) tools are available for a quantitative study of these problems?

In the second part of this survey we will treat systems where radiation effects or noise effects are important. Because of the stochastic emission of the radiation, radiative systems can be modelled by explicit stochastic dynamical systems. A straightforward way to extend deterministic systems to include noise effects and explicit stochastic phenomena is to write down stochastic differential equations. In this contribution we will illustrate some of the subtleties related to stochastic differential equations including Gaussian white

noise, and we will mention and illustrate some applications in accelerator physics.

In the third part of this review we will briefly mention and discuss the problems, which arise when one also takes into account the spin degree of freedom of the relativistic particles. We will describe the classical spin motion within the framework of the Thomas-Bargman-Michel-Telegdi (TBMT) equation. In particular we show how depolarizing phenomena like spin diffusion can be calculated.

Some details of the calculations have been relegated to the appendices.

This work cannot cover the whole subject exhaustively, we can only sketch the basic ideas and illustrate these ideas with simple (sometimes oversimplified) models. For many details we have to refer the reader to the references. Concerning the list of references we have tried to include a lot of information. However, due to the many topics treated in this review the list of cited research papers is by no means exhaustive. The choice we made is purely subjective and reflects our personal "taste".

The main aim of this review is to show in an informal way how tools and methods of dynamical systems theory (sometimes very old and well known in other fields) can be applied to various problems of particle motion in storage rings.

2 Hamiltonian dynamics

2.1 Hamiltonian for coupled synchro-betatron motion

In this section we will investigate the motion of charged particles under the influence of electromagnetic fields. The dynamics is governed by the well known Lorentz-equation, which we will write in canonical form.

Starting point is the following relativistic Lagrangian for a charged particle under the influence of an electromagnetic field described by the vector potential $\vec{A}$ and the scalar potential ϕ [13]

$$\mathcal{L} = -m_0 c^2 \sqrt{1 - \frac{\dot{\vec{r}}^2}{c^2}} + \frac{e}{c}(\dot{\vec{r}}\vec{A}) - e\phi \tag{4}$$

with

- e=elementary charge
- c=speed of light
- m_0=rest mass of the particle
- $\dot{\vec{r}}$=particle velocity .

Following the rules of analytical mechanics one changes to a Hamiltonian description of motion and one introduces the curvilinear coordinate system depicted in Figure 6 [14]. It consists of three unit vectors $\vec{e}_\tau$,$\vec{e}_x$, $\vec{e}_z$ attached to the design orbit of the storage ring. s is the pathlength along this trajectory. For simplicity we have assumed a plane reference orbit with horizontal curvature κ only. Using s as an independent variable and introducing difference variables with respect to an equilibrium particle on the design orbit one obtains in a certain gauge (see Appendix A)

$$\begin{aligned}
&\mathcal{H}(x, z, \tau, p_x, p_z, p_\tau; s) = \\
&\quad = -(1+\kappa x)\{(1+p_\tau)^2 - (p_x - \frac{e}{E_0}A_x)^2 - (p_z - \frac{e}{E_0}A_z)^2\}^{1/2} - \\
&\quad -(1+\kappa x)\cdot\frac{e}{E_0}\cdot A_\tau + (1+p_\tau)
\end{aligned} \tag{5}$$

where we have used

Figure 6: curvilinear coordinate system

- $v \approx c$ (ultrarelativistic particles)
- $\tau = s - ct$
- $p_\tau = \frac{\Delta E}{E_0}$
- E_0=energy of design particle.

The corresponding equations of motion are

$$\begin{aligned} \frac{d}{ds}\, x &= +\frac{\partial \mathcal{H}}{\partial p_x}; \qquad \frac{d}{ds}\, p_x = -\frac{\partial \mathcal{H}}{\partial x} \\ \frac{d}{ds}\, z &= +\frac{\partial \mathcal{H}}{\partial p_z}; \qquad \frac{d}{ds}\, p_z = -\frac{\partial \mathcal{H}}{\partial z} \\ \frac{d}{ds}\, \tau &= +\frac{\partial \mathcal{H}}{\partial p_\tau}; \qquad \frac{d}{ds}\, p_\tau = -\frac{\partial \mathcal{H}}{\partial \tau}\ . \end{aligned} \tag{6}$$

$\vec{A}(x,z,s) = (A_x(x,z,s), A_z(x,z,s), A_\tau(x,z,s))$ is the vector potential which is related to the external electromagnetic fields via

$$\vec{B} = \nabla \times \vec{A}$$

and

$$\vec{\varepsilon} = \frac{\partial \vec{A}}{\partial \tau}.$$

$\mathcal{H}$ describes the oscillations of a particle with respect to a design particle travelling with velocity c along the design trajectory. The oscillations in the transverse directions (x, z, p_x, p_z) are called *betatron oscillations* and the oscillations in the longitudinal direction (τ, p_τ) are called *synchrotron oscillations*. Some simple examples for the vector potential $\vec{A}(x, z, s)$ are shown below:

rf - cavity:

$$A_\tau = -\frac{L}{2\pi k} \cdot V_0 \cdot \cos(k\frac{2\pi}{L}\tau) \cdot \delta(s - s_0) \tag{7}$$

with

- V_0=peak voltage of cavity
- L=circumference of storage ring
- k=harmonic number
- $\delta(s - s_0)$=delta function (localized cavity)

bending (dipole) magnet:

$$\frac{e}{E_0}A_\tau = -\frac{1}{2}(1 + \kappa \cdot x) \tag{8}$$

with

- $\kappa = \frac{e}{E_0}B_z(x = z = 0)$=horizontal curvature of design orbit. B_z=z-component of magnetic field

quadrupole:

$$\frac{e}{E_0}A_\tau = \frac{1}{2}g_0 \cdot (z^2 - x^2) \tag{9}$$

with

- $g_0 = \frac{e}{E_0} \cdot (\frac{\partial B_z}{\partial x})_{x=z=0}$= focusing strength of quadrupole

multipole (sextupole):

$$\frac{e}{E_0} \cdot A_\tau = -\frac{1}{6} \cdot \lambda_0 \cdot (x^3 - 3xz^2) \tag{10}$$

with

- $\lambda_0 = \frac{e}{E_0} \cdot (\frac{\partial^2 B_z}{\partial x^2})_{x=z=0}$= strength of sextupole

multipole (octupole):

$$\frac{e}{E_0} \cdot A_\tau = \frac{1}{24} \cdot \mu_0 \cdot (z^4 - 6x^2z^2 + x^4) \tag{11}$$

with

- $\mu_0 = \frac{e}{E_0} \cdot (\frac{\partial^3 B_x}{\partial z^3})_{x=z=0}$=strength of octupole

Further examples for other types of electromagnetic fields can be found in [14].

Generally, by expanding the square root in equation (5) and the vector potential $\vec{A}(x,z,s)$ into a Taylor series around a reference orbit, various examples of nonlinear motion can be investigated.The linear part of the Hamiltonian is given by [15]

$$\begin{aligned}\mathcal{H}_0(x,z,\tau,p_x,p_z,p_\tau;s) = \ & \frac{1}{2}p_x^2 + \frac{1}{2}(\kappa^2(s) + g_0(s)) \cdot x^2 + \\ & + \frac{1}{2}p_z^2 - \frac{1}{2}g_0(s) \cdot z^2 - \frac{1}{2}V(s) \cdot \tau^2 - \kappa(s) \cdot x \cdot p_\tau \end{aligned} \tag{12}$$

where $V(s) = V_0 \cdot \delta_p(s - s_0)$ with $\delta_p(s-s_0) = \sum_{n=-\infty}^{n=+\infty} \delta(s-(s_0+n \cdot L))$ describes a localized cavity at position s_0 and where $g_0(s)$ characterizes the (periodic) focusing strength of the magnet system. $\mathcal{H}_0$ describes three coupled linear Floquet oscillators [15] (see also Appendix B).

Two simple examples of nonlinear motion are given below:

Example 1:Nonlinear Cavity

$$\begin{aligned}\mathcal{H}(x,z,\tau,p_x,p_z,p_\tau;s) = \ & \frac{1}{2}p_x^2 + \frac{1}{2}p_z^2 + \\ & + \frac{1}{2}g_0(s) \cdot (x^2 - z^2) + \frac{1}{2}\kappa^2(s) \cdot x^2 - \kappa(s) \cdot x \cdot p_\tau + V(s) \cdot \cos(\tau) \end{aligned} \tag{13}$$

Introducing the dispersion function D defined by

$$D''(s) = -(\kappa^2(s) + g_0(s)) \cdot D(s) + \kappa(s) \tag{14}$$

with

$$(.)' = \frac{d}{ds}$$

via the canonical transformation [16],[17],[18] (depending on the *old* coordinates x, z, τ and the *new* momenta $\bar{p}_x, \bar{p}_z, \bar{p}_\tau$)

$$\begin{aligned} F_2(x, z, \tau, \bar{p}_x, \bar{p}_z, \bar{p}_\tau; s) = \; & \bar{p}_x \cdot (x - \bar{p}_\tau \cdot D(s)) + \\ & + \bar{p}_\tau \cdot D'(s) \cdot x + \bar{p}_\tau \cdot \tau + \bar{p}_z \; \cdot z - \frac{1}{2} \cdot D(s) \cdot D'(s) \cdot \bar{p}_\tau^2 \end{aligned} \tag{15}$$

and the corresponding transformation rules,

$$\begin{cases} x = \bar{x} + \bar{p}_\tau \cdot D(s) \\ z = \bar{z} \\ \tau = \bar{\tau} + \bar{p}_x \cdot D(s) - \bar{x} \cdot D'(s) \end{cases} \tag{16}$$

$$\begin{cases} p_x = \bar{p}_x + \bar{p}_\tau \cdot D'(s) \\ p_z = \bar{p}_z \\ p_\tau = \bar{p}_\tau \end{cases} \tag{17}$$

one obtains the Hamiltonian in the *new* variables $(\bar{x}, \bar{z}, \bar{\tau}, \bar{p}_x, \bar{p}_z, \bar{p}_\tau)$ as follows

$$\begin{aligned} & \bar{\mathcal{H}}(\bar{x}, \bar{z}, \bar{\tau}, \bar{p}_x, \bar{p}_z, \bar{p}_\tau; s) = \\ & = \frac{1}{2}\bar{p}_x^2 + \frac{1}{2}(g_0(s) + \kappa^2(s)) \cdot \bar{x}^2 + \frac{1}{2}\bar{p}_z^2 - \frac{1}{2} g_0(s) \cdot \bar{z}^2 - \\ & - \frac{1}{2}\kappa(s) \cdot D(s) \cdot \bar{p}_\tau^2 + V(s) \cdot \cos(\bar{\tau} + D(s) \cdot \bar{p}_x - D'(s) \cdot \bar{x}). \end{aligned} \tag{18}$$

If there is no dispersion in the cavity region ($V \cdot D = 0$), and ($V \cdot D' = 0$) the synchrotron motion $(\bar{\tau}, \bar{p}_\tau)$ is completely decoupled from the betatron motion $(\bar{x}, \bar{z}, \bar{p}_x, \bar{p}_z)$ [19].

Example 2:multipole

As a second example of nonlinear motion we consider the influence of transverse multipole fields with the following Hamiltonian:

$$\mathcal{H}(x,z,p_x,p_z,s) = \frac{1}{2}p_x^2 + \frac{1}{2}p_z^2 - \frac{e}{E_0}\cdot A_\tau(x,z,s) \tag{19}$$

The equations of motion are given by

$$\begin{cases} \frac{d}{ds}\ x = p_x \\ \frac{d}{ds}\ z = p_z \\ \\ \frac{d}{ds}\ p_x = \frac{e}{E_0}\cdot\frac{\partial A_\tau}{\partial x} = -\frac{e}{E_0}\cdot B_z(x,z,s) \\ \\ \frac{d}{ds}\ p_z = \frac{e}{E_0}\cdot\frac{\partial A_\tau}{\partial z} = \frac{e}{E_0}\cdot B_x(x,z,s). \end{cases} \tag{20}$$

The magnetic field components B_x and B_z are usually expressed in terms of the skew and normal multipole expansion coefficients a and b according to

$$(B_z + iB_x) = B_0\cdot\sum_{n=2}^{\infty}(b_n + ia_n)\cdot(x+iz)^{n-1}. \tag{21}$$

It is an easy exercise to verify, that these simple examples (18) and (19) contain the standard map [20] [21]

$$\begin{cases} \bar{\tau}(n) = \bar{\tau}(n-1) + \bar{p}_\tau(n) \\ \bar{p}_\tau(n) = \bar{p}_\tau(n-1) + V\cdot\sin(\bar{\tau}(n-1)) \end{cases} \tag{22}$$

and the quadratic map of Hénon [22] (ν is the tune of the machine)

$$\begin{pmatrix} x(n+1) \\ p_x(n+1) \end{pmatrix} = \begin{pmatrix} \cos(2\pi\nu) & \sin(2\pi\nu) \\ -\sin(2\pi\nu) & \cos(2\pi\nu) \end{pmatrix}\begin{pmatrix} x(n) \\ p_x(n) \end{pmatrix} + \begin{pmatrix} x^2(n)\cdot\sin(2\pi\nu) \\ x^2(n)\cdot\cos(2\pi\nu) \end{pmatrix} \tag{23}$$

as special cases. These maps are extensively studied in nonlinear dynamics and show a very complex behaviour. Regular and chaotic motion is intricately mixed in phase space. For the quadratic map of Hénon this is illustrated in Figure 7 (created with the help of GIOTTO [23]).

Thus one can expect, that the original system as described by (5) also shows highly nontrivial behaviour.

In order to get a better understanding of this complex dynamical phase space pattern, we will briefly repeat some facts from the *qualitative* theory of dynamical systems with an emphasis on weakly perturbed integrable Hamiltonian systems.

Figure 7: Hénon map (23) for a tune of $\nu = 0..2502$

2.2 Qualitative theory of dynamical systems

Before we start with the qualitative discussion of Hamiltonian systems we will briefly sketch the theory for general dynamical systems. The aim of this short section is to give the reader a flavour of this fascinating branch of mathematics, to illustrate the ideas and to introduce some of the concepts and results that will be needed later. For a careful and exhaustive discussion and presentation we refer the reader to [24], [25], [26] and [27]. Dynamical systems are given by a set of ordinary differential equations [2]

$$\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x}) \tag{24}$$

[2] with some modifications similar considerations are valid for discrete systems (maps) of the form $\vec{x}(n+1) = \vec{f}(\vec{x}(n))$. Later we will see how to relate these discrete systems to time continuous differential equations

where $\vec{x} = (x_1, ..., x_n)^T$ specifies a vector in the $n-$ dimensional phase space. The mathematical problem is to investigate - qualitatively and quantitatively - the time evolution of $\vec{x}(t)$ for a given initial value $\vec{x}(0)$ and vector field $\vec{f} = (f_1, ...f_n)^T$, or stated differently to understand the phase flow of the system (24), T^t, which associates to each $\vec{x}(0)$ the corresponding vector $\vec{x}(t)$ at time t,

$$\vec{x}(t) = T^t \vec{x}(0). \tag{25}$$

Varying t one obtains a solution curve of (24), i.e. a curve in $n-$ dimensional phase space passing through $\vec{x}(0)$ with a tangent vector in each point $\vec{x}$ along the trajectory that is determined by $\vec{f}(\vec{x})$. In the following we will always assume, that our dynamical system has unique solutions, which can be assured, if, for example, $\vec{f}$ is continuously differentiable [28], [29].

Special questions of a qualitative study are:

- Is the motion described by (24) stable? i.e. does a solution, that starts near a given solution always stay near this solution or even approach it (asymptotic stability)? Stability studies require the investigation of neighbouring solutions and trajectories.

- Are there periodic solutions? i.e. are there solutions such that for fixed T (periodicity)

$$\vec{x}(t + T) = \vec{x}(t)? \tag{26}$$

- How does a given solution depend on parameters (external control parameters or initial conditions)?

- Are there other types of solutions (chaotic solutions, which - as we will see - depend sensitively on the initial conditions)?

These and other questions are studied in the framework of the qualitative theory of differential equations (dynamical systems), the foundations of which have been laid by H.Poincare more than a century ago [30].

Key issues of such a theory are:

- stationary solutions

which are determined by

$$\frac{d}{dt}\vec{x}(t) = 0 = \vec{f}(x). \tag{27}$$

Investigating the linearized motion around $\vec{x}_{F.P.}$ (a solution of the algebraic equation $\vec{f}(x) = 0$ (27))

$$\vec{x} \to \vec{x}_{F.P.} + \vec{y}$$

$$\frac{d}{dt}\vec{x}(t) = \frac{d}{dt}\vec{y}(t) = \vec{f}(\vec{x}_{F.P.} + \vec{y}) \approx \vec{f}(\vec{x}_{F.P.}) + \underline{Df}(\vec{x}_{F.P.}) \cdot \vec{y}(t) + ...$$

or

$$\frac{d}{dt}\vec{y}(t) = \underline{Df}(\vec{x}_{F.P.}) \cdot \vec{y}(t) \tag{28}$$

(with $\underline{Df}$ the Jacobian matrix resulting from the Taylor- expansion of (27)) one can extract information about the local stability of these stationary solutions or fixed points. Having determined the eigenvalues and eigenspaces (eigenvectors) of the linearized system the following statements can be proven and formulated as theorems [28], [31]: The fixed point solution of (24) is (asymptotically) stable if all the eigenvalues of the linear system (Jacobi matrix) fulfill (Re designates the real part of a complex number)

$$Re\lambda_i < 0$$

and the stationary solution is unstable if at least one of the eigenvalues fulfills

$$Re\lambda_i > 0.$$

Furthermore, for hyperbolic systems, i.e. systems where there is no eigenvalue with

$$Re\lambda_i = 0$$

the phase space dynamics of the linearized system near the fixed point looks locally the same as for the nonlinear system. This is the content of the Hartman-Grobman theorem which reads:

If $\vec{x}_{F.P.}$ is a hyperbolic fixed point of (24) then there is a continuous invertible map, h, defined on some neighbourhood of $\vec{x}_{F.P.}$ which takes orbits of the nonlinear flow to those of the linear flow. This map can be chosen so that the parametrization of orbits by time is preserved.

Another powerful theorem that is available for the study of fixed points is the so-called theorem for the stable and unstable invariant manifolds, which are - roughly speaking - the nonlinear extensions of the corresponding linear eigenspaces of a fixed point. We define:
stable invariant manifold (with U some neighbourhood of $\vec{x}_{F.P.}$)

$$W^s_{local} = \{\vec{y}\,\epsilon\, U \ with\ T^t\vec{y} \ \rightarrow \vec{x}_{F.P.}\ for\ all\ t \geq 0\} \tag{29}$$

unstable invariant manifold of $\vec{x}_{F.P.}$

$$W^u_{local} = \{\vec{y}\,\epsilon\, U \ with\ T^t\vec{y} \ \rightarrow \vec{x}_{F.P.}\ for\ all\ t \leq 0\} \tag{30}$$

The theorem then reads:
If $\vec{x}_{F.P.}$ is a hyperbolic fixed point of (24) and if E^s and E^u are the stable and unstable eigenspaces of the linearized system (with $Re\lambda_i < 0$ and $Re\lambda_i > 0$) then there exist local stable and unstable manifolds W^s_{local} and W^u_{local} of the same dimension as E^s and E^u respectively. These manifolds are (respectively) tangential to E^s and E^u at the fixed point $\vec{x}_{F.P.}$.
For an illustration of the situation see Figure 8. For non-hyperbolic (el-

Figure 8: illustration of stable manifold theorem

liptic) fixed points additional information is needed to determine stability. We will come back to this point later, when we investigate the dynamics of Hamiltonian systems.

Similar statements hold also for discrete systems (maps). However, there is one important difference. Whereas the stable and unstable invariant manifolds of hyperbolic fixed points of time continuous systems cannot intersect because of the assumed uniqueness, these manifolds can cross each other for discrete mappings. In this case discrete points are mapped to discrete points, and intersection points are mapped to intersection points. These intersection points or homoclinic points [3] are the source of chaotic and complex dynamics as we will see in the following, when we study the special features of Hamiltonian systems.

Other important issues of a qualitative theory are

- periodic orbits.

The determination of periodic orbits of a dynamical system i.e. finding solutions such that

$$\vec{x}(t) = \vec{x}(t+T)$$

is in general a very difficult task. Only few exact results are available - especially for higher-dimensional ($n > 2$) systems. One problem that often arises is the investigation of the stability of a given periodic orbit. This is facilitated by the Poincaré-return map, which reduces the time continuous dynamics to a discrete map. This map is constructed as follows. Let a plane Σ intersect the periodic trajectory transversely. The intersection of the periodic orbit with Σ is $\vec{x}_0$. Let us ask what happens to a point near to $\vec{x}_0$ namely $\vec{y}_0$. If the point $\vec{y}_0$ is close enough to $\vec{x}_0$ then, because of continuity, this neighbouring point will evolve in time such, that after a certain time T_1 it crosses Σ again, say in point $\vec{y}_1$. The map that associates $\vec{y}_1$ to $\vec{y}_0$ is called Poincaré-return map. The situation is illustrated in Figure 9.

From the picture it is clear that $\vec{x}_0$ is a fixed point of the map and the stability of the periodic orbit is reflected in the stability of this fixed point. Furthermore, the characteristics of this fixed point (elliptic or hyperbolic) can be used to characterize the periodic orbit.

Further issues of interest of a qualitative theory of dynamical systems are

[3] the possible intersections of the corresponding invariant manifolds of *different* hyperbolic fixed points are called heteroclinic points

- finding complex (chaotic) trajectories
- investigation of the dependence on parameters (bifurcation theory).

Figure 9: Poincaré-return map

After these remarks about general dynamical systems we will concentrate now on Hamiltonian systems. For these systems the right hand side of (24), $\vec{f}$, is determined by the Hamilton function $\mathcal{H}$, which depends on the generalized coordinates $\vec{q} = (q_1, ...q_n)^T$ and the generalized momentum vector $\vec{p} = (p_1, ...p_n)^T$. Without proof we mention two important characteristics of Hamiltonian dynamical systems: conservation of phase space volumes and symplectic structure of the phase flow [32].

Excellent and detailed reviews of the qualitative theory of Hamiltonian systems can be found in [20], [21],[33],[34],[35],[36].

The easiest way to investigate Hamiltonian systems is via a map. The general reduction of a Hamiltonian system to a nonlinear mapping has been a well-known procedure since Poincaré (1890).

Consider for example a two-dimensional Hamiltonian system without explicit time (or s-) dependence $\mathcal{H}(q_1, q_2, p_1, p_2)$. The corresponding phase space is four-dimensional, and since $\mathcal{H}$ itself is a constant of the motion, the physically accessible phase space is three-dimensional. Consider a surface Σ in this three-dimensional -not necessarily Euclidean- space as depicted for example in Figure 10. The bounded particle motion induced by the Hamiltonian $\mathcal{H}$ will generally intersect this surface in different points $(P_0, P_1, ...P_n...)$.

If one is not interested in the fine details of the orbit but only in the behaviour over longer time scales it is sufficient to consider the consecutive points $(P_0, P_1....)$ of intersection. These contain complete information on the Hamiltonian system. In this sense one has reduced the Hamiltonian dynamics to a mapping of Σ to itself, which is in general nonlinear (Poincaré surface of section technique). The Hamiltonian character is reflected in the symplectic structure of the map. Symplectic means that the Jacobian $\underline{J}$ of the map is a symplectic matrix with

$$\underline{J}^T \cdot \underline{S} \cdot \underline{J} = \underline{S} \tag{31}$$

where $\underline{J}^T$ is the transpose of $\underline{J}$ and where $\underline{S}$ is the symplectic unity

$$\begin{pmatrix} \underline{0} & \underline{1} \\ -\underline{1} & \underline{0} \end{pmatrix} \tag{32}$$

($\underline{1}$ designates the unit matrix). Similar mappings (stroboscopic maps) can also be derived for Hamiltonian systems with explicit periodic time (s-) dependence (this is normally the case in storage rings).

Another important fact and, after the work of Chirikov [21] one of the few beacons among an otherwise still dense mist of diverse phenomena, is the KAM-theorem (Kolmogorov,Arnold,Moser see for example [20]). We will only illustrate this theorem in the two-dimensional case and instead of concentrating on mathematical rigour we will discuss its physical implications.

Figure 10: Poincaré surface of section method

Consider first the bounded motion of a two-dimensional autonomous (not explicitly time (s-) dependent) Hamiltonian system which is integrable.

Roughly speaking, an n-dimensional system $\mathcal{H}(q_1, q_2, ..., q_n, p_1, p_2, ..., p_n)$ is integrable if there exists a canonical transformation to action-angle variables $(I_1, I_2, ..., I_n, \Theta_1, \Theta_2, ..., \Theta_n)$ such that the transformed Hamiltonian depends only on the n (constant) action variables $(I_1, I_2, ..., I_n)$ alone. For the two-dimensional case under consideration this implies, that $\mathcal{H}(q_1, q_2, p_1, p_2)$ is transformed into $\mathcal{H}(I_1, I_2)$ with the corresponding equations of motion

$$\begin{cases} \frac{d}{ds}\ I_1 = 0 \\ \frac{d}{ds}\ I_2 = 0 \\ \\ \frac{d}{ds}\ \Theta_1 = \frac{\partial \mathcal{H}}{\partial I_1} = \omega_1(I_1, I_2) = \text{const} \\ \\ \frac{d}{ds}\ \Theta_2 = \frac{\partial \mathcal{H}}{\partial I_2} = \omega_2(I_1, I_2) = \text{const.} \end{cases} \tag{33}$$

The motion is restricted to a two-torus, parametrized by the two angle variables Θ_1 and Θ_2, as depicted in Figure 11.

Figure 11: surface of section method for a two-dimensional integrable system

As surface of section one can choose the $(I_1 - \Theta_1)$ plane for Θ_2=constant. In this surface of section, which may be chosen to be just the plane of the page, the motion of the integrable two-dimensional system looks very simple. During the motion around the torus from one crossing of the plane to the next the radius of the torus (action variable) does not change (see (33))

$$I_1(n) = I_1(n-1) \tag{34}$$

and the angle Θ_1 changes according to (see (33))

$$\Theta_1(n) = \Theta_1(n-1) + \omega_1 \cdot T \tag{35}$$

where T is just the revolution time in Θ_2-direction from one intersection of the plane to the next, namely

$$T = \frac{2\pi}{\omega_2} \, . \tag{36}$$

Thus, for an integrable system one obtains the so-called twist-mapping

$$\begin{cases} I_1(n) = I_1(n-1) \\ \Theta_1(n) = \Theta_1(n-1) + 2\pi \cdot \alpha(I_1(n)). \end{cases} \tag{37}$$

The term $\alpha = \frac{\omega_1}{\omega_2}$ is called the winding number and is the ratio of the two frequencies of the system. In general α will depend on the actions. If α is irrational, the $\Theta_1(n)$ form a dense circle while if α is rational the $\Theta_1(n)$ close after a finite sequence of revolutions (periodic orbit or resonance).Thus, there are invariant curves (circles) under the mapping which belong to rational and irrational winding numbers. What happens now if a perturbation is switched on, i.e. if

$$\begin{cases} I_1(n) = I_1(n-1) + \varepsilon \cdot f(I_1(n), \Theta_1(n-1)) \\ \Theta_1(n) = \Theta_1(n-1) + 2\pi \cdot \alpha(I_1(n)) + \varepsilon \cdot g(I_1(n), \Theta_1(n-1)) \;\; ? \end{cases} \tag{38}$$

In particular, can one still find invariant curves? The KAM-theorem says that this is indeed the case if the following conditions are fulfilled

- perturbation must be weak
- $\alpha = \frac{\omega_1}{\omega_2}$ must be sufficiently irrational, i.e. $|\alpha - \frac{p}{q}| \geq \frac{k(\varepsilon)}{q^{2+\delta}}$ with p, q integers, $\delta > 0$ and $k(\varepsilon) \to 0$ for $\varepsilon \to 0$

together with some requirements of differentiability and periodicity for f and g. For further details see for example [20], [36]. Under these assumptions most of the unperturbed tori survive the perturbation although in slightly distorted form.

The rational and some nearby tori, however, are destroyed, only a finite number of fixed points of the rational tori survive - half of them are stable

(elliptic orbits around this fixed point), half of them are unstable (hyperbolic orbits).This is a consequence of the Poincaré-Birkhoff fixed point theorem [33], [37].

The hyperbolic fixed points with their stable and unstable branches, which generally intersect in the homoclinic points, see Figure 12, are the source of chaotic motion in phase space, i.e. motion which is extremely sensitive to the variation of initial conditions. From a historical point of view it is interesting to note, that these facts were already known by Poincaré 100 years ago [30], [33].

The motion around the elliptic fixed points can be considered as motion around a torus with smaller radius and the arguments used till now can be repeated on this smaller scale giving rise to the - now well known - schematic picture shown below (see Figure 13 ,"chaos-scenario" of weakly perturbed two-dimensional twist maps).

Figure 12: homoclinic intersections of stable and unstable branch of fixed point $\vec{x}_0$

Figure 13: chaos scenario

Thus, the phase space pattern of a weakly perturbed integrable two-

dimensional system looks extremely complicated. There are regular orbits confined to tori with chaotic trajectories delicately distributed among them. As the strength of the perturbation increases more and more KAM tori break up giving rise to larger and larger chaotic regions. This onset of large scale chaos has been the subject of many studies [20],[38]. For the standard map (see equation (22)) the situation is depicted in Figure 14.

One comment is pertinent at this point - two-dimensional systems are special in that the existence of KAM circles implies exact stability for orbits starting inside such an invariant curve. Since these trajectories cannot escape without intersecting the KAM tori, they are forever trapped inside.

The situation is much more complex and less well-understood for higher-dimensional systems like our storage ring (six-dimensional map, or three-dimensional explicitly s- dependent Hamiltonian system). The KAM theorem predicts three-tori in six-dimensional phase space, four-tori in eight-dimensional phase space etc. In this case chaotic trajectories can in principle always escape and explore all the accessible phase space although the motion can be obstructed strongly by existing tori. Chaotic regions can form a connected web along which the particle can diffuse, as has been demonstrated by Arnold (Arnold diffusion see for example [20], [38]).We will come back to this point later.

Figure 14: $(\bar{p}_\tau - \bar{\tau})$-phase space plot of the standard map (22) showing global chaos for $V = 3.3$

Figure 15 shows examples of regular and chaotic trajectories in a realistic

model of a storage ring [39],[40]. We have used the characteristic Lyapunov exponent λ to distinguish between regular and chaotic motion [20], [41],[42]:

$$\lambda = \lim_{t\to\infty, d(0)\to 0} \frac{1}{t} \cdot \ln(\frac{|d(t)|}{|d(0)|}) \tag{39}$$

Figure 15: regular and chaotic trajectories in a realistic model of a storage ring and the evolution of the distance of two adjacent initial conditions

$d(t)$ describes how the (Euclidean) distance between two adjacent phase space points evolves with time and $d(0)$ is the initial distance. In a chaotic region of phase space this distance will grow exponentially fast and a positive

Lyapunov-exponent λ is a quantitative measure for this separation. For the details of an explicit numerical calculation of the characteristic Lyapunov-exponents (for continuous and discrete dynamical systems) the reader is referred to [20], [41], [42].

Another method derived from the homoclinic structure of nearly integrable symplectic mappings is due to Melnikov, and this method belongs to one of the few analytical tools for investigating chaotic behaviour. It is applicable to dynamical systems of the form

$$\begin{cases} \frac{d}{ds}\ \vec{x}(s) = \vec{F}(\vec{x}) + \varepsilon \cdot \vec{G}(\vec{x}, s) \\ \vec{x} = (x_1, x_2)^T \in \mathbf{R^2} \\ s \in \mathbf{R} \\ \vec{F} = (F_1, F_2)^T \\ \vec{G} = (G_1, G_2)^T \end{cases} \tag{40}$$

where $\vec{F}$ usually describes a Hamiltonian system. The perturbation $\varepsilon\vec{G}$, which may also be weakly dissipative, is periodic in s, and the unperturbed system

$$\frac{d}{ds}\ \vec{x}(s) = \vec{F}(\vec{x}) \tag{41}$$

has a homoclinic orbit belonging to a saddle point $\vec{x}_0$. Homoclinic orbit means smooth joining of the stable and unstable branch of the saddle or hyperbolic fixed point (see Figure 16). The Melnikov method enables a kind of directed distance between the stable and unstable branch of the perturbed saddle $\vec{x}_0^{\varepsilon}$ to be calculated and thus allows the existence of homoclinic points to be predicted, a prerequisite of chaotic dynamics. A derivation of the Melnikov function and further details and applications can be found in [43] .

Figure 16: homoclinic orbit belonging to saddle $\vec{x}_0$

Remark:

High resolution 3D-colour graphics can be a very helpful tool for visualizing the dynamics of nonlinear four-dimensional mappings [40] [44]. Toy models like

$$\vec{x}(n+1) = \underline{R} \cdot \vec{x}(n) + \vec{f}(\vec{x}(n)) \tag{42}$$

with

$$\vec{x}(n) = \begin{pmatrix} x(n) \\ p_x(n) \\ z(n) \\ p_z(n) \end{pmatrix} \tag{43}$$

and

$$\underline{R}(\phi, \theta) = \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 & 0 \\ -\sin(\phi) & \cos(\phi) & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \\ 0 & 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix} \tag{44}$$

and

$$\left\{ \begin{array}{c} \vec{f}(\vec{x}(n)) = \begin{pmatrix} 0 \\ \frac{\partial f}{\partial x}(x(n+1), z(n+1)) \\ 0 \\ \frac{\partial f}{\partial z}(x(n+1), z(n+1)) \end{pmatrix} \\ (f = f(x,z)) \end{array} \right. \tag{45}$$

can help to get a better understanding of the break-up mechanism of invariant tori and the role periodic orbits play in this process [44],[45]. $((n+1)$ periodic orbits are defined by: $\vec{x}(n+1) = \underline{T}(\vec{x}(n)) = \vec{x}(0)$ where $\underline{T}$ is some nonlinear (symplectic) map).

In the last chapter we have seen that the single particle dynamics of a proton in a storage ring can be modelled by nonintegrable Hamiltonians. The qualitative theory we have briefly sketched predicts a very rich and complicated phase space structure - regular and chaotic regions are intricately mixed in phase space.

When applying these concepts to accelerators one is immediately faced with questions such as:
What is the relevance of chaos for the practical performance of a storage ring? How do KAM tori break up as the strength of the nonlinearity increases? Can we somehow estimate the size of the chaotic regions in phase space? What is the character of the particle motion in this region? Can it be described by diffusion-like models? Is it possible to calculate escape rates of the particle if it is in such a chaotic region of phase space?

A quantitative analysis of these and other questions makes extensive use of perturbation theory and numerical simulations of the system.

2.3 Perturbation theory

The main goal of perturbation theory is to study systems of the form

$$\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x},t) + \varepsilon\vec{g}(\vec{x},t) \tag{46}$$

where the solution of

$$\frac{d}{dt}\vec{x}(t) = \vec{f}(\vec{x},t)$$

is known (unperturbed problem) and where $\varepsilon\vec{g}(\vec{x},t)$ is a small perturbation. For example, one could be interested in a Taylor expanded system such as

$$\frac{d}{dt}\vec{x}(t) = \underline{A}\cdot\vec{x}(t) + \vec{v}(\vec{x}) \tag{47}$$

with $\vec{x} = (x_1, ...x_n)^T$, $\underline{A}$ an n x n matrix and $\vec{v} = (v_1, ..., v_n)^T$ with

$$v_i(\vec{x}) = \sum_{n_1,..,n_n} a_{i,n_1....n_n}\cdot x_1^{n_1}\cdots x_n^{n_n}. \tag{48}$$

The linear system is easily solved. However, what can be said about the perturbed (nonlinear) dynamics?

Perturbation theory usually consists of formal manipulations such as coordinate transformations to a set of new variables. In the ideal case this transformation makes the system exactly solvable, for example, if the transformed equation is linear in the new variables. However this is rarely the case. What one can hope for, in general, is that the transformed system is somehow easier to handle (in a way to be specified). These formal manipulations lead -as we will see- to subtle mathematical problems such as

- divergence of series and
- estimates of time scales over which the perturbative methods give valid results.

In the following discussion we will only mention these problems and we will illustrate some of the *formal* steps and basic results for weakly perturbed Hamiltonian systems. The treatment is far from being mathematically rigorous. Readers interested in a careful mathematical analysis of the validity of perturbation theory should consult, for example, [46], [47] or [48].

Before we enter into detail we will briefly repeat some facts from the linear theory of particle motion in storage rings (linear theory of synchro-betatron oscillations see [15], [49] and Appendix B).

In simple cases, as for example pure x- or z-motion without any coupling, the system is described by Floquet oscillators of the form

$$\mathcal{H}(q,p,s) = \frac{1}{2}\cdot p^2 + \frac{1}{2}\cdot g_0(s)\cdot q^2 \tag{49}$$

with $p = p_x,\ p_z,\ q = x,\ z$ and $g_0(s) = g_0(s+L)$ periodic function of circumference L.

It is well known that these Floquet type systems can be solved exactly. Using the optical functions $\alpha(s)$,$\beta(s)$ and $\gamma(s)$ defined by the following set of differential equations

$$\frac{d}{ds}\ \alpha(s) = -\gamma(s) + \beta(s)\cdot g_0(s) \tag{50}$$

$$\frac{d}{ds}\ \beta(s) = -2\cdot\alpha(s) \tag{51}$$

$$\frac{d}{ds}\ \gamma(s) = 2\cdot\alpha(s)\cdot g_0(s). \tag{52}$$

one can find a canonical transformation to action angle variables I and Θ such that the Hamiltonian in equation (49) is transformed into [49], [50]

$$\bar{\mathcal{H}}(\Theta, I) = \frac{2\pi\cdot Q}{L}\cdot I \tag{53}$$

with

$$Q = \frac{1}{2\pi} \cdot \int_0^L \frac{ds'}{\beta(s')} \tag{54}$$

(Q is the so-called tune of the machine) and

$$I = \frac{q^2}{2\beta(s)} \cdot \{1 + (\frac{\beta(s) \cdot p}{q} + \alpha(s))^2\} \tag{55}$$

(I is called Courant-Snyder invariant see [52]).

In realistic cases there is always some coupling between the different degrees of freedom and the situation is more complicated. In these cases machine physicists rely on the one-turn matrix $\underline{M}$ relating some initial state phase space vector $\vec{y}(s_{in})$ to the final state vector $\vec{y}(s_{fin})$ after one complete revolution around the ring

$$\vec{y}(s_{fin} = s_{in} + L) = \underline{M}(s_{in} + L, s_{in}) \cdot \vec{y}(s_{in}). \tag{56}$$

In general $\vec{y}$ is six-dimensional and consists of the phase space coordinates $x, z, \tau, p_x, p_z, p_\tau$. The linear one-turn map $\underline{M}(s_{in} + L, s_{in})$ contains all the information about the system. For example the stability of the particle motion depends on the eigenvalue spectrum of the (symplectic) matrix $\underline{M}$ [15] - stability is only guaranteed if the eigenvalues lie on the complex unit circle (see also Figure 17).

What happens now if we perturb such a linear system with some nonlinear terms? How can we extend the linear analysis to the nonlinear case?

In simple models we can start with a perturbed Hamiltonian

$$\mathcal{H}(q, p, s) = \mathcal{H}_0(q, p, s) + \varepsilon \cdot \mathcal{H}_1(q, p, s) \; . \tag{57}$$

Using the action angle variables of the unperturbed system one can apply conventional Hamiltonian perturbation theory [49], [50], [51], [53], which we will sketch in a moment. The advantages of such an approach are that -in low order of perturbation theory- one easily gets simple analytical expressions for interesting machine parameters of the perturbed system in terms of the unperturbed quantities. The price one has to pay, however, is an oversimplification of the problem. Realistic machines with all their nonlinearities

and perturbations are extremely complex and cannot be handled efficiently in such a way. In such a case one should try to extend the concept of the one-turn matrix to the nonlinear case. This "contemporary" approach of Hamiltonian-free perturbation theory for particle dynamics in storage rings has been strongly advanced by A.Dragt and E.Forest and a recent description can be found in [54], [55], [56] and [57].

Figure 17: eigenvalue spectrum of a six-dimensional symplectic matrix

As mentioned above the basic idea of perturbation theory (common to both the direct Hamiltonian (or time-continuous) approach and the contemporary Hamiltonian-free approach) is to find - in a way to be specified - new variables, such that the system becomes solvable or at least easier to handle in this new set.

Let us first consider the Hamiltonian (time-continuous) case. Various realizations exist for this kind of perturbation theory:Poincaré-von Zeipel [49], [50],[51],[53],[58], Lie methods [59], [60], and normal form algorithms [61],

[62],[63]. Here we will illustrate the Poincaré-von Zeipel method. Assume our Hamiltonian is of the form

$$\mathcal{H}(\vec{q},\vec{p}) = \mathcal{H}_0(\vec{q},\vec{p}) + \varepsilon \cdot \mathcal{H}_1(\vec{q},\vec{p}) \tag{58}$$

where the vectors for the coordinates and momenta $\vec{q}$ and $\vec{p}$ may have arbitrary dimension (3 in the storage ring case)

$$\vec{q} = \begin{pmatrix} q_1 \\ . \\ . \\ . \\ q_n \end{pmatrix}, \qquad \vec{p} = \begin{pmatrix} p_1 \\ . \\ . \\ . \\ p_n \end{pmatrix}. \tag{59}$$

Introducing the action angle variable vectors $\vec{I}$ and $\vec{\Theta}$ of the unperturbed system $\mathcal{H}_0$

$$\vec{I} = \begin{pmatrix} I_1 \\ . \\ . \\ . \\ I_n \end{pmatrix}, \qquad \vec{\Theta} = \begin{pmatrix} \Theta_1 \\ . \\ . \\ . \\ \Theta_n \end{pmatrix} \tag{60}$$

the Hamiltonian (58) can be rewritten in the form

$$\mathcal{H}(\vec{I},\vec{\Theta}) = \mathcal{H}_0(\vec{I}) + \varepsilon \cdot \mathcal{H}_1(\vec{I},\vec{\Theta}). \tag{61}$$

The problem would be trivial, if we could find a transformation to new variables $\vec{J} = (J_1 ... J_n)^T$ and $\vec{\psi} = (\psi_1 ... \psi_n)^T$ such that the transformed Hamiltonian depends only on the new action variables $J_1 ... J_n$ alone. Since most Hamiltonian systems are nonintegrable [33],[34] this cannot be done exactly. What one can achieve is to push the nonlinear perturbation to higher and higher orders in ε i.e. after a sequence of N canonical transformations

$$\mathcal{H}(\vec{I},\vec{\Theta}) = \mathcal{H}_0(\vec{I}) + \varepsilon \cdot \mathcal{H}_1(\vec{I},\vec{\Theta})$$

is transformed into a form given by [4]

$$\begin{cases} \bar{\mathcal{H}}(\vec{J}^{(N)}, \vec{\psi}^{(N)}) = \bar{\mathcal{H}}_0(\vec{J}^{(N)}) + \varepsilon^{N+1} \cdot R_N(\vec{\psi}^{(N)}, \vec{J}^{(N)}) \\ \bar{\mathcal{H}}_0(\vec{J}^{(N)}) = \sum_{i=0}^{N} \varepsilon^i \cdot \mathcal{H}_0^{(i)}(\vec{J}^{(N)}) \\ \mathcal{H}_0^{(0)}(\vec{J}^{(N)}) = \mathcal{H}_0(\vec{J}^{(N)}) \end{cases} \tag{62}$$

where

$$\vec{J}^{(N)} = \begin{pmatrix} J_1^{(N)} \\ . \\ . \\ . \\ J_n^{(N)} \end{pmatrix}, \qquad \vec{\psi}^{(N)} = \begin{pmatrix} \psi_1^{(N)} \\ . \\ . \\ . \\ \psi_n^{(N)} \end{pmatrix} \tag{63}$$

are the new variables after N transformations. Neglecting the remainder $\varepsilon^{N+1} \cdot R_N$ (which is of order ε^{N+1} i.e. one order higher than the first part in equation (62)) the system is then trivially solvable.

For example in first order of perturbation theory this is achieved by a canonical transformation (depending on the *old* coordinates $\vec{\Theta}$ and the *new* momenta $\vec{J}$)

$$F_2(\vec{\Theta}, \vec{J}) = \vec{\Theta} \cdot \vec{J} + \varepsilon \cdot S_1(\vec{\Theta}, \vec{J}) \tag{64}$$

where $S_1(\vec{\Theta}, \vec{J})$ is given by

$$S_1(\vec{\Theta}, \vec{J}) = -\frac{1}{i} \cdot \sum_{\vec{n} \neq 0} \frac{\mathcal{H}_{1,\vec{n}}(\vec{J})}{\vec{\omega} \cdot \vec{n}} \cdot \exp(i\vec{n} \cdot \vec{\Theta}). \tag{65}$$

$\vec{\omega}$ designates the frequency vector

$$\vec{\omega}(\vec{J}) = \frac{\partial \mathcal{H}_0(\vec{J})}{\partial \vec{J}} \tag{66}$$

[4] For Hamiltonian systems one can do even better. As shown by Kolmogorov and Arnold [46] one can push the phase dependence after N transformations not only to terms of order ε^{N+1} but even to terms of order ε^{2^N} (superconvergence)

and $\mathcal{H}_{1,\vec{n}}(\vec{J})$ is defined by the Fourier expansion of $\mathcal{H}_1(\vec{\Theta},\vec{J})$

$$\mathcal{H}_1(\vec{\Theta},\vec{J}) = \sum_{\vec{n}} \mathcal{H}_{1,\vec{n}}(\vec{J}) \cdot \exp(i\vec{n}\cdot\vec{\Theta}). \tag{67}$$

However, there is a serious problem concerning the convergence of our perturbative approach: even if we exclude the *nonlinear resonances* $\vec{n}\cdot\vec{\omega}=0$ in equation (65) the infinite sum always contains n_i's such that the denominator in equation (65) can become arbitrarily small (small divisor problem) making this whole enterprise very doubtful. Generally these expansions diverge. Nevertheless the hope is that these expansions can be useful as asymptotic series and that the new invariants $(J_1,...J_n)$ calculated in this way approximate in some sense our original system. However, for finite perturbations, there is no proof for the accuracy, if any, of such an approximation. So some care is always needed when one applies perturbation theories of this kind. A careful analysis of the convergence properties of (65) leads immediately to the heart of the KAM theory and requires sophisticated mathematical tools, which are far beyond the scope of this survey [46].

Before we discuss perturbative methods for discrete time systems (maps), we will briefly illustrate how single isolated resonances can be treated. In this case the dynamics can often be reduced to the well known pendulum Hamiltonian. The computational steps are very carefully described in [21] and [64]. We will illustrate these steps here with a simple model, namely a non-autonomous, one- degree of freedom system with explicit periodic time dependence. An extension to n degrees of freedom is rather straightforward and the details can be found in the above mentioned references [21], [64].

Starting point is a Hamiltonian of the form (periodically driven system with period Ω)

$$\mathcal{H}(I,\theta,t) = \mathcal{H}_0(I) + \varepsilon\cdot\sum_{n,m} V_{n,m}(I)\cdot\exp(in\theta+im\phi) \tag{68}$$

with $\phi=\Omega t$. The first step is

- fixing the nonlinear resonance of interest

for example,

$$l\omega(I^r) - k\Omega = 0 \tag{69}$$

with $\omega(I) = \frac{\partial \mathcal{H}_0}{\partial I}$, k, l fixed integers and I^r the resonant action defined by (69). The next step is

- transformation to a resonant coordinate system and shift of the origin of the action variable I with the help of the following canonical transformation

$$F_2(\theta, J, t) = (l\theta - k\phi) \cdot J + \theta \cdot I^r. \tag{70}$$

The corresponding transformation equations read

$$\psi = \frac{\partial F_2}{\partial J} = l\theta - k\phi \tag{71}$$

$$I = \frac{\partial F_2}{\partial \theta} = lJ + I^r \tag{72}$$

$$\frac{\partial F_2}{\partial t} = -k\Omega J. \tag{73}$$

In the new variables J, ψ the Hamiltonian (68) takes the form

$$\begin{aligned} \mathcal{H}(\psi, J, t) &= \mathcal{H}_0(lJ + I^r) + \\ &+ \varepsilon \sum_{m,n} V_{n,m}(lJ + I^r) \cdot \exp(i\frac{1}{l}(n\psi + (nk + ml)\phi)) - k\Omega J. \end{aligned} \tag{74}$$

For the next step we assume that $\phi = \Omega t$ is a fast variable compared to the (by definition) slowly varying resonance variable ψ so that we can

- average over ϕ.

This gives

$$\mathcal{H}(J, \psi) = \mathcal{H}_0(J) + \varepsilon \sum_p V_{p \cdot l, -p \cdot k}(J) \exp(ip\psi) - k\Omega J. \tag{75}$$

We assume that $V_{0,0} = 0$, $V_{-l,k} = V_{l,-k}$ and we absorb a factor 2 in the Fourier coefficients $V_{p \cdot l, -p \cdot k}$. Keeping only the lowest order Fourier harmonics we finally

- expand the Hamiltonian (75) around the resonant value I^r

and we obtain the well known pendulum Hamiltonian

$$\mathcal{H}(J,\psi) \approx \frac{J^2}{2M} + \varepsilon V_{l,-k}(I^r)\cos(\psi) \tag{76}$$

with

$$M^{-1} = l^2(\frac{\partial^2}{\partial I^2}\mathcal{H}_0(I))_{I=I^r}. \tag{77}$$

A phase space plot of the dynamics described by (76) is shown in Figure 18. The separatrix width of (76) defines the width of our resonance (69) namely

$$(\Delta I^r) = 2l(\varepsilon M V_{l,-k})^{\frac{1}{2}}. \tag{78}$$

This type of analysis has been used to investigate nonlinear synchro-betatron

Figure 18: phase space plot of nonlinear pendulum with separatrix

oscillations in [50] and further applications will be mentioned later when we consider the onset of large scale chaotic motion in phase space using Chirikovs resonance overlap criterion [21].

One remark should be made at this stage. Deriving (76) we have assumed that the unperturbed Hamiltonian $\mathcal{H}_0$ depends nonlinearly on the action variable I. In the case of a linear dependence (this is usually the case in accelerator applications) the validity of the computational steps described above has to be checked very carefully [21].

Let us now briefly illustrate the perturbative techniques for mappings [54],[65], [66],[67],[68],[69].

As mentioned already in the introduction, an accelerator acts as a nonlinear device and an initial state phase space vector $\vec{y}(s_{in})$ is nonlinearly related to the final state $\vec{y}(s_{fin})$ by a symplectic map or phase flow

$$\vec{y}(s_{fin}) = \mathcal{M}(\vec{y}(s_{in})) \ . \tag{79}$$

Let us assume this map can be Taylor expanded up to some order N with respect to $\vec{y}(s_{in})$

$$y_i(s_{fin}) = \sum_j \ A_{ij} \cdot y_j(s_{in}) + \sum_{jk} \ B_{ijk} \cdot y_j(s_{in}) \cdot y_k(s_{in}) + ... \tag{80}$$

with the transfer matrix or aberration coefficients A_{ij}, B_{ijk} etc. Because of the symplectic condition of $\mathcal{M}$ (Hamiltonian dynamics) these coefficients are strongly interrelated, and a truncation of the Taylor series usually results in a violation of symplecticity. Dragt and Finn [70] have shown that Lie algebraic techniques can be very convenient and efficient for parametrizing and handling maps like (80) (see also [71]). The factorization theorem [68] for example states that $\mathcal{M}$ can be expressed as a product of *Lie transforms*

$$\mathcal{M} = e^{:f_1:} \ \cdots \ e^{:f_k:} \ \cdot \cdot \tag{81}$$

where $: f_i :$ denotes a *Lie operator* related to a homogeneous polynomial of degree i in the variables $y_j(s_{in})$ and $: f :$ acts on the space of phase space functions g via the Poisson bracket operation of classical mechanics [53]

$$: f : g \equiv \{f, g\} \ . \tag{82}$$

Example

The map $e^{:\frac{a}{3}x^3:}$ gives the known expression for a sextupole in thin lens (kick) approximation [68], [69]:

$$\begin{pmatrix} x(s_{fin}) \\ p_x(s_{fin}) \end{pmatrix} = \begin{pmatrix} x(s_{in}) \\ p_x(s_{in}) + a \cdot x^2(s_{in}) \end{pmatrix} \tag{83}$$

because

$$\begin{cases} e^{:\frac{a}{3}x^3(s_{in}):}\ x(s_{in}) = x(s_{in}) \\ e^{:\frac{a}{3}x^3(s_{in}):}\ p_x(s_{in}) = p_x(s_{in}) + a \cdot x^2(s_{in}) \end{cases} \tag{84}$$

In principle one could now try to construct the one-turn map for a non-linear accelerator using these Lie algebraic tools. However, beyond an order $N = 3$ in the Taylor expansion, this becomes incredibly tedious and complicated. So we will discuss a more efficient way of obtaining Taylor expanded maps for one turn later.

The advantage of using maps in the form (81) is formal and lies in the Lie algebraic tools that are available for treating these systems. There is an elegant extension of the normal form theory to such cases [72]. The problem is -roughly stated- that given a map $\mathcal{M}$ one has to determine a map $\mathcal{A}$ such that

$$\mathcal{N} = \mathcal{A} \cdot \mathcal{M} \cdot \mathcal{A}^{-1} \tag{85}$$

is as simple as possible. Simple means that the action of the map is simple. We can easily illustrate this fact with the following map which describes the action of a single multipole in kick approximation [66],[73]

$$\begin{pmatrix} x(n+1) \\ p_x(n+1) \end{pmatrix} = \begin{pmatrix} \cos(2\pi \cdot Q) & \sin(2\pi \cdot Q) \\ -\sin(2\pi \cdot Q) & \cos(2\pi \cdot Q) \end{pmatrix} \cdot \begin{pmatrix} x(n) \\ p_x(n) + \varepsilon \cdot x^p(n) \end{pmatrix} \tag{86}$$

In complex notation $z = x + ip_x$ equation (86) can be rewritten as

$$z(n+1) = \exp(-i \cdot 2\pi \cdot Q) \cdot \{z(n) + \frac{i\varepsilon}{2^p} \cdot (z(n) + z^\star(n))^p\} \tag{87}$$

where $z^\star$ designates the complex conjugate of z.

Finding the map $\mathcal{A}$ implies that one transforms to a new set of variables

$$z \longrightarrow \xi \tag{88}$$

such that in the new variables equation (87) takes the following form

$$\xi(n+1) = \exp\{-i\Omega(\xi(n) \cdot \xi^\star(n))\}\ \xi(n) + \text{terms} \quad \geq \varepsilon^2. \tag{89}$$

Now, up to order ε, the action of the map is very simple - it is just a rotation in the ξ-plane with a frequency (winding number) Ω which depends on the distance from the origin (see Figure 19) [5]. This kind of perturbative analysis has been developed in detail in [54], [65] and has led to a powerful strategy for investigating the nonlinear motion of particles in storage rings. We will come back to this point after the description of numerical simulations in the next section.

Figure 19: normal form theory of maps

2.4 Numerical simulations and particle tracking

The main idea of numerical simulations is to track particles over many revolutions in realistic models of the storage ring and to observe the amplitude of the particle at a special point s_0 [74], [75], [76]. Given the initial amplitude $\vec{y}(s_0) = (x(s_0), z(s_0), \tau(s_0), p_x(s_0), p_z(s_0), p_\tau(s_0))$ one needs to know $\vec{y}(s_0 + n \cdot L)$ for n of the order of 10^9 (corresponding to a storage time of a particle of about 10 hours in HERA). Different methods and codes have been developed to evaluate $\vec{y}(s_0+n\cdot L)$. Among others there are COSY INFINITY [77],TEAPOT [78],MARYLIE [79],TRANSPORT [80], and RACETRACK [81]. We will not go into the details of these codes, we restrict ourselves to some general remarks and facts instead.

[5] As in the time continuous case, resonances need a special treatment [54], [65]

An ideal code should be fast and accurate. It should allow for six-dimensional phase space calculations, thus allowing all kinds of coupling between the synchrotron and betatron oscillations. Error simulations of the storage ring should be possible as well as the calculation of interesting physical quantities such as tunes, (perturbed) invariants, nonlinear resonance widths etc.

One way of achieving this is by naive element to element tracking.

Figure 20: beam line

One has to solve the corresponding equations of motion for each element (linear or nonlinear). In each case the symplectic structure of the underlying Hamiltonian system has to be preserved by using suitable symplectic integration schemes [82], [83]. Such a code would be accurate but also extremely slow especially for large colliders like HERA and the LHC. A modification of these element-to-element tracking codes is the so-called kick approximation.Nonlinear elements described for example by terms

$$\mathcal{H}_1 = \sum_{n,m} a_{n,m}(s) \cdot x^n \cdot z^m$$

in the Hamiltonian are replaced by

$$\mathcal{H}_1 = \sum_{n,m} \bar{a}_{n,m} \cdot x^n \cdot z^m \cdot \delta_p(s - s_i)$$

where s_i denotes the localization of the nonlinear kick. These codes can speed up the calculations considerably and they also preserve the symplectic structure of the underlying equations automatically. However, one has to check carefully the accuracy of this kind of approximation.

A typical numerical investigation of particle motion in nonlinear storage rings then comprises the following steps:

1. specification of the storage ring model by a Hamiltonian $\mathcal{H}$

2. numerical integration of the corresponding equations of motion for one complete revolution using symplectic integrators

3. extraction of the Taylor expanded form of the one-turn map (see equation (80)) from this calculation

4. use of this map for long-time tracking [6] and for a perturbative analysis (to get physical quantities of interest such as invariants, perturbed frequencies and tunes of the synchro-betatron oscillations, nonlinear resonance widths etc.).

Step 3 is elegantly solved by using the powerful automatic differentiation package (often called differential algebra approach) developed by M. Berz [85], [86] [87], [88], [89]. Figure 21 shows a flow chart for this approach.

Besides the symplectic problems with the Taylor expanded maps there are also some other problems related to tracking namely the unavoidable rounding errors of the computers and the limited CPU time. The rounding errors depend on the number system used by the compiler and they can also destroy the symplectic structure of the nonlinear mappings. Thus these rounding errors can simulate non-physical damping (anti-damping) effects [90]. In order to estimate the order of magnitude of these effects, one can switch to a higher precision structure in the computer hardware or software and observe the differences. Another way is to compare the differences between forward tracking of the particles and backward tracking [91].

The limited CPU time could be improved by developing special tracking processors [92], [93]. Special processors have been successfully used in celestial mechanics for studies of the long-time stability of the solar system [94].

Besides these technical problems there are also some physical problems related to the evaluation and interpretation of the tracking data. For example, fast instabilities with an exponential increase of amplitudes beyond a certain boundary (dynamic aperture) can easily be detected, whereas slow, diffusion-like processes which are very important for an understanding of the long-time dynamics are much more difficult to detect.

[6] for long-term tracking suitable symplectification algorithms are required [76], [84]

Figure 21: particle tracking

Nevertheless, tracking is the only way to obtain realistic estimates for the dynamic aperture up to $10^5 - 10^6$ revolutions, but it is very difficult and sometimes dangerous to extrapolate these data to longer times (10^9 revolutions or more). Furthermore, tracking is always very important for checking perturbative calculations because of the divergence problems in perturbation theories as mentioned above. We conclude this chapter on Hamiltonian systems with some final remarks.

2.5 Remarks

As mentioned already in the introduction, in accelerator physics one often tries to define different zones or regions corresponding to the importance of the nonlinearities. For small nonlinearities the accelerator behaves more or less like a linear element. A quantitative measure for this quasilinear behaviour is the so-called smear, a concept developed during the design studies for the SSC [95]. This quantity indicates how much the invariants of the linear machine are changed due to the nonlinear perturbations (see Figure 22). Another measure could be the amplitude dependence of the tunes. In the weakly nonlinear region one would expect that perturbation theory is the adequate theoretical tool.

Figure 22: concept of smear

For stronger nonlinearities the dynamics becomes more and more non-

linear and chaotic i.e. sensitively dependent on the initial conditions. A quantitative measure for the onset of large scale chaos can be derived from Chirikov's resonance overlap criterion.One estimates the resonance widths (see equation (78)) and resonance distances and the criterion roughly states that no KAM tori survive in the region where resonance overlap occurs which leads to a completely chaotic particle motion in this area of phase space. The formal steps for applying this criterion are carefully described in [21] , [64] and [96]. Direct application to the standard map (22) for example yields a critical nonlinearity parameter of $V \approx 2.47$, which is in qualitative agreement with numerical simulations (see Figure 14).

The problem of beam-beam interaction in storage rings is an example where this method has been applied extensively by Tennyson et. al. [97], [98], [99].

An interesting and important question is: how does the particle motion look in this extended chaotic region of phase space? Can it be described by a diffusion-like process and can probabilistic concepts be used successfully in this context [100], [101]? A microscopic and mathematically rigorous derivation of particle diffusion for non-integrable Hamiltonian systems is a very complicated problem (see also [102]) and cannot be presented in this review. What we can do here is to illustrate some of the ideas and techniques on a phenomenological level by considering several toy models (which are nevertheless relevant in some aspects of accelerator dynamics). At first we choose the following simple model [20], [21], [38]

$$\begin{aligned}\mathcal{H}(\psi_1,\psi_2,J_1,J_2,t) &= \\ &= \frac{1}{2}\cdot(J_1^2+J_2^2)+\varepsilon\cdot(\cos\psi_1-1)\cdot(1+\mu\cdot\sin\psi_2+\mu\cdot\cos t). \quad (90)\end{aligned}$$

which in extended phase space $(J_1, J_2, p, \psi_1, \psi_2, x = t)$ can be written as

$$\begin{aligned}\mathcal{K}(J_1,J_2,p,\psi_1,\psi_2,x) &= \\ &= \frac{1}{2}\cdot(J_1^2+J_2^2)+p+\varepsilon\cdot(\cos\psi_1-1)-\mu\cdot\varepsilon\cdot\sin\psi_2-\mu\cdot\varepsilon\cos x+ \\ &+\frac{\mu\cdot\varepsilon}{2}(\sin(\psi_2-\psi_1)+\sin(\psi_2+\psi_1)+\cos(\psi_1-x)+\cos(\psi_1+x)). \quad (91)\end{aligned}$$

$\mathcal{K}$ represents now an autonomous system in six-dimensional phase space.

The primary resonances of the system (91) and the corresponding resonance widths are given by

$$\begin{cases} \frac{d}{dt}\ \psi_1 \approx J_1 = 0; \ \text{ width} \sim \sqrt{\varepsilon} \\ \frac{d}{dt}\ \psi_2 \approx J_2 = 0; \ \text{ width} \sim \sqrt{\varepsilon \cdot \mu} \\ \frac{d}{dt}\ (\psi_1 \pm \psi_2) \approx J_1 \pm J_2 = 0; \ \text{ width} \sim \sqrt{\varepsilon \cdot \mu} \\ \frac{d}{dt}\ (\psi_1 \pm x) \approx J_1 \pm 1 = 0; \ \text{ width} \sim \sqrt{\varepsilon \cdot \mu}. \end{cases} \tag{92}$$

For small ε, μ the energy surface is approximated by

$$\mathcal{K}_0(J_1, J_2, p) \approx \frac{1}{2}(J_1^2 + J_2^2) + p \tag{93}$$

and the resonance zones are given approximately by the intersection of the resonance surfaces (92) with the unperturbed energy surface (93), see Figures 23, 24.

If $\varepsilon \gg \varepsilon \cdot \mu$, $J_1 = 0$ is the dominant resonance (guiding resonance). The motion (transport, diffusion) along the guiding resonance and the resonances which intersect it is called Arnold diffusion, see Figures 23, 24.

For $\mu = 0$ the Hamiltonian in equation (91) is integrable (nonlinear pendulum) and a constant of the motion. For $\mu \neq 0$ the Hamiltonian is nonintegrable and the separatrix of the $J_1 = 0$ resonance will be replaced by a chaotic layer. In this nonintegrable case we expect some diffusive variation of the energy [7]. One can calculate this variation approximately [20], [21], [38].

Using

$$\begin{aligned} &\Delta\mathcal{H}(J_1, J_2, \psi_1, \psi_2, t) = \\ &\quad = \int_{-\infty}^{\infty} dt \frac{d\mathcal{H}}{dt} = \int_{-\infty}^{\infty} \varepsilon \cdot \mu \sin t \cdot (1 - \cos \psi_1) dt \end{aligned} \tag{94}$$

[7] The dynamics described by (91) can be considered as stochastic pumping [20]. We treat (91) as a system of two two degree of freedom Hamiltonians $\mathcal{K}_1(J_1, J_2, \psi_1, \psi_2)$ and $\mathcal{K}_2(J_1, p, \psi_1, x)$. Solving the dynamics of the first system gives a chaotic layer around the $J_1 = 0$ resonance with a quasi-random evolution of the corresponding phase ψ_1, which when put in system 2 causes a diffusion-like motion of p.

(see equation (90)) and replacing $\psi_1(t)$ by the unperturbed separatrix expression $\psi_{1sx}(t)$ the evaluation of the resulting Melnikov-Arnold integral [21] gives

$$< (\Delta\mathcal{H})^2 > \approx 8\pi^2 \cdot \mu^2 \cdot \exp\{-\frac{\pi}{\sqrt{\varepsilon}}\}. \tag{95}$$

Figure 23: energy surface of unperturbed system (93)

Figure 24: projection of resonance curves on $(J_1 - J_2)$ plane

Equation (95) is an estimate for the short time variation of the energy of the system for trajectories deep inside the chaotic layer of the $J_1 = 0$ resonance.Further details can be found in [20], [21], [38].

As a second example we consider a system described by

$$\begin{aligned}\mathcal{H}(I_1, I_2, \theta_1, \theta_2, t) = \frac{I_1^2}{2} + \frac{I_2^2}{2} - \\ -\varepsilon\cos(\theta_1 + \lambda\sin(\Omega \cdot t)) - \mu\cos(\theta_1 - \theta_2)\end{aligned} \tag{96}$$

which has been discussed extensively by Chirikov et al. in [103] [104]. It consists of a phase modulated oscillator (I_1, θ_1) which is coupled to a second system (I_2, θ_2). This model describes another instability mechanism in Hamiltonian systems. Because of the (slow) modulation a set of closely spaced resonances can form which - depending on the chosen set of parameters- may overlap and form a broad chaotic region in phase space. As in the previous model the resulting stochastic 1-motion can couple to system 2 and can cause stochastic oscillations and diffusion-like motion. Chirikov et al. describe this kind of dynamics as follows: *The modulational diffusion can be regarded as representative of a larger class of dynamical instabilities, sometimes referred to as thick layer diffusion. This terminology is intended to characterize motion along the broad stochastic domains generated by the overlap of several resonances closely situated in the system phase space. A thick layer diffusion differs from Arnold diffusion both in structure and in the size of the stochastic components involved. Indeed, Arnold diffusion takes place within the narrow stochastic domains (thin layers) which unavoidably appear in the vicinity of separatrices of nonlinear resonances under the effect of arbitrary perturbation. Since thin layers exist for any perturbation strength, so does Arnold diffusion. In contrast, a thick layer can exist only in some suitable parameter range, so that the same condition will determine the onset of the associated diffusion....*

The above mentioned model plays an important role in modelling modulation effects in accelerators. It has been used to describe the influence of the harmonic content of the power supply ripple of the magnets in a storage ring [2], [105], [106], [107], [108]. In [2] for example, the influence of tune- modulation on the proton dynamics in nonlinear storage rings has been investigated. The emittance growth has been studied, and analytical estimates have been given for the drift and diffusion coefficients in HERA. It was demonstrated how various field nonlinearities in combination with tune modulation effects can drastically reduce the dynamic aperture of an accelerator. The analytical estimates for the local particle diffusion during luminosity operation are in very good agreement with the observed emittance growth in HERA see [2].

A good understanding of these diffusion-like phenomena in a storage ring is essential for the operation and performance of experiments such as HERA-B [109]. HERA-B is an experiment to study CP violation in the B(eauty)-system using an internal target at the HERA proton beam. A controled population of the proton beam halo would be highly desirable from the de-

tector designer's point of view [110].

Furthermore, these theoretical considerations have led to compensation schemes for dangerous harmonics of the power supply ripples. Details of these investigations and ideas can be found in [111], [112].

As mentioned already, an outstanding problem of accelerator physics is the long time stability of particle motion under the influence of various nonlinearities such as magnetic multipoles, rf fields and beam-beam forces.

In perturbation theory one usually approximates a nonintegrable system by an integrable (solvable) system. Whether this approximation really reflects the "reality" of the nonintegrable case has to be checked very carefully especially because integrable systems have no chaotic regions in phase space and because the dominant instability mechanisms are related to chaotic diffusion or transport [113]. For two-dimensional systems chaotic transport is in general only possible by breaking KAM tori (but see also [114]).For higher-dimensional systems the chaotic layers can form a connected web along which diffusion like motion is always possible.

For weakly perturbed integrable Hamiltonian systems stability for infinitely long time scales is guaranteed for initial conditions which are distributed on existing KAM-tori. These tori, however, form a complicated Cantor-like set in phase space. What one would like to have is a stability theory not necessarily for infinitely long times but for an *open* set of initial conditions (compatible with the injection conditions for the particle motion in a storage ring). Nekhoroshev [115] has developed such a theory which deals directly with the variation of the action variables during the perturbed motion

$$\mathcal{H}(\vec{I},\vec{\theta}) = \mathcal{H}_0(\vec{I}) + \varepsilon\mathcal{H}_1(\vec{I},\vec{\theta})$$

and which does not refer to the existence of invariant tori. Provided the Hamiltonian fulfills certain conditions he could prove that the action variables change only little over an exponentially long time scale, namely [116], [46]

$$\mid I(\vec{t}) - I(\vec{0}) \mid < \varepsilon^b$$

for times

$$0 \leq t \leq \frac{1}{\varepsilon} \cdot \exp(\frac{1}{\varepsilon^a}).$$

The coefficients a, b depend on the geometrical properties of the unperturbed Hamiltonian. A proof of this theorem leads immediately to the complicated problem of estimating remainders in perturbation series and giving time scales over which perturbative methods are valid. Further details and a lot of background information can be found in [46], [116].

Attempts to extend and apply these ideas to realistic accelerator conditions have been made by Warnock, Ruth and Turchetti [117], [118], [119]. An interesting approach has been chosen in [120], [121], where these ideas and interval arithmetic concepts have been used to give stability bounds on the particle motion in storage rings.

To extract information about the long time stability of particle motion from numerical simulations is also a difficult task as mentioned above. So-called survival plots (see Figure 25) [122] can be helpful in getting some insight into the problem.

Figure 25: survival plot for the SSC, i.e. number of turns of the particle before loss versus initial betatron amplitude of the particle

Since there are no exact solutions available for the complicated nonlinear

dynamics in storage rings and in order to check and test the theoretical concepts and tools described above, existing accelerators at DESY, FERMILAB, CERN and at the University of Indiana have been used for experimental investigations of the particle motion. These studies include the experimental determination and measurement of phase space plots, investigations of the dynamic aperture, and the beam blow-up and diffusion-like motion under various operational conditions. Summaries of these results and further details can be found in [123], [124], [125], [126], [127], [128], [129], [130], [131], [132], [133]. As an example of an experimentally observed phase space plot of a nonlinear machine we show Figure 26 [132].

Figure 26: measured transverse phase space plot near a $Q_x = 15/4$ nonlinear resonance in the IUCF

Figure 27 summarizes the status of the art of the nonlinear particle motion in storage rings.

In the next chapter we will investigate explicitly stochastic systems i.e. systems subject to noise or stochastic forces.

Figure 27: nonlinear dynamics of storage rings

3 Stochastic dynamics in storage rings

In the first part of this review we have shown how the single particle dynamics of proton storage rings can be described by nonintegrable Hamiltonians. These systems show a very complex dynamics - regular and chaotic motion is intricately mixed in phase space. We have mentioned the concept of Arnold diffusion and chaotic transport and we have asked whether probabilistic methods can be applied successfully in this context.

In the second part of our survey we want to investigate systems where probabilistic tools are necessary, because we want to study the influence of

stochastic forces and noise. In this case the equations of motion, which describe the dynamics, take the form

$$\frac{d}{dt}\,\vec{x}(t) = \vec{f}(\vec{x}, t; \vec{\xi}(t)) \tag{97}$$

or in the discrete time (mapping) case

$$\vec{x}(n+1) = \vec{f}(\vec{x}(n), \vec{\xi}(n)) \tag{98}$$

where $\vec{\xi}(t)$ or $\vec{\xi}(n)$ designates some explicit stochastic vector process with known statistical properties. Our aim will be to study the temporal evolution of $\vec{x}(t)$ or $\vec{x}(n)$ under the influence of these explicit stochastic forces. We will call this kind of (probabilistic) dynamics *stochastic dynamics* in contrast to the (deterministic) *chaotic dynamics* investigated in the first part of this review. Questions we want to answer in the following are:

Given the statistical properties of the random forces, what are the statistical properties of $\vec{x}(t)$ or $\vec{x}(n)$? How can we treat these systems mathematically? And how can we calculate, for example, average values $< x_i(t) >$ or correlations $< x_i(t)\, x_j(t') >$?

This part of the review is organized as follows. At first we will summarize some basic results of probability theory and the theory of stochastic processes. Then we will concentrate on stochastic differential equations and their use in accelerator physics problems. In the case that the fluctuating random forces are modelled by Gaussian white noise processes (which is quite often a very good approximation) we will illustrate the mathematical subtleties related to these processes.

Examples of stochastic differential equations are

1. Langevin equation approach to Brownian motion

$$\frac{d}{dt}\,v = -\eta \cdot v + \xi(t) \tag{99}$$

with v particle velocity, η friction coefficient and $\xi(t)$ fluctuating random force

2. stochastically driven harmonic oscillator

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \xi(t) \end{pmatrix} \tag{100}$$

3. spin diffusion or Brownian motion on the unit sphere [137] (see Figure 28)

$$\frac{d}{dt}\,\vec{S}(t) = \vec{H}(t) \times \vec{S}(t) \tag{101}$$

where $\vec{H}(t)$ denotes a fluctuating field.

Figure 28: Brownian motion on the unit sphere

As we will see later, single particle dynamics in accelerators is a rich source for stochastic differential equations. Before we start with a systematic study of these systems we have to repeat some basic facts.

3.1 Summary of mathematical facts

Fundamental concepts of probability theory are the *random experiment* (e.g. throwing dice) and the *probability space*. This space consists of the sample space Ω of outcomes ω of the random experiment, a sigma algebra of events $\mathcal{A}$ i.e. a family of sets defined over Ω such that

1. $\Omega \in \mathcal{A}$
2. for every $A_j \in \mathcal{A}$, $\bar{A}_j \in \mathcal{A}$
3. for $A_j, j = 1, 2, ...$ with $A_j \in \mathcal{A}$

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$$

(where $\bar{A}$ denotes the complement of A with respect to Ω) and a probability measure Pr defined over $\mathcal{A}$:

$$Pr : \mathcal{A} \longrightarrow [0, 1].$$

Pr is a measure for the frequency of the occurance of an event in $\mathcal{A}$ and it satisfies the following axioms:

1. $Pr(\phi) = 0$
2. $Pr(\Omega) = 1$
3. $Pr(A_i \cup A_j) = Pr(A_i) + Pr(A_j)$ for $A_i \cap A_j = \phi$

(ϕ is the empty set and designates the impossible event whereas Ω is the certain event).

The aim of probability theory is not the calculation of the probability measure of the underlying sample space Ω, but it is concerned with the calculation of new probabilities from given ones [134]. A rigorous treatment needs sophisticated measure theoretic concepts and is beyond the scope of this review. We will restrict ourselves to some basic facts and results which will be needed later. Detailed presentations of probability theory can be found in the references [134], [135], [136], [138], [139], [140], [141], [142]. In the following summary we will closely follow the book of Horsthemke and Lefever [134].

3.1.1 Random variables (r.v.)

The first notion we need is that of a random variable (r.v.) $\mathcal{X}$. A random variable is a function from the sample space Ω to $\mathbf{R}$ i.e. $\mathcal{X} : \Omega \longrightarrow \mathbf{R}$ with the property that

$$A = \{\omega | \mathcal{X}(\omega) \leq x\} \in \mathcal{A} \tag{102}$$

for all $x \in \mathbf{R}$, or equivalently

$$A = \mathcal{X}^{-1}(B) = \{\omega | \mathcal{X}(\omega) \in B\} \in \mathcal{A} \tag{103}$$

where $B \in \mathcal{B}$ and where $\mathcal{B}$ denotes the Borel field over $\mathbf{R}$ (the sigma algebra of Borel sets formed by all open and closed intervals of $\mathbf{R}$). (103) means that $\mathcal{X}$ is a measurable function and because of this property we can also define a probability measure (distribution law of the r.v.) on ($\mathbf{R}$,$\mathcal{B}$) via

$$P_{\mathcal{X}}(B) \stackrel{\text{def}}{=} Pr\{\mathcal{X}^{-1}(B)\}.$$

Although, in general, one should always distinguish carefully between the random variable $\mathcal{X}$ and x, i.e. the value $\mathcal{X}$ takes on $\mathbf{R}$, we shall use in the following the canonical representation of $\mathcal{X}$ where we identify the (general) probability space $(\Omega, \mathcal{A}, Pr)$ with $(\mathbf{R},\ \mathcal{B}, P_{\mathcal{X}})$. Furthermore, we will only consider continuous r.v. which can be characterized by probability density functions $p_{\mathcal{X}}(x)dx$ which are related to $P_{\mathcal{X}}(B)$ via

$$P_{\mathcal{X}}(B) = \int_B p_{\mathcal{X}}(x)dx.$$

Roughly stated $p_{\mathcal{X}}(x)dx$ gives the probability of finding the value $\mathcal{X}$ between x and $x + dx$ i.e.

$$p_{\mathcal{X}}(x)dx = P_{\mathcal{X}}(x \leq \mathcal{X} \leq x + dx).$$

Given this probability density one can define expectation value, moments and mean square deviation or variance of a r.v.:

1. expectation value of a random variable $\mathcal{X}$

$$E\{\mathcal{X}\} \equiv < \mathcal{X} > \equiv m_{\mathcal{X}} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x p_{\mathcal{X}}(x)dx. \tag{104}$$

2. moment of order r

$$E\{\mathcal{X}^r\} \equiv < \mathcal{X}^r > \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x^r p_{\mathcal{X}}(x)dx. \tag{105}$$

3. mean square deviation or variance

$$E\{(\delta\mathcal{X})^2\} \equiv E\{(\mathcal{X} - <\mathcal{X}>)^2\} \equiv \sigma^2 \stackrel{\text{def}}{=}$$
$$\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} (x - m_{\mathcal{X}})^2 p_{\mathcal{X}}(x) dx \tag{106}$$

As an example we consider the Gaussian distribution, which because of the central limit theorem (see the references), plays an important role in statistics. This distribution is defined by (see also Figure 29)

$$p_{\mathcal{X}}(x) = [(2\pi)^{1/2}\sigma]^{-1} \cdot \exp\{-\frac{(x-m)^2}{2\sigma^2}\}. \tag{107}$$

In this case equations (104), (105) and (106) yield

$$E\{\mathcal{X}\} = m \tag{108}$$

$$E\{(\delta\mathcal{X})^2\} = \sigma^2 \tag{109}$$

$$E\{(\delta\mathcal{X})^r\} = \begin{cases} 0, & \text{for } r \geq 1 \text{ odd} \\ (r-1)!! \cdot \sigma^r, & r \text{ even} \end{cases} \tag{110}$$

where we have used the following definition $(r-1)!! = 1 \cdot 3 \cdot 5 \cdot .... \cdot (r-1)$.

A Gaussian variable is thus completely specified by its first two moments.

Extending these considerations to the multivariable or random vector case $(\mathcal{X}_1, ...\mathcal{X}_n)$ requires the notion of joint probability densities i.e.

$$p_{\mathcal{X}_1 ... \mathcal{X}_n}(x_1, ...x_n) dx_1 ... dx_n =$$
$$= P_{\mathcal{X}_1 ... \mathcal{X}_n}(x_1 \leq \mathcal{X}_1 \leq x_1 + dx_1, \ ..., \ x_n \leq \mathcal{X}_n \leq x_n + dx_n). \tag{111}$$

Moments, cross correlations, covariance matrix etc can then be defined. For example in the two-dimensional case $\vec{\mathcal{X}} = (\mathcal{X}, \mathcal{Y})^T$ mixed moments are defined by

$$E\{\mathcal{X}^r \cdot \mathcal{Y}^p\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^r \cdot y^p p_{\mathcal{X}\mathcal{Y}}(x,y) dx dy. \tag{112}$$

Another important definition we will need is the conditional density function $p_{\mathcal{X}\mathcal{Y}}(x|y)$. Conditional probability $p(A|B)$ means the probability that

Figure 29: Gaussian distribution

event A will take place, knowing with certainty that another event B has occured. For random variables the corresponding density is given by

$$p_{\mathcal{X}\mathcal{Y}}(x|y) = \frac{p_{\mathcal{X}\mathcal{Y}}(x,y)}{p_{\mathcal{Y}}(y)} \tag{113}$$

or similarly

$$p_{\mathcal{X}\mathcal{Y}}(x,y) = p_{\mathcal{X}\mathcal{Y}}(x|y) \cdot p_{\mathcal{Y}}(y). \tag{114}$$

In the multivariable case one has accordingly

$$\begin{aligned} & p_{\mathcal{X}_1 \ldots \mathcal{X}_n}(x_1 \ldots x_n) = \\ & \quad = \; p_{\mathcal{X}_1 \ldots \mathcal{X}_n}(x_1|x_2 \ldots x_n) \cdot p_{\mathcal{X}_2 \ldots \mathcal{X}_n}(x_2|x_3 \ldots x_n) \cdot \\ & \quad \ldots p_{\mathcal{X}_{n-1},\mathcal{X}_n}(x_{n-1}|x_n) \cdot p_{\mathcal{X}_n}(x_n). \end{aligned} \tag{115}$$

3.1.2 Stochastic processes (s.p.)

Next we introduce stochastic processes (s.p.) with the following definition: a family of random variables indexed by a parameter t, $\mathcal{X}_t$, is called a random or stochastic process (t may be continuous or discrete). A stochastic process thus depends on two arguments (t, ω) where ω is again an element of the underlying sample space Ω. For fixed t, $\mathcal{X}_t$ is a random variable whereas

for fixed ω and continuous t , $\mathcal{X}_{(.)}(\omega)$ is a real valued function of t which is called realization or sample path of the s.p. (realizations are designated by $x(t)$),see Figure 30. Generally, stochastic processes $\mathcal{X}_t$ are defined by an

Figure 30: Stochastic processes

infinite hierarchy of joint distribution density functions

$$\begin{cases} p(x_1, t_1) \\ p(x_1, t_1; x_2, t_2) \\ \quad . \\ \quad . \\ \quad . \\ p(x_1, t_1; ...x_n, t_n) \\ \quad . \\ \quad . \end{cases} \tag{116}$$

and are a complicated mathematical object. A proper treatment requires the so-called stochastic calculus see for example [136].

In a similar way as for random variables we can define moments and correlation functions. For example

$$E\{\mathcal{X}_{t_1}\mathcal{X}_{t_2}\} =< x(t_1)x(t_2) >= \\ = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 p(x_1,t_1;x_2,t_2)dx_1 dx_2 \tag{117}$$

is called the two-time correlation function of the stochastic process $\mathcal{X}_t$. Higher order correlations are obtained analogously

$$E\{\mathcal{X}_{t_1}...\mathcal{X}_{t_n}\} =< x(t_1)...x(t_n) >= \\ = \int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} x_1...x_n \cdot p(x_1,t_1;...;x_n,t_n)dx_1...dx_n \tag{118}$$

One way of characterizing a stochastic process is by looking at its history or memory. The completely independent process is defined by

$$p(x_1,t_1;...;x_n,t_n) = \prod_{i=1}^{n} p(x_i,t_i) \tag{119}$$

i.e. only the one-time distribution density is needed to classify and determine this process.

The next simplest case is the so-called *Markov process*. It is defined by

$$p(x_n,t_n|x_{n-1},t_{n-1};..;x_1,t_1) = p(x_n,t_n|x_{n-1},t_{n-1}) \tag{120}$$

with

$$t_1 \leq t_2 \leq ... \leq t_n. \tag{121}$$

Equation (120) implies that if the present state is known, any additional information on the past history is totally irrelevant for predicting the (probabilistic) future evolution.

Markov processes are completely specified by the transition probability density $p(x_n,t_n|x_{n-1},t_{n-1})$ and the one-time probability density $p(x,t)$. Because of (120) and (115) we have

$$p(x_n,t_n;...;x_1,t_1) = \\ = p(x_n,t_n|x_{n-1},t_{n-1}) \cdot ... \cdot p(x_2,t_2|x_1,t_1) \cdot p(x_1,t_1). \tag{122}$$

The transition probability densities fulfill the following nonlinear functional relation (Chapman-Kolmogorov equation)

$$p(x_3,t_3|x_1,t_1) = \\ = \int_{-\infty}^{\infty} p(x_3,t_3|x_2,t_2)\cdot p(x_2,t_2|x_1,t_1)dx_2. \qquad (123)$$

Examples of stochastic processes are:

1. Gaussian stochastic processes $\mathcal{X}_t$.
   $\mathcal{X}_t$ is specified by

$$\left\{\begin{array}{c} p(x,t) \\ p(x_1,t_1;x_2,t_2) \\ . \\ . \\ . \\ p(x_1,t_1;...;x_n,t_n) \\ . \\ . \end{array}\right.$$

   If all the $m-$th order distributions are Gaussian i.e.

$$p(x_1,t_1;...;x_m,t_m) = \\ = (2\pi)^{-m/2}\cdot(\det\underline{\Lambda})^{-1/2}\cdot\exp\{-\frac{1}{2}(\vec{x}-\vec{m})^T\cdot\underline{\Lambda}^{-1}\cdot(\vec{x}-\vec{m})\}$$

   with $\vec{m}^T = (m_{\mathcal{X}}(t_1),...,m_{\mathcal{X}}(t_m))$ and $\underline{\Lambda} = \Lambda_{ij} = E\{(\mathcal{X}_{t_i} - m_{\mathcal{X}}(t_i))(\mathcal{X}_{t_j} - m_{\mathcal{X}}(t_j))\}$, $\mathcal{X}_t$ is called a Gaussian stochastic process.

2. The Wiener process $\mathcal{W}_t$ which plays an important role in probability theory and which is defined by:

$$p(w_n,t_n;...;w_0,t_0) = \prod_{i=0}^{n-1} p(w_{i+1},t_{i+1}|w_i,t_i)\cdot p(w_0,t_0) = \\ = \prod_{i=0}^{n-1}[2\pi(t_{i+1}-t_i)]^{-1/2}\cdot\exp\{-\frac{(w_{i+1}-w_i)^2}{2(t_{i+1}-t_i)}\}\cdot p(w_0,t_0). \qquad (124)$$

The Wiener process is an example for an independent increment process. This can be seen as follows: defining the increments

$$\Delta \mathcal{W}_i = \mathcal{W}_i - \mathcal{W}_{i-1} \tag{125}$$

and

$$\Delta t_i = t_i - t_{i-1} \tag{126}$$

we obtain

$$\begin{aligned} p(\Delta w_n, t_n; ...; \Delta w_1, t_1; w_0, t_0) = \\ = \prod_{i=1}^{n} (2\pi \Delta t_i)^{-1/2} \cdot \exp\{-\frac{(\Delta w_i)^2}{2\Delta t_i}\} \cdot p(w_0, t_0) \end{aligned} \tag{127}$$

i.e. the random variables $\Delta \mathcal{W}_{t_i}$ are statistically independent. Furthermore one calculates in this case

$$E\{\mathcal{W}_t\} = 0 \tag{128}$$

$$E\{\mathcal{W}_t \mathcal{W}_s\} = \min(t, s) \tag{129}$$

$$E\{\mathcal{W}_t^2\} = t. \tag{130}$$

A typical path of a Wiener process is shown in Figure 31. These paths are continuous but nowhere differentiable.

Figure 31: typical path of a Wiener process

3. Gaussian white noise process $\mathcal{Z}_t$. It is a completely random process with

$$p(z_1, t_1; ...; z_n, t_n) = \prod_{i=1}^{n} p(z_i) \tag{131}$$

i.e. independent values at every instant of time. It has

$$E\{\mathcal{Z}_t\} = 0 \tag{132}$$

and the two-time correlation function is given by

$$E\{\mathcal{Z}_t \mathcal{Z}_s\} = \delta(t-s) \ . \tag{133}$$

Since $\mathcal{Z}_t$ is Gaussian all odd correlations vanish automatically (see equation (110)) and the even correlations are given by [138]

$$E\{\mathcal{Z}_{t_1} \cdot \ \ ... \ \ \cdot \mathcal{Z}_{t_{2n}}\} = \sum_{P_i} \delta(t_{i_1} - t_{i_2}) \cdot \ \ ... \ \ \cdot \delta(t_{i_{2n-1}} - t_{i_{2n}}) \tag{134}$$

where the sum is taken over $(2n)!/(2^n \ n!)$ permutations. Gaussian white noise is a mathematical idealization and does not occur in nature. It plays a similar role in the theory of stochastic processes as the Dirac δ-function in functional analysis. One can show, that in a generalized sense [134] Gaussian white noise is the derivative of the Wiener process.

The last concept we need is that of a Markovian diffusion process. This is a Markov process with continuous sample paths. Diffusion processes play an important role in physics and in the context of stochastic differential equations with Gaussian white noise, as we will see in the next section. The temporal evolution of these diffusion processes is described by the so-called Fokker-Planck equation. This is a linear partial differential equation for the transition density $p(x, t|x_0, t_0)$ or the one-time density $p(x, t)$.

3.1.3 Stochastic differential equations (s.d.e.)

As mentioned already stochastic differential equations are the natural extension of deterministic systems, if one wants to include noise effects

$$\frac{d}{dt} \ \vec{x}(t) = \vec{f}(\vec{x}, t; \vec{\xi}(t)) \ .$$

Often $\vec{\xi}(t)$ is modelled by a Gaussian white noise process. This approximation is well justified if the fluctuating forces show only short-time correlations compared to other typical time-scales of the system. Introducing Gaussian white noise in dynamical systems is related to some mathematical problems which we want to illustrate now. In order to keep the notation as simple as possible we will restrict ourselves for the moment to scalar equations of motion (multiplicative stochastic processes [8]) of the form

$$\frac{d}{dt}\ \mathcal{X}_t = f(\mathcal{X}_t) + g(\mathcal{X}_t) \cdot \mathcal{Z}_t \tag{135}$$

where $\mathcal{Z}_t$ designates Gaussian white noise and where we have switched to the notation introduced above in order to make clear that we are treating stochastic processes.

Before we start our investigation let us repeat what Horsthemke and Lefever have written in this context:

The transition to Gaussian white noise sounds rather harmless but it is actually at this point that dangerous territory is entered, which contains hidden pitfalls and traps to ensnare the unwary theoretician ... if one succeeds in avoiding the various traps, either by luck, intuition or whatever, one captures a treasure which might be bane or boon: the white noise

We have mentioned already that the realizations of the Wiener process are continuous but nowhere differentiable, and the same holds also, for example, for the Ornstein-Uhlenbeck process which is defined by

$$\frac{d}{dt}\ \mathcal{V}_t = -\eta \mathcal{V}_t + \mathcal{Z}_t. \tag{136}$$

Thus (135) and (136) are no differential equations in the ordinary sense since $\frac{d}{dt}$ does not exist in the usual sense, and we have to ask how to interpret stochastic differential equations with Gaussian white noise. The mathematical problems are related to the irregular behaviour of white noise. We will not go through all the mathematical details, we only want to illustrate the subtleties, so that readers who are confronted with this problem are reminded of being careful when using stochastic differential equations with white noise. Excellent presentations of this problem can be found in [143],[144].

[8] the external noise is coupled in a multiplicative manner to $\mathcal{X}_t$ the statistical properties of which are sought

After these remarks we try to give a sense to the stochastic differential equation by rewriting it as an integral equation

$$\mathcal{X}_t = \mathcal{X}_0 + \int_0^t f(\mathcal{X}_s)ds + \int_0^t g(\mathcal{X}_s)\mathcal{Z}_s ds$$

which is equivalent to [9]

$$\mathcal{X}_t = \mathcal{X}_0 + \int_0^t f(\mathcal{X}_s)ds + \int_0^t g(\mathcal{X}_s)d\mathcal{W}_s. \tag{137}$$

As before $\mathcal{W}_s$ denotes the Wiener process. The second integral on the right hand side of equation (137) - a kind of a stochastic Stieltjes integral - is the main reason for the mathematical problems. Let us quote again Horsthemke and Lefever :

The problem is though a sense can be given to this integral and thus to the stochastic differential equation in spite of the extremely irregular nature of the white noise, there is no unique way to define it, precisely because white noise is so irregular. This has nothing to do with the different definitions of ordinary integrals by Riemann and Lebesgue. After all for the class of functions for which the Riemann integral as well as the Lebesgue integral can be defined, both integrals yield the same answer. The difference between the definitions for the above stochastic integral, connected with the names of Ito and Stratonovich , is much deeper; they give different results.

This difficulty can be illustrated as follows:

Consider a stochastic integral of the form

$$\mathcal{S}_t = \int_{t_0}^t \mathcal{W}_s d\mathcal{W}_s. \tag{138}$$

If (138) would be Riemann integrable the result would be

$$\mathcal{S}_t = \frac{1}{2}(\mathcal{W}_t^2 - \mathcal{W}_{t_0}^2). \tag{139}$$

[9] a careful analysis which requires the concept of generalized stochastic processes shows that Gaussian white noise is the derivative of the Wiener process only in a generalized sense. For more details the reader should consult [134]

As in the Riemann case we try to evaluate (138) by a limit of approximating sums of the form

$$\mathcal{S}_n = \sum_{i=1}^{n} \mathcal{W}_{\tau_i^{(n)}} \cdot (\mathcal{W}_{t_i^{(n)}} - \mathcal{W}_{t_{i-1}^{(n)}}) \tag{140}$$

with a partition of the interval $[t_0, t]$

$$t_0 = t_0^{(n)} < t_1^{(n)} < t_2^{(n)} < ... < t_{n-1}^{(n)} < t_n^{(n)} = t$$

and

$$\tau_i^{(n)} \in [t_{i-1}^{(n)}, t_i^{(n)}]$$

or

$$\tau_i^{(n)} = (1-\alpha) t_{i-1}^{(n)} + \alpha t_i^{(n)}$$

with $0 \leq \alpha \leq 1$.

Using the stochastic calculus (the proper calculus to treat stochastic processes as mentioned above) one can show that the limit of $\mathcal{S}_n$ for $n \to \infty$ depends on the evaluation points $\tau_i^{(n)}$ or α [134], [135]

$$\lim_{n\to\infty} \mathcal{S}_n = \frac{1}{2}(\mathcal{W}_t^2 - \mathcal{W}_{t_0}^2) + (\alpha - \frac{1}{2})(t - t_0) \quad . \tag{141}$$

Thus, the stochastic integral is no ordinary Riemann integral. However, an unambiguous definition of the integral can be given - and thus a consistent calculus is possible - if $\tau_i^{(n)}$ is fixed once and forever. Two choices are convenient

- $\alpha = 0$, Ito definition
- $\alpha = \frac{1}{2}$, Stratonovich definition.

Thus, a stochastic differential equation has always to be supplemented by a kind of interpretation rule for the stochastic integral. In both cases mentioned above one can show, that the solutions of the corresponding equations

exist and are Markovian diffusion processes - provided f and g fulfill certain smoothness conditions [134]. In the Ito case

$$(I) \quad d\mathcal{X}_t = f(\mathcal{X}_t, t)dt + g(\mathcal{X}_t, t)d\mathcal{W}_t \tag{142}$$

the corresponding Fokker-Planck equation for the transition probability density $p(x, t|x_0, t_0)$ reads

$$\begin{aligned} &\frac{\partial}{\partial t}p(x,t|x_0,t_0) = \\ &\quad = -\frac{\partial}{\partial x}[f(x,t)\cdot p(x,t|x_0,t_0)] + \frac{1}{2}\cdot\frac{\partial^2}{\partial x^2}[g^2(x,t)\cdot p(x,t|x_0,t_0)] \end{aligned} \tag{143}$$

whereas in the Stratonovich case

$$(S) \quad d\mathcal{X}_t = f(\mathcal{X}_t, t)dt + g(\mathcal{X}_t, t)d\mathcal{W}_t \tag{144}$$

the Fokker-Planck equation is given by

$$\begin{aligned} &\frac{\partial}{\partial t}p(x,t|x_0,t_0) = \\ &\quad = -\frac{\partial}{\partial x}[f(x,t)\cdot p(x,t|x_0,t_0)] + \frac{1}{2}\cdot\frac{\partial}{\partial x}[g(x,t)\cdot\frac{\partial}{\partial x}(g(x,t)p(x,t|x_0,t_0))] \end{aligned}$$

or equivalently by

$$\begin{aligned} &\frac{\partial}{\partial t}p(x,t|x_0,t_0) = \\ &\quad = -\frac{\partial}{\partial x}([f(x,t) + \frac{1}{2}\frac{\partial g(x,t)}{\partial x}\cdot g(x,t)]\cdot p(x,t|x_0,t_0)) + \\ &\quad +\frac{1}{2}\cdot\frac{\partial^2}{\partial x^2}[g^2(x,t)\cdot p(x,t|x_0,t_0)] \quad . \end{aligned} \tag{145}$$

Equations (143) and (145) have to be supplemented with the initial condition

$$p(x,t|x_0,t_0)_{t=t_0} = \delta(x - x_0)$$

and suitable boundary conditions for x.

The Ito calculus is mathematically more general but leads to unusual rules such as

$$\int_{t_0}^{t} \mathcal{W}_s d\mathcal{W}_s = \frac{1}{2}(\mathcal{W}_t^2 - \mathcal{W}_{t_0}^2) - \frac{1}{2}(t - t_0)$$

and some care is needed when one transforms from one process $\mathcal{X}_t$ to $\mathcal{R}_t = h(\mathcal{X}_t)$. For further details and a discussion of the relationship between Ito and Stratonovich approach (which preserves the "normal" rules of calculus) the reader is referred to the references.

Remarks:

1. The one-time probability density $p(x,t)$ of a Markovian diffusion process $\mathcal{X}_t$ also satisfies the Fokker-Planck equation (143) or (145).

2. In the case of purely additive noise where g does not depend on $\mathcal{X}_t$ there is no difference between the Ito and Stratonovich approach, so both stochastic differential equations define the same Markovian diffusion process.

3. Since Gaussian white noise is a mathematical idealization and can only approximately model real stochastic processes in nature, there is always the question how to interpret equation (135) in practical problems. In most physical cases one will rely on the Stratonovich interpretation as is suggested by a theorem due to Wong and Zakai [134] which roughly states :

   if we start with a phenomenological equation containing realistic noise $\mathcal{W}_t^{(n)}$ of the form

$$\frac{d}{dt}\,\mathcal{X}_t = f(\mathcal{X}_t) + g(\mathcal{X}_t) \cdot \frac{d}{dt}\,\mathcal{W}_t^{(n)} \tag{146}$$

where all the integrals can be interpreted in the usual (e.g. Riemann) sense and if we pass to the white noise limit

$$\mathcal{W}_t^{(n)} \longrightarrow \mathcal{W}_t \tag{147}$$

so that a stochastic differential equation of the form

$$\frac{d}{dt}\ \mathcal{X}_t = f(\mathcal{X}_t) + g(\mathcal{X}_t) \cdot \frac{d}{dt}\ \mathcal{W}_t \tag{148}$$

is obtained (remember that Gaussian white noise is the derivative of the Wiener process $\mathcal{Z}_t = \frac{d}{dt}\ \mathcal{W}_t$) the latter has to be interpreted as a Stratonovich equation.

4. The above considerations can be extended to the multivariable case where $\mathcal{X}_t$, $f(\mathcal{X}_t)$ and $\mathcal{W}_t$ have to be replaced by vector quantities and where $g(\mathcal{X}_t)$ has to be replaced by a matrix. Now,the stochastic differential equation takes the form

$$d\vec{\mathcal{X}}_t = \vec{f}(\vec{\mathcal{X}}_t, t)dt + \underline{g}(\vec{\mathcal{X}}_t, t)d\vec{\mathcal{W}}_t \ . \tag{149}$$

The Ito interpretation leads to a Fokker-Planck equation for the transition density $p(\vec{x}, t|\vec{x}_0, t_0)$ of the form

$$\begin{aligned}\frac{\partial}{\partial t}p(\vec{x}, t|\vec{x}_0, t_0) &= \\ &= -\sum_i\ \frac{\partial}{\partial x_i}[f_i(\vec{x}, t) \cdot p(\vec{x}, t|\vec{x}_0, t_0)] + \\ &+\frac{1}{2} \cdot \sum_{i,j} \frac{\partial}{\partial x_i} \cdot \frac{\partial}{\partial x_j}[\{\underline{g}(\vec{x}, t)\underline{g}^T(\vec{x}, t)\}_{ij} \cdot p(\vec{x}, t|\vec{x}_0, t_0)]\end{aligned} \tag{150}$$

whereas the Stratonovich interpretation gives

$$\frac{\partial}{\partial t}p(\vec{x},t|\vec{x}_0,t_0) =$$
$$= -\sum_i \frac{\partial}{\partial x_i}[f_i(\vec{x},t)\cdot p(\vec{x},t|\vec{x}_0,t_0)] +$$
$$+\frac{1}{2}\cdot\sum_{ijk}\frac{\partial}{\partial x_i}\{g_{ik}(\vec{x},t)\frac{\partial}{\partial x_j}[g_{jk}(\vec{x},t)\cdot p(\vec{x},t|\vec{x}_0,t_0)]\} \quad . \qquad (151)$$

Examples

1. Wiener process

$$d\mathcal{X}_t = d\mathcal{W}_t \qquad (152)$$

with the corresponding Fokker-Planck (diffusion) equation ($f = 0, g = 1$)

$$\frac{\partial}{\partial t}p(x,t|x_0,t_0) = \frac{1}{2}\cdot\frac{\partial^2}{\partial x^2}\cdot p(x,t|x_0,t_0) \qquad (153)$$

2. The Ornstein Uhlenbeck process (see also equation (99))

$$d\mathcal{V}_t = -\eta\mathcal{V}_t dt + \sigma\cdot d\mathcal{W}_t \qquad (154)$$

leads to the following Fokker-Planck equation ($f = -\eta, g = \sigma$)

$$\frac{\partial}{\partial t}p(v,t|v_0,t_0) =$$
$$= \frac{\partial}{\partial v}[\eta\cdot v\cdot p(v,t|v_0,t_0)] + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial v^2}\cdot p(v,t|v_0,t_0) \quad . \qquad (155)$$

3. Stochastically driven harmonic oscillator as an example of a multivariable system:

$$\begin{pmatrix} d\mathcal{X}_{1_t} \\ d\mathcal{X}_{2_t} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathcal{X}_{1_t}\, dt \\ \mathcal{X}_{2_t}\, dt \end{pmatrix} + \begin{pmatrix} 0 \\ d\mathcal{W}_t \end{pmatrix} \tag{156}$$

In this case one obtains as Fokker-Planck equation

$$\begin{aligned} &\frac{\partial}{\partial t} p(x_1, x_2, t | x_{1_0}, x_{2_0}, t_0) = \\ &\quad = -\frac{\partial}{\partial x_1}[x_2 \cdot p(x_1, x_2, t | x_{1_0}, x_{2_0}, t_0)] + \frac{\partial}{\partial x_2}[x_1 \cdot p(x_1, x_2, t | x_{1_0}, x_{2_0}, t_0)] + \\ &\quad + \frac{1}{2} \cdot \frac{\partial^2}{\partial {x_2}^2} p(x_1, x_2, t | x_{1_0}, x_{2_0}, t_0) \quad . \end{aligned} \tag{157}$$

The Fokker-Planck equation can be used to derive differential equations for the moments very quickly. We will illustrate this for the second order moments of the harmonic oscillator (156).

$$\frac{d}{dt} < x_1^2(t) >= \frac{d}{dt} \int_{-\infty}^{\infty} x_1^2 p(x_1, x_2, t) dx_1 dx_2 = \int_{-\infty}^{\infty} x_1^2 \frac{\partial}{\partial t} p(x_1, x_2, t) dx_1 dx_2$$

Using the explicit form of the Fokker-Planck equation on the right hand side we obtain after partial integration

$$\frac{d}{dt} < x_1^2(t) >= 2 \ < x_1(t)\, x_2(t) > . \tag{158}$$

Repeating this procedure for $< x_1(t)\, x_2(t) >$ and $< x_2^2(t) >$ one gets a closed system of equations for the second order moments , namely

$$\frac{d}{dt} < x_1^2(t) >= 2 \ < x_1(t)\, x_2(t) > \tag{159}$$

$$\frac{d}{dt} < x_1(t)\, x_2(t) >=< x_2^2(t) > - < x_1^2(t) > \tag{160}$$

$$\frac{d}{dt} < x_2^2(t) >= 1 - 2 \ < x_1(t)\, x_2(t) > . \tag{161}$$

Note that the average of the total energy of the harmonic oscillator varies linearly with t under the influence of Gaussian white noise

$$\frac{d}{dt} < E(t) >= \frac{d}{dt}\frac{1}{2}(< x_1^2(t) > + < x_2^2(t) >) = \frac{1}{2}.$$

The fact that the system of equations is closed is typical for linear systems like the harmonic oscillator. In general this scheme delivers an infinite hierarchy of higher order moments, which has to be truncated at a certain step in order to get a solution of the problem.

For linear systems one can also calculate the moments or correlations directly. Consider for example the following case

$$\frac{d}{dt}\vec{x}(t) = \underline{A} \cdot \vec{x}(t) + \vec{\xi}(t) \tag{162}$$

where $\vec{\xi(t)}$ is a random force (not necessarily Gaussian white noise). The formal solution of this equation is

$$\vec{x}(t) = \underline{M}(t,0) \cdot \vec{x}(0) + \int_0^t \ \underline{M}(t,t_1) \cdot \vec{\xi}(t_1) dt_1 \tag{163}$$

where $\underline{M}(t,t_1)$ is a solution of the matrix equation (Greens function)

$$\frac{d}{dt}\underline{M}(t,t_1) = \underline{A}\ \underline{M}(t,t_1). \tag{164}$$

The correlation function $< x_i(t)\, x_j(\tilde{t}) >$ is then given by

$$< x_i(t)\, x_j(\tilde{t}) >= \sum_{k,l} \ \int_0^t \int_0^{\tilde{t}} M_{i,k}(t,t_1) M_{j,l}(\tilde{t},t_2) < \xi_k(t_1)\xi_l(t_2) > dt_1 dt_2 \tag{165}$$

where the correlation function of the external noise is known. For example, applying this scheme to the harmonic oscillator under the influence of Gaussian white noise one gets immediately

$$< x_1^2(t) >= \frac{1}{2}t - \frac{1}{4}\sin(2t). \tag{166}$$

This way of calculating correlations and moments will be used later when we investigate the linear beam emittance matrix of an electron storage ring.

Finally, we want to make some remarks concerning non-Gaussian white noise. As mentioned already Gaussian white noise is an idealization and does not occur in nature. The question is how to treat systems under the influence of colored noise such as, for example, the Ornstein-Uhlenbeck noise $\mathcal{V}_t$. In this case one can use the following scheme, which we illustrate for a simple scalar equation of the form

$$\frac{d}{dt}\mathcal{X}_t = f(\mathcal{X}_t, t) + g(\mathcal{X}_t, t) \cdot \mathcal{V}_t. \tag{167}$$

The Ornstein-Uhlenbeck process $\mathcal{V}_t$ is defined by

$$\frac{d}{dt}\mathcal{V}_t = -\eta\mathcal{V}_t + \sigma\mathcal{Z}_{2_t}$$

with $\mathcal{Z}_{2_t}$ Gaussian white noise. Introducing the vector notation $\vec{\mathcal{Y}}_t = (\mathcal{X}_t, \mathcal{V}_t)^T$ and $\vec{\mathcal{Z}}_t = (\mathcal{Z}_{1_t}, \mathcal{Z}_{2_t})^T$ one gets with suitably defined quantities $\vec{f}$ and $\underline{g}$

$$\frac{d}{dt}\vec{\mathcal{Y}}_t = \vec{f}(\vec{\mathcal{Y}}_t, t) + \underline{g}(\vec{\mathcal{Y}}_t, t) \cdot \vec{\mathcal{Z}}_t \tag{168}$$

i.e. a vector stochastic differential equation of the form (149). The solution of this equation exists and is a vector Markovian diffusion process - provided $\vec{f}$ and $\underline{g}$ fulfill certain smoothness conditions. The disadvantage is that the corresponding Fokker-Planck equation includes an additional variable, v, namely $p(x, v, t|x_0, v_0, t_0)$ and thus requires the solution of a partial differential equation of higher dimension. The same trick can also be used for more complicated systems.

Summarizing, we can say that stochastic differential equations are the natural extension of deterministic systems if one wants to study the influence of noise. Often these noise processes are approximated by Gaussian white noise, a mathematical idealization, which has to be treated with great care. However, the mathematical subtleties related to stochastic differential equations with white noise are outweighed by the results which are available for these Markovian diffusion processes namely the Fokker-Planck equation [140].

In this -admittedly- very sketchy summary of mathematical results we could not consider advanced topics such as martingales and their great importance for Markov processes. Furthermore, we completely skipped the qualitative (stability) theory of stochastic differential equations. Detailed presentations of these topics can be found,for example, in [145], [146], [147].

In the next chapter we will illustrate some applications of stochastic processes in accelerator physics (electron and proton storage rings). Stochastic processes are used for modelling various noise effects. Since a realistic modelling of these noise phenomena is in general a very complicated problem and requires a detailed experimental and statistical analysis, we will often restrict our investigations to simple toy models, which show what is in principle possible in these systems.

3.2 Stochastic dynamics problems in accelerator physics

3.2.1 Electron storage rings

Radiative phenomena play an important role in electron storage rings. As mentioned already these devices are used extensively as synchrotron radiation sources. Because of the interesting physical properties (time structure and polarization) this kind of radiation is an ideal tool for solid state spectroscopy, for atomic and molecular spectroscopy, for x-ray structural analysis and it can be used for many other (also non-physical e.g. medical) applications [148].

Instead of studying these properties and applications further we will now investigate the influence of the radiation on the beam dynamics. This is a rather subtle point [148], [149]. On the one hand there is strong classical radiation damping due to the radiation losses [150], [151] [152], [153], and on the other hand, due to the quantum-like emission of the synchrotron light and due to the recoil the electrons experience during the emission process, there is a steady excitation of the particle motion which causes a trajectory broadening like in Brownian motion. Systems which show such quantum effects on a macroscale are often called "macro-atoms" in the Russian literature [148], [149]. Although an exact treatment requires a quantum mechanical calculation [154], in accelerator physics one often uses a semiclassical approach. In this approach the classical Lorentz-Dirac equation of the electron is supplemented by some explicit noise which simulates the quantum-like emission of

the radiation [150], [151], [155], [156], [157], [158], [159], [160], [161]. Using such a semiclassical theory the electron dynamics (including radiative effects) is governed by the following system of equations [158], [159], [162] (perturbed Hamiltonian)

$$\begin{aligned}
\frac{d}{ds}\, x &= +\frac{\partial \mathcal{H}}{\partial p_x}; \quad \frac{d}{ds}\, p_x = -\frac{\partial \mathcal{H}}{\partial x} - \frac{P(s)}{cE_0}\frac{\partial \mathcal{H}}{\partial p_x} \\
\frac{d}{ds}\, z &= +\frac{\partial \mathcal{H}}{\partial p_z}; \quad \frac{d}{ds}\, p_z = -\frac{\partial \mathcal{H}}{\partial z} - \frac{P(s)}{cE_0}\frac{\partial \mathcal{H}}{\partial p_z} \\
\frac{d}{ds}\, \tau &= +\frac{\partial \mathcal{H}}{\partial p_\tau}; \quad \frac{d}{ds}\, p_\tau = -\frac{\partial \mathcal{H}}{\partial \tau} - \frac{P(s)}{cE_0}\,[1 - \frac{\partial \mathcal{H}}{\partial p_\tau}]
\end{aligned} \tag{169}$$

with $\mathcal{H}$ the Hamiltonian of the coupled synchro-betatron motion (see (5)), and

$$P(s) = E_0^2(1+p_\tau)^2\,[c_1 b^2 + \sqrt{c_2}|b|^{\frac{3}{2}}\,\xi(s)] \tag{170}$$

$$\vec{b}(x,z,s) = \frac{e}{E_0}\vec{B}(x,z,s) \tag{171}$$

(E_0 design energy of storage ring)

$$c_1 = \frac{2r_e p_0^2}{3(m_0 c)^3}; \quad c_2 = \frac{55 r_e \hbar p_0^3}{24\sqrt{3}(m_0 c)^6} \tag{172}$$

and the white noise process

$$< \xi(s)\xi(s') >= \delta(s-s'); \quad < \xi(s) >= 0 \tag{173}$$

($p_0 = \frac{E_0}{c}$ and $r_e = \frac{e^2}{m_0 c^2}$). Using the equations for the magnetic fields of the different kinds of lenses the particle motion under the influence of radiation is then completely specified, and in shorthand notation one obtains

$$\frac{d}{ds}\,\vec{y}(s) = \vec{f}(\vec{y},s) + \underline{T}(\vec{y},s)\cdot\delta\vec{c}(s) \tag{174}$$

with $\vec{y}^T = (x,z,\tau,p_x,p_z,p_\tau)$. $\delta\vec{c}(s)$ designates a Gaussian white noise vector process.

Interesting physical questions are :

What are the average fluctuations of the particle around the closed orbit (beam emittances)? What is the particle distribution $p(\vec{y}, s)$ *i.e. what is the probability for finding the particle between* $\vec{y}$ *and* $\vec{y} + d\vec{y}$ *at location* s *?* [10] *Is there a stationary (or equilibrium) solution of this density i.e. what is* $\lim_{s\to\infty}\ p(\vec{y}, s)$ *? What is the particle lifetime in the finite vacuum chambers of the accelerator (time to hit the border) ?*

These questions have been extensively studied in the linear case [160], [151], [161], [163], [164]

$$\frac{d}{ds}\ \vec{y}(s) = (\underline{A}(s) + \delta\underline{A}(s)) \cdot \vec{y}(s) + \vec{c}(s) + \delta\vec{c}(s) \tag{175}$$

$\underline{A}$ designates the Hamiltonian part of the motion (six-dimensional coupled synchro-betatron oscillations) [15]; $\delta\underline{A}(s)$ describes the radiation damping ; $\delta\vec{c}$ is the fluctuating part of the radiative force and $\vec{c}$ denotes some additional field errors of the system. In this case one obtains compact and easily programmable expressions for the important beam parameters (beam emittances) [161], [163]. An outline of the calculations and a summary of the results can be found in Appendix C. Furthermore, the corresponding Fokker-Planck equation for the probability distribution can be solved exactly [165], [166], [167].

An investigation of the nonlinear system is much more complicated and is an active area of research. Nonlinear systems such as an octupole-dipole wiggler or the beam-beam interaction in electron storage rings have been analyzed by various authors [168], [169], [170], [171], [172], [173]. Let us consider the latter case in more detail. The main problem is to understand the motion of a test particle under the influence of the nonlinear electromagnetic fields of the counter rotating beam [174] (see Figure 32).

This so-called weak-strong model of the beam-beam interaction is mathematically described by the following set of equations [174] (perturbed Hamiltonian system) :

[10] the probability density can be interpreted as density distribution of a system of non-interacting particles

Figure 32: deflection of a test particle in beam A in the field of the counter rotating beam B

$$\begin{cases} \frac{d}{ds}\ x(s) = \frac{\partial}{\partial p_x}\ \mathcal{H}(x, z, p_x, p_z, s) \\ \frac{d}{ds}\ p_x(s) = -\frac{\partial}{\partial x}\ \mathcal{H}(x, z, p_x, p_z, s) - \gamma_x \cdot p_x + d_x \cdot \Gamma \\ \frac{d}{ds}\ z(s) = \frac{\partial}{\partial p_z}\ \mathcal{H}(x, z, p_x, p_z, s) \\ \frac{d}{ds}\ p_z(s) = -\frac{\partial}{\partial z}\ \mathcal{H}(x, z, p_x, p_z, s) - \gamma_z \cdot p_z + d_z \cdot \Gamma \end{cases} \tag{176}$$

The Hamiltonian $\mathcal{H}(x, z, p_x, p_z, s)$ consists of the linear part and the non-linear potential due to the beam-beam interaction

$$\begin{aligned} &\mathcal{H}(x, z, p_x, p_z, s) = \\ &\quad = \frac{p_x^2}{2} + K_x(s) \cdot \frac{x^2}{2} + \frac{p_z^2}{2} + K_z(s) \cdot \frac{z^2}{2} + U(x, z) \cdot \delta_p(s - s_0). \end{aligned} \tag{177}$$

This nonlinear term is given by [174]

$$U(x, z) = \frac{N_b \cdot r_e}{\gamma} \cdot \int_0^\infty \frac{1 - \exp\{-\frac{x^2}{2\sigma_x^2 + q} - \frac{z^2}{2\sigma_z^2 + q}\}}{(2\sigma_x^2 + q)^{1/2} \cdot (2\sigma_z^2 + q)^{1/2}} \cdot dq. \tag{178}$$

In equations (176), (177) and (178) we have used the following definitions : $\delta_p(s - s_0)$ periodic delta function, γ relativistic factor, r_e classical electron

radius, N_b number of particles in the counter rotating bunch, σ_x, σ_z rms beam sizes of the strong bunch. Radiation damping is described by the two damping constants γ_x, γ_z and the strength of the stochastic excitation Γ is denoted by d_x, d_z.

These equations have been used extensively in numerical simulations. These simulations are very helpful for understanding the complicated interplay of nonlinearity, damping and stochastic excitation in lepton colliders. Figure 33 [175] shows such a calculation. The combined effect of quantum fluctuations and nonlinearity can move a particle starting near the origin in phase space to a (nonlinear) resonance island before it is damped again and eventually pushed to another resonance nearby.

Figure 33: $(p_z - z)$ phase diagram of a simulation

In this context an interesting and important question is :

How does the distribution function $p(x, z, p_x, p_z, s)$ of a stochastic system like (176) evolve with s or time?

Since brute force numerical simulations are very CPU intensive and require the tracking of thousands of particles, it would be desirable to have a faster algorithm based directly on $p(x, z, p_x, p_z, s)$ and

$$p(x, z, p_x, p_z, s) = \int p(x, z, p_x, p_z, s|x', z', p'_x, p'_z, s_0)\; p(x', z', p'_x, p'_z, s_0) dx' dz' dp'_x dp'_z$$

where $p(x', z', p'_x, p'_z, s_0)$ designates the initial distribution at s_0. However, the stochastic differential equations (176) contain singular terms like the $\delta-$ functions which describe the localized beam-beam kicks. Therefore, there is no guarantee that the solution of these equations is a diffusion process and that the transition density of this process can be derived from a Fokker-Planck equation. Instead of smoothing the kicks however, one can use directly the discrete stochastic map which can be derived from (176). The suitable quantity is then the transition density for finding the system between $\vec{y}$ and $\vec{y} + d\vec{y}$ in phase space at time step n if it was at $\vec{y}'$ at time step m, namely $p(\vec{y}, n|\vec{y}', m)$. One could now try to derive an equation for this transition density in analogy to the Fokker-Planck equation in the time continuous case. However, in the following we will sketch a different method for calculating the temporal evolution of the density function which is based on an idea of Gerasimov and which gives much faster results than direct numerical simulations (see also [176]).

We will illustrate this approach with a simple two-dimensional model of the beam-beam interaction. The details are described in [177] and in a PhD thesis of Pauluhn [178].

The considered model is given by (see also [99]):

$$\begin{pmatrix} x(n+1) \\ p_x(n+1) \end{pmatrix} = \begin{pmatrix} \cos(2\pi Q) & \beta \cdot \sin(2\pi Q) \\ -\frac{1}{\beta} \cdot \sin(2\pi Q) & \cos(2\pi Q) \end{pmatrix} \cdot \begin{pmatrix} x(n) \\ \bar{p}_x(n) \end{pmatrix} \qquad (179)$$

Where $\bar{p}_x(n)$ is defined by

$$\bar{p}_x(n) = p_x(n) - \gamma_x p_x(n) + u(x(n)) + d_x \Gamma. \qquad (180)$$

Γ is now a random variable, Q is the tune of the storage ring, β is the beta-function at the interaction point and $u(x(n))$ is given by

$$u(x(n)) = -\frac{4\pi\xi_{B-B}}{\beta} \cdot x(n) \cdot \frac{1 - \exp(-\frac{x^2(n)}{2\sigma^2})}{\frac{x^2(n)}{2\sigma^2}} \tag{181}$$

with ξ_{B-B} beam-beam strength parameter [174].
The main steps of this algorithm are:

1. discretization of the two-dimensional phase space
2. use of the microscopic dynamics (see equation (179)) to calculate the transition rates A_{ij} between the discretized bins of the phase space
3. use of this (stochastic) transition matrix A_{ij} as macroscopic propagator for the time evolution of an initial particle distribution

Figure 34 shows how an initially constant and homogeneous distribution evolves with time (after 1000, 3000, 15000, 99000 turns respectively). These results are in excellent agreement with direct numerical simulations [178].

Spin dynamics in electron storage rings constitutes another interesting application of stochastic differential equations in beam dynamics [179], [180]. In this case the orbital equations of motion (174)

$$\frac{d}{ds}\ \vec{y}(s) = \vec{f}(\vec{y}, s) + \underline{T}(\vec{y}, s) \cdot \delta\vec{c}(s)$$

have to be supplemented by the spin equation of motion, the so-called TBMT equation (Thomas, Bargman, Michel, Telegdi see for example [13])

$$\frac{d}{ds}\ \vec{S} = \vec{\Omega}(\vec{y}) \times \vec{S} \tag{182}$$

where the field $\vec{\Omega}$ depends on the orbital degrees of freedom. We will come back to this point later.

Figure 34: evolution of the density function for the stochastic map (179) (here denoted by $\rho(x, p_x, t)$) as a function of time

3.2.2 Proton storage rings

Until now we have seen, that radiation is a natural source for noise in electron accelerators. However, there are also other sources for stochastic forces such as rf noise, power supply noise, random ground motion or restgas scattering. These influences can also be present in proton (Hamiltonian) systems and some of these sources could for example be included in the vector potential $\vec{A}$ of equation (5).

For example, a simple toy model to investigate the influence of random

ground motion is given by

$$\frac{d}{ds}\vec{y}(s) = \underline{A}(s)\ \vec{y}(s) + \delta\underline{A}(s)\ \vec{y}(s) + \delta\vec{b}(s) \tag{183}$$

where

$$\frac{d}{ds}\vec{y}(s) = \underline{A}(s)\ \vec{y}(s)$$

describes the unperturbed linear synchro-betatron motion and where $\delta\vec{b}(s)$ and $\delta\underline{A}(s)$ specify the random dipole errors and focusing errors due to the stochastic ground motion. This model can be solved easily by the methods we have introduced earlier. However, it is very difficult to determine the explicit statistical properties of these errors [181].

The influence of rf noise on the beam dynamics in storage rings is another example which has been investigated by several authors [182], [183], [184], [185], [186].In the smooth approximation (oscillator model) [17], where one averages $\kappa(s)D(s)$ and $V(s)$ over the circumference L of the storage ring one obtains the following Hamiltonian (see equation (18))

$$\begin{aligned}\bar{\mathcal{H}}(\bar{\tau}, \bar{p}_\tau) &= \\ &= -\frac{1}{2}\mu \cdot \bar{p}_\tau^2 + \frac{L}{2\pi k} \cdot \frac{e \cdot \bar{V}_0}{E_0} \cdot \cos(\frac{2\pi k}{L} \cdot \bar{\tau}) + \\ &+ \frac{L}{2\pi k} \cdot \frac{e \cdot \delta\bar{V}_0}{E_0} \cdot \cos(\frac{2\pi k}{L} \cdot \bar{\tau}) - \delta\bar{\tau} \cdot \frac{e \cdot \bar{V}_0}{E_0} \cdot \sin(\frac{2\pi k}{L} \cdot \bar{\tau})\end{aligned} \tag{184}$$

with

$$\mu = \frac{1}{L} \cdot \int_0^L \kappa(s) \cdot D(s) \cdot ds \tag{185}$$

and

$$\bar{V}_0 = \frac{1}{L} \cdot \int_0^L V(s) \cdot ds. \tag{186}$$

$\delta\bar{V}_0$ and $\delta\bar{\tau}$ denote the amplitude and phase noise (for example Gaussian white noise) of a cavity. The corresponding stochastic differential equations now read

$$\begin{cases} \frac{d}{ds}\,\bar{\tau} = -\mu\cdot\bar{p}_\tau \\ \frac{d}{ds}\,\bar{p}_\tau = \frac{e}{E_0}\cdot(\bar{V}_0+\delta\bar{V}_0)\cdot\sin(\frac{2\pi k}{L}\cdot\bar{\tau}) + \frac{e\cdot\bar{V}_0}{E_0}\cdot\frac{2\pi k}{L}\cdot\delta\bar{\tau}\cdot\cos(\frac{2\pi k}{L}\cdot\bar{\tau}) \end{cases} . \tag{187}$$

Using the action-angle variables of the unperturbed (integrable) system (without noise) and averaging the corresponding Fokker-Planck equation over the angle variables one obtains a very good approximation of the problem [182], [183], [185]. An alternative to this perturbative approach would be a numerical integration of the exact Fokker-Planck equation or a direct numerical treatment of the stochastic differential equations (187). Numerical methods to solve stochastic differential equations are described in detail in [187], [188]. The main steps of such a numerical treatment are

- Taylor expansion of the approximate solution in the step width h
- modelling of the noise process
- simulating a sufficient number of realizations for averaging

(see also [139]). These numerical tools have been used to calculate the bunch lengthening in electron storage rings due to rf noise in [178], [189].

Until now we have considered stochastically perturbed integrable Hamiltonian systems such as (183) (linear system) or such as (187) (nonlinear pendulum). However, most Hamiltonian models one encounters in accelerator physics are non-integrable, and the corresponding phase space dynamics shows a very rich structure : invariant KAM tori, nonlinear resonance islands, and extended chaotic regions. Such models are needed for investigating the influence of localized beam-beam kicks, localized cavities and other nonlinearities. Various tools have been developed to study the influence of random tune modulations or random terms in the beam-beam interaction of an accelerator [108], [190], [191], [192], [193].

The general problem consists in solving equations of the form

$$\mathcal{H}(\vec{\Theta},\vec{I},s) = \mathcal{H}_0(\vec{I}) + \varepsilon[\mathcal{H}_1(\vec{\Theta},\vec{I},s) + \vec{W}(\vec{\Theta},\vec{I},s)\cdot\vec{\xi}(s)] \tag{188}$$

where the stochastic excitation $\vec{\xi}(s)$ is defined by the following stochastic differential equation including white noise $\vec{\zeta}(s)$

$$\frac{d}{ds}\vec{\xi}(s) = \underline{D}\ \vec{\xi}(s) + \underline{F}\ \vec{\zeta}(s) \tag{189}$$

and where $\underline{D}, \underline{F}$ designate known matrices specifying the external noise. A complete solution of (188) is still missing and a lot of open questions remain, especially, if one is interested in the resonant case and the weak noise limit. The development of approximation schemes for such systems is an important task and at the same time one also needs suitable numerical tools for checking these approximations. Among these numerical methods quick and powerful solvers of the Fokker-Planck equation can be extremely helpful.

In the last two sections we have given a list of stochastic problems in the beam dynamics of accelerators. We have considered electron machines (dissipative systems) and proton machines (Hamiltonian systems). Tools like stochastic differential equations and the corresponding Fokker-Planck equation have been introduced and it has been shown how these concepts can be applied to practical problems. We have sketched a kind of "hybrid" method for discrete dissipative maps, which is based on analytical tools from the theory of Markov chains combined with a numerical evaluation of the stochastic matrices. In the Hamiltonian case we have mentioned some open questions. In this case it would be highly desirable to extend concepts (such as the one-turn map) which have been developed in the deterministic case to the stochastic case. For related problems see also [195], [196] and [197].

Since in stochastic problems one is often only interested in the moments of a dynamical quantity (average values, average fluctuations) it is also interesting to develop shemes (maps) directly for these moments. This approach has been advanced by K.Hirata, see for example [194]. Altogether, stochastic beam dynamics is an active and important area of research. Only few results are available.

4 Spin dynamics in storage rings

In the previous sections we have only considered the orbital dynamics of charged particles under the influence of external electromagnetic fields and radiation fields. However, particles like the electron or proton have also an internal degree of freedom, the spin. Controled use of this additional degree of freedom allows, for example, the measurement of the spin dependent structure functions of the proton and neutron (see HERMES experiment at HERA [198]) and thus allows the extraction of more detailed information from the experiments. Therefore, from the physical point of view it is highly desirable to have high energy polarized particle beams [199]. In general, this requires

- sources for polarized particles
- acceleration of these particles and
- preservation of the polarization degree over a period of several hours (luminosity run time).

In this short section we will discuss some of the problems and topics which are related to the single particle dynamics in storage rings. Instead of giving an exhaustive review of this field we will only illustrate the basic ideas and results. Much more information can be found in the proceedings of the various SPIN conferences (see, for example, [200]) or in the review article [201]. Furthermore, we will mainly concentrate on electron storage rings.

We have shown how radiation phenomena influence the orbital dynamics in this case, and we have sketched the semiclassical theory, where the quantum effects are simulated by explicit stochastic elements in the equations of motion. Furthermore we have seen how the equilibrium beam sizes arise from a balance between radiation damping and quantum excitations.

Radiative phenomena also strongly influence the spin dynamics in electron storage rings. In these machines there is a natural polarization mechanism due to spin-flip synchrotron radiation. The kinetics for this process can be modelled by a simple rate equation. Consider a system of n_0 spins and a reference direction specified by a homogeneous and constant magnetic field. The polarization degree along this direction is given by

$$P = \frac{n_+ - n_-}{n_0} \tag{190}$$

where n_+ designates the number of particles which are aligned parallel to the external magnetic field, and where n_- designates the number of spins which are aligned anti-parallel to this field. Due to spin-flip radiative processes n_+ and n_- will change in time and one obtains

$$\frac{d}{dt}P = (w_{-+} - w_{+-}) - (w_{-+} + w_{+-})\ P\ . \tag{191}$$

w_{+-}, w_{-+} are the spin-flip probabilities from parallel to anti-parallel and from anti-parallel to parallel alignment, respectively. The solution of this rate equation is

$$P(t) = P_{eq}(1 - e^{-\frac{t}{\tau_p}}) \tag{192}$$

with

$$P_{eq} = \frac{w_{-+} - w_{+-}}{w_{-+} + w_{+-}} \tag{193}$$

and

$$\tau_p = (w_{-+} + w_{+-})\ . \tag{194}$$

The flip probabilities have been calculated for the first time by Sokolov and Ternov using the Dirac equation for an electron in a constant and homogeneous magnetic field (for a summary of these results see [150], [202]) and with these results one obtains

$$P_{eq} = -92.38 \tag{195}$$

and

$$\frac{1}{\tau_p} = \frac{5\sqrt{3}e^2\gamma^5\hbar}{8m_0^2c^2R^3}\ . \tag{196}$$

R is the radius of the particle trajectory in the homogeneous magnetic field. Thus, as a consequence of the spin-flip radiation electrons become polarized anti-parallel to the direction of the bending field (transverse polarization). For example, at HERA the polarization time is about 40 min. at 27 GeV.

In a real storage ring the situation is much more complicated due to the weakly inhomogeneous and nonlinear fields of the various beam line elements

such as dipoles, quadrupoles, rf-fields, space charge fields of the counter rotating beam and multipoles. The derivation of a kinetic equation for the polarization of an ensemble of ultrarelativistic charged particles with spin in these realistic fields of an accelerator is a very complicated problem [203], [204], [205], [206], [207], [208] . Such a general kinetic description should not only include the radiative polarization mechanism but also all depolarizing influences. A treatment of this topic is beyond the scope of this review. In this work we restrict ourselves to the question: *What are the mechanisms that can cause considerable depolarization of an initially polarized beam?* .

As in the orbital theory we use a semiclassical theory which is based on the Thomas-Bargman-Michel-Telegdi (TBMT) equation. This equation is the relativistic covariant generalization of the Lamor equation and describes the classical spin (polarization) motion of a relativistic particle under the influence of electromagnetic fields. A derivation of this equation can be found in [13], [209] and [210]. Using the arclength s as independent variable one obtains (see also Appendix D)

$$\frac{d}{ds}\vec{S}(s) = \vec{\Omega}(\vec{y}(s)) \times \vec{S}(s) \tag{197}$$

where $\vec{\Omega}(\vec{y}(s))$ depends on the orbital degrees of freedom. (197) describes the coupled spin-orbit motion in an accelerator and is the starting point for depolarization investigations. The main depolarizing mechanism in a real storage ring with closed orbit (field) errors, with a coupling between the different oscillation modes and with various nonlinearities is due to the stochastic emission of synchrotron light in the orbital part of the motion. The recoil the electron experiences during the emission process induces an orbital motion of the particle and on this stochastically induced trajectory the particle spin sees perturbing fields which lead to a tilt away from the original polarization direction and thus to a depolarization. The situation we encounter is similar to the problem of Brownian motion on the unit sphere, or to the derivation of the Bloch equations [137].

Extending the six-dimensional orbit analysis to eight dimensions by adding the two independent spin degrees of freedom we can calculate spin diffusion effects in storage rings [179], [180], [211]. The magnitude of this diffusion effect is described by a depolarization time τ_d, which depends strongly on machine imperfections as mentioned above. Especially strong depolarization is expected, when the so-called spin tune $a\gamma$ fulfills the following resonance

condition

$$a\gamma = n \pm Q_{x,z,\tau} \tag{198}$$

with γ Lorentz factor, a anomalous magnetic moment of the electron, n integer number and $Q_{x,z,\tau}$ horizontal betatron tune, vertical betatron tune and synchrotron tune respectively. The main steps of the theory are outlined in Appendix D.

The kinetic equation derived from this simplified model then reads

$$\frac{d}{dt}P = \frac{1}{\tau_p}(P_{eq} - P) - \frac{1}{\tau_d}P$$

with the stationary polarization degree

$$P_\infty = \frac{P_{eq}}{1 + \frac{\tau_p}{\tau_d}} \ .$$

The simple model calculations sketched above have been very helpful for an understanding of the depolarization mechanisms in storage rings such as PETRA, DORIS and HERA [212], [213], [214] and they form the basis of various optimization and spin matching schemes [212], [215] [216], [217], [218]. The main idea of these schemes is to decouple the spin motion as much as possible from the orbit dynamics which is induced by the photon emission. Polarized beams have been investigated in many other electron storage rings too. In HERA these polarized beams are now used and handled routinely [219]. HERA is especially interesting because with the help of spin rotators one can obtain longitudinally polarized electron beams at the interaction point. A measurement of the polarization in HERA is shown in Figure 35 [220].

Before we mention the situation in proton storage rings we make two remarks:

1. from the TBMT equation one can derive simple toy models. For example in [221], [222] we investigated a two-dimensional spin vector (constrained to a plane) under the influence of a random perpendicular stochastic field $x(t)$ (diffusion on the circle see Figure 36) with the stochastic differential equations derived from the Hamiltonian

$$\mathcal{H}(\phi, S) = S\ x(t) \tag{199}$$

Figure 35: electron polarization in HERA

namely

$$\frac{d}{dt}S = 0$$

$$\frac{d}{dt}\phi = x(t)\ . \tag{200}$$

The fluctuating field $x(t)$ is determined by

$$\frac{d^2}{dt^2}x(t) + \alpha\frac{d}{dt}x(t) + \omega^2 x(t) = \sigma\zeta(t) \tag{201}$$

($\zeta(t)$ Gaussian white noise and α, ω, σ, constants). The model equations (200) and (201) have been used to investigate spin decoherence effects in electron storage rings.

2. T.Boyer [223] studied the motion of a classical spinning magnetic dipole under the influence of a random radiation field with the help of the following equation

$$\frac{d}{dt}\vec{S} = \alpha\vec{S}\ \times\ \vec{B}_0 - \frac{2}{3}c^{-3}\alpha^2\ \vec{S}\ \times\ \frac{d^3}{dt^3}\vec{S} + \alpha\ \vec{S}\ \times\ \vec{B}_R(t)\ . \tag{202}$$

The first part is the usual Lamor part in a constant magnetic field $\vec{B}_0$. The second part describes the radiation damping of a classical

Figure 36: diffusion on the circle

precessing magnetic moment and the third part is due to the random field. α is a constant and specifies the relation between the spin angular momentum $\vec{S}$ and the magnetic dipole moment $\vec{\mu}$

$$\vec{\mu} = \alpha \vec{S} \ .$$

As in the storage ring case one expects a stationary equilibrium for the orientation of the system as a consequence of the balance between damping and random fluctuating torques due to the random classical electromagnetic radiation.

We will end this short section on spin dynamics in storage rings with a few remarks about proton machines. Because of the negligible radiation effects the most efficient way to get polarized proton beams is to accelerate polarized particles and to preserve the polarization degree. This requires a careful study of the depolarizing mechanisms (resonances) of the deterministic spin-orbit dynamics described by the TBMT equation

$$\frac{d}{ds}\vec{S}(s) = \Omega(\vec{y}(s)) \times \vec{S}(s)$$

$$\frac{d}{ds}\vec{y}(s) = \underline{S}\frac{\partial \mathcal{H}(\vec{y}(s))}{\partial \vec{y}}$$

($\mathcal{H}$ Hamiltonian of the coupled synchro-betatron motion).

The question how the polarization of an ensemble of spins behaves in time in this case is a complicated problem - especially if one includes the full nonlinear orbit dynamics (chaos and nonlinear resonances) - and this question will not be discussed further here. More details about these topics and about the extension of the Hamiltonian description of the orbit motion to the coupled spin-orbit dynamics, the corresponding one-turn maps, and about suitable perturbative and numerical tools can be found, for example, in [224], [225], [226] and [227]

5 Summary and conclusions

The main aim of this review was to illustrate the problems of the linear and nonlinear beam dynamics in storage rings and to introduce various concepts and tools to study these systems. Although a lot of facts are known and powerful techniques - especially in the Hamiltonian case - have been developed, designing new accelerators (like the LHC) remains a challenging problem of nonlinear dynamics. The design of such a machine requires a lot of numerical simulations, the knowledge of the basic facts of the qualitative theory of dynamical systems and perturbative investigations. Besides these theoretical issues, future accelerator developments will always rely on the experience with existing machines and on special nonlinear dynamics experiments performed with these machines.

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A Derivation of the Hamiltonian for the Coupled Synchro- Betatron Motion

In this appendix we will sketch the derivation of the Hamiltonian of the coupled synchro-betatron motion [14].

Starting point is the following relativistic Lagrangian for a charged particle under the influence of an electromagnetic field specified by the vector-potential $\vec{A}(\vec{r},t)$ and the scalar potential $\phi(\vec{r},t)$

$$\mathcal{L} = -m_0c^2\sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}} + \frac{e}{c}(\dot{\vec{r}}\vec{A}) - e\phi.$$

We introduce the curvilinear coordinate system depicted in Figure 6 via

$$\vec{r} = \vec{r}_0(s) + \delta\vec{r}(s)$$

where $\vec{r}_0(s)$ specifies the position on the design trajectory (which we assume to ly in a plane, so that there is no torsion) and where s designates the arclength along this trajectory from a certain origin. $\delta\vec{r}(s)$ is defined by

$$\delta\vec{r}(s) = x\ \vec{e}_x + z\ \vec{e}_z$$

with

$$\frac{d}{ds}\vec{e}_x = \kappa(s)\vec{e}_\tau$$

and

$$\frac{d}{ds}\vec{e}_z = 0$$

where κ designates the curvature of the design trajectory and where $\vec{e}_x, \vec{e}_z, \vec{e}_\tau$ are orthogonal unit vectors related to this design orbit. Using this coordinate system one obtains

$$\dot{\vec{r}} = \vec{e}_\tau\dot{s}(1+\kappa\ x) + \dot{x}\ \vec{e}_x + \dot{z}\ \vec{e}_z$$

and in the new coordinates x, z, s the Lagrangian then becomes

$$\begin{aligned}\mathcal{L}(x,z,s,\dot{x},\dot{z},\dot{s},t) &= \\ &= -m_0c^2(1-\frac{1}{c^2}(\dot{x}^2+\dot{z}^2+\dot{s}^2(1+\kappa x)^2))^{\frac{1}{2}} + \\ &+\frac{e}{c}(\dot{x}A_x + \dot{z}A_z + (1+\kappa x)\dot{s}A_\tau) - e\phi. \end{aligned} \tag{203}$$

The Hamiltonian is given by

$$\mathcal{H} = p_x\dot{x} + p_z\dot{z} + p_s\dot{s} - \mathcal{L} \tag{204}$$

with the canonical momenta

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{m_0\dot{x}}{(1-\frac{\dot{\vec{r}}^2}{c^2})^{\frac{1}{2}}} + \frac{e}{c}A_x \tag{205}$$

$$p_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = \frac{m_0\dot{z}}{(1-\frac{\dot{\vec{r}}^2}{c^2})^{\frac{1}{2}}} + \frac{e}{c}A_z \tag{206}$$

$$p_s = \frac{\partial \mathcal{L}}{\partial \dot{s}} = \frac{m_0\dot{s}}{(1-\frac{\dot{\vec{r}}^2}{c^2})^{\frac{1}{2}}}(1+\kappa x)^2 + \frac{e}{c}(1+\kappa x)A_\tau. \tag{207}$$

Introducing these momenta into the Hamiltonian (204) one obtains

$$\begin{aligned}\mathcal{H}(x,z,s,p_x,p_z,p_s,t) &= \\ &= c[m_0^2c^2 + (p_x - \frac{e}{c}A_x)^2 + (p_z - \frac{e}{c}A_z)^2 + (\frac{p_s}{(1+\kappa x)} - \frac{e}{c}A_\tau)^2]^{\frac{1}{2}} \\ &+e\phi\end{aligned} \tag{208}$$

with the corresponding equations of motion

$$\begin{aligned}\frac{d}{dt}\,x &= +\frac{\partial \mathcal{H}}{\partial p_x}; \qquad \frac{d}{dt}p_x = -\frac{\partial \mathcal{H}}{\partial x} \\ \frac{d}{dt}\,z &= +\frac{\partial \mathcal{H}}{\partial p_z}; \qquad \frac{d}{dt}p_z = -\frac{\partial \mathcal{H}}{\partial z} \\ \frac{d}{dt}\,s &= +\frac{\partial \mathcal{H}}{\partial p_s}; \qquad \frac{d}{dt}p_s = -\frac{\partial \mathcal{H}}{\partial s}\end{aligned}$$

As the next step we introduce the arclength s along the design trajectory as independent variable [228]. This can be achieved by using the fact that the Hamilton equations of motion can be derived from a variational principle

$$\delta \int_{t_1}^{t_2} (\dot{x}p_x + \dot{z}p_z + \dot{s}p_s - \mathcal{H}(x,z,s,p_x,p_z,p_s,t))dt = 0 \tag{209}$$

and that with $dt = \frac{dt}{ds} ds$ one gets

$$\delta \int_{s_1}^{s_2} (x' p_x + z' p_z + t'(-\mathcal{H}) + p_s) ds = 0 \tag{210}$$

where we have put $(.)' = \frac{d}{ds}$. Comparing (209) and (210) we see that we can interprete $-p_s(x, z, t, p_x, p_z, -\mathcal{H}; s)$ as the new Hamilton-function with s as independent variable (for a more careful analysis see [228]). This Hamilton function which we designate by $\mathcal{K}$ is given by

$$\begin{aligned} &\mathcal{K}(x, z, t, p_x, p_z, -\mathcal{H}; s) = -p_s(x, z, t, p_x, p_z, -\mathcal{H}; s) = \\ &= -(1 + \kappa x)[\frac{(\mathcal{H} - e\phi)^2}{c^2} - m_0^2 c^2 - (p_x - \frac{e}{c} A_x)^2 - (p_z - \frac{e}{c} A_z)^2]^{\frac{1}{2}} - \\ &-(1 + \kappa x) \frac{e}{c} A_\tau \end{aligned} \tag{211}$$

and the corresponding equations of motion with s as independent variable read

$$\begin{aligned} \frac{d}{ds}\, x &= +\frac{\partial \mathcal{K}}{\partial p_x}; \qquad \frac{d}{ds}\, p_x = -\frac{\partial \mathcal{K}}{\partial x} \\ \frac{d}{ds}\, z &= +\frac{\partial \mathcal{K}}{\partial p_z}; \qquad \frac{d}{ds}\, p_z = -\frac{\partial \mathcal{K}}{\partial z} \\ \frac{d}{ds}\, t &= +\frac{\partial \mathcal{K}}{\partial(-\mathcal{H})}; \qquad \frac{d}{ds}\, (-\mathcal{H}) = -\frac{\partial \mathcal{K}}{\partial t} \end{aligned} \tag{212}$$

Choosing a gauge such that $\phi = 0$ and introducing the variables $-c\, t$ and $\hat{p_\tau}$ with $\hat{p_\tau} = \frac{E - E_0}{E_0}$ one obtains the following equations of motion

$$\begin{aligned} \frac{d}{ds}\, x &= +\frac{\partial \bar{\mathcal{K}}}{\partial \hat{p_x}}; \qquad \frac{d}{ds}\, \hat{p_x} = -\frac{\partial \bar{\mathcal{K}}}{\partial x} \\ \frac{d}{ds}\, z &= +\frac{\partial \bar{\mathcal{K}}}{\partial \hat{p_z}}; \qquad \frac{d}{ds}\, \hat{p_z} = -\frac{\partial \bar{\mathcal{K}}}{\partial z} \\ \frac{d}{ds}(-ct) &= +\frac{\partial \bar{\mathcal{K}}}{\partial \hat{p_\tau}}; \qquad \frac{d}{ds}\, \hat{p_\tau} = -\frac{\partial \bar{\mathcal{K}}}{\partial(-ct)} \end{aligned} \tag{213}$$

with $\hat{p_x} = \frac{c}{E_0} p_x, \hat{p_z} = \frac{c}{E_0} p_z$ and $\bar{\mathcal{K}} = \frac{c}{E_0} \mathcal{K}$

$\bar{\mathcal{K}}(x, z, -ct, \hat{p_x}, \hat{p_z}, \hat{p_\tau}; s) =$

$$= -(1+\kappa x)[(1+\hat{p_\tau})^2 - (\frac{m_0 c^2}{E_0})^2 - (\hat{p_x} - \frac{e}{E_0}A_x)^2 - (\hat{p_z} - \frac{e}{E_0}A_z)^2]^{\frac{1}{2}} -$$
$$-(1+\kappa x)\frac{e}{E_0}A_\tau; \ . \tag{214}$$

In the next step we introduce the difference variable $\tau = s - ct(s)$ via the generating function F_3 depending on the old momenta and new coordinates

$$F_3(\bar{x}, \bar{z}, \tau, \hat{p_x}, \hat{p_z}, \hat{p_\tau}, s) = -\hat{p_x}\bar{x} - \hat{p_z}\bar{z} - \hat{p_\tau}\tau + s\hat{p_\tau} + s \tag{215}$$

with the transformation equations $\bar{x} = x, \bar{z} = z, \bar{p_x} = \hat{p_x}, \bar{p_z} = \hat{p_z}$ and

$$(-ct) = -\frac{\partial F_3}{\partial \hat{p_\tau}} = \tau - s$$

$$\bar{p_\tau} = -\frac{\partial F_3}{\partial \tau} = \hat{p_\tau} \ .$$

Neglecting the term $(\frac{m_0 c^2}{E_0})^2$ which is very small for ultrarelativistic particles one finally obtains (5).

B Floquet oscillators

In this appendix we will summarize some results for Floquet oscillators of the form (49) [50]. We will consider the general case

$$\mathcal{H}_0(q,p,s) = \frac{1}{2}F(s)p^2 + R(s)qp + \frac{1}{2}G(s)q^2 \tag{216}$$

where $F(s), R(s), G(s)$ are periodic functions of s i.e $F(s+L) = F(s), R(s+L) = R(s), G(s+L) = G(s)$. The equations of motion can be written as

$$\frac{d}{ds}\begin{pmatrix} q(s) \\ p(s) \end{pmatrix} = \begin{pmatrix} R(s) & F(s) \\ -G(s) & -R(s) \end{pmatrix}\begin{pmatrix} q(s) \\ p(s) \end{pmatrix} \tag{217}$$

or in shorthand notation

$$\frac{d}{ds}\vec{y}(s) = \underline{A}(s)\vec{y}(s)\ . \tag{218}$$

The general solution of (217), (218) is given by

$$\vec{y}(s) = \underline{M}(s,s_0)\vec{y}(s_0) \tag{219}$$

where $\underline{M}(s,s_0)$ satisfies

$$\frac{d}{ds}\underline{M}(s,s_0) = \underline{A}(s)\underline{M}(s,s_0). \tag{220}$$

$\underline{M}(s,s_0)$ is a symplectic matrix i.e.

$$\underline{M}^T(s,s_0)\underline{S}\underline{M}(s,s_0) = \underline{S}$$

with the symplectic unity

$$\underline{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{221}$$

Proof: Introducing

$$\underline{B}(s) = \underline{M}^T(s,s_0)\ \underline{S}\ \underline{M}(s,s_0)$$

with $\underline{B}(s_0) = \underline{S}$ one obtains

$$\frac{d}{ds}\underline{B}(s) = (\frac{d}{ds}\underline{M}^T(s,s_0))\ \underline{S}\ \underline{M}(s,s_0) + \underline{M}^T(s,s_0)\underline{S}(\frac{d}{ds}\underline{M}(s,s_0)) \qquad (222)$$

or with (217)

$$\frac{d}{ds}\underline{B}(s) = \underline{M}^T(s,s_0)\ [\underline{A}^T\underline{S} + \underline{S}\underline{A}]\ \underline{M}(s,s_0) = 0\ . \qquad (223)$$

Thus $\underline{B}(s) = \text{const} = \underline{B}(s_0) = \underline{S}$ (q.e.d) . Following [52] the one-turn matrix $\underline{M}(s+L,s)$ can be parametrized by

$$\underline{M}(s+L,s) = \cos\mu\ \underline{1} + \sin\mu\ \underline{J}(s) \qquad (224)$$

with

$$\underline{J}(s) = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix}. \qquad (225)$$

Since $det(\underline{M}) = 1$ we have

$$\gamma\ \beta - \alpha^2 = 1\ \ . \qquad (226)$$

From the condition

$$\underline{M}(s+L,s) = \underline{M}(s+L,s_0+L)\ \underline{M}(s_0+L,s_0)\ \underline{M}(s_0,s) \qquad (227)$$

we also have (tr = trace of a matrix)

$$\cos\mu(s) = \frac{1}{2}tr\underline{M}(s+L,s) = \frac{1}{2}tr\underline{M}(s_0+L,s_0) = \cos\mu(s_0) \qquad (228)$$

i.e. $\mu = const$. From (224) and (227) we get

$$\underline{J}(s) = \underline{M}(s,s_0)\ \underline{J}(s_0)\ \underline{M}^{-1}(s,s_0) \qquad (229)$$

which can be used to prove that $\underline{J}(s)$ is periodic

$$\underline{J}(s_0+L) = \underline{M}(s_0+L,s_0)\ \underline{J}(s_0)\ \underline{M}^{-1}(s_0+L,s_0) = \underline{J}(s_0)\ . \qquad (230)$$

The (optical) functions $\alpha(s), \beta(s), \gamma(s)$ satisfy the following equations of motion

$$\frac{d}{ds}\alpha(s) = -\gamma(s)F(s) + \beta(s)G(s) \tag{231}$$

$$\frac{d}{ds}\beta(s) = 2(\beta(s)R(s) - \alpha(s)F(s)) \tag{232}$$

$$\frac{d}{ds}\gamma(s) = 2(\alpha(s)G(s) - \gamma(s)R(s)) \tag{233}$$

where we have used

$$\begin{aligned}
\frac{d}{ds}\underline{J}(s) &= \lim_{\Delta s\to 0}\frac{1}{\Delta s}[\underline{J}(s+\Delta s) - \underline{J}(s)] = \\
&= \lim_{\Delta s\to 0}\frac{1}{\Delta s}[\underline{M}(s+\Delta s, s)\,\underline{J}(s)\,\underline{M}^{-1}(s+\Delta s, s) - \underline{J}(s)] = \\
&= \lim_{\Delta s\to 0}\frac{1}{\Delta s}[(\underline{1} + \Delta s\underline{A}(s))\,\underline{J}(s)\,(\underline{1} - \Delta s\underline{A}(s)) - \underline{J}(s)] \\
&= \underline{A}(s)\,\underline{J}(s) - \underline{J}(s)\,\underline{A}(s).
\end{aligned} \tag{234}$$

Using these optical functions and the following generating function for a canonical transformation $(q,p) \to (I,\phi)$

$$F_1(q,\phi) = -\frac{q^2}{2\beta}\,[\tan(\phi+\phi_0) + \alpha] \tag{235}$$

one obtains

$$q(s) = \sqrt{2\beta(s)I}\cos(\phi(s)+\phi_0) \tag{236}$$

and

$$p(s) = -\sqrt{\frac{2I}{\beta(s)}}[\sin(\phi(s)+\phi_0) + \alpha(s)\cos(\phi(s)+\phi_0)] \tag{237}$$

where I is given by

$$I = \frac{1}{2}[\gamma\,q^2 + \beta\,p^2 + 2\alpha\,pq]. \tag{238}$$

The transformed Hamiltonian $\bar{\mathcal{H}}_0$ now reads

$$\bar{\mathcal{H}}_0(I,\phi) = \frac{F(s)}{\beta(s)}\, I \tag{239}$$

and the corresponding equations of motion are given by

$$\frac{d}{ds}\phi(s) = \frac{\partial \bar{\mathcal{H}}_0}{\partial I} = \frac{F(s)}{\beta(s)} \tag{240}$$

and

$$\frac{d}{ds} I = -\frac{\partial \bar{\mathcal{H}}_0}{\partial \phi} = 0\ . \tag{241}$$

Thus I is a constant of the motion. The Hamiltonian (239) can be simplified further by using a second canonical transformation $(I,\phi) \to (J,\psi)$ generated by

$$F_2(\phi, J) = J\,[\frac{2\pi Q}{L}\, s - \int_0^s d\hat{s} \frac{F(\hat{s})}{\beta(\hat{s})}] + \phi\, J \tag{242}$$

where Q is defined by

$$Q = \frac{1}{2\pi} \int_0^L d\hat{s} \frac{F(\hat{s})}{\beta(\hat{s})}\ . \tag{243}$$

The transformed Hamiltonian now reads

$$\bar{\bar{\mathcal{H}}}_0 = \frac{2\pi Q}{L}\, J \tag{244}$$

and the solutions of the corresponding equations of motion are

$$J = I = const \quad ; \quad \psi(s) = \psi(s_0) + \frac{2\pi Q}{L}(s - s_0)\ . \tag{245}$$

Using $\phi_0 = 0$, $\phi_0 = \frac{\pi}{2}$ and $I = \frac{1}{4}$ in (236) and (237) one obtains two independent solutions $\vec{u}_1(s)$ and $\vec{u}_2(s)$

$$\vec{u}_1(s) = \begin{pmatrix} \sqrt{\frac{\beta(s)}{2}} \cos\phi(s) \\ -\sqrt{\frac{1}{2\beta(s)}}[\sin\phi(s) + \alpha(s)\cos\phi(s)] \end{pmatrix} \tag{246}$$

$$\vec{u}_2(s) = \begin{pmatrix} \sqrt{\frac{\beta(s)}{2}} \sin\phi(s) \\ \sqrt{\frac{1}{2\beta(s)}}[\cos\phi(s) - \alpha(s)\sin\phi(s)] \end{pmatrix} \tag{247}$$

The general solution is then

$$\vec{u}(s) = c_1\ \vec{u}_1(s) + c_2\ \vec{u}_2(s)\ . \tag{248}$$

Given these solution vectors it is possible to determine μ. From

$$(\vec{u}_1(s+L), \vec{u}_2(s+L)) = \underline{M}(s+L,s)\ (\vec{u}_1(s), \vec{u}_2(s)) \tag{249}$$

we get

$$\underline{M}(s+L,s) = \underline{1}\cos[\phi(s+L) - \phi(s)] + \underline{J}(s)\sin[\phi(s+L) - \phi(s)] \tag{250}$$

and thus

$$\mu = \phi(s+L) - \phi(s)\ . \tag{251}$$

Using

$$\phi(s+L) - \phi(s) = \int_s^{s+L} d\hat{s}\frac{F(\hat{s})}{\beta(\hat{s})} = 2\pi Q \tag{252}$$

(see (240)) we can finally write

$$\mu = 2\pi Q\ . \tag{253}$$

For further developments it is also useful to introduce the solution vectors $\vec{v}_I$ and $\vec{v}_{-I}$

$$\vec{v}_I(s) = \vec{u}_1(s) - i\vec{u}_2(s) = \frac{1}{\sqrt{2\beta(s)}} e^{-i\phi(s)} \begin{pmatrix} \beta(s) \\ -[\alpha(s) + i] \end{pmatrix} \tag{254}$$

$$\vec{v}_{-I}(s) = \vec{u}_1(s) + i\vec{u}_2(s) = \vec{v}_I^*(s) \tag{255}$$

which are eigenvectors of the one turn matrix $\underline{M}(s+L,s)$

$$\underline{M}(s+L,s)\ \vec{v}_I(s) = \lambda_I \vec{v}_I(s) \tag{256}$$

$$\underline{M}(s+L,s)\,\vec{v}_{-I}(s) = \lambda_{-I}\vec{v}_{-I}(s) \tag{257}$$

with the eigenvalues

$$\lambda_I = e^{-i\,2\pi Q} \tag{258}$$

$$\lambda_{-I} = e^{+i\,2\pi Q} \tag{259}$$

and the "normalization" conditions

$$\vec{v}_I^{+}(s)\,\underline{S}\,\vec{v}_I(s) = -i \tag{260}$$

$$\vec{v}_{-I}^{+}(s)\,\underline{S}\,\vec{v}_{-I}(s) = +i\ . \tag{261}$$

($\vec{a}^{+}$ is defined by $(\vec{a}^{\star})^T$ and $\star$ means complex conjugation). Furthermore, Floquet theory applies and we have

$$\vec{v}_I(s) = \vec{\hat{v}}_I(s)e^{-i2\pi Q\frac{s}{L}}\ \ ;\ \ \vec{v}_{-I}(s) = \vec{\hat{v}}_{-I}(s)e^{+i2\pi Q\frac{s}{L}} \tag{262}$$

with

$$\vec{\hat{v}}_I(s+L) = \vec{\hat{v}}_I(s)\ \ ;\ \ \vec{\hat{v}}_{-I}(s+L) = \vec{\hat{v}}_{-I}(s)\ . \tag{263}$$

These relations will be useful in many calculations. Stability of the particle motion is only guaranteed if

$$Q,\ \mu\ \in \mathbf{R}\ . \tag{264}$$

C Linear beam emittance matrix for electron storage rings

In this appendix we will outline the calculation of the beam emittance matrix of an electron storage ring in the linear approximation i.e we seek a solution of (175)

$$\frac{d}{ds}\vec{y}(s) = (\underline{A}(s) + \underline{\delta A}(s))\vec{y}(s) + \vec{c}(s) + \delta\vec{c}(s) \ .$$

In the following we take $\vec{y}(s)$ in the form

$$\vec{y}^T(s) = (x, p_x, z, p_z, \tau, p_\tau) \ .$$

The fluctuating part in the linear approximation is given by

$$\delta\vec{c}^T(s) = (0, 0, 0, 0, 0, \delta c(s))$$

with the correlation function

$$< \delta c(s)\ \delta c(s') >= |\kappa^3(s)|\frac{55 r_e \gamma_0^5 \hbar}{24\sqrt{3} m_0 c}\delta(s - s') \ . \tag{265}$$

(For a complete list of all the other matrix elements of the different beam line elements the reader is refered to [15], [17], [161], [180]).

Due to the interplay between radiation damping and quantum excitation we expect some equilibrium fluctuations around a periodic closed orbit. We separate the general solution of (175) into two parts

- closed orbit part $\vec{y}_0(s)$ and
- fluctuation part around $\vec{y}_0(s)$ which we designate by $\tilde{\vec{y}}(s)$.

The closed orbit part satisfies

$$\frac{d}{ds}\vec{y}_0(s) = (\underline{A}(s) + \underline{\delta A}(s))\vec{y}_0(s) + \vec{c}(s) \tag{266}$$

and the fluctuation part fulfills

$$\frac{d}{ds}\tilde{\vec{y}}(s) = (\underline{A}(s) + \underline{\delta A}(s))\tilde{\vec{y}}(s) + \delta\vec{c}(s) \ . \tag{267}$$

For $\tilde{\vec{y}}(s)$ we make the following ansatz (variation of constants)

$$\tilde{\vec{y}}(s) = \sum_{k=I,II,III} [A_k(s)\vec{v}_k(s) + A_{-k}(s)\vec{v}_{-k}(s)] \tag{268}$$

where the $\vec{v}_k(s), \vec{v}_{-k}(s)$ form a system of linear independent solutions of the unperturbed system ($\underline{\delta A}(s) = 0, \delta\vec{c}(s) = 0$). These solutions can be obtained as eigenvectors of the one turn matrix $\underline{M}(s+L,s)$

$$\underline{M}(s+L,s)\ \vec{v}_k(s) = \lambda_k \vec{v}_k(s)\ . \tag{269}$$

It can be shown that the eigenvalues λ_k are independent of s that

$$\lambda_k\ \lambda_{-k} = 1$$

and that the eigenvectors fulfill the condition (see also Appendix B)

$$\vec{v}_k^{+}(s)\underline{S}\vec{v}_k(s) = \vec{v}_{-k}^{+}(s)\underline{S}\vec{v}_{-k}(s) \neq 0$$

$$\vec{v}_\mu^{+}(s)\underline{S}\vec{v}_\nu(s) = 0 \quad \text{otherwise.}$$

For a more detailed investigation of the eigenvalue spectrum of the general six-dimensional transfer maps of a linear accelerator see the references mentioned above and Figure 17.

Then, the functions $A_k(s)$ are determined by

$$\frac{d}{ds}A_k(s) = -iA_k(s)\frac{2\pi}{L}\delta Q_k + iv_{k5}^{\star}(s)\delta c(s) \tag{270}$$

where

$$\delta Q_k = \frac{1}{2\pi}\int_{s_0}^{s_0+L} d\tilde{s}\vec{v}_k^{+}(\tilde{s})\underline{S\delta A}(\tilde{s})\vec{v}_k(\tilde{s}) \tag{271}$$

is just the (complex) $Q-$shift of the k-th oscillation mode caused by the perturbation $\underline{\delta A}$.

Using these equations one obtains the stationary (periodic) second order moments of a linear electron storage ring in the form

$$< \tilde{y}_m(s)\tilde{y}_n(s) >^{st} = 2\sum_{k=I,II,III} < |A_k|^2 >^{st} Re[v_{km}(s)v_{kn}^{\star}(s)] \tag{272}$$

with

$$< |A_k|^2 >^{st} = \frac{1}{2\alpha_k} \int_{s_0}^{s_0+L} d\tilde{s}\ \omega(\tilde{s})\ |v_{k5}(\tilde{s})|^2 \tag{273}$$

and the "damping" constant

$$\alpha_k = \frac{i}{2} \int_{s_0}^{s_0+L} d\tilde{s}\ \vec{v}_k^{+}(\tilde{s})\ [\underline{S}\ \underline{\delta A}(\tilde{s}) + \underline{\delta A}^T(\tilde{s})\ \underline{S}]\ \vec{v}_k(\tilde{s}) \quad \text{for } k = I, II, III \tag{274}$$

$\omega(s)$ is the prefactor of the δ-function in equation (265) .

In the limit of a simple uncoupled storage ring these general results reduce to the well known expressions in [156] and in other textbooks on accelerator physics.

D Spin diffusion in electron storage rings

In this appendix we will outline the calculation of the depolarization time τ_d. We will only list the main steps, the details of the calculation can be found in [180].

Starting point of our investigation is the TBMT equation

$$\frac{d}{dt}\vec{S}(t) = \vec{\Omega}_{TBMT} \times \vec{S}(t)$$

with

$$\vec{\Omega}_{TBMT} = -\frac{e}{m_0 \gamma c}[(1+\gamma a)\vec{B} - \frac{a\gamma^2}{1+\gamma}\frac{1}{c^2}(\dot{\vec{r}}\vec{B})\cdot\dot{\vec{r}} - (a\gamma + \frac{\gamma}{1+\gamma})\dot{\vec{r}} \times \frac{\vec{\varepsilon}}{c}] \; .$$

(a anomalous magnetic moment and $\gamma = \frac{E}{m_0 c^2}$). Using the accelerator coordinate system and the independent variable s this equation can be rewritten in the form [229]

$$\frac{d}{ds}\vec{S}(s) = \underline{\Omega}(\vec{y}(s)) \; \vec{S}(s) \; .$$

If one restricts the investigation to the linear synchro-betatron motion, $\underline{\Omega}(\vec{y}(s))$ is linear in the orbital coordinates [229], and in analogy to the orbit dynamics one separates $\underline{\Omega}(\vec{y}(s))$ into two parts - a part depending only on the periodic closed orbit $\vec{y}_0(s)$ and a part depending on the fluctuating amplitude $\tilde{\vec{y}}(s)$

$$\underline{\Omega}(\vec{y}(s)) = \underline{\Omega}^{(0)}(\vec{y}_0(s)) + \underline{\omega}(\tilde{\vec{y}}(s)) \; . \tag{275}$$

The TBMT equation then reads

$$\frac{d}{ds}\vec{S}(s) = (\underline{\Omega}^{(0)}(\vec{y}_0(s)) + \underline{\omega}(\tilde{\vec{y}}(s))) \; \vec{S}(s) \; . \tag{276}$$

At first one solves the spin motion on the closed orbit $\vec{y}_0(s)$

$$\frac{d}{ds}\vec{S}_0(s) = \underline{\Omega}^{(0)}(\vec{y}_0(s)) \; \vec{S}_0(s) \; . \tag{277}$$

The general solution of equation (277) is a precession around a periodic axis. For the orthogonal one-turn (rotation) matrix $\underline{N}(s_0 + L, s_0)$ one obtains

$$\underline{N}(s_0 + L, s_0) \; \vec{r}_\mu(s_0) = \alpha_\mu \vec{r}_\mu(s_0) \quad \mu = 1, 2, 3 \; . \tag{278}$$

The eigenvalues α_μ are independent of the chosen position s_0 and are given by

$$\alpha_1 = 1 \tag{279}$$

$$\alpha_2 = e^{i2\pi\nu} \tag{280}$$

$$\alpha_3 = e^{-i2\pi\nu} \tag{281}$$

(ν real) and the corresponding eigenvectors $\vec{r}_\mu(s_0)$ can be used to construct a righthanded orthogonal coordinate system $\vec{n}_0(s_0), \vec{m}_0(s_0), \vec{l}_0(s_0)$ [229] via

$$\vec{r}_1(s_0) = \vec{n}_0(s_0) \tag{282}$$

$$\vec{r}_2(s_0) = \vec{m}_0(s_0) + i\vec{l}_0(s_0) \tag{283}$$

$$\vec{r}_3(s_0) = \vec{m}_0(s_0) - i\vec{l}_0(s_0) \tag{284}$$

$$\vec{n}_0(s_0) = \vec{m}_0(s_0) \times \vec{l}_0(s_0)\ . \tag{285}$$

Because of

$$\vec{n}_0(s) = \underline{N}(s, s_0)\ \vec{n}_0(s_0)$$

$$\vec{m}_0(s) = \underline{N}(s, s_0)\ \vec{m}_0(s_0)$$

$$\vec{l}_0(s) = \underline{N}(s, s_0)\ \vec{l}_0(s_0)$$

one can construct an orthogonal coordinate system at an arbitrary location s. Since $\vec{n}_0(s), \vec{m}_0(s), \vec{l}_0(s)$ do not form a periodic system

$$\vec{m}_0(s_0 + L) + i\vec{l}_0(s_0 + L) = \underline{N}(s_0 + L, s_0)\ [\vec{m}_0(s_0) + i\vec{l}_0(s_0)]$$

$$= e^{i2\pi\nu}[\vec{m}_0(s_0) + i\vec{l}_0(s_0)]$$

$$\neq \vec{m}_0(s_0) + i\vec{l}_0(s_0) \tag{286}$$

we introduce

$$\begin{pmatrix} \vec{m}(s) \\ \vec{l}(s) \end{pmatrix} = \begin{pmatrix} \cos[\psi(s)-\psi(s_0)] & \sin[\psi(s)-\psi(s_0)] \\ -\sin[\psi(s)-\psi(s_0)] & \cos[\psi(s)-\psi(s_0)] \end{pmatrix} \begin{pmatrix} \vec{m}_0(s) \\ \vec{l}_0(s) \end{pmatrix} \tag{287}$$

This system is periodic if the following condition is fulfilled

$$\psi(s_0+L)-\psi(s_0) = 2\pi\nu \ . \tag{288}$$

The vectors $\vec{n}_0(s), \vec{m}(s), \vec{l}(s)$ satisfy

$$\vec{n}_0(s) = \vec{m}(s) \times \vec{l}(s)$$

$$\frac{d}{ds}\vec{m}(s) = \underline{\Omega}^{(0)}(\vec{y}_0(s))\ \vec{m}(s) + \psi'(s)\vec{l}(s) \tag{289}$$

$$\frac{d}{ds}\vec{l}(s) = \underline{\Omega}^{(0)}(\vec{y}_0(s))\ \vec{l}(s) - \psi'(s)\vec{m}(s) \tag{290}$$

$$\frac{d}{ds}\vec{n}_0(s) = \underline{\Omega}^{(0)}(\vec{y}_0(s)) \ . \tag{291}$$

($\psi'(s) = \frac{d}{ds}\psi(s)$). Using this coordinate system we make the following ansatz for the solution of (276)

$$\vec{S}(s) = \sqrt{1-\alpha^2(s)-\beta^2(s)}\vec{n}_0(s) + \alpha(s)\vec{m}(s) + \beta(s)\vec{l}(s) \tag{292}$$

and in the linear approximation where one neglects terms which are nonlinear in $\alpha, \beta, \tilde{y}_i(s)$ one obtains

$$\frac{d}{ds}\alpha(s) = (\underline{\omega}(\tilde{\vec{y}}(s))\ \vec{n}_0(s)) \cdot \vec{m}(s) + \beta(s)\psi'(s) \tag{293}$$

$$\frac{d}{ds}\beta(s) = (\underline{\omega}(\tilde{\vec{y}}(s))\ \vec{n}_0(s)) \cdot \vec{l}(s) - \alpha(s)\psi'(s) \ . \tag{294}$$

Introducing the two-dimensional vector

$$\vec{\zeta}(s) = \begin{pmatrix} \alpha(s) \\ \beta(s) \end{pmatrix} \tag{295}$$

(293) and (294) can be combined into a single equation

$$\frac{d}{ds}\vec{\zeta}(s) = \underline{G_0}(s)\tilde{\vec{y}}(s) + \underline{D_0}(s)\vec{\zeta}(s) \tag{296}$$

where $\underline{G_0}$ describes the spin-orbit coupling [229] and where $\underline{D_0}$ is defined by

$$\underline{D_0}(s) = \begin{pmatrix} 0 & \psi'(s) \\ -\psi'(s) & 0 \end{pmatrix} \tag{297}$$

In the next step we introduce the eight-dimensional spin-orbit vector

$$\vec{u}(s) = \begin{pmatrix} \tilde{\vec{y}}(s) \\ \vec{\zeta}(s) \end{pmatrix} \tag{298}$$

and the TBMT equation finally reads

$$\frac{d}{ds}\vec{u}(s) = [\underline{\hat{A}}(s) + \underline{\delta\hat{A}}(s)]\vec{u}(s) + \delta\vec{\hat{c}}(s) \tag{299}$$

with the following definitions

$$\underline{\hat{A}}(s) = \begin{pmatrix} \underline{A}(s) & \underline{0} \\ \underline{G_0}(s) & \underline{D_0}(s) \end{pmatrix} \tag{300}$$

$$\underline{\delta\hat{A}}(s) = \begin{pmatrix} \underline{\delta A}(s) & \underline{0} \\ \underline{0} & \underline{0} \end{pmatrix} \tag{301}$$

$$\delta\vec{\hat{c}}(s) = \begin{pmatrix} \delta\vec{c}(s) \\ 0 \\ 0 \end{pmatrix} \tag{302}$$

$\underline{A}(s), \underline{\delta A}(s), \delta\vec{c}(s)$ describe the orbit dynamics see Appendix C. The solution of (299) is now in complete analogy to the calculation of the linear beam emittance matrix (see Appendix C)

$$\vec{u}(s) = \sum_{k=I,II,III,IV} [A_k(s)\vec{q}_k(s) + A_{-k}(s)\vec{q}_{-k}(s)] \; . \tag{303}$$

The $\vec{q}_k(s)$ form a system of linear independent solutions of the unperturbed system $\underline{\delta \hat{A}} = 0, \delta\vec{\hat{c}} = 0$ and can be obtained as eigenvectors of the eight-dimensinal transfer matrix

$$\underline{M}_{8\times 8}(s+L,s) = \begin{pmatrix} \underline{M}(s+l,s) & \underline{0}_{6\times 2} \\ \underline{G}(s+l,s) & \underline{D}(s+L,s) \end{pmatrix} \tag{304}$$

namely (see [229] for more details)

$$\underline{M}_{8\times 8}(s+L,s)\vec{q}_k(s) = \lambda_k \vec{q}_k(s) \ .$$

For $k = I, II, III$ the $\vec{q}_k(s)$ consist of the orbital vectors $\vec{v}_k(s)$ of Appendix C and a spin part $\vec{w}_k(s)$ with the eigenvalues λ_k of Appendix C (eigenvalues of the transfer map of the coupled synchro-betatron oscillations).

$$\vec{q}_k(s) = \begin{pmatrix} \vec{v}_k(s) \\ \vec{w}_k(s) \end{pmatrix} \tag{305}$$

$$\vec{w}_k(s) = -(\underline{D}(s+L,s) - \lambda_k \underline{1})^{-1}\underline{G}(s+L,s)\vec{v}_k(s) \tag{306}$$

$$\lambda_k = e^{-i2\pi Q_k} \ .$$

The fourth eigenvector $\vec{q}_{IV}(s)$ has only spin components

$$\vec{q}_{IV}(s) = \begin{pmatrix} \vec{0}_6 \\ \vec{w}_{IV}(s) \end{pmatrix} \tag{307}$$

$$\lambda_{IV} = e^{-i2\pi\nu}$$

$$\vec{w}_{IV}(s) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-i\psi(s)} \ . \tag{308}$$

(the vectors $\vec{q}_{-k}(s)$ are obtained by taking the complex conjugate of (305) and (307)). Using these results one can derive stochastic differential equations for $A_k(s)$, $k = I, II, III, IV$. These equations together with equation (303) allow the calculation of various moments of the coupled spin-orbit motion

such as the beam emittance matrix (see Appendix C) and the depolarization time τ_d which is related to $< |A_{IV}(s)|^2 >$ via [180]

$$\tau_d^{-1} = \frac{c}{L}[< |A_{IV}(s_0 + (N+1)L)|^2 > - < |A_{IV}(s_0 + NL)|^2 >] \; . \qquad (309)$$

After a lengthy calculation one obtains

$$\tau_d^{-1} = \frac{2c}{L} \int_{s_0}^{s_0+L} d\tilde{s}\omega(\tilde{s})([Im \sum_{k=I,II,III} (v_{k5}^*(\tilde{s})w_{k1}(\tilde{s}))]^2 + [Im \sum_{k=I,II,III} (v_{k5}^*(\tilde{s})w_{k2}(\tilde{s}))]^2) \; .$$

This expression was also obtained by A.Chao with a different method [179].

References

[1] P.Lefevre, T.Pettersson (ed) "The Large Hadron Collider - Conceptual Design" CERN/AC/95-05 (LHC) (1995)

[2] O.Brüning "An Analysis of the Long-Term Stability of the Particle Dynamics in Hadron Storage Rings" DESY 94-85 (1994)

[3] A. Piwinski "Recent Results from DORIS and PETRA" in **Nonlinear Dynamics and the Beam-Beam Interaction** AIP Conf. Proc 57 (1979)

[4] A.Sessler "Instabilities of Relativistic Particle Beams" in **International Conference on High Energy Accelerators Frascati 1965**, CNEN Roma (1966)

[5] W.Hardt,R.D.Kohaupt "On Bunch Lengthening and the Particle Distribution in Electron Storage Rings" DESY 77/20 (1977)

[6] A.W.Chao **Physics of Collective Beam Instabilities in High Energy Accelerators** J. Wiley (1993)

[7] W.Scandale, G.Turchetti (ed) **Nonlinear problems in future particle accelerators** World Scientific Press (1991)

[8] J.M.Jowett, M.Month, S.Turner (ed) **Nonlinear dynamics aspects of particle accelerators** Springer (1986)

[9] M.Month, A.G.Ruggiero, W.T.Weng (ed) **Stability of Particle Motion in Storage Rings** AIP Conf. Proc. 292 (1994)

[10] M.Berz, S.Martin, K.Ziegler (ed) **Nonlinear Problems in Accelerator Physics** Institute of Physics (1993)

[11] S.Chattopadhyay, M.Cornacchia, C.Pellegrini (ed) **Nonlinear Dynamics in Particle Accelerators: Theory and Experiments** AIP Conf. Proc. 344 (1995)

[12] P.Schmüser "Superconducting Magnets for Particle Accelerators" Rep. Prog. Phys. 54, 683 (1991)

[13] J.D.Jackson **Classical electrodynamics** John Wiley (1975)

[14] G.Ripken "Nonlinear canonical equations of coupled synchro-betatron motion and their solution within the framework of a nonlinear six-dimensional (symplectic) tracking program for ultrarelativistic protons" DESY 85-084 (1985)

[15] H.Mais, G.Ripken "Theory of coupled synchro-betatron oscillations" DESY M-82-05 (1982)

[16] T.Suzuki "Hamiltonian formulation for synchrotron oscillations and Sacherer's integral equation" Part. Accel. 12, 237 (1982)

[17] H.Mais, G.Ripken "Spin-orbit motion in a storage ring in the presence of synchrotron radiation using a dispersion formalism" DESY 86-029 (1986)

[18] C.J.A.Corsten, H.L.Hagedoorn "Simultaneous treatment of betatron and synchrotron motions in circular accelerators" Nucl. Instr. Methods 212, 37 (1983)

[19] A.Piwinski, A.Wrulich "Excitation of betatron-synchrotron resonances by a dispersion in the cavities" DESY 76-07 (1976)

[20] A.J.Lichtenberg, M.A.Lieberman **Regular and stochastic motion** Springer (1983)

[21] B.V.Chirikov "A universal instability of many-dimensional oscillator systems" Phys. Rep. 52, 263 (1979)

[22] M.Hénon "Numerical study of quadratic area-preserving mappings" Quart. Appl. Mathem. Vol XXVII, 291 (1969)

[23] G.Servizi, D.Bortolotti, E.Todesco, M.Giovannozzi, M.N.Vrahatis "GIOTTO: A code for the nonlinear analysis of area preserving mappings" Intern. J. Mod. Phys C6, 651 (1995)

[24] J.Guckenheimer, P.Holmes **Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields** Springer (1983)

[25] G.Jetschke **Mathematik der Selbstorganisation** Vieweg (1989)

[26] P.Glendinning **Stability, Instability and Chaos: an introduction to the theory of nonlinear differential equations** Cambridge University Press (1994)

[27] D.K.Arrowsmith, C.M.Place **An Introduction to Dynamical Systems** Cambridge University Press (1990)

[28] H.Amann **Gewoehnliche Differentialgleichungen** Walter de Gruyter (1983)

[29] P.Hartman **Ordinary Differential equations** John Wiley (1964)

[30] H. Poincaré **New Methods of Celestial Mechanics** Vols. 1, 2, 3 (edited and introduced by D.L.Goroff - originally published as **Les Méthodes Nouvelles de la Mécanique Celeste**) AIP (1991)

[31] M.W.Hirsch,S.Smale **Differential Equations, Dynamical Systems and Linear Algebra** Academic Press (1974)

[32] V.I. Arnold **Mathematical Methods of Classical Mechanics** Springer (1989)

[33] M.V.Berry "Regular and irregular motion" in **Topics in Nonlinear Dynamics - a Tribute to Sir Edward Bullard** AIP Conf. Proc. 46 (1978)

[34] R.H.G.Helleman "Self-generated chaotic behavior in nonlinear mechanics" in **Fundamental Problems in Statistical Mechanics** Vol. V North Holland (1980)

[35] M.Hénon "Numerical exploration of Hamiltonian systems" in **Chaotic Behavior of deterministic Systems** Les Houches 1981 North Holland (1983)

[36] J.Moser **Stable and random motions in dynamical systems** Princeton University Press (1973)

[37] M.Tabor **Chaos and Integrability in Nonlinear Dynamics** J.Wiley (1989)

[38] L.E.Reichl **The transition to chaos** Springer (1992)

[39] H.Mais, A.Wrulich, F.Schmidt "Studies of chaotic behaviour in HERA caused by transverse magnetic multipole fields" DESY M-85-08 (1985) and IEEE Trans.Nucl.Sci. 32, 2252 (1985)

[40] F.Schmidt "Untersuchungen zur dynamischen Akzeptanz von Protonenbeschleunigern und ihre Begrenzung durch chaotische Bewegung" DESY HERA 88-02 (1988)

[41] S.N.Rasband **Chaotic dynamics of nonlinear systems** John Wiley (1990)

[42] G.Benettin, L.Galgani, J.M.Strelcyn "Kolmogorov entropy and numerical experiments" Phys. Rev. A14, 2338 (1976)

[43] S.Wiggins **Introduction to applied nonlinear dynamical systems and chaos** Springer (1990)

[44] L.Michelotti private communication

[45] S.Becker, private communication and "Nichtlineare Dynamik in Zirkularbeschleunigern" Diploma thesis TU Berlin (1992)

[46] V.I.Arnold, V.V.Kozlov, A.I.Neishtadt "Mathematical Aspects of Classical and Celestial Mechanics" in **Encyclopaedia of Mathematical Sciences** Vol. 3 Springer (1988)

[47] J.A.Ellison, H.J.Shih "The Method of Averaging in Beam Dynamics" in **Accelerator Physics at the Superconducting Super Collider** AIP Conf. Proc. 326 (1995)

[48] K.R.Meyer, G.R.Hall **Introduction to Hamiltonian Dynamical Systems and the N-Body Problem** Springer (1992)

[49] R.D.Ruth "Single Particle Dynamics in Circular Accelerators" SLAC-PUB-4103 (1986)

[50] D.P.Barber, H.Mais, G.Ripken, F.Willeke "Nonlinear theory of synchro-betatron motion" DESY 86-147 (1986)

[51] Z.Parsa, S.Tepikian, E.Courant "Second Order Perturbation Theory for Accelerators" Part.Accel. 22, 205 (1987)

[52] E.D.Courant, H.S.Snyder "Theory of the alternating-gradient synchrotron" Ann. Phys. 3, 1 (1958)

[53] H.Goldstein **Classical mechanics** (second edition) Addison-Wesley (1980)

[54] E.Forest "A Hamiltonian-free description of single particle dynamics for hopelessly complex periodic systems" J. Math. Phys. 31, 1133 (1990)

[55] E.Forest, K.Hirata "A contemporary guide to beam dynamics" KEK-92-12 (1992)

[56] E.Forest, L.Michelotti, A.J.Dragt, J.S.Berg "The Modern Approach to Single-Particle Dynamics for Circular Rings" published in [9]

[57] M.Berz "Modern Map Methods for charged Particle Optics" Nucl.Instr.Methods A363, 100 (1995)

[58] A.Nayfeh **Perturbation methods** John Wiley (1973)

[59] A.Deprit "Canonical transformations depending on a small parameter" Cel. Mech. 1, 12 (1969)

[60] L.Michelotti "Moser like transformations using the Lie transform" Part.Accel. 16, 233 (1985)

[61] A.Nayfeh **Method of Normal Forms** J.Wiley (1993)

[62] F.G.Gustavson "On constructing formal integrals of a Hamiltonian system near an equilibrium point" The Astron. Journ. 71, 670 (1966)

[63] R.T.Swimm, J.B.Delos "Semiclassical calculations of vibrational energy levels for nonseparable systems using the Birkhoff-Gustavson normal form" J. Chem. Phys. 71, 1706 (1979)

[64] M.Tabor "The onset of chaotic motion in dynamical systems" in **Advances in Chemical Physics** Vol. XLVI (1981)

[65] A.Bazzani, E.Todesco, G.Turchetti, G.Servizi "A Normal Form Approach to the Theory of Nonlinear Betatronic Motion" CERN-94-02 (1994)

[66] G.Turchetti "Perturbative methods for Hamiltonian maps" in **Methods and Applications of Nonlinear Dynamics** World Scientific Press (1988)

[67] A.Bazzani, P.Mazzanti, G.Servizi, G.Turchetti "Normal forms for Hamiltonian maps and nonlinear effects in a particle accelerator" Il Nuovo Cimento 102B, 51 (1988)

[68] A.Dragt "Lectures on nonlinear orbit dynamics" in **Physics of High Energy Particle Accelerators** AIP Conf. Proc. 87 (1982)

[69] A.Dragt, F.Neri, G.Rangarajan, D.R.Douglas, L.M.Healy, R.D.Ryne "Lie algebraic treatment of linear and nonlinear beam dynamics" Ann. Rev. Nucl. Part. Sci. Vol. 38, 455 (1988)

[70] A.Dragt, J.Finn "Lie series and invariant functions for analytic symplectic maps" J.Math.Phys. 17, 2215 (1976)

[71] J.R.Cary "Lie transform perturbation theory for Hamiltonian systems" Phys. Rep. 79, 129 (1981)

[72] E.Forest "Normal form algorithm on non-linear symplectic maps" SSC-29 (1985)

[73] H.Mais, C.Mari unpublished notes "Lie algebraic methods for nonlinear symplectic maps"

[74] H.Mais, A.Wrulich, F.Schmidt "Particle tracking" DESY 86-024 (1986) and CERN Accelerator School, Oxford, 1985 CERN 87-03 (1987)

[75] E.Forest "Canonical integrators as tracking codes (or how to integrate perturbation theory with tracking)" in **Physics of Particle Accelerators** AIP Conf.Proc. 184 (1989)

[76] F.Willeke "Tutorial on Modern Tools of Particle Tracking" DESY M-94-12 (1994)

[77] M.Berz "Computational aspects of optics design and simulation: COSY INFINITY" Nucl.Instr. Methods A298, 473 (1990)

[78] L.Schachinger, R.Talman "TEAPOT:a thin element accelerator program for optics and tracking" Part.Accel. 22, 35 (1987)

[79] A.Dragt et al. "MARYLIE 3.0 a program for charged particle beam transport based on Lie algebraic methods" Dept. of Phys. Technical Report, University of Maryland (1987)

[80] K.L.Brown, D.C.Carey, C.Iselin, F.Rothacker "TRANSPORT, a computer program for designing charged particle beam transport systems" SLAC 91 and CERN 80-04

[81] A.Wrulich "RACETRACK:a computer code for the simulation of nonlinear particle motion in accelerators" DESY 84-026 (1984)

[82] E.Forest, R.D.Ruth "Fourth order symplectic integration" Physica D43, 105 (1990)

[83] F.Feng, Q.Meng-zhao "Hamiltonian algorithms for Hamiltonian systems and a comparative numerical study" Computer Phys. Commun. 65, 173 (1991)

[84] J.S.Berg, R.L.Warnock, R.D.Ruth, E.Forest "Construction of symplectic maps for nonlinear motion of particles in accelerators" Phys. Rev. E49, 722 (1994)

[85] M.Berz "Differential algebraic treatment of beam dynamics to very high orders including applications to space charge" in **Linear Accelerator and Beam Optics Codes** AIP Conf. Proc. 177 (1988)

[86] M.Berz "Arbitrary order description of arbitrary particle optical systems"
Nucl.Instr.Methods A298, 426 (1990)

[87] E.Forest, M.Berz, J.Irwin "Normal form methods for complicated periodic systems: a complete solution using differential algebra and Lie operators" Part. Accel. 24, 91 (1989)

[88] M.Berz "Differential algebraic description of beam dynamics to very high orders" Part.Accel. 24, 109 (1989)

[89] M.Berz "High order computation and normal form analysis of repetitive systems" in **The Physics of Particle Accelerators** AIP Conf. Proc. 249 (1992)

[90] R.V.Servranckx "Improved tracking codes: present and future" IEEE Trans. Nucl. Sci. NS-32, 2186 (1985)

[91] P. Wilhelm "Role of rounding errors in beam tracking calculations" Part. Accel. 19, 99 (1986)

[92] A.Wrulich "Tracking and special processors" DESY HERA 85-06 published in [8]

[93] M.Johnson, C.Avilez "Hardware Processor for Tracking Particles in an Alternating-Gradient Synchrotron" Part.Accel. 22, 231 (1987)

[94] J.F.Applegate, M.R.Douglas, Y.Gursel, P.Hunter, C.L.Seitz, G.J.Sussman "A digital orrery" IEEE Trans. on Comp. C34, 822 (1985)

[95] SSC-Conceptual Design, SSC-SR-2020 (1986)

[96] G.H.Walker, J.Ford "Amplitude Instability and Ergodic Behavior for Conservative Nonlinear Oscillator Systems" Phys.Rev. 188, 416 (1969)

[97] J.Tennyson "The dynamics of the beam-beam interaction" in **Physics of High Energy Particle Accelerators** AIP Conf. Proc. 87 (1982)

[98] J.Tennyson "The instability threshold for bunched beams in ISABELLE" in **Nonlinear Dynamics and the Beam-Beam Interaction** AIP Conf. Proc. 57 (1979)

[99] F.M.Izraelev "Nearly linear mappings and their applications" Physica D1, 243 (1980)

[100] S.I.Tzenov "Application of master equation technique for the study of nonlinear dynamics of particles in accelerators and storage rings" CERN SL/92-17 (1992)

[101] A.Gerasimov "The applicability of diffusion phenomenology to particle losses in hadron colliders" CERN SL/92-30 (1992)

[102] L.Chierchia, G.Gallavotti "Drift and Diffusion in Phase Space" Ann. Inst. Henri Poincaré 60, 1 (1994)

[103] B.V.Chirikov, M.A.Lieberman, D.L.Shepelyansky, F.M.Vivaldi "A Theory of Modulational Diffusion" Physica D14, 289 (1985)

[104] F.Vivaldi "Weak instabilities in many-dimensional Hamiltonian systems" Rev. Mod. Phys. 56, 737 (1984)

[105] O.Brüning "Diffusion in a FODO Cell due to Modulation Effects in the Presence of Non-Linear Fields" Part. Accel. 41, 133 (1993)

[106] F.Zimmermann "Emittance Growth and Proton Beam Lifetime in HERA" DESY 93-059 (1993)

[107] F.Zimmermann "Transverse Proton Diffusion" Part. Accel. 49, 67 (1995)

[108] A.Bazzani, S.Siboni, G.Turchetti "Diffusion in stochastically and periodically modulated Hamiltonian systems" in **Nonlinear Dynamics in Particle Accelerators: Theory and Experiments** AIP Conf. Proc. 344 (1995)

[109] HERA-B Proposal DESY-PRC 94/02 (1994)

[110] C.Hast et al. "Test of Internal Halo Targets in the HERA Proton Ring" DESY 94-119 (1994)

[111] O.Brüning, K.H.Mess, M.Seidel, F.Willeke "Measuring the Effect of an External Tune Modulation on the Particle Diffusion in the Proton Storage Ring of HERA" DESY HERA 94-01 (1994)

[112] O.Brüning "Compensation of Frequency Lines in the Power Spectrum of the Proton Loss Rate in HERA" in **Nonlinear Dynamics in Particle Accelerators: Theory and Experiments** AIP Conf. Proc. 344 (1995)

[113] S.Wiggins **Chaotic transport in dynamical systems** Springer (1992)

[114] G.M.Zaslavsky, R.Z.Sagdeev, D.A.Usikov, A.A.Chernikov **Weak chaos and quasi-regular patterns** Cambridge Univ. Press (1991)

[115] N.N.Nekhoroshev "An exponential estimate of the time of stability of nearly integrable Hamiltonian systems I" Uspekhi Mat. Nauk 32(6), 5 (1977), English translation: Russian Math. Surveys 32(6), 1 (1977)

[116] P.Lochac,C.Meunier **Multiphase Averaging for Classical Systems** Springer (1988)

[117] R.L.Warnock, R.D.Ruth "Stability of orbits in nonlinear mechanics for finite but very long times" published in [7]

[118] R.L.Warnock, R.D.Ruth "Long-term bounds on nonlinear Hamiltonian motion" Physica D56, 188 (1992)

[119] G.Turchetti "Nekhoroshev stability estimates for symplectic maps and physical applications" in **Number Theory and Physics** Springer (1990)

[120] G.H.Hoffstätter "Rigorous Bounds on Survival Times in Circular Accelerators and Efficient Computation of Fringe-Field Transfer Maps" DESY 94-242 (1994)

[121] M.Berz, G.Hoffstätter "Exact estimates of the long term stability of weakly nonlinear systems applied to the design of large storage rings" Interval Computations 2, 68 (1994)

[122] Y.Yan "Applications of differential algebra to single particle dynamics in storage rings" SSCL-500 (1991)

[123] A.Chao et al "Experimental investigation of nonlinear dynamics in the Fermilab Tevatron" Phys. Rev. Lett. 61, 2752 (1988)

[124] X.Altuna et al. "The 1991 dynamic aperture experiment at the CERN SPS" CERN-SL-91-043 (1991)

[125] D.D.Caussyn et al. "Experimental studies of nonlinear beam dynamics" Phys. Rev. A46, 7942 (1992)

[126] O.Brüning "Beam-Beam observations in HERA" Part. Accel. 50, 35 (1995)

[127] O.Brüning, W.Fischer, F.Schmidt, F.Willeke "A Comparison of Measured and Calculated Dynamic Aperture of the HERA Proton Ring at Injection Energy" DESY HERA 95-05

[128] M.Seidel "The Proton Collimation System of HERA" DESY 94-103 (1994)

[129] S.Y.Lee "Review of Nonlinear Beam Dynamics Experiments" published in [10]

[130] S.Y.Lee "Beam Dynamics Experiments at the IUCF Cooler Ring" in **Accelerator Physics at the Superconducting Super Collider** AIP Conf. Proc. 326 (1995)

[131] Y.Wang et al. "Effects of tune modulation on particles trapped in one-dimensional resonance islands" Phys. Rev. E49, 5697 (1994)

[132] A.Chao "Recent efforts on nonlinear dynamics" Int. J. Mod. Phys. A (Proc. Suppl.) 2B, 981 (1993)

[133] W.Fischer "An Experimental Study on the Long-Term Stability of Particle Motion in Hadron Storage Rings" DESY 95-235 (1995)

[134] W.Horsthemke, R.Lefever **Noise induced transitions** Springer (1984)

[135] C.W.Gardiner **Handbook of stochastic methods** Springer (1985)

[136] T.T.Soong **Random differential equations in science and engineering** Academic Press (1973)

[137] M.Lax "Classical noise IV: Langevin methods" Rev. Mod. Phys. 38, 541 (1966)

[138] N.G.van Kampen **Stochastic processes in physics and chemistry** North Holland (1981)

[139] J.Honerkamp **Stochastische dynamische Systeme** VCH Verlagsgesellschaft (1990)

[140] H.Risken **The Fokker Planck equation** Springer (1989)

[141] R.L.Stratonovich **Topics in the theory of random noise** Vols. 1,2 Gordon and Breach (1967)

[142] L.Arnold **Stochastische Differentialgleichungen** R.Oldenbourg (1973)

[143] N.G.van Kampen "Ito versus Stratonovich" J. Statist. Phys. 24, 175 (1981)

[144] R.E.Mortensen "Mathematical problems of modeling stochastic nonlinear dynamic systems" J. Statist. Phys. 1, 271 (1969)

[145] M.I.Freidlin, A.D.Wentzell **Random Perturbations of Dynamical Systems** Springer (1984)

[146] R.Z.Hasminskii **Stochastic Stability of Differential Equations** Sijthoff Noordhoff (1980)

[147] H.J.Kushner **Stochastic Stability and Control** Academic Press (1967)

[148] I.M.Ternov, V.V.Mikhailin, V.R.Khalilov **Synchrotron Radiation and its Applications** Harwood Acad. Publ. (1985)

[149] I.M.Ternov "Synchrotron radiation" Phys. Usp. 38, 409 (1995)

[150] A.A.Sokolov, I.M.Ternov **Synchrotron Radiation** Akademie Verlag (1968)

[151] A.A.Kolomensky, A.N.Lebedev **Theory of Cyclic Accelerators** North Holland (1966)

[152] F.Rohrlich **Classical Charged Particles** Addison-Wesley (1965)

[153] N.P.Klepikov "Radiation damping forces and radiation from charged particles" Sov. Phys. Usp. 28, 506 (1985)

[154] F.Gutbrod "Quantentheoretische Behandlung der Strahlungsrückwirkung relativistischer Elektronen in axialsymmetrischen Magnetfeldern" Zeitsch. Phys. 168, 177 (1962)

[155] M.Sands "Synchrotron oscillations induced by radiation fluctuations" Phys. Rev. 97, 470 (1955)

[156] M.Sands "The physics of electron storage rings - an introduction" SLAC-PUB 121 (1970)

[157] K.W.Robinson "Radiation effects in circular electron accelerators" Phys. Rev. 111, 373 (1958)

[158] J.Jowett "Introductory statistical mechanics for electron storage rings" in **Physics of Particle Accelerators** AIP Conf. Proc. 153 (1987)

[159] J.M.Jowett "Electron dynamics with radiation and nonlinear wigglers" in CERN Accelerator School, Oxford, 1985 CERN 87-03 (1987)

[160] C.Bernardini, C.Pellegrini "Linear theory of motion in electron storage rings" Ann. Phys. 46, 174 (1968)

[161] H.Mais, G.Ripken "Influence of the synchrotron radiation on the spin-orbit motion of a particle in a storage ring" DESY M-82-20 (1982)

[162] D.P.Barber, K.Heinemann, E.Karantzoulis, H.Mais, G.Ripken "The Influence of Damping in Nonlinear Fields on Electron Motion in Storage Rings" DESY M-94-09 (1994)

[163] A.W.Chao "Evaluation of beam distribution parameters in an electron storage ring" J. Appl. Phys. 50, 595 (1979)

[164] Y.H.Chin "Quantum lifetime" DESY 87-062 (1987)

[165] Yu.P.Virchenko, Yu.N.Grigorev "Equilibrium distribution of charged particles in the phase space of a cyclic accelerator" Ann. Phys. 209, 1 (1991)

[166] A.W.Chao, M.J.Lee "Particle distribution parameters in an electron storage ring" J. Appl. Phys. 47, 4453 (1976)

[167] D.P.Barber, K.Heinemann, H.Mais, G.Ripken "A Fokker-Planck treatment of stochastic particle motion within the framework of a fully coupled 6-dimensional formalism for electron-positron storage rings including classical spin motion in linear approximation" DESY 91-146 (1991)

[168] A.Hofmann, J.Jowett "Theory of the dipole-octupole wiggler" partI: "Phase oscillations" CERN/ISR-TH/81-23 partII: "Coupling of phase and betatron oscillations" CERN/ISR-TH/81-24 (1981)

[169] S.Kheifets "Application of the Green's function method to some nonlinear problems of an electron storage ring, Part IV : Study of a weak-beam interaction with a flat strong beam" Part. Accel. 15, 153 (1984)

[170] F.Ruggiero "Renormalized Fokker-Planck equation for the problem of the beam-beam interaction in electron storage rings" Ann. Phys. 153, 122 (1984)

[171] J.F.Schonfeld "Statistical mechanics of colliding beams" Ann. Phys. 160, 149 (1985)

[172] Y.H.Chin "Renormalized beam-beam interaction theory" KEK-Preprint 87-143a (1988)

[173] A.L.Gerasimov "Phase convection and distribution "tails" in periodically driven Brownian motion" Physica D41, 89 (1990)

[174] H.Mais, C.Mari "Introduction to beam-beam effects" DESY M-91-04 (1991)

[175] A.Piwinski "Dependence of the luminosity on various machine parameters and their optimization at PETRA" DESY 83-028 (1983)

[176] S.Milton "A different approach to beam-beam interaction simulation" in Proc. of 1991 Particle Accelerator Conference, IEEE (1991)

[177] A.Pauluhn, A.Gerasimov, H.Mais "A stochastic map for the one-dimensional beam-beam interaction" Int. J. Mod. Phys. A (Proc. Suppl.) 2B, 1091 (1993)

[178] A.Pauluhn "Stochastic beam dynamics in storage rings" DESY 93-198 (1993)

[179] A.W.Chao "Evaluation of radiative spin polarization in an electron storage ring" Nucl. Instr. Methods 180, 29 (1981)

[180] H.Mais, G.Ripken "Theory of spin-orbit motion in electron-positron storage rings - summary of results" DESY 83-062 (1983)

[181] R.Brinkmann, J.Rossbach "Observation of Closed Orbit Drift at HERA Covering Eight Decades of Frequency" Nucl.Instr.Methods A350, 8 (1994)

[182] G.Dôme, "Diffusion due to rf noise" in CERN Advanced Accelerator School Oxford, CERN 87-03 (1987)

[183] S.Krinsky, J.M.Wang "Bunch diffusion due to rf-noise" Part. Accel 12, 107 (1982)

[184] H.J.Shih, J.Ellison, B.Newberger, R.Cogburn "Longitudinal beam dynamics with rf noise" Part. Accel. 43, 159 (1994)

[185] A.Pauluhn "Some aspects of rf noise in storage rings" HERA 92-07 (1992)

[186] H.Mais unpublished notes "rf noise in storage rings"

[187] P.E.Kloeden, E.Platen **Numerical solution of stochastic differential equations** Springer (1992)

[188] P.E.Kloeden, E.Platen, H.Schurz **Numerical Solution of SDE Through Computer Experiments** Springer (1994)

[189] M.Seesselberg, H.P.Breuer, H.Mais, F.Petruccione, J.Honerkamp "Simulation of one-dimensional noisy Hamiltonian systems and their application to particle storage rings" Z. Phys. C62, 63 (1994)

[190] T.Sen, J.A.Ellison "Diffusion due to the beam-beam interaction and fluctuating fields in hadron colliders" (to be published)

[191] J.A.Ellison, T.Sen "Transverse beam dynamics with noise" (to be published)

[192] T.Sen, J.A.Ellison "Emittance growth due to tune fluctuations and the beam-beam interaction" (to be published)

[193] A.Bazzani, M.Giovannozzi, G.Turchetti "Stochastic Perturbation of the Linear Tune and Diffusion for Simple Lattice Models" in **Nonlinear Dynamics in Particle Accelerators: Theory and Experiment** AIP Conf. Proc. 344 (1995)

[194] K.Hirata "A Theory of Bunch Lengthening in Electron Storage Rings with Localized Wake Force Sources" Part. Accel 22, 57 (1987)

[195] A.Bazzani, S.Siboni, G.Turchetti "Diffusion in Hamiltonian Systems with a Small Stochastic Perturbation" Physica D76, 8 (1994)

[196] C.F.F.Karney, A.B.Rechester, R.B.White "Effect of noise on the standard mapping" Physica D4, 425 (1982)

[197] G.Györgyi, N.Tishby "Destabilization of islands in noisy Hamiltonian systems" Phys. Rev. A36, 4957 (1987)

[198] HERMES Collaboration "Technical Design Report" DESY-PRC 93/06 (1993)

[199] J.S.Bell "Polarized Particles for Accelerator Physicists" CERN 75-11 (1975)

[200] K.J.Heller, S.L.Smith (ed) **High Energy Spin Physics** AIP Conf Proc. 343 AIP (1995)

[201] B.W.Montague " Polarized Beams in High Energy Storage Rings" Phys. Rep. 113, 1 (1984)

[202] A.A.Sokolov, I.M.Ternov **Radiation from Relativistic Electrons** American Institute of Physics Translation Series (1986)

[203] Y.S.Derbenev, A.M.Kondratenko "Polarization kinetics of particles in storage rings" Sov.Phy.JETP 37, 968 (1973)

[204] Y.S.Derbenev, A.M.Kondratenko "Relaxation and the equilibrium state of polarization of electrons in storage rings" Sov.Phys.Dokl. 19, 438 (1975)

[205] J.D.Jackson "On understanding spin-flip synchrotron radiation and the transverse polarization of electrons in storage rings" Rev.Mod.Phys. 48, 417 (1976)

[206] S.R.Mane "Electron spin polarization in high energy storage rings" PhD Thesis Cornell University (1987)

[207] S.R.Mane "Equilibrium spin polarization in high energy storage rings" Phys.Rev. A36, 105, 120 (1987)

[208] V.N.Baier "Radiative polarization of electrons in storage rings" Sov. Phys. Usp. 14, 695 (1972)

[209] V.Bargman, L.Michel, V.L.Telegdi "Precession of the polarization of particles moving in a homogeneous electromagnetic field" Phys.Rev.Lett. 2, 435 (1959)

[210] H.-J.Meister "Die Änderung der Polarisation eines Strahles von Dirac- Teilchen bei der Wechselwirkung mit einem äusseren Feld" Bayer.Akad.Wiss. (1962)

[211] Y.S.Derbenev, A.M.Kondratenko " Diffusion of particle spins in storage rings" Sov.Phys.JETP 35, 230 (1972)

[212] R.Schmidt "Polarisationsuntersuchungen am Speicherring PETRA" DESY M-82-22 (1982)

[213] D.P.Barber et al. "Polarization measurements at PETRA" DESY M-83-06 (1983)

[214] D.P.Barber et al. " High spin polarization at the HERA electron storage ring" Nucl.Instr.Methods A338, 166 (1994)

[215] R.Rossmanith, R.Schmidt "Compensation of depolarizing effects in electron-positron storage rings" Nucl.Instr.Methods A236, 231 (1985)

[216] D.P.Barber, H.Mais, G.Ripken, R.Rossmanith "A general harmonic spin matching formalism for the suppression of depolarization caused by closed orbit distortion in electron storage rings" DESY 85-044 (1985)

[217] H.Mais, G.Ripken "Investigations of spin-orbit decoupling and elimination of depolarizing resonances in electron-positron storage rings" DESY M-84-04 (1984)

[218] K.Yokoya "Improvement of radiative polarization in plane storage rings" KEK 81-19 (1982)

[219] M.Düren "Transverse and longitudinal electron polarization at HERA" publ. in **High Energy Spin Physics** AIP Conf.Proc. 343 AIP (1995)

[220] D.P.Barber et al. "The first achievement of longitudinal spin polarization in a high energy electron storage ring" DESY 94-171 (1994)

[221] H.Mais "Diffusion on a circle" unpubl. notes (1993)

[222] D.P.Barber, M.Böge, K.Heinemann, H.Mais, G.Ripken "Spin decoherence in electron storage rings" DESY M-94-13 (1994)

[223] T.H.Boyer "Classical spinning magnetic dipole in classical electrodynamics with classical electromagnetic zero-point radiation" Phys.Rev A29, 2389 (1984)

[224] V.Balandin, N.Golubeva "Taylor map and calculation of equilibrium polarization direction for proton storage rings" Particle Accelerator Conference Washington USA (1993)

[225] V.Balandin, N.Golubeva "Investigation of spin motion in the booster lattice" TRIUMF Report TRI-DN-93-K236 (1993)

[226] V.Balandin, N.Golubeva "Nonlinear spin dynamics" Int. J. Mod. Phys. A (Proc.Suppl.) 2B, 998 (1993)

[227] M.Berz "Differential algebraic description and analysis of spin dynamics" in **High Energy Spin Physics** AIP Conf.Proc. 343 AIP (1995)

[228] H.C.Corben, P.Stehle **Classical Mechanics** R.E.Krieger Publ. Co. (1974)

[229] H.Mais, G.Ripken "Resonance Depolarization in Storage Rings by Time Dependent Electric and Magnetic Fields" DESY M-82-17 (1982)