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Blockspin transformations for finite temperature field theories with gauge fields

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Abstract

A procedure is proposed to study quantum field theories at zero or at finite temperature by a sequence of real space renormalization group (RG) or blockspin transformations. They transform to effective theories on coarser and coarser lattices. The ultimate aim is to compute constraint effective potentials, i.e. the free energy as a function of suitable order parameters. From the free energy one can read off the thermodynamic behaviour of the theory, in particular the existence and nature of phase transitions.

In a finite temperature field theory one begins with either one or a sequence of transformations which transform the original theory into an effective theory on a three-dimensional lattice. Its effectiv action has temperature dependent coefficients. Thereafter one may proceed with further blockspin transformations of the three-dimensional theory. Assuming a finite volume, this can in principle be continued until one ends with a lattice with a single site. Its effective action is the constraint effective potential.

In each RG-step, an integral over the high frequency part of the field, also called the fluctuation field, has to be performed. This is done by perturbation theory. It requires the knowledge of bare fluctuation field propagators and of interpolation operators which enter into the vertices. A detailed examination of these quantities is presented for scalar fields, abelian gauge fields and for Higgs fields, finite temperature is admitted.

The lattice perturbation theory is complicated because the bare lattice propagators are complicated. This is due to a partial loss of translation invariance in each step. Therefore the use of translation invariant cutoffs in place of a lattice is also discussed. In case of gauge fields this is only possible as a continuum version of the blockspin method.

Zusammenfassung

Es wird ein Verfahren vorgestellt, Quantenfeldtheorien auch bei endlicher Temperatur durch eine Folge von Blockspin Renormierungsgruppen Transformationen zu studieren. Dabei erhält man effektive Theorien, die auf immer gröberen Gittern leben. Das letztendliche Ziel ist die Berechnung des *constraint effective potential*, d.h. der freien Energie als Funktion von geeigneten Ordungsparametern. An der freien Energie lässt sich das thermodynamische Verhalten der Theorie ablesen, insbesondere die Existenz und Ordnung von Phasenübergängen.

Bei endlicher Temperatur startet man mit einer Transformation oder einer Folge von Transformationen, die die ursprüngliche Theorie in eine effektive drei-dimensionale Gittertheorie überführt. Deren effektive Wirkung hat temperaturabhängige Koeffizienten. Anschließend kann man mit weiteren Blockspin Transformationen fortfahren. Im Fall endlichen Volumens kann dies solange fortgesetzt werden bis man bei einem Gitter mit nur noch einem Gitterpunkt landet. Die zugehörige effektive Wirkung ist das *constraint effective potential*.

In jedem RG-Schritt müssen die hochfrequenten Feldanteile, das sog. Fluktuationsfeld, im Funktionalintegral ausintegriert werden. Dies wird hier störungstheoretisch gemacht, und erfordert die Kenntnis der nackten Feldpropagatoren und des Interpolationsoperators, der in die Vertices eingeht. Eine detailierte Untersuchung dieser Größen wird für das skalare Feld, das abelsche Eichfeld und für das Higgs-Feld vorgestellt, einschließlich der Situation bei endlicher Temperatur.

Die Gitterstörungstheorie ist kompliziert, weil die nackten Gitterpropagatoren kompliziert sind. Dies resultiert aus dem Verlust der vollen Translationsinvarianz bei den einzelnen RG-Schritten. Es werden deshalb auch translationsinvariante *cutoff*-Verfahren anstelle der Gitterregularisierung untersucht. Im Falle von Eichfeldern ist das nur als eine Kontinuumsversion des Blockspinverfahrens möglich.

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1 Introduction

The possibility of restoration of spontaneously broken symmetry in the electroweak theory at high temperature has recently led to a renewed interest in the electroweak phase transition [1].

Since the seminal and pioneering paper by Kirzhnits and Linde [2], considerable effort has been devoted to describe the rather involved processes occurring very close to the critical temperature. The effective potential has shown great usefulness. It gives the free energy as a function of the magnetization. There are perturbative computations combined with $1/N$ expansions for the effective potential [3], and also numerical results based on Monte Carlo simulations and on three dimensional reduced actions [4].

Nevertheless the perturbative calculations have shown to be plagued with problems which are to a large extent due to infrared divergences and are manifested in the appearance of spurious complex terms in the expansion for the effective potential. A great amount of work has been done to obtain meaningful expressions very close to the critical surface. Most of these attempts are related to the resummation of an infinite number of daisy and superdaisy diagrams which have to be taken into account in a consistent perturbative expansion, because they are of the same order of magnitude in the coupling constants [5].

Another very successful approach was proposed by Buchmüller et al. [6], who use an improved perturbation expansion for the effective potential, where the dynamically generated plasma masses are already included in the corresponding expressions from the beginning. These plasma masses are computed up to one loop order by self-consistent gap-equations, and have the effect of damping the infrared divergences. In this context, in [7] the absence of linear terms in the scalar field of the effective potential was explicitly verified for the abelian Higgs model, up to the order e^4 and λ^2, and proved to remain valid at higher orders. Such a linear term contains spurious infrared divergent contributions.

Despite the important and promising contributions to the understanding of the nature of the phase transition that have already been made, there are still ambiguities to be explained and higher order corrections to be included consistently to ensure the survival of the predictions to all orders.

We propose an alternative variational two-step method to study phase transitions at finite temperature in the electroweak theory, and in related models. It consists first in the calculation of a coarse grained free energy by perturbative methods, obtaining a three-dimensional perfect lattice action as a function of a block spin field. Their coefficients depend explicitly on the temperature. In a second step, the Feynman-Bogoliubov method can be used with this three-

dimensional lattice action to obtain the best quadratic approximation to the perfect action, and therefore the best split into a free part and an interaction. Using this, the constraint effective potential can be computed. To lowest order, the masses are obtained in this way as solutions of a gap equation whose formal solution is the sum of superdaisy diagrams for the three-dimensional lattice theory. They depend on temperature. Higher order corrections can be computed in principle. These masses are to be inserted into the expression for the constraint effective potential.

This procedure separates the UV- and IR-problems in a clean way. It allows to study the perfect three-dimensional lattice action directly by numerical simulations. One could for instance compute the constraint effective potential numerically.

The paper is organized as follows. In section 2 a general introduction to the idea of effective actions is given. In section 3 we introduce and compare three methods to separate a scalar field in a high and a low frequency part. The hard-soft invariance fixing method and the block spin transformation for the scalar fields at zero temperature are discussed in detail. The evaluation of the perfect scalar action at T=0 is explained. In section 4 we extend the procedures to the Maxwell theory at zero temperature. The block spin of Bałaban and Jaffe for the abelian gauge field is used [8]. The corresponding averaging operator and gauge fixing are discussed. A definition of the perfect lattice action appropriate for perturbative evaluation is given. In section 5 the procedure is extended to scalar electrodynamics. In section 6 the extension to finite temperature is outlined. The connection to dimensional reduction is explained. In section 7 most of the quantities which are needed for perturbative calculations are exhibited in momentum space. In section 8 an explicit computation of the leading terms of the perfect action for the ϕ^4-theory is shown. It is explained how to extend the results to finite temperature. The Feynman-Bogoliubov method is introduced and applied to the just obtained effective action. In section 10 and 11 we close with conclusion and outlook.

2 Effective actions

A low energy effective action is an approximative action designed to describe low energy phenomena. For some effective actions it is not even possible to calculate high energy observables, since the theory developes divergencies. In any case from a certain energy onwards the predictions of an effective action cannot be expected to agree with the experimental results. This sets the range of validity of the effective action. In the following we will always speak of a cutoff energy above which the effective action ceases to be valid.

One example for an effective action is the Fermi theory of weak interaction. It is valid only for energies so low, that one does not see the intermediate vector bosons, i.e. the cutoff is of the order of the W mass.

As one can see at this example an effective theory does not need to be renormalizable since it is not expected that the theory is valid for all energies. In general the cutoff can not be sent to infinity, because that would mean that the effective action describes the physics correctly for all energies. The degree of non-renormalizability of an effective action sets inherently the scale of its validity. A renormalizable effective action could in principle be valid up to infinite energies. It could be a fundamental action. A non-renormalizable effective action will develop divergencies in loop-integrals. If the cutoff is very low the structure of the theory should already be revealed in tree approximation because the phase space for the loop integrals is small.

An effective action needs to be strictly local. This shows that the appropriate degrees of freedom are chosen to describe the physical phenomena. The notion of locality depends on the energy scale via Heisenberg's uncertainty principle. If higher energies are used to probe the objects, the resolution in space is increased. This is an experimental possibility to set the scale of validity. Fermi's theory of weak interaction makes sense up to those energies, where one starts to observe non-local interactions of the fermions. Then one has to change the description, i.e. to introduce new degrees of freedom, the intermediate vector bosons, which propagate and mediate an interaction over a distance which is given by the inverse of their mass.

The new action can again be an effective action valid up to a higher cutoff. This process can be iterated and by no experiment one can be forced to choose an action which is valid for all energies since experiments are always done at finite energy. So the concept of an effective action is very natural.

There are two pricipally distinct ways to obtain a low energy effective action. The above description shows how effective actions are set up to explain the experimental data up to a cutoff. New experiments at higher energies require new effective actions valid up to a higher cutoff. An example for this approach is the developement of a theory for the weak interaction.

Our aim is to proceed from larger to smaller energies. We want to derive an effective action from a more fundamental one. This more fundamental action can be an effective action with a higher cutoff or a truly fundamental action valid for all energies. Then it must be renormalizable and live on the continuum. In the following we will call the lower energy action effective and the higher energy action fundamental, not distinguishing whether it is truly fundamental or itself an effective action.

In theories which are not asymptotically free, a genuine continuum theory may

not exist. In this case one has to start with a theory with a UV-cutoff. But if the cutoff is high enough, it is still possible to consider effective actions which describe the physics at much larger length scales than the inverse UV-cutoff, and which are approximately independent of the original UV-cutoff.

The process of lowering the energy consists in two parts. First we have to decide which degrees of freedom we want to use in the low energy action. Afterwards we have to get rid of all the others.

This approach to effective field theories was introduced into physics by Wilson [30]. It was first applied to statistical physics problems and afterward to Euclidian field theories. The blockspin approach introduced by Kadanoff is a very intuitive way to introduce the low energy degrees of freedom. Blockspins are averages of the fundamental fields over a certain space-time volume. They are defined on a (coarser) lattice than the fundamental fields. By this averaging process small scale fluctuations which correspond to high frequencies are eliminated.

If the fundamental theory is a continuum theory blockspins can be defined on the continuum, following Wetterich. The resulting effective action is then called an average action [27].

First we concentrate on the problem of choosing the low energy degrees of freedom. One has to determine the energy separating the two regimes and to choose the type of low energy field. This determines of course the type of the high energy field.

As mentioned above the effective action must be local. Grabowski has shown that the adequate choice of the low energy degrees of freedom is crucial to obtain locality. To choose the low energy field of the same type as the fundamental field is not necessarily a good choice. If the low energy behaviour of the theory is characterized by bound states of the fundamental degrees of freedom, the low energy field must be such a bound state. If one ignores this the low energy effective action will develop nonlocalities. This nonacceptable behaviour is a hint to the wrong choice of the low energy field. Therefore bound states are offered as low energy degrees of freedom to the theory at each separation step. If the effective action stays local if the bound states are integrated out again, one removes them. Otherwise they are kept [24].

On the other hand we want to be able to have low energy degrees of freedom of the same type as the fundamental fields. This means especially that an internal symmetry of the fundamental action must not be broken by the choice of the low energy fields. This is of special importance if one starts with a truly fundamental theory. The very first step of cutoff 'lowering' is here the intoduction of a cutoff at all, i.e. a regularization of the theory. To be forced to change the type of field in this process means that the fundamental theory

cannot be regularized.

Connected with these considerations is the question whether the separation of the field in low energy part and fluctuation part is performed in momentum or in coordinate space. In momentum space a sharp or mollified momentum cutoff is introduced. An infinitesimal cutoff lowering is possible thereby introducing a small parameter in which one can expand. In real space the low energy degrees of freedom are blockspins living on a lattice. Of course it is possible to go from momentum to coordinate space and vice versa with Fourier transformations. But in one of them the low energy field definition looks more natural. For chiral fermions it is not possible to use the lattice as a regulator without breaking the symmetry or introducing doublers. Here a momentum space formulation is adequate.

After we have decided which part of the fundamental field we want to keep, we need to integrate the other degrees of freedom.

Let us first suppose that this can be done exactly. Assuming that we introduce also effective observables the effective theory is equivalent to the fundamental one. It does not matter whether a low energy observable is calculated with the effective or fundamental action. There is no need to remove the momentum cutoff or to take the continuum limit. By integration of the fluctuation field the effective theory will gain new interactions which take into account the effect of the high frequency modes no more explicitly present in the effective theory. All UV-divergencies of the fundamental theory will show up in this part of the calculation. Therefore some of the effective couplings could be divergent. This must be avoided by adjustment of the bare parameters of the fundamental theory in such a way that the observables of the effective theory are finite. The fact that the adjustment of a finite number of bare parameters ensures the convergence of all observables is due to the renormalizability of the fundamental theory.

All effects of the high energy physics are now contained in the infinitely many coupling constants of the effective action. Since we can not demand the effective action to be renormalizable none of these couplings are ruled out for dimensional reasons. This does not result in divergencies, since the effective action is regulated by the cutoff. But any symmetry of the fundamental action which is preseved in the choice of the low energy field restricts the possible couplings in the effective action.

The procedure to calculate an effective action from a (more) fundamental one is called a renormalization group step. If one repeats these steps one obtains a renormalization group flow of effective actions. The physics is invariant under these renormalization group transformations. The renormalization group comes in different disguises. If one separates high and low energy regime in

real space with the help of coarser and coarser lattices, it is called Wilson's real space renormalization group [30].

In momentum space the cutoff can be lowered by an infinitesimal amount. The fluctuation integral can then be approximated by the leading term of a loop expansion. The RG equation can be differentiated with respect to the cutoff lowering and one obtains Polchinski's renormalization group equation [26].

Starting with a continuum theory exact integration results in a lattice theory without lattice artefacts. This is called a perfect lattice action. Observables calculated with the perfect lattice action are the continuum values. One does not have to take the continuum limit anymore.

We use the terminology 'perfect action' as a synonym for any accurate and manageable approximation to effective actions in the sense of Wilson. Hasenfratz and Niedermayer consider special approximations which are designed to be accurate near a fixed point.

In interacting theories it is not possible to do the fluctuation integral exactly. There are three steps to approximate it.

The first is very crude. One ignores the fluctuation integral completely. This is an approximation to zeroeth order in the fluctuation propagator. It is sufficient if one wants to use the effective theory as a regulated version of the fundamental action to perform calculations, and if one intends to remove the regulator afterwards.

An example for this is lattice regularization. The low energy degrees of freedom are the blockspins living on a lattice. The Laplacian is discretized using lattice derivatives, thereby next neighbour interactions are introduced. Lattice observables can be calculated numerically. They differ from their continuum counterparts by lattice artefacts. This happens, because the fluctuation field is ignored. If one uses the lattice as a method of regularization, one has to take the continuum limit afterwards. The physical correlation length is finite and so is the lattice spacing, however small it is. Hence one has to choose such lattice couplings for the simulation which render an infinite lattice correlation length. But with growing lattice correlation length it takes longer and longer to obtain statistically independent configurations for Monte Carlo measurements. One is plagued by critical slowing down.

Another example of the same kind is regularization by a momentum cutoff.

The next level of sophistication is to treat the fluctuation integral perturbatively. Thereby high energy effects are introduced to the effective action as modifications of the values of the coupling constants or as additional couplings. For lattice actions this is in the spirit of Symanzik's improvement pro-

gram which is designed to remove the lowest order lattice artefacts from the observables by adding corrections to the lattice Laplacian. In the continuum these alterations amount to the addition of higher derivatives to the kinetic operator.

Expansion can be done in the fundamental couplings, in the fluctuation propagator or in loop order.

In place of perturbation theory as will be used here, one can also use saddle point approximations which amount to a partial resummation of perturbation theory. This method is described in [23] for zero temperature, and can be adapted to the finite temperature situation in the same way as in the present paper. Some numerical results are also found in [23].

The fluctuation integral is IR-regulated by the same cutoff which UV-regulates the effective action. This facilitates the application of non-perturbetive methods. In particular one may sum certain classes of subdiagrams to infinite order. For example the Feynman-Bogoliubov method sums up all superdaisy diagrams.

3 Scalar theory

3.1 The scalar action

Before we start this program we review various procedures to introduce the low energy field. After explaining the relations between the different methods we study which procedures can be generalized to gauge fields. A scalar theory without internal symmetries, i.e. ϕ^4-theory, serves as a playground. The Euclidian action we consider in our short review is

$$S[\phi] = \frac{1}{2}\langle\phi, v^{-1}\phi\rangle + V[\phi] \quad , \tag{1}$$

where v^{-1} is the inverse free propagator and V a self-interaction. To be specific we choose $v^{-1} = -\Delta^{-1} + m^2$. The mass term is needed to render v^{-1} invertible. The interaction may contain another mass term.

In pertubation theory, calculations are based on the use of the generating functional

$$\begin{aligned} Z_0[j] &= \sqrt{\det v^{-1}} \int \mathcal{D}\phi e^{-\frac{1}{2}\langle\phi, v^{-1}\phi\rangle + \langle\phi, j\rangle} \\ &= \int d\mu_v[\phi] e^{\langle\phi, j\rangle} \end{aligned}$$

$$= e^{\frac{1}{2}\langle j,vj\rangle} \quad . \tag{2}$$

$d\mu_v[\phi] = \sqrt{\det v^{-1}}\mathcal{D}\phi e^{-\frac{1}{2}\langle\phi,v^{-1}\phi\rangle}$ is the normalized Gaussian measure with covariance v and j the current.

3.2 The convolution theorem for Gaussian measures

Since the propagator v exists, we can apply the convolution theorem for Gaussian measures. The field split can be induced by a propagator split into a soft and a hard part

$$v = v^s + v^h \quad . \tag{3}$$

Both propagators v^s and v^h must be invertible. In addition both must be positive in momentum space to ensure the convergence of the functional integrals.

The actual split can be of Pauli-Villars type

$$\frac{1}{k^2+m^2} = \frac{M^2-m^2}{(k^2+m^2)(k^2+M^2)} + \frac{1}{k^2+M^2} \tag{4}$$

or a mollified momentum cutoff following Rivasseau [10]

$$\frac{1}{k^2+m^2} = \frac{1}{k^2+m^2}e^{-\frac{k^2}{M^2}} + \frac{1}{k^2+m^2}(1-e^{-\frac{k^2}{M^2}}) \tag{5}$$

where the mass M sets the scale of the split.

A sharp momentum cutoff will not do because then the hard and soft propagator are not invertible anymore.

With any of these splits the generating functional of eq.(2) factorizes

$$Z_0[j] = e^{\frac{1}{2}\langle j,vj\rangle} = e^{\frac{1}{2}\langle j,v^s j\rangle}e^{\frac{1}{2}\langle j,v^h j\rangle} \quad . \tag{6}$$

From the different forms of $Z_0[j]$ we see

$$\frac{\delta^n}{\delta j^n}Z_0[j]\Big|_{j=0} = \langle\phi^n\rangle = \langle(\phi^s+\phi^h)^n\rangle \quad . \tag{7}$$

That means for the fields itself $\phi = \phi^s + \phi^h$. To deduce the field split from the propagator split both v^s and v^h must be invertible. This is the only condition on the propagator split.

$$e^{\frac{1}{2}\langle j,v^s j\rangle}e^{\frac{1}{2}\langle j,v^h j\rangle} = \sqrt{\det v^{s-1}}\int \mathcal{D}\phi^s\, e^{-\frac{1}{2}\langle\phi^s,v^{s^{-1}}\phi^s\rangle+\langle\phi^s,j\rangle}$$

$$\sqrt{\det v^{h^{-1}}} \int \mathcal{D}\phi^h e^{-\frac{1}{2}\langle \phi^h, v^{h^{-1}} \phi^h\rangle + \langle \phi^h, j\rangle}$$
$$= \int d\mu_{v^s}[\phi^s] e^{\langle \phi^s, j\rangle} \int d\mu_{v^h}[\phi^h] e^{\langle \phi^h, j\rangle} \tag{8}$$

The free part is now separated into contributions of the hard respective soft field only. The interaction connects both fields, since the same source j is coupled to the hard and the soft field.

No knowledge of the functional relation of ϕ^s to ϕ is obtained in this method.

3.3 The blockspin method

The second method to split the field is the blockspin method [29] used in real space renormalization group calculations [30]. The fundamental field ϕ lives either in the continuum or on a fine lattice. The block spin Φ is a block average of ϕ. It lives on a coarse lattice.

$$\Phi = C\phi \tag{9}$$

where C is called the averaging operator. During the averaging procedure all fine scale information of the fundamental field is lost. This implies that the averaging operator C has no inverse.

To perform the field split of the fundamental field ϕ we have to go back to the space it lives on, be it continuum or fine lattice. Following Gawedzki and Kupiainen [18] one splits the fundamental field ϕ into a low frequency part ϕ^s determined by the block spin Φ, and a high frequency or fluctuation field ϕ^h which has vanishing block average. This is done with the interpolation operator $\mathcal{A}$. The soft field is

$$\phi^s = \mathcal{A}\Phi = \mathcal{A}C\phi \tag{10}$$

and the hard field contains the remainder, especially the fluctuations

$$\phi^h = \phi - \phi^s = (1 - \mathcal{A}C)\phi \quad . \tag{11}$$

ϕ^h contains only fine scale information if its block average vanishes

$$C\phi^h = 0 \quad . \tag{12}$$

Then

$$C\phi^s = \Phi \quad \text{i.e.} \quad C\mathcal{A} = 1 \quad \text{i.e.} \quad \mathcal{A}C\mathcal{A}C = \mathcal{A}C \quad . \tag{13}$$

The operator $\mathcal{A}C$ is the projector to the space of low frequency fields, whereas $(1 - \mathcal{A}C)$ is the projector to the space of high frequency fields.

The use of projectors makes the blockspin method very clear as far as the fields are concerned. But projectors cannot be inverted so that the kinetic terms of the hard and soft field cannot be inverted either. They still posess a pseudoinverse, which is sufficient for our present purpose, but complicates the following calculations.

The hard and soft fields are defined as functions of the fundamental field. They are introduced into the generating functional via delta-functionals

$$Z_0[j] = \sqrt{\det v^{-1}} \int \mathcal{D}\phi \mathcal{D}\Phi \mathcal{D}\phi^s \mathcal{D}\phi^h \quad \delta(\Phi - C\phi)\delta(\phi^s - \mathcal{A}\Phi)\delta(\phi - \phi^s - \phi^h) \\ e^{-\frac{1}{2}\langle\phi, v^{-1}\phi\rangle + \langle\phi, j\rangle} \quad . \tag{14}$$

Now we have to decide whether to use the blockspin Φ or the background field ϕ^s as the low frequency variable. Both of them contain the same information. To keep Φ is more convenient for numerical simulations since Φ lives on a coarser lattice and one needs less computing time to sweep through it. For analytical calculations ϕ^s has the advantage to live on the same space as the fundamental field ϕ and the fluctuation field ϕ^h. We first keep the blockspin.

$$Z_0[j] = \sqrt{\det v^{-1}} \int \mathcal{D}\Phi \mathcal{D}\phi^h \quad \delta(\Phi - C\mathcal{A}\Phi - C\phi^h) \\ e^{-\frac{1}{2}\langle(\mathcal{A}\Phi+\phi^h), v^{-1}(\mathcal{A}\Phi+\phi^h)\rangle + \langle(\mathcal{A}\Phi+\phi^h), j\rangle} \tag{15}$$

The kinetic term contains the mixed term

$$\langle \phi^h, v^{-1}\mathcal{A}\Phi \rangle \quad . \tag{16}$$

This contradicts the idea of a field split because it is possible that a fluctuation field turns into a blockspin or vice versa without involving an interaction. With the proper choice of $\mathcal{A}$ one can ensure that this term vanishes. We recall that we have demanded $C\mathcal{A} = 1$ which made $\mathcal{A}C$ and $(1 - \mathcal{A}C)$ projectors. To avoid the mixed kinetic term they should be orthogonal with respect to the scalar product $\langle \cdot, v^{-1} \cdot \rangle$.

The kinetic term for the blockspin is

$$\frac{1}{2}\langle \Phi, \underbrace{\mathcal{A}^\dagger v^{-1} \mathcal{A}}_{=:u^{-1}} \Phi \rangle \tag{17}$$

with the blockspin propagator

$$u = CvC^\dagger \quad . \tag{18}$$

The interpolation kernel $\mathcal{A}$ which gives the projectors the correct othogonality is defined by

$$v^{-1}\mathcal{A} = C^\dagger u^{-1} \tag{19}$$

This is the Gawedzki-Kupiainen interpolation kernel, which minimizes the kinetic part of the action for a given blockspin [29]. For its derivation see appendix A.

That $u^{-1} = \mathcal{A}^\dagger v^{-1} \mathcal{A}$ is truly inverse to u can be shown using the definition of $\mathcal{A}$.

The generating functional now factorizes

$$\begin{aligned} Z_0[j] &= \sqrt{\det v^{-1}} \int \mathcal{D}\Phi e^{-\frac{1}{2}\langle \Phi, u^{-1}\Phi\rangle + \langle \mathcal{A}\Phi, j\rangle} \int \mathcal{D}\phi^h \delta(C\phi^h) e^{-\frac{1}{2}\langle \phi^h, v^{-1}\phi^h\rangle + \langle \phi^h, j\rangle} \qquad (20) \\ &= \sqrt{\det v^{-1}} \int \mathcal{D}\phi^s e^{-\frac{1}{2}\langle \phi^s, C^\dagger u^{-1} C\phi^s\rangle + \langle \phi^s, j\rangle} \int \mathcal{D}\phi^h \delta(C\phi^h) e^{-\frac{1}{2}\langle \phi^h, v^{-1}\phi^h\rangle + \langle \phi^h, j\rangle} \quad . \end{aligned}$$

We use $\Phi = C\phi^s$ and $\phi^s = \mathcal{A}\Phi$ to introduce the background field instead of the blockspin.

The kinetic operator for the background field is

$$v^{s^{-1}} = C^\dagger u^{-1} C = C^\dagger \mathcal{A}^\dagger v^{-1} \mathcal{A} C = v^{-1} \mathcal{A} C = C^\dagger \mathcal{A}^\dagger v^{-1} \quad . \qquad (21)$$

It is immediately clear that it cannot have a true inverse since it contains the projector $\mathcal{A}C$. But we can calculate the propagator v^s for the background field by evaluating the first integral in eq.(20). Note that the source j is not coupled to the blockspin, but to the background field $\phi^s = \mathcal{A}\Phi$.

$$\begin{aligned} Z_0^s[j] &= \int \mathcal{D}\Phi e^{-\frac{1}{2}\langle \Phi, u^{-1}\Phi\rangle + \langle \mathcal{A}\Phi, j\rangle} \qquad (22) \\ &= \frac{1}{\sqrt{\det u^{-1}}} e^{\frac{1}{2}\langle j, \mathcal{A} u \mathcal{A}^\dagger j\rangle} \qquad (23) \end{aligned}$$

We read off the background propagator and use again eq.(19) to write it in a more suggestive form

$$v^s = \mathcal{A} u \mathcal{A}^\dagger = \mathcal{A} C v C^\dagger \mathcal{A}^\dagger = \mathcal{A} C v = v C^\dagger \; \mathcal{A}^\dagger \quad . \qquad (24)$$

The background propagator is equal to the fundamental propagator preceded and/or followed by the projector to the low frequency fields.

$$\begin{aligned} v^{s^{-1}} v^s &= C^\dagger \mathcal{A}^\dagger v^{-1} v C^\dagger \; A^\dagger = C^\dagger \mathcal{A}^\dagger \\ v^s v^{s^{-1}} &= \mathcal{A} C v v^{-1} \mathcal{A} C = \mathcal{A} C \qquad (25) \end{aligned}$$

This shows that v^s is the pseudoinverse of $v^{s^{-1}}$. In the subspace of low frequency fields it is the true inverse.

The last step is to calculate the fluctuation propagator. Because of the delta-functional in the integral it is not possible to read off the kinetic term as for

the other fields. In place of the delta-function one could use a Gaussian

$$\delta_\kappa(C\phi^h) = \prod_{x\in\Lambda}\left(\frac{2\pi}{\kappa}\right)^{-\frac{1}{2}} e^{-\frac{1}{2}\kappa(C\phi^h(x))^2} \quad . \tag{26}$$

Hasenfratz and Niedermayer [20] pointed out that optimal locality properties of the perfect action L_Λ are obtained for a preferred finite value of κ. But for finite κ the relation $\Phi = C\phi$ is lost and only restored in the limit $\kappa \to \infty$. This blurrs the clearness of the real space renormalization.

For this reason we perform the integration using the Fourier representation of the delta-functional.

$$\begin{aligned} {Z_0}^h[j] &= \int \mathcal{D}\phi^h \delta(C\phi^h) e^{-\frac{1}{2}\langle\phi^h, v^{-1}\phi^h\rangle + \langle\phi^h, j\rangle} \\ &= \frac{1}{\sqrt{\det v^{-1}}}\frac{1}{\sqrt{\det u}} e^{\frac{1}{2}\langle j,(v-\mathcal{A}u\mathcal{A}^\dagger)j\rangle} \end{aligned} \tag{27}$$

The calculation is done in appendix A.

We can read off the fluctuation propagator

$$v^h = v - \mathcal{A}u\mathcal{A}^\dagger = v(1-C^\dagger\mathcal{A}^\dagger) = (1-\mathcal{A}C)v = (1-\mathcal{A}C)v(1-C^\dagger\mathcal{A}^\dagger) \tag{28}$$

It is equal to the fundamental propagator preceded and/or followed by the projector to the high frequency fields so that v^h cannot have a true inverse. In analogy to v^s we find its pseudoinverse

$$v^{h^{-1}} = (1-C^\dagger\mathcal{A}^\dagger)v^{-1}(1-\mathcal{A}C) = (1-C^\dagger\mathcal{A}^\dagger)v^{-1} = v^{-1}(1-\mathcal{A}C) \quad . \tag{29}$$

Because v^s and v^h have only pseudoinverses we obtain the following two identities

$$v = v^s + v^h \quad \text{and} \quad v^{-1} = v^{s^{-1}} + v^{h^{-1}} \quad . \tag{30}$$

Using the fluctuation propagator we can reexpress the fluctuation field

$$\phi^h = (1-\mathcal{A}C)\phi = v^h v^{-1}\phi \quad . \tag{31}$$

The same can be done for the background field

$$\phi^s = \mathcal{A}C\phi = v^s v^{-1}\phi \quad . \tag{32}$$

The generating functional factorizes

$$Z_0[j] = \sqrt{\det v^{-1}}\, {Z_0}^h[j] Z_0^s[j] = e^{\frac{1}{2}\langle j, v^h j\rangle + \frac{1}{2}\langle j, v^s j\rangle} \tag{33}$$

From eqs.(31) and (32) we see how the hard and soft fields are defined in terms of the fundamental field once the propagator split is given. But even then the

averaging procedure is not uniquely defined as any averaging operator C with $C\phi^h = 0$ and $C\phi^s \neq 0$ is allowed.

3.4 The invariance fixing method

The third method we present is introduced by Mitter and Valent [25]. It makes no reference to a lattice. We split the fundamental field ϕ into a hard and a soft part

$$\phi = \phi^s + \phi^h \quad . \tag{34}$$

The nomenclature will be justified later by the properties of the respective propagators. A second condition

$$F[\phi^s, \phi^h] = 0 \tag{35}$$

serves to make the split unique. The field split is implemented in the generating functional by insertion of following unity into the partition function

$$1 = \int \mathcal{D}\phi^s \mathcal{D}\phi^h \delta(\phi - \phi^s - \phi^h)\delta(F[\phi^s, \phi^h]) \det\left(\frac{\delta F}{\delta \phi^h} - \frac{\delta F}{\delta \phi^s}\right) \quad . \tag{36}$$

As eqs.(34) and (35) give the hard and the soft field in terms of the fundamental field this approach is similar to the blockspin method. Up to now the only difference is that we avoid the intermediate blockspin and concentrate immediately on the soft field which is of course the background field.

Now we change the order of integration. Integrating the fundamental field results in

$$\begin{aligned} Z_0[j] = \sqrt{\det v^{-1}} \int \; & \mathcal{D}\phi^s \mathcal{D}\phi^h \delta(F[\phi^s, \phi^h]) \det\left(\frac{\delta F}{\delta \phi^h} - \frac{\delta F}{\delta \phi^s}\right) \\ & e^{-\frac{1}{2}\langle(\phi^s+\phi^h), v^{-1}(\phi^s+\phi^h)\rangle} e^{\langle j, (\phi^s+\phi^h)\rangle} \quad . \end{aligned} \tag{37}$$

This form suggests the following alternative interpretation. We have split the fundamental field into a sum of two new ones. Since this is not unique the fundamental field and hence the action are invariant under the following hard-soft transformation.

$$\begin{aligned} \phi^h &\to \phi^h + \Lambda \\ \phi^s &\to \phi^s - \Lambda \quad . \end{aligned} \tag{38}$$

Λ is a field living on the same space as ϕ^s and ϕ^h. We call this new symmetry of the fundamental theory hard-soft invariance. One deals with this symmetry

as with a gauge symmetry, fixing it with the condition $F = 0$ eq.(35). Using the hard-soft transformation eq.(38) the determinant can be written as

$$\det(\frac{\delta F}{\delta \phi^h} - \frac{\delta F}{\delta \phi^s}) = \det 2\frac{\delta F}{\delta \Lambda} \quad . \tag{39}$$

This is the Faddeev-Popov determinant of the hard-soft fixing. To evaluate $Z_0[j]$ we proceed as in textbooks handling a gauge field action with the Faddeev-Popov trick.

We remind the reader that $\delta(F)$ need not to be fulfilled literarily if we use the invariance fixing interpretation. We could equally well use $\delta(F - B)$ with arbitrary B to obtain a field split. Of course inversion of this condition yields a functional connection of ϕ^s to ϕ dependent on B. In the final step we may integrate with some convergence-producing weight over all possible B. During this procedure we loose our previous knowledge of ϕ^s in terms of ϕ.

$$\begin{aligned} 1 &= \int \mathcal{D}B\mathcal{D}\phi^s\mathcal{D}\phi^h \delta(\phi - \phi^s - \phi^h) e^{-\frac{1}{2}\frac{1}{M^2}B^2} \delta(F[\phi^s, \phi^h] - B) \det(\frac{\delta F}{\delta \phi^h} - \frac{\delta F}{\delta \phi^s}) \\ &= \int \mathcal{D}\phi^s\mathcal{D}\phi^h \delta(\phi - \phi^s - \phi^h) e^{-\frac{1}{2}\frac{1}{M^2}F[\phi^s,\phi^h]^2} \det(\frac{\delta F}{\delta \phi^h} - \frac{\delta F}{\delta \phi^s}) \end{aligned} \tag{40}$$

The result is that the fixing term F is raised quadratically into the exponent. No limiting process for M is involved. The determinant is regarded as a hard-soft ghost kinetic term. It is this treatment of the second condition (35) on the two new fields which makes the difference to the blockspin method.

Eq.(40) is the unity which is inserted into the generating functional in the present approach

$$Z_0[j] = \int \mathcal{D}\phi^s\mathcal{D}\phi^h\mathcal{D}\bar{\eta}\mathcal{D}\eta e^{-\frac{1}{2}\langle(\phi^s+\phi^h),v^{-1}(\phi^s+\phi^h)\rangle - \frac{1}{2}\frac{1}{M^2}F^2[\phi^s,\phi^h] + \langle(\phi^s+\phi^h),j\rangle + \langle\bar{\eta},2\frac{\delta F}{\delta\Lambda}\eta\rangle} \tag{41}$$

$1/M^2$ is the hard-soft equivalent to the gauge parameter with the limit $M^2 \to \infty$ removing the fixing.

Now we have to choose F. The conditions on F can be grouped into necessary ones and those imposed only to ease further calculations. We list them below starting with the necessary ones.

- $\det(\frac{\delta F}{\delta \Lambda})$ must not be zero because otherwise F would be hard-soft invariant. Then $F = 0$ would not be the second condition for the split, but a restriction of the fundamental field $\phi = \phi^s + \phi^h$.
- $\frac{1}{2}\langle(\phi^s + \phi^h), v^{-1}(\phi^s + \phi^h)\rangle + \frac{1}{2}\frac{1}{M^2}\langle F[\phi^s, \phi^h], F[\phi^s, \phi^h]\rangle$ may not have a mixed term connecting the hard and soft field. Otherwise a hard field could turn into a soft field without interaction contradicting the idea of a field split.

– The resulting kinetic terms for the hard and soft fields must be invertible. The one for the hard field must be IR-regulated. The soft propagator must be UV-regulated, i.e. fall off faster than $1/k^2$.

Concerning the conditions for convenience we know that it is possible to split the fundamental field without invoking ghosts. This was demonstrated using the other two explained methods. Therefore it should be possible to find an F such that the ghosts decouple. We want $\det(\frac{\delta F}{\delta \Lambda})$ to be independent of the fields ϕ^s and ϕ^h so that F should be linear in ϕ^s and ϕ^h.

The linear Ansatz for F is the following

$$F[\phi^s, \phi^h] = F^s \phi^s + F^h \phi^h \quad \text{with} \quad \det(F^s - F^h) \neq 0 \quad . \tag{42}$$

We insert this into the kinetic term

$$\begin{aligned} \frac{1}{2}\langle(\phi^s + \phi^h), v^{-1}(\phi^s + \phi^h)\rangle + \frac{1}{2}\frac{1}{M^2}F^2 = & \frac{1}{2}\langle\phi^s, v^{-1}\phi^s\rangle + \frac{1}{2}\frac{1}{M^2}\langle\phi^s, F^{s\dagger}F^s\phi^s\rangle \\ & + \frac{1}{2}\langle\phi^h, v^{-1}\phi^h\rangle + \frac{1}{2}\frac{1}{M^2}\langle\phi^h, F^{h\dagger}F^h\phi^h\rangle \\ & + \langle\phi^s, v^{-1}\phi^h\rangle + \frac{1}{M^2}\langle\phi^s, F^{s\dagger}F^h\phi^h\rangle \end{aligned} \tag{43}$$

The separation condition is

$$v^{-1} + \frac{1}{M^2}F^{s\dagger}F^h = v^{-1} + \frac{1}{M^2}F^{h\dagger}F^s = 0 \quad . \tag{44}$$

For the following considerations we assume F^s, F^h and v^{-1} to be invertible. The separation condition (44) gives

$$\frac{1}{M^2}F^{s\dagger} = -v^{-1}F^{h-1} \qquad \text{and} \qquad \frac{1}{M^2}F^{h\dagger} = -v^{-1}F^{s-1} \tag{45}$$

leading to the following propagators

$$\begin{aligned} v^h &= (v^{-1} + \frac{1}{M^2}F^{h\dagger}F^h)^{-1} = (F^s - F^h)^{-1}F^s v \\ v^s &= (v^{-1} + \frac{1}{M^2}F^{s\dagger}F^s)^{-1} = -(F^s - F^h)^{-1}F^h v \end{aligned} \tag{46}$$

hence

$$v^s + v^h = v \quad . \tag{47}$$

For every linear choice of F with invertible F^s and F^h the sum of the new propagators is the fundamental one. For any given propagator split $v = v^s + v^h$ inversion of eqs.(46) leads to the coefficients F^h and F^s of the suitable fixing condition F.

The special solution we have in mind and which justifies to call v^h hard and v^s soft is

$$F^{s\dagger} = v^{-1} \qquad F^h = -M^2 \quad . \tag{48}$$

The resulting kinetic terms for the hard and soft fields are

$$\begin{aligned} v^{h^{-1}} &= v^{-1} + \frac{1}{M^2} F^{h\dagger} F^h = v^{-1} + M^2 \\ v^{s^{-1}} &= v^{-1} + \frac{1}{M^2} F^{s\dagger} F^s = v^{-1} + \frac{1}{M^2} (v^{-1})^2 \quad . \end{aligned} \tag{49}$$

Indeed the 'gauge parameter' M plays the role of a mass for the hard field. The hard propagator is IR-regulated with a mass term. The soft propagator is UV-regulated with higher derivatives. This corresponds to a propagator split of the Pauli-Villar type.

3.5 Comparison

To use the convolution theorem for Gaussian measures the fundamental propagator must exist. Moreover the hard and soft propagator must be invertible. This forbids to use a strict frequency separation, since this would imply the use of projectors. We have obtained the field split $\phi = \phi^s + \phi^h$, but we do not know ϕ^s in terms of ϕ within this approach.

In the blockspin approach we know $\phi = \phi^s + \phi^h$ beforehand. Both new fields and the blockspin Φ are defined in terms of the fundamental field. The field split is achieved by using projectors so that the method is complementary to the split of the Gaussian measure. During the calculation we obtain the propagators of blockspin, background and fluctuation field separately. In addition we find the identity $v^s + v^h = v$. It is thus shown that for every choice of the background field the propagator split used in the split of the Gaussian measure is regained. The difference is that here v^s and v^h are not truly invertible. Due to the use of pseudoinverses we have $v^{s^{-1}} + v^{h^{-1}} = v^{-1}$ as well. Alternative definitions of background and fluctuation fields can be given

$$\phi^s = v^s v^{-1} \phi \qquad \text{and} \qquad \phi^h = v^h v^{-1} \phi \quad . \tag{50}$$

This shows how background and fluctuation field are defined for a given propagator split. The blockspin can still be chosen at will subject only to the condition $C\phi^h = 0$.

The hard-soft invariance fixing method starts like the blockspin method with given functional relationships of ϕ^s and ϕ^h to ϕ. But these relations are lost during the course of the calculation because we use the Faddeev-Popov trick.

We know from the input $\phi = \phi^s + \phi^h$. Calculation shows the identity $v^s + v^h = v$. We end up with an elaborate way of obtaining the convolution theorem of the Gaussian measure. The advantage of this method compared to the convolution theorem of the Gaussian measure is that it can be adapted to the case when the fundamental propagator does not exist. If the fundamental propagator does exist both methods are equivalent.

Both blockspin method and hard-soft invariance fixing method are generalizations of the split of the Gaussian measure. But they are complementary since the propagator split is achieved with projectors in the former method and in an invertible way in the latter.

3.6 Explicit choice of the block spin

We start on the continuum R^4 of points z. The continuum is divided into blocks x identified with the sites at their centers. They may have different extensions in different directions. In this way a hyper-rectangular lattice Λ of lattice spacing L_μ in μ-direction is obtained. For the explicit calculations we use the following notation

$$\int\limits_{x\in\Lambda} = L_1 L_2 L_3 L_4 \sum_{n\in N^4} \qquad , \qquad x = (n_1 L_1, n_2 L_2, n_3 L_3, n_4 L_4). \tag{51}$$

With a scalar field $\phi(z)$ one associates a block spin $\Phi(x)$. Following Gawedzki and Kupiainen [18] we choose them as block averages

$$\Phi(x) = C\phi(x) = \operatorname*{av}_{z\in x} \phi(z) \quad . \tag{52}$$

The averaging operator has a kernel $C(x,z)$ which equals the properly normalized characteristic function $\chi_x(z)$ of the block x

$$C(x,z) = \prod_{\mu=1}^{4} \frac{1}{L_\mu} \chi_{x_\mu}(z_\mu) = \prod_{\mu=1}^{4} C(x_\mu, z_\mu) \tag{53}$$

with $\chi_{x_\mu}(z_\mu)$ the characteristic function in μ-direction.

Block propagator and kinetic term, interpolation operator, background and fluctuation propagator are all calculated using the general formulae of subsection 3.3.

3.7 'Blocking' to the continuum

Up to now the fundamental field could live on a fine lattice or on the continuum, but the blockspin had to live on a lattice. This has the disadvantage that the Poincaré invariance of the fundamental theory is broken down to its lattice counterpart. This problem can be cured if we choose blockspins defined on the continuum and a blocking operator respecting the full Poincaré invariance.

The simplest possible choice is to define the 'block'spin $\Phi(z)$ at any point $z \in R^4$ as the average over a hypersphere with radius L and center z. This choice is in the spirit of Wetterich et. al. [27]. It corresponds to a mollified momentum cutoff.

Despite the fact that the 'block'spin Φ lives now on the continuum, it is not identical to the background field ϕ^s. $\phi^s = \mathcal{A}C\phi$ is a projection of the fundamental field ϕ to low frequencies, i.e $\mathcal{A}C\phi^s = \phi^s$. This is different for the 'block'spin. $C\Phi \neq \Phi$, but every application of C smoothes the 'block'spin further.

The advantage of this peculiar choice of the 'block'spin preserving the Poincaré invariance is that it is possible to apply it to fermions. To bring fermions to the lattice is a difficult problem because of the Nielsen-Ninomiya no-go theorem. One is either plagued by doublers (staggered fermions) or has to introduce a mass term which breaks the chiral invariance (Wilson fermions). Both approaches make it difficult to take the continuum limes after the calculations are done.

One disadvantage of the continuum 'block'spin is that its notion has a tendency to get mixed up with that of the background field. Another one is that we cannot use it for dimensional reduction, as we do with the lattice blockspin when we deal with finite temperature, see section 6.

Moreover for numerical calculations one needs a lattice theory, and we want to be able to apply the here developed methods to a fundamental lattice theory. Therefore we use in the following always blockspins defined on the lattice. The switch to the continuum blockspin can be done at the very end of the derivation of the effective action.

3.8 The effective action

After all the preparations for the calculation of the effective action it remains to show how it can be evaluated perturbatively. Its definition is

$$\begin{aligned} e^{-S_{\rm eff}[\Phi]} &= \int \mathcal{D}\phi^h \delta(C\phi^h) e^{-S[\mathcal{A}\Phi+\phi^h]} \\ &= e^{-\frac{1}{2}\langle \mathcal{A}\Phi, v^{-1}\mathcal{A}\Phi\rangle} \int \mathcal{D}\phi^h \delta(C\phi^h) e^{-\frac{1}{2}\langle \phi^h, v^{-1}\phi^h\rangle - V[\mathcal{A}\Phi+\phi^h]} \\ &= e^{-\frac{1}{2}\langle \mathcal{A}\Phi, v^{-1}\mathcal{A}\Phi\rangle} e^{-V[\mathcal{A}\Phi+\frac{\delta}{\delta j}]} \int \mathcal{D}\phi^h \delta(C\phi^h) e^{-\frac{1}{2}\langle \phi^h, v^{-1}\phi^h\rangle + \langle \phi^h, j\rangle}\big|_{j=0} \\ &= e^{-\frac{1}{2}\langle \mathcal{A}\Phi, v^{-1}\mathcal{A}\Phi\rangle} e^{-V[\mathcal{A}\Phi+\frac{\delta}{\delta j}]} e^{+\frac{1}{2}\langle j, v^h j\rangle}\big|_{j=0} \quad . \end{aligned} \tag{54}$$

The effective potential, in the sense of the interaction part of the effective action, is given by the set of connected diagrams of the above expansion.

4 Maxwell theory

4.1 The Maxwell action

In the previous section we introduced three methods to split the fundamental field into a hard and a soft part and factorize the generating functional. Now we want to generalize the procedure to a situation with gauge symmetry. Maxwell theory is used as an illustrating example.

The Euclidian action is

$$S_M = -\frac{1}{4}\langle F_{\mu\nu}, F_{\mu\nu}\rangle = \frac{1}{2}\langle a_\mu, (\partial_\nu\partial_\mu - \partial^2\delta_{\mu\nu})a_\nu\rangle \quad . \tag{55}$$

Again we suppress the spacetime arguments. The action is invariant under the following gauge transformation

$$a_\mu \underset{gauge}{\longrightarrow} a_\mu - \partial_\mu\lambda \quad . \tag{56}$$

The invariance of S_M under gauge transformations gives rise to zero modes of the kinetic operator as we can see by applying the kinetic operator to a pure gauge

$$(\partial_\nu\partial_\mu - \partial^2\delta_{\mu\nu})\partial_\nu\lambda = (\partial_\nu\partial_\mu\partial_\nu - \partial^2\partial_\mu)\lambda = 0 \quad . \tag{57}$$

The kinetic operator cannot be inverted, i.e. the propagator does not exist without gauge fixing.

Hence the convolution theorem for Gaussian measures cannot be applied and so we have to look whether at least one of the other methods can be generalized.

4.2 Hard-soft invariance fixing

Like in the scalar case we split the fundamental field into two parts, one called hard, the other soft

$$a_\mu = a_\mu^h + a_\mu^s \quad . \tag{58}$$

As before the names are justified later. The action is invariant under the following transformations

$$a_\mu^h \underset{hs}{\longrightarrow} a_\mu^h + \Lambda_\mu \quad \text{and simultanously} \quad a_\mu^s \underset{hs}{\longrightarrow} a_\mu^s - \Lambda_\mu \quad . \tag{59}$$

This is the hard-soft transformation. Λ_μ is a vector field as are a_μ^h and a_μ^s. In addition there are now two independent gauge invariances of the action

$$\begin{aligned} a_\mu^h &\underset{gauge}{\longrightarrow} a_\mu^h - \partial_\mu \alpha \\ a_\mu^s &\underset{gauge}{\longrightarrow} a_\mu^s - \partial_\mu \beta \quad . \end{aligned} \tag{60}$$

The next step is to fix the hard-soft invariance by means of a fixing condition $F_\mu[a^s, a^h] = 0$. As we need one fixing condition for every spacetime coordinate we have written the fixing condition as a vector identity. This hard-soft fixing restricts also the allowed gauge transformations for the hard and soft field. Whatever gauge transformation is allowed for the hard field is forbidden for the soft field because otherwise it would serve as a hard-soft transformation.

The choice of the fixing condition gives restricted and different gauge transformation properties to the hard and the soft field. We want to obtain a low energy theory with the same symmetries as the fundamental one. In the literature [25] is suggested that only the soft field should transform as a gauge field, whereas the hard field should be invariant like a matter field in adjoint representation.

$$\begin{aligned} a_\mu^s &\underset{gauge}{\longrightarrow} a_\mu^s - \partial_\mu \lambda \\ a_\mu^h &\underset{gauge}{\longrightarrow} a_\mu^h \quad . \end{aligned} \tag{61}$$

This means that we have to look for such an F_μ that

$$\delta(F_\mu[a^s, a^h]) e^{-S_M[a^s + a^h]} \tag{62}$$

is invariant under any ordinary gauge transformation of the soft field but under no transformation of the hard field. One effect of these transformation properties is that the hard field can never be a pure gauge which are the zero modes of the kinetic operator of the Maxwell action. The hard field no longer contributes zero modes. Therefore the hard propagator exists and the hard field might be integrated to obtain a low energy effective action which is still gauge invariant by construction.

Now we come to the point of choosing $F_\mu[a^s, a^h]$. as in subsection 3.4 the conditions on F_μ are divided into a set of necessary conditions and those imposed to ease further calculations.

In complete analogy to the scalar problem we list the necessary conditions.

(i) $S_M[a^s + a^h] + \frac{1}{2}m^2\langle F_\mu, F_\mu\rangle$ must not have a mixed kinetic term since otherwise a hard field could transform into a soft field without interaction which contradicts the idea of a field split.
(ii) $F_\mu[a^s, a^h] = 0$ must be invariant under any gauge transformation of the soft field, whereas any attempt at a gauge transformation of the hard field must violate $F_\mu = 0$. This breaks down the gauge invariance of eq.(60) to the desired gauge transformation properties of eq.(61).
(iii) F_μ must be an independent second condition on the field split. This is not the case if F_μ acts only on the sum of the hard and soft field because then F_μ would be a restriction of the fundamental field a_μ. To ensure this independence $\det(\delta F_\mu/\delta\Lambda_\nu)$ must be non-zero.
(iv) The hard kinetic operator must be invertible without further amendments. It should involve some sort of IR-cutoff to justify its name.

For the soft kinetic operator we cannot demand invertibility since gauge transformations are still allowed for the soft field and create zero modes of the soft kinetic operator.

In the scalar case we had other methods for the field split and so we knew that a fixing condition decoupling the hard-soft ghosts must exist. This resticted our Ansatz to be linear in the hard and the soft field. Here we argue similarly. Since we start from a free theory, it should be possible to find such an F_μ that the hard-soft ghosts do not couple to both fields because otherwise integration of the ghosts would create an interaction. This means that F_μ must be linear either in a^s or in a^h.

After this long prologue we give a choice of F_μ that fulfills all the above conditions:

$$F_\mu = a^h_\mu + \frac{1}{m^2}\partial_\nu F^s_{\nu\mu} \tag{63}$$

is linear in both fields, hence the hard-soft ghosts decouple completely. We

show that the four necessary conditions are fulfilled.

(i) The mixed terms are

$$\frac{1}{2}a^h_\mu(\partial_\mu\partial_\nu - \partial^2\delta_{\mu\nu})a^s_\nu + \frac{1}{2}a^s_\mu(\partial_\mu\partial_\nu - \partial^2\delta_{\mu\nu})a^h_\nu$$
$$+\frac{1}{2}m^2 a^h_\mu\frac{1}{m^2}(\partial^2\delta_{\mu\nu} - \partial_\mu\partial_\nu)a^s_\nu + \frac{1}{2}m^2 a^s_\nu\frac{1}{m^2}(\partial^2\delta_{\mu\nu} - \partial_\mu\partial_\nu)a^h_\mu \quad (64)$$

and they cancel.

(ii) Since $F^s_{\nu\mu} = \partial_\nu a^s_\mu - \partial_\mu a^s_\nu$ is invariant under every gauge transformation of the soft field, so is F_μ. On the contrary every attempt to gauge transform the hard field violates $F_\mu = 0$.

(iii) The determinant of

$$\frac{\delta F_\mu}{\delta\Lambda_\nu} = \delta_{\mu\nu} + \frac{1}{m^2}(\partial_\mu\partial_\nu - \partial^2\delta_{\mu\nu}) \quad (65)$$

does not vanish.

(iv) The hard kinetic term is

$$\frac{1}{2}a^h_\mu(\partial_\mu\partial_\nu - \partial^2\delta_{\mu\nu} + m^2\delta_{\mu\nu})a^h_\nu \quad . \quad (66)$$

It has a mass as an IR-regulator which removes the zero modes.

The soft kinetic term is

$$\frac{1}{2}a^s_\mu[(\partial_\mu\partial_\nu - \partial^2\delta_{\mu\nu}) + \frac{1}{m^2}(\partial_\mu\partial_\rho - \partial^2\delta_{\mu\rho})(\partial_\rho\partial_\nu - \partial^2\delta_{\rho\nu})]a^s_\nu \quad . \quad (67)$$

It has a higher derivative part. Since it allows the same zero modes as the fundamental kinetic term it cannot be inverted without prior gauge fixing.

The hard kinetic operator is the kinetic operator of a massive vector field. It can be inverted to

$$D^h_{\mu\nu} = \frac{1}{k^2 + m^2}(\delta_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}) \quad . \quad (68)$$

The propagator does not fall off at all in its longitudinal part.

Up to now we have dealt with a free field theory where we could tolerate the bad behaviour of the hard propagator because the only effect of integrating the hard free field is an infinite constant. However as soon as we use the procedure for interacting theories and generalize the above choice of F_μ to the non-Abelian case we run into problems. Integrating the hard gauge field gives now rise to higher loop diagrams and with a propagator not falling off at all, the higher the loop order the more divergent the diagram will be. This is a disaster for the calculation of a low energy effective theory, because no

renormalization of a finite set of constants can can render the infinitely many coupling constants of the effective theory finite.

Choosing F_μ in such a way has wrecked the renormalizability of the theory.

To trace back the reason of the ill-behaving hard gauge field propagator is easier in the Abelian case. With covariant gauge fixing (with gauge parameter α) we could have obtained a fundamental propagator

$$D_{\mu\nu} = \frac{1}{k^2}(\delta_{\mu\nu} + \frac{1-\alpha}{\alpha}\frac{k_\mu k_\nu}{k^2}) \tag{69}$$

falling off as $1/k^2$.

How can it be that the hard propagator behaves worse than the fundamental one?

The answer is that we did not start with a gauge fixed theory. So there is no fundamental propagator to compare with. But from the above we can see that the gauge fixing is responsible for the good UV-behaviour of the longitudinal part. We cannot do the same for the hard field because there is no gauge invariance to be fixed. This observation brings us one step closer to the source of the trouble.

First of all we should stop calling the hard gauge field a 'gauge field' since the action together with the fixing condition allows no gauge transformation of the hard field. The hard field can righteously be called 'hard' because it has a mass to prevent IR-problems.

What about the soft gauge field a^s_μ? It is truely a gauge field, but not necessarily a soft one. The gauge transformation may again add hard components to this field so that the gauge transformation properties of eq.(61) are not useful for our purpose.

What we really want to achieve is

$$\begin{aligned} a^h_\mu &\underset{gauge}{\longrightarrow} a^h_\mu - (\partial_\mu\lambda)^h \\ a^s_\mu &\underset{gauge}{\longrightarrow} a^s_\mu - (\partial_\mu\lambda)^s \quad . \end{aligned} \tag{70}$$

The gauge transformation itself is split in such a way that $(\partial_\mu\lambda)^h$ is a hard and $(\partial_\mu\lambda)^s$ a soft contribution to the gauge transformation not interfering any more with the hard-soft split. If we could find a modified F_μ supporting these transformation properties we had a good starting point for the generalization of this method to the non-Abelian case.

The fundamental field is free, so integrating out the high frequency modes should not affect the low frequency field. This suggests a linear choice of F_μ

in both the hard and the soft field since any nonlinearity would give rise to interactions. In addition we know that the mixed part of $\frac{1}{2}m^2\langle F_\mu, F_\mu\rangle$ must cancel the mixed kinetic term and must therefore have the same symmetries as the fundamental action. It has to be invariant under gauge transformations as in eq.(60) with α and β independent of each other. On the other hand the hard part of $\frac{1}{2}m^2\langle F_\mu, F_\mu\rangle$ must only be invariant under a restricted choice of α, and contain a mass term. Within the present approach it seems impossible to fulfill all these requirements simultaneously.

4.3 Block spin for free Abelian gauge theory

Since the above method failed we try to generalize the block spin method to free Abelian gauge fields. In this section we use the Bałaban-Jaffe block spin transformation for the free Abelian gauge field at zero temperature [8].

Let the symbol ∂ denote the exterior derivative of a p-form and $\partial^\dagger$ its coderivative. For the 1-form a and the 0-form λ

$$\begin{aligned} \partial a &= \frac{1}{2}(\partial_\mu a_\nu - \partial_\nu a_\mu)\,\mathrm{d}z_\mu \wedge \mathrm{d}z_\nu \\ \partial^\dagger a &= \partial_\mu a_\mu \\ \partial^\dagger \lambda &= 0 \quad . \end{aligned} \tag{71}$$

The Laplacian is given by $(\partial\partial^\dagger + \partial^\dagger\partial)$. The Maxwell action of the electromagnetic field can be written as

$$S_M[a] = \frac{1}{2}\langle \partial a, \partial a\rangle \quad . \tag{72}$$

The perfect lattice action associated with this is defined by a formula analoguous to eq. (54) for the scalar field. There is one difference, however. In order to give meaning to the fluctuation integral some amount of gauge fixing is necessary. We wish to obtain a gauge covariant perfect lattice action and therefore to retain the freedom of gauge transformations on the lattice, i.e. one gauge degree of freedom per block. Hence global gauge fixing is not appropriate. Instead, gauge fixing is only used locally within each block.

4.3.1 Covariant averaging operator C

Given a vector potential $a(z) = a_\mu(z)\,\mathrm{d}z_\mu$ on the continuum we define a block spin $\mathbf{A}$ living on links b of the block lattice.

We use the alternative notation

$$\mathbf{A}[b] = \mathbf{A}_\mu(x) \tag{73}$$

where b is the link emanating from x in μ-direction.

The explicit blocking operation is the following: Given a link b from x in μ-direction and a point $z \in x$, let $\mathcal{C}_{z,\mu}$ be the straight path of length L_μ, i.e. one block lattice spacing, in μ-direction starting from z. The blocking procedure is defined by

$$\mathbf{A}[b] = \underset{z \in x}{\text{av}}\, a[\mathcal{C}_{z,\mu}] \tag{74}$$

$$a[\mathcal{C}_{z,\mu}] = \frac{1}{L_\mu} \int\limits_{\mathcal{C}_{z,\mu}} \mathrm{d}z'_\nu a_\nu(z') \quad . \tag{75}$$

This blocking procedure is covariant under gauge transformations in the following sense. If we transform a_μ according to

$$a_\mu(z) \to a_\mu(z) - \partial_\mu \lambda(z) \equiv a^\lambda_\mu(z) \tag{76}$$

then the block gauge field transforms as

$$\mathbf{A}_\mu(x) \to \mathbf{A}_\mu(x) - \nabla_\mu \Lambda(x) \tag{77}$$

with $\Lambda(x)$ the block average of the gauge transformations $\lambda(z)$.

To distinguish notationally between the averaging operators for gauge fields on the one hand, and for scalars and gauge transformations on the other we denote the latter by C_S from now on. Thus $\mathbf{A} = Ca$ and the gauge transformation blocked to the lattice is

$$\Lambda(x) = \underset{z \in x}{\text{av}}\, \lambda(z) \qquad \text{i.e.} \qquad \Lambda = C_S \lambda \quad . \tag{78}$$

We use the notation a^λ for the gauge transform of a, etc. In this notation the covariance property reads

$$Ca^\lambda = (Ca)^{C_S \lambda} \tag{79}$$

or written with space-time indices

$$C_{\mu\nu} \partial_\nu = \nabla_\mu C_S \quad . \tag{80}$$

To facilitate the comparison with the scalar case we give the explicit form of the gauge covariant averaging kernel

$$C_{\mu\nu}(x,z) = \delta_{\mu,\nu} \int dz'_\mu C_S(x_\mu, z'_\mu) C_S(z'_\mu + \frac{1}{2} L_\mu, z_\mu) \prod_{\rho \neq \mu} C_S(x_\rho, z_\rho) \tag{81}$$

where $C_S(x_\rho, z_\rho)$ is the one dimensional scalar averaging kernel. In μ-direction the covariant averaging kernel is not a simple step function but a convolution of two of them.

4.3.2 Covariant interpolation operator $\mathcal{A}$

A gauge covariant blocking operator ensures that the blockspin has a gauge invariant action under block gauge transformations. To evaluate the effective action we need to define the fluctuation field for given blockspin. This must be done in a covariant way. It is important to separate the background and fluctuation field in such way that gauge transformations respect this separation.

The background field is subject to non blocked gauge transformations. The same is true for the fluctuation field. Choosing the block operator has divided the set of gauge transformations into two groups. Those which leave the block gauge field $\mathbf{A}_\mu$ unaltered and the others. Since the background field depends only on the blockspin it must be invariant under gauge transformations of the first type. Because it lives on the fundamental space it must transform as a fundamental gauge fields under gauge transformations of the other type. The block average of the fluctuation field must vanish, and this must remain valid if we apply a gauge transformation to it. For transformations of the first type this is true by definition. Transformations with non vanishing block average may not be applied to a fluctuation field.

The definition for the interpolation operator is the following

$$v^{-1}\mathcal{A} = C^\dagger u^{-1} \quad . \tag{82}$$

Note that eq.(82) defining the interpolation kernel does not involve propagators. Therefore it is possible to remove the mixed kinetic term and thereby to define background and fluctuation field even if the propagators do not exist.

The fundamental kinetic operator is $v^{-1} = \partial^\dagger\partial$ and u^{-1} is the kinetic operator for the blockspin. From the form of v^{-1} it is immedialtely obvious that $\mathcal{A}$ is only determined up to a summand beginning with an exterior derivative ∂. This is the gauge freedom of $\mathcal{A}$.

To obtain the correct gauge transformation properties for background and fluctuation fields we look for an interpolation operator with the following covariance property analogous to that of the blocking operator displayed in eq.(80)

$$\mathcal{A}_{\nu\mu}\nabla_\mu = \partial_\nu \mathcal{A}_S \qquad \text{or} \qquad \mathcal{A}A^\Lambda = (\mathcal{A}A)^{\mathcal{A}_S\Lambda} \quad . \tag{83}$$

$\mathcal{A}_S$ is defined by eqs.(83).

The hard part of the calculation is to find any $\mathcal{A}$ which solves eq.(82) since v^{-1} has no inverse without gauge fixing. As we will see our laboriously preserved gauge invariance does not interfere with the existence of u^{-1} or $\mathcal{A}$. If we had a guess for $\mathcal{A}$ we could insert it into the eq.(82) and verify it.

We will show now how to obtain such a guess. In section 3.3 $\mathcal{A}$ is calculated in a straightforward way using the propagators v and u. Here these entities do not exist without gauge fixing.

To remedy this we adapt the procedure as follows. First we fix the gauge which will ensure the existence of all propagators and calculate all the quantities we need with the formulae of section 3.3. Of course the results then depend on the choice of the gauge fixing and on the gauge parameters. Afterwards we remove the gauge fixing, i.e. we perform a limiting procedure with the gauge parameters. Some quantities remain finite and others become divergent. The divergent quantities do definitely not exist without gauge fixing. Finally we use the finite quantities as the guess we mentioned above and look whether they fulfill the gauge invariant equations. This final step ensures that the results do not depend on the specific choice of gauge fixing we used in between.

In short we use the gauge fixing as a regulator which we remove afterwards. Therefore we do not need ghosts. Finally we show the independence of the results from the regulator scheme.

4.3.3 Gauge fixing within blocks

We want to disentangle the results of fixing the gauge within the block from those of the gauge fixing on the block lattice.

For this purpose we introduce the projector R onto those gauge transformation functions λ satisfying the constraint $C_S\lambda = 0$. These are the transformations which leave the block spin $\mathbf{A}_\mu$ invariant.

We recall from section 3 that the fluctuation field ϕ^h associated with a scalar field ϕ satisfies $C_S\phi^h = 0$ and can be obtained by applying a projector, see eq.(31)

$$\phi^h = (1 - \mathcal{A}^{(\phi)}C_S)\phi \quad . \tag{84}$$

$\mathcal{A}^{(\phi)}C_S$ is an orthogonal projector with respect to the scalar product furnished by the kinetic operator for ϕ.

We want to use covariant gauge fixing. The covariant gauge fixing condition acts on λ as $\Delta\lambda$. Using the Faddeev-Popov trick the gauge fixing condition is raised quadratically into the exponent. The kinetic operator of λ with respect to which we want the projector R to be orthogonal is therefore Δ^2 in place of $-\Delta$ as in the scalar case of section 3.

So we may write

$$R = 1 - \mathcal{A}^{(\lambda)}C_S \tag{85}$$

where C_S is the same averaging kernel for scalars as before, while $\mathcal{A}^{(\lambda)}$ is chosen to satisfy

$$v_{\Delta^2}^{-1}\mathcal{A}^{(\lambda)} = C_S^\dagger u_{\Delta^2}^{-1} \quad , \qquad u_{\Delta^2} = C_S v_{\Delta^2} C_S^\dagger \quad , \qquad v_{\Delta^2}^{-1} = \Delta^2 \tag{86}$$

in complete analogy to eq.(19).

From R we construct another projector $\mathcal{R} = \Delta R \Delta^{-1}$. $\mathcal{R}$ is the projector onto those $\Delta\lambda$ with $C_S\lambda = 0$. Hence the covariant gauge fixing restricted to act only within blocks is

$$\mathcal{R}\partial^\dagger a = 0 \quad . \tag{87}$$

There remains one gauge degree of freedom per block $\Lambda(x) = C_S\lambda(x)$ not affected by the fixing. It extends to a global gauge transformation per block, $\lambda(z) = \Lambda(x)$ for $z \in x$.

Now we calculate the projectors R and $\mathcal{R}$ explicitly. We insert the formula for $\mathcal{A}^{(\lambda)}$ in eq.(85)

$$R = 1 - \Delta^{-2} C_S^\dagger u_{\Delta^2}^{-1} C_S \tag{88}$$

and we see

$$\Delta^2 R = R^\dagger \Delta^2 \quad . \tag{89}$$

For the other projector we have

$$\mathcal{R} = \Delta R \Delta^{-1} = 1 - \Delta^{-1} C_S^\dagger u_{\Delta^2}^{-1} C_S \Delta^{-1} = \mathcal{R}^\dagger \tag{90}$$

Using eqs.(88) and (86) we see

$$C_S R = 0 \quad \text{and} \quad R\Delta^{-2} C_S^\dagger = 0 \quad . \tag{91}$$

and also with eq. (90)

$$\mathcal{R}\Delta^{-1} C_S^\dagger = \Delta R \Delta^{-2} C_S^\dagger = 0 \quad \text{and} \quad C_S \Delta^{-1} \mathcal{R} = C_S R \Delta^{-1} = 0 \quad . \tag{92}$$

4.3.4 Gauge fixed interpolation operator

With the help of the projector $\mathcal{R}$ we now introduce two gauge fixing terms, one forbidding only gauge transformations with vanishing block average and the other being just the complement of it.

The kinetic operator for the gauge field is then

$$\begin{aligned} v_{\alpha\beta}^{-1} &= \partial^\dagger\partial + \alpha\partial\mathcal{R}\partial^\dagger + \beta\partial(1-\mathcal{R})\partial^\dagger \\ &= \Delta + (\alpha-1)\partial\mathcal{R}\partial^\dagger + (\beta-1)\partial(1-\mathcal{R})\partial^\dagger \end{aligned} \tag{93}$$

The term proportional to α fixes the gauge transformations with vanishing block average and the term proportional to β fixes the remainder and only the remainder. Due to the use of the projector there exist no gauge transformations onto which two possibly contradicting conditions are imposed. The reason to use this complicated gauge fixing is that during the following calculations we will see that for some entities partial gauge fixing (i.e. $\beta = 0$ or $\alpha = 0$) is sufficient. This way we prepare the evaluation of the fluctuation propagator when the gauge is fixed only within the blocks.

Now all entities we need to apply the formulae derived in section 3 exist. We recall them

$$\begin{aligned} u_{\alpha\beta} &= C v_{\alpha\beta} C^\dagger \\ \mathcal{A}_{\alpha\beta} &= v_{\alpha\beta} C^\dagger u^{-1}_{\alpha\beta} \\ v^s_{\alpha\beta} &= \mathcal{A}_{\alpha\beta} C v_{\alpha\beta} = v_{\alpha\beta} C^\dagger \mathcal{A}^\dagger_{\alpha\beta} = \mathcal{A}_{\alpha\beta} C v_{\alpha\beta} C^\dagger \mathcal{A}^\dagger_{\alpha\beta} \\ \Gamma_{\alpha\beta} &= v_{\alpha\beta} - v^s_{\alpha\beta} = (1 - \mathcal{A}_{\alpha\beta} C) v_{\alpha\beta} \quad . \end{aligned} \tag{94}$$

The suffix reminds us of the values of the two gauge parameters used in the evaluation of the operators.

Before we begin the calculations we summarize our expectations concerning the question which of the quantities remain finite if we remove one or the other part of the gauge fixing. Blockspin and fluctuation fields are introduced without reference to any propagator. Kinetic operators exist without gauge fixing, i.e. we expect well defined u^{-1}_{00},$v^{s^{-1}}_{00}$ and Γ^{-1}_{00}. The interpolation kernel $\mathcal{A}$ is defined in eq.(82) via $v^{-1}\mathcal{A} = C^\dagger u^{-1}$. Only kinetic operators are involved hence $\mathcal{A}_{00}$ should exist. The fundamental propagator needs full gauge fixing. Neither $v_{\alpha 0}$ nor $v_{0\beta}$ nor v_{00} exist. Blockspin and background field propagator should not mind whether the gauge within a block is fixed or not, but they need gauge fixing on the coarse scale. So for $u_{\alpha\beta}$ and $v^s_{\alpha\beta}$ it is possible to choose $\alpha = 0$ but not $\beta = 0$. For the fluctuation propagator it should be just the other way round. We expect $\beta = 0$ to be possible but not $\alpha = 0$.

The calculations we have to do are tedious but straightforward. In their full beauty they are banned to appendix B. Their result is summarized in eqs.(105).

The gauge fixed fundamental propagator is

$$v_{\alpha\beta} = \Delta^{-1} - (1 - \frac{1}{\alpha})\partial\Delta^{-1}\mathcal{R}\Delta^{-1}\partial^\dagger - (1 - \frac{1}{\beta})\partial\Delta^{-1}(1-\mathcal{R})\Delta^{-1}\partial^\dagger \quad . \tag{95}$$

To verify this we use the projector properties of $\mathcal{R}$ and $(1-\mathcal{R})$ and

$$\begin{aligned} \partial\partial = 0 \qquad &, \qquad \partial^\dagger\partial^\dagger = 0 \\ \partial\Delta = \Delta\partial \qquad &, \qquad \partial^\dagger\Delta^{-1} = \Delta^{-1}\partial^\dagger \quad . \end{aligned} \tag{96}$$

For the calculation of $u_{\alpha\beta}$ we need the covariance property of the blocking operator $C\partial = \nabla C_S$, and $C_S\Delta^{-1}\mathcal{R} = 0$ from the explicit calculation of the projector $\mathcal{R}$. This yields

$$u_{\alpha\beta} = Cv_{\alpha\beta}C^\dagger = C\Delta^{-1}C^\dagger - (1-\frac{1}{\beta})\nabla \underbrace{C_S\Delta^{-2}C_S^\dagger}_{u_{\Delta^2}}\nabla^\dagger \quad . \tag{97}$$

Here we already see that $u_{\alpha\beta}$ is independent of α. We see as well that $u_{\alpha\beta}$ does not know about the projector $\mathcal{R}$. It looks as if it stems from a fundamental propagator with one term fixing the gauge without distinguishing between high and low frequencies.

For notational ease we introduce abbreviations for operators which act on functions on the block lattice

$$\begin{aligned} () &= (C\Delta^{-1}C^\dagger) \\ [] &= [\nabla^\dagger()^{-1}\nabla] \\ \{_\beta\} &= \{[]^{-1} - (1-\frac{1}{\beta})^{-1}[]^{-1}u_{\Delta^2}^{-1}[]^{-1}\} \quad . \end{aligned} \tag{98}$$

[] and $\{_\beta\}$ act on scalars and () is the block gauge field propagator for the special choice of the parameter $\beta = 1$. Their inverses can be computed in momentum space. Note that only $\{_\beta\}$ is gauge parameter dependent, and that the limit $\beta \to 0$ can be taken to be

$$\{_0\} = []^{-1} \tag{99}$$

whereas $\{_\beta\}$ is divergent for $\beta \to 1$. If one is interested in the special case $\beta = 1$ one inserts this in eq.(95) of the fundamental gauge fixed propagator and applies the formulae of eq.(94) directly. The calculations are much shorter than in the general case and pose no problems.

The inverse block propagator is

$$u_{\alpha\beta}^{-1} = ()^{-1} - ()^{-1}\nabla[]^{-1}\{_\beta\}^{-1}[]^{-1}\nabla^\dagger()^{-1} \quad . \tag{100}$$

Now we can calculate the interpolation kernel

$$\mathcal{A}_{\alpha\beta} = v_{\alpha\beta}C^\dagger u_{\alpha\beta}^{-1} = \Delta^{-1}C^\dagger u_{\alpha\beta}^{-1} + \partial\Delta^{-2}C_S^\dagger u_{\Delta^2}^{-1}[]^{-1}\{_\beta\}^{-1}[]^{-1}\nabla^\dagger()^{-1} \quad . \tag{101}$$

The interpolation kernel is also independent of α. The same must then be true for the background field. Contrary to u_{00} and v_{00}, $\mathcal{A}_{00}$ does exist. Neither $\mathcal{A}_{\alpha\beta}$ nor the propagator of the background field contain the projector $\mathcal{R}$ explicitly. We expect that the fluctuation propagator exists even for $\beta = 0$. Therefore the background propagator must contain a term proportional to $1/\beta$ and $1-\mathcal{R}$ so that the divergence of $v_{\alpha\beta}$ in the limit $\beta \to 0$ is completely contained in the background propagator and the fluctuation propagator stays finite.

We recall from the explicit calculation of the projector

$$1 - \mathcal{R} = \Delta^{-1} C_S^\dagger u_{\Delta^2}^{-1} C_S \Delta^{-1} \quad . \tag{102}$$

The following expression emerges as the result of the calculation of the background propagator.

$$\begin{aligned} v^s_{\alpha\beta} &= \mathcal{A}_{\alpha\beta} C v_{\alpha\beta} = v_{\alpha\beta} C^\dagger u^{-1}_{\alpha\beta} C v_{\alpha\beta} \\ &= \Delta^{-1} C^\dagger ()^{-1} C \Delta^{-1} - (1 - \frac{1}{\beta}) \partial \Delta^{-1} (1 - \mathcal{R}) \Delta^{-1} \partial^\dagger \\ &- \left(\partial \Delta^{-2} C_S^\dagger u_{\Delta^2}^{-1} - \Delta^{-1} C^\dagger ()^{-1} \nabla \right) []^{-1} \{_\beta\}^{-1} []^{-1} \left(u_{\Delta^2}^{-1} C_S \Delta^{-2} \partial^\dagger - \nabla^\dagger ()^{-1} C \Delta^{-1} \right) \end{aligned} \tag{103}$$

Here we finally see that the divergent term $\sim (1 - 1/\beta)$ is exactly the one already present in the fundamental propagator. Therefore it is obvious that the fluctuation propagator cannot contain it and is indeed finite for $\beta \to 0$ as expected. The result for the fluctuation propagator reads

$$\begin{aligned} \Gamma_{\alpha\beta} &= v_{\alpha\beta} - v^s_{\alpha\beta} \\ &= \Delta^{-1} - \Delta^{-1} C^\dagger ()^{-1} C \Delta^{-1} - (1 - \frac{1}{\alpha}) \partial \Delta^{-1} \mathcal{R} \Delta^{-1} \partial^\dagger \\ &+ \left(\partial \Delta^{-2} C_S^\dagger u_{\Delta^2}^{-1} - \Delta^{-1} C^\dagger ()^{-1} \nabla \right) []^{-1} \{_\beta\}^{-1} []^{-1} \left(u_{\Delta^2}^{-1} C_S \Delta^{-2} \partial^\dagger - \nabla^\dagger ()^{-1} C \Delta^{-1} \right) \quad . \end{aligned} \tag{104}$$

The reason that the fluctuation propagator is β-dependent at all i.e. it feels the gauge fixing on the coarse scale is that $\mathcal{A}_{\alpha\beta}$ is chosen to separate the kinetic term including the gauge fixing.

The dependence of the fluctuation propagator on β is disturbing. If we want a gauge invariant effective action we have to take the limit $\beta \to 0$. This removes the gauge fixing term and any β-dependence of the loop integrations. In the course of the further calculations we might want to reintroduce gauge fixing for the degrees of freedom which are next to be integrated out. But the β-dependence of the loop integrations is lost and cannot be restored. There are two possibilities to explain this. The first one is that the result of the loop integrations is β-independent despite the fact that the fluctuation propagator is β-dependent. The second and threatening possible explanation is that something is wrong with taking the limits. This needs to be ruled out.

In addition the fluctuation propagator depends on the choice of the gauge fixing within blocks and on the associated gauge parameter α. It has to be shown that the effective action obtained by integration of the fluctuation field does not depend on α. If the integration could be done exactly the Slavnov-Taylor identities guaranteed this. If approximations are used one has to show the independence of the effective action on α explicitly. This is the standard situation in all perturbative calculations in gauge theories.

Finally as promised the summary of the results with gauge fixing.

$$u_{\alpha\beta} = C\Delta^{-1}C^\dagger - (1-\frac{1}{\beta})\nabla u_{\Delta^2}\nabla^\dagger \tag{105}$$

$$\mathcal{A}_{\alpha\beta} = \Delta^{-1}C^\dagger u^{-1}_{\alpha\beta} + \partial\Delta^{-2}C_S^\dagger u^{-1}_{\Delta^2}[]^{-1}\{_\beta\}^{-1}[]^{-1}\nabla^\dagger()^{-1}$$

$$\begin{aligned} v^s_{\alpha\beta} &= \Delta^{-1}C^\dagger()^{-1}C\Delta^{-1} - (1-\frac{1}{\beta})\partial\Delta^{-1}(1-\mathcal{R})\Delta^{-1}\partial^\dagger \\ &- \left(\partial\Delta^{-2}C_S^\dagger u^{-1}_{\Delta^2} - \Delta^{-1}C^\dagger()^{-1}\nabla\right)[]^{-1}\{_\beta\}^{-1}[]^{-1}\left(u^{-1}_{\Delta^2}C_S\Delta^{-2}\partial^\dagger - \nabla^\dagger()^{-1}C\Delta^{-1}\right) \end{aligned}$$

$$\begin{aligned} \Gamma_{\alpha\beta} &= \Delta^{-1} - \Delta^{-1}C^\dagger()^{-1}C\Delta^{-1} - (1-\frac{1}{\alpha})\partial\Delta^{-1}\mathcal{R}\Delta^{-1}\partial^\dagger \\ &+ \left(\partial\Delta^{-2}C_S^\dagger u^{-1}_{\Delta^2} - \Delta^{-1}C^\dagger()^{-1}\nabla\right)[]^{-1}\{_\beta\}^{-1}[]^{-1}\left(u^{-1}_{\Delta^2}C_S\Delta^{-2}\partial^\dagger - \nabla^\dagger()^{-1}C\Delta^{-1}\right) \end{aligned}$$

4.3.5 Covariant interpolator

To be on the safe side we go back to the quantities $v^{-1}_{\alpha\beta}$, $u^{-1}_{\alpha\beta}$ and $\mathcal{A}_{\alpha\beta}$ take the limits $\alpha \to 0$ and $\beta \to 0$ and show explicitly that v^{-1}_{00}, u^{-1}_{00} and $\mathcal{A}_{00}$ fulfill eq.(82)

$$\begin{aligned} v^{-1}_{00} &= \partial^\dagger\partial \\ \mathcal{A}_{00} &= \Delta^{-1}C^\dagger u^{-1}_{00} + \partial\Delta^{-2}C_S^\dagger u^{-1}_{\Delta^2}[]^{-1}\nabla^\dagger()^{-1} \\ u^{-1}_{00} &= ()^{-1} - ()^{-1}\nabla[]^{-1}\nabla^\dagger()^{-1} \quad . \end{aligned} \tag{106}$$

It is obvious from the gauge invariance of v^{-1}_{00} that the second summand of $\mathcal{A}_{00}$ is not determined by eq.(82). This is the gauge invariance of the interpolation kernel

$$\begin{aligned} v^{-1}_{00}\mathcal{A}_{00} &= \partial^\dagger\partial\Delta^{-1}C^\dagger u^{-1}_{00} \\ &= (1-\partial\partial^\dagger\Delta^{-1})C^\dagger u^{-1}_{00} \\ &= (C^\dagger - \Delta^{-1}\partial\partial^\dagger C^\dagger)u^{-1}_{00} \\ &= (C^\dagger - \Delta^{-1}\partial C_S^\dagger\nabla^\dagger)u^{-1}_{00} \\ &= C^\dagger u^{-1}_{00} \end{aligned} \tag{107}$$

since

$$\nabla^\dagger u^{-1}_{00} = \nabla^\dagger(()^{-1} - ()^{-1}\nabla[]^{-1}\nabla^\dagger()^{-1}) = 0 \quad . \tag{108}$$

One sees immediately that

$$\mathcal{A}_{00}\nabla = \Delta^{-1}C^\dagger \underbrace{u^{-1}_{00}\nabla}_{0} + \partial\Delta^{-2}C_S^\dagger u^{-1}_{\Delta^2}[]^{-1}\underbrace{\nabla^\dagger()^{-1}\nabla}_{[]}$$

$$= \partial \underbrace{\Delta^{-2} C_S^\dagger u_{\Delta^2}^{-1}}_{=:\mathcal{A}_S} \quad . \tag{109}$$

The above defined $\mathcal{A}_S$ is identical to $\mathcal{A}^{(\lambda)}$ of eqs.(85) and (86) and fulfills $C_S \mathcal{A}_S = 1$. We use it to rewrite the projectors R and $\mathcal{R}$

$$\begin{aligned} R &= 1 - \mathcal{A}_S C_S \\ \mathcal{R} &= \Delta R \Delta^{-1} = 1 - \Delta \mathcal{A}_S C_S \Delta^{-1} = \mathcal{R}^\dagger \quad . \end{aligned} \tag{110}$$

Explicit calculation confirms as well

$$\begin{aligned} C\mathcal{A}_{00} &= \underbrace{C\Delta^{-1}C^\dagger}_{()} u_{00}^{-1} + C\partial\Delta^{-2} C_S^\dagger u_{\Delta^2}^{-1} []^{-1} \nabla^\dagger ()^{-1} \\ &= 1 - \nabla []^{-1} \nabla^\dagger ()^{-1} + \nabla \underbrace{C_S \Delta^{-2} C_S^\dagger}_{u_{\Delta^2}} u_{\Delta^2}^{-1} []^{-1} \nabla^\dagger ()^{-1} \\ &= 1 - \nabla []^{-1} \nabla^\dagger ()^{-1} + \nabla []^{-1} \nabla^\dagger ()^{-1} \\ &= 1 \quad . \end{aligned} \tag{111}$$

We use eq.(92), the explicit form of $\mathcal{A}_{00}$ from eq.(106) and eq.(108) to show that $\mathcal{A}_{00}$ fulfills the block Landau gauge condition

$$\begin{aligned} \mathcal{R}\partial^\dagger \mathcal{A}_{00} &= \mathcal{R}\partial^\dagger \Delta^{-1} C^\dagger u_{00}^{-1} + \mathcal{R} \underbrace{\partial^\dagger \partial}_{\Delta} \underbrace{\Delta^{-2} C_S^\dagger u_{\Delta^2}^{-1} []^{-1} \nabla^\dagger ()^{-1}}_{\text{0-form if applied to } a} \\ &= \mathcal{R}\Delta^{-1}\partial^\dagger C^\dagger u_{00}^{-1} + \underbrace{\mathcal{R}\Delta^{-1} C_S^\dagger}_{0} u_{\Delta^2}^{-1} []^{-1} \nabla^\dagger ()^{-1} \\ &= \mathcal{R}\Delta^{-1} C_S^\dagger \nabla^\dagger u_{00}^{-1} = 0 \quad . \end{aligned} \tag{112}$$

This property of $\mathcal{A}_{00}$ is not a relict of the gauge fixing in the course of its calculation. $\mathcal{A}_{00}$ and hence the background field fulfill the block Landau gauge condition despite the fact that $\mathcal{A}_{00}$ is gauge covariant. The reason is that $\mathcal{R}\partial^\dagger$ acts on $\mathcal{A}_{00}$ as a projector to high frequency fields whereas $\mathcal{A}_{00}C$ is a projector to low frequencies.

The gauge transformation for the background field a^s is

$$a^s = \mathcal{A}_{00} C a \underset{gauge}{\longrightarrow} \mathcal{A}_{00} C (a - \partial\lambda) = a^s - \partial \mathcal{A}_S C_S \lambda = a^s - \partial(1 - \mathcal{R})\lambda \tag{113}$$

and by subtraction it follows for the fluctuation field

$$a^h \underset{gauge}{\longrightarrow} a^h - \partial\mathcal{R}\lambda \quad . \tag{114}$$

We have now successfully factorized the generating functional into a high and low frequency contribution without fixing the gauge.

$$Z[j] = \int \mathcal{D}a e^{-S_M[a] + \langle a, j \rangle}$$

$$= \int \mathcal{D}A e^{-\frac{1}{2}\langle A, \mathcal{A}^\dagger v_{00}^{-1} \mathcal{A} A\rangle + \langle j, \mathcal{A} A\rangle} \int \mathcal{D}a^h \delta(Ca^h) e^{-\frac{1}{2}\langle a^h, v_{00}^{-1} a^h\rangle + \langle j, a^h\rangle} \quad (115)$$

4.3.6 The fluctuation propagator

To obtain the effective action we need to perform the fluctuation integral. The integrand is invariant under gauge transformations

$$a \underset{gauge}{\longrightarrow} a + \partial\lambda \quad \text{with} \quad C_S\lambda = 0 \quad . \qquad (116)$$

Gauge transfomations with nonvanishing block average are excluded since they violate $Ca = 0$. Before we can perform the integration it is necessary to remove also the gauge freedom of eq.(116).

We have chosen the blocking and interpolation operators in such a way that the blockspin and the background field still possess a restricted gauge invariance. If we now apply a gauge fixing indiscriminately to the frequency of the gauge transformations the desired gauge invariance of the blockspin is destroyed.

To avoid this we have already constructed the projector $\mathcal{R}$. It ensures that only the gauge degrees of freedom λ with $C\lambda = 0$, i.e. vanishing block averages of λ, are fixed but the freedom of performing global gauge transformations on each block remains. This freedom is reflected by the fact that $\exp\{-S_{\text{eff}}[\mathbf{A}]\}$ is invariant under gauge transformations $\mathbf{A} \longrightarrow \mathbf{A}^\Lambda = \mathbf{A} - \nabla\Lambda$, where Λ is a scalar function on the lattice and ∇ is the lattice exterior derivative.

As before the block gauge fixing condition is $\mathcal{R}\partial^\dagger a = 0$, which can be raised with the Faddeev-Popov trick to the exponent. This time it is a real gauge fixing which will not be removed after the calculations. Therefore we should add a ghost term to the exponent. But since the gauge fixing condition is linear in the hard field the ghosts decouple and we suppress them. So the fluctuation integral is defined as

$$\begin{aligned} Z^h[j] \;&=\; \int \mathcal{D}a^h \delta(Ca^h) e^{-\frac{1}{2}\langle a^h, (v_{00}^{-1} + \alpha\partial\mathcal{R}\partial^\dagger) a^h\rangle + \langle j, a^h\rangle} \\ &=: \; e^{\frac{1}{2}\langle j, \Gamma j\rangle} \quad . \end{aligned} \qquad (117)$$

The last expression is the definition of the fluctuation propagator Γ we are looking for. The integral is only defined due to the delta-function. The partly gauge fixed kinetic operator $v_{\alpha 0}^{-1} = v_{00}^{-1} + \alpha\partial\mathcal{R}\partial^\dagger$ is not invertible since it has the zero modes $\partial\lambda$ with $C_S\lambda \neq 0$. The constraint $\delta(Ca^h)$ serves two purposes. It forces the field a^h to be a fluctuation field with block average zero and furthermore it excludes the remaining zero modes of $v_{\alpha 0}^{-1}$ from the range of integration.

From the preceding calculations we guess that $\Gamma = \Gamma_{\alpha 0}$. To prove this we must

show

$$\Gamma_{\alpha 0} v_{\alpha 0}^{-1} = (1 - \mathcal{A}_{00} C) \quad \text{and} \quad C\Gamma_{\alpha 0} = 0 \quad . \tag{118}$$

For both purposes we use the general formulae (94) and adapt them to the case $\beta = 0$.

$$\begin{aligned} \Gamma_{\alpha\beta} &= (1 - \mathcal{A}_{\alpha\beta} C) v_{\alpha\beta} \\ \Gamma_{\alpha\beta} v_{\alpha\beta}^{-1} &= (1 - \mathcal{A}_{\alpha\beta} C) \\ C\Gamma_{\alpha\beta} &= 0 \end{aligned} \tag{119}$$

for $\beta = 0$ this results in

$$\begin{aligned} \Gamma_{\alpha 0} v_{\alpha 0}^{-1} &= (1 - \mathcal{A}_{\alpha 0} C) \\ C\Gamma_{\alpha 0} &= 0 \end{aligned} \tag{120}$$

and since $\mathcal{A}_{\alpha\beta}$ is independent of α

$$\Gamma_{\alpha 0} v_{\alpha 0}^{-1} = (1 - \mathcal{A}_{00} C) \quad . \tag{121}$$

The fluctuation propagator in special gauges can be taken from eq.(104) for special values of the block gauge parameter α. The block Landau gauge (BLG) is $\alpha \to \infty$ and taking the limit gives no problems. In addition to the properties of the fluctuation propagator for finite α one can show that

$$\partial^\dagger \Gamma_{BLG} = 0 \quad . \tag{122}$$

4.3.7 The effective action

Starting with a fundamental action of the form $S_M[a] + S_I[a]$ we insert our results into an equation of the type of eq.(54). In applications the interaction $S_I[a]$ may depend on other fields as well. This is the case for scalar electrodynamics which we treat in the following section 5. We obtain the gauge invariant effective action

$$e^{-S_{\text{eff}}[\mathbf{A}]} = \int Da \delta(Ca - \mathbf{A}) e^{-S_M[a] - S_I[a] - \frac{1}{2}\alpha \langle \partial^\dagger a, \mathcal{R} \partial^\dagger a \rangle} \quad . \tag{123}$$

With the field split $a = \mathcal{A}_{00}\mathbf{A} + a^h$ the Maxwell part of the action is separated. The gauge fixing part acts only onto the fluctuation field. The only interaction is via S_I

$$e^{-S_{\text{eff}}[\mathbf{A}]} = e^{-S_M[\mathcal{A}_{00}\mathbf{A}]} \int Da^h \delta(Ca^h) e^{-S_M[a^h] - S_I[\mathcal{A}_{00}\mathbf{A} + a^h] - \frac{1}{2}\alpha \langle \partial^\dagger a^h, \mathcal{R} \partial^\dagger a^h \rangle} \tag{124}$$

The integration of the fluctuation field can be done perturbatively

$$e^{-S_{\text{eff}}[\mathbf{A}]} = e^{-S_M[\mathcal{A}_{00}\mathbf{A}] - S_I[\mathcal{A}_{00}\mathbf{A} + \frac{\delta}{\delta j}]} \int Da^h \delta(Ca^h) e^{-S_M[a^h] + \langle j, a^h \rangle - \frac{1}{2}\alpha \langle \partial^\dagger a^h, \mathcal{R} \partial^\dagger a^h \rangle}$$

$$= e^{-S_M[\mathcal{A}_{00}\mathbf{A}]-S_I[\mathcal{A}_{00}\mathbf{A}+\frac{\delta}{\delta j}]}e^{\frac{1}{2}\langle j,\Gamma j\rangle} \quad . \tag{125}$$

In the free case this reduces to

$$\begin{aligned} e^{-S_{\text{eff}}[\mathbf{A}]} &= e^{-S_M[\mathcal{A}_{00}\mathbf{A}]} \\ &= e^{-\frac{1}{2}\langle\partial\mathcal{A}_{00}\mathbf{A},\partial\mathcal{A}_{00}\mathbf{A}\rangle} \\ &= e^{-\frac{1}{2}\langle\mathbf{A},\mathcal{A}_{00}^{\dagger}\partial^{\dagger}\partial\mathcal{A}_{00}\mathbf{A}\rangle} \\ &= e^{-\frac{1}{2}\langle\mathbf{A},u_{00}^{-1}\mathbf{A}\rangle} \quad . \end{aligned} \tag{126}$$

5 Scalar electrodynamics

5.1 The action of scalar electrodynamics

The action is defined by

$$S[a,\phi] = S_M[a] + \frac{1}{2}\langle\phi,(-D_\mu^\dagger[a]D_\mu[a])\phi\rangle + V[\phi] \quad . \tag{127}$$

The scalar self-interaction is

$$V[\phi] = \int\limits_z \left(\frac{1}{2}m_0^2\phi\phi^* + \frac{g}{4!}(\phi\phi^*)^2\right) \quad . \tag{128}$$

ϕ denotes a complex Higgs field and

$$D_\mu[a] = \partial_\mu - a_\mu \tag{129}$$

the covariant derivative.

The action is invariant under the following simultaneous gauge transformations

$$a_\mu \underset{gauge}{\longrightarrow} a_\mu^\lambda = a_\mu - \partial_\mu\lambda \quad \text{and} \quad \phi \underset{gauge}{\longrightarrow} \phi^\lambda = \phi e^\lambda \quad . \tag{130}$$

To evaluate the low energy effective action we need to define blockspin and/or background field and fluctuation fields. The definitions for the gauge fields can be taken literarily from the preceding section 4. Since the Higgs field is not invariant under gauge transformations we cannot simply use the results of section 3 for it.

We will show two different ways to deal with it. The first regards the Higgs field as a scalar field with a gauge field dependent kinetic operator. The proceeding is analogous to section 3. The transformation properties for block,

background and fluctuation Higgs fields follow from the choice of averaging and interpolation operator.

The second way regards the Higgs field as subject to gauge transformations. Its emphasis is to make the gauge transformation properties of block, background and fluctuation Higgs field compatible with those of block, background and fluctuation Maxwell fields evaluated in section 4. The choice of averaging and interpolation operator satisfies this demand.

5.2 Block spin transformation for the Higgs field at zero temperature

Here we explain the first way to choose the averaging operator. For a given Higgs field $\phi(z)$ we wish to define a block Higgs field $\Phi(x)$ on the lattice Λ in a gauge covariant way. The blocking procedure for the ϕ^4-theory cannot be used as it stands because it is not gauge covariant. In order to maintain gauge covariance, we must use averaging kernels depending on the gauge field a.

We denote the averaging operator for the Higgs field by $C_H[a]$. and choose the block operator to be linear with respect to the Higgs field, therefore we can write it as an integral operator

$$\Phi(x) = (C_H[a]\phi)(x) = \int_z C_H[a](x,z)\phi(z) \quad . \tag{131}$$

In the ϕ^4-theory we used an averaging kernel which was constant on blocks and vanished outside. A constant is the lowest eigenvector of the Laplacian with Neumann boundary conditions on block boundaries. The natural generalization to the gauge covariant situation is as follows [1].

Let $-\Delta_a = -(\partial - a)^\dagger(\partial - a)$ denote the covariant Laplacian and $-\Delta_a^{N,x}$ the covariant Laplacian with Neumann boundary conditions on the block boundary of x. We demand that

$$-\Delta_a^{N,x} C_H[a]^\dagger(z,x) = \varepsilon_0[a](x) C_H[a]^\dagger(z,x) \tag{132}$$

and

$$C_H[a]^\dagger(z,x) = 0 \quad \text{for} \qquad z \notin x \tag{133}$$

where $\varepsilon_0[a](x)$ is the lowest eigenvalue of $-\Delta_a^{N,x}$. In addition we impose the normalization condition

$$C_H[a] C_H[a]^\dagger = 1 \quad . \tag{134}$$

[1] Here we deviate from the work of Bałaban, Jaffe and Imbrie [13]

This leaves the freedom of multiplying $C_H[a]$ with an a-dependent phase factor

$$C_H[a]^\dagger(z,x) \to C_H[a]^\dagger(z,x)\eta[a](x) \quad . \tag{135}$$

It follows from the gauge covariance of the eigenvalue problem (132) that under a gauge transformation

$$C_H[a^\lambda]\phi^\lambda = (C_H[a]\phi)^\Lambda \quad . \tag{136}$$

Λ is a gauge transformation on the lattice which depends on λ and on the choice of conventions to fix C_H uniquely. The freedom in the choice of an a-dependent phase factor may be exploited to demand that

$$\Lambda(x) = (C_S\lambda)(x) \tag{137}$$

where C_S is the scalar averaging kernel introduced in section 3. Eq.(136) parallels the gauge covariance property of the blocking procedure for the gauge field.

We can compute the averaging kernel $C_H[a]$ as a solution of the eigenvalue equation (132) by standard quantum mechanical perturbation theory. A prototype of such a computation is found in reference [16].

5.3 Interpolation kernel and fluctuation propagator for the Higgs field

The interpolation kernel $\mathcal{A}_H[a]$ for the Higgs field will also depend on the gauge field a because the averaging kernel does. In order to split the kinetic term for the Higgs field, we closely follow the arguments in section 3. First we adapt our notation and call the gauge dependent kinetic operator of the fundamental Higgs field

$$v_H[a]^{-1} = -D_\mu^\dagger[a]D_\mu[a] \quad . \tag{138}$$

The interpolation kernel is then defined via

$$v_H[a]^{-1}\mathcal{A}_H[a] = C_H[a]^\dagger u_H[a]^{-1} \tag{139}$$

for some choice of $u_H[a]^{-1}$, and if

$$C_H[a]\mathcal{A}_H[a] = 1 \tag{140}$$

the last condition implies that

$$u_H[a] = C_H[a]v_H[a]C_H[a]^\dagger \quad \text{and} \qquad u_H[a]^{-1} = \mathcal{A}_H[a]^\dagger v_H[a]^{-1}\mathcal{A}_H[a] \tag{141}$$

See appendix A for the derivation of these formulae in the scalar case.

We can now split the fundamental Higgs field into background and fluctuation fields

$$\phi = \phi^s + \phi^h = \mathcal{A}_H[a]\Phi + \phi^h \quad . \tag{142}$$

The fluctuation field propagator for the Higgs field is

$$\Gamma_H[a] = v_H[a] - \mathcal{A}_H[a]C_H[a]v_H[a]C_H[a]^\dagger\mathcal{A}_H[a^\dagger] \tag{143}$$

where $v_H[a] = -\Delta_a^{-1}$ is the full gauge covariant free massless propagator for the Higgs field.

The interpolation operator and the gauge covariant fluctuation field propagator can also be computed by standard quantum mechanical perturbation theory [16].

5.4 The perfect action of scalar electrodynamics: Representation as a functional integral with gauge fixed fluctuation fields

We use again the projector $\mathcal{R}$ of section 4 to define the fluctuation integral. The essential point making this possible is that both delta-functions

$$\delta(Ca - \mathbf{A}) \qquad \text{and} \qquad \delta(C_H[a]\phi - \Phi) \tag{144}$$

are invariant under gauge transformations $a \to a^\lambda$ and $\phi \to \phi^\lambda$ which obey the constraint $C_S\lambda = 0$. Proceeding as before we obtain

$$\begin{aligned} e^{-S_{\text{eff}}(\Phi,\mathbf{A})} = \int \mathcal{D}a \int \mathcal{D}\phi \;\; & \delta(Ca - \mathbf{A})\delta(C_H[a]\phi - \Phi) \\ & e^{-S_M[a] - S_{\text{gf}}[a] - \frac{1}{2}\langle\phi, v_H[a]^{-1}\phi\rangle - V[\phi]} \end{aligned} \tag{145}$$

where

$$S_{\text{gf}}[a] = \frac{1}{2}\alpha\langle\partial^\dagger a, \mathcal{R}\partial^\dagger a\rangle \quad . \tag{146}$$

Now we split the fields into background and fluctuation part

$$\phi = \mathcal{A}_H[a]\Phi + \phi^h \qquad \text{and} \qquad a = \mathcal{A}\mathbf{A} + a^h \quad . \tag{147}$$

The expression for the perfect action becomes

$$\begin{aligned} e^{-S_{\text{eff}}[\Phi,\mathbf{A}]} = \int \mathcal{D}a^h \int \mathcal{D}\phi^h \delta(Ca^h)\delta(C_H[a]\phi^h) \\ \exp\{-S_M[\mathcal{A}\mathbf{A}] - S_M[a^h] - S_{\text{gf}}[a^h] \\ -\frac{1}{2}\langle\Phi, \underbrace{\mathcal{A}_H^\dagger[\mathcal{A}\mathbf{A} + a^h]v_H[\mathcal{A}\mathbf{A} + a^h]^{-1}\mathcal{A}_H[\mathcal{A}\mathbf{A} + a^h]}_{u_H[\mathcal{A}\mathbf{A}+a^h]^{-1}}\Phi\rangle \end{aligned}$$

$$-\frac{1}{2}\langle\phi^h, v_H[\mathcal{A}\mathbf{A}+a^h]^{-1}\phi^h\rangle - V[\mathcal{A}_H[\mathcal{A}\mathbf{A}+a^h]\Phi+\phi^h]\} \quad (148)$$

We used the fact that $S_{\rm gf}[\mathcal{A}\mathbf{A}+a^h] = S_{\rm gf}[a^h]$ since $\mathcal{R}\partial^\dagger\mathcal{A}=0$, see eq.(112).

We separate the terms of zeroeth order in a^h and ϕ^h

$$\begin{aligned} u_H[\mathcal{A}\mathbf{A}+a^h]^{-1} &= u_H[\mathcal{A}\mathbf{A}]^{-1} + E[\mathbf{A},a^h] \\ V[\mathcal{A}_H[\mathcal{A}\mathbf{A}+a^h]\Phi+\phi^h] &= V[\mathcal{A}_H[\mathcal{A}\mathbf{A}]\Phi] + W[\mathbf{A},a^h,\Phi,\phi^h] \quad . \end{aligned} \quad (149)$$

The effective action becomes

$$S_{\rm eff}[\Phi,\mathbf{A}] = S_{M,{\rm eff}}[\mathbf{A}] - \frac{1}{2}\langle\Phi, u_H[\mathcal{A}\mathbf{A}]^{-1}\Phi\rangle + V[\mathcal{A}_H[\mathcal{A}\mathbf{A}]\Phi] + \tilde{S}_{\rm eff}[\Phi,\mathbf{A}] \quad (150)$$

with $\tilde{S}_{\rm eff}[\Phi,\mathbf{A}]$ the result of the fluctuation integral

$$\begin{aligned} e^{-\tilde{S}_{\rm eff}[\Phi,\mathbf{A}]} = \int \mathcal{D}a \int \mathcal{D}\phi\delta(Ca-\mathbf{A})\delta(C_H[a]\phi-\Phi) \\ \exp\{-S_M[a^h] - S_{\rm gf}(a^h) - \frac{1}{2}\langle\phi^h, v_H[a]^{-1}\phi^h\rangle \\ -\frac{1}{2}\langle\Phi, E[\mathbf{A},a^h]\Phi\rangle - W[\mathbf{A},a^h,\Phi,\phi^h]\} \end{aligned} \quad (151)$$

with E and W defined in eq.(149). To zeroeth order in the fluctuation field propagators $\tilde{S}_{\rm eff}[\Phi,\mathbf{A}]=0$ after subtracting a constant.

5.5 Alternative definition of the block operator

An alternative way to define averaging and interpolation operators for the Higgs field concentrates on its behaviour under gauge transformations. We recall from section 4 the transformation properties of the various gauge fields

$$\begin{aligned} a_\mu &\underset{gauge}{\longrightarrow} a_\mu^\lambda = a_\mu - \partial_\mu\lambda \\ \mathbf{A}_\mu &\underset{gauge}{\longrightarrow} \mathbf{A}_\mu^\Lambda = \mathbf{A}_\mu - \nabla_\mu\Lambda \\ a_\mu^s &\underset{gauge}{\longrightarrow} a_\mu^{s\,\lambda^s} = a_\mu^s - \partial_\mu\lambda^s \\ a_\mu^h &\underset{gauge}{\longrightarrow} a_\mu^{h\,\lambda^h} = a_\mu^h - \partial_\mu\lambda^h \end{aligned} \quad (152)$$

with the definitions of the gauge fields

$$\mathbf{A} = Ca \quad , \quad a^s = \mathcal{A}\mathbf{A} = \mathcal{A}Ca \quad , \quad a^h = a - a^s = (1-\mathcal{A}C)a \quad (153)$$

and the gauge transformations

$$\Lambda = C_S\lambda \quad , \quad \lambda^s = \mathcal{A}_S\Lambda = \mathcal{A}_SC_S\lambda \quad , \quad \lambda^h = \lambda - \lambda^s = (1-\mathcal{A}_SC_S)\lambda \quad (154)$$

For the Higgs fields we demand gauge transformation properties analogous to those for the gauge fields

$$\begin{aligned} \phi &\underset{gauge}{\longrightarrow} \phi^\lambda = \phi e^\lambda \\ \Phi &\underset{gauge}{\longrightarrow} \Phi^\Lambda = \Phi e^\Lambda \\ \phi^s &\underset{gauge}{\longrightarrow} \phi^{s\,\lambda^s} = \phi^s e^{\lambda^s} \quad . \end{aligned} \tag{155}$$

This suggests the following averaging and interpolation procedures

$$\begin{aligned} \Phi(x) &= (C_H[\phi])(x) = e^{\int_z C_S(x,z) \ln \phi(z)} \\ \phi^s(z) &= (\mathcal{A}_H[\Phi])(z) = e^{\int_x \mathcal{A}_S(z,x) \ln \Phi(x)} \quad . \end{aligned} \tag{156}$$

Inserting the gauge transformation property for the fundamental Higgs field we obtain the desired properties for blockspin and background field.

The branch of the logarithm must be chosen self-consistently. Since we want to be able to expand in a small fluctuation field, we suppose that there exists some smooth field ψ interpolating the blockspin Φ, so that ϕ is close to ψ. Since $\psi(z)$ shall not vary much within a block, we approximate it with its value in the block center $\psi(\hat{x})$. We now precisize the definition of the blockspin to

$$\Phi(x) = \psi(\hat{x}) e^{\int_z C_S(x,z) \ln \psi(\hat{x})^{-1} \phi(z)} \quad . \tag{157}$$

This way we use ψ as a reference field to determine the branch of the logarithm. The definition of the background field must contain the same reference field.

$$\phi^s(z) = \psi(z) e^{\int_x \mathcal{A}_S(z,x) \ln \psi(\hat{x})^{-1} \Phi(x)} \quad . \tag{158}$$

In a non-perturbative approach, one cannot rely on the existence of some smooth field ψ to determine the branch of the logarithm. One has to split the field space in a small field region, where the above considerations remain valid, and a large field region where the above blockspin definition cannot be applied. This is discussed in the work of Bałaban [8].

The difference to the operators $C_H[a]$ and $\mathcal{A}_H[a]$ of the subsections 5.2 and 5.3 is that here the operators do not depend on the gauge field a and act nonlinearly on the Higgs field.

From $C_S \mathcal{A}_S = 1$ we infer

$$C_H[\phi^s] = C_H[\mathcal{A}_H[\Phi]] = \Phi \quad . \tag{159}$$

The blockspin is completely defined by the background field. Since the averaging operator is not linear, but commutes with multiplication, it is reasonable

to introduce the fluctuation field as

$$\phi = \phi^s e^{\phi^h} \quad . \tag{160}$$

Applying the averaging operator to this identity yields

$$C_H[e^{\phi^h}] = 1 \qquad \text{i.e.} \qquad C_S \phi^h = 0 \quad . \tag{161}$$

The last equation is the reason to define the fluctuation field in the exponential, so that it makes sense to expand around vanishing fluctuation field to evaluate the fluctuation integral. With this definition the gauge transformation property of the fluctuation field is

$$\phi^h \underset{gauge}{\longrightarrow} \phi^h + \lambda^h \quad . \tag{162}$$

Note that ϕ^h is complex as is ϕ whereas λ and λ^h are purely imaginary functions.

We are now prepared to define the effective block action.

$$\begin{aligned} e^{-S_{\text{eff}}[\Phi,\mathbf{A}]} = \int \mathcal{D}a\mathcal{D}\phi \quad & \delta(\Phi - C_H[\phi])\delta(\mathbf{A} - Ca) \\ & e^{-S_M[a]-S_{\text{gf}}[a]-\frac{1}{2}\langle\phi, v_H[a]^{-1}\phi\rangle - V[\phi]} \end{aligned} \tag{163}$$

$v_H[a]^{-1}$ is the covariant kinetic term of the Higgs field given in eq.(138). Since blockspin, background and fluctuation Higgs fields are defined nonlinearly it is advisable to be careful with their introduction to keep track of the functional determinants connected with these nonlinearities. Therefore we insert background and fluctuation fields with their defining delta-functions into the definiton of the effective action

$$\begin{aligned} e^{-S_{\text{eff}}[\Phi,\mathbf{A}]} = \int \quad & \mathcal{D}a\mathcal{D}a^s\mathcal{D}a^h\delta(\mathbf{A} - Ca)\delta(a^s - \mathcal{A}\mathbf{A})\delta(a^h + a^s - a) \\ & \mathcal{D}\phi\mathcal{D}\phi^s\mathcal{D}\phi^h\delta(\Phi - C_H[\phi])\delta(\phi^s - \mathcal{A}_H[\Phi])\delta(\phi^h + \ln\phi^s - \ln\phi) \\ & e^{-S_M[a]-S_{\text{gf}}[a]-\frac{1}{2}\langle\phi, v_H[a]^{-1}\phi\rangle - V[\phi]} \quad . \end{aligned} \tag{164}$$

The next step is to change the order of integration to remove the fundamental fields ϕ and a. Since $\delta(\phi^h + \ln\phi^s - \ln\phi)$ is nonlinear in the fundamental Higgs field ϕ this gives rise to a functional determinant

$$\det\left(\frac{\delta\phi(z)}{\delta\ln\phi(z')}\right) = \det\left(\delta(z-z')\phi(z)\right) = \det\left(\mathbf{1}\phi\right) \quad . \tag{165}$$

For the integration of the gauge field a this complication does not exist.

$$\begin{aligned} e^{-S_{\text{eff}}[\Phi,\mathbf{A}]} = \int \quad & \mathcal{D}a^s\mathcal{D}a^h\delta(\mathbf{A} - Ca^s - Ca^h)\delta(a^s - \mathcal{A}\mathbf{A}) \\ & \mathcal{D}\phi^s\mathcal{D}\phi^h\delta(\Phi - C_H[\phi^s]C_H[e^{\phi^h}])\delta(\phi^s - \mathcal{A}_H[\Phi])\det(\mathbf{1}\phi^s e^{\phi^h}) \end{aligned}$$

$$e^{-S_M[a^s+a^h]-S_{\rm gf}[a^s+a^h]-\frac{1}{2}\langle\phi^s e^{\phi^h},v_H[a^s+a^h]^{-1}\phi^s e^{\phi^h}\rangle-V[\phi^s e^{\phi^h}]} \qquad (166)$$

We now integrate the background field. This is possible without introduction of a further functional determinant

$$e^{-S_{\rm eff}[\Phi,\mathbf{A}]} = \int \mathcal{D}a^h\delta(Ca^h)\mathcal{D}\phi^h\delta(\Phi - C_H[\mathcal{A}_H[\Phi]]C_H[e^{\phi^h}])\det(\mathbf{1}\mathcal{A}_H[\Phi]e^{\phi^h}) \qquad (167)$$
$$e^{-S_M[\mathcal{A}\mathbf{A}+a^h]-S_{\rm gf}[\mathcal{A}\mathbf{A}+a^h]-\frac{1}{2}\langle\mathcal{A}_H[\Phi]e^{\phi^h},v_H[\mathcal{A}\mathbf{A}+a^h]^{-1}\mathcal{A}_H[\Phi]e^{\phi^h}\rangle-V[\mathcal{A}_H[\Phi]e^{\phi^h}]}$$

We simplify this result using eq.(159) and $\det = e^{\rm tr\ln}$. Writing the trace as the appropriate integral we obtain

$$\begin{aligned}
\delta(\Phi - C_H[\mathcal{A}_H[\Phi]]C_H[e^{\phi^h}])\det(\mathcal{A}_H[\Phi]e^{\phi^h}) &= \delta(C_H[e^{\phi^h}])\det(\mathbf{1}\mathcal{A}_H[\Phi])\det(\mathbf{1}e^{\phi^h}) \\
&= \delta(e^{C_S\phi^h})e^{\int_z \mathcal{A}_S\ln\Phi}e^{\int_z\phi^h} \\
&= \delta(C_S\phi^h)\frac{1}{\det(\mathbf{1}e^{C_S\phi^h})}e^{\int_x C_S\mathcal{A}_S\ln\Phi}e^{\int_x C_S\phi^h} \\
&= \delta(C_S\phi^h)e^{\int_x\ln\Phi} \\
&= \delta(C_S\phi^h)\det(\mathbf{1}\Phi) \qquad (168)
\end{aligned}$$

The unit matrix $\mathbf{1}$ must be choosen according to the space-time in which the following diagonal element is defined.

In addition we again use $S_{\rm gf}[\mathcal{A}\mathbf{A}+a^h] = S_{\rm gf}[a^h]$ to obtain

$$e^{-S_{\rm eff}[\Phi,\mathbf{A}]} = \det(\mathbf{1}\Phi)e^{-S_M[\mathcal{A}\mathbf{A}]}\int \mathcal{D}a^h\delta(Ca^h)\mathcal{D}\phi^h\delta(C_S\phi^h) \qquad (169)$$
$$e^{-S_M[a^h]-S_{\rm gf}[a^h]-\frac{1}{2}\langle\mathcal{A}_H[\Phi]e^{\phi^h},v_H[\mathcal{A}\mathbf{A}+a^h]^{-1}\mathcal{A}_H[\Phi]e^{\phi^h}\rangle-V[\mathcal{A}_H[\Phi]e^{\phi^h}]} \quad .$$

To evaluate the fluctuation integral perturbatively we expand e^{ϕ^h} in the fluctuation field ϕ^h.

Finally we compare the result of this alternative blockspin definition in eq.(169) with the effective action of eq.(148). They differ only in the Higgs field part of the action. The former definition has the advantage that the operators $C_H[a]$ and $\mathcal{A}_H[a]$ are linear, hence they have an integral representation. This is not the case for the nonlinear operators C_H and $\mathcal{A}_H$. The advantage of using the latter is that the interaction between Higgs and gauge fields remains restricted to the covariant kinetic operator of the Higgs field and does not invade the self-interaction of the Higgs as it has happened in eq.(148).

Moreover if we deals with non-Abelian gauge fields we have to use nonlinear blockspin definitions anyway [23].

6 Adaption to finite temperature

6.1 Notational preliminaries

At finite temperature $\mathrm{T} > 0$ we have periodicity in time with period $\beta = 1/\mathrm{T}$. The extension L_4 of blocks in time direction must be chosen commensurate with β i.e. $\beta = NL_4$. A great simplification results if we chose $L_4 = \beta$ so that only one block fits in time direction. The lattice Λ then becomes three dimensional. Our three-dimensional fields still have the same dimension as the original four-dimensional ones. This could be remedied by a rescaling by $\beta^{(1/2)}$. We prefer not to do so because without rescaling we can keep track of the temperature dependence with dimensional considerations. Since we do not rescale the block fields in position space the appropriate three-dimensional integration measure contains a factor β and the three-dimensional delta-functions a corresponding factor $1/\beta$.

Continuum coordinates are denoted by z, lattice coordinates by x. In the finite temperatur case it makes sense to distinguish the space coordinates from the time coordinate

$$\begin{aligned} z &= (\vec{z}, z_4) \\ x &= (\vec{x}, x_4) = (n_1 L_1, n_2 L_2, n_3 L_3, n_4 L_4) \end{aligned} \tag{170}$$

If $L_4 = \beta$ then $x = (\vec{x}, 0)$. Despite the fact that in the latter case x_4 is always equal to 0 we do not drop this superfluous coordinate because our theory is not truly three-dimensional.

Integration over coordinate space is a summation in case of the lattice. For the sake of readability we define a symbolic notation:

$$\begin{aligned} &\mathrm{T} = 0 \text{ continuum} && \int_z := \int_{\vec{z}} \int_{-\infty}^{\infty} dz_4 = \int_{-\infty}^{\infty} d^4 z \\ &\mathrm{T} > 0 \text{ continuum} && \int_z := \int_{\vec{z}} \int_0^{\beta} dz_4 = \int_{-\infty}^{\infty} d^3 z \int_0^{\beta} dz_4 \\ &\mathrm{T} = 0 \text{ lattice} && \int_x := \int_{\vec{x}} L_4 \sum_{n_4 \in Z} = L_1 L_2 L_3 L_4 \sum_{n \in Z^4} \\ &\mathrm{T} > 0 \text{ lattice} && \int_x := \int_{\vec{x}} L_4 \sum_{n_4=0}^{N-1} = L_1 L_2 L_3 L_4 \sum_{\vec{n} \in Z^3} \sum_{n_4=0}^{N-1} \\ &\mathrm{T} > 0 \text{ 3-dim lattice} && \int_x := \int_{\vec{x}} \beta = \beta L_1 L_2 L_3 \sum_{\vec{n} \in Z^3} \quad . \end{aligned} \tag{171}$$

with $N = \beta/L_4$.

Remark that in the special case $\beta = L_4$ a factor β remains as a relict of the fourth dimension. This reflects the fact that the blockfield has the same canonical dimension as the fundamental field and this remains true even if we block one direction completely.

Correspondingly we subsume delta-functions on the continuum and Kronecker symbols on the lattice under the same notation $\delta\{\}$

$$\begin{aligned}
&\text{T} = 0 \text{ continuum} \qquad \delta\{z - z'\} := \delta(\vec{z} - \vec{z}')\delta(z_4 - z_4') = \prod_{i=1}^{4} \delta(z_i - z_i') \\
&\text{T} > 0 \text{ continuum} \qquad \delta\{z - z'\} := \delta(\vec{z} - \vec{z}') \sum_n \delta(z_4 - z_4' - \beta n) \\
&\text{T} = 0 \text{ lattice} \qquad \delta\{x - x'\} := \delta(\vec{x} - \vec{x}') \frac{1}{L_4} \delta_{n_4, n_4'} = \prod_{i=1}^{4} \frac{1}{L_i} \delta_{n_i, n_i'} \\
&\text{T} > 0 \text{ lattice} \qquad \delta\{x - x'\} := \delta(\vec{x} - \vec{x}') \frac{1}{L_4} \sum_m \delta_{n_4 - n_4', mN} = \prod_{i=1}^{3} \frac{1}{L_i} \delta_{n_i, n_i'} \frac{1}{L_4} \sum_m \delta_{n_4 - n_4', mN} \\
&\text{T} > 0 \text{ 3-dim lattice} \qquad \delta\{x - x'\} := \delta(\vec{x} - \vec{x}') \frac{1}{\beta} = \prod_{i=1}^{3} \frac{1}{L_i} \delta_{n_i, n_i'} \frac{1}{\beta} \quad .
\end{aligned} \tag{172}$$

For finite extension of the coordinate space the delta-functions are periodized. In the special case $L_4 = \beta$ the factor $1/\beta$ is the remaining coefficient of the necessarily fulfilled Kronecker delta in time direction.

If one uses corresponding integrations and delta-functions

$$\int_z \delta\{z - z'\} = 1 \qquad \text{and} \qquad \int_x \delta\{x - x'\} = 1 \tag{173}$$

is true for all five cases.

6.2 Scholium

Our aim is to adapt the method of block spin transformations to the treatment of quantum field theories at finite temperature. To prepare for this we first recall the relation between propagators at zero and at finite temperature.

At finite temperature $\text{T} = 1/\beta$ a field theory lives on an Euclidean space time which is periodic in time direction with period $\beta = 1/\text{T}$, i.e. on a tube. Propagators admit a random walk representation. The difference between zero and finite temperature stems from the possibility that these random walks may wind around the tube several times. Accordingly, the finite temperature

propagator v_{T} is obtained from v_0 by periodization in time

$$v_{\mathrm{T}}(\vec{z}, z_4, \vec{z}', z_4') = \sum_{n \in Z} v_0(\vec{z}, z_4 + n\beta, \vec{z}', z_4') \quad . \tag{174}$$

The same method is used to obtain the block propagator for finite temperature from the zero-temperature block propagator. On the other hand we want to calculate u_{T} from v_{T} using a thermalized averaging operator C_{T}

$$\begin{aligned} u_{\mathrm{T}}(\vec{x}, x_4, \vec{x}', x_4') &= \sum_{n \in Z} (C_0 v_0 C_0^\dagger)(\vec{x}, x_4 + n\beta, \vec{x}', x_4') \\ &= (C_{\mathrm{T}} v_{\mathrm{T}} C_{\mathrm{T}}^\dagger)(\vec{x}, x_4, \vec{x}', x_4') \quad . \end{aligned} \tag{175}$$

This equation determines how the averaging operator must be thermalized, see appendix C.

$$C_{\mathrm{T}}(\vec{x}, x_4, \vec{z}, z_4) = \sum_{n \in Z} C_0(\vec{x}, x_4 + n\beta, \vec{z}, z_4) \tag{176}$$

The same arguments hold for other operator products. We conclude that all operators for finite temperature are gained by periodization from their zero temperature counterparts.

6.3 Scalar theory at finite temperature

The averaging kernel of eq.(53) factorizes into parts for each direction. According to eq.(176) the kernel for the finite temperature averaging operator is obtained by periodization in time of the zero temperature kernel. Therefore it is convenient to factorize the zero temperature kernel in a space and a time part

$$C_0(\vec{x}, x_4, \vec{z}, z_4) = C_{\mathrm{time}}(x_4, z_4) C_{\mathrm{space}}(\vec{x}, \vec{z}) = \frac{1}{L_4} \chi_{x_4}(z_4) C_{\mathrm{space}}(\vec{x}, \vec{z}) \tag{177}$$

$$\begin{aligned} C_T(\vec{x}, 0, \vec{z}, z_4) &= \sum_{n \in Z} C_{\mathrm{time}}(n\beta, z_4) C_{\mathrm{space}}(\vec{x}, \vec{z}) \\ &= \sum_{n \in Z} \frac{1}{\beta} \chi_{n\beta}(z_4) C_{\mathrm{space}}(\vec{x}, \vec{z}) \\ &= \frac{1}{\beta} C_{\mathrm{space}}(\vec{x}, \vec{z}) \end{aligned} \tag{178}$$

where we have used the fact that z_4 must lie in exactly one block.

The factor for the blocking in time direction is altered from $1/L_4 \chi_{x_4}(z_4)$ to $1/\beta$. It is now independent of the fine scale time coordinate z_4. This facilitates the calculation of the block propagator u_{T}.

Due to the translational invariance of the fundamental propagator v_0 one of the time integrations is trivial and gives a factor β, see appendix C

$$\begin{aligned} u_{\mathrm{T}}(\vec{x},0,\vec{y},0) &= (C_{\mathrm{T}} v_{\mathrm{T}} C_{\mathrm{T}}^{\dagger})(\vec{x},0,\vec{y},0) \\ &= \frac{1}{\beta} \int\limits_{\vec{z}} \int\limits_{\vec{z}'} C_{\mathrm{space}}(\vec{x},\vec{z}) \int\limits_{-\infty}^{\infty} dz_4 v_0(\vec{z},z_4,\vec{z}',0) C_{\mathrm{space}}^{\dagger}(\vec{z}',\vec{y}) \\ &=: \frac{1}{\beta} u_{FT}(\vec{x},\vec{y}) \quad . \end{aligned} \tag{179}$$

Since we have a single block in time direction the factor $1/\beta$ contains the only temperature dependence. $u_{FT}(\vec{x},\vec{y})$ is the temperature- and time-independent part of the finite temperature block propagator.

Taking care of the β factors in the integration measure and the delta-function we find for the block kinetic term

$$u_{\mathrm{T}}^{-1}(\vec{x},0,\vec{y},0) = \frac{1}{\beta} u_{FT}^{-1}(\vec{x},\vec{y}) \quad . \tag{180}$$

We use these simplified (due to $\beta = L_4$) averaging operator and block kinetic term to obtain a time independent interpolation kernel, see appendix C

$$\begin{aligned} \mathcal{A}_{\mathrm{T}}(\vec{z},z_4,\vec{x},0) &= \int\limits_{\vec{z}'} \int\limits_{0}^{\beta} dz_4' \beta \int\limits_{\vec{x}'} v_{\mathrm{T}}(\vec{z},z_4,\vec{z}',z_4') \frac{1}{\beta} C_{\mathrm{space}}^{\dagger}(\vec{z}',\vec{x}') \frac{1}{\beta} u_{FT}^{-1}(\vec{x}',\vec{x}) \\ &= \mathcal{A}_{\mathrm{T}}(\vec{z},0,\vec{x},0) =: \frac{1}{\beta} \mathcal{A}_{FT}(\vec{z},\vec{x}) \quad . \end{aligned} \tag{181}$$

Again the factor $1/\beta$ is the only temperature dependence, and $\mathcal{A}_{FT}$ is time independent.

It is immediately clear that

$$\begin{aligned} \mathcal{A}_{\mathrm{T}} C_{\mathrm{T}}(\vec{z},z_4,\vec{z}',z_4') &= \beta \int\limits_{\vec{x}} \mathcal{A}_{\mathrm{T}}(\vec{z},z_4,\vec{x},0) C_{\mathrm{T}}(\vec{x},0,\vec{z}',z_4') \\ &= \frac{1}{\beta} \int\limits_{\vec{x}} \mathcal{A}_{FT}(\vec{z},\vec{x}) C_{\mathrm{space}}(\vec{x},\vec{z}') \end{aligned} \tag{182}$$

is time independent as well as are all background quantities.

The above simplifications, i.e. time independence of C_{T} and $\mathcal{A}_{\mathrm{T}}$ and the simple structure of the temperature dependence, result from the fact that we use a single block in time direction. As mentioned above the resulting lattice is only three-dimensional, so time does not exist anymore. The temperature dependence can be inferred from dimensional considerations.

6.4 Gauge fields at finite temperature

As in the scalar case the operators are adapted to finite temperature by periodization.

In principle we could perform the same calculations as in the scalar case. However the space-time indices μ, ν and indications of possible gauge fixing (indices α, β of section 4) in addition to the subscripts indicating finite temperature, scalar averaging operator and spatial part of whatever averaging operator would make these calculations extremely ugly.

Therefore we go again through the arguments for the scalar case written explicitly in appendix C to extract the necessary conditions for their validity.

These are two conditions. The first is that the fundamental zero temperature propagator must be translationally invariant in time. This is of course the case for the gauge field propagator. The second is that the finite temperature averaging kernel is time independent and has a factor $1/\beta$ as sole temperature dependence. We now show this explicitly.

We recall the covariant averaging kernel from section 4

$$C_{\mu\nu}(x,z) = \delta_{\mu,\nu} \prod_{i \neq \mu} C(x_i, z_i) \int_{z'_\mu} C(x_\mu, z'_\mu) C(z'_\mu + \tfrac{1}{2} L_\mu, z_\mu) \quad . \tag{183}$$

Before we periodize in time we factorize it in a time and a space part. To do that we must consider two different cases. The first is $\mu = 4$

$$C_{44}(\vec{x}, x_4, \vec{z}, z_4) = \int_{-\infty}^{\infty} dz'_4 C(x_4, z'_4) C(z'_4 + \tfrac{1}{2} L_4, z_4) C_{\text{space}}(\vec{x}, \vec{z}) \tag{184}$$

where $C_{\text{space}}(\vec{x}, \vec{z})$ is the same as in the scalar case.

The second case is $\mu \neq 4$

$$C_{\mu\mu}(\vec{x}, x_4, \vec{z}, z_4) = C(x_4, z_4) \prod_{i \neq 4, i \neq \mu} C(x_i, z_i) \int_{z'_\mu} C(x_\mu, z'_\mu) C(z'_\mu + \tfrac{1}{2} L_\mu, z_\mu) \tag{185}$$

Periodization involves only the first factor in both cases. Again we use only one block in time direction i.e. we set $x_4 = 0$ and $L_4 = \beta$.

For the second case we need

$$\sum_{n \in Z} C(n\beta, z_4) = \sum_{n \in Z} \frac{1}{\beta} \chi_{n\beta}(z_4) = \frac{1}{\beta} \tag{186}$$

and for the first case

$$\begin{aligned}
&\sum_{n\in Z}\int_{-\infty}^{\infty} dz_4' C(n\beta, z_4') C(z_4' + \tfrac{1}{2}\beta, z_4) \\
&= \int_{-\infty}^{\infty} dz_4' \underbrace{\sum_{n\in Z} C(n\beta, z_4')}_{\frac{1}{\beta}} C(z_4' + \tfrac{1}{2}\beta, z_4) \\
&= \frac{1}{\beta}\int_0^{\beta} dz_4' \underbrace{\sum_{n\in Z} C(z_4' + n\beta + \tfrac{1}{2}\beta, z_4)}_{\frac{1}{\beta}} \\
&= \frac{1}{\beta} \quad . \qquad (187)
\end{aligned}$$

The finite temperature covariant averaging kernels are for $\mu \neq 4$

$$C_{\mathrm{T}\ \mu\mu}(\vec{x}, 0, \vec{z}, z_4) = \frac{1}{\beta} \prod_{i\neq 4 \mathrm{T} i\neq\mu} C(x_i, z_i) \int_{z_\mu'} C(x_\mu, z_\mu') C(z_\mu' + \tfrac{1}{2}L_\mu, z_\mu) \quad (188)$$

and for $\mu = 4$

$$C_{\mathrm{T}\ 44}(\vec{x}, 0, \vec{z}, z_4) = \frac{1}{\beta} C_{\mathrm{space}}(\vec{x}, \vec{z}) \quad . \qquad (189)$$

We conclude that formulae similar to eqs.(179) and (181) hold for the gauge field case as well.

But we can see something else from the explicit calculation. The finite temperature kernel in time direction is the finite temperature scalar kernel. It has lost its covariance. This is no problem since on the lattice there is no time direction and hence there can be no lattice derivative in time direction of a lattice gauge function. Because of the anisotropy, the time component of the lattice gauge fields $\mathbf{A}_4$ has lost its gauge field property. The dependence of the resulting action on $\mathbf{A}_4$ and on $\mathbf{A}_i$, $i = 1, 2, 3$ is different. The $\mathbf{A}_4$ field behaves like an extra scalar field in the effective three-dimensional theory.

6.5 Scalar electrodynamics at finite temperature

We wish to adopt the considerations of the preceding subsections to scalar electrodynamics. The blocking of the gauge field at finite temperature was discussed before. So we are only concerned with the Higgs sector.

First we consider the gauge field dependent linear averaging and interpolation operators. Because of this gauge field dependence a covariant momentum space description does not exist, and we cannot apply exactly the same periodization procedure as for a purely scalar theory. Nevertheless, the transition to finite temperature by periodization in time is straightforward. The formulae of the preceding section remain valid as they stand when they are properly interpreted.

Since averaging and integration operators are integral operators the arguments of subsection 6.2 can be applied. All operators and propagators are adapted to finite temperature by periodization in time.

For the gauge field dependent covariant propagator $v_H[a]$ this is done as in eq.(174).

Finding $C_{H,\mathrm{T}}$ is more involved since here the time dependence is not explicitly known. Instead it is contained in the definition of the gauge field dependent Laplacian with Neumann boundary conditions whose eigenfunction to the lowest eigenvalue we chose as $C_H^\dagger$.

Here the periodization must take place in the definition of the Laplacian. We regard the blocks as coming equipped with periodic boundary conditions in time direction. That means we deal immediately with the three-dimensional lattice with $L_4 = \beta$. The Neumann boundary conditions apply only to those boundaries of the block which remain, after periodic boundary conditions in time direction are imposed.

This finite temperature Laplacian defines the finite temperature block operator as the zero temperature Laplacian the zero temperature block operator.

All other operators can be obtained from these two explicitly thermalized ones using the zero temperature relations of section 5.3 for the finite temperature operators.

Since the temperature dependence of $C_{H,\mathrm{T}}$ is more complicated than that of the scalar block operator, we can not extract it as simple factors of β. This implies a hidden temperature dependence also for the block Higgs field.

The effective action is defined as before, with $C_{H,\mathrm{T}}$ substituted for C_H , etc.. It lives on a three-dimensional lattice Λ.

The gauge field independent nonlinear averaging and interpolatoin operators C_H and $\mathcal{A}_H$ are adapted to finite temperature by the substitution $C_S \to C_{S,\mathrm{T}}$ and $\mathcal{A}_S \to \mathcal{A}_{S,\mathrm{T}}$ in their respective definitions. The fundamental gauge field dependent propagator $v_H[a]$ is periodized in time as before. Now we insert these finite temperature quantities into eq.(169) and continue as in the zero

temperature case keeping in mind that all time integrations are restricted to the range from 0 to β.

7 Propagators, averaging operators, and interpolators in momentum space

7.1 More notational preliminaries

Before we calculate the various Fourier transforms, we give a short review of the definition of Fourier transformations for operators on different supports. In the previous sections we dealt with functions and operators living either on the continuum or on a lattice Λ with lattice spacing L_i in i-direction.

As explained in section 6 at finite temperature we have periodicity in time with period $\beta = 1/\mathrm{T}$. For our simplifying choice $L_4 = \beta$ the lattice Λ becomes three-dimensional.

The type of coordinate space determines the type of momentum space. To the infinite continuum as coordinate space belongs the infinite continuum as momentum space. An infinite lattice with lattice constant L as coordinate space has as corresponding momentum space the continuous first Brillouin zone (BZ) $]-\frac{\pi}{L}, \frac{\pi}{L}]$. This is shown in any textbook of solid state physics. As shown in subsection 6.2 the result of periodization in time with period β is discretization in energy with lattice constant $(2\pi)/\beta$. If a lattice coordinate is periodized the corresponding momentum component is restricted to a discrete first BZ. If our block lattice is three-dimensional, so is the momentum space, i.e. the fourth component is always zero. But as in coordinate space factors of β remain to remind us that we did not rescale the block field.

We will call momenta restricted to the first Brillouin zone p and nonrestricted momenta k. Sometimes the decomposition of not confined momenta k into a part k_{BZ} lying in the first BZ and a discrete summand which shifts the momentum to the other Brillouin zones is convenient

$$k = k_{BZ} + \frac{2\pi}{L} n \quad . \tag{190}$$

where both k_{BZ} and n are uniquely determined by k. Again we define a uniform notation for integration

$$\mathrm{T} = 0 \text{ continuum} \qquad \int\limits_k := \int\limits_{\vec{k}} \int\limits_{-\infty}^{\infty} \frac{dk_4}{2\pi} = \int\limits_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4}$$

$$\text{T} > 0 \text{ continuum} \qquad \int\limits_k := \int\limits_{\vec{k}} \frac{1}{\beta} \sum_{n=-\infty}^{\infty} = \int\limits_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} \frac{1}{\beta} \sum_{n=-\infty}^{\infty}$$

$$\text{T} = 0 \text{ lattice} \qquad \int\limits_p := \int\limits_{\vec{p}} \int\limits_{-\frac{\pi}{L_4}}^{\frac{\pi}{L_4}} \frac{dp_4}{2\pi} = \prod_{i=1}^{4} \int\limits_{-\frac{\pi}{L_i}}^{\frac{\pi}{L_i}} \frac{dp_i}{2\pi}$$

$$\text{T} > 0 \text{ lattice} \qquad \int\limits_p := \int\limits_{\vec{p}} \frac{1}{\beta} \sum_{n=0}^{N-1} = \prod_{i=1}^{3} \int\limits_{-\frac{\pi}{L_i}}^{\frac{\pi}{L_i}} \frac{dp_i}{2\pi} \frac{1}{\beta} \sum_{n=0}^{N-1}$$

$$\text{T} > 0 \text{ 3-dim lattice} \qquad \int\limits_p := \int\limits_{\vec{p}} \frac{1}{\beta} = \prod_{i=1}^{3} \int\limits_{-\frac{\pi}{L_i}}^{\frac{\pi}{L_i}} \frac{dp_i}{2\pi} \frac{1}{\beta} \tag{191}$$

and for the delta-function in momentum space

$$\text{T} = 0 \text{ continuum} \qquad \delta\{k-k'\} := (2\pi)^3 \delta(\vec{k}-\vec{k}') 2\pi \delta(k_4 - k_4') = (2\pi)^4 \prod_{i=1}^{4} \delta(k_i - k_i')$$

$$\text{T} > 0 \text{ continuum} \qquad \delta\{k-k'\} := (2\pi)^3 \delta(\vec{k}-\vec{k}')\, \beta \delta_{n,n'} = (2\pi)^3 \prod_{i=1}^{3} \delta(k_i - k_i')\, \beta \delta_{n,n'}$$

$$\text{T} = 0 \text{ lattice} \qquad \delta\{p-p'\} := (2\pi)^4 \prod_{i=1}^{4} \sum_{n_i} \delta(p_i - p_i' - \frac{2\pi}{L} n_i)$$

$$\text{T} > 0 \text{ lattice} \qquad \delta\{p-p'\} := (2\pi)^3 \prod_{i=1}^{3} \sum_{n_i} \delta(p_i - p_i' - \frac{2\pi}{L} n_i)\, \beta \sum_m \delta_{n-n',mN}$$

$$\text{T} > 0 \text{ 3-dim lattice} \qquad \delta\{p-p'\} := (2\pi)^3 \prod_{i=1}^{3} \sum_{n_i} \delta(p_i - p_i' - \frac{2\pi}{L} n_i)\, \beta \quad . \tag{192}$$

The factors and periodizations are chosen such that

$$f(k) = \int\limits_{k'} \delta\{k-k'\} f(k') \qquad \text{and} \qquad f(p) = \int\limits_{p'} \delta\{p-p'\} f(p') \quad . \tag{193}$$

Moreover with the above defined symbolic notation we regain the familiar Fourier representation of the delta-function

$$\begin{aligned} \delta\{k\} &= \int\limits_z e^{-ikz} \qquad \text{and} \qquad \delta\{p\} = \int\limits_x e^{-ipx} \\ \delta\{z\} &= \int\limits_k e^{ikz} \qquad \text{and} \qquad \delta\{x\} = \int\limits_p e^{ipx} \quad . \end{aligned} \tag{194}$$

One can show the validity of these formulae with the above definitions and the formulae for finite and infinite geometric series.

In the following it is often necessary to give up the symbolic notation during the course of the calculation and to use the explicit expressions.

7.2 Translational invariance

Translationally invariant operators are trivially to invert in momentum space, moreover the successive application of translationally invariant operators in coordinate space reduces to the multiplication of their Fourier transforms in momentum space. Here most of the operators are only lattice translational invariant, which means that their Fourier transforms have a simpler structure than that of a non-invariant operator, but are more complicated than that of a fully translational invariant operator.

To keep the formulae simple we write them for the one dimensional case. The generalization to four dimensions must allow different lattice constants in different directions.

An operator kernel $O(z,z')$ has the Fourier representation

$$O(z,z') = \int\limits_k \int\limits_{k'} e^{ikz+ik'z'} \tilde{O}(k,k') \quad . \tag{195}$$

If O is translational invariant $O(z,z') = O(z+a,z'+a)$. Then the momenta are restricted by

$$\frac{2\pi}{a} m = k + k' = k_{BZ} + k'_{BZ} + \frac{2\pi}{L} n + \frac{2\pi}{L} n' \tag{196}$$

with the decomposition of momenta.

If z and z' are both in the continuum there are two types of translational invariance. Full translational invariance allows any $a \in R$. The only solution for the momenta is $n + n' = 0$ and $k_{BZ} + k'_{BZ} = 0$ hence $k + k' = 0$. For $a \in \Lambda$ there are solutions for every $m \in N$. They are $k_{BZ} + k'_{BZ} = 0$ and $n + n' = m$. Hence the sum of the momenta is only fixed up to jumps from one Brillouin zone to another.

If $z \in R$ and $z' \in \Lambda$ (or the other way round) one of the momenta is not restricted to the first brillouin zone and $k + k' = (2\pi/L)m$ can be fulfilled for $m \neq 0$ as $k_{BZ} + k'_{BZ} = 0$ and $n = m$ (or $n' = m$).

If z and z' are both block lattice coordinates k and k' are both restricted to the first Brillouin zone hence n and n' are zero. The solution $k_{BZ} + k'_{BZ} = 0$ exists only for $m = 0$. That means for operators living entirely on the block lattice, lattice translational invariance is full translational invariance.

There are two possibilities to use this relation of the momenta. One can either display the delta-function explicitly in the Fourier transform, or one can use it to restrict the range of integration in the Fourier representation of the translational invariant operator. As an example we suppose the operator O to be lattice translational invariant

$$\begin{aligned} O(z,z') &= \int_k \int_{k'} e^{ikz+ik'z'} \tilde{O}(k,k') \\ &= \int_k \int_{k'} e^{ikz+ik'z'} \underbrace{\sum_n (2\pi)^4 \delta(k+k'-\frac{2\pi}{L}n) O(k,k')}_{\tilde{O}(k,k')} \\ &= \sum_n \int_k e^{ik(z-z')+i\frac{2\pi}{L}nz'} O(k,-k+\frac{2\pi}{L}n) \end{aligned} \tag{197}$$

We prefer the notation of the second line with the openly displayed delta-function. The missing tilde over $O(k,k')$ denotes that the momentum delta-function is not any more implicitly contained in the transform.

7.3 Concatenation of two operators

In concatenations the resulting operator has the translation invariance property common to both parent operators. In general the Fourier transform of the resulting operator is not the product of the Fourier transforms of the single operators but

$$(\widetilde{AB})(k,k'') = \int_{k'} \tilde{A}(k,k')\tilde{B}(-k',k'') \quad . \tag{198}$$

If the operators are translational invariant the equation simplifies. To show this we use the explicit expressions for the momentum delta-functions. For example if A is fully and B lattice translational invariant we can observe how the delta-function corresponding to the lattice translational invariance of the product appears

$$\begin{aligned} (\widetilde{AB})(k,k'') &= \int_{k'} (2\pi)^4 \delta(k+k') \sum_n (2\pi)^4 \delta(-k'+k''+\frac{2\pi}{L}n) A(k,k')B(-k',k'') \\ &= \sum_n (2\pi)^4 \delta(k+k''+\frac{2\pi}{L}n) \underbrace{A(k,-k)B(k,k'')}_{(AB)(k,k'')} \quad . \end{aligned} \tag{199}$$

7.4 Fourier transformed operators of the scalar model at zero temperature

Now we are prepared to calculate the Fourier transforms of all the operators we used in section 3. We have to evaluate only two of them, the transforms of the fundamental kinetic term and of the averaging operator, from their respective coordinate representation. All the others are constructed from these two and their Fourier transforms can be obtained from the momentum representation of the former using the rule for concatenation.

The fundamental kinetic term $v^{-1}(z,z') = \delta\{z-z'\}(-\Delta + m^2)$ is continuum translational invariant hence

$$\tilde{v}^{-1}(k,k') = \delta\{k+k'\}v^{-1}(k,k') = \delta\{k+k'\}(k^2+m^2) \quad . \tag{200}$$

Inversion in momentum space is trivial

$$\tilde{v}(k,k') = \delta\{k+k'\}v(k,k') = \delta\{k+k'\}\frac{1}{(k^2+m^2)} \quad . \tag{201}$$

7.4.1 The averaging kernel

We consider the kernel $C(x,z)$ of the averaging operator C which is given in eq.(53).

$$C(x,z) = \frac{1}{\prod_{i=1}^4 L_i}\chi_x(z) = \int\limits_k \int\limits_p e^{ikz+ipx}\tilde{C}(p,k) \tag{202}$$

Here the integrals are four-dimensional, and $kz = \prod_{i=1}^4 k_i z_i$. Because C is lattice translational invariant under simultaneous translations of z and x by lattice vectors in Λ, its Fourier transform includes a delta-function relating the momenta p and k. The product form of C allows to perform the Fourier transformation independently for each dimension so that the actual calculation can be done in one dimension

$$\begin{aligned}
\tilde{C}(p,k) &= \int\limits_x \int\limits_z e^{-ipx-ikz}\frac{1}{L}\chi_x(z) \\
&= \frac{1}{L}\int\limits_x e^{-ipx}\int\limits_{x-\frac{L}{2}}^{x+\frac{L}{2}} dz e^{-ikz} \\
&= \frac{1}{L}\int\limits_x e^{-ipx}\frac{1}{-ik}e^{-ikx}\underbrace{[e^{-ik\frac{L}{2}} - e^{ik\frac{L}{2}}]}_{-2i\sin\frac{kL}{2}}
\end{aligned}$$

$$= \underbrace{\int_x e^{-i(p+k)x}}_{\delta\{p+k\}} \underbrace{\frac{2}{Lk}\sin\frac{kL}{2}}_{C(p,k)} \quad . \tag{203}$$

Note that the emerging delta-function is to be understood in the symbolic sense since it is the result of an integration in symbolic notation. The translation to the explicit calculation is

$$\int_x e^{-i(p+k)x} = L\sum_{n=-\infty}^{\infty} e^{i(p+k)Ln} = 2\pi\sum_{n=-\infty}^{\infty}\delta(p+k-\frac{2\pi}{L}n) = \delta\{p+k\} \tag{204}$$

p lies in the first Brillouin zone and because the delta-function is periodized k may take values outside of it.

The four-dimensional result is

$$\tilde{C}(p,k) = \prod_{i=1}^{4}\tilde{C}(p_i,k_i) \tag{205}$$

7.4.2 The block propagator and block kinetic operator

The block spin propagator was defined in eq.(18) as $u = CvC^\dagger$. In momentum space this results in

$$\tilde{u}(p,p') = \int_k\int_{k'}\tilde{C}(p,k)\tilde{v}(k,k')\tilde{C}^\dagger(k',p') \quad . \tag{206}$$

Here all the Fourier transforms on the right hand side contain delta-functions. We use them to obtain

$$\begin{aligned}\tilde{u}(p,p') &= \delta\{p+p'\}\int_k \frac{1}{k^2} \mid C(p,k) \mid^2 \delta\{p+k\} \\ &= \delta\{p+p'\}\sum_n \frac{1}{(p-\frac{2\pi}{L}n)^2} \mid C(p,-p+\frac{2\pi}{L}n) \mid^2 \\ &=: \delta\{p+p'\}u(p,-p) \quad . \end{aligned} \tag{207}$$

We note that

$$u(p,-p) \sim \frac{1}{p^2} \qquad \text{for} \qquad p^2 \to 0 \tag{208}$$

because only the $n = 0$ term in eq.(207) is singular at $p = 0$.

The kinetic operator of the effective action is obtained as the inverse of the lattice translational invariant block propagator u. In momentum space the

inversion is trivial:

$$\tilde{u}^{-1}(p,p') = \delta\{p+p'\}\frac{1}{u(p,-p)} \quad . \tag{209}$$

7.4.3 The interpolation kernel, the background and fluctuation propagators

We recall the definitions of these operators in coordinate space in eqs.(19), (24) and (30)

$$\begin{aligned} \mathcal{A} &= vC^{\dagger}u^{-1} \\ v^s &= \mathcal{A}Cv \\ v^h &= v - v^s \quad . \end{aligned} \tag{210}$$

All of them are only lattice translational invariant, despite the fact that v^s and v^h live on the continuum.

The kernel $\mathcal{A}(z,x)$ has as Fourier transform

$$\begin{aligned} \tilde{\mathcal{A}}(k,p) &= \int_{k'}\int_{p'} \tilde{v}(k,k')\tilde{C}^*(k',p')\tilde{u}^{-1}(p',p) \\ &= \delta\{k+p\}\underbrace{v(k,-k)C^*(p,k)\frac{1}{u(-p,p)}}_{\mathcal{A}(k,p)} \quad . \end{aligned} \tag{211}$$

The momentum representation of the background propagator is

$$\begin{aligned} \tilde{v}^s(k,k') &= \int_{p}\int_{k''} \tilde{\mathcal{A}}(k,p)\tilde{C}(p,k'')\tilde{v}(k'',k') \\ &= \int_{p}\int_{k''} \delta\{k+p\}\delta\{k''+p\}\delta\{k''+k'\}\mathcal{A}(k,p)C(p,k'')v(k'',k') \\ &= \delta\{k+k'_{BZ}\}\underbrace{\mathcal{A}(k,-k_{BZ})C(k'_{BZ},-k')v(-k',k')}_{v^s(k,k')} \quad . \end{aligned} \tag{212}$$

The symbolic delta-function $\delta\{k+k'_{BZ}\}$ is to be read as periodized so that k can take values outside of the first BZ. To obtain this we have used the fact that

$$\delta(k+p-\frac{2\pi}{L}m) = \delta(k_{BZ}+\frac{2\pi}{L}n+p-\frac{2\pi}{L}m) \tag{213}$$

for a given k, i.e. given k_{BZ} and n both momenta p and $(2\pi/L)m$ are determined since p is confined to the first Brillouin zone. $\tilde{v}^s(k,k')$ does not contain the ordinary continuum delta-function $\delta\{k+k'\}$ because $v^s(z,z')$ is not continuum- but only lattice translational invariant.

The fluctuation propagator in momentum space is given by

$$\begin{aligned}\tilde{v}^h(k,k') &= \tilde{v}(k,k') - \tilde{v}^s(k,k') \\ &= \delta\{k+k'\}v(k,k') - \delta\{k+k'_{BZ}\}v^s(k,k') \\ &= \delta\{k+k'_{BZ}\}\underbrace{[\delta_{n,0}v(k,k') - v^s(k,k')]}_{v^h(k,k')} \quad . \end{aligned} \tag{214}$$

7.5 Scalar fields at finite temperature

To adjust to the finite temperature situation we need to periodize the zero temperature quantities in time in the manner described in section 6.

We exhibit the result of this periodization for the Fourier representation of a the fundamental propagator v

$$\begin{aligned} & v_T(\vec{z},z_4,\vec{z}',z'_4) = \sum_{n\in Z} v_0(\vec{z},z_4+n\beta,\vec{z}',z'_4) \\ &= \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{d^4k'}{(2\pi)^4} e^{i\vec{k}\vec{z}+ik_4z_4+i\vec{k}'\vec{z}'+ik'_4z'_4}\tilde{v}_0(\vec{k},k_4,\vec{k}',k'_4)\sum_{n\in Z} e^{in\beta k_4} \end{aligned} \tag{215}$$

One may use Poisson's resummation formula

$$\sum_{n\in Z} e^{in\beta k_4} = \sum_{n\in Z} \frac{2\pi}{\beta}\delta(k_4 - \frac{2\pi}{\beta}n) \quad . \tag{216}$$

As a result for the momentum component k_4 the finite temperature Fourier expansion is obtained as a sum over Matsubara frequencies $(2\pi/\beta)n$. Since all operators are invariant under translation by β in time direction all Fourier transfomed contain

$$\sum_{n\in Z} \delta(k_4 + k'_4 - \frac{2\pi}{\beta}) \quad . \tag{217}$$

Through this the restriction of k_4 to the Matsubara frequencies is imposed on k'_4 as well

$$\begin{aligned} & v_T(\vec{z},z_4,\vec{z}',z'_4) \\ &= \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3}\frac{1}{\beta}\sum_{n\in Z}\int_{-\infty}^{\infty} \frac{d^3k'}{(2\pi)^3}\frac{1}{\beta}\sum_{n'\in Z} e^{i\vec{k}\vec{z}+i\frac{2\pi}{\beta}nz_4+i\vec{k}'\vec{z}'+i\frac{2\pi}{\beta}n'z'_4}\tilde{v}_0(\vec{k},\frac{2\pi}{\beta}n,\vec{k}',\frac{2\pi}{\beta}n') \quad . \end{aligned} \tag{218}$$

The momentum components corresponding to the periodized coordinate are discretized with lattice constant $2\pi/\beta$. We anticipated this in our symbolic

notation so that we can write

$$v_T(\vec{z}, z_4, \vec{z}', z_4') = \int\limits_k \int\limits_{k'} e^{ikz+ik'z'} \tilde{v}_0(k, k') \tag{219}$$

for the finite temperature case as well. We only have to keep in mind what kind of momentum space is appropriate in this formula.

In case of a lattice operator the momentum is in addition confined to the first BZ. If we have only one block in time direction, i..e. $L_4 = \beta$, the fourth lattice momentum component is set to zero. This is the effect of the dimensional reduction. The factor $1/\beta$ in the measure of integration and the factor β in the definition of the symbolic delta-function are the remainders of the fourth dimension.

To summarize, all the effect of the finite temperature on the operators is incorporated in the integration measure in momentum space. The kernels of the operators remain unchanged, provided one never forgets which values of the fourth momentum component are allowed.

7.6 Free Abelian gauge field at zero temperature

In analogy to the scalar theory there are two operators whose Fourier transforms we have to calculate from the coordinate representation: the fundamental kinetic operator and the averaging operator.

In addition there exist the projectors R and $\mathcal{R}$ defined in subsection 4.3.3. They need the momentum representation of $v_{\Delta^2}^{-1}$ and the scalar averaging kernel for their evaluation in Fourier space.

7.6.1 The kinetic terms

Since both kinetic operators are fully translational invariant their respective Fourier representations are momentum conserving.

In coordinate space

$$\begin{aligned} v_{00}^{-1}(z, z') &= \delta(z - z')(\partial_\mu \partial_\nu - \partial^2 \delta_{\mu,\nu}) \\ v_{\Delta^2}^{-1}(z, z') &= \delta(z - z')\Delta^2 \quad . \end{aligned} \tag{220}$$

Their Fourier transforms are

$$\tilde{v}_{00}^{-1}(k, k') = \delta(k + k')(k^2 \delta_{\mu,\nu} - k_\mu k_\nu) \quad , \tag{221}$$

which is not invertible due to gauge invariance and

$$\tilde{v}^{-1}_{\Delta^2}(k,k') = \delta(k+k')k^4 \quad . \tag{222}$$

7.6.2 The gauge covariant averaging kernel

As the last building block we need the Fourier transform of the averaging operator C for Abelian gauge fields. Its coordinate space kernel is given in eq.(81) as

$$C_{\mu\nu}(x,z) = \delta_{\mu,\nu} \int\limits_{z'} C_S(x_\mu, z'_\mu) C_S(z'_\mu + \frac{1}{2}L_\mu, z_\mu) \prod_{\rho\neq\mu} C_S(x_\rho, z_\rho) \quad . \tag{223}$$

Like the scalar block operator it factorizes in the different directions. For three directions we can directly use the scalar result. To evaluate the Fourier transform in μ-direction we use

$$C_S(z'_\mu + \frac{1}{2}L_\mu, z_\mu) = C_S(z'_\mu, z_\mu - \frac{1}{2}L_\mu) \tag{224}$$

and then apply the formula for concatenation of operators. The resulting expression is

$$\tilde{C}_{\mu\nu}(k,p) = \delta\{k+p\}\delta_{\mu\nu} \exp^{-i\frac{k_\nu L_\nu}{2}} \frac{2}{L_\nu k_\nu} \sin\frac{k_\nu L_\nu}{2} C_S(k,p) \tag{225}$$

where $C_S(k,p)$ is the scalar kernel of eq.(205), and no sum over ν is implied.

8 Computation of effective actions by perturbation theory

We know from eq.(54) that the effective actions are obtained as logarithms of partition functions of auxiliary field theories whose free propagator ist the fluctuation field propagator Γ and which have Φ-dependent coupling constants

$$e^{-V_{\rm eff}[\Phi]} = \int \mathcal{D}\phi^h \delta(C\phi^h) e^{-\frac{1}{2}\langle\phi^h, v^{-1}\phi^h\rangle - V[\mathcal{A}\Phi+\phi^h]} \quad . \tag{226}$$

The fluctuation integral can be evaluated perturbatively

$$\begin{aligned} e^{-V_{\rm eff}[\Phi]} &= \int \mathcal{D}\phi^h \delta(C\phi^h) e^{-\frac{1}{2}\langle\phi^h, v^{-1}\phi^h\rangle - V[\mathcal{A}\Phi+\frac{\delta}{\delta j}] + \langle\phi^h, v^{-1}\phi^h\rangle}\big|_{j=0} \\ &= e^{-V[\mathcal{A}\Phi+\frac{\delta}{\delta j}]} e^{\frac{1}{2}\langle j,\Gamma j\rangle}\big|_{j=0} \quad . \end{aligned} \tag{227}$$

This form is appropriate if one wants to obtain $V_{\rm eff}[\Phi]$ as an expansion in the interaction. If one is interested in its expansion in powers of the fluctuation propagator Γ it is easier to use an alternative formula.

The perturbative expansion to all orders is given by Wick's theorem in the form

$$\begin{aligned} e^{-V_{\rm eff}[\Phi]} &= e^{\frac{1}{2}\langle \frac{\delta}{\delta\phi^h}, \Gamma \frac{\delta}{\delta\phi^h}\rangle} e^{-V[\mathcal{A}\Phi+\phi^h]}\big|_{\phi^h=0} \\ &= e^{\frac{1}{2}\langle \frac{\delta}{\delta\mathcal{A}\Phi}, \Gamma \frac{\delta}{\delta\mathcal{A}\Phi}\rangle} e^{-V[\mathcal{A}\Phi]} \quad . \end{aligned} \tag{228}$$

That both equations give the same result can be seen by the following short consideration

$$\begin{aligned} f(\frac{d}{dx})g(x)|_{x=0} &= \sum_n f_n (\frac{d}{dx})^n \sum_m g_m x^m|_{x=0} \\ &= \sum_n f_n g_n (\frac{d}{dx})^n x^n \\ &= \sum_n f_n g_n n! \\ &= g(\frac{d}{dy}) f(y)|_{y=0} \end{aligned} \tag{229}$$

since the result is symmetric in f and g.

In both expansions the effective potential itself contains only the contributions corresponding to the connected diagrams.

The effective action at finite temperature is given by the same equation provided one substitutes

$$\begin{aligned} C &\to C_{\rm T} \\ v^{-1} &\to v_{\rm T}^{-1} \\ \mathcal{A} &\to \mathcal{A}_{\rm T} \\ \Gamma &\to \Gamma_{\rm T} \end{aligned} \tag{230}$$

The effective interaction, including mass terms, is temperature dependent because the fluctuation field propagator $\Gamma_{\rm T}$ and the interpolation kernel $\mathcal{A}_{\rm T}$ are both temperature dependent. This temperature dependence is weak and disappears in a zeroth order local approximation.

8.1 Leading terms in the perturbative expansion of the perfect action for ϕ^4-theory

Let us consider ϕ^4-theory with bare mass m_0

$$\begin{aligned} S_0[\phi] &= \frac{1}{2}\langle \partial_\mu\phi, \partial_\mu\phi\rangle \\ V[\phi] &= \int_z \left(\frac{1}{2}m_0^2\phi^2 + \frac{g}{4!}\phi^4\right) + \text{wave function renormalization term.} \end{aligned} \tag{231}$$

It is the same action we used in section 3 but here the mass term is considered as a part of the interaction.

Of course without a mass term the fundamental kinetic operator will have zero modes and the fundamental propagator is in momentum space divergent for $k \to 0$. In coordinate space it does not decay exponentially. How to circumvent this problem we have learned in section 4, where we had to impose a regulator to define the fundamental gauge field propagator. Here we could introduce an auxiliary mass as a regulator into the fundamental kinetic term and remove this mass at the very end of the calculations. The in this way obtained fluctuation propagator remains finite for $k \to 0$, because as a fluctuation propagator it is IR-regulated by definition.

Inserting the field split $\phi = \phi^s + \phi^h$, we obtain

$$V[\phi^s + \phi^h] = U_{\rm cl}[\Phi] + \sum_{n=1}^{4} \int\limits_z \frac{1}{n!} g_n[\phi^s] \phi^h(z)^n \tag{232}$$

with

$$\begin{aligned} g_1 &= \frac{g}{3!}\phi^s(z)^3 + m_0^2 \phi^s(z) \\ g_2 &= \frac{g}{2!}\phi^s(z)^2 + m_0^2 \\ g_3 &= g\phi^s(z) \\ g_4 &= g \\ \phi^s &= \int\limits_x \mathcal{A}(z,x)\Phi(x) \\ U_{\rm cl}[\Phi] &= V[\mathcal{A}\Phi] \end{aligned} \tag{233}$$

$V_{\rm eff}$ can be calculated in a loop expansion. When the starting point is a lattice theory, one can use Mayer expansions instead. They are convergent for weak coupling, and the asymptotic expansion of individual terms in powers of the bare coupling constant contains infinite sets of diagrams [21].

We write the perturbative expansion of $V_{\rm eff}$ as

$$V_{\rm eff}[\Phi] = U_{\rm cl}[\Phi] + \sum_{N \geq 1} V_{\rm eff}^{(N)}[\Phi] \tag{234}$$

where $V_{\rm eff}^{(N)}$ scales as γ^N when the fluctuation field propagator $\Gamma \longrightarrow \gamma\Gamma$.

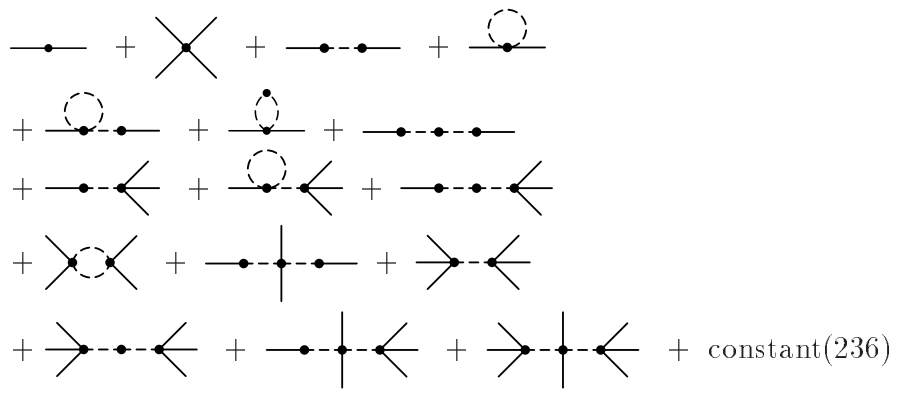
Let us compute $V_{\rm eff}^{(1)}$.

$$V_{\rm eff}^{(1)} = \frac{1}{2!} \int\limits_z (\frac{g}{2!}\phi^s(z)^2 + m_0^2)\Gamma(z,z)$$

$$-\frac{1}{2!}\int_z \int_z{}' (\frac{g}{3!}\phi^s(z)^3 + m_0^2\phi^s(z))\Gamma(z,z')(\frac{g}{3!}\phi^s(z')^3 + m_0^2\phi^s(z')) \quad (235)$$

Higher order calculations can be found in the literature [23], [31].

The effective potential up to second order in the fluctuation propagator is displayed in appendix D and given in diagrammatic form in eq.(236). Factors ϕ^s are indicated by solid external lines and a fluctuation field propagator by a dotted inner line.

$V_{\text{eff}} =$  + constant (236)

For computer simulations it is appropriate to consider V_{eff} as a function of $\Phi(x)$. For analytical computations it is more convenient to regard it as a function of $\phi^s(z)$.

The fluctuation field propagator $\Gamma(z,z')$, decays exponentially with distance $|z-z'|$ with decay length one block lattice spacing. That is, it decays with $|z_i - z_i'|$ with decay length L_i. Similarly, $\mathcal{A}(z,x)$ decays exponentially in $|z_i - x_i|$ with decay length L_i. As a result, each term in V_{eff} is local in Φ modulo exponential tails with decay length of one lattice spacing.

By a process of partial integration, or rather summation, one can exhibit each term in V_{eff} as a sum of local terms and small remainders which represent the exponential tails (see appendix E). The terms with coefficients of dimensions up to $(\text{mass})^{-2}$ are

$$V_{\text{eff}}[\Phi] = \int_x \left(\tfrac{1}{2}m^2\Phi^2 + \delta_z(\nabla_\mu\Phi)^2 + \frac{g_r}{4!}\Phi^4 + \frac{g_6}{6!}\Phi^6 + \tilde{\gamma}(\nabla_\mu\Phi)^2\Phi^2 + \cdots\right) \quad (237)$$

Again the adaption to finite temperature is made with the substitutions of eq.(230). Then all the coefficients are T-dependent and finite. Since $u_{\text{FT}}^{-1}(p) \sim p^2$ as $p \to 0$, we may write the finite temperature expansion equally well in

Fig. 1. temperature dependent divergent diagrams

the form

$$V_{\mathrm{eff,T}}[\Phi] = \int\limits_x \left(\tfrac{1}{2}m_{\mathrm{T}}^2\Phi^2 + \frac{g_{r,\mathrm{T}}}{4!}\Phi^4 + \frac{g_{6,\mathrm{T}}}{6!}\Phi^6\right) + \frac{\beta}{2}\int\limits_{\vec{x}}\int\limits_{\vec{y}} \Phi(x)u^{-1}_{\mathrm{FT}}(x,y)\Phi(y)\left(2\delta_{z,\mathrm{T}} + \gamma_{\mathrm{T}}\Phi(x)^2\right) + \cdots \qquad (238)$$

8.2 Note on the cancellation of temperature dependent UV-divergent diagrams

The UV-convergence of a quantum field theory concerns its local behaviour and is therefore not temperature dependent. If the proper choice of counter terms makes the theory finite at zero temperature, then also at finite temperature. The cancellation occurs order by order in perturbation theory. This is well known.

Problems can occur when one sums selected classes of diagrams. We do not propose to do so when deriving the perfect action, but stay strictly within the realm of standard perturbation theory. So there can be no problem.

It is nevertheless appropriate to point out that there do exist individual diagrams with temperature dependent UV-divergent pieces. They cancel. We give an example.

Let us split the fluctuation propagator Γ into a static part (heavy line) Γ_{stat} and a non static part Γ_{ns} (wavy line). The non static part represents random walks which wind several times around the tube; it is not singular at coinciding arguments: $\Gamma_{\mathrm{ns}}(z,z') \leq \infty$. But it is not zero and is T-dependent. As a result, the diagrams in fig.(1) are both logarithmically UV-divergent, with a temperature dependent coefficient $\Gamma_{\mathrm{ns}}(z,z')$. They must cancel. δg is the logarithmically divergent coupling constant counter term to second order in g.

9 Selfconsistent improvements of perturbation theory: Gap equations

9.1 The Feynman-Bogoliubov Method

To obtain information on the nature of a phase transition at finite temperature, one wants to compute the effective potential, i.e. the free energy as a function of the magnetization

$$M = \int_z \phi(z) = \int_x \Phi(x) \quad . \tag{239}$$

Alternatively, one may compute the constraint effective potential which gives the probability distribution of M.

To do so we should apply nonperturbative methods to solve the lattice theory. One of these methods consists in the solution of gap equations. It is very old and known as the Feynman-Bogoliubov method [14].

The gap equations have 'perturbative' solutions which come from summations of superdaisy diagrams. In principle they might also have other solutions. We wish to examine this method in order to see what will be the effect of terms like $(\nabla_\mu \Phi)^2 \Phi^2$ etc. in the perfect lattice action.

Let us make it clear that it is not sufficient to find the solutions of these gap equations. In order to justify the perturbative calculation of the perfect action it will be necessary to investigate also the stability properties of the solutions of the gap equations against small perturbations of the lattice action. We will comment on this, but a thorough treatment of this question is beyond the scope of this thesis.

Given an action $S[\Phi]$ of some Euclidean field theory, one seeks an optimal quadratic approximation $S_{\text{free}}[\Phi]$ around which to expand

$$S_{\text{free}}[\Phi] = \frac{1}{2} \int_x \int_y \Phi(x) J(x,y) \Phi(y) \tag{240}$$

By Peierls inequality [14], the partition functions obey the inequality

$$\ln(Z) \geq \ln(Z_{\text{free}}) - \langle S - S_{\text{free}} \rangle_{\text{free}} \tag{241}$$

for any choice of the S_{free}. Herein, $\langle\rangle_{\text{free}}$ is the expectation value in the theory with action S_{free}. The right hand side is the first order approximation to the left hand side in the perturbative expansion around S_{free}. The optimal choice of S_{free} is that which makes the right hand side of (241) maximal. In other

words, it makes the first order approximation as good as possible. It is asserted that there exists always a unique optimal choice of S_{free}. It is not asserted that the optimal choice is necessarily a good one. The optimal J is determined by the extreme value condition

$$\langle \frac{\delta^2 S}{\delta\Phi(x)\delta\Phi(y)} \rangle_{\text{free}} = J(x,y). \tag{242}$$

This is equivalent to the condition that the right hand side of (241) is maximal, i.e.

$$\frac{\delta}{\delta J} \left(\ln(Z_{\text{free}}) - \langle S - S_{\text{free}} \rangle_{\text{free}} \right) = 0 \tag{243}$$

When applied to the standard Φ^4 action, this produces the gap equation whose perturbative solution is the sum of superdaisy diagrams (see below).

Let us consider the gap equation which results from the more complicated finite temperature lattice action.

$$S[\Phi] = \frac{\beta}{2} \langle \Phi, u_{\text{FT}}^{-1} \Phi \rangle + V_{\text{eff,T}}[\Phi] \tag{244}$$

with $V_{\text{eff,T}}$ from equation (238).

We obtain

$$\begin{aligned} \frac{\delta^2 S[\Phi]}{\delta\Phi(x)\delta\Phi(y)} &= \beta u_{\text{FT}}^{-1}(x,y) \left(1 + 2\delta_{z,\text{T}} + 3\gamma_{\text{T}} \Phi(x)^2 \right) \\ &+ \beta\delta(x-y) 3\gamma_{\text{T}} \Phi(x) (u_{\text{FT}}^{-1}\Phi)(x) \\ &+ \delta(x-y) \left(m_{\text{T}}^2 + \frac{g_{r,\text{T}}}{2!}\Phi(x)^2 + \frac{g_{6,\text{T}}}{4!}\Phi(x)^4 \right) \end{aligned} \tag{245}$$

The expectation value in the theory with action $S_{\text{free}} = \int \frac{1}{2}\Phi J \Phi$ is

$$\begin{aligned} \langle \frac{\delta^2 S[\Phi]}{\delta\Phi(x)\delta\Phi(y)} \rangle_{\text{free}} &= \beta u_{\text{FT}}^{-1}(x,y) \left(1 + 2\delta_{z,\text{T}} + 3\gamma_{\text{T}} J^{-1}(0) \right) \\ &+ \beta\delta(x-y) 3\gamma_{\text{T}} (u_{\text{FT}}^{-1} J^{-1})(0) \\ &+ \delta(x-y) \left(m_{\text{T}}^2 + \frac{g_{r,\text{T}}}{2} J^{-1}(0) + \frac{g_{6,\text{T}}}{2^2 2!} J^{-1}(0)^2 \right) \end{aligned} \tag{246}$$

where $J^{-1}(0) = J^{-1}(x,x)$ is independent of x by translation invariance, assuming that we seek a translation invariant solution. Other solutions could be of interest.

The gap equation (242) can be solved by the Ansatz

$$J(x,y) = A u_{\text{FT}}^{-1}(x,y) + B\delta(x-y) \tag{247}$$

Inserting the Ansatz results in two transcendental equations for A, B whose solutions depend on the coefficients $\beta, \delta_z, \gamma, g_i$.

Basically, the inclusion of the $(\nabla_\mu \Phi)^2 \Phi^2$-term results in a system of two equations for mass and wave functions renormalization. In standard Φ^4-theory there is only one equation for the mass.

9.2 Self-consistent calculation of the effective action

For a given blockspin Φ one chooses a background field ϕ^s which minimizes the full action. This corresponds to a nonlinear choice of the interpolator $\mathcal{A}$.

The fluctuation integral is performed by a saddle point method with blockspin-dependent saddle point, but only after the action is normal orderd with respect to the fundamental propagator. If one performs consecutive renormalization group transformations the effective action of one step will be the fundamental one of the next. So normal ordering with respect to the fundamental propagator is required anew after each evaluation of the effective action.

This method developed by Griessl, Mack, Palma and Xylander [23] preserves the stability properties of the Boltzmann factor.

10 Conclusions

We compared three methods to perform a field split of a scalar field into a high and a low frequency part. We showed that the blockspin method and the hard-soft invariance fixing method are two complementary generalizations of the split of the Gaussian measure. The blockspin method is a real space method. It uses projectors to separate the fields and therefore the operators containing these projectors do not have inverses anymore but only pseudoinverses. The hard-soft invariance fixing method is a momentum space method. It introduces the split as a Pauli-Villar regularization for the hard field and as a regularization with higher derivatives for the soft field. It renders invertible operators.

Both methods could be adapted to preserve the gauge invariance if applied to an abelian gauge field. However, the hard-soft invariance fixing method destroyed the perturbative renormalizability of the fundamental theory which ruled out its use for perturbative calculations of the effective action.

The adaption of the blockspin method required an intermediate gauge fixing as a means of regularization which could be removed at the end. A new set of

projectors were introduced to split the gauge transformations themselves into a soft and a hard part.

We applied the blockspin method to scalar electodynamics and presented two ways to adapt it to the needs of the Higgs field. One involved gauge field dependent projectors the other nonlinear operators.

The generalization to finite temperature was done by restricting the time to a finite range with periodic boundary conditions. The operators were adapted to finite temperature by periodization in time. Within the developed formalism we could mimick dimensional reduction by choosing a block lattice with a single block in time direction. The temperature dependence of the block lattice quantities could then simply be inferred by purely dimensional considerations.

To prepare for perturbative calculations of the effective action we calculated the Fourier transforms of the various operators. Here adaption to finite temperature resulted in a discretization of energy to the Matsubara frequencies.

Applying the blockspin method to a scalar ϕ^4 action we presented the expansion in the fluctuation propagator up to second order. We explained why the UV-divergencies must be temperature independent and how they are removed by counterterms of the fundamental theory. We discussed possible resummations to approximate the effective action non-perturbatively.

11 Outlook

There are several points which still need to be studied. One of them is the inclusion of non-abelian gauge fields into the outlined formalism. To deal with them requires the use of nonlinerar averaging and interpolation operators. Closely related to this problem is the question whether an perturbative expansion in the fluctuation field makes sense. It might be better to use non-perturbative approximations if one evaluates the fluctuation integral numerically. These questions are addressed in [23] and are under further study.

A next source of trouble is the inclusion of fermions. In section 3 we have already mentioned that by a special choice of the averaging operator the blockspin method can work entirely in the continuum, but then one cannot use MC methods to solve the effective action. One has to think about a hybrid approach. First all fields of the theory could be separated into background and fluctuation fields using the continuum 'block'spins. This would guarantee a consistent implementation of the cutoff. Fluctuation fields must then be integrated out. The fermionic background fields could then be handled in the continuum, whereas the other background fields might be blocked to a lattice

without lowering the cutoff further. It needs careful considerations to show that this way gauge fields and fermions interacting via minimal coupling are treated consistently. To facilitate the problem one could start again with an abelian gauge field, attacking QED this way.

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A Interpolator for scalar fields

The Ansatz for the interpolation operator

$$v^{-1}\mathcal{A} = C^\dagger \tilde{\mathcal{A}} \tag{A.1}$$

removes the mixed kinetic term since $C\phi^h = 0$.

The kinetic term for the blockspin is

$$\frac{1}{2}\langle \Phi, \underbrace{\mathcal{A}^\dagger v^{-1}\mathcal{A}}_{=:u^{-1}} \Phi\rangle \quad . \tag{A.2}$$

The definition of u^{-1} helps us to get rid of the auxiliary operator $\tilde{\mathcal{A}}$

$$u^{-1} := \mathcal{A}^\dagger v^{-1}\mathcal{A} = \mathcal{A}^\dagger C^\dagger \tilde{\mathcal{A}} = \tilde{\mathcal{A}} \tag{A.3}$$

hence

$$v^{-1}\mathcal{A} = C^\dagger u^{-1} \quad \text{with} \quad u^{-1} := \mathcal{A}^\dagger v^{-1}\mathcal{A} \quad \text{and} \quad u = CvC^\dagger \quad . \tag{A.4}$$

The derivation of the fluctuation propagator is done using the Fourier representation of the δ-functional

$$\begin{aligned}
{Z_0}^h[j] &= \int \mathcal{D}\phi^h \delta(C\phi^h)\ \mathrm{e}^{-\frac{1}{2}\langle \phi^h, v^{-1}\phi^h\rangle + \langle \phi^h, j\rangle} \\
&= \int \mathcal{D}\phi^h \mathcal{D}\sigma\ \mathrm{e}^{-\frac{1}{2}\langle \phi^h, v^{-1}\phi^h\rangle + \langle \phi^h, j\rangle + i\langle \sigma, C\phi^h\rangle} \\
&= \frac{1}{\sqrt{\det v^{-1}}} \int \mathcal{D}\sigma\ \mathrm{e}^{-\frac{1}{2}\langle (C^\dagger\sigma - ij), v(C^\dagger\sigma - ij)\rangle} \\
&= \frac{1}{\sqrt{\det v^{-1}}}\ \mathrm{e}^{\frac{1}{2}\langle j, vj\rangle} \int \mathcal{D}\sigma\ \mathrm{e}^{-\frac{1}{2}\langle \sigma, CvC^\dagger\sigma\rangle + i\langle \sigma, Cvj\rangle} \\
&= \frac{1}{\sqrt{\det v^{-1}}}\ \mathrm{e}^{\frac{1}{2}\langle j, vj\rangle}\ \frac{1}{\sqrt{\det(CvC^\dagger)}}\ \mathrm{e}^{-\frac{1}{2}\langle j, vC^\dagger (CvC^\dagger)^{-1} Cvj\rangle} \quad .
\end{aligned} \tag{A.5}$$

From eq.(A.4) we know

$$vC^\dagger = \mathcal{A}u \qquad \text{and} \qquad Cv = u\mathcal{A}^\dagger \quad . \tag{A.6}$$

Using the definition of the blockspin propagator

$$u = CvC^\dagger \tag{A.7}$$

the above expression simplifies to

$${Z_0}^h[j] = \frac{1}{\sqrt{\det v^{-1}}}\frac{1}{\sqrt{\det u}}\ \mathrm{e}^{\frac{1}{2}\langle j, (v - \mathcal{A}u\mathcal{A}^\dagger) j\rangle} \quad . \tag{A.8}$$

B Propagators and interpolation kernel in the gauge fixed case

Given

$$v_{\alpha\beta}^{-1} = \Delta + (\alpha-1)\partial\mathcal{R}\partial^\dagger + (\beta-1)\partial(1-\mathcal{R})\partial^\dagger \tag{B.1}$$

one verifies explicitly that

$$v_{\alpha\beta} = \Delta^{-1} - (1-\frac{1}{\alpha})\partial\Delta^{-1}\mathcal{R}\Delta^{-1}\partial^\dagger - (1-\frac{1}{\beta})\partial\Delta^{-1}(1-\mathcal{R})\Delta^{-1}\partial^\dagger \tag{B.2}$$

is its inverse.

For this one uses the projector properties of $\mathcal{R}$ and $(1-\mathcal{R})$ and

$$\begin{aligned} \partial\partial = 0 \quad &, \quad \partial^\dagger\partial^\dagger = 0 \\ \partial\Delta = \Delta\partial \quad &, \quad \partial^\dagger\Delta^{-1} = \Delta^{-1}\partial^\dagger \quad . \end{aligned} \tag{B.3}$$

One computes

$$\begin{aligned} v_{\alpha\beta}^{-1}v_{\alpha\beta} &= \Delta\Delta^{-1} - (1-\frac{1}{\alpha})\Delta\partial\Delta^{-1}\mathcal{R}\Delta^{-1}\partial^\dagger - (1-\frac{1}{\beta})\Delta\partial\Delta^{-1}(1-\mathcal{R})\Delta^{-1}\partial^\dagger \\ &+ (\alpha-1)\partial\mathcal{R}\partial^\dagger\Delta^{-1} - (\alpha-1)(1-\frac{1}{\alpha})\partial\mathcal{R}\partial^\dagger\partial\Delta^{-1}\mathcal{R}\Delta^{-1}\partial^\dagger \\ &- (\alpha-1)(1-\frac{1}{\beta})\partial\mathcal{R}\partial^\dagger\partial\Delta^{-1}(1-\mathcal{R})\Delta^{-1}\partial^\dagger \\ &+ (\beta-1)\partial(1-\mathcal{R})\partial^\dagger\Delta^{-1} - (\beta-1)(1-\frac{1}{\alpha})\partial(1-\mathcal{R})\partial^\dagger\partial\Delta^{-1}\mathcal{R}\Delta^{-1}\partial^\dagger \\ &- (\beta-1)(1-\frac{1}{\beta})\partial(1-\mathcal{R})\partial^\dagger\partial\Delta^{-1}(1-\mathcal{R})\Delta^{-1}\partial^\dagger \\ &= 1 - (1-\frac{1}{\alpha})\partial\mathcal{R}\Delta^{-1}\partial^\dagger - (1-\frac{1}{\beta})\partial(1-\mathcal{R})\Delta^{-1}\partial^\dagger \\ &+ (\alpha-1)\partial\mathcal{R}\partial^\dagger\Delta^{-1} + (\beta-1)\partial(1-\mathcal{R})\partial^\dagger\Delta^{-1} \\ &- (\alpha-1)(1-\frac{1}{\alpha})\partial\mathcal{R}\Delta^{-1}\partial^\dagger - (\beta-1)(1-\frac{1}{\beta})\partial(1-\mathcal{R})\Delta^{-1}\partial^\dagger \\ &= 1 + \underbrace{[(\alpha-1) - (1-\frac{1}{\alpha}) - (\alpha-1)(1-\frac{1}{\alpha})]}_{0}\partial\mathcal{R}\Delta^{-1}\partial^\dagger \\ &+ \underbrace{[(\beta-1) - (1-\frac{1}{\beta}) - (\beta-1)(1-\frac{1}{\beta})]}_{0}\partial(1-\mathcal{R})\Delta^{-1}\partial^\dagger \\ &= 1 \quad . \end{aligned} \tag{B.4}$$

The block propagator is

$$u_{\alpha\beta} = C v_{\alpha\beta} C^\dagger = C\Delta^{-1}C^\dagger - (1-\frac{1}{\beta})\nabla \underbrace{C_S \Delta^{-2} C_S^\dagger}_{u_{\Delta^2}} \nabla^\dagger \quad . \tag{B.5}$$

We use the abbreviations

$$\begin{aligned} () &= (C\Delta^{-1}C^\dagger) \\ [] &= [\nabla^\dagger ()^{-1} \nabla] \\ \{_\beta\} &= \{[]^{-1} - (1-\frac{1}{\beta})^{-1}[]^{-1} u_{\Delta^2}^{-1} []^{-1}\} \end{aligned} \tag{B.6}$$

with the identities

$$\begin{aligned} ([]\{_\beta\} - 1) &= 1 - (1-\frac{1}{\beta})^{-1} u_{\Delta^2}^{-1}[]^{-1} - 1 \\ &= -(1-\frac{1}{\beta})^{-1} u_{\Delta^2}^{-1}[]^{-1} \end{aligned} \tag{B.7}$$

and

$$\begin{aligned} []^{-1}\{_\beta\}^{-1} &= 1 + \left([]^{-1} - \{_\beta\}\right)\{_\beta\}^{-1} \\ &= 1 + (1-\frac{1}{\beta})^{-1}[]^{-1} u_{\Delta^2}^{-1} []^{-1}\{_\beta\}^{-1} \quad . \end{aligned} \tag{B.8}$$

The inverse block propagator is

$$\begin{aligned} u_{\alpha\beta}^{-1} &= \underbrace{(C\Delta^{-1}C^\dagger)^{-1}}_{()^{-1}} + (1-\frac{1}{\beta})()^{-1}\nabla\{u_{\Delta^2}^{-1} - (1-\frac{1}{\beta})^{-1} \underbrace{[\nabla^\dagger()^{-1}\nabla]}_{[]}\}^{-1}\nabla^\dagger()^{-1} \\ &= ()^{-1} - ()^{-1}\nabla[]^{-1} \underbrace{\{[]^{-1} - (1-\frac{1}{\beta})^{-1}[]^{-1}u_{\Delta^2}^{-1}[]^{-1}\}^{-1}}_{\{_\beta\}^{-1}} []^{-1}\nabla^\dagger()^{-1} \quad . \end{aligned} \tag{B.9}$$

This is again verified explicitly

$$\begin{aligned} u_{\alpha\beta}u_{\alpha\beta}^{-1} &= \{() - (1-\frac{1}{\beta})\nabla u_{\Delta^2}\nabla^\dagger\}\{()^{-1} - ()^{-1}\nabla[]^{-1}\{_\beta\}^{-1}[]^{-1}\nabla^\dagger()^{-1}\} \\ &= 1 + \nabla\{-(1-\frac{1}{\beta})u_{\Delta^2} - []^{-1}\{_\beta\}^{-1}[]^{-1} + (1-\frac{1}{\beta})u_{\Delta^2}\underbrace{\nabla^\dagger()^{-1}\nabla}_{[]}[]^{-1}\{_\beta\}^{-1}[]^{-1}\}\nabla^\dagger()^{-1} \\ &= 1 + \nabla\{-(1-\frac{1}{\beta})u_{\Delta^2}(1 - \{_\beta\}^{-1}[]^{-1}) - []^{-1}\{_\beta\}^{-1}[]^{-1}\}\nabla^\dagger()^{-1} \\ &= 1 + \nabla\{-(1-\frac{1}{\beta})u_{\Delta^2} \underbrace{([]\{_\beta\} - 1)}_{-(1-\frac{1}{\beta})^{-1}u_{\Delta^2}^{-1}[]^{-1}} - []^{-1}\}\{_\beta\}^{-1}[]^{-1}\nabla^\dagger()^{-1} \\ &= 1 \quad . \end{aligned} \tag{B.10}$$

Now we come to the interpolation kernel

$$\begin{aligned}
\mathcal{A}_{\alpha\beta} &= v_{\alpha\beta}C^\dagger u_{\alpha\beta}^{-1} \\
&= (\Delta^{-1} - (1-\frac{1}{\beta})\partial\Delta^{-2}\partial^\dagger)C^\dagger u_{\alpha\beta}^{-1} \\
&= (\Delta^{-1}C^\dagger - (1-\frac{1}{\beta})\partial\Delta^{-2}C_S^\dagger\nabla^\dagger)u_{\alpha\beta}^{-1} \\
&= \Delta^{-1}C^\dagger u_{\alpha\beta}^{-1} - (1-\frac{1}{\beta})\partial\Delta^{-2}C_S^\dagger\nabla^\dagger[()^{-1} - ()^{-1}\nabla[]^{-1}\{_\beta\}^{-1}[]^{-1}\nabla^\dagger()^{-1}] \\
&= \Delta^{-1}C^\dagger u_{\alpha\beta}^{-1} - (1-\frac{1}{\beta})\partial\Delta^{-2}C_S^\dagger[1 - \{_\beta\}^{-1}[]^{-1}]\nabla^\dagger()^{-1} \\
&= \Delta^{-1}C^\dagger u_{\alpha\beta}^{-1} - (1-\frac{1}{\beta})\partial\Delta^{-2}C_S^\dagger \underbrace{[[]\{_\beta\} - 1]}_{-(1-\frac{1}{\beta})u_{\Delta^2}^{-1}[]^{-1}} \{_\beta\}^{-1}[]^{-1}\nabla^\dagger()^{-1} \\
&= \Delta^{-1}C^\dagger u_{\alpha\beta}^{-1} + \partial\Delta^{-2}C_S^\dagger u_{\Delta^2}^{-1}[]^{-1}\{_\beta\}^{-1}[]^{-1}\nabla^\dagger()^{-1} \quad . \qquad \text{(B.11)}
\end{aligned}$$

The background field propagator is

$$\begin{aligned}
v_{\alpha\beta}^s &= \mathcal{A}_{\alpha\beta}Cv_{\alpha\beta} = v_{\alpha\beta}C^\dagger u_{\alpha\beta}^{-1}Cv_{\alpha\beta} \\
&= \{\Delta^{-1} - (1-\frac{1}{\beta})\partial\Delta^{-2}\partial^\dagger\}C^\dagger u_{\alpha\beta}^{-1}C\{\Delta^{-1} - (1-\frac{1}{\beta})\partial\Delta^{-2}\partial^\dagger\} \\
&= (1-\frac{1}{\beta})^2\partial\Delta^{-2}C_S^\dagger\left([]\{_\beta\} - 1\right)\{_\beta\}^{-1}C_S\Delta^{-2}\partial^\dagger \\
&- (1-\frac{1}{\beta})\partial\Delta^{-2}C_S^\dagger\left([]\{_\beta\} - 1\right)\{_\beta\}^{-1}[]^{-1}()^{-1}\nabla^\dagger C\Delta^{-1} \\
&- (1-\frac{1}{\beta})\Delta^{-1}C^\dagger\nabla()^{-1}\left([]\{_\beta\} - 1\right)\{_\beta\}^{-1}[]^{-1}C_S\Delta^{-2}\partial^\dagger \\
&+ \Delta^{-1}C^\dagger\left(()^{-1} - \nabla()^{-1}[]^{-1}\{_\beta\}^{-1}[]^{-1}()^{-1}\nabla^\dagger\right)C\Delta^{-1} \quad . \qquad \text{(B.12)}
\end{aligned}$$

Using the identity (B.7) we see that only the first summand is divergent for $\beta = 0$.

$$\begin{aligned}
v_{\alpha\beta}^s = &-(1-\frac{1}{\beta})\partial\Delta^{-2}C_S^\dagger u_{\Delta^2}^{-1}[]^{-1}\{_\beta\}^{-1}C_S\Delta^{-2}\partial^\dagger \\
&+ \partial\Delta^{-2}C_S^\dagger u_{\Delta^2}^{-1}[]^{-1}\{_\beta\}^{-1}[]^{-1}\nabla^\dagger()^{-1}C\Delta^{-1} \\
&+ \Delta^{-1}C^\dagger()^{-1}\nabla[]^{-1}\{_\beta\}^{-1}[]^{-1}u_{\Delta^2}^{-1}C_S\Delta^{-2}\partial^\dagger \\
&+ \Delta^{-1}C^\dagger\left(()^{-1} - ()^{-1}\nabla[]^{-1}\{_\beta\}^{-1}[]^{-1}\nabla^\dagger()^{-1}\right)C\Delta^{-1} \qquad \text{(B.13)}
\end{aligned}$$

We use the identity (B.8) to separate the divergent term in the first line and obtain

$$\begin{aligned}
v_{\alpha\beta}^s = &-(1-\frac{1}{\beta})\partial\Delta^{-2}C_S^\dagger u_{\Delta^2}^{-1}C_S\Delta^{-2}\partial^\dagger \\
&- \partial\Delta^{-2}C_S^\dagger u_{\Delta^2}^{-1}[]^{-1}\{_\beta\}^{-1}[]^{-1}u_{\Delta^2}^{-1}C_S\Delta^{-2}\partial^\dagger
\end{aligned}$$

$$
\begin{aligned}
&+ \partial\Delta^{-2}C_S^\dagger u_{\Delta^2}^{-1}[]^{-1}\{_\beta\}^{-1}[]^{-1}\nabla^\dagger()^{-1}C\Delta^{-1} \\
&+ \Delta^{-1}C^\dagger()^{-1}\nabla[]^{-1}\{_\beta\}^{-1}[]^{-1}u_{\Delta^2}^{-1}C_S\Delta^{-2}\partial^\dagger \\
&- \Delta^{-1}C^\dagger()^{-1}\nabla[]^{-1}\{_\beta\}^{-1}[]^{-1}\nabla^\dagger()^{-1}C\Delta^{-1} \\
&+ \Delta^{-1}C^\dagger()^{-1}C\Delta^{-1} \\
= &- (1-\frac{1}{\beta})\partial\Delta^{-1}(1-\mathcal{R})\Delta^{-1}\partial^\dagger \\
&- \left(\partial\Delta^{-2}C_S^\dagger u_{\Delta^2}^{-1} - \Delta^{-1}C^\dagger()^{-1}\nabla\right)[]^{-1}\{_\beta\}^{-1}[]^{-1}\left(u_{\Delta^2}^{-1}C_S\Delta^{-2}\partial^\dagger - \nabla^\dagger()^{-1}C\Delta^{-1}\right) \\
&+ \Delta^{-1}C^\dagger()^{-1}C\Delta^{-1} \quad . \qquad \text{(B.14)}
\end{aligned}
$$

Here we finally see that the divergent term is exactly the one already present in the fundamental propagator. So it will cancel when the two propagators are subtracted from each other to construct the fluctuation propagator.

C Thermalization of concatenated operators

Since the difference between zero and finite temperature is only the extension of the time direction we handle space and time coordinates in a different way in this appendix. Space coordinates are supressed whereever possible. Integration over them is written in symbolic notation. Time coordinates are always explicitly written, integration over them is written with limits so that the reader can follow how periodization in time and restricted and unrestricted time integration conspire to give finite temperature results.

$$
\begin{aligned}
& u_{\mathrm{T}}(x_4, x_4') \\
&= \sum_{n\in Z} u_0(x_4 + n\beta, x_4') \\
&= \sum_{n\in Z} (C_0 v_0 C_0^\dagger)(x_4 + n\beta, x_4') \\
&= \int\limits_{\vec{z}} \int\limits_{\vec{z}'} \sum_{n\in Z} \int\limits_{-\infty}^{\infty} dz_4 \int\limits_{-\infty}^{\infty} dz_4' C_0(x_4 + n\beta, z_4) v_0(z_4, z_4') C_0^\dagger(z_4', x_4') \\
&= \int\limits_{\vec{z}} \int\limits_{\vec{z}'} \sum_{n\in Z} \sum_{m\in Z} \sum_{l\in Z} \int\limits_0^\beta dz_4 \int\limits_0^\beta dz_4' C_0(x_4 + n\beta, z_4 + m\beta) v_0(z_4 + m\beta, z_4' + l\beta) C_0^\dagger(z_4' + l\beta, x_4') \\
&= \int\limits_{\vec{z}} \int\limits_{\vec{z}'} \sum_{n\in Z} \sum_{m\in Z} \sum_{l\in Z} \int\limits_0^\beta dz_4 \int\limits_0^\beta dz_4' C_0(x_4 + (n-m)\beta, z_4) v_0(z_4 + (m-l)\beta, z_4') C_0^\dagger(z_4' + l\beta, x_4') \\
&= \int\limits_{\vec{z}} \int\limits_{\vec{z}'} \sum_{n'\in Z} \sum_{m'\in Z} \sum_{l\in Z} \int\limits_0^\beta dz_4 \int\limits_0^\beta dz_4' C_0(x_4 + n'\beta, z_4) v_0(z_4 + m'\beta, z_4') C_0^\dagger(z_4' + l\beta, x_4') \\
&= \int\limits_{\vec{z}} \int\limits_{\vec{z}'} \int\limits_0^\beta dz_4 \int\limits_0^\beta dz_4' \underbrace{\sum_{n'\in Z} C_0(x_4 + n'\beta, z_4)}_{C_{\mathrm{T}}(x_4, z_4)} \underbrace{\sum_{m'\in Z} v_0(z_4 + m'\beta, z_4')}_{v_{\mathrm{T}}(z_4, z_4')} \underbrace{\sum_{l\in Z} C_0^\dagger(z_4' + l\beta, x_4')}_{C_{\mathrm{T}}^\dagger(z_4', x_4')} \\
&= (C_{\mathrm{T}} v_{\mathrm{T}} C_{\mathrm{T}}^\dagger)(x_4, x_4')
\end{aligned}
\tag{C.1}
$$

To get from line 5 to line 6 we can use the lattice translation invariance of C_0 and v_0 since β is always a multiple of the lattice spacing in time direction.

For the calculation of the temperature dependence of the block propagator we use the simple thermalized averaging operator with only one block in time direction. Due to the translation invariance of the fundamental propagator v_0 one of the time integrations is trivial and gives a factor β

$$\begin{aligned}
u_{\mathrm{T}}(0,0) &= (C_{\mathrm{T}} v_{\mathrm{T}} C_{\mathrm{T}}^{\dagger})(0,0) \\
&= \int_{\vec{z}} \int_{\vec{z}'} \int_0^\beta dz_4 \int_0^\beta dz_4' \frac{1}{\beta} C_{\mathrm{space}} \sum_{n \in Z} v_0(z_4 + n\beta, z_4') \frac{1}{\beta} C_{\mathrm{space}}^{\dagger} \\
&= \int_{\vec{z}} \int_{\vec{z}'} \frac{1}{\beta} C_{\mathrm{space}} \int_{-\infty}^{\infty} dz_4 \int_0^\beta dz_4' v_0(z_4, z_4') \frac{1}{\beta} C_{\mathrm{space}}^{\dagger} \\
&= \int_{\vec{z}} \int_{\vec{z}'} \frac{1}{\beta} C_{\mathrm{space}} \int_{-\infty}^{\infty} dz_4 \int_0^\beta dz_4' v_0(z_4 - z_4', 0) \frac{1}{\beta} C_{\mathrm{space}}^{\dagger} \\
&= \int_{\vec{z}} \int_{\vec{z}'} \frac{1}{\beta} C_{\mathrm{space}} \int_{-\infty}^{\infty} dz_4'' \int_0^\beta dz_4' v_0(z_4'', 0) \frac{1}{\beta} C_{\mathrm{space}}^{\dagger} \\
&= \frac{1}{\beta} \int_{\vec{z}} \int_{\vec{z}'} \int_{-\infty}^{\infty} dz_4 C_{\mathrm{space}}(\vec{x}, \vec{z}) \int_{-\infty}^{\infty} dz_4 v_0(z_4, 0) C_{\mathrm{space}}^{\dagger}(\vec{z}', \vec{y}) \\
&= : \frac{1}{\beta} u_{FT}(\vec{x}, \vec{y}) \quad . \qquad \text{(C.2)}
\end{aligned}$$

The factor $\frac{1}{\beta}$ is the only temperature dependence.

We now present the calculation of the finite temperature interpolator

$$\begin{aligned}
\mathcal{A}_{\mathrm{T}}(z_4, 0) &= \int_{\vec{z}'} \int_{\vec{x}'} \int_0^\beta dz_4' \beta v_{\mathrm{T}}(z_4, z_4') \frac{1}{\beta} C_{\mathrm{space}}^{\dagger} \frac{1}{\beta} u_{FT}^{-1} \\
&= \int_{\vec{z}'} \int_{\vec{x}'} \int_0^\beta dz_4' \beta \sum_{n \in Z} v_0(z_4, z_4' - n\beta) \frac{1}{\beta} C_{\mathrm{space}}^{\dagger} \frac{1}{\beta} u_{FT}^{-1} \\
&= \frac{1}{\beta} \int_{\vec{z}'} \int_{\vec{x}'} \int_{-\infty}^{\infty} dz_4' v_0(z_4, z_4') C_{\mathrm{space}}^{\dagger} u_{FT}^{-1} \\
&= \frac{1}{\beta} \int_{\vec{z}'} \int_{\vec{x}'} \int_{-\infty}^{\infty} dz_4' v_0(0, z_4') C_{\mathrm{space}}^{\dagger} u_{FT}^{-1} \\
&= \mathcal{A}_{\mathrm{T}}(0,0) =: \frac{1}{\beta} \mathcal{A}_{FT} \quad . \qquad \text{(C.3)}
\end{aligned}$$

Again the factor $\frac{1}{\beta}$ is the only temperature dependence.

D The effective potential for ϕ^4-theory up to second order in the fluctuation propagator

$$
\begin{aligned}
V_{\text{eff}}[\phi^s] = & \frac{1}{2}m_0^2 \int_z \phi^s(z)^2 + \frac{g}{4!}\int_z \phi^s(z)^4 \\
& - \frac{1}{2}m_0^4 \int_{z_1,z_2} \phi^s(z_1)\Gamma(z_1,z_2)\phi^s(z_2) + \frac{g}{4}\int_z \phi^s(z)^2\Gamma(z,z) \\
& - \frac{1}{2}m_0^2 g \int_{z_1,z_2} \phi^s(z_1)\Gamma(z_1,z_1)\Gamma(z_1,z_2)\phi^s(z_2) \\
& - \frac{1}{4}m_0^2 g \int_{z_1,z_2} \phi^s(z_1)^2\Gamma(z_1,z_2)^2 \\
& + \frac{1}{2}m_0^6 \int_{z_1,z_2,z_3} \phi^s(z_1)\Gamma(z_1,z_2)\Gamma(z_2,z_3)\phi^s(z_3) \\
& - \frac{1}{3!}m_0^2 g \int_{z_1,z_2} \phi^s(z_1)\Gamma(z_1,z_2)\phi^s(z_2)^3 \\
& - \frac{1}{2}\frac{g^2}{3!} \int_{z_1,z_2} \phi^s(z_1)\Gamma(z_1,z_1)\Gamma(z_1,z_2)\phi^s(z_2)^3 \\
& + \frac{1}{3!}m_0^4 g \int_{z_1,z_2,z_3} \phi^s(z_1)\Gamma(z_1,z_2)\Gamma(z_2,z_3)\phi^s(z_3)^3 \\
& - \frac{g^2}{2^4} \int_{z_1,z_2} \phi^s(z_1)^2\Gamma(z_1,z_2)^2\phi^s(z_2)^2 \\
& + \frac{1}{4}m_0^4 g \int_{z_1,z_2,z_3} \phi^s(z_1)\Gamma(z_1,z_2)\phi^s(z_2)^2\Gamma(z_2,z_3)\phi^s(z_3) \\
& - \frac{g^2}{2\cdot 3!^2} \int_{z_1,z_2} \phi^s(z_1)^3\Gamma(z_1,z_2)\phi^s(z_2)^3 \\
& + \frac{m_0^2 g^2}{2\cdot 3!^2} \int_{z_1,z_2,z_3} \phi^s(z_1)^3\Gamma(z_1,z_2)\Gamma(z_2,z_3)\phi^s(z_3)^3 \\
& + \frac{1}{12}m_0^2 g^2 \int_{z_1,z_2,z_3} \phi^s(z_1)\Gamma(z_1,z_2)\phi^s(z_2)^2\Gamma(z_2,z_3)\phi^s(z_3)^3 \\
& + \frac{g^3}{4\cdot 3!^2} \int_{z_1,z_2,z_3} \phi^s(z_1)^3\Gamma(z_1,z_2)\phi^s(z_2)^2\Gamma(z_2,z_3)\phi^s(z_3)^3 \\
& + \frac{1}{2}m_0^2 \int_z \Gamma(z,z) - \frac{1}{4}m_0^4 \int_{z_1,z_2} \Gamma(z_1,z_2)\Gamma(z_2,z_1) + \frac{1}{8}g\int_z \Gamma(z,z)^2
\end{aligned}
$$

E Partial integration of nonlocal terms in the effective action to obtain a sum of local and irrelevant terms

Consider a term which is quadratic in the field such as

$$I_2[\Phi] = \int_{x_1} \int_{x_2} \rho_2(x_2 - x_1)\Phi(x_2)\Phi(x_1) \tag{E.1}$$

We are interested in situations where $\rho_2(x)$ falls off exponentially with decay length one lattice spacing. In this case the sum can be rewritten in the form

$$I_2[\Phi] = \mu^2 \int_{x_1} \Phi(x_1)^2 + z_{\mu\nu} \int_{x_1} \nabla_\mu \Phi(x_1) \nabla_\nu \Phi(x_1) + \text{irrelevant term} \tag{E.2}$$

where the irrelevant term is of the form

$$\gamma_{\mu\nu\rho\sigma} \int_{x_1} \int_{x_2} \nabla_\mu \nabla_\nu \Phi(x_1) \rho_2'(x_2 - x_1) \nabla_\rho \nabla_\sigma \Phi(x_2) \tag{E.3}$$

ρ_2' also decays exponentially with distance $x_2 - x_1$. The coefficients are

$$\mu^2 = \int_x \rho_2(x) \tag{E.4}$$

$$z_{\mu\nu} = -\tfrac{1}{2} \int_x x_\mu x_\nu \rho_2(x) \tag{E.5}$$

Because of the exponential falloff of ρ_2, its Fourier transform $\tilde{\rho}_2(p)$ is holomorphic in a strip. Because of the presence of a lattice, it is a periodic and even function of p. Therefore

$$\tilde{\rho}_2(p) = \mu^2 + z_{\mu\nu} \sin p_\mu \sin p_\nu + \gamma_{\mu\nu\rho\sigma} \sin p_\mu \sin p_\nu \sin p_\rho \sin p_\sigma \, \tilde{\rho}_2'(p) \tag{E.6}$$

where $\tilde{\rho}_2'(p)$ is also holomorphic, periodic and even.

$$\mu^2 = \tilde{\rho}_2(0) \tag{E.7}$$

$$z_{\mu\nu} = \tfrac{1}{2} \frac{\partial}{\partial p_\mu} \frac{\partial}{\partial p_\nu} \tilde{\rho}_2(p)|_{p=0} \tag{E.8}$$

Inserting back one gets eq.(E.2).

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