Revised Version DESY 96-176

gr-qc/9608066 v4

On the Classification of Decoherence Functionals

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gr-qc/9608066 v4 19 Dec 1996

ABSTRACT

The basic ingredients of the *consistent histories approach to quantum mechanics* are the space of histories and the space of decoherence functionals. In this work we extend the classification theorem for decoherence functionals proven by Isham, Linden and Schreckenberg to the case where the space of histories is the lattice of projection operators on an arbitrary separable or non-separable complex Hilbert space of dimension greater than two.

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1 Introduction

The consistent histories approach to quantum mechanics has attracted much interest in the last years. The consistent histories approach has enriched and deepened our understanding of nonrelativistic quantum mechanics and in particular of the interpretation of standard Hilbert space quantum mechanics. There is also hope that the consistent histories approach may be a guide towards the construction of history theories generalizing standard Hilbert space quantum mechanics. This hope is supported by the observation that general quantum history theories exhibit a much richer structure than standard quantum mechanics [1, 2].

The consistent histories approach to nonrelativistic quantum mechanics has been inaugurated in a seminal paper by Griffiths [3] and further developed by Griffiths [4], by Omnès [5]-[7], by Isham [1, 2], by Isham and Linden [8], by Isham, Linden and Schreckenberg [9], by Schreckenberg [10], by Pulmannová [11] and by this author [12]. In a series of interesting publications Gell-Mann and Hartle [13] have studied quantum cosmology and the path integral approach to relativistic quantum field theory in the framework of consistent histories. Further important developments and a critical examination of the consistent histories approach can be found in the work by Dowker and Kent [14] and Kent [15].

The consistent histories approach asserts that quantum mechanics provides a realistic description of *individual* quantum mechanical systems, regardless of whether they are open or closed and regardless of whether they are observed or not. Probabilities are thought of as measures of propensities. To avoid confusion it should be stressed that the term *realistic description* is not meant here in the sense of determinism or hidden variable theories.
The basic ingredients in the consistent histories approach are the space of histories on the one hand and the space of decoherence functionals on the other hand. The histories are identified with the general possibilities or properties of a quantum system. In a somewhat different language histories may be said to represent *temporal events* or simply *events*. The probabilities associated with histories are interpreted as measures of the tendency that certain *histories* will be realized in a single system. The assignment of probabilities to certain histories is only admissible when these histories belong to a common Boolean lattice of histories which satisfies some consistency condition [1, 3, 4, 5, 7, 12].

In standard Hilbert space quantum mechanics the state of some quantum mechanical system comprises all probabilistic predictions of quantum mechanics for the system in question. This idea of the notion of state can be carried over to general quantum history theories: it is in this sense that decoherence functionals can be said to represent the *state* of a system described by a quantum history theory.

To get insight into the possible structure of general history theories it is worthwhile to study the (algebraic) structure of the space of decoherence functionals for general quantum history theories in some detail. In particular — as also stressed by Isham, Linden and Schreckenberg [9] — the classification of decoherence functionals is an important problem. It is equivalent

to the classification of states in quantum history theories. As is well-known, the analogous problem in standard quantum mechanics has been completely solved by Gleason [16, 17].

This work is organized as follows: in Section 2 the classification theorem for decoherence functionals is formulated and proved. Section 2.1 is devoted to an exposition of some necessary basic definitions and propositions. In Section 2.2 we give an alternative proof for the classification theorem for decoherence functionals in the case when the set of histories is the set of projection operators on a finite-dimensional complex Hilbert space. This classification theorem has first been proven by Isham, Linden and Schreckenberg [9]. Our proof is based on methods used by Cooke, Keane and Moran in their elementary proof of Gleason's theorem [18] and differs from the proof given by Isham, Linden and Schreckenberg in that we do not use Gleason's theorem directly. However, our proof makes use of a theorem due to Wright which is in turn based on the solution of the Mackey-Gleason problem [19, 20]. In Section 2.3 we consider the case that the space of histories is the set $\mathcal{P}(\mathfrak{H})$ of projection operators on some infinite-dimensional separable or non-separable complex Hilbert space $\mathfrak{H}$ and extend the classification theorem for decoherence functionals to this case. It is perhaps worthwhile to mention that our result is also valid if we identify the set of histories with the set of effects $\mathfrak{E}(\mathfrak{H})$ on some Hilbert space (as done in [12]) since every ultraweakly continuous normal decoherence functional on $\mathcal{P}(\mathfrak{H})$ can be uniquely extended to a functional on $\mathfrak{E}(\mathfrak{H})$, as shown in Corollary 3 in Section 2.4. Section 3 presents our summary.

Throughout this work we will make use of Dirac's well-known ket and bra notation to denote vectors in Hilbert space and dual vectors in the dual Hilbert space respectively.

2 The Classification Theorem

2.1 Preliminaries

Consider a history theory where the set of histories can be identified with the set $\mathcal{P}(\mathfrak{H})$ of projection operators on some separable or non-separable complex Hilbert space $\mathfrak{H}$. A functional $d : \mathcal{P}(\mathfrak{H}) \times \mathcal{P}(\mathfrak{H}) \to \mathbb{C}, (h, k) \longmapsto d(h, k)$ will be called a DECOHERENCE FUNCTIONAL ON $\mathcal{P}(\mathfrak{H})$ if the following conditions are satisfied for all $h, h', k \in \mathcal{P}(\mathfrak{H})$:

- $d(h, h) \in \mathbb{R}$, and $d(h, h) \geq 0$.
- $d(h, k) = d(k, h)^*$.
- $d(1, 1) = 1$ and $d(0, h) = 0$.
- $d(h \vee h', k) = d(h, k) + d(h', k)$, whenever $h \perp h'$.

By definition every decoherence functional is finitely additive in both arguments. We say that a decoherence functional d is σ-ADDITIVE in both arguments whenever

- $d\left(\bigvee_{i\in\mathcal{I}} h_i, k\right) = \sum_{i\in\mathcal{I}} d(h_i, k)$,

whenever $\{h_i\}_{i\in\mathcal{I}}$ is a countable set of mutually orthogonal histories and k is an arbitrary history. We say that a decoherence functional d is σ-SUMMABLE in both arguments if d is σ-additive and $\sum_{i\in\mathcal{I}} d(h_i,k)$ converges absolutely for all k and all countable families $\{h_i\}_{i\in\mathcal{I}}$ of mutually orthogonal histories. Moreover, we say that a decoherence functional d is COMPLETELY ADDITIVE in both arguments if for any set $\{h_j\}_{j\in\mathcal{J}}$ of mutually orthogonal histories and for all k the family $\{d(h_i,k)\}$ is summable and

- $d\left(\bigvee_{j\in\mathcal{J}} h_j, k\right) = \sum_{j\in\mathcal{J}} d(h_j,k)$.

Every decoherence functional d can be used to define *consistent Boolean sublattices* of $\mathcal{P}(\mathfrak{H})$ such that d induces a probability measure on these consistent Boolean lattices. Clearly, a probability measure induced by a decoherence functional d on a consistent Boolean lattice $\mathcal{B}_c$ is σ-additive if and only if d is σ-additive in both arguments on $\mathcal{B}_c$ and is completely additive if and only if d is completely additive in both arguments on $\mathcal{B}_c$.

Throughout this work $\mathcal{B}(\mathfrak{H})$ denotes the set of all bounded operators on a complex Hilbert space $\mathfrak{H}$ and $\mathcal{S}(\mathfrak{H})$ denotes the set of all unit vectors in the complex Hilbert space $\mathfrak{H}$. Moreover, $\mathfrak{E}(\mathfrak{H})$ denotes the set of all effect operators on $\mathfrak{H}$, i.e., the set of all Hermitean operators E on $\mathfrak{H}$ with $0 \leq E \leq 1$. The set $\mathfrak{E}(\mathfrak{H})$ carries the structure of a D-poset [21]. In [12] it has been shown that $\mathfrak{E}(\mathfrak{H})$ can be supplied with countably many different D-poset structures. However, all the D-poset structures on $\mathfrak{E}(\mathfrak{H})$ considered in [12] are isomorphic and thus it is enough to consider the *canonical* D-poset structure on $\mathfrak{E}(\mathfrak{H})$. We recall that the canonical D-poset structure on $\mathfrak{E}(\mathfrak{H})$ is given by a partially defined addition $\oplus$ on $\mathfrak{E}(\mathfrak{H})$: for $e_1, e_2 \in \mathfrak{E}(\mathfrak{H})$ the expression $e_1 \oplus e_2$ is defined if (and only if) $e_1 + e_2 \leq 1$ by $e_1 \oplus e_2 := e_1 + e_2$.

If $\mathfrak{T}$ is a topology on $\mathcal{P}(\mathfrak{H})$, then a decoherence functional d is called BI-CONTINUOUS WITH RESPECT TO THE TOPOLOGY $\mathfrak{T}$ if d is continuous in both arguments with respect to the topology $\mathfrak{T}$. In the present work we use the standard nomenclature for topologies on $\mathcal{B}(\mathfrak{H})$, see, e.g., [17]. We say that a decoherence functional d is ultraweakly bi-continuous if d is continuous in both arguments with respect to the ultraweak operator topology on $\mathcal{P}(\mathfrak{H})$. Every ultraweakly bi-continuous decoherence functional is obviously also continuous with respect to every stronger topology but not vice versa. The results in Section 2.3 are formulated for ultraweakly bi-continuous decoherence functionals. Since for norm bounded sequences of operators the notions of weak and ultraweak convergence coincide, it is clear that the results in Section 2.3 below are also valid for weakly bi-continuous decoherence functionals. However, the classification theorem for decoherence functionals on infinite-dimensional Hilbert spaces is in general not valid for decoherence functionals which are bi-continuous with respect to a stronger topology than the ultraweak topology.

Remark 1 Let $\mathfrak{H}$ denote a complex Hilbert space with $\dim(\mathfrak{H}) > 2$. Wright [19, 20] has proven the important general result that a bounded decoherence functional d on $\mathcal{P}(\mathfrak{H})$ can be uniquely extended to a bilinear bounded functional $\mathcal{D}$ on $\mathcal{B}(\mathfrak{H})$. The extension $\mathcal{D}$ is continuous in both arguments with respect to the norm topology on $\mathcal{B}(\mathfrak{H})$ and thus every bounded decoherence functional d is necessarily bi-continuous with respect to the topology induced on $\mathcal{P}(\mathfrak{H})$

by the norm on $\mathcal{B}(\mathfrak{H})$.

Let $\mathfrak{H}$ denote a complex Hilbert space. We denote by $\mathfrak{H}_0$ the everywhere dense subset of $\mathfrak{H} \otimes \mathfrak{H}$ generated by the simple vectors of the form $|\phi\rangle \otimes |\psi\rangle$, where $|\phi\rangle, |\psi\rangle \in \mathfrak{H}$. That is, $\mathfrak{H}_0$ contains all finite linear combinations of simple vectors. For all $|\Psi_0\rangle \in \mathfrak{H}_0$ we denote the one-dimensional projection operator onto $|\Psi_0\rangle$ by P_{Ψ_0}. We denote the set of all such projection operators by $\mathcal{P}(\mathfrak{H}_0)$. Moreover, we denote the set of all projection operators in $\mathcal{P}(\mathfrak{H} \otimes \mathfrak{H})$ which can be written as a finite sum $\sum_j P_{\Psi_j}$ of pairwise orthogonal projection operators $P_{\Psi_j} \in \mathcal{P}(\mathfrak{H}_0)$ by $\mathcal{P}_{fin}(\mathfrak{H}_0)$. The set of all projection operators in $\mathcal{P}(\mathfrak{H} \otimes \mathfrak{H})$ which can be written as a ultraweakly converging sum $\sum_j P_{\Psi_j}$ of pairwise orthogonal projection operators $P_{\Psi_j} \in \mathcal{P}(\mathfrak{H}_0)$ will be denoted by $\mathcal{P}_\infty(\mathfrak{H}_0)$. If $\mathfrak{H}$ is finite-dimensional, then obviously $\mathfrak{H}_0 = \mathfrak{H} \otimes \mathfrak{H}$.

Proposition 1 *Let $\mathfrak{H}$ be a complex Hilbert space with $\dim(\mathfrak{H}) > 2$ and let d denote a bounded decoherence functional on $\mathcal{P}(\mathfrak{H})$, then d can be uniquely extended to a functional $\widehat{d} : \mathcal{P}_{fin}(\mathfrak{H}_0) \to \mathbb{C}$ satisfying $\widehat{d}(h \otimes k) = d(h,k)$ for all $h, k \in \mathcal{P}(\mathfrak{H})$. Moreover, $\widehat{d}$ is additive for orthogonal projection operators, i.e., $\widehat{d}(P_1 + P_2) = \widehat{d}(P_1) + \widehat{d}(P_2)$ for all $P_1, P_2 \in \mathcal{P}_{fin}(\mathfrak{H}_0)$ with $P_1 \perp P_2$. If $\mathfrak{H}$ is finite-dimensional, then $\widehat{d}$ is bounded.*

Lemma 1 *Let $E_1, ..., E_{n+m}, F_1, ..., F_{m+n} \in \mathcal{B}(\mathfrak{H})$, then $(\sum_{i=1}^{n} E_i \otimes F_i) + (\sum_{i=n+1}^{n+m} E_i \otimes F_i) = 0$ if and only if there is an $(n+m) \times (n+m)$ complex matrix $[c_{ik}]$ such that*

$$\begin{aligned} \sum_{i=1}^{n+m} c_{ik} E_i &= 0, (k = 1, ..., n+m), \\ \sum_{k=1}^{n+m} c_{ik} F_k &= F_i, (i = 1, ..., n+m). \end{aligned}$$

The assertion of Lemma 1 is exactly Proposition 11.1.8 (i) in [22].

Proof of Proposition 1: For simple projection operators of the form $h \otimes k$ with $h, k \in \mathcal{P}(\mathfrak{H})$ we define $\widehat{d}(h \otimes k) := d(h,k)$. We denote by $\mathfrak{H}_0$ the everywhere dense subset of $\mathfrak{H} \otimes \mathfrak{H}$ generated by the simple vectors of the form $|\phi\rangle \otimes |\psi\rangle$. That is, $\mathfrak{H}_0$ contains all finite linear combinations of simple vectors. Let $|\Psi_0\rangle \in \mathcal{S}(\mathfrak{H}_0) := \mathcal{S}(\mathfrak{H} \otimes \mathfrak{H}) \cap \mathfrak{H}_0$, then $|\Psi_0\rangle$ can be written as $|\Psi_0\rangle = \sum_{j=1}^{N} \kappa_j |\phi_j\rangle \otimes |\psi_j\rangle$, with $|\phi_j\rangle, |\psi_j\rangle \in \mathcal{S}(\mathfrak{H})$ for all j. Denote the projection operator on $|\Psi_0\rangle$ by $P_{\Psi_0} = |\Psi_0\rangle\langle\Psi_0|$ and define $\widehat{d}(P_{\Psi_0}) := \sum_{i=1}^{N} \sum_{j=1}^{N} \kappa_i \kappa_j^* \mathcal{D}\left(|\phi_i\rangle\langle\phi_j|, |\psi_i\rangle\langle\psi_j|\right)$ where $\mathcal{D}$ denotes the unique extension of d to $\mathcal{B}(\mathfrak{H})$ mentioned in Remark 1. Lemma 1 implies that the such defined $\widehat{d}(P_{\Psi_0})$ is independent of the particular representation of $|\Psi_0\rangle \in \mathcal{S}(\mathfrak{H}_0)$ chosen (see Proposition 11.1.8 (ii) in [22]). Thus we have extended d to a functional on the set of projection operators of the form P_{Ψ_0}. Now let $P_M \in \mathcal{P}_{fin}(\mathfrak{H}_0)$. Then by definition P_M can be written as a finite sum $P_M = \sum_{j=1}^{M} P_{\Psi_j}$ of mutually orthogonal projection operators $P_{\Psi_j} \in \mathcal{P}(\mathfrak{H}_0)$. We

define $\widehat{d}(P_M) := \sum_{j=1}^{M} \widehat{d}(P_{\Psi_j})$. Again Lemma 1 implies that $\widehat{d}(P_M)$ is well-defined and independent of the particular decomposition of P_M chosen. If $\mathfrak{H}$ is n-dimensional, then $\widehat{d}$ is bounded by n^4K, where K is a bound of $\mathcal{D}$. ■

We say that a bounded decoherence functional d is PROPER if the unique extension $\widehat{d}$: $\mathcal{P}_{fin}(\mathfrak{H}_0) \to \mathbb{C}$ from Proposition 1 is bounded. Let $P \in \mathcal{P}_\infty(\mathfrak{H}_0)$ and $P = \sum_{j\in J} p_j$ one of its representations with $p_j \in \mathcal{P}(\mathfrak{H}_0)$. If d is proper, then the net $\left\{ \widehat{d}\left(\sum_{j\in J_0} p_j\right) | J_0 \subset J, J_0 \text{ finite } \right\}$ possesses a converging subsequence. We say that a bounded decoherence functional d is NORMAL if $\widehat{d}$ can be unambiguously extended to a completely additive functional $\widetilde{d} : \mathcal{P}_\infty(\mathfrak{H}_0) \to \mathbb{C}$.

We introduce some notations and terminology: A map $m : \mathcal{P}(\mathfrak{H}) \to \mathbb{R}$ such that

$$m(0) = 0, \tag{1}$$

$$m\left(\bigvee_{i\in\mathcal{I}} p_i\right) = \sum_{i\in\mathcal{I}} m\left(p_i\right), \tag{2}$$

whenever $\{p_i\}_{i\in\mathcal{I}}$ is a system of mutually orthogonal projection operators in $\mathcal{P}(\mathfrak{H})$ is said to be (i) a FINITELY ADDITIVE SIGNED MEASURE, (ii) a SIGNED MEASURE, or (iii) a COMPLETELY ADDITIVE SIGNED MEASURE if Equation 2 holds for every (i) finite, (ii) countable, or (iii) arbitrary index set $\mathcal{I}$ respectively. A finitely additive signed measure is said to be JORDAN if it can be written as a difference of two positive finitely additive measures.

A map $f : \mathcal{S}(\mathfrak{H}) \to \mathbb{R}$ is called a FRAME FUNCTION if there is a constant $\omega \in \mathbb{R}$ such that for every orthonormal basis $\{|h_i\rangle\}$ of $\mathfrak{H}$

$$\sum_i f\left(|h_i\rangle\right) = \omega$$

is satisfied. The constant ω is called the WEIGHT of the frame function f. A frame function f on $\mathfrak{H}$ is called BOUNDED if $\sup\{|f(|h\rangle)| : |h\rangle \in \mathcal{S}(\mathfrak{H})\} < \infty$. A frame function f on $\mathfrak{H}$ is called REGULAR if there is a Hermitean operator T_f on $\mathfrak{H}$ such that f can be written as $f\left(|h\rangle\right) = \langle h|T_f|h\rangle$, for all $|h\rangle \in \mathcal{S}(\mathfrak{H})$, where $\langle\cdot|\cdot\rangle$ denotes the inner product in $\mathfrak{H}$.
There is a duality between completely additive signed measures on $\mathcal{P}(\mathfrak{H})$ and frame functions on $\mathfrak{H}$: let m be a completely additive signed measure on $\mathcal{P}(\mathfrak{H})$ and denote for every $|h\rangle \in \mathcal{S}(\mathfrak{H})$ the projection operator onto $|h\rangle$ by $P_h = |h\rangle\langle h|$, then $f_m\left(|h\rangle\right) := m(P_h), |h\rangle \in \mathcal{S}(\mathfrak{H})$, defines a frame function f_m on $\mathfrak{H}$ with weight $\omega_m = m(1)$. Conversely, let f be a frame function on $\mathfrak{H}$. Let $P \in \mathcal{P}(\mathfrak{H})$ and let $\{P_i\}$ be a decomposition of P into mutually orthogonal one-dimensional projection operators. Denote by $|P_i\rangle$ the unit vector in $\mathfrak{H}$ onto which the ith projector P_i projects, then $m_f(P) := \sum_i f\left(|P_i\rangle\right)$ defines a completely additive signed measure m_f on $\mathfrak{H}$. It is easy to see that $f_{m_f} = f$.

Proposition 2 *For any integer $n > 2$ let $\mathfrak{H}_n$ be an n-dimensional complex Hilbert space. Then every bounded frame function on $\mathfrak{H}_n$ is regular.*

Proposition 3 *Let $\mathfrak{H}$ be an infinite-dimensional complex Hilbert space. Then any frame function on $\mathfrak{H}$ is bounded and regular.*

For the proof of Propositions 2 and 3 we refer the reader to [17], Chapter 3.

Now let $\mathfrak{K}$ be a dense linear subspace of $\mathfrak{H}$. Define $\mathcal{S}(\mathfrak{K}) := \mathcal{S}(\mathfrak{H}) \cap \mathfrak{K}$. A map $g : \mathcal{S}(\mathfrak{K}) \to \mathbb{R}$ is said to be a FRAME TYPE FUNCTION ON $\mathfrak{H}$ if the following conditions are satisfied

- the family $\{g(|h_i\rangle)\}$ is summable for every orthonormal system $\{|h_i\rangle\}$ in $\mathfrak{K}$;
- for any finite-dimensional subspace $\mathcal{K}_0$ of $\mathfrak{K}$, the restriction $g|_{\mathcal{S}(\mathcal{K}_0)}$ of g to $\mathcal{S}(\mathcal{K}_0)$ is a frame function on $\mathcal{K}_0$.

Now we have the following important result due to Dorofeev and Sherstnev

Proposition 4 *Let $\mathfrak{K}$ be a dense linear subspace of an infinite-dimensional complex Hilbert space $\mathfrak{H}$ and let $g : \mathcal{S}(\mathfrak{K}) \to \mathbb{R}$ be a frame type function on $\mathfrak{H}$. Then g is bounded and there is a unique Hermitean trace class operator T_g on $\mathfrak{H}$ such that $g(|h\rangle) = \langle h|T_g|h\rangle$ for all $|h\rangle \in \mathcal{S}(\mathfrak{K})$.*

A proof of this proposition can be found in [17], Section 3.2.4.

2.2 The finite-dimensional case

Theorem 1 *If the dimension $\dim(\mathfrak{H})$ of a complex Hilbert space $\mathfrak{H}$ satisfies $2 < \dim(\mathfrak{H}) < \infty$, then there is a one-one correspondence between bounded decoherence functionals d on $\mathcal{P}(\mathfrak{H})$ and trace class operators $\mathfrak{X}$ on $\mathfrak{H} \otimes \mathfrak{H}$ according to the rule*

$$d(h,k) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h \otimes k\mathfrak{X}) \tag{3}$$

with the restrictions that

- $\mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h \otimes k\mathfrak{X}) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(k \otimes h\mathfrak{X}^\dagger)$ *for all $h, k \in \mathcal{P}(\mathfrak{H})$;*
- $\mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h \otimes h\mathfrak{X}) \geq 0$ *for all $h \in \mathcal{P}(\mathfrak{H})$;*
- $\mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(\mathfrak{X}) = 1$.

In particular, every such decoherence functional is uniformly bi-continuous.

Theorem 1 has first been proven by Isham, Linden and Schreckenberg in [9]. Theorem 1 is **not** valid if $\dim(\mathfrak{H}) = 2$.

Proof: Consider the finite-dimensional Hilbert space $\mathfrak{H} \otimes \mathfrak{H}$. According to Proposition 1 every bounded decoherence functional d on $\mathfrak{H}$ can be extended to a bounded functional $\widehat{d} : \mathcal{P}(\mathfrak{H}\otimes\mathfrak{H}) \to \mathbb{C}$. Hence, the real part $\Re\widehat{d}$ of $\widehat{d}$ induces a bounded frame function $f_{\Re d}$ on $\mathfrak{H} \otimes \mathfrak{H}$ by

$$f_{\Re d}(|\Psi_0\rangle) := \Re\widehat{d}(P_{\Psi_0}),$$

for all $|\Psi_0\rangle \in \mathcal{S}(\mathfrak{H}\otimes\mathfrak{H})$. Similarly, the imaginary part $\Im\widehat{d}$ of $\widehat{d}$ induces a bounded frame function $f_{\Im d}$ on $\mathfrak{H}\otimes\mathfrak{H}$ by $f_{\Im d}\left(|\Psi_0\rangle\right) := \Im\widehat{d}(P_{\Psi_0})$, for all $|\Psi_0\rangle \in \mathcal{S}(\mathfrak{H}\otimes\mathfrak{H})$. Therefore, according to Proposition 2 both $f_{\Re d}$ and $f_{\Im d}$ are regular. This proves the existence of two Hermitean operators $\mathfrak{X}_{\Re d}$ and $\mathfrak{X}_{\Im d}$ on $\mathfrak{H}\otimes\mathfrak{H}$ such that $\widehat{d}$ can be written as $\widehat{d}(P_M) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(P_M(\mathfrak{X}_{\Re d} + i\mathfrak{X}_{\Im d}))$, for all $P_M \in \mathcal{P}_{fin}(\mathfrak{H}\otimes\mathfrak{H})$. In particular, it follows $d(h,k) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h\otimes k\mathfrak{X}_d)$ for all $h,k \in \mathcal{P}(\mathfrak{H})$ where we have set $\mathfrak{X}_d := \mathfrak{X}_{\Re d} + i\mathfrak{X}_{\Im d}$. The remaining assertions of the Theorem are now straightforward. ■

Theorem 1 shows that in the finite-dimensional case a decoherence functional d on $\mathcal{P}(\mathfrak{H})$ is ultraweakly bi-continuous if and only if d is bi-continuous with respect to the uniform (or operator norm) topology on $\mathcal{P}(\mathfrak{H})$. Moreover, since in the finite-dimensional case the weak and the ultraweak topology on $\mathcal{P}(\mathfrak{H})$ coincide, an ultraweakly bi-continuous decoherence functional d is also weakly bi-continuous.

2.3 The infinite-dimensional case

Theorem 2 *Let $\mathfrak{H}$ be a complex Hilbert space with dimension greater than two,* $\dim(\mathfrak{H}) > 2$*, then there is a one-one correspondence between normal completely additive decoherence functionals d on $\mathcal{P}(\mathfrak{H})$ and trace class operators $\mathfrak{X}$ on $\mathfrak{H}\otimes\mathfrak{H}$ according to the rule*

$$d(h,k) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h\otimes k\mathfrak{X}) \tag{4}$$

with the restrictions that

- $\mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h\otimes k\mathfrak{X}) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(k\otimes h\mathfrak{X}^\dagger)$ *for all $h,k \in \mathcal{P}(\mathfrak{H})$;*
- $\mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h\otimes h\mathfrak{X}) \geq 0$ *for all $h \in \mathcal{P}(\mathfrak{H})$;*
- $\mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(\mathfrak{X}) = 1$.

Proof: If $\mathfrak{H}$ is finite-dimensional, then the assertion of the theorem has already been proven in Theorem 1. Let $\mathfrak{H}$ be infinite-dimensional and let d denote a normal decoherence functional on $\mathcal{P}(\mathfrak{H})$. Notice, that the requirement of complete additivity in the theorem is redundant. As above we denote by $\mathfrak{H}_0$ the everywhere dense linear subspace of $\mathfrak{H}\otimes\mathfrak{H}$ generated by the simple vectors of the form $|\phi\rangle\otimes|\psi\rangle$. From Proposition 1 we know that d can be uniquely extended to a functional $\widehat{d} : \mathcal{P}_{fin}(\mathfrak{H}_0) \to \mathbb{C}$. The real part $\Re\widehat{d}$ of $\widehat{d}$ induces a frame type function $g_{\Re d} : \mathcal{S}(\mathfrak{H}_0) \to \mathbb{R}$ on $\mathfrak{H}\otimes\mathfrak{H}$ by

$$g_{\Re d}\left(|\Psi_0\rangle\right) := \Re\widehat{d}(P_{\Psi_0}),$$

for all $|\Psi_0\rangle \in \mathcal{S}(\mathfrak{H}_0)$. Similarly, the imaginary part $\Im\widehat{d}$ of $\widehat{d}$ induces a frame type function $g_{\Im d} : \mathcal{S}(\mathfrak{H}_0) \to \mathbb{R}$ on $\mathfrak{H}\otimes\mathfrak{H}$ by $g_{\Im d}\left(|\Psi_0\rangle\right) := \Im\widehat{d}(P_{\Psi_0})$, for all $|\Psi_0\rangle \in \mathcal{S}(\mathfrak{H}_0)$. According to Proposition 4 both $g_{\Re d}$ and $g_{\Im d}$ are regular. This proves the existence of two Hermitean operators $\mathfrak{X}_{\Re d}$ and $\mathfrak{X}_{\Im d}$ on $\mathfrak{H}\otimes\mathfrak{H}$ such that $\widehat{d}$ can be written as $\widehat{d}(P_M) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(P_M(\mathfrak{X}_{\Re d} + i\mathfrak{X}_{\Im d})) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(P_M\mathfrak{X}_d)$, for all $P_M \in \mathcal{P}_{fin}(\mathfrak{H}_0)$ where we have set $\mathfrak{X}_d := \mathfrak{X}_{\Re d} + i\mathfrak{X}_{\Im d}$. It is clear now, that the real part $\Re\widehat{d}$ of $\widehat{d}$ can be extended to a Jordan completely additive signed measure $\Re\bar{d}$ on $\mathfrak{H}\otimes\mathfrak{H}$ given by

$\Re\bar{d}(P) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(P\mathfrak{X}_{\Re d})$ for all $P \in \mathcal{P}(\mathfrak{H}\otimes\mathfrak{H})$. The imaginary part $\Im\widehat{d}$ of $\widehat{d}$ can be extended to a Jordan completely additive signed measure $\Im\bar{d}$ on $\mathfrak{H}\otimes\mathfrak{H}$ given by $\Im\bar{d}(P) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(P\mathfrak{X}_{\Im d})$ for all $P \in \mathcal{P}(\mathfrak{H}\otimes\mathfrak{H})$. Hence $\widehat{d}$ can be extended to a completely additive complex valued measure $\bar{d}$ on $\mathfrak{H}\otimes\mathfrak{H}$ given by

$$\bar{d}(P) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(P\mathfrak{X}_d),$$

for all $P \in \mathcal{P}(\mathfrak{H}\otimes\mathfrak{H})$. Since d is finitely additive, it follows that $d(h,k) = \bar{d}(h\otimes k)$ for all finite-dimensional $h,k \in \mathcal{P}(\mathfrak{H})$. Obviously $\bar{d}$ is ultraweakly continuous. By assumption d is completely additive in both arguments. Now let $h,k \in \mathcal{P}(\mathfrak{H})$ denote two arbitrary projection operators on $\mathfrak{H}$. Consider the projection operator $h\otimes k$ on $\mathfrak{H}\otimes\mathfrak{H}$. Then there is a family of mutually orthogonal one-dimensional projection operators $\{h_i\}_{i\in I}$ such that $h = \sum_i h_i$ in the weak operator topology and a family of mutually orthogonal one-dimensional projection operators $\{k_l\}_{l\in L}$ such that $k = \sum_l k_l$ in the weak operator topology, compare, e.g., [23] Section I.5.12. Obviously $h\otimes k = \sum_{i,l} h_i\otimes k_l$ in the weak operator topology. Consider the net $\mathcal{N}$ consisting of all finite sums of the form $\left(\sum_{i\in\widetilde{I}} h_i\right)\otimes\left(\sum_{l\in\widetilde{L}} k_l\right)$ where $\widetilde{I}$ is a finite subset of I and $\widetilde{L}$ is a finite subset of L. Proposition I.5.12.IX in [23] implies that for every trace class operator T on $\mathfrak{H}\otimes\mathfrak{H}$ there exist at most countable subsets $I_T \subset I$ and $L_T \subset L$ such that $\mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h\otimes kT) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}\left(\left(\sum_{i\in I_T} h_i\right)\otimes\left(\sum_{l\in L_T} k_l\right)T\right)$. Hence, for all $\epsilon > 0$ and all trace class operators T on $\mathfrak{H}\otimes\mathfrak{H}$ there exist finite subsets $\widetilde{I}_{T,\epsilon}$ of I and $\widetilde{L}_{T,\epsilon}$ of L such that $\left|\mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h\otimes kT) - \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}\left(\left(\sum_{i\in\widetilde{I}} h_i\right)\otimes\left(\sum_{l\in\widetilde{L}} k_l\right)T\right)\right| < \epsilon$ for all finite $\widetilde{I} \supset \widetilde{I}_{T,\epsilon}$ and all finite $\widetilde{L} \supset \widetilde{L}_{T,\epsilon}$. We conclude that the net $\mathcal{N}$ defined above converges to $h\otimes k$ in the ultraweak topology. Clearly, the net of complex numbers $\left\{\bar{d}\left(\left(\sum_{i\in\widetilde{I}} h_i\right)\otimes\left(\sum_{l\in\widetilde{L}} k_l\right)\right)\right\}_{\widetilde{I},\widetilde{L}}$, where $\widetilde{I}$ and $\widetilde{L}$ run through all finite subsets of I and L respectively, converges to $\bar{d}(h\otimes k)$. Similarly, the net $\left\{d\left(\sum_{i\in\widetilde{I}} h_i, \sum_{l\in\widetilde{L}} k_l\right)\right\}_{\widetilde{I},\widetilde{L}}$, where $\widetilde{I}$ and $\widetilde{L}$ run through all finite subsets of I and L respectively, converges to $d(h,k)$. Since $\bar{d}\left(\left(\sum_{i\in\widetilde{I}} h_i\right)\otimes\left(\sum_{l\in\widetilde{L}} k_l\right)\right) = d\left(\sum_{i\in\widetilde{I}} h_i, \sum_{l\in\widetilde{L}} k_l\right)$ for every finite $\widetilde{I} \subset I$ and $\widetilde{L} \subset L$, we conclude that $\bar{d}(h\otimes k) = d(h,k)$. In particular, it follows $d(h,k) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h\otimes k\mathfrak{X}_d)$ for all $h,k \in \mathcal{P}(\mathfrak{H})$. The remaining assertions of the Theorem are now straightforward. ■

We say that a decoherence functional d on the set $\mathcal{P}(\mathfrak{H})$ of projectors on the Hilbert space $\mathfrak{H}$ is REGULAR if there is a trace class operator $\mathfrak{X}_d$ on $\mathfrak{H}\otimes\mathfrak{H}$ such that d can be written as $d(h,k) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h\otimes k\mathfrak{X}_d)$ for all $h,k \in \mathcal{P}(\mathfrak{H})$. In this case we say that $\mathfrak{X}_d$ defines a REGULAR REPRESENTATION of d.

Moreover, we say that a decoherence functional d on the set $\mathcal{P}(\mathfrak{H})$ of projectors on the Hilbert space $\mathfrak{H}$ is QUASI-REGULAR if there is a bounded operator $\mathfrak{X}_d$ on $\mathfrak{H}\otimes\mathfrak{H}$ (not necessarily of trace class) such that d can be written as $d(h,k) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h\otimes k\mathfrak{X}_d)$ for all finite-dimensional $h,k \in \mathcal{P}(\mathfrak{H})$. In this case we say that $\mathfrak{X}_d$ defines a QUASI-REGULAR REPRESENTATION of d.

We say that a decoherence functional d on the set $\mathcal{P}(\mathfrak{H})$ of projectors on the Hilbert space $\mathfrak{H}$ is σ-QUASI-REGULAR if there is a bounded operator $\mathfrak{X}_d$ on $\mathfrak{H}\otimes\mathfrak{H}$ (not necessarily of trace class) such that the following condition is satisfied: given any two projection operators $h,k \in \mathcal{P}(\mathfrak{H})$ projecting onto separable subspaces of $\mathfrak{H}$ and arbitrary decompositions $\{h_i\}$ of h and $\{k_j\}$ of k into mutually orthogonal one-dimensional projection operators $h = \sum_i h_i$ and $k = \sum_j k_j$, then

the sum $\sum_{i,j} \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h_i \otimes k_j \mathfrak{X}_d)$ is well-defined and independent of the particular decompositions considered and equals $d(h,k)$, i.e., $d(h,k) = \sum_{i,j} \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h_i \otimes k_j \mathfrak{X}_d) = \sum_{i,j} \langle h_i \otimes k_j | \mathfrak{X}_d | h_i \otimes k_j \rangle$. In this case we say that $\mathfrak{X}_d$ defines a σ-QUASI-REGULAR REPRESENTATION of d.
We say that a decoherence functional d on the set $\mathcal{P}(\mathfrak{H})$ of projectors on the Hilbert space $\mathfrak{H}$ is PSEUDO-REGULAR if there is a bounded operator $\mathfrak{X}_d$ on $\mathfrak{H} \otimes \mathfrak{H}$ (not necessarily of trace class) such that the following condition is satisfied: given any two projection operators $h, k \in \mathcal{P}(\mathfrak{H})$ projecting onto arbitrary subspaces of $\mathfrak{H}$ and given arbitrary decompositions $\{h_i\}$ of h and $\{k_j\}$ of k into mutually orthogonal one-dimensional projection operators $h = \sum_i h_i$ and $k = \sum_j k_j$, then the family $\{\sum_{i,j} \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h_i \otimes k_j \mathfrak{X}_d)\}$ is summable and its sum is independent of the particular decompositions considered and equals $d(h,k)$, i.e., $d(h,k) = \sum_{i,j} \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h_i \otimes k_j \mathfrak{X}_d) = \sum_{i,j} \langle h_i \otimes k_j | \mathfrak{X}_d | h_i \otimes k_j \rangle$. In this case we say that $\mathfrak{X}_d$ defines a PSEUDO-REGULAR REPRESENTATION of d.

Corollary 1 *Let $\mathfrak{H}$ be an infinite-dimensional complex Hilbert space. Then every completely additive normal decoherence functional d on $\mathcal{P}(\mathfrak{H})$ is ultraweakly bi-continuous and regular.*

In the following propositions the requirement that the decoherence functional is normal is weakened.

Proposition 5 *Let $\mathfrak{H}$ be an infinitely dimensional complex Hilbert space, then for every proper completely additive decoherence functional d on $\mathcal{P}(\mathfrak{H})$ there exists a unique Hilbert-Schmidt operator $\mathfrak{X}_d$ on $\mathfrak{H} \otimes \mathfrak{H}$ (not necessarily of trace class) defining a pseudo-regular representation of d.*

Proof: Denote by $\widehat{d}$ the extension of d from Proposition 1. Since $\widehat{d}$ is proper, it follows by a standard argument (see, e.g., the first part of the proof of Theorem 3.2.21 in [17]) that there exist uniquely determined bounded Hermitean operators $\mathfrak{X}_{\Re d}$ and $\mathfrak{X}_{\Im d}$ on $\mathfrak{H} \otimes \mathfrak{H}$ such that $\Re\widehat{d}$ and $\Im\widehat{d}$ can be written as $\Re\widehat{d}(P_{\Psi_0}) = \langle \Psi_0 | \mathfrak{X}_{\Re d} | \Psi_0 \rangle$ and $\Im\widehat{d}(P_{\Psi_0}) = \langle \Psi_0 | \mathfrak{X}_{\Im d} | \Psi_0 \rangle$ for all $|\Psi_0\rangle \in \mathcal{S}(\mathfrak{H}_0)$. Denote by $\{\varphi_i\}$ and $\{\chi_j\}$ two complete systems of mutually orthogonal one-dimensional projection operators on $\mathfrak{H}$ and by $\{|\varphi_i\rangle\}$ and $\{|\chi_j\rangle\}$ the corresponding orthonormal bases. Since d is completely additive, it follows $\sum_{i,j} |\langle \varphi_i \otimes \chi_j | \mathfrak{X}_{\Re d} | \varphi_i \otimes \chi_j \rangle|^2 = \sum |\Re d(\varphi_i, \chi_j)|^2 \leq \left(\sum |\Re d(\varphi_i, \chi_j)|\right)^2 < \infty$. Hence $\mathfrak{X}_{\Re d}$ is a Hilbert-Schmidt operator. Similarly, $\mathfrak{X}_{\Im d}$ is a Hilbert-Schmidt operator. Then define $\mathfrak{X}_d \equiv \mathfrak{X}_{\Re d} + i\mathfrak{X}_{\Im d}$. ■

Proposition 6 *Let $\mathfrak{H}$ be an infinitely dimensional complex Hilbert space, then for every proper finitely additive decoherence functional d on $\mathcal{P}(\mathfrak{H})$ there exists a unique bounded operator $\mathfrak{X}_d$ on $\mathfrak{H} \otimes \mathfrak{H}$ (not necessarily of trace class) such that d can be written as $d(h,k) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h \otimes k \mathfrak{X}_d)$ for all finite-dimensional $h, k \in \mathcal{P}(\mathfrak{H})$.*

In other words: every proper decoherence functional admits a *unique* quasi-regular representation.

Proposition 7 *Let $\mathfrak{H}$ be an infinitely dimensional complex Hilbert space, then for every proper σ-additive decoherence functional d on $\mathcal{P}(\mathfrak{H})$ there exists a unique bounded operator $\mathfrak{X}_d$ on $\mathfrak{H} \otimes \mathfrak{H}$ (not necessarily of trace class) defining a σ-quasi-regular representation of d. If $\mathfrak{H}$ is separable and if d is σ-summable, then $\mathfrak{X}_d$ is a Hilbert-Schmidt operator.*

The proofs of Proposition 6 and Proposition 7 are analoguous to the proof of Theorem 2 and Proposition 5.

Corollary 2 *Let $\mathfrak{H}$ be an infinite-dimensional complex separable Hilbert space. Then every σ-summable normal decoherence functional d on $\mathcal{P}(\mathfrak{H})$ is ultraweakly bi-continuous and regular.*

To sum up: Theorem 1 asserts that every bounded decoherence functional d on the set of projection operators on a finite-dimensional Hilbert space of dimension greater than two is regular. Theorem 2 shows that every normal completely additive decoherence functional d on the set of projection operators of an infinite-dimensional Hilbert space is regular. And Proposition 7 shows among others that every σ-summable normal decoherence functional d on the set of projection operators of an infinite-dimensional separable Hilbert space is regular. Obviously every regular decoherence functional is ultraweakly bi-continuous and hence σ-summable for separable Hilbert spaces and completely additive for non-separable Hilbert spaces.

2.4 Effect Histories

In [12] it has been argued that in a general history theory the space of histories should be identified with the set of effects $\mathfrak{E}(\mathfrak{H})$ on some Hilbert space $\mathfrak{H}$. The following Corollary 3 shows that there is a one-one correspondence between ultraweakly continuous normal decoherence functionals $d : \mathcal{P}(\mathfrak{H}) \times \mathcal{P}(\mathfrak{H}) \to \mathbb{C}$ and ultraweakly continuous normal decoherence functionals $d : \mathfrak{E}(\mathfrak{H}) \times \mathfrak{E}(\mathfrak{H}) \to \mathbb{C}$ as defined in [12]. Corollary 3 is an easy consequence of Theorem 2.

Corollary 3 *Let d denote a completely additive normal decoherence functional on $\mathcal{P}(\mathfrak{H})$, $\dim(\mathfrak{H}) \geq 2$, then d can be uniquely extended to an ultraweakly bi-continuous bounded functional $\widetilde{d} : \mathfrak{E}(\mathfrak{H}) \times \mathfrak{E}(\mathfrak{H}) \to \mathbb{C}$. $\widetilde{d}$ satisfies $\widetilde{d}(e,e) \in \mathbb{R}; \widetilde{d}(e,e) \geq 0; \widetilde{d}(e,f) = \widetilde{d}(f,e)^*; \widetilde{d}(1,1) = 1$ and $\widetilde{d}(0,e) = 0$, for all $e, f \in \mathfrak{E}(\mathfrak{H})$. Moreover, $\widetilde{d}$ is additive with respect to the canonical D-poset structure on $\mathfrak{E}(\mathfrak{H})$, i.e., $\widetilde{d}(e_1 \oplus e_2, f) = \widetilde{d}(e_1, f) + \widetilde{d}(e_2, f)$, whenever $e_1 \oplus e_2$ is well-defined.*

3 Summary

In this work we have proven a classification theorem for decoherence functionals on the set $\mathcal{P}(\mathfrak{H})$ of projection operators on an arbitrary finite- or infinite-dimensional separable or non-separable complex Hilbert space $\mathfrak{H}$ with dimension greater than two. In the finite-dimensional case we have seen that there is a one-to-one correspondence between bounded decoherence functionals on $\mathcal{P}(\mathfrak{H})$ and certain trace class operators $\mathfrak{X}$ on $\mathfrak{H} \otimes \mathfrak{H}$. The conditions $\mathfrak{X}$ has to satisfy are listed in Theorem 1. This result has first been proven by Isham, Linden and Schreckenberg [9]. If $\mathfrak{H}$ is an infinite-dimensional separable Hilbert space, we have shown that there is a one-to-one correspondence between normal (σ-summable) decoherence functionals on $\mathcal{P}(\mathfrak{H})$ and certain trace class operators $\mathfrak{X}$ on $\mathfrak{H} \otimes \mathfrak{H}$. The conditions $\mathfrak{X}$ has to satisfy are listed in Theorem 2. If $\mathfrak{H}$ is an infinite-dimensional non-separable Hilbert space, we have seen that there is a one-to-one correspondence between normal (completely additive) decoherence functionals on $\mathcal{P}(\mathfrak{H})$ and

certain trace class operators $\mathfrak{X}$ on $\mathfrak{H} \otimes \mathfrak{H}$. The conditions $\mathfrak{X}$ has to satisfy are listed in Theorem 2.
In addition, we have seen that if $\mathfrak{H}$ is an arbitrary Hilbert space with $\dim(\mathfrak{H}) > 2$, then every proper decoherence functional on $\mathcal{P}(\mathfrak{H})$ admits a unique quasi-regular representation, every proper σ-additive decoherence functional admits a unique σ-quasi-regular representation and every proper completely additive decoherence functional admits a unique pseudo-regular representation.
There are many decoherence functionals which are not covered by Theorem 2 and the subsequent propositions. It would be particularly interesting to learn more about the general structure and properties of the quasi-regular representations of decoherence functionals and about representations for non-normal and non-proper decoherence functionals. These topics deserve further investigation.

Acknowledgments

I am grateful to Professor Frank Steiner for his support of my work. I would like to thank Dr. Stephan Schreckenberg for stimulating discussions. Financial support given by Deutsche Forschungsgemeinschaft (Graduiertenkolleg für theoretische Elementarteilchenphysik) is also gratefully acknowledged. I am grateful to Professor J.D.M. Wright for his insightful comments on a previous version of this paper.

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