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FIXED POINT RESOLUTION IN CONFORMAL FIELD THEORY

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Abstract.
We summarize recent progress in the understanding of fixed point resolution for conformal field theories. Fixed points in both coset conformal field theories and non-diagonal modular invariants which describe simple current extensions of chiral algebras are investigated. A crucial rôle is played by the mathematical structures of twining characters and orbit Lie algebras.

Slightly extended version of a talk given by J. Fuchs at the XXI International Colloquium on Group Theoretical Methods in Physics (Goslar, Germany, July 1996)

1 Introduction

Attempts to analyze quantum field theories beyond perturbation theory typically face enormous problems. For two-dimensional conformal field theories this task is certainly easier than for higher-dimensional models, but even for the most extensively studied Wess–Zumino–Witten (WZW) models based on simple Lie algebras a satisfactory field theory interpretation is available only for a few of the known non-diagonal modular invariants. Among the more severe obstacles there is the so-called *fixed point resolution* problem, which arises when the orbits with respect to a symmetry have unequal sizes. (This kind of problem occurs in many other areas of physics and mathematics, too.)

In this talk we report on recent progress in fixed point resolution for various classes of conformal field theories which are based on WZW theories, namely coset theories [1] and simple current extensions of WZW theories [2]. These results imply a Verlinde formula for WZW models on non-simply connected groups, and also led us to a conjecture concerning simple current extensions of arbitrary conformal field theories [2]. Moreover, they should play a rôle in the treatment of orbifolds of the type discussed in V.G. Kac's talk at this conference. The mathematical structures emerging in our analysis of fixed point resolution are those of twining characters and orbit Lie algebras [3, 4], which are explained in more detail in the talk by C. Schweigert [5].

2 Coset conformal field theories

The chiral algebra of a WZW theory is the semi-direct sum of the Virasoro algebra and an untwisted affine Kac–Moody [6] Lie algebra $\mathfrak{g}$. The well-established representation theory of affine Lie algebras makes such theories amenable to detailed study; they have therefore become prototypes and building blocks of two-dimensional conformal field theory. The coset construction exploits the representation theory of affine algebras also for more complicated models. The basic idea is to consider embeddings $\mathfrak{g}' \hookrightarrow \mathfrak{g}$, where both algebras are direct sums of untwisted affine Lie algebras and Heisenberg algebras at some level(s) k. When the embedding of $\mathfrak{g}'$ into $\mathfrak{g}$ is induced by an embedding $\bar{\mathfrak{g}}' \hookrightarrow \bar{\mathfrak{g}}$ of the respective horizontal subalgebras, then the difference of the two Virasoro algebras $\mathcal{V}\mathrm{ir}_{\mathfrak{g}}$ and $\mathcal{V}\mathrm{ir}_{\mathfrak{g}'}$ (which are obtained via the affine Sugawara construction for $\mathfrak{g}$ and $\mathfrak{g}'$), i.e. the Lie algebra with generators $\dot{L}_m = L_m - L'_m$, is again a Virasoro algebra.

It is, however, an open problem whether for *any* embedding $\bar{\mathfrak{g}}' \hookrightarrow \bar{\mathfrak{g}}$ this prescription of a Virasoro algebra can be complemented in such a way that one gets a consistent conformal field theory – called the *coset theory* and briefly denoted by '$(\bar{\mathfrak{g}}/\bar{\mathfrak{g}}')_k$' – and if so, whether that theory is unique. To decide these questions, one must in particular construct the (maximally extended) chiral algebra $\dot{\mathcal{W}}$ of $(\bar{\mathfrak{g}}/\bar{\mathfrak{g}}')_k$, as well as the spectrum of the theory, i.e. tell which modules (representation spaces) – of the coset chiral algebra $\dot{\mathcal{W}}$, or at least of the coset Virasoro algebra $\dot{\mathcal{V}}\mathrm{ir}$ – appear.

Concerning the spectrum, one first notes that even though $\dot{\mathcal{V}}\mathrm{ir}$ acts on the chiral state space $\mathcal{H}_{\mathfrak{g}}$ of the WZW theory for $\mathfrak{g}$, i.e. on the direct sum $\mathcal{H}_{\mathfrak{g}} = \bigoplus_\Lambda \mathcal{H}_\Lambda$ of all inequivalent unitary irreducible highest weight modules $\mathcal{H}_\Lambda$ of $\mathfrak{g}$ at level k, $\mathcal{H}_{\mathfrak{g}}$ is *not* the state space $\dot{\mathcal{H}}$

of the coset theory. The crucial point is that $\mathfrak{g}'$ commutes with $\dot{\mathcal{V}}\mathrm{ir}$, so that when retaining the full space $\mathcal{H}_{\mathfrak{g}}$, infinitely many spin zero fields, namely all $\mathfrak{g}'$-currents and their $\mathfrak{g}'$-descendants, would be present. This disaster is avoided by imposing the gauge principle that $\mathcal{H}_{\mathfrak{g}}$-vectors differing only by the action of $\mathfrak{g}'$ represent one and the same state in $\dot{\mathcal{H}}$. Thus the elements of $\dot{\mathcal{H}}$ are $\mathfrak{g}'$-*orbits* of vectors in $\mathcal{H}_{\mathfrak{g}}$ rather than individual $\mathcal{H}_{\mathfrak{g}}$-vectors. An obvious ansatz is then to regard the 'multiplicity spaces' $\mathcal{H}_{(\Lambda;\Lambda')}$ that appear in the decomposition

$$\mathcal{H}_\Lambda = \bigoplus\nolimits_{\Lambda'} \mathcal{H}_{(\Lambda;\Lambda')} \otimes \mathcal{H}_{\Lambda'} \tag{1}$$

of the $\mathfrak{g}$-modules $\mathcal{H}_\Lambda$ into irreducible $\mathfrak{g}'$-modules $\mathcal{H}_{\Lambda'}$ as the $(\bar{\mathfrak{g}}/\bar{\mathfrak{g}}')_k$-modules, and correspondingly the branching functions $b_{(\Lambda;\Lambda')}$, which are defined by the decomposition

$$\chi_\Lambda = \sum_{\Lambda'} b_{(\Lambda;\Lambda')}\, \chi_{\Lambda'} \tag{2}$$

of irreducible $\mathfrak{g}$-characters [1]

$$\chi_\Lambda := \mathrm{tr}_{\mathcal{H}_\Lambda}(q^{L_0 - C/24}) \tag{3}$$

into irreducible $\mathfrak{g}'$-characters $\chi_{\Lambda'} := \mathrm{tr}_{\mathcal{H}_{\Lambda'}}(q^{L'_0 - C'/24})$, as the characters of the coset theory.

However, in general the redundancy is larger than the obvious gauge symmetry $\mathfrak{g}'$, so that this construction is incomplete. This manifests itself in selection rules, i.e. the branching functions $b_{(\Lambda;\Lambda')}$ for certain pairs $(\Lambda;\Lambda')$ vanish, and equivalences, i.e. multiplicity spaces $\mathcal{H}_{(\Lambda;\Lambda')}$ for distinct pairs $(\Lambda;\Lambda')$ are isomorphic modules. In particular the putative vacuum module occurs several times. For the same reasons which force us to divide out the $\mathfrak{g}'$-action, this implies that a primary field of $(\bar{\mathfrak{g}}/\bar{\mathfrak{g}}')_k$ is not associated to an individual pair $(\Lambda;\Lambda')$ of integrable highest weights, but rather to an appropriate equivalence class $[(\Lambda;\Lambda')]$ of such pairs. This projection to classes $[(\Lambda;\Lambda')]$ is known as *field identification*.

Both selection rules and field identification are conveniently described by using the concept of the *identification group* G_{id} [7]. The elements $(J;J')$ of G_{id} are primary fields of the tensor product theory $\mathfrak{g}\oplus(\mathfrak{g}')^*$ (the star indicates that one must use the complex conjugate $\mathrm{SL}(2,\mathbb{Z})$-representation) of unit quantum dimension (known as *simple currents*) and of integral spin; they act via the fusion product. The selection rules are equivalent to the vanishing of the so-called monodromy charges $Q_{(J;J')}$ of any allowed branching function with respect to all $(J;J')\in G_{\mathrm{id}}$. (Here $Q_{(J;J')}((\Lambda;\Lambda')) = Q_J(\Lambda) - Q_{J'}(\Lambda')$, where $Q_J(\Lambda)$ is the combination $Q_J(\Lambda) = \Delta_\Lambda + \Delta_J - \Delta_{J\star\Lambda}$ of conformal weights.) Moreover, the equivalence classes in the field identification are precisely the orbits of G_{id},

$$[(\Lambda;\Lambda')] = \{\,(\Upsilon;\Upsilon') \mid \Upsilon = J\Lambda,\ \Upsilon' = J'\Lambda' \text{ for some } (J;J')\in G_{\mathrm{id}}\,\}\,. \tag{4}$$

As long as all G_{id}-orbits have a common length, taking one branching function out of each G_{id}-orbit and combining them diagonally yields a modular invariant spectrum, and the modular S-matrix $\dot{S}$ is given by the restriction to orbits of the S-matrix $S\otimes(S')^*$ of $\mathfrak{g}\oplus(\mathfrak{g}')^*$.

[1] For notational simplicity here and below we only write the $\dot{\mathcal{V}}\mathrm{ir}$-specialized characters. But all our results hold for the full characters. Note that the branching functions are indeed functions of q rather than numbers.

3 Fixed point resolution in coset theories

While the construction presented above works in many cases, in general it is still insufficient. Indeed, as soon as *fixed points* – that is, orbits $[(\Lambda;\Lambda')]$ with non-maximal size (i.e. smaller than the size $|G_{\rm id}|$ of the orbit $[(0;0)]$ that yields the vacuum module) – are present, taking precisely one representative out of each orbit does not give a modular invariant spectrum. On the other hand, there does exist a modular invariant sesquilinear combination Z of (non-zero) branching functions, namely

$$\tilde{Z} = \sum_{\substack{[(\Lambda;\Lambda')] \\ Q=0}} |G_{\Lambda;\Lambda'}| \cdot \Big| \sum_{(J;J')\in G_{\rm id}/G_{\Lambda;\Lambda'}} b_{(J;J')\,(\Lambda;\Lambda')} \Big|^2 , \tag{5}$$

which is modular invariant. Here $G_{\Lambda;\Lambda'}$ is the *stabilizer* of $(\Lambda;\Lambda')$, i.e. the subgroup

$$G_{\Lambda;\Lambda'} = \{(J;J')\in G_{\rm id} \,|\, J\Lambda = \Lambda,\ J'\Lambda' = \Lambda'\} \tag{6}$$

of those currents which leave $(\Lambda;\Lambda')$ fixed; its order $|G_{\Lambda;\Lambda'}|$ is the quotient of N by the size $N_{\Lambda;\Lambda'}$ of the orbit $[(\Lambda;\Lambda')]$.

However, regarding the branching functions $b_{(\Lambda;\Lambda')}$ appearing in (5) as irreducible $(\bar{\mathfrak{g}}/\bar{\mathfrak{g}}')_k$-characters would lead to the conclusion that fractional multiplicities occur in Z, which does not allow for any sensible interpretation of Z as a partition function. The difficulty in interpreting Z constitutes the *fixed point problem* of coset theories. This is indeed a severe problem, as it seems to prevent us from gaining control over the coset theory $(\bar{\mathfrak{g}}/\bar{\mathfrak{g}}')_k$ by completely understanding it in terms of the underlying WZW theories based on $\mathfrak{g}$ and $\mathfrak{g}'$. The solution to this problem ([1]; earlier work is summarized in [7]) tells us that $(\bar{\mathfrak{g}}/\bar{\mathfrak{g}}')_k$ is still fully controlled by the affine algebras $\mathfrak{g}$ and $\mathfrak{g}'$, but that one needs to implement additional novel structures [3, 4] associated to Kac–Moody algebras.

The basic observation is that the coset modules $\mathcal{H}_{(\Lambda;\Lambda')}$ for fixed points are *not* irreducible and hence must be *resolved* into irreducible subspaces. For the full two-dimensional theory obtained by combining its two chiral halves, this means that not all states that one would naively expect are present, i.e. an additional projection takes place on the space of states. To construct the submodules, we study the intertwiners between isomorphic spaces $\mathcal{H}_{(\Lambda;\Lambda')}$. The simple currents J and J' act on the weights of $\mathfrak{g}$ and $\mathfrak{g}'$ by maps $\omega^\star$ and $\omega'^\star$ (i.e. $J\Lambda = \omega^\star(\Lambda)$, $J'\Lambda' = \omega'^\star(\Lambda')$) which are induced by certain outer automorphisms ω and ω' of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. [5]. Using these automorphisms we are able to construct a map $\dot{\tau}_{(J;J')}\colon\ \mathcal{H}_{(\Lambda;\Lambda')} \to \mathcal{H}_{(J\Lambda;J'\Lambda')}$ which obeys $[\dot{\tau}_{(J;J')}, \dot{L}_m] = 0$, i.e. intertwines the action of $\dot{\mathcal{V}}\mathrm{ir}$.[2]

If $(\Lambda;\Lambda')$ is fixed by $(J;J')$, then $\dot{\tau}_{(J;J')}$ is an *endo*morphism, so that we can decompose $\mathcal{H}_{(\Lambda;\Lambda')}$ as

$$\mathcal{H}_{(\Lambda;\Lambda')} = \bigoplus_{\Psi} \mathcal{H}^{\Psi}_{(\Lambda;\Lambda')} \tag{7}$$

[2] For technical reasons concerning the precise relation between ω and ω', the fixed point resolution works literally as sketched here only for so-called 'generalized diagonal coset theories'. But it is to be expected that our ideas generalize to all coset theories except for very special classes. The known exceptions (conformal embeddings and so-called maverick cosets) all appear at levels $k = 1$ or $k = 2$.

into eigenspaces of $\dot{\tau}_{(J;J')}$. The label Ψ takes values in the character group $G^\star_{(\Lambda;\Lambda')}$ of the (abelian) group $G_{(\Lambda;\Lambda')}$.
Now the first crucial result of [1] is:
• The eigenspaces $\mathcal{H}^\Psi_{(\Lambda;\Lambda')}$ are submodules of $\mathcal{H}_{(\Lambda;\Lambda')}$, and they can consistently be assumed to be irreducible. This implies in particular that the characters of the resolved fixed points are the characters $b^\Psi_{(\Lambda;\Lambda')}$ of the eigenspaces $\mathcal{H}^\Psi_{(\Lambda;\Lambda')}$.
Although the definition of the eigenspaces $\mathcal{H}^\Psi_{(\Lambda;\Lambda')}$ is rather implicit, we can still compute their characters $b^\Psi_{(\Lambda;\Lambda')}$, due to our second result:
• The Fourier transforms $b^{[J;J']}_{(\Lambda;\Lambda')} := \sum_\Psi \Psi((J;J'))\, b^\Psi_{(\Lambda;\Lambda')}$ of the characters $b^\Psi_{(\Lambda;\Lambda')}$ with respect to $G^\star_{(\Lambda;\Lambda')}$ are generalized character-valued indices, obtainable as the trace

$$b^{[J;J']}_{(\Lambda;\Lambda')}(q) = \mathrm{tr}_{\mathcal{H}_{(\Lambda;\Lambda')}}(\dot{\tau}_{(J;J')}\, q^{\dot{L}_0-\dot{C}/24})\,. \tag{8}$$

• Moreover, these functions arise in the expansion

$$\chi^{[J]}_\Lambda = \sum_{\Lambda'} b^{[J;J']}_{(\Lambda;\Lambda')}\, \chi^{[J']}_{\Lambda'} \tag{9}$$

of the *twining characters* [3]

$$\chi^{[J]}_\Lambda := \mathrm{tr}_{\mathcal{H}_\Lambda}(\tau_J\, q^{L_0-C/24}) \tag{10}$$

of $\mathfrak{g}$ with respect to the twining characters $\chi^{[J']}_{\Lambda'} := \mathrm{tr}_{\mathcal{H}_{\Lambda'}}(\tau_{J'}\, q^{L'_0-C'/24})$ of $\mathfrak{g}'$. We therefore call them the *twining branching functions* associated to $(J;J')$.
• We can then compute the twining branching functions from the representation theory of $\mathfrak{g}$ and $\mathfrak{g}'$: the twining characters coincide with ordinary characters of the so-called orbit Lie algebras $\breve{\mathfrak{g}}$ and $\breve{\mathfrak{g}}'$ [3, 4, 5], so that $b^{[J;J']}_{(\Lambda;\Lambda')}$ can be calculated as the ordinary branching function $\breve{b}_{\breve{\Lambda},\breve{\Lambda}'}$, satisfying $\breve{\chi} = \sum_{\breve{\Lambda}'} \breve{b}_{\breve{\Lambda},\breve{\Lambda}'}\breve{\chi}'$, of the coset construction $(\breve{\mathfrak{g}}/\breve{\mathfrak{g}}')_{\breve{k}}$ of orbit Lie algebras.

It follows in particular that twining branching functions carry a unitary representation of SL(2,$\mathbb{Z}$). Their modular S-matrix $S^{[J;J']}$ is the tensor product of the relevant S-matrices $S^{[J]}$ and $S^{[J']}$ for the twining characters of $\mathfrak{g}$ respectively $\mathfrak{g}'$, which in turn are the ordinary S-matrices of the orbit Lie algebras $\breve{\mathfrak{g}}$ and $\breve{\mathfrak{g}}'$. By expressing the coset characters $b^\Psi_{(\Lambda;\Lambda')}$ as inverse Fourier transforms of the $b^{[J;J']}_{(\Lambda;\Lambda')}$, we can then determine the S-matrix $\dot{S}$ for the coset theory $(\bar{\mathfrak{g}}/\bar{\mathfrak{g}}')_k$. We find

$$\dot{S}_{([(\Lambda;\Lambda')],\Psi),([(\Upsilon;\Upsilon')],\tilde{\Psi})} = \frac{|G_{\mathrm{id}}|}{|G_{(\Lambda;\Lambda')}|\cdot|G_{(\Upsilon;\Upsilon')}|} \sum_{\substack{(J;J')\in \\ G_{(\Lambda;\Lambda')}\cap G_{(\Upsilon;\Upsilon')}}} \Psi^*((J;J'))\, S^{[J;J']}_{(\Lambda;\Lambda'),(\Upsilon;\Upsilon')} \tilde{\Psi}((J;J'))\,. \tag{11}$$

Using the modular properties of ordinary and twining characters of $\mathfrak{g}$ and $\mathfrak{g}'$, we can show that $\dot{T}$ – which is just the T-matrix of $\mathfrak{g}\oplus(\mathfrak{g}')^*$ restricted to orbits – and $\dot{S}$ generate a unitary representation of SL(2,$\mathbb{Z}$), and that $\dot{S}$ is symmetric and squares to an order-two permutation. Finally, in all the (many) cases that we have checked explicitly, $\dot{S}$ gives rise, via the Verlinde formula, to non-negative integral fusion coefficients.

4 Fixed point resolution for simple current extensions

Many non-diagonal modular invariants $\mathcal{Z}$ can be interpreted as the diagonal invariant for an extension of the chiral algebra by a group $G_{\rm xt}$ of integer spin simple currents. They can be written as

$$\mathcal{Z} = \sum_{[a]:\, Q(a)=0} |G_a| \cdot \Big| \sum_{J\in G_{\rm xt}/G_a} \chi_{Ja} \Big|^2, \tag{12}$$

where χ_a denotes the irreducible characters of the original 'unextended' theory, the first sum is over all $G_{\rm xt}$-orbits $[a]$ with zero monodromy charges Q [7], and $G_a \subseteq G_{\rm xt}$ is the stabilizer $G_a = \{J \in G_{\rm xt} \,|\, Ja = a\}$. As in the case of coset theories, we want to know the spectrum of the putative conformal field theory that has this extended chiral algebra.

For non-fixed points, i.e. $|G_a| = 1$, one can simply interpret $\mathcal{X}_a = \sum_{J\in G_{\rm xt}} \chi_{Ja}$ as an irreducible character of the extended theory. Similarly, for $|G_a| = 2$ or 3 the orbit $[a]$ corresponds to precisely $|G_a|$ irreducible modules in the extended theory whose characters $\mathcal{X}$ are identical with respect to the unextended algebra. But for $|G_a| \geq 4$ it can happen that the prefactor $|G_a|$ or part of it must be interpreted as belonging to the extended character rather than as a multiplicity, so that there are as many potential interpretations of $|G_a|$ as there are ways of writing it as a sum of squares. The task to determine the meaning of the prefactors $|G_a| \geq 4$ (and thereby find the spectrum) constitutes the fixed point resolution problem for simple current extensions.

The main results of [2] are a prescription on how to interpret factors $|G_a| \geq 4$ and a formula for the S-matrix $\mathcal{S}$ which acts on the characters $\mathcal{X}$ of the extended theory. They are obtained by solving the constraint that $\mathcal{S}$ is symmetric and together with $\mathcal{T}$ (which coincides with T 'on orbits') generates a unitary representation of SL(2,$\mathbb{Z}$), supplemented by the working hypotheses that the order of successive extensions does not matter and that the characters of all resolved fixed points coming from an orbit $[a]$ are identical with respect to the unextended algebra ('fixed point homogeneity'). Since the first condition is necessary, but not sufficient for the existence of a consistent conformal field theory, our results are still conjectures. Note that fixed point homogeneity means that taking (part of) $|G_a|$ into the characters can only occur when $|G_a|$ is a square.[3]

Our result for the S-matrix closely resembles the formula (11) for $\dot{S}$:

$$\mathcal{S}_{(a,\Psi),(b,\tilde\Psi)} = \sqrt{\mathcal{I}_a \mathcal{I}_b}\, \frac{|G_{\rm id}|}{|G_a|\cdot|G_b|} \sum_{J\in U_a\cap U_b} \Psi(J)\, S^{[J]}_{a,b}\, \tilde\Psi(J)^* \,. \tag{13}$$

But it involves a new ingredient, the *untwisted stabilizers* $U_a \subseteq G_a$. Namely, to each $J \in G_{\rm xt}$ we associate a unitary matrix $S^{[J]}$ which vanishes on non-fixed points of J and, when restricted to fixed points, satisfies $(S^{[J]})^4 = \mathbb{1}$ and $(S^{[J]}T)^3 = (S^{[J]})^2$, as well as $S^{[J^{-1}]} = (S^{[J]})^{\rm t}$. Further, for any simple current K that is local with respect to J one defines a complex number $F_a(K,J)$ by the relation $S^{[J]}_{Ka,b} = F_a(K,J)\, {\rm e}^{2\pi{\rm i}Q_K(b)} S^{[J]}_{ab}$ between S-matrix elements. Then $U_a := \{J \in G_a \,|\, F_a(K,J) = 1 \ \text{ for all } K \in G_a\}$.

[3] Resolutions which violate fixed point homogeneity sometimes also exist; e.g. the simple current invariant of A_4 at level 5 has a fixed point with $|G_a| = 5$ for which $|G_a|$ splits as $5 = 1^2 + 2^2$.

In (13), Ψ and $\tilde\Psi$ are characters of U_a and of U_b, respectively, and $\mathcal{I}_a$ is the index $\mathcal{I}_a = |G_a|/|U_a|$, which is always a square. For the invariant (12) this means that each fixed point $[a]$ is resolved into $|U_a|$ distinct fields, which are labelled by the characters of U_a. Again we checked in many cases that the formula (13) leads to non-negative integral fusion coefficients.

For WZW models, the matrices $S^{[J]}$ can be identified (up to a known over-all phase) with the modular S-matrices of the associated orbit Lie algebras. For other conformal field theories we must postulate the existence of the relevant matrices $S^{[J]}$. We conjecture that they do not constitute a new structure, but rather coincide with the matrices that describe the modular S-transformation on the space $\mathcal{F}$ of chiral one-point blocks on the torus with insertion of the relevant simple current $J(z)$. Indeed, our $S^{[J]}$ share all known properties of these matrices, including certain recently deduced [8] trace relations. Moreover, in [9] it has been shown that the untwisted stabilizers appear naturally in the framework of one-point blocks, where they possess a cohomological interpretation related to a basis choice for the space $\mathcal{F}$ of one-point blocks.

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