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Consistent Histories:

Mathematical and Structural Foundations

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I'm preachin' dis sermon to show it ain't necessarily so.

[from "Porgy and Bess" by George Gershwin]

CONTENTS

PREFACE vii

1 INTRODUCTION 1

Part I STANDARD QUANTUM MECHANICS 9

2 OPERATIONAL QUANTUM PHYSICS 11
- 2.1 States in Quantum Mechanics 11
- 2.2 Observables in Quantum Mechanics 12
- 2.3 The Projection Postulate 15
- 2.4 Quantum Properties and Quantum Events 16

3 CONSISTENT HISTORIES 19
- 3.1 Standard Consistent Histories 19
- 3.2 The Consistent Histories Interpretation 27
- 3.3 Weak Points and Criticisms 32
- 3.4 Mixed History States 34

4 CONSISTENT EFFECT HISTORIES 41
- 4.1 Homogeneous Effect Histories 41
- 4.2 Inhomogeneous Effect Histories 43

5 LOCALITY AND CONSISTENT HISTORIES 49
- 5.1 The Principle of Locality 49
- 5.2 The Schlieder-Hellwig-Kraus Projection Postulate 51
- 5.3 An Evolutionary Picture Based on Quantum Histories 55
- 5.4 Local Hilbert Spaces 62
- 5.5 Concluding Remarks 64

Part II GENERAL QUANTUM HISTORY THEORIES **67**

6 AXIOMS FOR QUANTUM HISTORY THEORIES **69**

7 GLEASON TYPE THEOREMS FOR DECOHERENCE FUNCTIONALS **73**

7.1 Introduction 73
7.2 Preliminaries 74
7.3 The Finite Dimensional Case 76
7.4 The Infinite Dimensional Case 77
7.5 Effect Histories 80
7.6 Concluding Remarks 81

8 CONCLUSIONS AND OUTLOOK **83**

A MISCELLANEOUS DEFINITIONS AND RESULTS **85**

A.1 Posets and Lattices 85
A.2 D-posets and Orthoalgebras 87
A.3 The D-poset $\mathfrak{E}(\mathfrak{H})$ 90
A.4 An Alternative Characterization of Boolean Lattices 91
A.5 Hilbert Lattices 92

B GLEASON'S AND WRIGHT'S THEOREMS **95**

B.1 Preliminaries 95
B.2 Gleason's Theorem 97
B.3 Wright's Theorem 97

C THE TENSOR PRODUCT FORM OF THE STANDARD DECOHERENCE FUNCTIONAL **99**

D CONTINUOUS TENSOR PRODUCTS **101**

E JOINT POSITION-MOMENTUM OBSERVABLES **103**

E.1 Galilean Systems of Covariance in Nonrelativistic Phase Space 103
E.2 Lorentzian Systems of Covariance in Relativistic Phase Space 104

REFERENCES **107**

PREFACE

Declaration

Unless explicitly otherwise stated, the results presented in this dissertation are original. Where the work of other authors has been used as a starting point, to complete the discussion or to put the results in a broader context, this is clearly marked. The references may be found on page 107.

Part of the material contained in this thesis is based on articles published in *Int. J. Theor. Phys.* **35** (1996), 1581, *J. Math. Phys.* **37** (1996), 5368, and in DESY 96-176.

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Abstract

In this thesis some recent results and developments in the history approach to quantum theory are presented. The history approach is also known under the name "consistent histories approach" or "decoherent histories approach" to quantum theory. There exist several competing closely related variants of the consistent histories formulation in the literature. The approach in the present thesis is closest to the approaches of Isham on the one hand and of Omnès and Griffiths on the other hand. It will be discussed in this thesis that the history approach provides a novel way to look upon quantum theories and provides us with a new interpretation of nonrelativistic Hilbert space quantum mechanics. In the first part of this work we will deal with the history formulation of nonrelativistic Hilbert space quantum mechanics. The consistent histories approach is usually formulated using the standard notions of observable and state. We will argue that the natural notion of an observable in quantum mechanics is that of a *positive-operator-valued measure* (POV measure) and will show that the consistent history formalism can be generalized to incorporate POV measures in a natural and simple way. Part I concludes with an attempt to formulate a Lorentz covariant version of the history approach in Minkowski space as a scheme for a quantum theory of events. In the second part of this thesis we will discuss Isham's general quantum history theories and will prove Gleason type classification theorems for decoherence functionals in the case that the space of histories is the set of projection operators on some Hilbert space.

Zusammenfassung

In dieser Arbeit werden einige jüngere Ergebnisse und Entwicklungen im sogenannten "Geschichten-Zugang zur Quantentheorie" vorgestellt. Der "Geschichten-Zugang" zur Quantentheorie wird in der Originalliteratur auch als "consistent histories approach to quantum mechanics" oder als "decoherent histories approach to quantum mechanics" bezeichnet. Von den verschiedenen jedoch eng verwandten publizierten Varianten des "Geschichten-Zuganges" liegen insbesondere die Ideen von Isham einerseits und von Omnès und Griffiths andererseits der vorliegenden Untersuchung zugrunde. Wir werden erörtern, daß der "Geschichten-Zugang" zur Quantentheorie eine neuartige Auffassung der Quantentheorie und außerdem eine neue Interpretation der nichtrelativistischen Hilbertraum-Quantenmechanik ermöglicht. Im ersten Teil dieser Arbeit wird die "Geschichten-Umformulierung" der nichtrelativistischen Hilbertraum-Quantenmechanik diskutiert. Der "Geschichten-Zugang" wird gewöhnlich mit Hilfe der üblichen Begriffe der quantenmechanischen Observable und des quantenmechanischen Zustandes formuliert. Wir werden jedoch erörtern, daß es vom physikalischen Standpunkt aus natürlich ist, positive operatorwertige Maße (POW-Maße) als Observable in der Hilbertraum Quantenmechanik anzusehen. Es wird gezeigt, daß der "Geschichten-Zugang" zur Quantenmechanik zu einem die POW-Maße enthaltenden Formalismus in natürlicher und einfacher Weise verallgemeinert werden kann. Teil I dieser Arbeit schließt mit einem Vorschlag für eine Lorentz kovariante Formulierung des "Geschichten-Zuganges" im Minkowski-Raum aufgefaßt als ein Versuch einer Quantentheorie für "Ereignisse." Der zweite Teil dieser Arbeit ist Ishams verallgemeinerten "Geschichten" Quantentheorien gewidmet. Für den Fall, daß die "Geschichten" durch Projektionsoperatoren auf einem Hilbertraum gegeben sind, beweisen wir einige Darstellungs- und Klassifikationstheoreme für Dekohärenz-Funktionale.

1

INTRODUCTION

All' eins und sinnlos alles, alles, alles !
Wozu das ganze Schaffen, Freuen, Leiden, Leben ?
Ob ich's vermöchte ? – Nein, ach nein ! Wozu,
Wozu das alles – ach wozu – – – wozu ?
[from "Palestrina" by Hans Pfitzner]

This is a work about the foundations of quantum mechanics. Ever since the invention of quantum mechanics many scientists – physicists, mathematicians and philosophers – have thought about the problem of how the physical world could possibly be how quantum mechanics says it is. This is clearly (at least in part) a metaphysical problem. In quantum physics, it is the problem of *interpreting* quantum mechanics. From a point of view of a philosopher, the problem of interpreting quantum mechanics is an interesting problem in its own right. In contrast, a physicist is mainly interested in the question whether a given theory is empirically adequate and successful. Quantum mechanics enjoys an overwhelming amount of empirical success and it has shown an ever increasing range of applicability. The spectacular empirical success of quantum mechanics as a physical theory is based on the interpretative rule known as *Born's rule.* [1] Born's rule in its simplest form states that the probability to find a system initially in the state φ in the state ψ is given by $p_\varphi(\psi) = |(\varphi, \psi)|^2$. Thus, Born's rule translates the abstract assertions of the theory (in terms of state vectors etc.) to empirically testable statements in terms of probabilities. The probabilistic predictions of quantum mechanics based on Born's rule (or at least some of them) can (in principle) be tested by comparing them with the relative frequencies of the various data in a long series of trials. [2] Born's rule (or some appropriate generalized formulation thereof) is contained or reproduced in any more extensive interpretation of quantum mechanics known to the author. Quantum mechanics admits many different interpretations, e.g., the Copenhagen type interpretations, [3] the many worlds interpretation, the modal interpretations, the quantum event interpretation [85] and the statistical interpretation [7] to name only a few of them. [4] However, all these

[1] For a historical account of Born's rule we refer to the book by Jammer [94].

[2] This does neither mean that probabilities should or must be interpreted only as approximate relative frequencies nor that probabilistic statements always (and exclusively) refer to ensembles of (similarly prepared) systems (as in the so-called *statistical ensemble interpretation*, see Ballentine [7]).

[3] I use the phrase *Copenhagen type interpretation* to denote collectively the different variants of the Copenhagen or orthodox interpretation, to wit, the body of ideas which is with some justice also often called "orthodoxy" and which is usually associated with the names of Bohr, Heisenberg, von Neumann, Pauli, Born, Jordan and others. More or less rudimentary accounts of the Copenhagen interpretation can be found in almost all textbooks on quantum mechanics. The best account of the ideas of the *Copenhagen school* ever written is perhaps the highly recommendable book by David Bohm [14] (see in particular Chapters 6-8,22-23). Bohm's book will serve as our general reference of the Copenhagen interpretation. Different variants of the Copenhagen interpretation may also be found in the recommendable books by von Neumann [127] and Scheibe [155] and also in [81]. Our cumulative nomenclature ignores the differences in the various versions of the Copenhagen type interpretations given by different authors; for an overview the reader is referred to the monographs by Jammer [94, 95].

[4] Since I share the view that a theory and its interpretation are an indivisible unit, it would be more appropriate to speak of *Copenhagen quantum mechanics, many-worlds quantum mechanics* and *modal quantum mechanics* etc. as different theories. However, since several differing variants of these interpretations can

interpretations contain or reproduce Born's rule in some way or other and thus the empirical content of quantum mechanics is remarkably independent of the interpretation adopted. Different interpretations do not change the empirical content of quantum mechanics.
So, why another work about the foundations of quantum mechanics ?

From a point of view of a physicist new work on the foundations of quantum mechanics will be worth while only if it offers something new in content or perspective.

I will argue briefly in the sequel that the consistent histories approach to quantum mechanics – which is the main subject of this thesis – indeed offers something new as well in content as in perspective.

The investigation of how quantum mechanics is to be interpreted and what quantum theory is really telling us about the deeper nature of physical reality is a slippery business. This is so because interpretations add no empirical content to the theory interpreted and accordingly we have no empirical criterion to make a decision between competing interpretations. Due to the authority of the founders of quantum mechanics and due to the lack of a sensible alternative interpretation, the Copenhagen interpretation of quantum mechanics has for many years acquired the status of a dogma. This goes as far as that in some quantum mechanics textbooks the Copenhagen interpretation is treated as if it were the only conceivable interpretation of quantum mechanics. Since the interpretation is undoubtedly part of quantum mechanics it is striking to see how little space is devoted to it and how uncritically the assertions of the Copenhagen interpretation are adopted in many of the standard textbooks (with notable exceptions, including the book by David Bohm [14]. The recommendable lecture notes by Chris Isham [88] may also serve as a supplementary text to the standard textbooks of quantum mechanics). In some texts the conceptual problems of the interpretation are swept under the carpet and are claimed to be pseudoproblems due to our (admittedly) limited ability to understand the quantum concepts in an imaginative or pictorial way. For instance in the book by Schwabl [162] we find the following typical passage: *"Die begrifflichen Schwierigkeiten [...] sind keine Schwäche der Quantenmechanik, die noch an keine Gültigkeitsgrenzen gestoßen ist, sondern* ***ausschließlich*** *eine Schwäche unseres Denkvermögens"* (my emphasis). [5] In the otherwise excellent book by Thirring we find the following passage [165]: *"Da die Quantenmechanik experimentell glänzend bestätigt ist, sprechen die zahlreichen [...] Paradoxa* ***nur*** *gegen das an klassischen Objekten geschulte menschliche Denken"* (my emphasis). In fact, the Copenhagen interpretation is plagued with problems, mysteries and anthropomorphisms. The main problem of the Copenhagen interpretation is the understanding and description of the measuring process. Usually, the time evolution of the wavefunction of two interacting physical systems is unitary and governed by the Schrödinger equation. Consider a macroscopic measuring apparatus designed to measure the value of some observable A. Accordingly, if a suitable quantum system interacts with this measuring apparatus, then the (unitarily timedeveloped)

be found in the literature and since there is a large overlap between the different interpretations (there is for instance a Copenhagen variant of the modal interpretation [166]), such a nomenclature would be quite confusing.

[5] In the later editions of his book Schwabl has replaced the term *Denkvermögens* by *Vorstellungsvermögens*. To avoid misunderstanding, I wish to stress that I fully agree with the first part of the sentence (before *sondern*).

state of the apparatus-object system after the interaction will in general (e.g., if the quantum system is initially *not* in an Eigenstate of the observable A) be a superposition of macroscopically distinguishable states. Even if some kind of decoherence mechanism is assumed to be present, the state of the apparatus after the interaction will in general be a mixture of macroscopically distinguishable states [57, 173, 174, 175]. In contrast, the time evolution during a measurement process contains a nonunitary collapse. This is due to the fact that every measurement has a definite outcome, the measuring result. The problem is that the measurement process is also a physical interaction between the measuring apparatus and the measuring object and that we have no intrinsic criterion to distinguish ordinary interactions from measurement situations. In its most general form this problem is called the *objectification problem* of quantum mechanics [24]. This problem assumes different faces within different variants of the Copenhagen interpretation. We now describe briefly the ubiquitous solution of the Copenhagen type interpretation to the objectification problem. According to this solution, one has to abandon the idea that macroscopic objects – at least when they serve as measuring apparatuses – can be described by quantum mechanics. Instead one has to introduce the following two assumptions (cited from [14], Chapter 23)

1. Quantum theory presupposes a classical level and the correctness of classical concepts in describing this level.
2. The classically definite aspects of large-scale systems cannot be deduced from the quantum-mechanical relationships of assumed small-scale elements. Instead, classical definiteness and quantum potentialities complement each other in providing a complete description of the system as a whole.

These two assumptions clearly contradict the widespread belief that quantum concepts are more fundamental than classical concepts and that the classical theory is merely a special case of the quantum theory. If one wants to maintain the latter view, then clearly the Copenhagen interpretation must be altered or abandoned.
It is beyond the scope of the present work to discuss and criticize the Copenhagen interpretation and its problems in detail. The following problems are closely related to the objectification problem: the problem of the status of the observer (in particular: the observer-mind problem and the relation between objective facts and human knowledge), the so-called Heisenberg cut and the above discussed state vector reduction due to our measurements (the projection postulate). However, there is no interpretation of quantum mechanics totally free of problems and to the best of my knowledge a satisfactory solution of the objectification problem is at present not known. It seems as if metaphysical problems of the kind just mentioned can never all be solved by solely changing one's metaphysical presuppositions. That said, I do believe, though, that it is important for a physicist to be aware of the multitude of metaphysical presuppositions and beliefs compatible with quantum mechanics. Metaphysical dogmas in theoretical physics and 'wrong' metaphysical research programmes may be an obstacle for further progress and may hinder thus new developments. This is one of the main reasons why it is important to question the pervasive Copenhagen type interpretation of quantum mechanics and to investigate alternative interpretations of quantum mechanics.

The *consistent* or *decoherent histories approach* to quantum theory is a fresh, novel attempt to formulate a generalization of standard quantum mechanics. The consistent histories approach introduces new concepts into quantum mechanics and is structurally different from all other approaches to quantum mechanics. Nonrelativistic quantum mechanics in its standard formulation is not a theory which describes the dynamical evolution pattern of events in time, but it is a theory which gives probabilities to the various possibilities or events. The history approach to quantum mechanics can be looked upon as an attempt to remedy this situation by introducing time sequences of possibilities (or events) as a rough substitute for dynamical processes.

Among others the consistent histories scheme provides a novel framework for the interpretation of standard Hilbert space quantum mechanics. This new interpretation has been developed mainly by Robert Griffiths and Roland Omnès. In this interpretation quantum mechanics is asserted to be a theory describing individual (microscopic and macroscopic) systems and their real properties regardless of whether the systems are open or closed and regardless of whether there is an external observer or not. That is, it is asserted that quantum mechanics provides an objective description of physical phenomena. There is no fundamental observer-system split and no reduction of the wave-packet induced by external measurements. The concept of measurement does not retain the central and fundamental status it possesses in the Copenhagen type interpretations of quantum mechanics. This philosophy is particularly interesting for quantum cosmology. In the philosophy of science the consistent histories interpretation of quantum mechanics would be called a *realistic* (but indeterministic) interpretation. According to the point of view of philosophical realism a physical theory is a description of entities, structures and processes which really exist or occur in the real physical world regardless of whether there are observers present or not. In contrast, an *instrumentalist* would say that all physical theories are in a sense phenomenological and that the purpose of a physical theory is solely to provide an economical tool for making predictions. According to this point of view the concepts and structures in a physical theory do not correspond to some real entities or structures behind the phenomena. It should be stressed, however, that a *realistic* interpretation does not necessarily interpret the theory in naive classical terms. Formulated differently: philosophical realism does neither entail determinism nor the realism of classical physics. A realist would not claim that all of reality can be described using classical terminology and classical concepts. Indeed, for instance the Kochen-Dieks interpretation [34, 35, 36, 105] of quantum mechanics is a realistic but indeterministic interpretation not using any classical pictures. In the interpretation based on the consistent histories approach to quantum mechanics the wave function (or more generally the state) is not interpreted as physical wave which is really there, but as comprising all propensities and tendencies inherent in the system in question. It is in this sense that the state can be considered real in the consistent histories approach. In contrast, the Copenhagen interpretation in its pure form is not a realistic interpretation of quantum mechanics. However, it is also not a purely instrumentalistic interpretation of quantum mechanics. In the Copenhagen type interpretation realistic and instrumentalistic point of views are mixed to a certain extend (depending on the

author). [6] An example for a purely instrumentalistic interpretation of quantum mechanics is the statistical ensemble interpretation [7].

In the consistent histories approach probabilities are thought of as *propensities* reflecting the tendency that certain events will take place or that certain properties will be realized (upon repetition).

So, one important point in favour of the consistent histories approach is that it can be used as a framework for a *realistic* interpretation of quantum mechanics. This kind of interpretation was clearly not envisaged by the fathers of quantum mechanics. But there is more to consistent histories.
The consistent histories approach incorporates radically new concepts whose introduction is, however, well motivated by standard nonrelativistic quantum mechanics. Although the consistent histories approach is accompanied by a radical change in the basic concepts of quantum theory, the mathematical framework of standard quantum mechanics and quantum field theory can be to a large extent retained. The new concepts of the consistent histories approach allow for a reassessment of several conceptual problems of quantum physics in the framework of general quantum history theories. Honestly, it must be said that the development of general quantum history theories is still in its infancy.

In standard Hamiltonian quantum mechanics the time variable is fixed from the outset as the variable conjugate to the Hamiltonian. One important new ingredient in the consistent histories scheme is the notion of *history*. In nonrelativistic quantum mechanics a history in its simplest form is simply a time sequence of events. However, in standard quantum mechanics probabilities are solely associated with events at some fixed time. In contrast, in the consistent histories approach probabilities are associated with complete histories. [7] Moreover, in general a history is a more general object than simply a sequence of single-time events. The notion of history can be generalized and general quantum history theories can be studied. The idea to investigate general quantum history theories has first been put forward by Chris Isham in Ref. [86]. Isham characterizes general quantum history theories by a list of axioms abstracted from the mathematical structure of the standard consistent histories formalism. Isham opens his first important paper on the subject [86] by remarking that a *"good illustration for the great difficulty encountered when attempting to construct a fully comprehensive account of quantum gravity is the striking lack of anything remotely resembling an 'axiomatic' scheme for such a theory. The point at stake is not one of mathematical rigour* ***per se****, but rather the need for a clear statement about what one is really trying to do. This uncertainty is reflected in a wide range of declared views on how to approach the subject and, indeed, on what would constitute a successful resolution of the problem."* In Ref. [87] he writes that

[6] There is an unhappy history of incomplete and confusing accounts of the Copenhagen interpretation. Our general reference is the excellent book by Bohm [14]. In this reference the realistic and the instrumentalistic aspects of the Copenhagen interpretation are clearly worked out. From Bohr's original writings it can be seen that Bohr has aimed to describe individual systems. His principle of complementarity is not necessarily anti-realistic. Dieks [35] and Omnès [137] have with justice stressed that the modal and the consistent histories interpretation are not in conflict with many of Bohr's ideas.

[7] This is the reason why it is meaningless to talk about reductions of the state at the times of measurements in the realm of histories.

there is *"the possibility that the decoherent histories programme could provide a framework for solving certain technical or structural problems that arise in quantum gravity. An example is the 'problem of time' that features prominently in canonical quantum gravity. One plausible conclusion from the extensive discussion of this issue is that the conventional notion of time applies only in some semi-classical limit: a conclusion that, if true, must throw doubt on the entire standard quantum formalism, depending as it does on certain* prima facie *views on the nature of time. One way of tackling this issue could be with the aid of a suitably generalised notion of a space-time history.*
However, of even greater importance perhaps is the question of whether quantum ideas should apply to space-time itself in addition to the metric or other fields that it carries. The inappropriateness of conventional quantum ideas becomes particularly apparent if one tries to develop non-continuum models of space-time involving, say, quantised point-set topologies. As with the conceptual problems of quantum cosmology, the challenge posed by issues of this type goes well beyond the question of which particular approach to quantum gravity (for example: superstring theory; canonical quantisation) is 'correct' by suggesting the need for a radical reappraisal of quantum theory itself."
I consider the present thesis a modest contribution to the research programme described by Isham.

An ardent supporter of the orthodox view will inevitably find a large part of the present thesis almost completely meaningless. Since there is no empirical criterion to decide between alternative interpretations, I think we should at least be open for new ideas. I would regard it already as a success if I can convince the orthodox-minded reader that the philosophy beyond the consistent histories approach is as good and as problematic as is the orthodox point of view. One reason why I am particularly fond of the history approach is because the interpretation of quantum mechanics based on consistent histories is to a large extent formalized and is based on only a few simple principles. Unfortunately there is no room in this thesis to illustrate this assertion appropriately. The reader should consult, e.g., the book by Omnès [137].

Perhaps some remarks to the "history" of the consistent effect histories approach are in order here. When I first tried to construct a generalization of the consistent histories approach taking into account POV measures, the result, see Ref. [151], was utterly complicated and deterrently complex. The mathematics used was by no means canonical. At that time, I thought that the complications were due to the lack of a general theory of coexistent effects in quantum mechanics and expressed my hope in Ref. [151] that with such a theory it would be possible to improve (and simplify) my approach and results. However, what was really missing was – as I now know – a theorem subsequently published by Wright and a theorem due to Foulis and Bennett. Both are stated in the appendices of this thesis. The consistent effect histories approach presented in my second paper [152] invokes Wright's theorem [171] and was accordingly much simpler. Admittedly, the definition of "admissible Boolean lattices" was somewhat artificial, but there was real progress. While I prepared this thesis I reconsidered the whole subject and looked through the literature again and fortunately I rediscovered the paper by Foulis and Bennett [47]. Their theorem can be used to remove the "last" artificial element contained in the formulation of the consistent effect histories approach in Ref. [152].

It should be stressed, however, that the approach presented in Chapter 4 of this thesis is equivalent to the approach in Ref. [152]. I think it is fair now to say that the consistent effect histories formalism as presented in this thesis is as natural and as coherent as the original formalism of the standard consistent histories approach. In any case it is so simple now that the casual reader will surely have problems to see that the results of [151] are nothing but special cases of the formulation in this thesis. It would be unduely confusing to present the original results of Ref. [151] here in detail. I shall refrain from discussing them in this thesis. The interested reader is referred to Ref. [151].

It is not the purpose of this work to discuss all issues touched upon in this introduction in full detail. The reader is referred to the references for a fuller account. The interested reader is particularly advised to consult the following original publications: [8], [9], [23], [24], [34], [35], [57], [62], [85], [86], [87], [94], [95], [96], [110], [137], [141], [168].
This thesis is organized as follows: Part I is devoted to the history formulation of standard Hilbert space quantum mechanics. Chapter 2 is devoted to a general discussion of the notion of state and observable in quantum mechanics. The general notion of observable discussed there is perhaps not too well known. In the last two sections of Chapter 2 we discuss some issues which motivate and illustrate the motivation of the consistent histories formalism which is presented in Chapter 3. We discuss the most general formulation of the consistent histories approach to quantum mechanics compatible with the notion of observable as self-adjoint operator which is already somewhat more general than the versions to be found in the literature. In Chapter 4 we present the generalized effect history approach which incorporates the generalized notion of observable put forward in Chapter 2. In Chapter 5 an attempt is made to reconcile the principles of locality and causality with the consistent histories approach. Part II is devoted to Isham's general quantum history theories which are introduced in Chapter 6. In Chapter 7 some representation and classification theorems for decoherence functionals are proven. Chapter 8 presents the summary and the conclusion. Some background material has been collected in Appendices A, B, D and E. These appendices recall some well-known facts to establish terminology, and also present some less standard results. The material presented in the Appendices A and B is used at many places in this thesis (often without further notice). Before proceeding with Chapter 2, the reader is invited to go rapidly through the Appendices A and B. The material presented in Appendix D is only used in Chapter 5.

Notations and Conventions

Throughout this work we will make use of Dirac's well-known ket and bra notation to denote vectors in Hilbert space and dual vectors in the dual Hilbert space respectively.

Throughout this work $\mathfrak{H}$ denotes some Hilbert space, $\mathcal{P}(\mathfrak{H})$ denotes the lattice of all projection operators on $\mathfrak{H}$, $\mathcal{B}(\mathfrak{H})$ denotes the set of all bounded operators on $\mathfrak{H}$ and $\mathcal{S}(\mathfrak{H})$ denotes the set of all unit vectors in the complex Hilbert space $\mathfrak{H}$. Moreover, $\mathfrak{E}(\mathfrak{H})$ denotes the set of all effect operators on $\mathfrak{H}$, i.e., the set of all Hermitean operators E on $\mathfrak{H}$ with $0 \leq E \leq 1$. The space of all positive trace class operators on $\mathfrak{H}$ with trace one will be denoted by $\mathcal{T}_1^+(\mathfrak{H})$.

PART I

STANDARD QUANTUM MECHANICS

2

OPERATIONAL QUANTUM PHYSICS

Nun sag'! Nichts weißt du, was ich dich frage:
jetzt meld' was du weißt;
denn etwas mußt du doch wissen.
[from "Parsifal" by Richard Wagner]

2.1 STATES IN QUANTUM MECHANICS

In standard Hilbert space quantum mechanics the *state* of some quantum mechanical system at time t comprises all probabilistic predictions of quantum mechanics at time t for the system in question.
Let $\mathcal{S}$ denote a quantum mechanical system with associated Hilbert space $\mathfrak{H}$. In standard quantum mechanics the set of all possible states of a quantum mechanical system is given by the set of all density operators on $\mathfrak{H}$. [1] It is often stated that the *pure states*, i.e., states of the form $\varrho = |\psi\rangle\langle\psi|$, provide the most detailed possible description of the system in question. Accordingly, such states are often referred to as states of maximum information. In contrast, *mixed* or *nonpure states*, i.e., states which are not one-dimensional projection operators, are often said to provide incomplete descriptions or to be states of less than maximal information. This is due to the fact that every density operator ϱ has a decomposition as $\varrho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ where $\{|\psi_i\rangle\}$ denotes an orthonormal system in $\mathfrak{H}$. It is at first sight tempting to interpret ϱ as a mixture of pure states $|\psi_i\rangle\langle\psi_i|$ with weights p_i. However, the decomposition of ϱ is unique if and only if ϱ has no degenerate Eigenvalue. Moreover, any density operator ϱ admits infinitely many convex decompositions into (possibly nonorthogonal) pure states. Accordingly, mixed states **do not** admit an *ignorance interpretation* contrary to what is suggested by the nomenclature. According to the ignorance interpretation of mixed states, an individual system $\mathcal{S}$ prepared in the state ϱ is actually in one of the component states $|\psi_i\rangle$ with probability p_i. That the ignorance interpretation of mixed states is problematic was already recognized by Fano [44]. [2] Hence, in any interpretation which asserts that quantum mechanics describes individual systems it is natural to treat pure and nonpure states on the same footing and not to regard pure states to be more fundamental than nonpure states. This is in accordance with Gleason's Theorem (cf. Appendix B) which can be used as another argument in favour of this assertion.

Consequently, there is a widespread agreement that states should be identified with density operators on Hilbert space. However, another logical possibility would be to adopt a more restricted notion of state according to which states are identified with rays in Hilbert space. I.e., the notion of state is restricted to what has above been called 'pure state' and the notion

[1] For simplicity we consider only systems without superselection rules.
[2] In practical experimental situations, the "ignorance interpretation" of mixed states does provide, however, an intuitive and often useful way to (naively) think about *ensembles* of systems.

of 'mixed state' is completely discarded. The price to be paid for this is that in certain situations there may be systems with which no pure state can be associated. Such situations can be easily imagined. Consider for instance a system which is part of a compound system. Assume that the compound system is described by some pure state. Then in general there will be no pure state associated with the subsystem.

The notion of *isolated system* is clearly an idealization. Real physical systems (for instance in the laboratory) are never totally isolated from their environments. However, rejecting the idea of mixed state ultimately leads to the conclusion that quantum mechanics is in general only applicable to isolated systems not interacting with their environments. If some real physical system is initially in a pure state, then due to the interaction with its environment this pure state will in general develop to a mixed state. In general only the overall time development of the system plus its environment is unitary.

The identification of states with density operators is also forced upon us by Gleason's theorem (cf. Appendix B) and by the requirement that self-adjoint operators represent quantum mechanical observables [167]. The notion of density operator plays also a central role in the theory of the decoherence process [57].

We see that the restriction of the notion of state to rays in Hilbert space is extremely unnatural. It is deeply rooted in the formalism and the structure of Hilbert space quantum mechanics that the states should be identified with general density operators. Nevertheless, it is a logically possible – although arguably artificial – point of view that only one-dimensional projection operators should be identified with states. We will not, however, adopt this latter point of view in this work. The reason why we dwell on this point is that the situation for states is analogous to the situation for observables. However, in contrast to the situation for states in the case of observables the natural notion of observable inherent in the quantum mechanical formalism is not generally accepted.

2.2 OBSERVABLES IN QUANTUM MECHANICS

The term "observable" already suggests that an observable is something which can be observed. (This terminology is appropriate in the Copenhagen interpretation of quantum mechanics, whereas in realistic interpretations the term "beable" or the term "speakable" would be more appropriate.) Every observable $\mathcal{O}$ has a certain range $\Omega_{\mathcal{O}}$ of possible values. We will not impose any restriction, whatsoever, on $\Omega_{\mathcal{O}}$. Quantum mechanics is a probabilistic theory. Hence, an observable $\mathcal{O}$ is fully determined by the specification of a σ-algebra $\mathcal{F}_{\varrho,\mathcal{O}}$ of subsets of $\Omega_{\mathcal{O}}$ and a probability measure $p_{\varrho,\mathcal{O}} : \mathcal{F}_{\varrho,\mathcal{O}} \to [0,1]$ for *every* state ϱ. It is natural to assume that $\mathcal{F}_{\varrho,\mathcal{O}}$ is independent of the state ϱ and to write $\mathcal{F}_{\mathcal{O}}$.

The positive and bounded operators E on $\mathfrak{H}$, satisfying

$$0 \leq E \leq 1,$$

are called EFFECT OPERATORS or briefly EFFECTS and the set of all effects on the Hilbert space $\mathfrak{H}$ will be denoted by $\mathfrak{E}(\mathfrak{H})$.
A GENERALIZED OBSERVABLE $\mathcal{O}$ is now a positive-operator-valued (POV) measure on some measurable space $(\Omega_\mathcal{O}, \mathcal{F}_\mathcal{O})$, i.e., a map $\mathcal{O} : \mathcal{F}_\mathcal{O} \to \mathfrak{E}(\mathfrak{H})$ with the properties:

- $\mathcal{O}(A) \geq \mathcal{O}(\emptyset)$, for all $A \in \mathcal{F}_\mathcal{O}$;
- Let $\{A_i\}$ be a countable set of disjoint sets in $\mathcal{F}_\mathcal{O}$, then $\mathcal{O}(\cup_i A_i) = \sum_i \mathcal{O}(A_i)$, the series converging ultraweakly;
- $\mathcal{O}(\Omega_\mathcal{O}) = 1$.

Generalized observables are also called EFFECT-VALUED MEASURES.

Given any state ϱ of $\mathcal{S}$, then every generalized observable $\mathcal{O}$ induces a probability measure $p_{\varrho,\mathcal{O}}$ on the measurable space $(\Omega_\mathcal{O}, \mathcal{F}_\mathcal{O})$ by

$$p_{\varrho,\mathcal{O}} : \mathcal{F}_\mathcal{O} \to [0,1], p_{\varrho,\mathcal{O}}(A) := \mathrm{tr}_\mathfrak{H}(\mathcal{O}(A)\varrho).$$

The number $p_{\varrho,\mathcal{O}}(A)$ is interpreted as the probability that the observable $\mathcal{O}$ assumes a value in the set $A \subset \Omega_\mathcal{O}$ in the state ϱ.
Conversely, let $p_\varrho : \mathcal{F} \to [0,1]$ be a probability measure on some measurable space $(\Omega, \mathcal{F})$. It is natural to assume that the map $\varrho \mapsto p_\varrho$ preserves the convex structure of the space of all states, i.e., $p_{\Sigma_i w_i \varrho_i} = \sum_i w_i p_{\varrho_i}$ whenever $0 \leq w_i \leq 1$ with $\sum_i w_i = 1$. Then $p_{|\psi\rangle\langle\psi|}(A)$ induces a unique bounded symmetric bilinear form t_A on $\mathfrak{H}$ for every $A \in \mathcal{F}$. From Riesz's theorem it follows that there exists a unique bounded operator $T(A)$ such that $p_{|\psi\rangle\langle\psi|}(A) = \mathrm{tr}_\mathfrak{H}(T(A)|\psi\rangle\langle\psi|)$. It is easy to check that the map $A \mapsto T(A)$ is indeed a positive-operator-valued measure on $(\Omega, \mathcal{F})$. Thus, we have seen that POV measures are the most general notion of observable compatible with the probabilistic structure of quantum mechanics. The argument which has led us to this conclusion is also valid if we restrict the notion of state to one-dimensional subspaces of Hilbert space (pure states).

In standard texts on quantum mechanics observables are identified with self-adjoint operators on $\mathfrak{H}$. As a consequence of the spectral theorem, these ordinary observables (associated with self-adjoint operators on $\mathfrak{H}$) are then in one-to-one correspondence with projection-operator-valued Borel measures on the real line $\mathbb{R}$, to wit, with maps $\mathcal{O}_s : \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathfrak{H})$, such that $\mathcal{O}_s(\mathbb{R}) = 1$ and $\mathcal{O}_s(\cup_i K_i) = \sum_i \mathcal{O}_s(K_i)$ for every pairwise disjoint sequence $\{K_i\}_i$ in $\mathcal{B}(\mathbb{R})$ (the series converging in the ultraweak topology). $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra of $\mathbb{R}$ and $\mathcal{P}(\mathfrak{H})$ denotes the set of projection operators on $\mathfrak{H}$. This notion of observable is usually motivated by the requirement that the *expectation value* of every observable should be real. This argument per se does not exclude more general concepts of observables as considered here. Our above discussion shows however that the notion of generalized observable is deeply rooted within Hilbert space quantum mechanics and is naturally and almost automatically forced upon us by the mathematical formalism of quantum mechanics.

Quantum mechanics is totally consistent without POV measures. Quantum mechanics is also consistent without the notion of mixed state. As discussed in the last section, the price to be

paid for the abandonment of the notion of mixed state is that in certain situations there are quantum mechanical systems with which no state can be associated. Analogously, abandoning POV measures would have the consequence that in certain measurement situations there would be no observable which actually is measured.

An example for such a measurement situation is easily imagined. This example is essentially due to Ludwig [115]. Consider a measuring device $\mathcal{M}$ consisting of a detector $\mathcal{D}$ and some test particle $\mathcal{T}$ and assume that the detector $\mathcal{D}$ is designed to measure the value of some ordinary observable $\mathcal{O}_{\mathcal{T}} : \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathfrak{H}_{\mathcal{T}})$ associated with the test particle $\mathcal{T}$. An appropriately prepared incident physical system $\mathcal{Z}$ (e.g., a particle) interacts first with $\mathcal{T}$ and then the detector $\mathcal{D}$ measures the value of the observable $\mathcal{O}_{\mathcal{T}}$ of the particle $\mathcal{T}$. To obtain the observable $\mathcal{O}_{\mathcal{Z}}$ measured by the device $\mathcal{M}$ one has to apply the unitary transformation given by the S-matrix S of the interaction between $\mathcal{Z}$ and $\mathcal{T}$ in an appropriate way to the observable measured by $\mathcal{D}$. Let $\varrho_{\mathcal{Z}}$ denote the initial state of $\mathcal{Z}$ and $\varrho_{\mathcal{T}}$ denote the initial state of $\mathcal{T}$. Then $\mathcal{O}_{\mathcal{Z}}$ can be expressed through $S, \varrho_{\mathcal{T}}$ and $\mathcal{O}_{\mathcal{T}}$ as

$$\mathcal{O}_{\mathcal{Z}} : \mathcal{B}(\mathbb{R}) \to \mathfrak{E}(\mathfrak{H}_{\mathcal{Z}}), \mathcal{O}_{\mathcal{Z}}(A) := \mathrm{tr}_{\mathfrak{H}_{\mathcal{T}}} \left((1 \otimes \varrho_{\mathcal{T}}) S^{\dagger} (1 \otimes \mathcal{O}_{\mathcal{T}}(A)) S \right),$$

where $\mathfrak{H}_{\mathcal{T}}$ denotes the Hilbert space of the test particle $\mathcal{T}$ and $\mathfrak{H}_{\mathcal{Z}}$ denotes the Hilbert space of $\mathcal{Z}$. For realistic physical S-matrices S the expression $\mathrm{tr}_{\mathfrak{H}_{\mathcal{T}}} \left((1 \otimes \varrho_{\mathcal{T}}) S^{\dagger} (1 \otimes \mathcal{O}_{\mathcal{T}}(A)) S \right)$ is in general not a projection operator but an effect operator on $\mathfrak{H}_{\mathcal{Z}}$. The observable $\mathcal{O}_{\mathcal{Z}}$ is so chosen as to conform with the following "probability invariance" requirement

$$\mathrm{tr}_{\mathfrak{H}_{\mathcal{Z}} \otimes \mathfrak{H}_{\mathcal{T}}} \left(S(\varrho_{\mathcal{Z}} \otimes \varrho_{\mathcal{T}}) S^{\dagger} (1 \otimes \mathcal{O}_{\mathcal{T}}(A)) \right) = \mathrm{tr}_{\mathfrak{H}_{\mathcal{Z}}} \left(\varrho_{\mathcal{Z}} \mathcal{O}_{\mathcal{Z}}(A) \right),$$

for all $A \subset \mathcal{B}(\mathbb{R})$.
Therefore whether a measuring device measures the value of a generalized observable or of an ordinary observable associated with some self-adjoint operator may depend on an arbitrary cut between the system and the apparatus. This argument can be formalized, see Ref. [110]. In summary, we see that if we abandon the idea that POV measures represent the observables and stick only to self-adjoint operators, then it cannot be said that in the above example of a measuring process the measuring apparatus $\mathcal{M}$ measures the value of some observable of $\mathcal{Z}$.

In the last thirty years or so many examples for POV measures have been discussed in the literature, which show that POV measures are also useful. In the sequel I will only briefly mention a few particularly interesting examples

- The problem of finding the 'phase' observable canonically conjugate to the number operator for a harmonic oscillator has a long history. A satisfactory self-adjoint operator representing 'phase' has not been found and possibly does not exist [156]. However, in terms of POV measures the problem of defining a 'phase' observable has a simple solution unifying various approaches to the problem. This can be found in Ref. [22, 23]. In this reference Busch, Grabowski and Lahti consider the following POV measure M_o on the Borel sets of $[0, 2\pi[$:

$$M_o : \mathcal{B}\left([0, 2\pi[\right) \to \mathfrak{E}(\mathfrak{H}), I \mapsto M_o(I) := \sum_{m,n} \frac{|n\rangle\langle m|}{2\pi} \int_I \exp(i(n-m)\varphi) d\varphi,$$

where $\mathfrak{H}$ denotes the Fock space of the harmonic oscillator. Busch et al. put forward the suggestion that M_o represents the appropriate quantum phase observable for the harmonic oscillator. The POV measure M_o is canonically conjugate to the number operator N in the following sense:

$$e^{iN\varphi}M_o(I)e^{-iN\varphi} = M_o(I+\varphi),$$

where $I+\varphi := \{\varphi'+\varphi \bmod 2\pi \mid \varphi' \in I\}$. The relation of M_o and various previous proposals to the quantum phase problem is discussed in detail in Ref. [22].

- The simultaneous unsharp specification of several noncommuting observables can conveniently be described with the aid of POV measures. An unsharp position-momentum observable is described in the book by Davies [30] and can also be found in Appendix E.1 of this thesis.

Many further examples and arguments in favour of POV measures can be found in the monographs by Davies [30] and Busch, Grabowski and Lahti [23] and references therein.

Summarizing our discussion in the last two sections, we have seen that the most general and natural notions of state and observable in quantum mechanics are density operators on Hilbert space and POV measures, respectively. These notions are particularly reasonable in the context of realistic interpretations of quantum mechanics. I have further briefly argued that these notions are not only compatible with quantum mechanics but are in fact useful. We will adopt these notions throughout this work.

2.3 THE PROJECTION POSTULATE

In this subsection we make a brief digression to the intricacies of the orthodoxy.

One of the central concepts in the usual formulations of the orthodox interpretation of quantum mechanics is the concept of *reduction of the wave packet* as a reaction to measurements performed on the system in question. This concept is closely related to the idea of *ideal* measurement. A measurement is said to be ideal if it is repeatable (loosely speaking an ideal measurement minimizes the disturbance of the state due to the measurement). The projection postulate can be formulated as follows: consider a quantum system in the state ϱ. Suppose an ideal measurement of the ordinary observable $\mathcal{A}$ is performed on the system and suppose that the Eigenvalue a is found upon this measurement, then the state of the system undergoes the nonunitary transformation as a response to the measurement

$$\varrho \mapsto \frac{P(a)\varrho P(a)}{\mathrm{tr}(P(a)\varrho)},$$

where $P(a)$ denotes the projection operator onto the Eigenspace corresponding to the Eigenvalue a in the spectral decomposition of $\mathcal{A}$. This is the well-known *Lüders-von Neumann*

projection postulate. It can essentially be traced back to von Neumann [127] but the formulation given here is due to Lüders [117]. The objectification problem of quantum mechanics can now reformulated as the problem of how the reduction of the state can be understood as a *physical process*.

The projection postulate can be generalized to sequential measurements. The resulting formula has seemingly been first given by Wigner [170]. Suppose that a quantum system initially in the state ϱ is exposed to a succession of ideal measurements and suppose that at time t_1 the observable $\mathcal{A}_1$ is measured with the result a_1, at time t_2 the observable $\mathcal{A}_2$ is measured with the result a_2 and so on, then the final state after the last measurement is

$$\frac{P(a_n)\cdots P(a_2)P(a_1)\varrho P(a_1)P(a_2)\cdots P(a_n)}{\mathrm{tr}(P(a_n)\cdots P(a_2)P(a_1)\varrho P(a_1)P(a_2)\cdots P(a_n))}.$$

Notice that we are using the Heisenberg picture and that accordingly the projection operators are time dependent. For notational simplicity the time dependence is suppressed.

In the consistent histories approach the projection postulate and its underlying philosophy is rejected. However, in its construction the following heuristic principle is adopted: *The results of measurement theory – in particular the Lüders-von Neumann rule for the wave function collapse caused by measurements and its generalization for sequential measurements (the Wigner formula) – have to be a* ***consequence*** *of the formalism of the consistent histories approach.*

2.4 QUANTUM PROPERTIES AND QUANTUM EVENTS

Consider propositions of the form "The value of the observable $\mathcal{O}$ is in the set $A \in \mathcal{F}_{\mathcal{O}}$." If $\mathcal{O}$ is an ordinary observable we can associate the projection operator $\mathcal{O}(A)$ with this proposition. If $\mathrm{tr}(\mathcal{O}(A)\varrho) = 1$ for some state ϱ, then we say that the proposition corresponding to $\mathcal{O}(A)$ is **true** in the state ϱ or briefly that $\mathcal{O}(A)$ is true in the state ϱ. [3] Clearly, there may be another observable $\mathcal{O}'$ such that $\mathcal{O}(A) = \mathcal{O}'(A')$ for some $A' \in \mathcal{F}_{\mathcal{O}'}$. Hence, the propositions associated with $\mathcal{O}(A)$ and $\mathcal{O}'(A')$ can only simultaneously be true. This observation has led von Neumann to the notion of *property* of a quantum mechanical system. According to von Neumann, there is a one-to-one correspondence between properties of quantum mechanical systems and projection operators on Hilbert space. Birkhoff and von Neumann have studied in their classic work [12] the structure of the space of all projection operators of a quantum system. [4] They proved that this space carries the structure of a nondistributive and hence non-Boolean lattice; for more details see Appendix A.4.

In classical physics meaningful propositions about a system are typically of a similar form: "The value of the observable $\mathcal{A}$ is in the set A," for some appropriately chosen subset A of

[3] We will discuss the meaning of the notion of truth briefly but in more depth in the next chapter.

[4] Notice, that the original terminology in von Neumann [127] and Birkhoff and von Neumann [12] is in the spirit of the Copenhagen interpretation. I will not use it in toto.

$\mathbb{R}$. Since observables in classical physics are real-valued functions on phase space $\mathfrak{P}$, we can associate with every meaningful proposition the subset

$$\mathfrak{P}(\mathcal{A} \in A) := \{\mathfrak{p} \in \mathfrak{P} \mid \mathcal{A}(\mathfrak{p}) \in A\}$$

of phase space. In contrast to the quantum case, the set of all propositions about a classical physical system is isomorphic to the space of all subsets of the classical phase space, specifically it constitutes a Boolean lattice. [5] Birkhoff and von Neumann argued that the characteristic feature which distinguishes the quantum mechanical propositional calculus from the classical propositional calculus is the breakdown of the distributive law in the quantum case. Indeed, as far as I can see, all paradoxa in quantum mechanics which are not concomitants of the objectification problem or of the "quantum entanglement" problem [88] can be ultimately traced back to the nondistributivity of the lattice of all projection operators on Hilbert space.

The Hilbert lattice of all projection operators does not satisfy all requirements which might intuitively be associated with a space of properties. This has been pointed out, e.g., by Ludwig [116]. The situation is even worse, when we consider the set of propositions about a quantum mechanical system and take into account the full set of generalized observables as introduced in the last section. In this case the space of all "properties" of a quantum system is isomorphic to the space of all effect operators on Hilbert space. The latter space carries the structure of a D-poset, see Appendix A. The use of the term "property" with regard to effect operators is absolutely counterintuitive. I shall thus refrain from using such a terminology. Instead, I will use a different terminology better suited to the structure of the space of effect operators and a realistic attitude.

Two propositions are said to be *equivalent* if they correspond to the same effect operator. Hence, there is a one-to-one correspondence between effect operators on Hilbert space and equivalence classes of propositions. I shall say that the equivalence classes of propositions represent the possible *events* which may occur in the physical system in question. The idealized notion of *real event* as proposed here describes irreversible transitions from the possible to the actual. The occurrence of individual events in the real world is postulated here. The notion of event is clearly an idealization, but hopefully a useful one. The notion of event is not assumed to be restricted to the macroscopic realm. Quantum events are assumed to take place also at a microscopic scale. Accordingly, quantum mechanics is viewed as a fundamentally stochastic theory describing the evolving pattern of events taking place in the real world. This evolutionary picture of physics has been recently put forward by Haag [71, 72]. According to Haag, a *real event* is the appropriate notion to replace the notion of "measurement result" in a realistic approach. A history is a sequence of causally tied events. It is appropriate to quote some lines from Ref. [72]: "There is an evolving pattern of events. At any stage the past consists of the part which has been realized, the future is open and allows possibilities for new events. Altogether we have a growing graph or, using another mathematical language, a growing category whose objects are the events and whose (directed) arrows are the causal links. We shall assume further that the relation to space-time is provided by the events.

[5] I do neither discuss here the question of whether some equivalence relation on the space of propositions should be taken into account nor how such equivalence relation should be chosen.

Each event marks roughly a region in space-time, the extension and sharpness of which depends on the nature of the event."

The notion of event as proposed above is also closely related to the earlier proposal by Griffiths [66]. While the philosophy beyond my proposal is essentially taken over from Ref. [71], the new ingredient here is my suggestion that the events in standard Hilbert space quantum mechanics are represented by the effect operators on Hilbert space. This proposal is inspired by the notion of generalized observable, see Section 2.2.

We will adopt the evolutionary point of view particularly in Chapter 5 in our attempt to build a consistent histories scheme compatible with Einstein causality. We anticipate already here that we deviate somewhat from Haag's original proposal in order to reconcile his evolutionary picture with the basic assumptions of the consistent histories approach. Specifically, we will assume that there are several alternative but equally objective evolving patterns of events. But this will be discussed in more detail later.

3

CONSISTENT HISTORIES

Der alte Weltenmeister
Der ohne Namen ist; der gleichfalls untertan
Uraltem Wort am Rand der Ewigkeit.
Er schafft sein Werk wie du das deine,
Er schmiedet Ringe sich, Figuren, Steine
Zu der schimmernden Kette der Zeiten
Der Weltbegebenheiten
[from "Palestrina" by Hans Pfitzner]

3.1 STANDARD CONSISTENT HISTORIES

3.1.1 Homogeneous Histories

We consider a quantum mechanical system $\mathcal{S}$ without superselection rules represented by a separable complex Hilbert space $\mathfrak{H}$ and a Hamiltonian operator H. Every physical state of the considered system is mathematically represented by a density operator on $\mathfrak{H}$, i.e., a linear, positive, trace-class operator on $\mathfrak{H}$ with trace 1. We denote the set of all trace-class operators on $\mathfrak{H}$ by $\mathcal{T}(\mathfrak{H})$ and the set of all density operators on $\mathfrak{H}$ by $\mathcal{T}(\mathfrak{H})_1^+$. The time evolution is governed by the unitary operator $U(t',t) = \exp(-i(t'-t)H/\hbar)$ which maps states at time t into states at time t' and satisfies $U(t'',t')U(t',t) = U(t'',t)$ and $U(t,t) = 1$.

In standard nonrelativistic quantum mechanics the observables are identified with PV Borel measures on the real line and, accordingly, to every sequence of measurement outcomes there corresponds a sequence of projection operators. Led by the heuristic principle stated in Section 2.3, the basic idea in the consistent histories approach is to abstract such *histories* from their concrete realization as a sequence of measurement outcomes and to think of histories as independent entities in their own right. Histories are then, loosely speaking, sequences of projection operators on $\mathfrak{H}$. This idea can be formalized as follows.

Definition 1 *A* HOMOGENEOUS HISTORY *is a map* $h : \mathbb{R} \to \mathcal{P}(\mathfrak{H}), t \mapsto h_t$. *We call* $t_i(h) := \inf(t \in \mathbb{R} \cup \{-\infty, \infty\} \mid h_t \neq 1)$ *the* INITIAL *and* $t_f(h) := \sup(t \in \mathbb{R} \cup \{-\infty, \infty\} \mid h_t \neq 1)$ *the* FINAL TIME *of* h, *respectively. Furthermore, the* SUPPORT OF h *is given by* $\mathfrak{s}(h) := \{t \in \mathbb{R} \mid h_t \neq 1\}$. *If* $\mathfrak{s}(h)$ *is finite, countable or uncountable, then we say that* h *is a* FINITE, COUNTABLE *or* UNCOUNTABLE HISTORY *respectively. The space of all homogeneous histories will be denoted by* $\mathcal{H}(\mathfrak{H})$, *the space of all finite homogeneous histories by* $\mathcal{H}_{fin}(\mathfrak{H})$ *and the space of all finite homogeneous histories with support* S *by* $\mathcal{H}_S(\mathfrak{H})$.

In this work we focus attention on finite histories. Infinite or even continuous histories are much more difficult to handle, see [91]. In the following we will identify every homogeneous history h with the string of its nontrivial projection operators, i.e., we write $h \simeq \{h_{t_k}\}_{t_k \in \mathfrak{s}(h)}$.

Furthermore, to every finite homogeneous history $h \in \mathcal{H}_{fin}(\mathfrak{H})$ we associate its CLASS OPERATOR WITH RESPECT TO THE FIDUCIAL TIME t_0

$$\begin{aligned} C_{t_0}(h) &:= U(t_0,t_n)h_{t_n}U(t_n,t_{n-1})h_{t_{n-1}}...U(t_2,t_1)h_{t_1}U(t_1,t_0) && (3.1)\\ &= U(t_0,t_i(h))h_{t_n}(t_n)h_{t_{n-1}}(t_{n-1})...h_{t_1}(t_1)U(t_i(h),t_0), && (3.2)\end{aligned}$$

where we have defined the Heisenberg picture operators

$$h_{t_k}(t_k) := U(t_k,t_i(h))^\dagger h_{t_k}U(t_k,t_i(h))$$

with respect to the initial time $t_i(h)$ of h.

The following definition is motivated by Wigner's formula and the heuristic principle mentioned in Section 2.3.

Definition 2 *Let the state of a quantum mechanical system at time t_0 be given by the density operator $\varrho(t_0)$. For every pair h and k of finite homogeneous histories we define the* DECOHERENCE WEIGHT OF h AND k *by*

$$\mathsf{d}_\varrho(h,k) := \mathrm{tr}\left(C_{t_0}(h)\varrho(t_0)C_{t_0}(k)^\dagger\right). \tag{3.3}$$

The functional $\mathsf{d}_\varrho : \mathcal{H}_{fin}(\mathfrak{H}) \times \mathcal{H}_{fin}(\mathfrak{H}) \to \mathbb{C}, (h,k) \mapsto \mathsf{d}_\varrho(h,k)$ will be called the HOMOGENEOUS DECOHERENCE FUNCTIONAL ASSOCIATED WITH THE STATE ϱ.

In view of the heuristic principle stated in Section 2.3 it is natural to attempt to interpret the value $d_\varrho(h,h)$ as the "probability" of the homogeneous history h. The problem to be addressed in this section is the problem of whether this interpretation makes sense, to wit, the problem of how and in what sense d_ϱ can be extended to a probability functional.

Following Isham [86], we will proceed in several steps. The first question to be addressed is the problem of what the appropriate mathematical representations of the grammatical connectives "and", "or" and "not" in the realm of histories are, so that under appropriate circumstances we can talk about propositions like "the history h is realized or (and) the history k is realized" etc.

In the second step (see Theorem 1), the functional d_ϱ will be extended to the space of generalized ("composed") history propositions, since as it stands, say, the expression $\mathsf{d}_\varrho(h \text{ or } k, h \text{ or } k)$ is not defined. Unfortunately (or fortunately ?), we will see that the resulting extension d_ϱ is not additive, i.e., in general one has $d_\varrho(h \text{ or } k, h \text{ or } k) \neq \mathsf{d}_\varrho(h,h) + \mathsf{d}_\varrho(k,k)$.

Thus, in the last step we formulate the necessary and sufficient condition a set of histories has to satisfy in order that d_ϱ induces a probability measure on it, see Section 3.1.3. Only for those sets $\mathcal{C}$ of histories satisfying the *consistency condition* the interpretation of the number $d_\varrho(h,h)$ as probability of the history $h \in \mathcal{C}$ is justified.

3.1.2 Inhomogeneous Histories

For every finite subset S of $\mathbb{R}$ we can consider the Hilbert tensor product $\otimes_{t\in S}\mathfrak{H}$ and the algebra $\mathcal{B}^{\otimes}_S(\mathfrak{H})$ of bounded linear operators on $\otimes_{t\in S}\mathfrak{H}$. It was pointed out by Isham [86] that for any fixed S there is an injective (but not surjective) correspondence σ_S between finite histories with support S and elements of $\mathcal{B}^{\otimes}_S(\mathfrak{H})$ given by

$$\sigma_S : \mathcal{H}_S(\mathfrak{H}) \to \mathcal{B}^{\otimes}_S(\mathfrak{H}), h \simeq \{h_{t_k}\}_{t_k\in S} \mapsto \otimes_{t_k\in S} h_{t_k}\,. \tag{3.4}$$

The finite homogeneous histories with support S can therefore be identified with projection operators on $\otimes_{t\in S}\mathfrak{H}$. Hence, without the risk of confusion, we will almost everywhere in this work briefly write h instead of $\sigma_S(h)$. The set of all projection operators on $\otimes_{t\in S}\mathfrak{H}$ will in the sequel be denoted by $\mathcal{P}^{\otimes}_S(\mathfrak{H})$. However, not all projection operators in $\mathcal{P}^{\otimes}_S(\mathfrak{H})$ have the form $\sigma_S(h)$ for some $h \in \mathcal{H}_S(\mathfrak{H})$.
If a homogeneous history vanishes for some $t_0 \in \mathbb{R}$, i.e., $h_{t_0} = 0$, then we say that h is a ZERO HISTORY. All zero histories are collectively denoted by 0, slightly abusing the notation.

Definition 3 *Let $h,k \in \mathcal{H}(\mathfrak{H})$. We say that k is* COARSER THAN *h if $h_t \leq k_t$ for all $t \in \mathbb{R}$ and write $h \leq k$. If furthermore $h \neq k$, then we write $h < k$. The set $\mathcal{H}(\mathfrak{H})$ equipped with the relation $\leq$ is a partially ordered set.*

Definition 4 *Two homogeneous histories h and k are said to be* DISJOINT *if there is some $t \in \mathbb{R}$ such that $h_t k_t = 0$.*

The identification of finite homogeneous histories with support S with projection operators on $\otimes_{t\in S}\mathfrak{H}$ allows for the introduction of a much broader class of histories. To this end we recall the well-known fact that the set $\mathcal{P}(\mathfrak{H})$ of projection operators on a Hilbert space $\mathfrak{H}$ carries the structure of an orthocomplemented complete lattice, see Appendix A.4.

Definition 5 *Let S be a finite subset of $\mathbb{R}$, then we call the space $\mathcal{K}_S(\mathfrak{H}) := \mathcal{P}^{\otimes}_S(\mathfrak{H})$ of projection operators on $\otimes_{t\in S}\mathfrak{H}$ the* SPACE OF FINITE INHOMOGENEOUS HISTORIES WITH SUPPORT *S. The direct limit of the directed system $\{\mathcal{P}^{\otimes}_S(\mathfrak{H}) \mid S \subset \mathbb{R}$ finite$\}$ will be called* THE SPACE OF ALL FINITE INHOMOGENEOUS HISTORIES WITH ARBITRARY SUPPORT *and will be denoted by $\mathcal{K}_{fin}(\mathfrak{H})$ or by $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$.*

The definition of the direct limit of a directed system of D-posets can be found in Appendix A.2.4. The lattice operations on $\mathcal{P}^{\otimes}_S(\mathfrak{H})$ induce corresponding operations on the finite homogeneous histories in $\mathcal{H}_{fin}(\mathfrak{H})$, which are explicitly described in the following remarks. These operations are the sought mathematical representations of the grammatical connectives "and", "or" and "not" in the history formalism.

Remark 1 *Let $h,k \in \mathcal{H}_{fin}(\mathfrak{H})$ be two finite homogeneous histories, then the* JOIN *$h \vee k$ of h and k is defined to be the unique finite history with support $\mathfrak{s}(h) \cup \mathfrak{s}(k)$ which is represented in $\mathcal{P}^{\otimes}_{\mathfrak{s}(h)\cup\mathfrak{s}(k)}(\mathfrak{H})$ by $\left(\otimes_{t_i\in\mathfrak{s}(h)} h_{t_i}\right) \bigvee \left(\otimes_{s_j\in\mathfrak{s}(k)} k_{s_j}\right)$. The history $h \vee k$ is in general not*

a homogeneous but an inhomogeneous history. The JOIN $\bigvee_j h_j$ *of any finite sequence* $\{h_j\}$ *of pairwise disjoint homogeneous histories is analogously defined to be the unique finite history with support* $\bigcup_j \mathfrak{s}(h_j)$ *which is represented in* $\mathcal{P}^{\otimes}_{\cup_j \mathfrak{s}(h_j)}(\mathfrak{H})$ *by* $\bigvee_j \left(\otimes_{t_i \in \mathfrak{s}(h_j)} h_{t_i}\right)$.

Remark 2 *Let* $h, k \in \mathcal{H}_{fin}(\mathfrak{H})$ *be two finite homogeneous histories, then the* MEET $h \wedge k$ *of* h *and* k *satisfies that* $(h \wedge k)_t := h_t \wedge k_t$ *is the projection operator on the intersection of the ranges of* h_t *and* k_t *for all* $t \in \mathbb{R}$. *The meet operation maps pairs of finite homogeneous histories to a finite homogeneous history.*

Remark 3 *Let* h *be a finite homogeneous history with support* $\mathfrak{s}(h)$, *then* $\neg h$ *is the unique history with support* $\mathfrak{s}(h)$ *which in* $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ *is represented by* $1 - \bigotimes_{t \in \mathfrak{s}(h)} h_t$. *We call* $\neg h$ *the* NEGATION *of* h. *The negation* $\neg h$ *of a finite homogeneous history* h *may be inhomogeneous. Obviously the negation satisfies* $h \vee \neg h = 1$ *and* $h \wedge \neg h = 0$. *It is clear that* $\neg h$ *is uniquely determined by these two conditions.*

Lemma 1 *Let* S *be a finite subset of* $\mathbb{R}$, *then the set* $\mathcal{K}_S(\mathfrak{H})$ *is an orthocomplemented complete lattice.*

Remark 4 *The join, meet and orthocomplementation operations on* $\mathcal{K}_S(\mathfrak{H})$ *(where* S *is a finite subset of* $\mathbb{R}$*) and on* $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ *are denoted by the same symbols (slightly abusing the notation).*

In [86, 128] it is explained how to imbed $\mathcal{K}_S(\mathfrak{H})$ into an infinite tensor product of operator algebras and how to furnish the latter with a Hilbert lattice structure.

Definition 6 *Two (possibly inhomogeneous) finite histories* h *and* k *are said to be* DISJOINT *if* $h \leq \neg k$, *where* $\leq$ *is the partial order on* $\mathcal{K}_{\mathfrak{s}(h) \cup \mathfrak{s}(k)}(\mathfrak{H})$. *We write* $h \perp k$.

Lemma 2 *Let* h *and* k *denote two disjoint finite histories, then* $h \wedge k = 0$.

Remark 5 *For every finite* $S \subset \mathbb{R}$ *the meet, join and orthocomplementation operations on* $\mathcal{P}^{\otimes}_{S}(\mathfrak{H})$ *induce a meet, join and an orthocomplementation operation on* $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ *respectively which will be denoted by the same symbols.*

Definition 7 *Let* $\mathcal{A}$ *denote a finite collection* $\{h_k\}$ *of histories in* $\mathcal{K}_{fin}(\mathfrak{H})$. *Then* $\mathcal{A}$ *is said to be* DISJOINT *if each pair of histories in* $\mathcal{A}$ *is disjoint.* $\mathcal{A}$ *is said to be* COMPLETE *if* $\bigvee_k h_k = 1$.

Let $S \subset \mathbb{R}$. A history $h \in \mathcal{H}_S(\mathfrak{H})$ is called a SIMPLE HISTORY if h_t is a projection operator on a one dimensional subspace of $\mathfrak{H}$ for every $t \in S$. Denote the set of finite inhomogeneous

histories which can be generated from the set of simple homogeneous histories in $\mathcal{H}_S(\mathfrak{H})$ by the application of a finite number of $\vee$ operations by $\mathcal{F}_S(\mathfrak{H})$. The class operators can be extended to $\mathcal{F}_S(\mathfrak{H})$ by requiring that C_{t_0} is additive in the following sense

$$C_{t_0}(h \vee k) := C_{t_0}(h) + C_{t_0}(k) \text{ whenever } h \perp k \tag{3.5}$$

$$C_{t_0}(\neg h) := 1 - C_{t_0}(h). \tag{3.6}$$

These definitions are compatible with the lattice theoretical identities $\neg(h \vee k) = (\neg h) \wedge (\neg k)$ and $\neg(h \wedge k) = (\neg h) \vee (\neg k)$. Notice, that Equation 3.6 is a consequence of Equation 3.5. The fiducial time t_0 can be chosen completely arbitrary.

Using Equation 3.5, the homogeneous decoherence functional can in an obvious way be extended to the set $\mathcal{F}_S(\mathfrak{H})$.

However, a much stronger result is true. The homogeneous decoherence functional d_ϱ from Definition 2 can be extended to a functional defined for arbitrary pairs of inhomogeneous histories. Moreover, this extension is σ-orthoadditive with respect to the join operation in $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$.
To see this, let S be a finite subset of $\mathbb{R}$ and define $\mathfrak{H}(S) := \otimes_{t\in S}\mathfrak{H}$. Isham, Linden and Schreckenberg [93] have proven that for every ϱ there exists a unique trace class operator $\mathfrak{X}_\varrho$ on $\mathfrak{H}(S) \otimes \mathfrak{H}(S)$ such that d_ϱ can be written as

$$\mathsf{d}_\varrho(h,k) = \mathrm{tr}_{\mathfrak{H}(S)\otimes\mathfrak{H}(S)}(h \otimes k\mathfrak{X}_\varrho),$$

provided $\mathfrak{s}(h) \subset S$ and $\mathfrak{s}(k) \subset S$. A proof for this assertion can be found in Appendix C. Thus we have

Theorem 1 *The homogeneous decoherence functional* $\mathsf{d}_\varrho : \mathcal{H}_{fin}(\mathfrak{H}) \times \mathcal{H}_{fin}(\mathfrak{H}) \to \mathbb{C}, (h,k) \mapsto \mathsf{d}_\varrho(h,k)$ *can be uniquely extended to a* DECOHERENCE FUNCTIONAL $d_\varrho : \mathcal{P}^{\otimes}_{fin}(\mathfrak{H}) \times \mathcal{P}^{\otimes}_{fin}(\mathfrak{H}) \to \mathbb{C}$. *Let* h, h' *and* k *denote finite histories. The decoherence functional* d_ϱ *satisfies for all* $h, h', k \in \mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$

- $d_\varrho(h,h) \in \mathbb{R}$ *and* $d_\varrho(h,h) \geq 0$.
- $d_\varrho(h,k) = d_\varrho(k,h)^*$.
- $d_\varrho(1,1) = 1$.
- $d_\varrho(h \vee h', k) = d_\varrho(h,k) + d_\varrho(h',k)$, *whenever* $h \perp h'$.
- $d_\varrho(0,h) = 0$, *for all* h.

The proof for the first item $d_\varrho(h,h) \geq 0$ and for the second item $d_\varrho(h,k) = d_\varrho(k,h)^*$ can be found in Appendix C. It is clear that d_ϱ is also σ-orthoadditive on $\mathcal{P}^{\otimes}_S(\mathfrak{H})$ for every finite $S \subset \mathbb{R}$.

Now fix h and vary ϱ in $p_\varrho(h) := d_\varrho(h,h)$. Then $p_\varrho(h)$ is a positive, linear, bounded functional on $\mathcal{T}(\mathfrak{H})_1^+$. Therefore it follows from Gleason's theorem that there exists a unique effect $F(h) \in \mathfrak{E}(\mathfrak{H})$ such that

$$d_\varrho(h,h) = \mathrm{tr}[\varrho F(h)].$$

Obviously

$$F(h) = C_{t_0}(h)^\dagger C_{t_0}(h).$$

We call $F(h)$ the effect associated with the finite history h. The map $F : \mathcal{H}_{fin}(\mathfrak{H}) \to \mathfrak{E}(\mathfrak{H})$ is many-to-one and therefore information about h is lost when considering $F(h)$ instead of h.

3.1.3 Consistent Sets of Histories

Definition 8 *Let h and k be two disjoint histories in $\mathcal{P}^\otimes_{fin}(\mathfrak{H})$. The histories h and k are said to be* PRECONSISTENT WITH RESPECT TO THE STATE ϱ *if* $\mathrm{Re}\, d_\varrho(h,k) = 0$. *Any collection $\mathcal{C}$ of histories in $\mathcal{P}^\otimes_{fin}(\mathfrak{H})$ is said to be* PRECONSISTENT WITH RESPECT TO THE STATE ϱ *if every pair of disjoint histories in $\mathcal{C}$ is preconsistent with respect to the state ϱ.*
Let $\mathcal{C}'$ be a Boolean lattice of histories in $\mathcal{P}^\otimes_{fin}(\mathfrak{H})$ with respect to the meet, join and orthocomplementation induced from $\mathcal{P}^\otimes_{fin}(\mathfrak{H})$ (see Remark 7) such that $0_{\mathcal{C}'} = 0 \in \mathcal{P}^\otimes_{fin}(\mathfrak{H})$. The unit in $\mathcal{C}'$ will be denoted by $1_{\mathcal{C}'}$. Such a Boolean lattice $\mathcal{C}'$ of histories in $\mathcal{P}^\otimes_{fin}(\mathfrak{H})$ is said to be CONSISTENT WITH RESPECT TO THE STATE ϱ *if $\mathcal{C}'$ is preconsistent with respect to the state ϱ.*

The reader may wonder what is meant by the "zero" and the "unit" in $\mathcal{P}^\otimes_{fin}(\mathfrak{H})$. Technically, $\mathcal{P}^\otimes_{fin}(\mathfrak{H})$ may be thought of as the direct limit of the directed set $\{\mathcal{P}^\otimes_S(\mathfrak{H}) | S \subset \mathbb{R}, S \text{ finite }\}$. According to Appendix A.2.3 this direct limit carries the structure of a D-poset (in fact, it also carries the structure of a lattice) and it is understood that the zero history and the unit history of $\mathcal{P}^\otimes_{fin}(\mathfrak{H})$ are the zero and the unit in this D-poset structure, respectively.

The condition $\mathrm{Re}\, d_\varrho(h,k) = 0$ is often expressed in physical terms by saying that the events h and k have vanishing interference in the state ϱ.
The notion of consistency is important because it is the key to a probability interpretation of the numbers $d_\varrho(h,h)$ for some (pre-)consistent sets of histories. The Definition 8 of a consistent set of histories is the minimal necessary requirement that d_ϱ induces a probability measure on the consistent set in question as we will discuss more fully below.
Note that our above terminology in Definition 8 differs somewhat from the terminology used by other authors. Further, some authors discuss more severe conditions. These authors call a pair h, k of histories *weakly decoherent* if it satisfies $\mathrm{Re}\, d_\varrho(h,k) = 0$ and *mediumly decoherent* if it satisfies $d_\varrho(h,k) = 0$. Recently, Kent [102] has argued in favour for a new criterion, *ordered consistency*, which excludes some counterintuitive retrodictive conclusions otherwise derivable in the consistent histories formalism [101] (it is important, however, to stress that these counterintuitive conclusions (in the terminology to be introduced below in Section 3.2) can only be derived in different incompatible frameworks [26, 27, 28]).
There can be found other related notions of decoherence and consistency in the literature,

see, e.g., [53, 45, 57]. However, since Definition 8 represents the mathematically minimal requirement we shall stick to it and will not consider the stronger conditions. Let us recall that usually a probability space is defined to be a triple $(\Omega, \mathcal{A}, p)$, where Ω is an arbitrary set, $\mathcal{A}$ is a Boolean σ-algebra of subsets of Ω and p is a probability measure on $\mathcal{A}$. This can be generalized as follows

Definition 9 *Let $\mathcal{L}$ be a partially ordered set and $\mathcal{B} \subset \mathcal{L}$ be a Boolean lattice. A nonnegative valuation $m : \mathcal{B} \to \mathbb{R}^+$ on $\mathcal{B}$ which is additive*

$$m\left[\bigvee_{k=1}^{N} \alpha_k\right] = \sum_{k=1}^{N} m[\alpha_k], \text{ if } \alpha_k \wedge \left(\bigvee_{i=1}^{k-1} \alpha_i\right) = 0, \text{ for every } k < N,$$

is called a FINITE MEASURE ON $\mathcal{B}$. *If $\mathcal{B}$ is a Borel lattice, then N may be taken to be ∞. In this case m is σ-additive. If $\mathcal{B}$ is not a Borel lattice, then N is always finite. If furthermore $m[1_{\mathcal{B}}] = 1$, then m is called a* PROBABILITY MEASURE ON $\mathcal{B}$ *and the triple $(\mathcal{L}, \mathcal{B}, m)$ is called a* PROBABILITY LATTICE.

A Borel lattice is a Boolean σ-lattice [11].

Theorem 2 *Let $\mathcal{C} \subset \mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ be a Boolean lattice. If $\mathcal{C}$ is preconsistent with respect to the state ϱ, then the triple $(\mathcal{P}^{\otimes}_{fin}(\mathfrak{H}), \mathcal{C}, p_\varrho)$ is a probability lattice, where p_ϱ is defined by*

$$p_\varrho : \mathcal{C} \to \mathbb{R}^+, p_\varrho(h) := \frac{d_\varrho(h,h)}{d_\varrho(1_{\mathcal{C}}, 1_{\mathcal{C}})}. \tag{3.7}$$

The proof is straightforward.

In the literature it is often tacitly assumed that the preconsistent set of histories under consideration forms (or generates) a Boolean lattice so that a probability interpretation of the diagonal values of the decoherence functional makes sense.
The probability defined by Equation 3.7 can for finite homogeneous histories be interpreted as conditional probability, namely as the probability of the sequence of the propositions $h_{t_f} = h_{t_k}, ..., h_{t_{k-j}}$ given that the sequence of propositions $h_{t_{k-j-1}}, ..., h_{t_0}$ is realized.

Lemma 3 *Let $(\mathcal{P}^{\otimes}_{fin}(\mathfrak{H}), \mathcal{C}, p_\varrho)$ be the probability lattice from Theorem 2, where p_ϱ is defined by Equation 3.7, then for all $h, k \in \mathcal{C}$*

- $0 \leq p_\varrho(h) \leq 1$.
- $p_\varrho(h \vee k) + p_\varrho(h \wedge k) = p_\varrho(h) + p_\varrho(k)$.

- $p_\varrho(h) \leq p_\varrho(k)$ *whenever* $h \leq k$.

Corollary 1 *Let* $\mathcal{C} \subset \mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ *be a Boolean lattice. Then* $\mathcal{C}$ *is a preconsistent set of histories w.r.t. the state* ϱ *if and only if* $0_{\mathcal{C}} = 0 \in \mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ *and if every pair* h, k *of histories in* $\mathcal{C}$ *satisfies*

$$d_\varrho(h \vee k, h \vee k) + d_\varrho(h \wedge k, h \wedge k) = d_\varrho(h, h) + d_\varrho(k, k). \tag{3.8}$$

Remark 6 *We notice that* d_ϱ *induces also probability functionals on sets of histories which are not Boolean lattices. Let* $\mathcal{C}$ *be a preconsistent set of pairwise disjoint histories, then* $m_\varrho : \mathcal{C} \to \mathbb{R}^+, m_\varrho(h) := d_\varrho(h,h) / \left(\sum_{k \in \mathcal{C}} d_\varrho(k,k)\right)$ *is an additive functional on* $\mathcal{C}$ *and* $m_\varrho(h)$ *can be interpreted as probability of* $h \in \mathcal{C}$. *However, since* $\mathcal{C}$ *generates a Boolean sublattice of* $\mathcal{K}_{fin}(\mathfrak{H})$ *on which* d_ϱ *induces a probability measure extending* m_ϱ, *it is enough to consider Boolean algebras of histories.*

3.1.4 Relatively Consistent Sets of Histories

In Definition 8 the unit $1_{\mathcal{C}}$ in a consistent Boolean lattice of histories $\mathcal{C}$ was allowed to differ from the unit in $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$. This provides a sensible framework for a description of a quantum system only when it is known or assumed a priori that the history $1_{\mathcal{C}}$ is true or realized. Hence, all probabilities appearing in the last section have to be understood as relative to the history $1_{\mathcal{C}}$, which may of course also be chosen as the unit in $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$.

In this subsection we briefly consider the "converse" situation that there is some history which is known or assumed to be **not** realized. To wit, we must allow for a more general notion of consistent Boolean lattice of histories also including Boolean lattices $\mathcal{C}$ for which the zero history $0_{\mathcal{C}}$ differs from $0 \in \mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$.

Definition 10 *Any collection* $\mathcal{C}$ *of histories in* $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ *is said to be* RELATIVELY CONSISTENT WITH RESPECT TO THE STATE ϱ *if* $\mathcal{C}$ *is a Boolean algebra (with respect to the meet, join and orthocomplementation in* $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ *(see Remark 7) and with unit* $1_{\mathcal{C}}$ *and zero* $0_{\mathcal{C}}$*) and if the following* ***relative consistency condition*** *is satisfied for all* $h, k \in \mathcal{C}$

$$\text{Re } d_\varrho(h \ominus 0_{\mathcal{C}}, k \ominus 0_{\mathcal{C}}) = 0 \quad \text{whenever } h \wedge k = 0_{\mathcal{C}}, \tag{3.9a}$$

$$d_\varrho(h, h) \geq d_\varrho(0_{\mathcal{C}}, 0_{\mathcal{C}}), \tag{3.9b}$$

with strict inequality in Equation 3.9b *if* $h = 1_{\mathcal{C}}$.

The set $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ carries a D-poset structure and in Definition 10 the symbol $\ominus$ denotes this D-poset structure.

Theorem 3 *Let $\mathcal{C} \subset \mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ be a Boolean lattice. If $\mathcal{C}$ is relatively consistent with respect to the state ϱ, then the triple $(\mathcal{P}^{\otimes}_{fin}(\mathfrak{H}), \mathcal{C}, p_\varrho)$ is a probability lattice, where p_ϱ is defined by*

$$p_\varrho : \mathcal{C} \to \mathbb{R}^+, p_\varrho(h) := \frac{d_\varrho(h,h) - d_\varrho(0_\mathcal{C}, 0_\mathcal{C})}{d_\varrho(1_\mathcal{C}, 1_\mathcal{C}) - d_\varrho(0_\mathcal{C}, 0_\mathcal{C})}. \tag{3.10}$$

The proof is straightforward.

Corollary 2 *Let $\mathcal{C} \subset \mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ be a Boolean lattice. Then $\mathcal{C}$ is a relatively consistent set of histories w.r.t. the state ϱ if and only if every pair h, k of histories in $\mathcal{C}$ satisfies*

$$d_\varrho(h \vee k, h \vee k) + d_\varrho(h \wedge k, h \wedge k) = d_\varrho(h,h) + d_\varrho(k,k) \tag{3.11}$$

and $d_\varrho(h,h) \geq d_\varrho(0_\mathcal{C}, 0_\mathcal{C})$ for all $h \in \mathcal{C}$ with strict inequality if $h = 1_\mathcal{C}$.

3.2 THE CONSISTENT HISTORIES INTERPRETATION

... è un facile vangelo
Che fa la vita vaga
Ma che intristisce il cor
[from "Madama Butterfly" by Giacomo Puccini/Giacosa/Illica]

The interpretation of quantum mechanics based on the consistent histories approach has been developed by Robert Griffiths and Roland Omnès. In this subsection I briefly summarize the most important aspects of the version of this interpretation adopted in this thesis. The version presented here differs in some minor points from the original expositions by Omnès and by Griffiths (which differ themselves in some aspects). And it differs slightly from the version given in [151]. Of course, I do not claim that Omnès and Griffiths will necessarily fully agree with my presentation.

At the heart of the consistent histories interpretation lies the following philosophical maxim: Whether some assertion or proposition is meaningful or not depends upon the context and the framework into which the assertion is placed. Accordingly, one has always be careful not only to specify the assertion itself but also its context. Propositions are always contextual. In the consistent histories approach this principle is carefully obeyed. According to the consistent histories approach in quantum mechanics probabilistic predictions and state histories are only meaningful with respect to a consistent set of histories. *Per se* state histories and probabilistic predictions have no meaning.

The consistent histories interpretation of quantum mechanics is a realistic interpretation of quantum mechanics. Quantum mechanics is asserted to be a theory describing individual systems regardless of whether they are open or closed and regardless of whether they are

observed or not. The basic ingredients in the formalism of the consistent histories approach are the space of histories on the one hand and the space of decoherence functionals on the other hand.

In standard Hilbert space quantum mechanics the state of some quantum mechanical system comprises all probabilistic predictions of quantum mechanics for the system in question. In the words of Popper [141] *"the real state of a physical system, at any moment, may be conceived as the sum total of its dispositions–or its potentialities, or possibilities, or propensities."* This idea of the notion of state can be carried over to the history formulation of quantum mechanics: it is in this sense that decoherence functionals can be said to represent the *state* of a system described by the history version of quantum mechanics. This notion of state of a system has a peculiar transtemporal meaning. This is in particular true for the general history theories discussed in Part II of this thesis.

In the last section we have seen that in standard quantum mechanics it is natural from a mathematical point of view and for aesthetic reasons to identify the space of histories with $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$. Whereas the homogeneous histories in $\mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$ admit a direct physical interpretation in terms of sequences of single time "events," there is no such immediate interpretation available for inhomogeneous histories. I tentatively suggest, however, that the latter may be interpreted as representatives of *unsharp quantum events*, i.e, events which cannot be associated with some fixed time, but which are smeared out in time. This proposal is supported by the following example which is adapted (rather stolen) from [4]. In this reference Aharonov and Albert consider so-called *multiple-time observables*. For definiteness, consider a spin-$\frac{1}{2}$ particle and the following object

$$\sigma_{zx}(t_1, t_2) := \sigma_z(t_1) + \sigma_x(t_2),$$

where $\sigma_z(t_1)$ denotes the self-adjoint operator representing the single-time spin observable in z direction in the Heisenberg picture at time t_1 and where $\sigma_x(t_2)$ denotes the single-time spin observable in x direction in the Heisenberg picture at time t_2. (The argument in [4] does not presuppose that the particle is free and is for instance also valid if the particle interacts with an external magnetic field.) By considering an appropriate *Gedankenexperiment*, Aharonov and Albert argue that the value of this multiple-time observable can be measured without measuring $\sigma_z(t_1)$ or $\sigma_x(t_2)$ individually. In the consistent histories approach the notion of measurement has no fundamental status, but as a working hypothesis we take seriously the idea that objects like $\sigma_{zx}(t_1, t_2)$ are physically meaningful also in the consistent histories approach and that they represent a new kind of observable. Now, when we ask the question what kind of history corresponds to the proposition "the value of the observable $\sigma_{zx}(t_1, t_2)$ is 0" we see that the corresponding history is of the form $P_{z,\uparrow}(t_1) \otimes P_{x,\downarrow}(t_2) + P_{z,\downarrow}(t_1) \otimes P_{x,\uparrow}(t_2)$ in an obvious notation. Clearly, this is in general an *inhomogeneous history*. This example can obviously be generalized to much more general situations involving also n-time observables. It follows that the propositions associated with general Aharonov-Albert type multiple-time observables are in general inhomogeneous histories. This gives a physical meaning to (at least some of) the inhomogeneous histories introduced in Section 3.1. The discussion substantiates the proposal that inhomogeneous histories correspond to events which are spread out or unsharp in time.

In the logical formulation the most general propositions about a quantum mechanical system which have a physical meaning are identified with finite (or at least countably infinite) history propositions. Other statements about a system which cannot be cast into the framework of history propositions are not considered to be meaningful and hence are excluded from consideration. Histories may be said to represent the possible *temporal events* (or simply *events*) which may occur. The probabilities associated with histories are considered to be objective entities in their own right and are interpreted as measures of the tendency or propensity of an individual system to realize certain *histories*. The probability measure on a consistent Boolean algebra of history propositions induced by the decoherence functional (according to Theorem 2) defines in this consistent Boolean algebra two logical relations, namely an implication and an equivalence relation between histories. A history proposition h is said to IMPLY a history proposition k if the conditional probability $p_\varrho(k|h) \equiv \frac{p_\varrho(h\wedge k)}{p_\varrho(h)}$ is well-defined and equal to one. Two history propositions h and k are said to be EQUIVALENT if h implies k and vice versa.
It is straightforward to show that the notion of implication is independent of the consistent Boolean lattice employed. If the history h implies the history k (w.r.t. ϱ) in some consistent Boolean lattice of histories, then h implies k (w.r.t. ϱ) in every consistent Boolean lattice containing h and k.
The notion of implication and of equivalence of histories can be formulated also for relatively consistent Boolean lattices. However, in this case both notions depend manifestly on the relatively consistent Boolean lattice employed (see also Definition 18 below). This important point has to be kept in mind.
The universal rule of interpretation of quantum mechanics can now be formulated as

Rule 1 (Omnès) *Propositions about quantum mechanical systems should solely be expressed in terms of history propositions. Every description of an isolated quantum mechanical system should be expressed in terms of finite history propositions belonging to a common (relatively) consistent Boolean algebra of histories. Every reasoning relating several propositions should be expressed in terms of the logical relations induced by the probability measure from Theorem* 2 *or Theorem* 3 *in that Boolean algebra.*

We briefly discuss the significance of the different requirements in this rule. It is well-known that every Boolean lattice is isomorphic to the algebra of clopen subsets of some topological space, see Theorem 8. Hence, by requiring that every description and every reasoning should be *within* a fixed Boolean lattice, all paradoxa and inconsistencies in quantum mechanics which are due to the nondistributivity of the lattice of projection operators have been expelled from the language of quantum mechanics by one single simple rule. The requirement that the Boolean lattice is consistent is technical and due to the fact that a physically sensible probability functional can only be defined on consistent Boolean lattices. There is a strong methodological argument supporting the Rule 1, i.e., the requirement of simplicity and economy of principles. There is a vast literature discussing examples illustrating the Rule 1. We refer in particular to Omnès' extensive book [137] and to Griffiths [62, 63] and Omnès [130]-[136].

In standard quantum mechanics texts, by and large, no attention is paid to the important issue under which conditions probability assertions and combinations of probability assertions are justified and meaningful. Consistent historians are more careful. According to Rule 1 the assignment of probabilities to certain histories is only admissible when these histories belong to a common Boolean lattice of histories which satisfies the consistency condition. The philosophy underlying this assertion is as follows (adapted from Griffiths [66]): We assert that for a quantum mechanical system there are several incompatible (or complementary) *frameworks* for its theoretical description in terms of possible events and for making logical inferences about possible events or about time sequences of possible events. All different frameworks (or in Griffiths' terminology *topics of conversation*) are similarly objective. That is, the symmetrical treatment of several incompatible frameworks in the mathematical formalism of quantum mechanics is not broken in the interpretation and (as is asserted in the interpretation) also not in the physical reality in the following sense: *it is the integral objective physical situation (for instance, but not necessarily, an experimental arrangement) which determines the framework that should be used for the description and reasoning.* The propensity that some particular event occurs depends upon the quantum system itself and upon the integral physical situation. Particularly, in measurement situations the result of the measurement depends upon the object under study and upon the mode of observation. However, for isolated individual systems, we are free to choose the framework we want. The multitude of alternative frameworks reflects the peculiar nature of quantum reality. It is in a sense as if with each alternative framework we inspect another degree of freedom of the system of which there are in principle an infinite supply.
It is easy to derive counterintuitive and even contradictory conclusions [100] by using different incompatible frameworks for the description of a given system. Thus we are led to the conclusion that the nature of quantum reality cannot be easily imagined and can only in part be described by using ordinary language. The last two sentences of my work [151] were intended to express this conclusion :*"The microscopic world is unanalyzable. Quantum physics as viewed by the logical interpretation provides no model for what 'actually happens.'* " In that work I have stressed the intimate connection between quantum mechanical descriptions and ordinary language and pointed out that the logical rule has to be understood as a 'semantical' rule which systematizes the language of quantum mechanics. It clearly states what the assertions and predictions of quantum mechanics are and it once and for all makes sure, whether a reasoning or an implication is allowed or not. The causal relationship between different histories is coded into a logical relationship.

The notions of *truth* and of *reality* are also framework dependent. We say that a history h is true or an *element of reality* in the state ϱ if $d_\varrho(h,h) = 1$. However, it is important to notice that the assertion that a history h is true (or real) is only meaningful with respect to a ϱ-consistent Boolean lattice of history propositions. This statement clearly unravels that the assertion that the consistent histories approach is a " realistic" interpretation must not be understood in a naive (to wit, classical) spirit. The alleged "quantum reality" is quite unusual and has rather strange features. Recently, Isham [89] has proposed that the possible truth values which can be associated with a history actually lie in a Heyting algebra. [1] The

[1] A *Heyting algebra* H is a lattice with universal bounds with an additional binary operation $\Rightarrow$ with the property that for $x, y, z \in H, x \wedge y \leq z$ if and only if $x \leq (y \Rightarrow z)$ [118].

physical idea beyond this proposal is briefly as follows: consider a Boolean lattice of history propositions $\mathcal{B}$ which is not consistent with respect to d_ϱ. Accordingly, no history in $\mathcal{B}$ can be said to be true. Nevertheless there may be a coarse-graining of $\mathcal{B}$, i.e., a Boolean sublattice of $\mathcal{B}$, which is ϱ-consistent. This consistent sublattice may contain 'true' histories h. Isham argues that the truth value associated with such histories in the fine-grained description provided by $\mathcal{B}$ should account for the fact that in a coarse-grained description these histories are true. The formalization of this idea has led Isham to his proposal that the possible truth values associated with histories lie in a Heyting algebra.

Strictly speaking, in the history version of quantum mechanics the concept of *state history* and the notorious collapse of the state need not be considered. In fact, the notion of absolute state history is fundamentally absent in the consistent histories approach. When a certain history has been found, a state history can be introduced in retrospect in order to associate an evolutionary picture with the history. If one is interested in the probability that some history h_1 is first realized and followed by another history h_2, then the appropriate procedure is to enlarge the framework containing h_1 to a set of histories containing $h_1 \circ h_2$, checking the consistency condition for the enlarged framework and finally (if the consistency check has been passed) to compute the probability through the decoherence functional. However, it is instructive to propose some formal state transformation rule corresponding to the realization of some history. This rule is said to be formal since it is not asserted to correspond to some physical process. This formal rule should coincide with the Lüders-von Neumann projection postulate for homogeneous single time histories. In standard quantum mechanics the state of the system is completely specified by a density operator ϱ which corresponds to the operator $\mathfrak{X}_\varrho$ in the history approach. Inspection of the explicit formula for $\mathfrak{X}_\varrho$ given in Appendix C suggests that when the history h is realized the history state $\mathfrak{X}_\varrho$ has to be transformed according to the following rule

$$\mathfrak{X}_\varrho \mapsto \widetilde{\mathfrak{X}}^h_\varrho := \frac{(1\otimes h)\mathfrak{X}_\varrho(h\otimes 1)}{\mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}\left((1\otimes h)\mathfrak{X}_\varrho(h\otimes 1)\right)}.$$

It is easy to check that the transformed state defines a decoherence functional. If h is a homogeneous single time history, then $\widetilde{\mathfrak{X}}^h_\varrho = \mathcal{N}_h(1\otimes h)\mathfrak{X}_\varrho(h\otimes 1) = \mathfrak{X}_{h\varrho h/\mathrm{tr}(\varrho h)}$, to wit, our above formal state transformation proposal reduces to the Lüders-von Neumann postulate as required ($\mathcal{N}_h$ is a normalization constant). However, for general histories the transformed operator $\widetilde{\mathfrak{X}}^h_\varrho$ is in general not of the form $\mathfrak{X}_{\varrho'}$ for some ϱ'. This already indicates that in the history version of quantum mechanics also more general operators than those of the form $\mathfrak{X}_\varrho$ define decoherence functionals and represent states of history systems. This proposal will be discussed in more detail below (in particular in Part II of this thesis where the 'allowed' operators are specified). A state transformation rule as proposed above may be also useful and enlarge the possibilities for making predictions in the history formalism when considering *branch dependent* consistent sets of histories. Consider a framework $\mathcal{F}_1$ including the alternative histories h_1 and h_2. Assume that there is a second (not necessarily consistent) set of histories $\mathcal{F}_2$ strictly later than $\mathcal{F}_1$. Then it may happen that $\mathcal{F}_2$ is consistent provided that h_1 has been realized whereas $\mathcal{F}_2$ is not consistent when the history h_2 has been realized. In other words $\mathcal{F}_2$ is consistent with respect to the state $\widetilde{\mathfrak{X}}^{h_1}_\varrho$ but not with respect to the state $\widetilde{\mathfrak{X}}^{h_2}_\varrho$. In the consistent histories approach without a proposal of a state transformation rule

such branch dependent frameworks have to be dismissed.
In the evolutionary picture of quantum physics proposed by Haag [71] and briefly discussed in Section 2.4 branch dependent histories are of central importance. The reason for this is simply that past events have influence on future events. Thus in any attempt to reconcile the consistent histories approach with Haag's evolutionary picture a state transformation rule as proposed above is needed. We consider this issue in more detail in Chapter 5 below.

In summary, we assert that for every physical system there are elements of physical reality which cannot be combined either in constructing a theoretical description or in making logical inferences about them. *Such complementary elements of reality are not independent.* The exact form of the framework for the theoretical description and for making logical inferences was specified above in Rule 1 for the standard logical interpretation of quantum mechanics and in Rule 2 below for the generalized logical interpretation developed in Chapter 4.

Concluding this subsection, I want to stress that the above remarks regarding the philosophy underlying the consistent histories approach must be taken *cum grano salis.* The philosophical and interpretational beliefs and presuppositions differ from author to author and will presumably also be subject to change in the future. What is important and stands fast – quite independent of the details of the interpretation – is the richer mathematical formalism and the new concepts of the consistent histories approach, which are a fruitful development in the foundations of quantum theory.

3.3 WEAK POINTS AND CRITICISMS

Und das ist eben schade
Das ist das riesig Fade.
Und darum ist es nichts damit
Und darum ist das alles Kitt !
[from "Die Dreigroschenoper" by Bertolt Brecht/Kurt Weill]

In this section I will briefly discuss some shortcomings of the consistent histories approach and some objections which have been raised in the literature. I make no attempt, however, to give an comprehensive account of all criticisms and objections raised by other authors. The objections that have been raised against the consistent histories approach can be roughly divided into two categories. In the first category we find all the objections against the metaphysics and the philosophy underlying the consistent histories approach. The objections in the second category argue that the consistent histories approach is *physically* incomplete or inappropriate.

Almost all objections that have been raised and discussed in the literature belong to the first "metaphysical" category. Admittedly, the philosophy of the consistent histories approach flies in the face of much textbook wisdom about the meaning of quantum theory. Therefore it is perhaps worthwhile to mention some prototypical objections against consistent histories. I do

this with considerable trepidation. This kind of discussion is bottomless. Therefore I will be brief and leave a more thorough discussion to the philosophers of science. I will mention only those objections which I consider important. Each objection is followed by a brief answer or comment.

- Objection 1: The notion of "event" cannot be reconciled with the quantum mechanical evolution law (i.e., the Schrödinger equation).
  This objection has been raised in [57], Chapter 2. It cannot be overemphasized that the state (or wave function) has no beable status in the consistent histories approach. The basic interpretational entity in the consistent histories approach is the notion of event. Therefore, the question of how these events – which have beable status – are related to the allegedly fundamental dynamics of the wave function – which has no beable status – is meaningless in the consistent histories philosophy.

- Objection 2: Omnès and Griffiths introduce a new empirical kind of logic. Since logic is a priori given, this new logic is meaningless.
  This objection has been raised, e.g., in [57], Chapter 2. I think that it is based on a misinterpretation of Omnès' and Griffiths' writings. Omnès' so-called 'logical' rule has nothing to do with introducing 'a new empirical kind of metalogic,' but simply introduces a convenient way of speaking: the logical rule is neither regarded as a scientific law of human thought nor intended to modify the logical structure of our mathematical reasoning, but it is rather confined to the systems of quantum mechanical propositions to which probabilities may be ascribed by the theory.

- Objection 3: The times in the decoherence functional at which "events" occur are "chosen" by hand by the physicist. Therefore, "reality" has meaning only with respect to some given consistent set of histories and (in this sense) "events" are only constructs of thought. This objection has been taken from Chapter 5 in [57]. I must admit that I have no answer to the first part of this objection and that it represents a weak (or at least unsatisfactory) point in the consistent histories interpretation. As discussed above, the second sentence is correct and one of the hallmarks of the consistent histories interpretation. I can only remind the reader that this objection is closely related to the measurement problem of quantum mechanics (see Problem 1, below). As far as I can see, there remains only the vague hope that a solution of the measurement problem of quantum mechanics may shed new light upon this problem.

The interested reader can find more metaphysical discussions of this and other related kind in the literature. Foremost d'Espagnat has raised repeatedly metaphysically motivated objections against the consistent histories approach, see, e.g., Ref. [31, 32, 33]. Answers to his objections by Griffiths and by Omnès can be found in Ref. [64, 135, 137]. Be all this as it may, I now turn to some physical shortcomings of the consistent histories approach.

- Problem 1: The consistent histories approach in its present form does not solve the objectification problem (or measurement problem) of quantum mechanics. The question

to be addressed is why and under which circumstances events take place. To the best of my knowledge this question has not been tackled in the realm of consistent histories and it is doubtful that this problem assumes a simple and straightforward answer in the consistent histories approach. The consistent histories approach shares this feature with every other known approach to quantum mechanics.

- Problem 2: An external time parameter is needed already to formulate the standard consistent histories approach. Moreover, the projection postulate discussed in Chapter 2 is intrinsically nonrelativistic. Although the consistent histories approach contains no collapse of the wave function as a dynamical process, the form of the decoherence functional is determined by the heuristic principle from Section 2.3. Hence, the form of the probabilities associated with histories is fundamentally nonrelativistic. To reconcile the consistent histories approach with relativity and particularly with causality and locality, the form of the probabilities has presumably to be changed.

I will comment on the last problem. That the Lüders-von Neumann projection postulate is incompatible with causality has already been recognized in the early days of quantum mechanics by Landau and Peierls [111]. It is tempting to replace the Lüders-von Neumann projection postulate by the relativistically invariant projection postulate due to Schlieder [157] and to Hellwig and Kraus [84]. However, their postulate has subsequently been criticized by Aharonov and Albert [3]-[4]. We will argue below however, that the arguments raised by Aharonov and Albert against the Schlieder-Hellwig-Kraus postulate can be circumvented in the consistent histories approach. Therefore we will conclude that a modified version of the Schlieder-Hellwig-Kraus postulate can be essentially retained and used to formulate a relativistically invariant version of the consistent histories approach. However, before this can be done in a physically sensible way the consistent histories approach must be generalized to POV measures.

Before concluding this subsection it is necessary to mention the important works by Dowker and Kent [37, 38] and Kent [99, 100, 101, 102, 103]. In their penetrating analysis Dowker and Kent have reconsidered diverse aspects of the consistent histories approach and have shown that several proposals and statements in the original works by Omnès and Gell-Mann and Hartle cannot be uphold. It is beyond the scope of this thesis to discuss any details.

3.4 MIXED HISTORY STATES

The notion of state in quantum mechanics has an inevitably probabilistic and statistical character. The *probability* that a certain property of the quantum mechanical system represented by the projection operator P will be realized or will be found upon measurement is in the state ρ given by $\mathrm{tr}(\rho P)$, where tr denotes the trace operation in $\mathfrak{H}$. Thus for any quantum mechanical system its state ρ comprises all probabilistic predictions of quantum mechanics for the system in question. This feature will in the sequel be regarded as the defining property of the notion of a quantum mechanical state. There is a basic duality between the set of possible

states and the set of observables of a quantum mechanical system. If we restrict the latter, then in general the former will become larger and vice versa. When we identify the set of all quantum mechanical observables with the self-adjoint operators on $\mathfrak{H}$, then it is a consequence of Gleason's Theorem [60, 39] that the most general notion of state in quantum mechanics is given by the set of density matrices. As explained in the previous sections, in the propensity interpretation quantum mechanical probabilities are not only viewed as representing relative frequencies obtained in a series of measurements on some ensemble of appropriately prepared systems, but as describing *single* quantum mechanical systems. Probabilities are considered as measures of the propensity or tendency of a possibility to realize itself upon repetition. The state of a single physical system may be conceived as "sum total of its dispositions – or its potentialities, or possibilities, or propensities" (cf. Popper [140]).
This idea can be carried over without change to the consistent histories approach. In the consistent histories approach the general possibilities or properties of a quantum system are identified with complete histories and probabilities are associated with histories. These probabilites are interpreted as measure of the tendency that certain *histories* will be realized in a single system. Thus from a formal point of view it would be totally acceptable to have a more general notion of (history) state in this theory. Such a general state should comprise all propensities inherent in an individual system to realize certain histories. Accordingly, in history quantum theories the *history state* of the system in question is given by and identified with the decoherence functional of the system.

As emphasized also in the previous sections the state of a quantum mechanical system depends upon the *integral physical situation,* and thus particularly on the past history and the preparation of the system. Furthermore, the interpretation that mixed states describe statistical ensembles of pure quantum mechanical ensembles cannot be uphold if states are assigned to single systems.

In this section we will briefly explain that the formalism of the consistent histories approach allows for a new class of mixed history states which are not given by bi-additive decoherence functionals on the space of all histories, but rather by generalized decoherence functionals which are bi-additive only on some subspace of the space of all histories.

Since probabilistic statements are usually tested by performing experiments on ensembles of systems, it is natural to adopt the following heuristic principle: [2]

The set of meaningful states of an ensemble of quantum mechanical systems in the frequency interpretation of probability is equal to the set of meaningful states of an individual quantum mechanical system in the propensity interpretation of probability.

For simplicity we assume in this section that the single-time Hilbert space $\mathfrak{H}$ is separable. Firstly, fix an arbitrary finite subset S of $\mathbb{R}$ and consider only finite histories with support S and only consistent Boolean sublattices of $\mathcal{P}_S^{\otimes}(\mathfrak{H})$. Since the dimension of $\bigotimes_{t\in S}\mathfrak{H}$ is at most countably infinite, every Boolean sublattice of $\mathcal{P}(\bigotimes_{t\in S}\mathfrak{H})$ is atomic and the atoms are the

[2] To my impression this (or a similar) principle is almost always tacitly assumed to hold in every realistic interpretation of quantum mechanics.

elements of an at most countably infinite orthogonal decomposition of the unit in $\mathcal{B}(\bigotimes_{t\in S}\mathfrak{H})$ as a finite ($\aleph < \infty$) or countably infinite ($\aleph = \infty$) sum

$$1_{\otimes_{t\in S}\mathfrak{H}} = \sum_{n=1}^{\aleph} P_{S,n}$$

of pairwise orthogonal projection operators $P_{S,n} \in \mathcal{P}(\bigotimes_{t\in S}\mathfrak{H})$. Vice versa every such decomposition of the unit $1_{\otimes_{t\in S}\mathfrak{H}}$ induces a Boolean sublattice of $\mathcal{P}_S^{\otimes}(\mathfrak{H})$. That is, there is a one-one correspondence between Boolean sublattices $\mathcal{B}$ of $\mathcal{P}_S^{\otimes}(\mathfrak{H})$ containing the unit operator $1_{\otimes_{t\in S}\mathfrak{H}}$ and at most countably infinite orthogonal projective decompositions $\mathcal{D} = \{P_{S,n}\}_n$ of the unit operator on $\bigotimes_{t\in S}\mathfrak{H}$. We will write $\mathcal{B}(\mathcal{D})$ and $\mathcal{D}(\mathcal{B})$ respectively.

A probability measure $\mathfrak{P}$ on a Boolean sublattice $\mathcal{B}(\mathcal{D})$ of $\mathcal{P}_S^{\otimes}(\mathfrak{H})$ is determined by its values on the corresponding decomposition $\mathcal{D}(\mathcal{B})$, i.e., by the collection of numbers $\mathfrak{P}(\mathcal{D})$. We write $\lambda_{S,n} := \mathfrak{P}(P_{S,n})$ and recognize that $\mathfrak{P}$ can be represented as $\mathfrak{P}(s) = \mathrm{tr}(\tilde{\rho}s)$, for all $s \in \mathcal{B}$, where $\tilde{\rho} := \sum_{n=1}^{\aleph} \lambda_{S,n} P_{S,n}$.

Every such probability measure $\mathfrak{P}$ on $\mathcal{B}$ represents a particular selection (e.g., preparation) procedure to which the single quantum system in question or some ensemble of systems is subjected. If the quantum system is initially in the state ρ, then $\mathfrak{P}$ has to satisfy $\mathfrak{P}(P_{S,n}) = 0$ whenever $d_\rho(P_{S,n}, P_{S,n}) = 0$. Otherwise $\mathfrak{P}$ would select dynamically impossible histories.

In standard quantum mechanics the systems are usually assumed to be subjected to selection procedures at a single time. By contrast, we allow for history (or multi-time) selection procedures, that is to say, our selection procedure selects systems for which a certain history is realized with probability one or systems for which certain histories are realized with certain probabilities. A special example are the so-called pre- and postselected quantum systems which have first been studied by Aharonov, Bergmann and Lebowitz [1] in their time-symmetric formulation of quantum mechanics.

Assume now that $\mathcal{B}$ is a consistent Boolean lattice of histories (with respect to some given state ϱ) and write $\mathcal{B} = \mathcal{C}$. As mentioned above the consistency functional d_ρ induces a probability measure $p_{\rho,\mathcal{C}}$ on $\mathcal{C}$. However, this probability measure does not account for the history selection procedure represented by $\mathfrak{P}$. To take it into account we have to replace $p_{\rho,\mathcal{C}}$ by the following probability measure $\mathbf{P}_{\rho,S,\mathcal{B},\tilde{\rho}}$ on $\mathcal{C}$

$$\begin{aligned}
\mathbf{P}_{\rho,S,\mathcal{B},\tilde{\rho}}(s) &:= \sum\nolimits_{n=1}^{\aleph}{}^{*}\ \lambda_{S,n} p_{\rho,\mathcal{C}}(s \wedge P_{S,n} | P_{S,n}) \\
&= \sum_{n=1}^{\aleph}{}^{*}\ \mathfrak{P}(P_{S,n}) p_{\rho,\mathcal{C}}(s \wedge P_{S,n} | P_{S,n}) \\
&= \sum_{n=1}^{\aleph}{}^{*}\ \mathfrak{P}(P_{S,n}) \frac{p_{\rho,\mathcal{C}}(s \wedge P_{S,n})}{p_{\rho,\mathcal{C}}(P_{S,n})}, \text{ for } s \in \mathcal{C}.
\end{aligned}$$

Here the star indicates that the sum is over all n such that $p_{\rho,\mathcal{C}}(P_{S,n}) \neq 0$.

The consistency conditions valid in $\mathcal{C}$ guarantee that $\mathbf{P}_{\rho,S,\mathcal{B},\tilde{\rho}}$ is indeed a probability measure on $\mathcal{C}$. We conclude that on a given consistent Boolean sublattice $\mathcal{C}$ of $\mathcal{P}_S^{\otimes}(\mathfrak{H})$ there are certain *"pure"* probability measures $p_{\rho,\mathcal{C}}$ characterized by a density matrix ρ on $\mathfrak{H}$, whereas there are moreover *"mixed"* probability measures given by $\mathbf{P}_{\rho,S,\mathcal{B},\tilde{\rho}}$. In the frequency interpretation of probability $\mathbf{P}_{\rho,S,\mathcal{B},\tilde{\rho}}$ may be considered as representing a statistical ensemble such that in the jth subensemble the history $P_{S,j}$ is realized respectively and such that the relative weight of the jth subensemble is $\lambda_{S,j}$.

The formula defining $\mathbf{P}_{\rho,S,\mathcal{B},\tilde{\rho}}$ looks similar to Griffiths' recently proposed *refinement rule* [66]. Indeed, when we specialize to a finite dimensional Hilbert space and set $\rho = 1$, then the above equation reduces to Griffiths' refinement rule. Our treatment not only clarifies the status of this refinement rule in the consistent histories approach, but also generalizes it to the case of an infinitely dimensional Hilbert space. In the infinite-dimensional case the unit operator is not of trace class and therefore in general $\rho \neq 1$. We conclude that the refinement rule is by no means a fundamental scientific law incorporating the dynamical content of quantum mechanics, but instead it solely represents the only sensible way (compatible with the heuristic principle stated above) to introduce a probability measure on a given consistent Boolean lattice of histories taking into account a specific selection procedure. In particular, it is not surprising that (as shown by Griffiths) the Born rule for transition probabilities can be derived from the refinement rule.

A slight generalization of the above introduced probability measure $\mathbf{P}_{\rho,S,\mathcal{B}},\tilde{\rho}$ on $\mathcal{C}$ suggests itself. We introduce the GENERALIZED CONSISTENCY FUNCTIONAL $\mathbf{D}_{\rho,S,\tilde{\rho},\mathcal{B}}$ with respect to $(\rho, S, \tilde{\rho}, \mathcal{B})$ by

$$\mathbf{D}_{\rho,S,\tilde{\rho},\mathcal{B}}(h,k) := \sum_{n=1}^{\aleph} {}^*\mathfrak{P}(P_{S,n}) \frac{d_\rho(h \wedge P_{S,n}, P_{S,n} \wedge k)}{d_\rho(P_{S,n}, P_{S,n})}, \tag{3.12}$$

for all $h,k \in \mathcal{P}_{fin}^{\otimes}(\mathfrak{H})$. The star indicates that the sum is over all n such that $d_\rho(P_{S,n}, P_{S,n}) \neq 0$. The so-defined generalized decoherence functional is obviously not bi-additive on all of $\mathcal{P}_{fin}^{\otimes}(\mathfrak{H})$ but it is bi-additive on the Boolean algebra $\mathcal{B}$.

A Boolean sublattice $\mathcal{G}$ of $\mathcal{B}$ is said to be GENERALIZED CONSISTENT WITH RESPECT TO $(\rho, S,\tilde{\rho}, \mathcal{B})$ if $\mathbf{D}_{\rho,S,\tilde{\rho},\mathcal{B}}(1_\mathcal{G}, 1_\mathcal{G}) \neq 0$ and if for every pair h,k of disjoint histories in $\mathcal{G}$ the `generalized consistency condition` $\mathrm{Re}\, \mathbf{D}_{\rho,S,\tilde{\rho},\mathcal{B}}(h,k) = 0$ is satisfied. If this is the case, then we set

$$\tilde{\mathbf{P}}_{\mathcal{G},\rho,S,\tilde{\rho},\mathcal{B}}(h) := \frac{\mathbf{D}_{\rho,S,\tilde{\rho},\mathcal{B}}(h,h)}{\mathbf{D}_{\rho,S,\tilde{\rho},\mathcal{B}}(1_\mathcal{G}, 1_\mathcal{G})}.$$

$\tilde{\mathbf{P}}_{\mathcal{G},\rho,S,\tilde{\rho},\mathcal{B}}$ defines a probability measure on $\mathcal{G}$. This probability measure can be used to define logical rules of inference in $\mathcal{G}$ in the usual way (conditional probability equal to one) and the extension of Omnès' logical rule [137] to the present context is trivial.

If $\mathcal{G} \subset \mathcal{B}$ is a Boolean lattice of histories which is consistent with respect to ρ in the ordinary sense, the $\mathcal{G}$ is also generalized consistent with respect to $(\rho, S,\tilde{\rho}, \mathcal{B})$ for arbitrary $(S,\tilde{\rho}, \mathcal{B})$ (provided $\mathbf{D}_{\rho,S,\tilde{\rho},\mathcal{B}}(1_\mathcal{G}, 1_\mathcal{G}) \neq 0$).

However, a generalized consistent (with respect to $(\rho, S, \tilde{\rho}, \mathcal{S})$) Boolean lattice $\mathcal{G}$ will in general not be a consistent Boolean lattice with respect to ρ.

Thus we conclude that generalized consistent Boolean lattices of histories $\mathcal{G}$ supply new frameworks for quantum description and quantum reasoning not present in the standard consistent histories approach.

Since in the observed macroscopic part of nature time symmetry is *de facto* absent, the idea of history selection procedures (or as a special case pre- and post selection) for *individual* quantum systems may seem to be somewhat absurd and our notion of generalized state is at least somewhat unusual. Since in every actual sequential measurement performed on a single system there is only one of the possible histories realized, it is immediately clear that it is impossible to determine (retrodictively) the mixed generalized state unambiguously by performing sequential measurements. However, it is worth recalling that already in standard quantum mechanics the state ρ of a single system — if unknown — cannot be determined (retrodictively) by performing measurements on that system, see [20] and references therein. Thus the requirement that the state of a single quantum system should be determinable by performing measurements on the system is incompatible with the known results of measurement theory and must be abandoned. The state of a quantum mechanical system can in principle only be known on the basis of various further experimental data or on the basis of theoretical models assumed or known to describe the past history of the system in question or both. Thus it is clear that our generalized quantum states are as meaningful in this respect as are the states in standard quantum mechanics.

The pre- and postselection of quantum systems first studied by Aharonov, Bergmann and Lebowitz [1] is a special case of history selection. These authors have stressed that "from a formal point of view we may legitimately specify any selection that could be performed with physical equipment, however complex". This statement – if taken seriously – ultimately requires to consider history selection procedures.

To summarize, in this section we have investigated the question what the most general notion of state is in the consistent histories approach compatible with the usual formulations of measurement theory. The relevant aspects of measurement theory have been formulated as heuristic principles. We found a new class of mixed states in quantum history theories corresponding to history selection procedures. These generalized states allowed us to introduce generalized consistency functionals and generalized frameworks for quantum description and quantum reasoning.

These generalized frameworks are the generalized consistent Boolean lattices of histories. The assertions obtained in these new frameworks can be verified by performing measurements on ensembles of quantum systems, where — of course — these ensembles of quantum systems are subjected to the fixed history selection criteria. The results of reasoning in such frameworks may be predictive or retrodictive or a mixture of both as the case may be. In the case of history selections the ensemble has to be selected on the basis of required outcomes of specified observations at specified times. Notice that histories which are dynamically possible

may be excluded by this selection procedure. It is important to notice that the propensity interpretation is in accordance with this procedure, since in this interpretation the probability is a measure of the tendency of a certain outcome to realize itself *upon repetition*.

4

CONSISTENT EFFECT HISTORIES

Ach, mit solchen Advokaten
ist man übel oft beraten,
und fürwahr, man braucht Geduld !
Statt daß jetzt die Sach' beendet,
hat's noch schlimmer sich gewendet
[from "Die Fledermaus" by Johann Strauß/Haffner & Genée]

4.1 HOMOGENEOUS EFFECT HISTORIES

In Section 3.1 we have introduced the standard consistent histories formalism. One of the basic ingredients of this formalism is the proposal that the observables in quantum mechanics have to be identified with self-adjoint operators on Hilbert space.

We have argued in Chapter 2 that Hilbert space quantum mechanics allows for a much richer notion of observable. Therefore it is worthwhile to study whether the consistent histories approach can be generalized to incorporate POV measures. From the outset I want to stress that I do not claim that the resulting extension can be used in any concrete application of quantum mechanics to make qualitatively new predictions different from the predictions of quantum mechanics. This might be the case, but I know of no example. The problem investigated in this chapter is mainly a matter of principle in the foundations of quantum mechanics. It is the question of whether both the consistent histories approach and the concept of generalized observable can be uphold together or whether they are incompatible.

I will argue that the consistent histories approach can indeed be generalized.

Definition 11 *A* HOMOGENEOUS EFFECT HISTORY (OF THE FIRST KIND) *is a map* $u : \mathbb{R} \to \mathfrak{E}(\mathfrak{H}), t \mapsto u_t$. *The* SUPPORT OF u *is given by* $\mathfrak{s}(u) := \{t \in \mathbb{R} \mid u_t \neq 1\}$. *If* $\mathfrak{s}(u)$ *is finite, countable or uncountable, then we say that* u *is a* FINITE, COUNTABLE *or* UNCOUNTABLE EFFECT HISTORY *respectively. The space of all homogeneous effect histories (of the first kind) will be denoted by* $\mathbb{E}(\mathfrak{H})$, *the space of all finite homogeneous effect histories (of the first kind) by* $\mathbb{E}_{fin}(\mathfrak{H})$ *and the space of all finite homogeneous effect histories (of the first kind) with support* S *by* $\mathbb{E}_S(\mathfrak{H})$. *All homogeneous effect histories for which there exists at least one* $t \in \mathbb{R}$ *such that* $u_t = 0$ *are collectively denoted by* 0, *slightly abusing the notation.*

The choice of the decoherence weight in Chapter 3 was motivated by the heuristic principle from Section 2.3 and the form of the Lüders-von Neumann projection postulate. The state transformation formula of the Lüders-von Neumann projection postulate can be generalized to the measurement of POV measures. Assume that upon a measurement of the observable $\mathcal{O}$ the value of $\mathcal{O}$ is found to be in the set $A \in \mathcal{F}_{\mathcal{O}}$, then the generalized Lüders-von Neumann

state transformation prescription is

$$\varrho \mapsto \frac{\sqrt{\mathcal{O}(A)}\varrho\sqrt{\mathcal{O}(A)}}{\mathrm{tr}(\mathcal{O}(A)\varrho)}.$$

In general, this prescription does not correspond to an ideal or repeatable measurement of $\mathcal{O}$ of course. Anyhow, in the consistent histories approach we are looking for something different. We are looking for a rule assigning probabilities to sequences of quantum events (represented by some homogeneous effect history) subject to the following conditions:

- If the effect history u is degenerate, i.e., if its support contains exactly one time point $\mathfrak{s}(u) = \{t\}$, then the probability assigned to the "history" u should be equal to the standard quantum mechanical probability $\mathrm{tr}(u_t\varrho)$ of the effect u_t in the state ϱ.
- The probability assigned to ordinary histories should coincide with the probability assigned in the standard consistent histories theory.

This shows that the following definitions are sensible.

The class operator C_{t_0} defined above for finite ordinary homogeneous histories can be extended to homogeneous finite effect histories $u \in \mathbb{E}_{fin}(\mathfrak{H})$

$$\begin{aligned} C_{t_0}(u) &:= U(t_0,t_n)\sqrt{u_{t_n}}U(t_n,t_{n-1})\sqrt{u_{t_{n-1}}}...U(t_2,t_1)\sqrt{u_{t_1}}U(t_1,t_0) && (4.1)\\ &= U(t_0,t_i(u))\sqrt{u_{t_n}}(t_n)\sqrt{u_{t_{n-1}}}(t_{n-1})...\sqrt{u_{t_1}}(t_1)U(t_i(u),t_0), && (4.2)\end{aligned}$$

where we have defined the Heisenberg picture operators

$$\sqrt{u_{t_k}}(t_k) := U(t_k,t_i(u))^\dagger\sqrt{u_{t_k}}U(t_k,t_i(u))$$

with respect to the initial time $t_i(u)$ of u.

Definition 12 *For every pair u and v of finite homogeneous effect histories (of the first kind) we define the* DECOHERENCE WEIGHT *of u and v by*

$$\mathsf{d}_\varrho(u,v) := \mathrm{tr}\left(C_{t_0}(u)\varrho(t_0)C_{t_0}(v)^\dagger\right).$$

The functional $\mathsf{d}_\varrho : \mathbb{E}_{fin}(\mathfrak{H})\times\mathbb{E}_{fin}(\mathfrak{H}) \to \mathbb{C}, (u,v) \mapsto \mathsf{d}_\varrho(u,v)$ will be called the HOMOGENEOUS DECOHERENCE FUNCTIONAL ASSOCIATED WITH THE STATE *ϱ.*

As in the standard consistent histories approach we wish to interpret the value $\mathsf{d}_\varrho(u,u)$ as the probability of u. There immediately arises a serious difficulty with this decoherence functional. At first sight it seems difficult (if not impossible) to construct a natural mathematical structure on the space of effect histories such that the decoherence functional is additive in both arguments. Without this structure a consistency condition generalizing Equation 3.8 cannot even be formulated and an interpretation of $\mathsf{d}_\varrho(u,u)$ as probability seems to be impossible. These questions are investigated in the next section.

4.2 INHOMOGENEOUS EFFECT HISTORIES

The notion of inhomogeneous effect history is introduced in close analogy to the parallel concepts in the standard consistent histories approach. However, under which circumstances some inhomogeneous effect histories can be interpreted as composition of homogeneous histories by the grammatical connectives "and", "or" and "not" – which was a major motivation for the definition of the notion inhomogeneous history in Chapter 3 – will be studied only at the end of this section.

The map σ_S given by Equation 3.4 can be extended to a map

$$\sigma_{fin} : \mathbb{E}_{fin}(\mathfrak{H}) \to \mathcal{B}^{\otimes}_{fin}(\mathfrak{H}), u \simeq \{u_{t_k}\}_{t_k \in \mathfrak{s}(u)} \mapsto \otimes_{t_k \in \mathfrak{s}(u)} u_{t_k}, \tag{4.3}$$

where $\mathcal{B}^{\otimes}_{fin}(\mathfrak{H})$ denotes the disjoint union of all $\mathcal{B}^{\otimes}_{S}(\mathfrak{H}), S \subset \mathbb{R}$ finite. The map σ_{fin} is neither injective nor surjective. However, $d_\varrho(u,v)$ depends on u and v only through $\sigma_{fin}(u)$ and $\sigma_{fin}(v)$. From a mathematical point of view it thus seems to be natural to define the notion of *inhomogeneous* effect history as follows:

Definition 13 *Let S be a finite subset of $\mathbb{R}$, then we call the space $\mathfrak{E}^{\otimes}_{S}(\mathfrak{H}) := \mathfrak{E}(\otimes_{t\in S}\mathfrak{H})$ of effect operators on $\otimes_{t\in S}\mathfrak{H}$ the* SPACE OF FINITE INHOMOGENEOUS EFFECT HISTORIES WITH SUPPORT S. *The direct limit of the directed system $\{\mathfrak{E}^{\otimes}_{S}(\mathfrak{H}) \mid S \subset \mathbb{R}$ finite$\}$ will be called* THE SPACE OF ALL FINITE INHOMOGENEOUS EFFECT HISTORIES WITH ARBITRARY SUPPORT *and will be denoted by $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$. The elements in $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ will also be called* EFFECT HISTORY PROPOSITIONS.

The construction of $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ is described in detail in Appendix A.3. The homogeneous elements in $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ represent equivalence classes of homogeneous effect histories. In this work we will carefully distinguish between homogeneous effect histories as defined in Definition 11 and homogeneous elements in $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$. For clarity of exposition we will call the former *homogeneous effect history of the first kind* or (where no confusion can arise) simply homogeneous effect histories, whereas the latter will be called *homogeneous effect histories of the second kind.*
All the $\mathfrak{E}^{\otimes}_{S}(\mathfrak{H})$, $S \subset \mathbb{R}$ and $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ carry several isomorphic distinct D-poset structures, as discussed in Appendix A.3.

In this work we use a recent result of J.D. Maitland Wright [171] which implies that the decoherence functional d_ϱ as defined above on the space of homogeneous effect histories (of the second kind) can indeed be extended to a functional on the space of inhomogeneous effect histories with the desired properties. Wright's Theorem can be found in Appendix B.

Since the set $\mathcal{B}(\mathfrak{H})$ of bounded operators on a Hilbert space $\mathfrak{H}$ with dimension greater than 2 is a von Neumann algebra (of type I), Theorem 13 can be applied to the decoherence functional $d_\varrho : \mathcal{K}_{fin}(\mathfrak{H}) \times \mathcal{K}_{fin}(\mathfrak{H}) \to \mathbb{C}, (h,k) \mapsto d_\varrho(h,k) := \mathrm{tr}\left(C_{t_0}(h)\varrho(t_0)C_{t_0}(k)^\dagger\right)$ defined in Chapter 3 above. Thus for every finite subset $S \subset \mathbb{R}$ there is a unique bounded bilinear functional $\widetilde{d}_{\varrho,S}$ on $\mathcal{B}^{\otimes}_{S}(\mathfrak{H}) \times \mathcal{B}^{\otimes}_{S}(\mathfrak{H})$ extending the decoherence functional d_ϱ restricted to $\mathcal{P}^{\otimes}_{S}(\mathfrak{H}) \times \mathcal{P}^{\otimes}_{S}(\mathfrak{H})$.

The restriction $\mathcal{D}_{\varrho,S}$ of $\widetilde{d}_{\varrho,S}$ to $\mathfrak{E}_S^{\otimes}(\mathfrak{H}) \times \mathfrak{E}_S^{\otimes}(\mathfrak{H})$ is a bounded functional which is additive in both arguments with respect to the canonical D-poset structure on $\mathfrak{E}_S^{\otimes}(\mathfrak{H})$. The collection of all such functionals $\mathcal{D}_{\varrho,S}$ for any finite $S \subset \mathbb{R}$ induces a bounded functional $\mathcal{D}_{\varrho}$ on $\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H}) \times \mathfrak{E}_{fin}^{\otimes}(\mathfrak{H})$ which is additive in both arguments with respect to the canonical D-poset structure on $\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H})$.

Let $\alpha \in \mathbb{Q}, \alpha > 0$. Consider the function $q_\alpha : (\mathfrak{E}(\mathfrak{H}), \oplus_\alpha) \times (\mathfrak{E}(\mathfrak{H}), \oplus_\alpha) \to (\mathfrak{E}(\mathfrak{H}), \oplus) \times (\mathfrak{E}(\mathfrak{H}), \oplus), q_\alpha(E_1, E_2) := (Q_\alpha^{-1}(E_1), Q_\alpha^{-1}(E_2))$, where Q_α denotes the D-poset isomorphism introduced in Appendix A.3. The functional $\mathcal{D}_{\varrho,S,\alpha} := \mathcal{D}_{\varrho,S} \circ q_\alpha$ is a bounded complex valued functional defined on $(\mathfrak{E}_S^{\otimes}(\mathfrak{H}), \oplus_\alpha) \times (\mathfrak{E}_S^{\otimes}(\mathfrak{H}), \oplus_\alpha)$. The functional $\mathcal{D}_{\varrho,S,\alpha}$ is additive in both arguments with respect to the D-poset structure $\oplus_\alpha$ on $\mathfrak{E}(\mathfrak{H})$.

The collection of all functionals $\mathcal{D}_{\varrho,S,\alpha}$ for any finite $S \subset \mathbb{R}$ and fixed α induces a bounded functional $\mathcal{D}_{\varrho,\alpha}$ on $(\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H}), \oplus_\alpha) \times (\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H}), \oplus_\alpha)$ which is additive in both arguments with respect to the D-poset structure $\oplus_\alpha$ on $\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H})$. All the functionals $\mathcal{D}_{\varrho,\alpha}$ are called DECOHERENCE FUNCTIONALS WITH RESPECT TO THE STATE ϱ.

Comparison with the decoherence weight defined in Definition 12 shows that it is the decoherence functional $\mathcal{D}_{\varrho,2}$ which coincides with d_ϱ when restricted to homogeneous effect histories. So, the "physical" value of α is $\alpha = 2$.

In view of our above remarks and due to the pairwise isomorphy of all $(\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H}), \oplus_\alpha)$, we conclude that the value of $\alpha \in \mathbb{Q}^+$ can be chosen at will. Indeed, it can be easily seen by inspection that the Foulis-Bennett Theorem in Appendix A.4 and all subsequent definitions in this chapter are forminvariant under the D-poset isomorphisms Q_α. It has only to be kept in mind that the "physical" form of the decoherence weight corresponds to $\alpha = 2$. In the rest of this chapter we will work with the value $\alpha = 1$. All results obtained in the $\alpha = 1$ "representation" can easily be shifted isomorphically to the "physical" $\alpha = 2$ case.

The Foulis-Bennett Theorem allows for an intrinsic definition of the notion of Boolean sublattice of the D-poset $(\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H}), \oplus)$. [1]

Definition 14 *A sub-D-poset $(\mathfrak{B}, \oplus)$ of $(\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H}), \oplus)$ is said to be a* BOOLEAN SUBLATTICE OF $(\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H}), \oplus)$ *if $\mathfrak{B}$ satisfies the coherence law and the law of compatibility.*

We also say that a Boolean sublattice of $(\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H}), \oplus)$ is a *Boolean lattice of effect histories.* The reader may wish to recall the definition of the notion of sub-D-poset given in Appendix A.2. It is important to remember that the universal bounds $0_{\mathfrak{B}}, 1_{\mathfrak{B}}$ of $\mathfrak{B}$ do not necessarily

[1] In [152] I have given a seemingly more artificial definition of the notion of "admissible Boolean sublattice." However, an application of the Foulis-Bennett Theorem shows that the every admissible Boolean lattice as defined in [152] is isomorphic as a D-poset to a Boolean sublattice of $(\mathfrak{E}_{fin}^{\otimes}(\mathfrak{H}), \oplus)$ as defined here. The isomorphism is denoted by $\mathfrak{M}$ in [152].

coincide with the universal bounds $0, 1$ of $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ and that the D-poset structure on $\mathfrak{B}$ induced by the D-poset structure on $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ does depend on $1_{\mathfrak{B}}$ and $0_{\mathfrak{B}}$.

Remark 7 *Since $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ is a D-poset, $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ is in particular a partially ordered set. However, for two elements $e_1, e_2 \in \mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ the supremum $e_1 \vee e_2$ and the infimum $e_1 \wedge e_2$ not necessarily exist. That is, $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ is not a lattice. But there exists a partially defined join operation denoted by $\vee$ and a partially defined meet operation denoted by $\wedge$. Strictly speaking a* SUBLATTICE $\mathcal{L}$ OF $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ *is a subset $\mathcal{L} \subset \mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ such that $\mathcal{L}$ endowed with the restrictions of $\vee$ and $\wedge$ to $\mathcal{L}$ is a lattice. It makes thus sense to speak of sublattices of $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$. However, it is important to notice that a Boolean sublattice of $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ is not necessarily a sublattice of $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ in this sense.*

Our target is to generalize Omnès' logical rule and thus to single out the appropriate subsets of $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ on which the decoherence functional $\mathcal{D}_{\varrho,1}$ induces a probability measure and on which a description and a reasoning involving inhomogeneous effect histories compatible with 'common sense' can be defined.

Definition 15 *A Boolean sublattice $(\mathfrak{B}, \oplus)$ of $(\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H}), \oplus)$ is called* CONSISTENT W.R.T. ϱ *if $0 \in \mathfrak{B}$ and if for every pair of disjoint elements $b_1, b_2 \in \mathfrak{B}$ (i.e., elements satisfying $b_1 \wedge_{\mathfrak{B}} b_2 = 0$) the* ***consistency condition*** $\mathrm{Re}\, \mathcal{D}_{\varrho,1}(b_1, b_2) = 0$ *is satisfied.*

Theorem 4 *Let $(\mathfrak{B}, \oplus)$ be a consistent Boolean lattice of effect histories. Then the decoherence functional $\mathcal{D}_{\varrho,1}$ induces a probability functional $p_{\varrho,\mathfrak{B}}$ on $\mathfrak{B}$ by*

$$b \mapsto p_{\varrho,\mathfrak{B}}(b) \equiv \frac{\mathcal{D}_{\varrho,1}(b,b)}{\mathcal{D}_{\varrho,1}(1_{\mathfrak{B}}, 1_{\mathfrak{B}})}.$$

The proof is straightforward.

Definition 16 *An effect history proposition $e_1 \in \mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ is said to* IMPLY *an effect history proposition $e_2 \in \mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ in the state ϱ if there exists a consistent Boolean sublattice $\mathfrak{B}$ of $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ containing e_1 and e_2 and if the conditional probability $p_{\varrho,\mathfrak{B}}(e_2|e_1) \equiv \frac{p_{\varrho,\mathfrak{B}}(e_1 \wedge_{\mathfrak{B}} e_2)}{p_{\varrho,\mathfrak{B}}(e_1)}$ is well-defined and equal to one. We write $e_1 \Longrightarrow_{\varrho} e_2$. Two history propositions e_1 and e_2 are said to be* EQUIVALENT *if e_1 implies e_2 and vice versa. We write $e_1 \Longleftrightarrow_{\varrho} e_2$.*

Remark 8 *The so-defined notions of implication and equivalence of effect histories are certainly framework independent. However, there is a major difference between Definition* 16 *and the parallel notions in the standard consistent histories formalism. In the latter case the conditional probability $p_{\varrho}(k \mid h)$ is independent of the ϱ-consistent Boolean lattice chosen. In*

the consistent effect histories approach the situation is different. To understand this, consider two ϱ-consistent Boolean sublattices $\mathfrak{B}$ and $\mathfrak{B}'$ of $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ both containing the effect history propositions e_1 and e_2. In general $e_1 \wedge e_2$ does not exist in $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ and correspondingly in general $e_1 \wedge_{\mathfrak{B}} e_2 \neq e_1 \wedge_{\mathfrak{B}'} e_2$ (even if $e_1 \wedge e_2$ exists in $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$). To wit, even if $e_1 \Longrightarrow e_2$, there may be a consistent Boolean lattice $\mathfrak{B}_1$ containing e_1 and e_2 such that $p_{\varrho,\mathfrak{B}_1}(e_2 \mid e_1)$ is not one or is not well-defined. However, it seems reasonable to define $e_1 \Longrightarrow_{\varrho} e_2$ if there exists a consistent Boolean sublattice $\mathfrak{B}$ of $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ containing e_1, e_2 and some further element $e_3 \in \mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ satisfying $e_1 \geq e_3$ and $e_2 \geq e_3$ such that $\frac{p_{\varrho,\mathfrak{B}}(e_3,e_3)}{p_{\varrho,\mathfrak{B}}(e_1,e_1)}$ is well-defined in $\mathfrak{B}$ and equal to one. But this is exactly what has been done in Definition 16.

Relatively Consistent Effect Histories

We proceed now as in Section 3.1.4 and introduce the notion of relatively consistent Boolean lattices of effect histories.

Definition 17 *A Boolean sublattice $(\mathfrak{B}, \oplus)$ of $(\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H}), \oplus)$ is called* RELATIVELY CONSISTENT W.R.T. *ϱ if the following* ***relative consistency condition*** *is satisfied for all $u, v \in \mathfrak{B}$:*

$$\text{Re}\, \mathcal{D}_{\varrho,1}(u \ominus 0_{\mathfrak{B}}, v \ominus 0_{\mathfrak{B}}) = 0 \quad \text{whenever } u \wedge_{\mathfrak{B}} v = 0_{\mathfrak{B}}, \tag{4.4a}$$

$$\mathcal{D}_{\varrho,1}(u,u) \geq \mathcal{D}_{\varrho,1}(0_{\mathfrak{B}}, 0_{\mathfrak{B}}), \tag{4.4b}$$

with strict inequality in Equation 4.4b *if $u = 1_{\mathfrak{B}}$.*

Theorem 5 *Let $(\mathfrak{B}, \oplus)$ be a relatively consistent Boolean lattice of effect histories. Then the decoherence functional $\mathcal{D}_{\varrho,1}$ induces a probability functional $p_{\varrho,\mathfrak{B}}$ on $\mathfrak{B}$ by*

$$p_{\varrho} : \mathfrak{B} \to \mathbb{R}^+, p_{\varrho}(u) := \frac{\mathcal{D}_{\varrho,1}(u,u) - \mathcal{D}_{\varrho,1}(0_{\mathfrak{B}}, 0_{\mathfrak{B}})}{\mathcal{D}_{\varrho,1}(1_{\mathfrak{B}}, 1_{\mathfrak{B}}) - \mathcal{D}_{\varrho,1}(0_{\mathfrak{B}}, 0_{\mathfrak{B}})}. \tag{4.5}$$

The proof is straightforward.

Definition 18 *An effect history proposition $e_1 \in \mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ is said to* IMPLY *an effect history proposition $e_2 \in \mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ in the state ϱ* ***relative to the effect history proposition*** *e_0 if there exists a relatively consistent Boolean sublattice $\mathfrak{B}$ of $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ containing e_1 and e_2 such that $e_0 = 0_{\mathfrak{B}}$ and such that the conditional probability $p_{\varrho,\mathfrak{B}}(e_2|e_1) \equiv \frac{p_{\varrho,\mathfrak{B}}(e_1 \wedge_{\mathfrak{B}} e_2)}{p_{\varrho,\mathfrak{B}}(e_1)}$ is well-defined and equal to one. We write $e_1 \Longrightarrow_{\varrho,e_0} e_2$. Two history propositions e_1 and e_2 are said to be* RELATIVELY EQUIVALENT *with respect to ϱ and relative to e_0 if $e_1 \Longrightarrow_{\varrho,e_0} e_2$ and vice versa. We write $e_1 \Longleftrightarrow_{\varrho,e_0} e_2$.*

The Generalized Rule of Interpretation

The generalized universal rule of interpretation of quantum mechanics can now simply be formulated as

Rule 2 *Propositions about quantum mechanical systems should solely be expressed in terms of effect history propositions. Every description of an isolated quantum mechanical system should be expressed in terms of finite effect history propositions belonging to a common (relatively) consistent Boolean sublattice of effect histories. Every reasoning relating several propositions should be expressed in terms of the logical relations induced by the probability measure from Theorem* 4 *in that Boolean algebra.*

It is clear that Rule 2 is indeed a generalization of Rule 1.

Superselection Rules

Our above discussion was restricted to the case when no superselection rules are present. Superselection rules can easily be included into the consistent histories formalism. Here we restrict ourselves to discrete superselection rules.

We first recall the notion of superselection rule in standard quantum mechanics. Assume that for the quantum system under consideration there are pairwise orthogonal projection operators $\{P_\nu\}$ with $\sum_\nu P_\nu = 1$. The $\{P_\nu\}$ are said to define a superselection rule if the following stipulations are true:

- A generalized observable $\mathcal{O}$ represents a physical quantity only if every effect operator occurring in its range commutes with every P_ν.
- A density operator ϱ represents a physical state only if ϱ commutes with every P_ν.

The subspaces $P_\nu\mathfrak{H}$ are called the *coherent subspaces* of $\mathfrak{H}$. We add the usual assumption that the time evolution respects the superselection rule, i.e., leaves the coherent subspaces invariant.

In the history formulation one considers *homogeneous superselection histories* of the form $\mathbf{P}_{\{\nu_i\}} := P_{\nu_1} \otimes P_{\nu_2} \otimes \cdots \otimes P_{\nu_n}$. The standard history state represented by the operator $\mathfrak{X}_\varrho$ gives nonvanishing probabilities only to those superselection histories which satisfy $\nu_1 = \nu_2 = \cdots = \nu_n$. We write $\mathbf{P}_\nu := P_\nu \otimes P_\nu \otimes \cdots \otimes P_\nu$. *Homogeneous superselection rules* may now be incorporated into a quantum history theory by requiring the following stipulations:

- An effect history E represents a physical history only if E commutes with every $\mathbf{P}_{\{\nu\}}$.

- An operator $\mathfrak{X}_\varrho$ represents a physical state only if $(1 \otimes \mathbf{P}_{\nu_1})\mathfrak{X}_\varrho(\mathbf{P}_{\nu_2} \otimes 1) = 0$ for all $\nu_1 \neq \nu_2$.

It is immediately clear that the formalism of the consistent histories approach allows for a more general notion of superselection rule. *Inhomogeneous superselection rules* in quantum history theories are determined by a complete set of finite pairwise orthogonal ordinary (possibly inhomogeneous) histories $\{H_\nu\}$ and by the following two stipulations:

- An effect history E represents a physical history only if E commutes with every H_ν.
- An operator $\mathfrak{X}_\varrho$ represents a physical state only if it satisfies the *medium decoherence condition* $d_\varrho(H_{\nu_1}, H_{\nu_2}) = \mathrm{tr}\,((1 \otimes H_{\nu_1})\mathfrak{X}_\varrho(H_{\nu_2} \otimes 1)) = 0$ whenever $H_{\nu_1} \neq H_{\nu_2}$.

Homogeneous superselection rules can already be fully understood at the level of single-time Hilbert spaces in ordinary quantum mechanics. In contrast, inhomogeneous superselection rules cannot even be formulated in ordinary quantum mechanics. However, at the time of writing the author knows of no candidate for an inhomogeneous superselection rule in nature (not to say an example).

5

LOCALITY AND CONSISTENT HISTORIES

Was du begehrst, in Büchern
las ich's nicht.
Doch hört ich oft, daß Ungewöhnliches
sich nur im Ungewöhnlichen entschleiert.
[from "Das Wunder der Heliane" by Erich Wolfgang Korngold/Hans Müller]

5.1 THE PRINCIPLE OF LOCALITY

Every real measurement performed upon a physical system is accompanied by a (possibly highly imprecise) localization of the system (or some part of it) in some (possibly very large but) finite region of space-time. That is, every measurable effect E takes place in a finite region $\mathsf{R}(E)$ of space-time. Using a more realistic language, only localized "events," i.e., events occurring in a finite space-time region, can be observed. [1] In other words, real measurements and state preparations have an inevitably quasilocal character. [2] Accordingly, in this chapter we exclude all observables which refer to infinitely extended regions of space-time.

Quantum mechanics contains nonlocal features. Attempts to reconcile the principle of locality with quantum field theory have led to the so-called axiomatic or algebraic approaches to quantum field theory, [68, 72]. In classical physics the principle of locality states that there is no action at a distance. In special relativity the principle of locality is sharpened to Einstein's principle of causality. In quantum field theory the principle of locality is usually implemented by proposing that observables can be associated with finite space-time regions and by requiring that spacelike separated observables are always compatible.

The consistent histories formalism as presented in the previous chapters is clearly in conflict with the principle of locality. It is the subject of the present chapter to investigate whether a "local" version of the consistent histories approach can be accomplished.

As discussed in detail in Chapter 3, one of the basic assumptions of the consistent histories approach is that the form of the quantum mechanical probabilities determined by the Wigner formula and the Lüders-von Neumann projection postulate is still meaningful even when the interpretation of the collapse of the state as a physical process is abandoned. Hence the form of the probabilities in "history" quantum mechanics is equal to the form of the probabilities in "collapse" quantum mechanics.

According to the orthodox Lüders-von Neumann postulate, in a measurement the alleged collapse of the wave function will instantaneously change the wave function in all space. This

[1] Some comments about the notion of localizability in relativistic quantum theories can be found in Section 5.5 below.

[2] The word *quasilocal* is used to indicate that quantum nonlocalities referring to infinitely extended space-time regions are dismissed whereas quantum nonlocalities are allowed for in finite regions of space-time.

implies, for instance, that the probabilities given by nonrelativistic quantum mechanics to the possible outcomes of several local measurements depend in general on the order in which the measurements are performed even if the measurement regions are spacelike separated. Moreover, the Lüders-von Neumann projection postulate gives no unambiguous prescription for the state change if several spacelike measurements are performed simultaneously. The prescription of the Lüders-von Neumann postulate is clearly in conflict with the principle of locality. Consider for instance a particle which is initially localized in a finite region of space. Suppose that at time t_p a measurement of the momentum of the particle is carried out. According to the Lüders-von Neumann postulate the initially localized wave packet will instantaneously 'collapse' into a momentum eigenfunction. If at time $t' > t_p$ a position measurement is performed, there is a nonzero probability to find the particle in every region spacelike separated from the initial localization region. Hence, apparently momentum measurements can be used to move the particle around at arbitrarily high velocities. Since the form of the probabilities is retained in the consistent histories approach, the situation is essentially the same in the consistent histories approach. In a sense, the situation is even worse. Whereas the breakdown of causality occurs in "collapse" enhanced quantum mechanics 'only' as a response to certain measurements, according to the realistic attitude of the consistent histories philosophy such nonlocal "events" are considered to be *real* !

Since the state has no beable status in the consistent histories approach an instantaneous "collapse" of the state in all of space is not *per se* problematic. What is problematic is that the Lüders-von Neumann postulate gives nonzero probabilities to events which are irreconcilable with Einstein causality.

In this chapter I will put forward a scheme for formulating the consistent histories programme in Minkowski space in a way respecting the principle of locality. I want to stress right from the beginning that this goal can only be achieved in a physically sensible way when POV measures are taken into account. The reason for this is that the 'nonlocal' observables have to be replaced by appropriate sets of localized "unsharp" observables (POV measures). In particular the momentum observable of a single particle has to be replaced by a net of local "unsharp" joint position-momentum observables associated with finite regions of space or space-time.

Although the 'local' scheme for consistent histories developed in this chapter is in some aspects reminiscent to the structure of algebraic approach to quantum field theory and although the scheme (or some modified version thereof) might possibly eventually also be applied to quantum field theory, I want to emphasize that the investigation of the present chapter is primarily and quite generally intended to be an investigation in the general structure of relativistic quantum *history* theories. The purpose of this section is twofold. The first objective is to put forward the idea that local quantum theories making use of the notion of localizability in *open* space-time regions is best formulated in terms of the concepts of the history approach. Secondly, it is attempted to construct for this framework a Lorentz covariant generalization of the orthodox Lüders-von Neumann reduction postulate. The purpose of this section is **not** to develop a novel axiomatic scheme for quantum theories. It will be argued below that

the proposed formalism can be imbedded without essential changes into the known algebraic approaches.

5.2 THE SCHLIEDER-HELLWIG-KRAUS PROJECTION POSTULATE

Historically, Landau and Peierls [111] were the first to investigate the conflict between the principle of locality and the projection postulate in quantum mechanics (although they stated the problem somewhat differently). They claimed to have shown that can as a matter of principle nonlocal observables such as the momentum of a particle cannot be measured accurately in the relativistic domain. Specifically, they claimed that in the relativistic domain "new uncertainty relations" like $\Delta p \Delta t \sim \hbar/c$ arise. This relation stipulates that in a momentum measurement of duration Δt the momentum p cannot be measured with higher accuracy than $\Delta p \sim \frac{\hbar}{c\Delta t}$. The derivation of this 'uncertainty relation' relies heavily upon the so called 'pragmatic' interpretation of the energy-time uncertainty relation. However, as discussed by Busch [19] this interpretation of the energy-time uncertainty relation is invalid. Moreover, Aharonov and Albert have encountered a *Gedankenexperiment* directly invalidating the Landau-Peierls uncertainty relation [2].

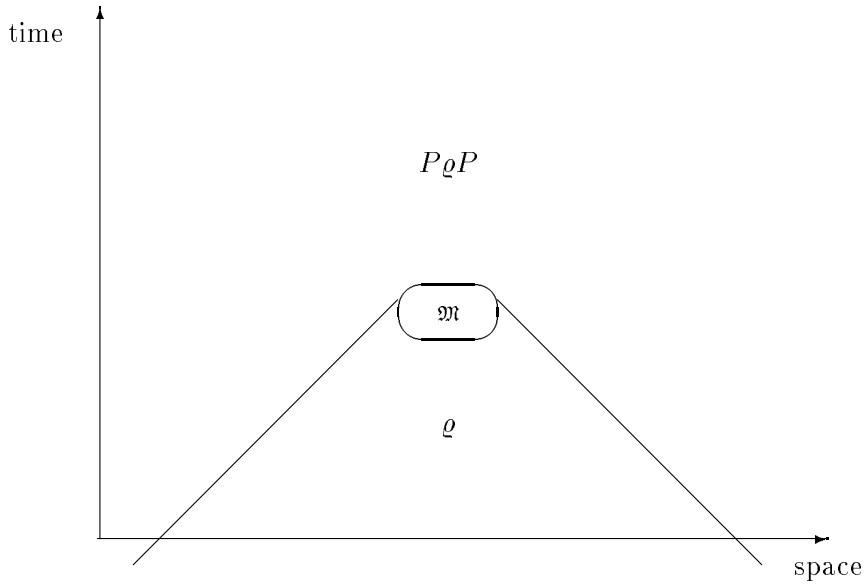

Figure 5.1

More than thirty years after Landau and Peierls [111], Schlieder [157] and then Hellwig and Kraus [84] investigated the problem in the realm of local quantum field theory. These works have been written in the context of the orthodox interpretation of quantum theories according to which a collapse of the state occurs a consequence of external measurements performed on the system. For the sake of the argument we will also adopt this point of view in the rest of

this section. Schlieder, Hellwig and Kraus observed that the Lüders-von Neumann postulate is not a Lorentz covariant prescription and disobeys Einstein causality. They only assumed the following general structure usually adopted in local quantum field theory.

- There is a global Hilbert space $\mathfrak{H}$ associated with the system in question.
- There exists a Lorentz covariant net of local observables with accompanying local "properties" (represented by projection operators on $\mathfrak{H}$).
- Observables (or properties) associated with spacelike separated regions commute.

To reinstate Einstein causality, they proposed that the relativistic reduction process occurs along the backward light cone of the measurement region. This prescription is obviously manifestly Lorentz covariant. We will refer to this prescription as to the *backward Schlieder-Hellwig-Kraus projection postulate*. It is illustrated in Figure 5.1.

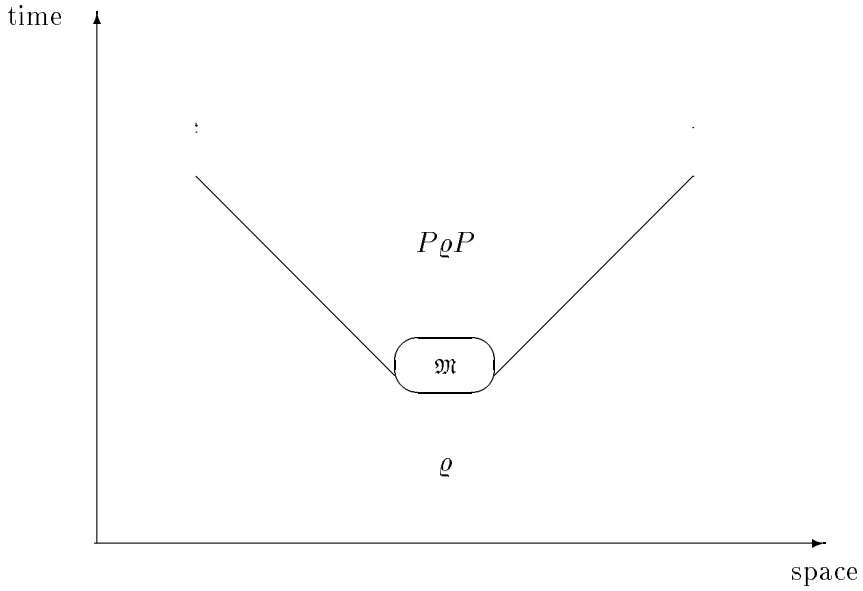

Figure 5.2

Suppose that the property P is measured in the measurement region $\mathfrak{M}$. Then the backward Schlieder-Hellwig-Kraus postulate stipulates that in the backward light cone of the measurement region $\mathfrak{M}$ the state is the initially prepared state ϱ. In the forward light cone and in the side cone of $\mathfrak{M}$ the state is given by the reduced density operator $P\varrho P/\mathrm{tr}(P\varrho P)$.

The *forward Schlieder-Hellwig-Kraus projection postulate* can be formulated analogously. According to this postulate the reduction process occurs along the forward light cone of the region in which the measurement takes place. [3] It is illustrated in Figure 5.2.

If several local measurements are performed, then neither the backward nor the forward Schlieder-Hellwig-Kraus postulate produce any ambiguities, since properties corresponding to spacelike separated space-time regions commute. The state history is illustrated in Figure 5.3 for the forward Schlieder-Hellwig-Kraus postulate in the case that two measurements are performed on the system in spacelike separated regions $\mathfrak{M}_1$ and $\mathfrak{M}_2$ with the results P_1 and P_2 respectively. Locality requires that $P_1 P_2 = P_2 P_1$ and accordingly the state assignment in the intersection of the forward light cones of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ is unambiguous.

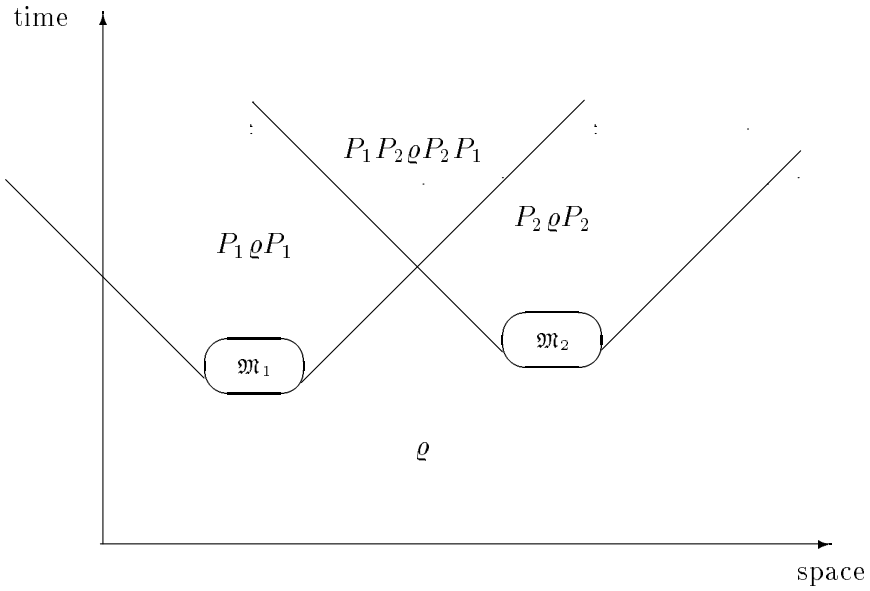

Figure 5.3

If the state of a quantum system is assumed to represent the *knowledge* of local observers, then – since no information can be transmitted faster than light – it seems to be sensible to adopt the forward Schlieder-Hellwig-Kraus postulate to describe the state collapse. (It is insignificant for the argument whether the state represents the actual or 'only' the potential knowledge of local observers.) To illustrate this, suppose that two local observers perform spacelike separated localization experiments $\mathfrak{M}_1$ and $\mathfrak{M}_2$ as depicted in Figure 5.3. If the state represents the knowledge of local observers, then both observers have to use the initially prepared state ϱ to make predictions.

In contrast, anyone who believes that the state of a system represents an objective description of the quantum system in question (and has in particular nothing to do with the (actual

[3] It should be stressed that Hellwig and Kraus considered solely the backward postulate, whereas Schlieder considered also the forward postulate.

or potential) knowledge of local observers) will prefer the backward Schlieder-Hellwig-Kraus postulate. A standard argument in favour of the backward Schlieder-Hellwig-Kraus postulate is that only the backward Schlieder-Hellwig-Kraus postulate can account for long-range correlations between space-like separated regions appropriately. In the context of the example of the two localization experiments depicted in Figure 5.3, assume that the local observer at $\mathfrak{M}_1$ localizes the particle in the region $\mathfrak{M}_1$, then it is clear that the observer at $\mathfrak{M}_2$ will not detect the particle in the region $\mathfrak{M}_2$. Clearly the forward Schlieder-Hellwig-Kraus postulate is inappropriate for this situation. This is in essence the reason why Hellwig and Kraus preferred the backward postulate and discarded the forward postulate.

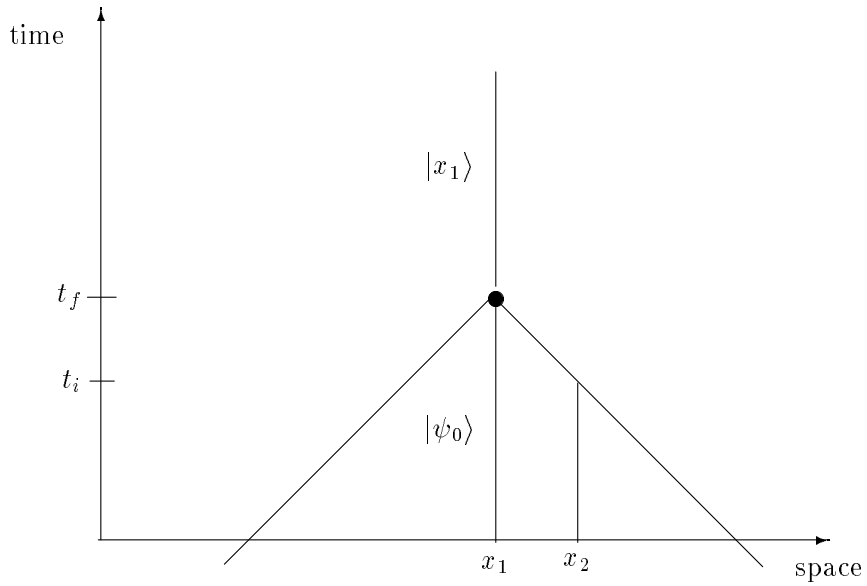

Figure 5.4

In contrast to the above described situation, in the realm of the consistent histories approach and its realistic attitude a modified version of the forward Schlieder-Hellwig-Kraus postulate can be used. This will be discussed in Section 5.3.

In a penetrating analysis Aharonov and Albert [2, 3, 4, 5] have argued that both the forward and the backward Schlieder-Hellwig-Kraus projection postulate cannot be uphold in a genuine relativistic theory of measurement. Their argument centres around the fact that it is as a matter of principle possible to measure (at least some) nonlocal properties of quantum systems in arbitrarily short time intervals. The Schlieder-Hellwig-Kraus postulate does not associate a unique state vector to every spacelike hypersurface. Therefore no quantum mechanical probability can unambiguously associated with nonlocal observables on some spacelike hypersurfaces. This clearly invalidates the forward Schlieder-Hellwig-Kraus postulate since quantum mechanics would then make no *predictions* for measurements of some nonlocal

observables. For the backward Schlieder-Hellwig-Kraus postulate the situation is not that clear. Aharonov and Albert have given an illustrative example which unravels the kind of problems arising here.

They consider a one particle system initially prepared in the state $|\psi_0\rangle = |x_1\rangle + |x_2\rangle$, where $|x_1\rangle$ and $|x_2\rangle$ denote wave packets sharply localized at x_1 and x_2 respectively. Suppose that at time t_f the particle is found in a measurement at x_1, then the state history as prescribed by the backward Schlieder-Hellwig-Kraus postulate is depicted in Figure 5.4. One sees by mere inspection that the Schlieder-Hellwig-Kraus postulate does not conserve probabilities. The overall probability to find the particle somewhere at time t, with $t_i < t < t_f$, is $1/2$.

In the Copenhagen spirit one may argue that all problems with the backward Schlieder-Hellwig-Kraus postulate arise in connection with *retrodictive* statements and that this may be taken only as a further indication that retrodictive statements in quantum mechanics are meaningless and should be avoided whereas the backward Schlieder-Hellwig-Kraus postulate can possibly be retained. If one takes a more realistic attitude (such as in the consistent histories approach), then the above example clearly invalidates the backward Schlieder-Hellwig-Kraus postulate.

In the consistent histories approach where state histories are only meaningful with respect to some fixed framework it is possible to cope with all problems of the (forward) Schlieder-Hellwig-Kraus postulate mentioned above. This will be discussed in the next section.

5.3 AN EVOLUTIONARY PICTURE BASED ON QUANTUM HISTORIES

5.3.1 Prolegomenon

In this section I will describe a general scheme for a Lorentz covariant generalization of formalism of the consistent histories approach. The aim of this section is to develop an evolutionary picture for quantum histories. The histories are interpreted as *events*. The approach presented in the present section mixes ideas from the consistent histories approach with ideas from the evolutionary picture of physics. Some ideas of both approaches, however, cannot be maintained. One of the basic assumptions of the approach is that all events take place in finite space-time regions. Events have therefore an inevitably quasilocal character. This is in contrast to some approaches to the consistent histories scheme where it is assumed that histories represent a possible sequence of actual events as they may happen in the actual world. Consequently, in the latter approaches histories can in principle be infinitely extended in time. The very notion of history makes use of the concept of event at a fixed time. In order to achieve a Lorentz covariant formulation it is necessary, however, to allow events to be spread out in time *and* space. The reason for this is, that if an event takes place in a bounded region of space at a fixed time in some Lorentz frame, then in another Lorentz transformed reference frame the event will in general only be spread out on a certain spacelike hypersurface but cannot be associated with a certain fixed time. Hence, from the point of view

of the consistent histories approach it is natural to identify events with general (homogeneous or inhomogeneous) histories.

The foregoing is of course neither meant to imply that events are necessarily pointlike nor that they do not exhibit nonlocal features. Indeed, events may be nonlocal as long as the nonlocality refers only to finite (but possibly large) regions of space-time. For instance, if $|\alpha\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2)$ represents a correlated (entangled) spin-0 state of two spacelike separated spin-$\frac{1}{2}$ particles, then the projector $P = |\uparrow\rangle_1|\downarrow\rangle_2\,{}_1\langle\uparrow|_2\langle\downarrow|$ corresponds to some possible nonlocal event.

In the next subsection we specify a framework for a Lorentz covariant consistent events approach (as it should be called) in an axiomatic way. The reader familiar with axiomatic field theory will recognize the structural similarity between the axioms listed and the Haag-Kastler axioms of local quantum field theory. However, since this is no work about quantum field theory, we will require only some very general physical assumptions, following from the requirements of causality and Lorentz covariance. It is not presupposed that the system under consideration is a relativistic quantized field. Accordingly, some (Wightman) axioms which are indispensable for a full-fledged quantum field theory (like, e.g., existence of a physical vacuum, positivity of the energy-momentum operator etc.) shall not be discussed. In particular, we do not make any assumptions whatsoever about the concrete realization and form of the local observables and properties. Specifically, we do not assume that the local properties arise from local observables which are local functions (e.g., polynomials) of the field operators of some quantum field theory. However, it is understood that the sets of local properties contain physically interesting properties, such as position, momentum, spin, isospin and so forth. Moreover, we assume that no fermionic degrees of freedom are present.

In their seminal paper [68] Haag and Kastler formulated their axiomatic approach to quantum field theory in terms of states and operations. In the axiomatic approach to quantum field theory it is usually assumed that to each bounded open space-time region O there is associated a $*$-algebra (e.g., a W^*- or C^*-algebra) $\mathcal{A}(\mathsf{O})$. Usually it is assumed that $\mathcal{A}(\mathsf{O})$ is (or can be represented as) an algebra of operators on Hilbert space. The ordinary local observables in O are then identified with the self-adjoint operators in $\mathcal{A}(\mathsf{O})$. Haag and Kastler assumed that the algebra of all observables has the texture of a net of local operator algebras

$$\mathsf{O} \mapsto \mathcal{A}(\mathsf{O})$$

on $\mathcal{M}$. They identified states with positive linear forms over the algebra $\mathcal{A}$ of all observables on $\mathcal{M}$. They considered in particular the case that the net of local algebras consists of abstract C^*-algebras and argued that two quantum theories with Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are physically equivalent whenever we have a faithful representation of the same abstract algebra on both Hilbert spaces. In the present thesis we will not follow that route and instead assume from the outset that a fixed Hilbert space is given and that the net of local algebras consists of von Neumann-algebras on the given Hilbert space $\mathfrak{H}$. Moreover, we will identify states only with density operators on that Hilbert space (or in the history approach with decoherence functionals). What is important for our purposes is, however, that Haag and Kastler put forward the idea to interpret the elements in $\mathcal{A}(\mathsf{O})$ as representing physical

operations performed in the space-time region O on the system. The general form of such operations in Hilbert space quantum mechanics has subsequently been investigated by Hellwig and Kraus [82, 83, 108], see also [110]. Their results show that every operation can be written as $\varrho \mapsto \sum_{k\in K} A_k \varrho A_k^*$ where $\{A_k\}$ is a countable family of bounded operators on $\mathfrak{H}$. This family is not unique. Moreover, to every operation there corresponds an effect operator given by $E = \sum_{k\in K} A_k A_k^*$. This correspondence is obviously many-to-one. The physical interpretation of this ambiguity is that every operation represents a measuring process, that every measuring process measures an effect and that there is more than one measuring process for every effect. The results of Hellwig and Kraus can easily be generalized to local quantum field theories, where only local operations corresponding to local measuring processes are meaningful. The measured effect is then represented by a local effect operator, see Ref. [108].

In the more realistic attitude of the consistent histories approach the notion of operation looses its central status. It is the effect which constitutes the fundamental ingredient in the theory. The "effect" operators should be called "event" operators in this approach, but we will stick to the usual terminology. Instead of states and operations a realistic local quantum theory should be formulated in terms of states and local events (or effects). It is this what is attempted in the present section.

From this point of view it is natural to assume that the net of D-posets of events on $\mathcal{M}$ proposed in Section 5.3.2 below arises from the net of local operator algebras $\mathsf{O} \mapsto \mathcal{A}(\mathsf{O})$ on $\mathcal{M}$ in the way sketched above. In the operationally oriented Haag-Kastler approach the local algebra $\mathcal{A}(\mathsf{O})$ may looked upon as a basin for families of operators $\{A_k\}, A_k \in \mathcal{A}(\mathsf{O})$, for making up the local operations. In a more realistic approach it is still sensible to assume that $\mathfrak{E}_\mathsf{O} \subset \mathcal{A}(\mathsf{O})$ as a working hypothesis.

It must be stressed that although it is assumed that $\mathfrak{E}_\mathsf{O} \subset \mathcal{A}(\mathsf{O})$, the elements in $\mathfrak{E}_\mathsf{O}$ and in $\mathcal{A}(\mathsf{O})$ are usually interpreted differently. An operator $E \in \mathfrak{E}_\mathsf{O}$ represents an event in the space-time region O whereas the same operator $E \in \mathcal{A}(\mathsf{O})$ is usually asserted to represent an ordinary observable in the space-time region O.

If one insists to interpret the local algebras as algebras of all local observables, then it is also possible but somewhat more complicated to incorporate generalized observables which are POV measures. As emphasized repeatedly above, the ordinary observables of quantum mechanics (identified with bounded self-adjoint operators) are in one-to-one correspondence with Borel PV measures on the real line. In contrast, POV measures cannot unambiguously be represented by bounded self-adjoint operator on Hilbert space. However, due to a theorem by Naimark, it is known that Borel POV measures on the real line are in one-to-one correspondence with maximally symmetric operators on Hilbert space. [4] This result indicates that in order to incorporate generalized observables into this axiomatic framework and to recast the net of D-posets into a net of local operator algebras, it is necessary to allow for local operator algebras of *unbounded* Hilbert space operators. The local observables are then identified with the maximally symmetric operators in $\mathcal{A}(\mathsf{O})$. The local D-posets $\mathfrak{E}_\mathsf{O}$ associated

[4] The details of the argument can be found for instance in the monograph by Busch et al. [23]. Naimark's theorem can for instance be found in the appendix of Reference [148].

with the open set $\mathsf{O} \subset \mathcal{M}$ can then be obtained by first rewriting the maximally symmetric operators in $\mathcal{A}(\mathsf{O})$ as Borel POV measures on $\mathbb{R}$ and second by recalling that the range of a POV measure generates a D-poset and third by taking $\mathfrak{E}_{\mathsf{O}}$ as the D-poset generated by all ranges of the POV measures in $\mathcal{A}(\mathsf{O})$. Notice, that the assertion that already the union of the ranges of all local POV measures carries the structure of a D-poset would be a nontrivial proposal which has to be proven on a case-by-case basis for the local operator algebras in question. However, the latter situation would be much more satisfactory and we suggest that only nets of local algebras satisfying the latter requirement should be taken into account.

In the next subsection we will "forget" the more general structure given by the net of local algebras and focus our attention to the net of local events.

5.3.2 Nets of D-posets

Consider Minkowski space $\mathcal{M} = \mathbb{R}^4$ endowed with the usual Lorentz metric $\eta_{\mu\nu}$ with signature $(+,-,-,-)$. The Minkowski space is assumed to be furnished with standard topology which equals the Euclidean topology on $\mathbb{R}^4$. We proceed to state our basic axioms for a Lorentz covariant consistent histories formalism in Minkowski space $\mathcal{M}$.

A. There is a Hilbert space $\mathfrak{H}(\mathcal{M})$ carrying a unitary ray representation $U(a,\Lambda)$ of the Poincaré group.

B. For each bounded open space-time region $\mathsf{O} \subset \mathcal{M}$ there is a sub-D-poset $\mathfrak{E}_{\mathsf{O}} \subset \mathfrak{E}(\mathfrak{H}(\mathcal{M}))$ of local *events*. That is, the *net of D-posets*

$$\mathsf{O} \mapsto \mathfrak{E}_{\mathsf{O}}$$

constitutes the intrinsic mathematical description of the formalism. It is assumed that every local D-poset $\mathfrak{E}_{\mathsf{O}}$ contains the unit operator $1 \in \mathfrak{E}(\mathfrak{H})$. The local D-posets are assumed to satisfy $\mathfrak{E}_{\mathsf{O}} \subset \mathfrak{E}_{\mathsf{O}'}$ whenever $\mathsf{O} \subset \mathsf{O}'$.

C. [Lorentz covariance] $U(a,\Lambda)\mathfrak{E}_{\mathsf{O}}U^{\dagger}(a,\Lambda) = \mathfrak{E}_{\Lambda\mathsf{O}+a}$, where $\Lambda\mathsf{O}+a$ denotes the Lorentz transform of the region O under the Lorentz transformation (a,Λ).

D. [Locality] Local events $E \in \mathfrak{E}_{\mathsf{O}}$ and $F \in \mathfrak{E}_{\mathsf{O}'}$ commute, i.e., $EF = FE$, if the space-time regions O and O' are spacelike with respect to each other.

It it an operational fact that every real measurement of some physical observable is accompanied by some (possibly rough) localization of the system (or parts of it) in question. In our realistic approach this fact is mirrored in the requirement that *events* are always localized in some finite (but possibly very large) region of space-time. In every fixed Lorentz reference frame we define the support of an event $E \in \mathfrak{E}(\mathfrak{H})$ in terms of the net of D-posets by [5]

$$\mathfrak{w}(E) := \cap_i \{\mathsf{O}_i \mid E \in \mathfrak{E}_{\mathsf{O}_i}\}.$$

[5] Let $\mathcal{B}$ be an arbitrary Borel subset of $\mathcal{M}$. For convenience, we denote symbolically by $\mathfrak{E}_{\mathcal{B}}$ the set of all effect operators E satisfying $E \in \mathfrak{E}_{\mathsf{O}}$ for every open set $\mathsf{O} \supset \mathcal{B}$ and $E \notin \mathfrak{E}_{\mathsf{O}}$ for every open set $\mathsf{O} \subset \mathcal{B}$.

In other words $\mathfrak{w}(E)$ satisfies $E \in \mathfrak{E}_{\mathfrak{w}(E)}$ and it is the smallest bounded Borel space-time region with this property. Whenever O_1 and O_2 are disjoint space-time regions, we assume that the local D-posets satisfy the following additivity property

$$\mathfrak{E}_{\mathsf{O}_1 \cup \mathsf{O}_2} = \mathfrak{E}_{\mathsf{O}_1} \oplus \mathfrak{E}_{\mathsf{O}_2},$$

where $\mathfrak{E}_{\mathsf{O}_1} \oplus \mathfrak{E}_{\mathsf{O}_2}$ denotes the D-poset generated by $\mathfrak{E}_{\mathsf{O}_1}$ and $\mathfrak{E}_{\mathsf{O}_2}$, i.e., the D-poset consisting of all well-defined sums of elements of $\mathfrak{E}_{\mathsf{O}_1}$ and $\mathfrak{E}_{\mathsf{O}_2}$.

In the consistent histories approach the state is assumed to be given by a decoherence functional. In standard quantum mechanics the decoherence functional is uniquely determined by the usual state ϱ and by the unitary time evolution operator. Accordingly, it is denoted by d_ϱ. It has been shown by Isham, Linden and Schreckenberg [93] (see also Chapter 3 and Appendix C) that d_ϱ can be written in the form $d_\varrho(h,k) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h \otimes k\mathfrak{X}_\varrho)$ with some appropriate trace class operator $\mathfrak{X}_\varrho$ on $\mathfrak{H} \otimes \mathfrak{H}$. Motivated by this fact, we assume in this section that

E. the state of a quantum system in our local history scheme is given by a trace class operator $\mathfrak{X}$ on $\mathfrak{H}(\mathcal{M}) \otimes \mathfrak{H}(\mathcal{M})$ satisfying the properties

   - $\mathfrak{X}^\dagger = M\mathfrak{X}M$;
   - $\mathrm{tr}_{\mathfrak{H}(\mathcal{M})\otimes\mathfrak{H}(\mathcal{M})}(\mathfrak{X}) = 1$;
   - $\mathrm{tr}_{\mathfrak{H}(\mathcal{M})\otimes\mathfrak{H}(\mathcal{M})}\left(h \otimes h(\mathfrak{X} + \mathfrak{X}^\dagger)\right) \geq 0$,

   where M denotes the reversal operator $M : \mathfrak{H}(\mathcal{M}) \otimes \mathfrak{H}(\mathcal{M}) \to \mathfrak{H}(\mathcal{M}) \otimes \mathfrak{H}(\mathcal{M})$ defined by $M|h_1\rangle \otimes |h_2\rangle := |h_2\rangle \otimes |h_1\rangle$ and extended by linearity.

5.3.3 Consistent Events

The next task is to investigate under what conditions a description of the quantum system and reasoning is unambiguously possible in terms of local events and sequences of local events. The requirements to be satisfied are as follows. First, it should be possible to define a probability measure on the consistent set of histories in a reasonable way. Second, it should be possible to unambiguously associate with every element of a consistent set of histories a state history in all of $\mathcal{M}$. [6] This state history should be Lorentz covariant. We allow for the possibility that the 'state' may be 0 in some regions of $\mathcal{M}$. In this case the probability of the corresponding history is likewise equal to 0. As we have seen already for the standard consistent histories formalism, to every history there corresponds some state history. As we will see below in the relativistic context state histories are *framework dependent*, i.e., depend on the consistent set

[6] As always in the history approach to quantum mechanics, a notion of *absolute* state history is fundamentally absent. The notion of state history can only be introduced in retrospect and *relative* to a given framework when finding (or considering) a situation that a certain history or sequence of histories is realized in order to reconcile it with the ideas of the evolutionary picture. (I am grateful to S. Schreckenberg for this clear statement (private conversation).)

chosen: A fixed history of events in $\mathcal{M}$ corresponds to possibly different state histories in $\mathcal{M}$ in different frameworks.

Since probabilistic statements are meaningful only with respect to some fixed framework, framework dependent state histories represent no restriction of the applicability of the formalism. We again stress that the framework dependence of the state history is closely related to the status of the state as an unobservable or non-beable.

As anticipated in Chapters 2 and 3 we will adopt a point of view which is similar to Haag's evolutionary picture. The reader may wish to recall our brief discussion in Section 2.4. However, we enhance Haag's proposal by some additional proposal as follows: we assert that there are several alternative or complementary evolving patterns of events which must all be considered equally objective. There is no notion of *absolute* state history and no absolute evolving pattern of events. These two concepts are meaningful only with respect to some framework when a certain history from this framework is considered. This is the consistent historians usual attitude applied to the evolutionary view of physics.

Suppose the state of the system in question in a certain open space-time region R is given by the operator $\mathfrak{X}$. The state $\mathfrak{X}$ represents the memory of the system of the past realized events in the causal past of R. This operator is framework (or context) independent and is assumed to represent the sum total of all propensities inherent in the system at the instant in question. Moreover, suppose that some event represented by some effect operator E occurs. Then the state has to be transformed according to the rule specified in Section 3.2

$$\mathfrak{X} \mapsto \widetilde{\mathfrak{X}}^E := \frac{(1 \otimes \sqrt{E})\mathfrak{X}(\sqrt{E} \otimes 1)}{\mathrm{tr}_{\mathfrak{H}(\mathcal{M})\otimes\mathfrak{H}(\mathcal{M})}\Big((1 \otimes \sqrt{E})\mathfrak{X}(\sqrt{E} \otimes 1)\Big)}. \tag{5.1}$$

In the evolutionary picture of physics the arrow of time is put in by hand by asserting that the future is open and that past events are subsumed in the state $\mathfrak{X}$. To make predictions for possible future events in some bounded open space-time region R the state has to be chosen as to incorporate all events in the causal past of the region R in question. Past events which are spacelike separated from the region R and which are not causally connected to R (e.g., through nonlocal quantum entanglement) may or may not be taken into account.

By locality the effect operators representing spacelike separated events commute. Hence, it is a consequence of the transformation rule 5.1 that for making predictions in a certain open space-time region R it is indeed only necessary to take into account all past events in the causal past of R.

The definition of a framework and the details of the state transformation are discussed next.

Consider a bounded open space-time region $\mathsf{R} \subset \mathcal{M}$. The region R is not assumed to have a topologically simple structure: R may have holes or may be the union of several disjoint sets and so on. Assume that the state of the system in the region R is unambiguously given by the operator $\mathfrak{X}_\mathsf{R}$ on $\mathfrak{H}(\mathcal{M}) \otimes \mathfrak{H}(\mathcal{M})$. As usual $\mathfrak{X}_\mathsf{R}$ is assumed to be of trace class and to define a decoherence functional d_R in the usual way.

A subset $\mathfrak{B}_{\mathsf{R}}$ of $\mathfrak{E}_{\mathsf{R}}$ is said to be a CONSISTENT SET OF EVENTS FOR THE REGION R if

- R is the union of the supports of all elements in $\mathfrak{B}_{\mathsf{R}}$.
- $(\mathfrak{B}_{\mathsf{R}}, \oplus)$ is a Boolean sub-D-poset of $(\mathfrak{E}_{\mathsf{R}}, \oplus)$.
- Re $d_{\mathsf{R}}(E_i, E_j) = \text{Re tr}_{\mathfrak{H}(\mathcal{M})\otimes\mathfrak{H}(\mathcal{M})}(E_i \otimes E_j \mathfrak{X}_{\mathsf{R}}) = 0$, for all disjoint elements $E_i, E_j \in \mathfrak{B}_{\mathsf{R}}$.

A specification of a bounded open space-time region R and of a consistent set of possible events for R will in the sequel also be called an R-FRAMEWORK whenever an unambiguous state can be associated with the region R. The decoherence functional d_{R} defines then a probability measure on consistent Boolean sub-D-posets of $(\mathfrak{E}_{\mathsf{R}}, \oplus)$ in the usual way. The details can be found in Chapters 3 and 4 and need not be repeated here.

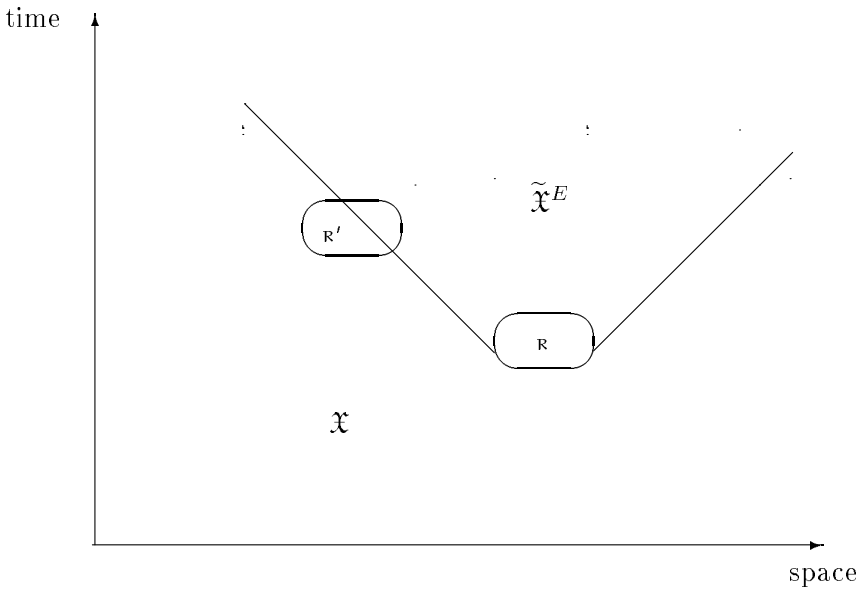

Figure 5.5

Consider some R-framework. Suppose that the event represented by the effect operator E has occurred. Then the transformation rule Equation 5.1 is supplemented by the stipulation that the state $\mathfrak{X}_{\mathsf{R}}$ has to be replaced by $\widetilde{\mathfrak{X}}^E_{\mathsf{R}} := \mathcal{N}^E(1 \otimes \sqrt{E})\mathfrak{X}_{\mathsf{R}}(\sqrt{E} \otimes 1)$ *in the forward light cone of the region* R ($\mathcal{N}^E$ is a normalization constant). This stipulation may be thought of as a modified version of the forward Schlieder-Hellwig-Kraus postulate. However, it differs from the original Schlieder-Hellwig-Kraus postulate in some important respects. First, the state is replaced in the complete forward light cone of R by the transformed state and not only in the forward light cone of the event which eventually has occurred. Second, it is not

possible to unambiguously associate a state with space-time regions crossing the boundary of the forward lightcone of some past events. Sets of possible events in such a space-time regions are doomed to be *inconsistent*. This is illustrated in Figure 5.5. Here the boundary of the forward light cone of the region R intersects the region R'. Suppose that in region R the event E has occurred. Then it is not possible to unambiguously associate a state with the region R'. Hence, all subsets of events $\mathfrak{B}_{\mathsf{R}'}$ of $\mathfrak{E}_{\mathsf{R}'}$ are a priori inconsistent.

If one insists in considering events in R', then one has to construct different frameworks. Specifically, instead of R' one has to consider the open space-time region $\mathsf{R}'' := \mathsf{R} \cup \mathsf{R}'$ and to construct consistent sets of events for the region R''. In the situation sketched in Figure 5.5 it is possible to associate unambiguously the state $\mathfrak{X}$ with the space-time region R''.

To summarize, in this section I have argued that the concepts and the ideas of the consistent histories approach are in a natural way suited to the evolutionary picture of local quantum physics and put forward a proposal for a consistent local event approach to quantum mechanics respecting locality and Einstein causality. However, a weak point of the proposed scheme is that the scheme is based on the assumption that space-time has the structure of Minkowski space $\mathcal{M}$. Minkowski space serves as the a priori given arena in which events take place. Moreover, Einstein causality is foreign to the formalism and has to be put in by brute force by requiring that a realized event can only have causal influence on future events lying strictly inside its forward light cone. It is of course of some interest whether these shortcomings can be overcome in more sophisticated schemes. In the next section we will describe a possible scheme where no external Schlieder-Hellwig-Kraus type rule has to be proposed. It is the state $\mathfrak{X}$ itself which carries all causal relations and all information about quantum entanglement between different space-time regions.

5.4 LOCAL HILBERT SPACES

Isham and Linden [91] have argued that from the consistent historians point of view it is reasonable to consider not only discrete histories (as done in Chapters 3 and 4) but also *continuous* histories. They argued that the continuous tensor product $\otimes_t \mathfrak{H}_t$ of the one-time Hilbert spaces $\mathfrak{H}_t$ over $\mathbb{R}$ (provided this continuous tensor product can be given an exact meaning) provides the appropriate mathematical framework for continuous histories. [7] In the context of special relativity, however, the space and time coordinates should be treated on the same footing. Therefore one is led quite naturally to the concept of a continuous tensor product of Hilbert spaces over $\mathcal{M}$ or to the accompanying notion of "local Hilbert space." To wit, in the sequel we assume the Axioms A-C and E supplemented by the following

A.1 The Hilbert space $\mathfrak{H}(\mathcal{M})$ is a continuous tensor product over $\mathcal{M}$, i.e, for every Borel subset $A \subset \mathcal{M}$ there is a "local" Hilbert space $\mathfrak{H}(A)$ such that all conditions of Definition

[7] Since every two Hilbert spaces of the same cardinality are isomorphic, every infinite dimensional Hilbert space $\mathfrak{H}$ is isomorphic to $\exp \mathfrak{K}$ for some appropriate Hilbert space $\mathfrak{K}$ and hence the continuous tensor product of $\mathfrak{H}$ over $\mathbb{R}$ can be defined as $\exp \int_{\mathbb{R}} \mathfrak{K}_t \, dt$, see Appendix D.

35 in Appendix D are satisfied. Moreover, we require that the continuous tensor product structure of $\mathfrak{H}(\mathcal{M})$ respects the unitary ray representation of the Poincaré group on $\mathfrak{H}(\mathcal{M})$, i.e., that $U(a,\Lambda)\mathfrak{H}(A) = \mathfrak{H}(\Lambda A + a)$ in an obvious notation.

B.1 For every bounded space-time region O the D-poset $\mathfrak{E}_{\mathsf{O}}$ is given by the set of all effect operators on $\mathfrak{H}(\mathsf{O})$, i.e., $\mathfrak{E}_{\mathsf{O}} = \mathfrak{E}(\mathfrak{H}(\mathsf{O}))$.

Symbolically, we associate with each space-time point x^μ a Hilbert space $\mathfrak{H}(x^\mu)$ and we write $\mathfrak{H}(\mathsf{O}) = \otimes_{x^\mu \in \mathsf{O}} \mathfrak{H}(x^\mu)$ for every Borel subset $\mathsf{O} \subset \mathcal{M}$. The reader should keep in mind, however, that this is only a convenient suggestive but symbolical notation. Its precise meaning is given by Definition 35. Since in this picture every space-time point refers to different degrees of freedom of the system, every two event operators $E_1 \in \mathfrak{E}_{\mathsf{O}_1}$ and $E_2 \in \mathfrak{E}_{\mathsf{O}_2}$ commute whenever O_1 and O_2 are spacelike separated. What is more, E_1 and E_2 commute whenever O_1 and O_2 are disjoint regardless of whether they are spacelike or timelike separated. So, the principle of locality (Axiom D) is satisfied trivially in the present setting.

Axiom A.1 can be weakened in several ways. There are good arguments that the assumptions in A.1 are overidealizations. For instance, Axiom A.1 can be weakened by requiring that only for every open subset $\mathsf{O} \subset \mathcal{M}$ there is a local Hilbert space $\mathfrak{H}(\mathsf{O})$. It is perhaps more sensible to require only that $\mathfrak{H}(\mathcal{M})$ is a quasi-continuous tensor product (or quasi-factorizable) over the open sets of $\mathcal{M}$ or over some other class of subsets of $\mathcal{M}$. Accordingly, the Axioms A.1 and B.1 (as the Axioms A-E) are not considered rigid axioms but tentative working hypotheses which might be subject to change if necessary.

In the present scheme it is assumed that the state $\mathfrak{X}$ determines whether two disjoint space-time regions are entangled or correlated. It is well-known that the state in ordinary quantum mechanics may entangle properties of a system in separated regions of space but at a fixed time. A standard example is the following spin 0 state of two separated spin-$\frac{1}{2}$ particles

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow, x_1\rangle \otimes |\downarrow, x_2\rangle - |\downarrow, x_1\rangle \otimes |\uparrow, x_2\rangle \right),$$

where for instance $|\uparrow, x_2\rangle$ denotes the one particle state sharply localized at x_2 with spin up. In a Lorentz covariant quantum theory which treats space and time coordinates on the same footing it is thus natural to assume that the state $\mathfrak{X}$ carries not only the information about entanglement between different regions of space at a fixed time but also about all entanglement between different space-time regions regardless of whether they are spacelike or timelike separated. We next define the concept of quantum entanglement between different space-time regions in terms of the postulated net of D-posets of events. We say that two space-time regions R_1 and R_2 are ENTANGLED if there exist operators $E \in \mathfrak{E}_{\mathsf{R}_1}$ and $F \in \mathfrak{E}_{\mathsf{R}_2}$ such that

$$\mathrm{tr}_{\mathfrak{H}(\mathcal{M})\otimes\mathfrak{H}(\mathcal{M})} \left((E \otimes F) \otimes (E \otimes F)\mathfrak{X} \right) \neq \mathrm{tr}_{\mathfrak{H}(\mathcal{M})\otimes\mathfrak{H}(\mathcal{M})} (E \otimes E\mathfrak{X}) \mathrm{tr}_{\mathfrak{H}(\mathcal{M})\otimes\mathfrak{H}(\mathcal{M})} (F \otimes F\mathfrak{X}).$$

The central problem is now to construct a physical $\mathfrak{X}$ compatible with locality and causality. This difficult problem will not be attacked in the present thesis and must be left to the open research problems.

Instead it is perhaps worthwhile to call to mind again the way the state $\mathfrak{X}$ entangles properties at different times in the Galilean invariant standard quantum history theory discussed in Chapter 3. There we have only considered histories with finite support. We have identified the space of all finite histories with support S with $\mathcal{P}_S^{\otimes}(\mathfrak{H})$. Equivalently, histories are in one-to-one correspondence with projection operators on the tensor product Hilbert space $\mathfrak{H}(\mathcal{M}) = \otimes_{t\in S}\mathfrak{H}_t$. The state in the standard quantum history formalism was given by a trace class operator $\mathfrak{X}_\varrho$ on $\mathfrak{H}(\mathcal{M}) \otimes \mathfrak{H}(\mathcal{M})$. Since $\mathfrak{X}_\varrho$ is of trace class, it is in particular bounded. The state $\mathfrak{X}_\varrho$ is uniquely determined by the usual quantum state ϱ at some fiducial time and by the unitary time evolution operator $U(t_2, t_1)$. For simplicity we assume here $U = 1$, i.e., the Hamiltonian of the system is trivial. Then the state $\mathfrak{X}_\varrho$ can be represented in the standard tensor product form explicitly stated in Appendix C:

$$\mathfrak{X}_\varrho = \sum_{i_1,\ldots,i_{2n}} \omega_{i_{2n}} \quad \Big\{ |e^n_{i_n}\rangle\langle e^{n-1}_{i_{n-1}}| \otimes |e^{n-1}_{i_{n-1}}\rangle\langle e^{n-2}_{i_{n-2}}| \otimes \cdots \otimes |e^1_{i_1}\rangle\langle e^{2n}_{i_{2n}}| \otimes$$
$$\otimes |e^{n+1}_{i_{n+1}}\rangle\langle e^n_{i_n}| \otimes |e^{n+2}_{i_{n+2}}\rangle\langle e^{n+1}_{i_{n+1}}| \otimes \cdots \otimes |e^{2n}_{i_{2n}}\rangle\langle e^{2n-1}_{i_{2n-1}}| \Big\},$$

where $\varrho = \sum_i \omega_i |e^\varrho_i\rangle\langle e^\varrho_i|$ and where the orthonormal bases $\{|e^k_{i_k}\rangle\}$, $k \in \{1, ..., 2n-1\}$ are completely arbitrary, whereas necessarily $|e^{2n}_i\rangle = |e^\varrho_i\rangle$ for all i. Recall that every factor in the tensor product corresponds to one time slice. This representation of the operator $\mathfrak{X}_\varrho$ is a simple but presumably typical example of how the state $\mathfrak{X}_\varrho$ entangles events at different times.

5.5 CONCLUDING REMARKS

The content of the previous Section 5.4 is mainly a research programme proposal. It remains to be seen whether it can be further developed.

I do not consider the axioms stated in Section 5.3.2 as rigid axioms. The have the status of working hypotheses which might be subject to change in future developments if necessary. This is also true for the quite lofty scheme proposed in Section 5.4.

Wightman showed at the end of Section 2 of [169] that there exists no Lorentz covariant PV measure on the Borel sets of Minkowski space $\mathcal{M}$ provided that the Hamiltonian is bounded from below. He concluded that *"a sensible notion of localizability in space-time does not exist."* However, it is possible to construct Lorentz covariant POV measures on $\mathcal{B}(\mathcal{M})$ (so-called *systems of covariance*).
In Section 5.1 it has been argued that the assumption that the momentum can be measured sharply in standard quantum mechanics leads to problems with Einstein causality. Moreover, in any real measurement of momentum the system under consideration is also localized. Hence, in any local approach to quantum theory the sharp momentum observable has to be replaced by some appropriate "unsharp" joint position-momentum observable, to wit by a POV measure. One may have the impression that this procedure leads to an asymmetry between position and momentum observables in any local quantum theory, in particular in

the consistent event approach put forward in Section 5.3. However, there is a theorem due to Hegerfeldt [77, 79] according to which in relativistic quantum mechanics the assumption that a particle can be localized in a bounded region of space cannot be reconciled with Einstein causality. An initially localized particle develops instantaneously infinite tails. This result has subsequently been generalized [80, 78, 79]. [8] In summary, one can state that no relativistic theory of quantum particles allows for a consistent relativistically covariant notion of localizability in finite regions of space. [9] Hence, the assumption that a relativistic particle can be localized in a bounded region of space and the assumption that a relativistic particle can be localized in a bounded region of momentum space lead both to problems with Einstein causality. Another alternative would be to abandon or modify Einstein causality. This possibility has not been considered in the present work where we stick to Einstein causality. Accordingly, in relativistic quantum mechanics the position observable and the momentum observable have both to be replaced by appropriate joint position-momentum observables (which are POV measures). It thus seems as if in any local relativistic quantum theory it makes only sense to speak of localization in phase space and that momentum and position can only be measured (or realized) unsharply together. This is particularly true for any approach in which the idealized notion of localization in space at a fixed time-slice is replaced by the notion of localization in *open* space-time regions. It is clear that an assertion like *"a particle is first localized in the open space-time region* O_1 *and then in the (strictly later) space-time region* O_2*"* is in general not only an assertion about the position of the particle but indirectly also about its momentum. For all these reasons it is natural to postulate that in relativistic quantum theories only joint "unsharp" position-momentum observables are meaningful. Wightman's result stated above shows that in relativistic quantum theories these joint position-momentum observables are necessarily POV measures on the relativistic phase space. A family of examples for such Lorentz covariant POV measures on the relativistic phase space of a spinless particle with mass m has been constructed by Ali and Prugovečki [6, 144, 145] and can be found in Appendix E of this thesis.

There are a few attempts to incorporate ideas from the consistent histories approach into the axiomatic formulation of quantum field theory. In particular, the article by Blencowe [13] should be mentioned. Sorkin [164] has discussed some problems formulating a relativistic projection postulate compatible with locality. It is instructive to compare Blencowe's approach with our approach. Quite generally. it can be said that Blencowe stays closer to the usual formulation of axiomatic quantum field theory whereas our scheme is a somewhat lofty aggregation of physically motivated axioms. Roughly, Blencowe considers a space-time which can be foliated with surfaces which can be unambiguously parametrized by a time parameter (e.g., globally hyperbolic manifolds). A history is then specified by

- a (finite) sequence of bounded open space-time regions O_{t_i} which can be unambiguously labelled by an increasing sequence of times $t_1 < t_2 < \cdots < t_n$.
- an observable $A_{t_i} \in \mathcal{A}(\mathsf{O}_{t_i})$ for every t_i.

[8] Hegerfeldt's theorem relies heavily on the assumption that the Hamiltonian is bounded from below.

[9] For instance, the well-known Newton-Wigner operators [129] do not lead to covariant probability densities and lead to violations of Einstein causality in accordance with Hegerfeldt's general theorem.

- a subset $I_{t_i} \subset \mathbb{R}$ of possible values of the observable A_{t_i} for every t_i.

The main differences between Blencowe's and our approach are in detail

- Blencowe uses a a different (more general) notion of state.
- Blencowe's notion of consistency is much more restrictive than the notion employed in this thesis.
- Blencowe states no prescription generalizing the Lüders-von Neumann postulate in the relativistic domain. Dowker and Kent [37, 38] have shown that consistent sets of histories in general cannot be extended infinitely. So, Blencowe's theory must be criticized on the ground that in general consistency breaks down after a finite number of time steps (although this has not yet been shown for Blencowe's general formalism, there are good reasons to suppose that Dowker and Kent's negative result carries over also to Blencowe's approach). Therefore in general (presumably in most cases) Blencowe's theory makes no sensible predictions after a finite number of time steps.

PART II

GENERAL QUANTUM HISTORY THEORIES

6

AXIOMS FOR QUANTUM HISTORY THEORIES

Let us review
Lesson eleven
Paragraph two,
Axiom seven.

Once one dismisses
The rest of all possible worlds,
One finds that this is
The best of all possible worlds.
[from "Candide" by Leonard Bernstein]

In this chapter I state the general axioms for a generalized quantum theory based on our generalized history concept. By the term *generalized history quantum theory* I mean a novel algebraic scheme for building quantum history theories which are more general than the standard consistent histories formalism from Part I. This algebraic scheme will be presented in this chapter. It's general axioms as stated here are introduced ad hoc. They are abstracted from the structure of the standard consistent histories formalism which I have discussed in Chapters 3 and 4. Hence, it is clear that the standard consistent histories formalism can be cast into the framework presented in this chapter.
The idea to study general quantum history theories has been first put forward by Isham and this chapter parallels the discussion in Ref. [86].

1. The space $\mathfrak{U}$ of general history propositions.
   - The space $\mathfrak{U}$ carries a canonical D-poset structure denoted by $\oplus$.
     Isham and Linden [90] have suggested that the mathematical structure on $\mathfrak{U}$ is that of an orthoalgebra. One may also consider the case that $\mathfrak{U}$ is an orthocomplemented lattice or has an even stronger structure (e.g., a complete orthocomplemented lattice).
   - In the consistent effect history formalism in Chapter 4 the space $\mathfrak{U}$ is given by $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$.
2. The space $\mathcal{U}$ of history filters or homogeneous histories.
   - $\mathcal{U}$ is the space of the basic physical properties of a physical system with a direct physical interpretation. An element of $\mathcal{U}$ is a time-ordered sequence of single-time propositions about the system. There exists a map F mapping the elements of $\mathcal{U}$ to a D-poset $\mathfrak{E}$. $\mathfrak{E}$ can be interpreted as the set of (equivalence classes of) single-time propositions.
   - In Chapter 4 of this thesis $\mathcal{U}$ equals the space of homogeneous effect histories of the first kind $\mathcal{U} = \mathbb{E}_{fin}(\mathfrak{H})$, cf. Definition 11; $\mathfrak{E}$ is given by $\mathfrak{E}(\mathfrak{H})$ and F is given by $F(u) = C_{t_0}(u)^{\dagger} C_{t_0}(u)$.

- $\mathcal{U}$ is a partially ordered set with unit history 1 and null history 0.
- There exists an order preserving map $\tau : \mathcal{U} \to \mathfrak{U}$, i.e., $\tau(\mathcal{U}) \subset \mathfrak{U}$.
  - In this work τ is given by σ_{fin}.
- $\mathcal{U}$ is a partial semigroup with composition law $\circ$, cf. Ref. [86]. $a \circ b$ is well-defined if $t_f(a) < t_i(b)$. In this case we say that a PROCEEDS b or that b FOLLOWS a. Further, $1 \circ a = a \circ 1 = a$ and $a \circ 0 = 0 \circ a = 0$. If $a \circ b$ is defined, then $a \circ b = a \wedge b$, in particular the right hand side is well-defined.
- The partial ordering on $\mathcal{U}$ induces a partial unary operation $\neg$ (complementation) and two partial binary operations $\wedge$ and $\vee$ (meet and join) on $\mathcal{U}$.

3. The space of decoherence functionals.

   - A decoherence functional is a map $d : \mathfrak{U} \times \mathfrak{U} \to \mathbb{C}$ which satisfies for all $\alpha, \alpha', \beta \in \mathfrak{U}$
     - $d(\alpha, \alpha) \in \mathbb{R}$ and $d(\alpha, \alpha) \geq 0$.
     - $d(\alpha, \beta) = d(\beta, \alpha)^*$.
     - $d(1, 1) = 1$.
     - $d(0, \alpha) = 0$, for all α.
     - $d(\alpha_1 \oplus \alpha_2, \beta) = d(\alpha_1, \beta) + d(\alpha_2, \beta)$ for all $\alpha_1, \alpha_2, \beta \in \mathfrak{U}$ for which $\alpha_1 \oplus \alpha_2$ is well-defined.

4. The physical interpretation.

   - The physically interesting subsets of $\mathfrak{U}$ are the Boolean sublattices $\mathcal{B}$ of $\mathfrak{U}$ (see Definition 14).
   - The decoherence functional induces a probability measure on the consistent (with respect to the decoherence functional) Boolean sublattices of $\mathfrak{U}$. In the context of some consistent Boolean sublattice of $\mathfrak{U}$ one can interpret $d(\alpha, \alpha)$ as the probability of the history α.
   - On the consistent Boolean sublattices of $\mathfrak{U}$ the decoherence functional defines a partial logical implication which allows to make logical inferences.
   - The description of a physical system and reasoning in terms of elements of $\mathcal{U}$ (homogeneous effect histories of the first kind) is only indirectly possible by using the map $\tau : \mathcal{U} \to \mathfrak{U}$.
   - While homogeneous effect histories have a direct physical interpretation in terms of time sequences of physical properties, inhomogeneous (effect) histories have no such direct interpretation. In accordance with what has been said in Chapter 3, we tentatively suggest, however, that they may be interpreted as representatives of *unsharp quantum events*, i.e, events which cannot be associated with some fixed time, but which are smeared out in time.

Besides the structures of the space of histories and the space of decoherence functionals which are believed to be present also in more general history theories than nonrelativistic

quantum mechanics, Isham [86] also describes structures arising in the history formulation of nonrelativistic quantum mechanics which may be an artefact of that theory and may be meaningless in more general quantum history theories. Isham's axioms, as stated above, are not the most general structure possible. There is great freedom to add further axioms to the list or to remove some (note that we have already omitted some of Isham's original axioms).

In the local consistent event scheme outlined in Chapter 5, the above axioms are supplemented by the following specializations:

S0. There is an a priori given space-time $\mathfrak{S}$ which serves as the arena of physics in which events take place. This space-time is assumed to be endowed with a causal structure so that it makes sense to talk about spacelike and timelike separated events etc.

S1. The space $\mathfrak{U}$ of general history propositions is assumed to be generated by some net of local D-posets over some appropriate class of subsets of $\mathfrak{S}$. Only local or quasilocal events are assumed to be physically meaningful.

S2. The notion of homogeneous history is only locally applicable and is not Lorentz covariant. A history which is homogeneous in some Lorentz frame may be inhomogeneous in another Lorentz frame and vice versa.

S4. The formalism needs to be augmented by a prescription of how the decoherence functional has to be changed when an event has occurred and the notion of consistency has to be chosen in accordance with this prescription.

The physically interesting subsets of $\mathfrak{U}$ are then only consistent Boolean sublattices of the local D-posets of $\mathfrak{U}$.

Comments:

Of course, the underlying space-time $\mathfrak{S}$ with its causal structure assumed in Axiom S0 is an artefact from the classical theory of general relativity. In a fully quantum history theory the space-time degrees of freedom should also be quantized, so that Axiom S0 has only the status of a working hypothesis.

Since the concept of single-time event is not a Lorentz covariant concept but only a Galilean invariant concept, the notion of homogeneous history is presumably an artefact of the nonrelativistic Galilean invariant standard consistent histories formalism and has no fundamental status in any Lorentz covariant generalization of the consistent histories scheme.

It is possible that the ingredients 1-4 for general quantum history theories represent only an intermediate step and that in a theory including "quantum" gravity more general and sophisticated concepts and structures are necessary. As just mentioned for instance the notion of homogeneous history is a prime candidate to be dropped in a Lorentz covariant formulation.

In general quantum history theories it may be too difficult to handle the full space of all decoherence functionals. Usually states represented by some operators are easier to handle. In Chapter 5 the full space of decoherence functionals has not been considered. Instead only those decoherence functionals arising from certain trace class operators have been dealt with. Hence, in concrete quantum history theories it may be useful to represent states by certain operators.

7

GLEASON TYPE THEOREMS FOR DECOHERENCE FUNCTIONALS

Du siehst, mein Sohn,
zum Raum wird hier die Zeit.
[from "Parsifal" by Richard Wagner]

7.1 INTRODUCTION

To get insight into the possible structure of general history theories it is worthwhile to study the (algebraic) structure of the space of decoherence functionals for general quantum history theories in some detail. In particular — as also stressed by Isham, Linden and Schreckenberg [93] — the classification of decoherence functionals is an important problem. It is equivalent to the classification of states in quantum history theories. As is well-known, the analogous problem in standard quantum mechanics has been completely solved by Gleason [60, 39].

Gleason's Theorem shows that in standard quantum mechanics states (i.e., σ-orthoadditive probability measures on $\mathcal{P}(\mathfrak{H})$) can be one-to-one mapped to density operators on Hilbert space. Since operators are in many respects easier to handle than decoherence functionals, it is of some interest whether there is a one-to-one correspondence between decoherence functionals and certain operators.

In the present chapter I consider the case that the space of histories is given by the set of projection operators on some Hilbert space. This choice is motivated by the results from Chapter 3, see in particular Theorem 1 where it has been shown that the standard consistent histories formalism can be written in this form.

This chapter is organized as follows: in Section 7.3 and Section 7.4 the classification theorem for decoherence functionals is formulated and proved. Section 7.2 is devoted to an exposition of some necessary basic definitions and propositions. In Section 7.4 I give an alternative proof for the classification theorem for decoherence functionals in the case when the set of histories is the set of projection operators on a finite-dimensional complex Hilbert space. This classification theorem has first been proven by Isham, Linden and Schreckenberg [93]. My proof is based on methods used by Cooke, Keane and Moran in their elementary proof of Gleason's theorem [29] and differs from the proof given by Isham, Linden and Schreckenberg in that we do not use Gleason's theorem directly. However, my proof makes use of a theorem due to Wright which is in turn based on the solution of the Mackey-Gleason problem [171, 17]. In Section 7.5 we consider the case that the space of histories is the set $\mathcal{P}(\mathfrak{H})$ of projection operators on some infinite-dimensional separable or non-separable complex Hilbert space $\mathfrak{H}$ and extend the classification theorem for decoherence functionals to this case. It is perhaps worthwhile to mention that our result is also valid if we identify the set of histories with the

set of effects $\mathfrak{E}(\mathfrak{H})$ on some Hilbert space (as done in [151]) since every ultraweakly continuous normal decoherence functional on $\mathcal{P}(\mathfrak{H})$ can be uniquely extended to a functional on $\mathfrak{E}(\mathfrak{H})$, as shown in Corollary 5 in Section 7.6.

7.2 PRELIMINARIES

As already stated repeatedly in this chapter I consider history theories where the set of histories can be identified with the set $\mathcal{P}(\mathfrak{H})$ of projection operators on some separable or non-separable complex Hilbert space $\mathfrak{H}$. The decoherence functionals on $\mathcal{P}(\mathfrak{H})$ are then the functionals $d : \mathcal{P}(\mathfrak{H}) \times \mathcal{P}(\mathfrak{H}) \to \mathbb{C}, (h, k) \longmapsto d(h, k)$ subject to the following conditions for all $h, h', k \in \mathcal{P}(\mathfrak{H})$:

- $d(h, h) \in \mathbb{R}$, and $d(h, h) \geq 0$.
- $d(h, k) = d(k, h)^*$.
- $d(1, 1) = 1$ and $d(0, h) = 0$.
- $d(h \vee h', k) = d(h, k) + d(h', k)$, whenever $h \perp h'$.

By definition every decoherence functional is finitely orthoadditive in both arguments. We say that a decoherence functional d is σ-ADDITIVE in both arguments whenever

- $d\left(\bigvee_{i\in\mathcal{I}} h_i, k\right) = \sum_{i\in\mathcal{I}} d(h_i, k)$,

whenever $\{h_i\}_{i\in\mathcal{I}}$ is a countable set of mutually orthogonal histories and k is an arbitrary history. We say that a decoherence functional d is σ-SUMMABLE in both arguments if d is σ-additive and $\sum_{i\in\mathcal{I}} d(h_i, k)$ converges absolutely for all k and all countable families $\{h_i\}_{i\in\mathcal{I}}$ of mutually orthogonal histories. Moreover, we say that a decoherence functional d is COMPLETELY ADDITIVE in both arguments if for any set $\{h_j\}_{j\in\mathcal{J}}$ of mutually orthogonal histories and for all k the family $\{d(h_i, k)\}$ is summable and

- $d\left(\bigvee_{j\in\mathcal{J}} h_j, k\right) = \sum_{j\in\mathcal{J}} d(h_j, k)$.

Analogously to the recipe discussed in Chapter 3 every decoherence functional d can be used to define *consistent Boolean sublattices* of $\mathcal{P}(\mathfrak{H})$ such that d induces a probability measure on these consistent Boolean lattices. Clearly, a probability measure induced by a decoherence functional d on a consistent Boolean lattice $\mathcal{B}_c$ is σ-additive if and only if d is σ-additive in both arguments on $\mathcal{B}_c$ and is completely additive if and only if d is completely additive in both arguments on $\mathcal{B}_c$.

If $\mathfrak{T}$ is a topology on $\mathcal{P}(\mathfrak{H})$, then a decoherence functional d is called BI-CONTINUOUS WITH RESPECT TO THE TOPOLOGY $\mathfrak{T}$ if d is continuous in both arguments separately with respect to the topology $\mathfrak{T}$. In the present work we use the standard nomenclature for topologies on $\mathcal{B}(\mathfrak{H})$, see, e.g., [39]. We say that a decoherence functional d is ultraweakly bi-continuous if d is continuous in both arguments with respect to the ultraweak operator topology on $\mathcal{P}(\mathfrak{H})$. Every ultraweakly bi-continuous decoherence functional is obviously also continuous with respect to every stronger topology but not vice versa. The results in Section 7.4 are formulated for ultraweakly bi-continuous decoherence functionals. Since for norm bounded sequences of operators the notions of weak and ultraweak convergence coincide, it is clear that the results in Section 7.4 below are also valid for weakly bi-continuous decoherence functionals. However, the classification theorem for decoherence functionals on infinite-dimensional Hilbert spaces is in general not valid for decoherence functionals which are bi-continuous with respect to a stronger topology than the ultraweak topology.

According to Wright's Theorem (see Appendix B) every bounded decoherence functional d on $\mathcal{P}(\mathfrak{H})$ can be uniquely extended to a bilinear bounded functional $\mathcal{D}$ on $\mathcal{B}(\mathfrak{H})$ provided $\dim(\mathfrak{H}) > 2$. The extension $\mathcal{D}$ is continuous in both arguments with respect to the norm topology on $\mathcal{B}(\mathfrak{H})$ and thus every bounded decoherence functional d is necessarily bi-continuous with respect to the topology induced on $\mathcal{P}(\mathfrak{H})$ by the norm on $\mathcal{B}(\mathfrak{H})$.

We denote by $\mathfrak{H}_0$ the everywhere dense subset of $\mathfrak{H} \otimes \mathfrak{H}$ generated by the simple vectors of the form $|\phi\rangle \otimes |\psi\rangle$, where $|\phi\rangle, |\psi\rangle \in \mathfrak{H}$. That is, $\mathfrak{H}_0$ contains all finite linear combinations of simple vectors. For all $|\Psi_0\rangle \in \mathfrak{H}_0$ we denote the one-dimensional projection operator onto $|\Psi_0\rangle$ by P_{Ψ_0}. We denote the set of all such projection operators by $\mathcal{P}(\mathfrak{H}_0)$. Moreover, we denote the set of all projection operators in $\mathcal{P}(\mathfrak{H} \otimes \mathfrak{H})$ which can be written as a finite sum $\sum_j P_{\Psi_j}$ of pairwise orthogonal projection operators $P_{\Psi_j} \in \mathcal{P}(\mathfrak{H}_0)$ by $\mathcal{P}_{fin}(\mathfrak{H}_0)$. The set of all projection operators in $\mathcal{P}(\mathfrak{H} \otimes \mathfrak{H})$ which can be written as a ultraweakly converging sum $\sum_j P_{\Psi_j}$ of pairwise orthogonal projection operators $P_{\Psi_j} \in \mathcal{P}(\mathfrak{H}_0)$ will be denoted by $\mathcal{P}_\infty(\mathfrak{H}_0)$. If $\mathfrak{H}$ is finite-dimensional, then obviously $\mathfrak{H}_0 = \mathfrak{H} \otimes \mathfrak{H}$.

Proposition 1 *Let $\mathfrak{H}$ be a complex Hilbert space, $\dim(\mathfrak{H}) > 2$, and let d denote a bounded decoherence functional on $\mathcal{P}(\mathfrak{H})$, then d can be uniquely extended to a functional $\widehat{d} : \mathcal{P}_{fin}(\mathfrak{H}_0) \to \mathbb{C}$ satisfying $\widehat{d}(h \otimes k) = d(h,k)$ for all $h, k \in \mathcal{P}(\mathfrak{H})$. Moreover, $\widehat{d}$ is additive for orthogonal projection operators, i.e., $\widehat{d}(P_1 + P_2) = \widehat{d}(P_1) + \widehat{d}(P_2)$ for all $P_1, P_2 \in \mathcal{P}_{fin}(\mathfrak{H}_0)$ with $P_1 \perp P_2$. If $\mathfrak{H}$ is finite-dimensional, then $\widehat{d}$ is bounded.*

Lemma 4 *Let $E_1, ..., E_{n+m}, F_1, ..., F_{m+n} \in \mathcal{B}(\mathfrak{H})$, then $(\sum_{i=1}^{n} E_i \otimes F_i) + (\sum_{i=n+1}^{n+m} E_i \otimes F_i) = 0$ if and only if there is an $(n+m) \times (n+m)$ complex matrix $[c_{ik}]$ such that*

$$\begin{aligned} \sum_{i=1}^{n+m} c_{ik} E_i &= 0, (k = 1, ..., n+m), \\ \sum_{k=1}^{n+m} c_{ik} F_k &= F_i, (i = 1, ..., n+m). \end{aligned}$$

The assertion of Lemma 4 is exactly Proposition 11.1.8 (i) in [98].

Proof of Proposition 1: For simple projection operators of the form $h \otimes k$ with $h, k \in \mathcal{P}(\mathfrak{H})$ we define $\widehat{d}(h \otimes k) := d(h,k)$. We denote by $\mathfrak{H}_0$ the everywhere dense subset of $\mathfrak{H} \otimes \mathfrak{H}$ generated by the simple vectors of the form $|\phi\rangle \otimes |\psi\rangle$. That is, $\mathfrak{H}_0$ contains all finite linear combinations of simple vectors. Let $|\Psi_0\rangle \in \mathcal{S}(\mathfrak{H}_0) := \mathcal{S}(\mathfrak{H} \otimes \mathfrak{H}) \cap \mathfrak{H}_0$, then $|\Psi_0\rangle$ can be written as $|\Psi_0\rangle = \sum_{j=1}^{N} \kappa_j |\phi_j\rangle \otimes |\psi_j\rangle$, with $|\phi_j\rangle, |\psi_j\rangle \in \mathcal{S}(\mathfrak{H})$ for all j. Denote the projection operator on $|\Psi_0\rangle$ by $P_{\Psi_0} = |\Psi_0\rangle\langle\Psi_0|$ and define $\widehat{d}(P_{\Psi_0}) := \sum_{i=1}^{N} \sum_{j=1}^{N} \kappa_i \kappa_j^* \mathcal{D}\left(|\phi_i\rangle\langle\phi_j|, |\psi_i\rangle\langle\psi_j|\right)$ where $\mathcal{D}$ denotes the unique extension of d to $\mathcal{B}(\mathfrak{H})$ mentioned in Remark 1. Lemma 4 implies that the such defined $\widehat{d}(P_{\Psi_0})$ is independent of the particular representation of $|\Psi_0\rangle \in \mathcal{S}(\mathfrak{H}_0)$ chosen (see Proposition 11.1.8 (ii) in [98]). Thus we have extended d to a functional on the set of projection operators of the form P_{Ψ_0}. Now let $P_M \in \mathcal{P}_{fin}(\mathfrak{H}_0)$. Then by definition P_M can be written as a finite sum $P_M = \sum_{j=1}^{M} P_{\Psi_j}$ of mutually orthogonal projection operators $P_{\Psi_j} \in \mathcal{P}(\mathfrak{H}_0)$. We define $\widehat{d}(P_M) := \sum_{j=1}^{M} \widehat{d}(P_{\Psi_j})$. Again Lemma 4 implies that $\widehat{d}(P_M)$ is well-defined and independent of the particular decomposition of P_M chosen. If $\mathfrak{H}$ is n-dimensional, then $\widehat{d}$ is bounded by $n^4 K$, where K is a bound of $\mathcal{D}$. ■

We say that a bounded decoherence functional d is PROPER if the unique extension $\widehat{d} : \mathcal{P}_{fin}(\mathfrak{H}_0) \to \mathbb{C}$ from Proposition 1 is bounded. Let $P \in \mathcal{P}_{\infty}(\mathfrak{H}_0)$ and $P = \sum_{j \in J} p_j$ one of its representations with $p_j \in \mathcal{P}(\mathfrak{H}_0)$. If d is proper, then the net $\{\widehat{d}\left(\sum_{j \in J_0} p_j\right) | J_0 \subset J, J_0 \text{ finite }\}$ possesses a converging subsequence. We say that a bounded decoherence functional d is NORMAL if $\widehat{d}$ can be unambiguously extended to a completely additive functional $\widetilde{d} : \mathcal{P}_{\infty}(\mathfrak{H}_0) \to \mathbb{C}$.

7.3 THE FINITE DIMENSIONAL CASE

Theorem 6 *If the dimension* $\dim(\mathfrak{H})$ *of a complex Hilbert space* $\mathfrak{H}$ *satisfies* $2 < \dim(\mathfrak{H}) < \infty$, *then there is a one-one correspondence between bounded decoherence functionals* d *on* $\mathcal{P}(\mathfrak{H})$ *and trace class operators* $\mathfrak{X}$ *on* $\mathfrak{H} \otimes \mathfrak{H}$ *according to the rule*

$$d(h,k) = \mathrm{tr}_{\mathfrak{H} \otimes \mathfrak{H}}(h \otimes k\mathfrak{X}) \tag{7.1}$$

with the restrictions that

- $\mathrm{tr}_{\mathfrak{H} \otimes \mathfrak{H}}(h \otimes k\mathfrak{X}) = \mathrm{tr}_{\mathfrak{H} \otimes \mathfrak{H}}(k \otimes h\mathfrak{X}^{\dagger})$ *for all* $h, k \in \mathcal{P}(\mathfrak{H})$;
- $\mathrm{tr}_{\mathfrak{H} \otimes \mathfrak{H}}(h \otimes h\mathfrak{X}) \geq 0$ *for all* $h \in \mathcal{P}(\mathfrak{H})$;
- $\mathrm{tr}_{\mathfrak{H} \otimes \mathfrak{H}}(\mathfrak{X}) = 1$.

In particular, every such decoherence functional is uniformly bi-continuous.

Theorem 6 has first been proven by Isham, Linden and Schreckenberg in [93].
Proof: Consider the finite-dimensional Hilbert space $\mathfrak{H} \otimes \mathfrak{H}$. According to Proposition 1 every bounded decoherence functional d on $\mathfrak{H}$ can be extended to a bounded functional $\widehat{d}$: $\mathcal{P}(\mathfrak{H} \otimes \mathfrak{H}) \to \mathbb{C}$. Hence, the real part $\Re\widehat{d}$ of $\widehat{d}$ induces a bounded frame function $f_{\Re d}$ on $\mathfrak{H} \otimes \mathfrak{H}$ by

$$f_{\Re d}\left(|\Psi_0\rangle\right) := \Re\widehat{d}(P_{\Psi_0}),$$

for all $|\Psi_0\rangle \in \mathcal{S}(\mathfrak{H} \otimes \mathfrak{H})$. Similarly, the imaginary part $\Im\widehat{d}$ of $\widehat{d}$ induces a bounded frame function $f_{\Im d}$ on $\mathfrak{H} \otimes \mathfrak{H}$ by $f_{\Im d}\left(|\Psi_0\rangle\right) := \Im\widehat{d}(P_{\Psi_0})$, for all $|\Psi_0\rangle \in \mathcal{S}(\mathfrak{H} \otimes \mathfrak{H})$. Therefore, according to Proposition 6 both $f_{\Re d}$ and $f_{\Im d}$ are regular. This proves the existence of two Hermitean operators $\mathfrak{X}_{\Re d}$ and $\mathfrak{X}_{\Im d}$ on $\mathfrak{H}\otimes\mathfrak{H}$ such that $\widehat{d}$ can be written as $\widehat{d}(P_M) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(P_M(\mathfrak{X}_{\Re d}+i\mathfrak{X}_{\Im d}))$, for all $P_M \in \mathcal{P}_{fin}(\mathfrak{H} \otimes \mathfrak{H})$. In particular, it follows $d(h,k) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h \otimes k\mathfrak{X}_d)$ for all $h, k \in \mathcal{P}(\mathfrak{H})$ where we have set $\mathfrak{X}_d := \mathfrak{X}_{\Re d} + i\mathfrak{X}_{\Im d}$. The remaining assertions of the Theorem are now straightforward. ■

Theorem 1 shows that in the finite-dimensional case a decoherence functional d on $\mathcal{P}(\mathfrak{H})$ is ultraweakly bi-continuous if and only if d is bi-continuous with respect to the uniform (or operator norm) topology on $\mathcal{P}(\mathfrak{H})$. Moreover, since in the finite-dimensional case the weak and the ultraweak topology on $\mathcal{P}(\mathfrak{H})$ coincide, an ultraweakly bi-continuous decoherence functional d is also weakly bi-continuous.

7.4 THE INFINITE DIMENSIONAL CASE

Theorem 7 *Let $\mathfrak{H}$ be a complex Hilbert space with dimension greater than one,* $\dim(\mathfrak{H}) > 2$*, then there is a one-one correspondence between normal completely additive decoherence functionals d on $\mathcal{P}(\mathfrak{H})$ and trace class operators $\mathfrak{X}$ on $\mathfrak{H} \otimes \mathfrak{H}$ according to the rule*

$$d(h,k) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h \otimes k\mathfrak{X}) \tag{7.2}$$

with the restrictions that

- $\mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h \otimes k\mathfrak{X}) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(k \otimes h\mathfrak{X}^\dagger)$ *for all* $h, k \in \mathcal{P}(\mathfrak{H})$*;*
- $\mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h \otimes h\mathfrak{X}) \geq 0$ *for all* $h \in \mathcal{P}(\mathfrak{H})$*;*
- $\mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(\mathfrak{X}) = 1$.

Proof: If $\mathfrak{H}$ is finite-dimensional, then the assertion of the theorem has already been proven in Theorem 6. Let $\mathfrak{H}$ be infinite-dimensional and let d denote a normal decoherence functional on $\mathcal{P}(\mathfrak{H})$. Notice, that the requirement of complete additivity in the theorem is redundant. As above we denote by $\mathfrak{H}_0$ the everywhere dense linear subspace of $\mathfrak{H} \otimes \mathfrak{H}$ generated by the simple vectors of the form $|\phi\rangle \otimes |\psi\rangle$. From Proposition 1 we know that d can be uniquely

extended to a functional $\widehat{d} : \mathcal{P}_{fin}(\mathfrak{H}_0) \to \mathbb{C}$. The real part $\Re\widehat{d}$ of $\widehat{d}$ induces a frame type function $g_{\Re d} : \mathcal{S}(\mathfrak{H}_0) \to \mathbb{R}$ on $\mathfrak{H} \otimes \mathfrak{H}$ by

$$g_{\Re d}(|\Psi_0\rangle) := \Re\widehat{d}(P_{\Psi_0}),$$

for all $|\Psi_0\rangle \in \mathcal{S}(\mathfrak{H}_0)$. Similarly, the imaginary part $\Im\widehat{d}$ of $\widehat{d}$ induces a frame type function $g_{\Im d} : \mathcal{S}(\mathfrak{H}_0) \to \mathbb{R}$ on $\mathfrak{H}\otimes\mathfrak{H}$ by $g_{\Im d}(|\Psi_0\rangle) := \Im\widehat{d}(P_{\Psi_0})$, for all $|\Psi_0\rangle \in \mathcal{S}(\mathfrak{H}_0)$. According to Proposition 8 both $g_{\Re d}$ and $g_{\Im d}$ are regular. This proves the existence of two Hermitean operators $\mathfrak{X}_{\Re d}$ and $\mathfrak{X}_{\Im d}$ on $\mathfrak{H}\otimes\mathfrak{H}$ such that $\widehat{d}$ can be written as $\widehat{d}(P_M) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(P_M(\mathfrak{X}_{\Re d}+i\mathfrak{X}_{\Im d})) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(P_M\mathfrak{X}_d)$, for all $P_M \in \mathcal{P}_{fin}(\mathfrak{H}_0)$ where we have set $\mathfrak{X}_d := \mathfrak{X}_{\Re d} + i\mathfrak{X}_{\Im d}$. It is clear now, that the real part $\Re\widehat{d}$ of $\widehat{d}$ can be extended to a Jordan completely additive signed measure $\Re\bar{d}$ on $\mathfrak{H} \otimes \mathfrak{H}$ given by $\Re\bar{d}(P) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(P\mathfrak{X}_{\Re d})$ for all $P \in \mathcal{P}(\mathfrak{H}\otimes\mathfrak{H})$. The imaginary part $\Im\widehat{d}$ of $\widehat{d}$ can be extended to a Jordan completely additive signed measure $\Im\bar{d}$ on $\mathfrak{H}\otimes\mathfrak{H}$ given by $\Im\bar{d}(P) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(P\mathfrak{X}_{\Im d})$ for all $P \in \mathcal{P}(\mathfrak{H} \otimes \mathfrak{H})$. Hence $\widehat{d}$ can be extended to a completely additive complex valued measure $\bar{d}$ on $\mathfrak{H} \otimes \mathfrak{H}$ given by

$$\bar{d}(P) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(P\mathfrak{X}_d),$$

for all $P \in \mathcal{P}(\mathfrak{H} \otimes \mathfrak{H})$. Since d is finitely additive, it follows that $d(h,k) = \bar{d}(h \otimes k)$ for all finite-dimensional $h, k \in \mathcal{P}(\mathfrak{H})$. Obviously $\bar{d}$ is ultraweakly continuous. By assumption d is completely additive in both arguments. Now let $h, k \in \mathcal{P}(\mathfrak{H})$ denote two arbitrary projection operators on $\mathfrak{H}$. Consider the projection operator $h \otimes k$ on $\mathfrak{H} \otimes \mathfrak{H}$. Then there is a family of mutually orthogonal one-dimensional projection operators $\{h_i\}_{i\in I}$ such that $h = \sum_i h_i$ in the weak operator topology and a family of mutually orthogonal one-dimensional projection operators $\{k_l\}_{l\in L}$ such that $k = \sum_l k_l$ in the weak operator topology, compare, e.g., [126] Section I.5.12. Obviously $h \otimes k = \sum_{i,l} h_i \otimes k_l$ in the weak operator topology. Consider the net $\mathcal{N}$ consisting of all finite sums of the form $\left(\sum_{i\in\widetilde{I}} h_i\right) \otimes \left(\sum_{l\in\widetilde{L}} k_l\right)$ where $\widetilde{I}$ is a finite subset of I and $\widetilde{L}$ is a finite subset of L. Proposition I.5.12.IX in [126] implies that for every trace class operator T on $\mathfrak{H} \otimes \mathfrak{H}$ there exist at most countable subsets $I_T \subset I$ and $L_T \subset L$ such that $\mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h \otimes kT) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}\left(\left(\sum_{i\in I_T} h_i\right) \otimes \left(\sum_{l\in L_T} k_l\right) T\right)$. Hence, for all $\epsilon > 0$ and all trace class operators T on $\mathfrak{H} \otimes \mathfrak{H}$ there exist finite subsets $\widetilde{I}_{T,\epsilon}$ of I and $\widetilde{L}_{T,\epsilon}$ of L such that $\left|\mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h \otimes kT) - \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}\left(\left(\sum_{i\in\widetilde{I}} h_i\right) \otimes \left(\sum_{l\in\widetilde{L}} k_l\right) T\right)\right| < \epsilon$ for all finite $\widetilde{I} \supset \widetilde{I}_{T,\epsilon}$ and all finite $\widetilde{L} \supset \widetilde{L}_{T,\epsilon}$. We conclude that the net $\mathcal{N}$ defined above converges to $h \otimes k$ in the ultraweak topology. Clearly, the net of complex numbers $\left\{\bar{d}\left(\left(\sum_{i\in\widetilde{I}} h_i\right) \otimes \left(\sum_{l\in\widetilde{L}} k_l\right)\right)\right\}_{\widetilde{I},\widetilde{L}}$, where $\widetilde{I}$ and $\widetilde{L}$ run through all finite subsets of I and L respectively, converges to $\bar{d}(h \otimes k)$. Similarly, the net $\left\{d\left(\sum_{i\in\widetilde{I}} h_i, \sum_{l\in\widetilde{L}} k_l\right)\right\}_{\widetilde{I},\widetilde{L}}$, where $\widetilde{I}$ and $\widetilde{L}$ run through all finite subsets of I and L respectively, converges to $d(h,k)$. Since $\bar{d}\left(\left(\sum_{i\in\widetilde{I}} h_i\right) \otimes \left(\sum_{l\in\widetilde{L}} k_l\right)\right) = d\left(\sum_{i\in\widetilde{I}} h_i, \sum_{l\in\widetilde{L}} k_l\right)$ for every finite $\widetilde{I} \subset I$ and $\widetilde{L} \subset L$, we conclude that $\bar{d}(h\otimes k) = d(h,k)$. In particular, it follows $d(h,k) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h \otimes k\mathfrak{X}_d)$ for all $h, k \in \mathcal{P}(\mathfrak{H})$. The remaining assertions of the Theorem are now straightforward. ■

We say that a decoherence functional d on the set $\mathcal{P}(\mathfrak{H})$ of projectors on the Hilbert space $\mathfrak{H}$ is REGULAR if there is a trace class operator $\mathfrak{X}_d$ on $\mathfrak{H} \otimes \mathfrak{H}$ such that d can be written as

$d(h,k) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h \otimes k\mathfrak{X}_d)$ for all $h, k \in \mathcal{P}(\mathfrak{H})$. In this case we say that $\mathfrak{X}_d$ defines a REGULAR REPRESENTATION of d.
Moreover, we say that a decoherence functional d on the set $\mathcal{P}(\mathfrak{H})$ of projectors on the Hilbert space $\mathfrak{H}$ is QUASI-REGULAR if there is a bounded operator $\mathfrak{X}_d$ on $\mathfrak{H} \otimes \mathfrak{H}$ (not necessarily of trace class) such that d can be written as $d(h,k) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h \otimes k\mathfrak{X}_d)$ for all finite-dimensional $h, k \in \mathcal{P}(\mathfrak{H})$. In this case we say that $\mathfrak{X}_d$ defines a QUASI-REGULAR REPRESENTATION of d.
We say that a decoherence functional d on the set $\mathcal{P}(\mathfrak{H})$ of projectors on the Hilbert space $\mathfrak{H}$ is σ-QUASI-REGULAR if there is a bounded operator $\mathfrak{X}_d$ on $\mathfrak{H} \otimes \mathfrak{H}$ (not necessarily of trace class) such that the following condition is satisfied: given any two projection operators $h, k \in \mathcal{P}(\mathfrak{H})$ projecting onto separable subspaces of $\mathfrak{H}$ and arbitrary decompositions $\{h_i\}$ of h and $\{k_j\}$ of k into mutually orthogonal one-dimensional projection operators $h = \sum_i h_i$ and $k = \sum_j k_j$, then the sum $\sum_{i,j} \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h_i \otimes k_j\mathfrak{X}_d)$ is well-defined and independent of the particular decompositions considered and equals $d(h,k)$, i.e., $d(h,k) = \sum_{i,j} \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h_i \otimes k_j\mathfrak{X}_d) = \sum_{i,j}\langle h_i \otimes k_j|\mathfrak{X}_d|h_i \otimes k_j\rangle$. In this case we say that $\mathfrak{X}_d$ defines a σ-QUASI-REGULAR REPRESENTATION of d.
We say that a decoherence functional d on the set $\mathcal{P}(\mathfrak{H})$ of projectors on the Hilbert space $\mathfrak{H}$ is PSEUDO-REGULAR if there is a bounded operator $\mathfrak{X}_d$ on $\mathfrak{H} \otimes \mathfrak{H}$ (not necessarily of trace class) such that the following condition is satisfied: given any two projection operators $h, k \in \mathcal{P}(\mathfrak{H})$ projecting onto arbitrary subspaces of $\mathfrak{H}$ and given arbitrary decompositions $\{h_i\}$ of h and $\{k_j\}$ of k into mutually orthogonal one-dimensional projection operators $h = \sum_i h_i$ and $k = \sum_j k_j$, then the family $\{\sum_{i,j} \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h_i \otimes k_j\mathfrak{X}_d)\}$ is summable and its sum is independent of the particular decompositions considered and equals $d(h,k)$, i.e., $d(h,k) = \sum_{i,j} \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h_i \otimes k_j\mathfrak{X}_d) = \sum_{i,j}\langle h_i \otimes k_j|\mathfrak{X}_d|h_i \otimes k_j\rangle$. In this case we say that $\mathfrak{X}_d$ defines a PSEUDO-REGULAR REPRESENTATION of d.

Corollary 3 *Let $\mathfrak{H}$ be an infinite-dimensional complex Hilbert space. Then every completely additive normal decoherence functional d on $\mathcal{P}(\mathfrak{H})$ is ultraweakly bi-continuous and regular.*

In the following propositions the requirement that the decoherence functional is normal is weakened.

Proposition 2 *Let $\mathfrak{H}$ be an infinitely dimensional complex Hilbert space, then for every proper completely additive decoherence functional d on $\mathcal{P}(\mathfrak{H})$ there exists a unique Hilbert-Schmidt operator $\mathfrak{X}_d$ on $\mathfrak{H} \otimes \mathfrak{H}$ (not necessarily of trace class) defining a pseudo-regular representation of d.*

Proof: Denote by $\widehat{d}$ the extension of d from Proposition 1. Since $\widehat{d}$ is proper, it follows by a standard argument (see, e.g., the first part of the proof of Theorem 3.2.21 in [39]) that there exist uniquely determined bounded Hermitean operators $\mathfrak{X}_{\Re d}$ and $\mathfrak{X}_{\Im d}$ on $\mathfrak{H} \otimes \mathfrak{H}$ such that $\Re\widehat{d}$ and $\Im\widehat{d}$ can be written as $\Re\widehat{d}\,(P_{\Psi_0}) = \langle\Psi_0|\mathfrak{X}_{\Re d}|\Psi_0\rangle$ and $\Im\widehat{d}\,(P_{\Psi_0}) = \langle\Psi_0|\mathfrak{X}_{\Im d}|\Psi_0\rangle$ for all $|\Psi_0\rangle \in \mathcal{S}(\mathfrak{H}_0)$. Denote by $\{\varphi_i\}$ and $\{\chi_j\}$ two complete systems of mutually orthogonal one-dimensional projection operators on $\mathfrak{H}$ and by $\{|\varphi_i\rangle\}$ and $\{|\chi_j\rangle\}$ the corresponding orthonormal bases. Since d is completely additive, it follows $\sum_{i,j} |\langle\varphi_i \otimes \chi_j|\mathfrak{X}_{\Re d}|\varphi_i \otimes \chi_j\rangle|^2 =$

$\sum |\Re d(\varphi_i, \chi_j)|^2 \leq (\sum |\Re d(\varphi_i, \chi_j)|)^2 < \infty$. Hence $\mathfrak{X}_{\Re d}$ is a Hilbert-Schmidt operator. Similarly, $\mathfrak{X}_{\Im d}$ is a Hilbert-Schmidt operator. Then define $\mathfrak{X}_d \equiv \mathfrak{X}_{\Re d} + i\mathfrak{X}_{\Im d}$. ■

Proposition 3 *Let $\mathfrak{H}$ be an infinitely dimensional complex Hilbert space, then for every proper finitely additive decoherence functional d on $\mathcal{P}(\mathfrak{H})$ there exists a unique bounded operator $\mathfrak{X}_d$ on $\mathfrak{H} \otimes \mathfrak{H}$ (not necessarily of trace class) such that d can be written as $d(h,k) = \mathrm{tr}_{\mathfrak{H}\otimes\mathfrak{H}}(h \otimes k\mathfrak{X}_d)$ for all finite-dimensional $h, k \in \mathcal{P}(\mathfrak{H})$.*

In other words: every proper decoherence functional admits a *unique* quasi-regular representation.

Proposition 4 *Let $\mathfrak{H}$ be an infinitely dimensional complex Hilbert space, then for every proper σ-additive decoherence functional d on $\mathcal{P}(\mathfrak{H})$ there exists a unique bounded operator $\mathfrak{X}_d$ on $\mathfrak{H} \otimes \mathfrak{H}$ (not necessarily of trace class) defining a σ-quasi-regular representation of d. If $\mathfrak{H}$ is separable and if d is σ-summable, then $\mathfrak{X}_d$ is a Hilbert-Schmidt operator.*

The proofs of Proposition 3 and Proposition 4 are analogous to the proof of Theorem 7 and Proposition 2.

Corollary 4 *Let $\mathfrak{H}$ be an infinite-dimensional complex separable Hilbert space. Then every σ-summable normal decoherence functional d on $\mathcal{P}(\mathfrak{H})$ is ultraweakly bi-continuous and regular.*

To sum up: Theorem 6 asserts that every bounded decoherence functional d on the set of projection operators on a finite-dimensional Hilbert space of dimension greater than two is regular. Theorem 7 shows that every normal completely additive decoherence functional d on the set of projection operators of an infinite-dimensional Hilbert space is regular. And Proposition 4 shows among others that every σ-summable normal decoherence functional d on the set of projection operators of an infinite-dimensional separable Hilbert space is regular. Obviously every regular decoherence functional is ultraweakly bi-continuous and hence σ-summable for separable Hilbert spaces and completely additive for non-separable Hilbert spaces.

7.5 EFFECT HISTORIES

In Chapter 4 I have argued that in a general history theory the space of histories should be identified with the set of effects $\mathfrak{E}(\mathfrak{H})$ on some Hilbert space $\mathfrak{H}$. The following Corollary 5 shows that there is a one-one correspondence between ultraweakly continuous normal decoherence functionals $d : \mathcal{P}(\mathfrak{H}) \times \mathcal{P}(\mathfrak{H}) \to \mathbb{C}$ and ultraweakly continuous normal decoherence functionals $d : \mathfrak{E}(\mathfrak{H}) \times \mathfrak{E}(\mathfrak{H}) \to \mathbb{C}$ as defined in [151]. Corollary 5 is an easy consequence of Theorem 7.

Corollary 5 *Let d denote a completely additive normal decoherence functional on $\mathcal{P}(\mathfrak{H})$, then d can be uniquely extended to an ultraweakly bi-continuous bounded functional $\widetilde{d} : \mathfrak{E}(\mathfrak{H}) \times \mathfrak{E}(\mathfrak{H}) \to \mathbb{C}$. $\widetilde{d}$ satisfies $\widetilde{d}(e,e) \in \mathbb{R}; \widetilde{d}(e,e) \geq 0; \widetilde{d}(e,f) = \widetilde{d}(f,e)^*; \widetilde{d}(1,1) = 1$ and $\widetilde{d}(0,e) = 0$, for all $e, f \in \mathfrak{E}(\mathfrak{H})$. Moreover, $\widetilde{d}$ is additive with respect to the canonical D-poset structure on $\mathfrak{E}(\mathfrak{H})$, i.e., $\widetilde{d}(e_1 \oplus e_2, f) = \widetilde{d}(e_1, f) + \widetilde{d}(e_2, f)$, whenever $e_1 \oplus e_2$ is well-defined.*

7.6 CONCLUDING REMARKS

The problem of representing decoherence functionals by operators is of foremost interest in the history approach. As emphasized repeatedly above, the decoherence functional can be looked upon as representing the (transtemporal) state of a quantum system. However, it turns out that decoherence functionals often are somewhat clumsy to work with and it would be easier to have an operator representing the state. An example for such a situation is the transformation rule Equation 5.1 in Section 5 which could only be formulated on the level of operators.

In the case of finite-dimensional Hilbert space the classification problem for decoherence functionals has been completely solved by Isham, Linden and Schreckenberg. The generalization of the theorem provided in the present chapter works only if certain strong boundedness conditions are imposed. Nonnormal decoherence functionals have no regular representation by some trace class operator. It is therefore of interest what can be said when the strong boundedness condition imposed on the decoherence functionals is weakened. The problem is closely related to the problem under which circumstances bilinear Hermitean forms J on Hilbert space are of the regular form $J(u,v) = J_A(u,v) = (A^{1/2}u, A^{1/2^*}v)$ for all $u, v \in D(J)$ and for some self-adjoint or symmetric operator A. In the general case (to wit, when J is not semibounded from below) no complete answer to this problem is known (as far as I know). Some partial results, particularly those of McIntosh [120, 121, 122] or those of Ylinen [172] can be used to formulate highly technical conditions on the decoherence functionals under which they are (quasi-) regular. This procedure is, however, from a physicists perspective not really insightful and, moreover, lacks the yearned generality.

Another recent development seems to be more promising. Let $\mathcal{C}(\mathfrak{H})$ denote the ideal of compact operators in $\mathcal{B}(\mathfrak{H})$. Let d denote a bounded bilinear decoherence functional and $\mathcal{D}$ its unique extension to a bounded bilinear functional on $\mathcal{B}(\mathfrak{H}) \times \mathcal{B}(\mathfrak{H})$. By the fundamental property of the algebraic tensor product, there exists a unique linear functional δ on the algebraic tensor product $\mathcal{C}(\mathfrak{H}) \otimes_{alg} \mathcal{C}(\mathfrak{H})$ such that $\mathcal{D}(b,c) = \delta(b \otimes c)$. It can be shown that the decoherence functional d is normal if and only if δ is bounded. This can be used to give an alternative and more elegant proof of the results presented in this chapter. [1] Moreover, it is possible to replace Proposition 6 by a slightly more general proposition. This points will be discussed in detail in a forthcoming publication.

[1] I am grateful to an anonymous referee for pointing this (and other things) out to me.

8

CONCLUSIONS AND OUTLOOK

Ecco la fine del mio cammin ...

[from "Otello" by Giuseppe Verdi/Arrigo Boito]

The virtues of the consistent histories approach are the naturalness of the concepts and a lucid clarity of the structure of the quantum mechanical formalism. I know of no approach to nonrelativistic quantum mechanics which unravels the conceptual and interpretational problems of quantum mechanics as clearly and as systematically as does the consistent histories approach. Isham's approach to general quantum history theories opens the gate for "timeless" quantum history theories which are candidates for quantum theories respecting (at least some of) the principles of general relativity. It should be stressed that the histories in general history theories may much more general and also completely different mathematical objects than those down-to-earth quantum histories which have been discussed in the present thesis. It is of course not claimed that in more general history theories all axioms which have been abstracted from the nonrelativistic quantum theory remain valid. At the present time one can only speculate, however, which axioms have to be added, modified or dropped completely.

The most serious shortcoming of the history approach at the present time is that "it has not enabled us yet to construct or even characterize a specific theory. It has given a frame and a language not a theory." [1] The central problem is to construct a physically sensible operator $\mathfrak{X}$ on some appropriate Hilbert or Banach space or even on a more general vector space. Our considerations in Chapter 5 may be considered as a first step in such a construction for quantum theories on Minkowski space. But much more work is necessary. We are presently lacking both physical principles and intuition to construct physically sensible general history theories. This is the reason why our discussion in Section 5.4 had to stop at a very unsatisfactory stage. It is probable that symmetry principles will play an important role in the further developments of general quantum history theories. This important topic has not been discussed in the present thesis. First investigations have led to the notion of *history group* introduced by Isham and Linden in Reference [91]. Stephan Schreckenberg has recently investigated the general symmetries of decoherence functionals [159, 160].

Another approach which has not been discussed in this thesis is the Gell-Mann–Hartle approach [48, 49, 50, 51, 52, 53, 74, 75, 76]. In a series of interesting publications Gell-Mann and Hartle have studied quantum cosmology and the path integral approach to relativistic quantum field theory in the framework of consistent histories. Gell-Mann and Hartle consider the history approach as a convenient setting to discuss the quantum mechanics of closed systems, i.e., most generally the universe. In this way they hope to circumvent some of the conceptual and mathematical difficulties inherent in quantum cosmology. They propose to use histories to describe the events that have taken place and that will take place in the universe. They pay special attention to what they call *quasiclassical domains*. [They prefer now the term

[1] This citation has been taken from the book by Haag [72], where it appears in a completely different context in the last chapter in which Haag dwells upon the weak points of the algebraic approach to quantum field theory.

quasiclassical realm [53].] Loosely speaking a quasiclassical domain is an exhaustive set of mutually exclusive coarse-grained alternative histories which can be ascribed probabilities and whose individual histories are described largely by alternative values of a limited set of quasiclassical variables (at a succession of times) which exhibit patterns of classical correlation in time subject to quantum mechanical fluctuations. In order to make pre- and retrodiction one needs information about the initial state of the universe and about past events. Since the human sensory perceptions are rather limited, we can deal in general only with sets of alternative coarse-grained histories. Furthermore, it is only possible to ascribe probabilities to suitably coarse-grained histories provided some consistency conditions are satisfied. The research in this direction is faced with severe difficulties, as discussed by Dowker and Kent [37, 38] and Kent [99, 100, 101, 102, 103].

In this work we studied the consistent histories approach to quantum mechanics from different perspectives and presented some recent developments and results in this approach. However, there is a host of open problems related to the history approach. Some of them have already discussed in the main text above. I conclude this thesis by briefly summarizing the problems which I consider particularly important.

- The interpretational questions discussed in Chapter 3 are currently under debate. But as discussed above this is an issue of great delicacy.
- As just stated the main problem of the history approach is to construct physically sensible candidates for general quantum history theories.
- The consistent histories approach suggests several mathematical problems. For instance, the structure of the space of decoherence functionals and of the space of all histories has to be investigated. When the space of all histories is a Hilbert space this problem has been investigated in Chapter 7. The analogous problem for arbitrary von Neumann algebras or C^*-algebras has so far not been investigated. This would be of interest also for the research programme from Chapter 5. Some of the mathematical concepts which have been used in this thesis were introduced only very recently (for instance: D-posets). It is important to deepen our understanding of this mathematical structures which presumably plays also a role in more general history theories.

This modest list of possible research problems is by no means complete. The reader is invited to make up his own mind.

In any case, the consistent histories approach is presently one of the prime candidates for a conceptual framework for a generalized timeless quantum theory. It is worth to be studied further.

A

MISCELLANEOUS DEFINITIONS AND RESULTS

Behutsam ! Hier ist alles alt
und gespenstig
[from "Die tote Stadt" by Erich Wolfgang Korngold]

A.1 POSETS AND LATTICES

Definition 19 *Let $\mathcal{P}$ be a nonempty set. A binary relation $\leq$ on $\mathcal{P}$ is called a* PARTIAL ORDER *if for all $x, y, z \in \mathcal{P}$ the following conditions are satisfied*

- $x \leq x$ *(reflexivity)*
- *if* $x \leq y$ *and* $y \leq x$, *then* $x = y$ *(antisymmetry)*
- *if* $x \leq y$ *and* $y \leq z$, *then* $x \leq z$ *(transitivity)*

In this case the pair $(\mathcal{P}, \leq)$ is called a PARTIALLY ORDERED SET *or a* POSET. *A poset is said to have* UNIVERSAL BOUNDS *if there exist two elements $0, 1 \in \mathcal{P}$ such that $0 \leq x \leq 1$ for all $x \in \mathcal{P}$.*

If $x \leq y$ for some elements x, y in a poset, then we also say that x is smaller than y, that x is contained in y or that y is greater than x.

Definition 20 *Let $\mathcal{P}$ be a poset. A non-zero element $a \in \mathcal{P}$ is called an* ATOM *if $y \leq a$ implies $y = 0$ or $y = a$. $\mathcal{P}$ is said to be* ATOMIC *if for every non-zero element $x \in \mathcal{P}$ there is an atom a with $a \leq x$.*

Definition 21 *A* LATTICE *is a poset $\mathcal{L}$ any two of whose elements x, y have a greatest lower bound or "meet" denoted by $x \wedge y$ and a least upper bound or "join" denoted by $x \vee y$. A lattice is called* COMPLETE *if each of its subsets $\mathcal{L}_1$ has a greatest lower bound and a least upper bound. A lattice $\mathcal{L}$ is said to be a* σ-LATTICE *if each countable subset of $\mathcal{L}$ has a greatest lower bound and a least upper bound.*

Definition 22 *We say that a lattice $\mathcal{L}$ is* DISTRIBUTIVE *if for all $x, y, z \in \mathcal{L}$ the following identity is satisfied*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z). \tag{A.1a}$$

Lemma 5 *A lattice $\mathcal{L}$ is* DISTRIBUTIVE *if and only if for all $x, y, z \in \mathcal{L}$ the following identity is satisfied*

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \tag{A.1b}$$

Definition 23 *A lattice $\mathcal{L}$ is* MODULAR *if for all $x, y, z \in \mathcal{L}$*

$$x \vee (y \wedge z) = (x \vee y) \wedge z, \text{ whenever } x \leq y. \tag{A.2}$$

Let $\mathcal{L}$ be a lattice with universal bounds. An element $x' \in \mathcal{L}$ is said to be a COMPLEMENT *of $x \in \mathcal{L}$ if $x \wedge x' = 0$ and $x \vee x' = 1$. If any element of $\mathcal{L}$ has at least one complement, then $\mathcal{L}$ is said to be* COMPLEMENTED. *A lattice $\mathcal{L}$ is said to be* RELATIVELY COMPLEMENTED *if all its intervals (i.e. sublattices of the form $a \leq x \leq b$, with $a, b \in \mathcal{L}$) are complemented.*

Definition 24 *A mapping $\perp\colon \mathcal{P} \to \mathcal{P}$ is said to be an* ORTHOCOMPLEMENTATION *on a poset $\mathcal{P}$ with universal bounds if for all $x, y \in \mathcal{P}$*

- $(x^{\perp})^{\perp} = x$,
- *if $x \leq y$, then $y^{\perp} \leq x^{\perp}$,*
- $x \vee x^{\perp} = 1$.

In this case the pair $(\mathcal{P}, \perp)$ is called an ORTHOCOMPLEMENTED POSET. *A lattice $\mathcal{L}$ with an orthocomplementation is called an* ORTHOLATTICE.

Definition 25 *An ortholattice $\mathcal{L}$ which satisfies for all $x, y \in \mathcal{L}$*

$$x \vee (x^{\perp} \wedge y) = y, \text{ whenever } x \leq y. \tag{A.3}$$

is called ORTHOMODULAR.

Remark 9 *Any modular ortholattice is orthomodular.*

Definition 26 *A* BOOLEAN LATTICE *is a complemented distributive lattice.*

Throughout this thesis we will use the terms *Boolean lattice* and *Boolean algebra* synonymously. For a justification we refer the reader to the monograph by Birkhoff [11], Section I.10.

Remark 10 *Complements are unique in distributive lattices.*

Theorem 8 (Stone) *Every Boolean lattice is isomorphic to the algebra of clopen subsets of some topological space.*

A.2 D-POSETS AND ORTHOALGEBRAS

A.2.1 Basic Definitions

Definition 27 *A* DIFFERENCE POSET *or* D-POSET *is a partially ordered set D with greatest element 1 and with a partial binary operation $\ominus : D_2 \to D$, where $D_2 \subset D \times D$, such that*

- *$b \ominus a$ is defined if and only if $a \leq b$ for all $a, b \in D$,*
- *$b \ominus a \leq b$ for all $a \leq b$,*
- *$b \ominus (b \ominus a) = a$, for all $a \leq b$,*
- *$a \leq b \leq c \Rightarrow c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.*

Difference posets have been introduced by Kôpka and Chovanec [106] and have been further studied in [40, 41, 42, 43, 107, 147].

Proposition 5 *Let $(D, \ominus)$ be a D-poset. Then*

- *D has universal bounds, given by 1 and $0 := 1 \ominus 1$;*
- *$a \ominus 0 = a$, for all $a \in D$;*
- *$a \ominus a = 0$, for all $a \in D$;*
- *If $a, b \in D$ with $a \leq b$, then $b = a$ if and only if $b \ominus a = 0$;*
- *If $a, b \in D$ with $a \leq b$, then $a = 0$ if and only if $b \ominus a = b$;*
- *If $a, b, c \in D$ with $a \leq b \leq c$, then $b \ominus a \leq c \ominus a$ and $a \leq c \ominus (b \ominus a)$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$ and $(c \ominus (b \ominus a)) \ominus a = (c \ominus b)$;*
- *If $a, b, c \in D$ with $b \leq c$ and with $a \leq c \ominus b$, then $b \leq c \ominus a$ and $(c \ominus b) \ominus a = (c \ominus a) \ominus b$.*

Definition 28 *A set D with two special elements $0, 1 \in D$ supplied with a partially defined associative and commutative operation $\oplus : D_2' \to D$, where $D_2' \subset D \times D$, is called an* EFFECT ALGEBRA *if*

- *For every $a \in D$ there exists a unique $a' \in D$ such that $a \oplus a'$ is defined and $a \oplus a' = 1$,* ***[orthosupplementation law]***
- *If $1 \oplus b$ is defined, then $b = 0$, for all $b \in D$.* ***[Zero-One law]***

An effect algebra D is called an ORTHOALGEBRA *if furthermore*

- *If $b \oplus b$ is defined, then $b = 0$, for all $b \in D$.* ***[consistency law]***

Effect algebras have been introduced by Foulis and Bennett [10, 47]. Whenever $a \oplus b$ is well-defined for $a, b \in D$, then we write $a \perp b$. Let $(D, \ominus)$ be a D-poset. Define

$$a \oplus b := 1 \ominus ((1 \ominus a) \ominus b),$$

whenever the right hand side is well-defined. Then $\oplus$ is a well-defined partial binary operation on D and $(D, \oplus)$ is an effect algebra. Conversely, let $(D, \oplus)$ be an effect algebra. Define

$$b \ominus a := (a \oplus b')',$$

whenever the right hand side is well-defined. Then $\ominus$ is a well-defined partially binary operation on D. Further, define $a \leq b$ for $a, b \in D$ if there exists $c \in D$ such that $c \perp a$ and $a \oplus c = b$. Then $(D, \ominus)$ is a D-poset. Therefore the notions of D-poset and effect algebra are equivalent and we will use *both* terms synonymously in this work.

Definition 29 *Let $(D, \oplus)$ be a D-poset. A* PROBABILITY MEASURE ON D *is a map $p : D \to \mathbb{R}^+$ satisfying $p(1) = 1$ and $p(a \oplus b) = p(a) + p(b)$, whenever $a \oplus b$ is well-defined.*

A.2.2 Sub-D-Posets

Let $(D, \ominus)$ be a D-poset. Then it is natural to say that a subset $S \subset D$ is a sub-D-poset of D if

- $1 \in S$;
- $b \ominus a \in S$, for all $a, b \in S$ with $a \leq b$.

However, we need a more general notion of sub-D-poset. Consider a subset S of D. Then the partial order on D induces a partial order on S. We assume that S possesses universal bounds 0_S and 1_S not necessarily coinciding with the universal bounds $0, 1$ of D. In this case we define a partial binary operation $\ominus_S : D_2 \cap (S \times S) \to D$ by

$$b \ominus_S a := 0_S \oplus (b \ominus 0_S) \ominus (a \ominus 0_S),$$

for all $a, b \in S$ with $a \leq b$.

Definition 30 *Let S be a subset of the D-poset $(D, \ominus)$ with universal bounds $0_S, 1_S$. Then the pair $(S, \ominus_S)$ is called a* SUB-D-POSET *of D if $b \ominus_S a \in S$ for all $a, b \in S$ with $a \leq b$.*

It is easy to verify that a sub-D-poset of a D-poset is a D-poset in its own right. The operation $\oplus_S$ dual to $\ominus_S$ is given by

$$\begin{aligned} a \oplus_S b &:= 1_S \ominus_S ((1_S \ominus_S a) \ominus_S b) \\ &= (a \ominus 0_S) \oplus (b \ominus 0_S) \oplus 0_S, \end{aligned}$$

whenever the right hand sides are well defined. In the main text we shall drop the subscript S in the symbols $\ominus_S$ and $\oplus_S$ and simply write $(S, \ominus)$ for a sub-D-poset of $(D, \ominus)$. This abuse of notation can be justified by the observation that $\ominus_S$ is unambiguously determined by $\ominus$ and by the universal bounds of S.

A.2.3 Tensor products of D-posets

The tensor product of a pair of D-posets is defined by a universal mapping property, cf. [40].

Definition 31 *Let D_1, D_2, ..., D_n and L denote D-posets.*

- *A mapping $\phi : D_1 \to L$ is said to be a* MORPHISM *if (i) $\phi(1) = 1$, (ii) $a \perp b$ implies $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$, for all $a, b \in D_1$.*
- *A mapping $\phi : D_1 \to L$ is said to be a* MONOMORPHISM *if (i) ϕ is a morphism and (ii) $\phi(a) \perp \phi(b)$ implies $a \perp b$, for all $a, b \in D_1$.*
- *A mapping $\phi : D_1 \to L$ is said to be an* ISOMORPHISM *if ϕ is a surjective monomorphism.*
- *A mapping $\theta : D_1 \times D_2 \to L$ is said to be a* SEMI-BIMORPHISM *if*
  - *$a, b \in D_1$ with $a \perp b$ implies $\theta(a, c) \perp \theta(b, c)$ and $\theta(a \oplus b, c) = \theta(a, c) \oplus \theta(b, c)$ for all $c \in D_2$;*
  - *$c, d \in D_2$ with $c \perp d$ implies $\theta(a, c) \perp \theta(a, d)$ and $\theta(a, c \oplus d) = \theta(a, c) \oplus \theta(a, d)$ for all $a \in D_1$;*
- *A mapping $\theta : D_1 \times D_2 \to L$ is said to be a* BIMORPHISM *if it is a semi-bimorphism and $\theta(1, 1) = 1$.*
- *A mapping $\theta : D_1 \times D_2 \times \cdots \times D_n \to L$ is said to be a* MULTIMORPHISM *if it is a semi-bimorphism in every pair of its arguments and if $\theta(1, ..., 1) = 1$.*

Definition 32 *Let $D_1, ..., D_n$ denote a family of n D-posets. We say that a pair $(\mathfrak{D}, \Theta)$ consisting of a D-poset $\mathfrak{D}$ and a multimorphism $\Theta : D_1 \times \cdots \times D_n \to \mathfrak{D}$ is a* TENSOR PRODUCT *of the family $D_1, ..., D_n$ if the following conditions are satisfied*

- *If L is a D-poset and $\theta_L : D_1 \times \cdots \times D_n \to L$ a multimorphism, then there exists a morphism $\beta : \mathfrak{D} \to L$ such that $\theta_L = \beta \circ \Theta$;*

- *Every element of $\mathfrak{D}$ is a finite orthogonal sum of elements of the form $\Theta(a_1, ..., a_n)$ with $a_i \in D_i$, for all i.*

We also write $\mathfrak{D} = \bigotimes_{i=1}^{n} D_i = D_1 \otimes \cdots \otimes D_n$ and $\Theta = \otimes$.

Dvurečenskij [40] has proven that two D-posets D_1 and D_2 admit a tensor product if and only if there is a difference poset L for which there is a bimorphism $\theta_L : D_1 \times D_2 \to L$. The definition of the tensor product of D-posets and Dvurečenskij's theorem can straightforwardly be extended to any finite number of D-posets, see [147].

A.2.4 Directed Systems and Direct Limits

Definition 33 *Let $(\mathfrak{T}, \leq)$ be a partially ordered set. A* $\mathfrak{T}$-DIRECTED SYSTEM *of D-posets is a family $D_{\mathfrak{T}} := \{D_t, t \in \mathfrak{T}\}$ of D-posets supplied with a family of morphisms $f_{ts} : D_t \to D_s, t, s \in \mathfrak{T}$, defined iff $t \leq s$, such that*

- *$f_{tt} = id_{D_t}$, for all $t \in \mathfrak{T}$;*
- *If $t \leq s \leq r$ in $\mathfrak{T}$, then $f_{sr} f_{ts} = f_{tr}$.*

Let $D_{\mathfrak{T}}$ be a $\mathfrak{T}$-directed system of D-posets. Then a D-poset $\mathfrak{L}$ supplied with a family of morphisms $\{f_t : D_t \to \mathfrak{L}\}_{t \in \mathfrak{T}}$ is called the DIRECT LIMIT *of $D_{\mathfrak{T}}$ if*

- *If $t \leq s$ in $\mathfrak{T}$, then $f_s f_{ts} = f_t$;*
- *If D is a D-poset supplied with a set of morphisms $\{g_t : D_t \to D, t \in \mathfrak{T}\}$, then there exists a unique morphism $g : \mathfrak{L} \to D$, such that $g f_t = g_t$, for all $t \in \mathfrak{T}$.*

The direct limit of a directed system of D-posets always exists [147].

A.3 THE D-POSET $\mathfrak{E}(\mathfrak{H})$

The set $\mathfrak{E}(\mathfrak{H})$ of all effect operators on a Hilbert space $\mathfrak{H}$ [with the scalar product denoted by $(\cdot, \cdot)$] can be organized into a D-poset by defining a partial ordering $\leq$ and a partial binary operation $\ominus$ on D by $A \leq B$ if $(Ax, x) \leq (Bx, x)$ for all $x \in \mathfrak{H}$ and $C = B \ominus A$ if $(Bx, x) - (Ax, x) = (Cx, x)$ for all $x \in \mathfrak{H}$. This D-poset structure can be alternatively characterized by the partial sum $\oplus$, where $E_1 \oplus E_2$ is defined if $E_1 + E_2 \leq 1$ by $E_1 \oplus E_2 := E_1 + E_2$. We refer to the so defined D-poset structure on $\mathfrak{E}(\mathfrak{H})$ as the CANONICAL D-POSET STRUCTURE.

The D-poset structure on $\mathfrak{E}(\mathfrak{H})$ is not unique. It is possible to define a countably infinite family of D-poset structures on $\mathfrak{E}(\mathfrak{H})$. Let α be a rational number with $\alpha > 0$ and define

$$A \oplus_\alpha B := \left(A^{1/\alpha} + B^{1/\alpha}\right)^\alpha, \text{ for all } A, B \in \mathfrak{E}(\mathfrak{H}) \text{ satisfying } A^{1/\alpha} + B^{1/\alpha} \leq 1.$$

That these expressions are well-defined is a consequence of the work of Langer [113]. In particular it follows from Proposition 2 in [113] that E^α is well-defined and that E^α is itself an effect operator for all $E \in \mathfrak{E}(\mathfrak{H})$ and all $\alpha \in \mathbb{Q}, \alpha > 0$. The pair $(\mathfrak{E}(\mathfrak{H}), \oplus_\alpha)$ is a D-poset for every $\alpha > 0$. However, all these D-poset structures are isomorphic. The D-poset isomorphisms are given by $Q_\alpha : (\mathfrak{E}(\mathbb{H}), \oplus) \to (\mathfrak{E}(\mathbb{H}), \oplus_\alpha), u \mapsto u^\alpha$.

Let in the sequel $\mathfrak{T}$ denote the set of all finite subsets of $\mathbb{R}$ partially ordered by set inclusion. For every $t \in \mathbb{R}$ set $\mathfrak{E}(\mathfrak{H})_t := \mathfrak{E}(\mathfrak{H})$ and for every $T = \{t_1, ..., t_n\} \in \mathfrak{T}$ set $\mathfrak{E}(\mathfrak{H})_T := \mathfrak{E}\left(\otimes_{t \in T} \mathfrak{H}_t\right)$ where $\mathfrak{H}_t := \mathfrak{H}$ for all $t \in \mathbb{R}$. Then it has been shown in [147] that for every $T \subset S \in \mathfrak{T}$ there exists a morphism $f_{TS} : \mathfrak{E}(\mathfrak{H})_T \to \mathfrak{E}(\mathfrak{H})_S$ such that $\{\mathfrak{E}(\mathfrak{H})_T, T \in \mathfrak{T}\}$ supplied with $\{f_{TS}, T \subset S \in \mathfrak{T}\}$ is a $\mathfrak{T}$-directed system. Let, e.g., $T = \{t_1, t_3\} \subset S = \{t_1, t_2, t_3\}$, then $f_{TS}(A \otimes B) = A \otimes 1 \otimes B$.
Therefore the direct limit of $\{(\mathfrak{E}(\mathfrak{H})_T, \oplus), T \in \mathfrak{T}\}$ exists and will be denoted by $(\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H}), \oplus)$. $\mathfrak{E}(\mathfrak{H})_{\mathfrak{T}}$ can be constructed as follows: consider the disjoint union $\cup_{T \in \mathfrak{T}} \mathfrak{E}(\mathfrak{H})_T$ and call two elements h_1, h_2 of $\cup_{T \in \mathfrak{T}} \mathfrak{E}(\mathfrak{H})_T$ EQUIVALENT if there exist $T_1, T_2, T_{12} \in \mathfrak{T}$ such that $h_1 \in T_1 \subset T_{12}$, $h_2 \in T_2 \subset T_{12}$ and such that $f_{T_1 T_{12}}(h_1) = f_{T_2 T_{12}}(h_2)$. Then $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$ is the quotient space of $\cup_{T \in \mathfrak{T}} \mathfrak{E}(\mathfrak{H})_T$ by the such defined equivalence relation. It is easy to extend the D-poset structures on $\mathfrak{E}(\mathfrak{H})_T$, for $T \in \mathfrak{T}$ to a D-poset structure on $\mathfrak{E}^{\otimes}_{fin}(\mathfrak{H})$.

A.4 AN ALTERNATIVE CHARACTERIZATION OF BOOLEAN LATTICES

Boolean lattices can be characterized as special D-posets. The following Theorem is due to Foulis and Bennett [47]

Theorem 9 *Let $(\mathcal{B}, \oplus)$ be a D-poset such that for all $a, b, c \in \mathcal{B}$:*

- *If $a \oplus b$, $b \oplus c$ and $a \oplus c$ are defined, then $(a \oplus b) \oplus c$ is defined.* ***[coherence law]***
- *For all $a, b \in \mathcal{B}$, there exist $c, d, e \in \mathcal{B}$ such that $d \oplus e$ and $c \oplus (d \oplus e)$ are defined, $a = c \oplus e$ and $b = d \oplus e$.* ***[law of compatibility]***

Then $\mathcal{B}$ can be organized into a Boolean algebra in one and only one way, so that

- $0 \leq b \leq 1$ *for all $b \in \mathcal{B}$,*

- *$a \oplus b$ is defined if and only if $a \wedge b = 0$,*
- *$a \wedge b = 0$ implies $a \oplus b = a \vee b$.*

Conversely, let $\mathcal{B}'$ denote a Boolean lattice, then a partial binary operation $\oplus_{\mathcal{B}'}$ is defined by $a \oplus_{\mathcal{B}'} b := a \vee b$ whenever $a \wedge b = 0$. The pair $(\mathcal{B}', \oplus_{\mathcal{B}'})$ is then a D-poset satisfying the coherence law and the law of compatibility.

We will call a D-poset satisfying both the coherence law and the law of compatibility a BOOLEAN D-POSET.

If $(\mathcal{B}, \oplus)$ is a Boolean sub-D-poset of $(\mathfrak{E}(\mathfrak{H}), \oplus)$, then $(Q_\alpha(\mathcal{B}), \oplus_\alpha)$ is a Boolean sub-D-poset of $(\mathfrak{E}(\mathfrak{H}), \oplus_\alpha)$ such that for all $a, b \in \mathcal{B}$

- $Q_\alpha(a \wedge b) = Q_\alpha(a) \wedge_\alpha Q_\alpha(b)$;
- $Q_\alpha(a \vee b) = Q_\alpha(a) \vee_\alpha Q_\alpha(b)$;
- $Q_\alpha(a') = Q_\alpha(a)'$.

A.5 HILBERT LATTICES

It is well-known that there is a one-to-one correspondence between closed subspaces of a Hilbert space $\mathfrak{H}$ and projection operators on $\mathfrak{H}$. In this work we will freely switch between both pictures and identify each projection operator with the subspace onto which it projects. We denote by $\mathcal{P}(\mathfrak{H})$ the set of all projection operators on the Hilbert space $\mathfrak{H}$. For all $p_1, p_2 \in \mathcal{P}(\mathfrak{H})$ one defines

- $p_1 \leq p_2$ if p_1 projects on a subspace of the range of p_2, ($\leq$ defines a partial order on $\mathcal{P}(\mathfrak{H})$),
- the join $p_1 \vee p_2$ of p_1 and p_2 to be the projection operator which projects on the smallest closed subspace of $\mathfrak{H}$ which contains the subspaces $p_1\mathfrak{H}$ and $p_2\mathfrak{H}$,
- the meet $p_1 \wedge p_2$ of p_1 and p_2 to be the projection operator which projects on the intersection of $p_1\mathfrak{H}$ and $p_2\mathfrak{H}$ and
- the orthocomplementation $\neg p_1$ of p_1 to be the projection operator which projects on the orthogonal subspace of $p_1\mathfrak{H}$ in $\mathfrak{H}$.

Analogously, for any family $\{p_i\}_i$ we define the meet $\wedge_i p_i$ to be the projection operator projecting onto $\cap_i p_i\mathfrak{H}$ and the join $\vee_i p_i$ the be the projection operator onto the closure of $\cup_i p_i\mathfrak{H}$. Then we have

Theorem 10 *$\mathcal{P}(\mathfrak{H})$ is a complete, atomic, orthomodular lattice. The atoms are the projection operators onto the one-dimensional subspaces of $\mathfrak{H}$. If $\mathfrak{H}$ is finite dimensional, then $\mathcal{P}(\mathfrak{H})$ is modular.*

B

GLEASON'S AND WRIGHT'S THEOREMS

Un secret à nos yeux
est caché dans ce jour;
que pensier ? cherchons bien;
[from "Les Huguenots" by Giacomo Meyerbeer/Eugène Scribe]

B.1 PRELIMINARIES

The famous Gleason Theorem has become a cornerstone in the axiomatic foundations of Hilbert space quantum mechanics. Gleason's Theorem characterizes the σ-additive measures on $\mathcal{P}(\mathfrak{H})$ (= states) for a real or complex Hilbert space $\mathfrak{H}$ of dimension greater than two. A map $m : \mathcal{P}(\mathfrak{H}) \to \mathbb{R}$ such that

$$m(0) = 0, \tag{B.1}$$

$$m\left(\bigvee_{i\in\mathcal{I}} p_i\right) = \sum_{i\in\mathcal{I}} m\left(p_i\right), \tag{B.2}$$

whenever $\{p_i\}_{i\in\mathcal{I}}$ is a system of mutually orthogonal projection operators in $\mathcal{P}(\mathfrak{H})$ is said to be (i) a FINITELY ADDITIVE SIGNED MEASURE, (ii) a SIGNED MEASURE, or (iii) a COMPLETELY ADDITIVE SIGNED MEASURE on $\mathcal{P}(\mathfrak{H})$ if Equation B.2 holds for every (i) finite, (ii) countable, or (iii) arbitrary index set $\mathcal{I}$ respectively. A finitely additive signed measure on $\mathcal{P}(\mathfrak{H})$ is said to be JORDAN if it can be written as a difference of two positive finitely additive measures.
If m is positive on $\mathcal{P}(\mathfrak{H})$, that is, $m(P) \geq 0$, for all P, then m is said to be (i) a FINITELY ADDITIVE MEASURE, (ii) a MEASURE, or (iii) a COMPLETELY ADDITIVE MEASURE on $\mathcal{P}(\mathfrak{H})$ if Equation B.2 holds for every (i) finite, (ii) countable, or (iii) arbitrary index set $\mathcal{I}$ respectively.
If a positive m satisfies $m(1) = 1$, then m is called (i) a FINITELY ADDITIVE STATE, (ii) a STATE, or (iii) a COMPLETELY ADDITIVE STATE on $\mathcal{P}(\mathfrak{H})$ if Equation B.2 holds for every (i) finite, (ii) countable, or (iii) arbitrary index set $\mathcal{I}$ respectively.
We recall that $\mathcal{S}(\mathfrak{H})$ denotes the set of all unit vectors in the Hilbert space $\mathfrak{H}$. A map $f : \mathcal{S}(\mathfrak{H}) \to \mathbb{R}$ is called a FRAME FUNCTION if there is a constant $\omega \in \mathbb{R}$ such that for every orthonormal basis $\{|h_i\rangle\}$ of $\mathfrak{H}$

$$\sum_i f\left(|h_i\rangle\right) = \omega$$

is satisfied. The constant ω is called the WEIGHT of the frame function f. A frame function f on $\mathfrak{H}$ is called BOUNDED if $\sup\{|f(|h\rangle)| : |h\rangle \in \mathcal{S}(\mathfrak{H})\} < \infty$. A frame function f on $\mathfrak{H}$ is called REGULAR if there is a Hermitean operator T_f on $\mathfrak{H}$ such that f can be written as $f\left(|h\rangle\right) = \langle h|T_f|h\rangle$, for all $|h\rangle \in \mathcal{S}(\mathfrak{H})$, where $\langle\cdot|\cdot\rangle$ denotes the inner product in $\mathfrak{H}$.
There is a duality between completely additive signed measures on $\mathcal{P}(\mathfrak{H})$ and frame functions

on $\mathfrak{H}$: let m be a completely additive signed measure on $\mathcal{P}(\mathfrak{H})$ and denote for every $|h\rangle \in \mathcal{S}(\mathfrak{H})$ the projection operator onto $|h\rangle$ by $P_h = |h\rangle\langle h|$, then $f_m(|h\rangle) := m(P_h), |h\rangle \in \mathcal{S}(\mathfrak{H})$, defines a frame function f_m on $\mathfrak{H}$ with weight $\omega_m = m(1)$. Conversely, let f be a frame function on $\mathfrak{H}$. Let $P \in \mathcal{P}(\mathfrak{H})$ and let $\{P_i\}$ be a decomposition of P into mutually orthogonal one-dimensional projection operators. Denote by $|P_i\rangle$ the unit vector in $\mathfrak{H}$ onto which the ith projector P_i projects, then $m_f(P) := \sum_i f(|P_i\rangle)$ defines a completely additive signed measure m_f on $\mathfrak{H}$. It is easy to see that $f_{m_f} = f$.

Proposition 6 *For any integer $n > 2$ let $\mathfrak{H}_n$ be an n-dimensional complex Hilbert space. Then every bounded frame function on $\mathfrak{H}_n$ is regular.*

Proposition 7 *Let $\mathfrak{H}$ be an infinite-dimensional complex Hilbert space. Then any frame function on $\mathfrak{H}$ is bounded and regular.*

For the proof of Propositions 6 and 7 we refer the reader to [39], Chapter 3.

Now let $\mathfrak{K}$ be a dense linear subspace of $\mathfrak{H}$. Define $\mathcal{S}(\mathfrak{K}) := \mathcal{S}(\mathfrak{H}) \cap \mathfrak{K}$. A map $g : \mathcal{S}(\mathfrak{K}) \to \mathbb{R}$ is said to be a FRAME TYPE FUNCTION ON $\mathfrak{H}$ if the following conditions are satisfied

- the family $\{g(|h_i\rangle)\}$ is summable for every orthonormal system $\{|h_i\rangle\}$ in $\mathfrak{K}$;
- for any finite-dimensional subspace $\mathcal{K}_0$ of $\mathfrak{K}$, the restriction $g|_{\mathcal{S}(\mathcal{K}_0)}$ of g to $\mathcal{S}(\mathcal{K}_0)$ is a frame function on $\mathcal{K}_0$.

Now we have the following important result due to Dorofeev and Sherstnev. A proof of this proposition can be found in [39], Section 3.2.4.

Proposition 8 *Let $\mathfrak{K}$ be a dense linear subspace of an infinite-dimensional complex Hilbert space $\mathfrak{H}$ and let $g : \mathcal{S}(\mathfrak{K}) \to \mathbb{R}$ be a frame type function on $\mathfrak{H}$. Then g is bounded and there is a unique Hermitean trace class operator T_g on $\mathfrak{H}$ such that $g(|h\rangle) = \langle h|T_g|h\rangle$ for all $|h\rangle \in \mathcal{S}(\mathfrak{K})$.*

B.2 GLEASON'S THEOREM

Gleason's Theorem is now a consequence of Proposition 6 and Proposition 7

Theorem 11 (Gleason) *Let $\mathfrak{H}$ be a separable Hilbert space of dimension greater than two and let m be a state on $\mathcal{P}(\mathfrak{H})$. Then there exists a unique Hermitean trace class operator ϱ with trace one such that $m(P) = \mathrm{tr}_{\mathfrak{H}}(P\varrho)$ for all $P \in \mathcal{P}(\mathfrak{H})$.*

The original proof given by Gleason uses the representation theory of $O(3)$ and is rather complicated. An elementary proof of Gleason's Theorem has been given by Cooke, Keane and Moran [29]. This proof can also be found in the monographs [39, 85, 146]. With the aid of Proposition 8 Gleason's Theorem can be generalized to nonseparable Hilbert spaces [39]

Theorem 12 *Let $\mathfrak{H}$ be a separable or nonseparable Hilbert space of dimension greater than two and let m be a completely additive signed measure on $\mathcal{P}(\mathfrak{H})$. Then there exists a unique Hermitean trace class operator T on $\mathfrak{H}$ such that $m(P) = \mathrm{tr}_{\mathfrak{H}}(TP)$ for all $P \in \mathcal{P}(\mathfrak{H})$. Hence, m is Jordan and bounded.*

For generalizations of Gleason's Theorem to arbitrary von Neumann algebras we refer the reader to Maeda [119] and Bunce and Wright [16, 17].

B.3 WRIGHT'S THEOREM

If A is a von Neumann algebra, let $\mathcal{P}(A)$ denote the set of projectors in A. A function $d : \mathcal{P}(A) \times \mathcal{P}(A) \to \mathbb{C}$ is called a DECOHERENCE FUNCTIONAL , if

- $d(p_1 \oplus p_2, q) = d(p_1, q) + d(p_2, q)$, whenever p_1 and p_2 are mutually orthogonal;
- $d(p, q)^* = d(q, p)$;
- $d(p, p) \geq 0$;
- $d(1, 1) = 1$.

Theorem 13 *Let A be a von Neumann algebra with no type I_2 direct summand. Let $d : \mathcal{P}(A) \times \mathcal{P}(A) \to \mathbb{C}$ be a decoherence functional . If d is bounded, then d extends to a unique bounded bilinear functional $\widetilde{d}$ on $A \times A$. Furthermore, d is continuous when $\mathcal{P}(A)$ is equipped with the topology induced by the norm of A. Also, $d(u, v)^* = d(v^*, u^*)$.*

A proof of this theorem can be found in Wright's original article [171].

C

THE TENSOR PRODUCT FORM OF THE STANDARD DECOHERENCE FUNCTIONAL

I never could follow the last act
of The Marriage of Figaro.
And this is even worse ...
[from "The Ghosts of Versailles" by John Corigliano/William F. Hofman]

In this appendix we give some remarks on the proof of Theorem 1.

Since the state ϱ is a density operator, there exists an orthonormal basis $\{|e_i^\varrho\rangle\}_i$ of $\mathfrak{H}$ and positive numbers $\{\omega_i\}$ such that $\sum_i \omega_i = 1$ such that ϱ can be written as $\varrho = \sum_i \omega_i |e_i^\varrho\rangle\langle e_i^\varrho|$. Now let h, k be homogeneous finite histories and let $m := \#\mathfrak{s}(h)$ and $n := \#\mathfrak{s}(k)$.

For every $k \in \mathbb{N}$, denote by $\otimes_{al}^k \mathfrak{H}$ the k-fold algebraic tensor product of $\mathfrak{H}$, the set of all finite sums of homogeneous vectors of the form $\phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_k$, where $\phi_j \in \mathfrak{H}$. Define an operator $S_k : \otimes_{al}^k \mathfrak{H} \to \otimes_{al}^k \mathfrak{H}$ by $S_k(\phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_k) := \phi_2 \otimes \cdots \otimes \phi_k \otimes \phi_1)$ and extend by linearity. Moreover, define an operator $R_k : \otimes_{al}^k \mathfrak{H} \to \otimes_{al}^k \mathfrak{H}$ by $R_k(\phi_1 \otimes \phi_2 \otimes \cdots \phi_{k-1} \otimes \phi_k) := \phi_k \otimes \phi_{k-1} \otimes \cdots \otimes \phi_2 \otimes \phi_1)$ and extend by linearity. In the sequel we assume without loss of generality $\mathfrak{s}(h) = \mathfrak{s}(k)$. This can always be achieved by inserting the $1 \in \mathcal{P}(\mathfrak{H})$ appropriately. In particular, $m = n$. We denote the unit operator on $\otimes^k \mathfrak{H}$ by 1_k.

A straightforward computation now shows that $\mathsf{d}_\varrho(h, k)$ can be written as

$$\mathsf{d}_\varrho(h,k) = \mathrm{tr}_{\otimes^{2n}\mathfrak{H}} \left((h \otimes k)(U^\dagger_{t_1,t_2,\ldots,t_n} \otimes U^\dagger_{t_1,t_2,\ldots,t_n}) \mathfrak{Y}_\varrho (U_{t_1,t_2,\ldots,t_n} \otimes U_{t_1,t_2,\ldots,t_n}) \right),$$

where

$$U_{t_1,t_2,\ldots,t_n} := U(t_0,t_1) \otimes U(t_0,t_2) \otimes \cdots \otimes U(t_0,t_n),$$

with some fiducial time t_0 and where

$$\mathfrak{Y}_\varrho = (R_n \otimes 1_n)(1_{2n-1} \otimes \varrho) S_{2n} (R_n \otimes 1_n).$$

Inserting resolutions of the identity for 1_{2n-1} shows that there exist $2n$ orthonormal bases $\{|e_{i_k}^k\rangle\}$, $k \in \{1, \ldots, 2n\}$, of $\mathfrak{H}$ such that

$$\mathfrak{Y}_\varrho = \sum_{i_1,\ldots,i_{2n}} \omega_{i_{2n}} \quad \Big\{ |e^n_{i_n}\rangle\langle e^{n-1}_{i_{n-1}}| \otimes |e^{n-1}_{i_{n-1}}\rangle\langle e^{n-2}_{i_{n-2}}| \otimes \cdots \otimes |e^1_{i_1}\rangle\langle e^{2n}_{i_{2n}}| \otimes$$
$$\otimes |e^{n+1}_{i_{n+1}}\rangle\langle e^n_{i_n}| \otimes |e^{n+2}_{i_{n+2}}\rangle\langle e^{n+1}_{i_{n+1}}| \otimes \cdots \otimes |e^{2n}_{i_{2n}}\rangle\langle e^{2n-1}_{i_{2n-1}}| \Big\}.$$

The orthonormal bases $\{|e_{i_k}^k\rangle\}$, $k \in \{1, \ldots, 2n-1\}$ are completely arbitrary, whereas necessarily $|e_i^{2n}\rangle = |e_i^\varrho\rangle$ for all i.

Obviously, $\mathfrak{Y}_\varrho$ is a trace class operator on $\otimes^{2n}\mathfrak{H}$.

Now, using the abbreviation $\mathcal{U}_n := U_{t_1,t_2,\ldots,t_n}$, $\mathsf{d}_\varrho(h,k)$ can also be written as

$$\mathsf{d}_\varrho(h,k) = \sum_{i_1,\ldots,i_{2n}} \omega_{i_{2n}} \quad \Big\{ \langle e^{n-1}_{i_{n-1}} \otimes e^{n-2}_{i_{n-2}} \otimes \cdots \otimes e^{2n}_{i_{2n}} | \mathcal{U}_n h \mathcal{U}_n^\dagger | e^n_{i_n} \otimes e^{n-1}_{i_{n-1}} \otimes \cdots \otimes e^1_{i_1} \rangle \times$$
$$\times \langle e^n_{i_n} \otimes e^{n+1}_{i_{n+1}} \otimes \cdots \otimes e^{2n-1}_{i_{2n-1}} | \mathcal{U}_n k \mathcal{U}_n^\dagger | e^{n+1}_{i_{n+1}} \otimes e^{n+2}_{i_{n+2}} \otimes \cdots \otimes e^{2n}_{i_{2n}} \rangle \Big\} .$$

By choosing $e^{n-j}_{n-j} = e^{n+j}_{i_{n+j}}$ for all $j \in \{0,\ldots,n-1\}$, one immediately sees from this representation that $\mathsf{d}_\varrho(h,k) = \mathsf{d}_\varrho(k,h)^*$ and that also the extension d_ϱ of d_ϱ satisfies $d_\varrho(h,k) = d_\varrho(k,h)^*$ and $d_\varrho(h,h) \geq 0$ for arbitrary inhomogeneous histories $h,k \in \mathcal{P}^{\otimes}_{fin}(\mathfrak{H})$.

D

CONTINUOUS TENSOR PRODUCTS

Es kann sein, daß die Dinge,
die wir gut nennen, sehr schlecht sind, und die
Dinge, die wir schlimm nennen, sehr gut sind.
Wir wissen von nichts etwas ...
[from "Salome" by Richard Strauss (Lachmann/Wilde)]

This Appendix is included for the convenience of the reader. It contains the definition of continuous tensor products of Hilbert spaces and a few remarks. The source for this Appendix is the book by Parthasarathy and Schmidt [139]. More details can be found in this book. Closely related is also the monograph by Guichardet [67]. An example for a continuous tensor product of Hilbert spaces is the familiar symmetric Fock space $\exp \mathfrak{H}$ of a Hilbert space $\mathfrak{H}$ when $\mathfrak{H}$ is a direct integral of a family of Hilbert spaces. This space is also called the exponential Hilbert space of $\mathfrak{H}$ and is discussed in more detail in [104]. However, for our purposes in this thesis the concept of exponential Hilbert space is not needed.

Definition 34 *Let Ω be a topological space and let $\mathcal{B}(\Omega)$ be the Borel σ-algebra of Ω. The measurable space $(\Omega, \mathcal{B}(\Omega))$ is said to be* STANDARD *if it is Borel isomorphic to the measurable space $([0,1], \mathcal{B}([0,1]))$.*

Theorem 14 *Let X an uncountable complete separable metric space, then $(X, \mathcal{B}(X))$ is Borel equivalent to $([0,1], \mathcal{B}([0,1]))$.*

In particular $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is a standard measurable space for all $n \in \mathbb{N}$. A proof of the theorem can for instance be found in the book by Royden [149], Chapter 15.4.

Definition 35 *Let Ω be a topological space and let $\mathcal{B}(\Omega)$ be the Borel σ-algebra of Ω. Let $(\mathfrak{H}(A), A \in \mathcal{B}(\Omega))$ be a family of Hilbert spaces satisfying the following: for any two disjoint Borel sets $A, B \in \mathcal{B}(\Omega)$ there is a unitary map $U(A,B)$ from $\mathfrak{H}(A) \otimes \mathfrak{H}(B)$ onto $\mathfrak{H}(A \cup B)$ such that*

$$U(A \cup B, C)\{U(A,B) \otimes Id(C)\} = U(A, B \cup C)\{Id(A) \otimes U(B,C)\},$$

for any three disjoint Borel sets $A, B, C \in \mathcal{B}(\Omega)$. Then the family $(\mathfrak{H}(A), A \in \mathcal{B}(\Omega))$ is said to be FACTORIZABLE. *If $(\Omega, \mathcal{B}(\Omega))$ is a standard measurable space and if $\mathfrak{H}(\{\omega\})$ is one dimensional for every single point set $\{\omega\}, \omega \in \Omega$, then $\mathfrak{H}(\Omega)$ is said to be a* CONTINUOUS TENSOR PRODUCT OVER Ω.

The main example used in this thesis is the case when $\Omega = \mathbb{R}^4$. In our "local" version of the consistent histories approach we assume in Chapter 5 that there is a Hilbert space

$\mathfrak{H}(\mathbb{R}^4)$ on Minkowski space which is a continuous tensor product of "local" Hilbert spaces $\mathfrak{H}(A), A \in \mathcal{B}(\mathbb{R}^4)$.

The nomenclature in Definition 35 can be motivated by the following consideration (stolen from [139]). Let $\mathfrak{H}$ be a Hilbert space. Define the symmetrized n-fold tensor product of $\mathfrak{H}$ with itself

$$\mathfrak{H}_n := (\otimes_{i=1}^n \mathfrak{H})_s$$

and set $\mathfrak{H}_0 := \mathbb{C}$. Then the EXPONENTIAL HILBERT SPACE $\exp \mathfrak{H}$ of $\mathfrak{H}$ is defined by

$$\exp \mathfrak{H} := \oplus_{n=0}^{\infty} \mathfrak{H}_n .$$

For every vector $h \in \mathfrak{H}$ we set

$$\exp h := 1 \oplus h \oplus \frac{1}{\sqrt{2}} h \otimes h \oplus \frac{1}{\sqrt{3!}} h \otimes h \otimes h \oplus ....$$

Clearly, $\exp h \in \exp \mathfrak{H}$.

Let $(\Omega, \mathcal{A}, \mu)$ a measure space and $\{\mathfrak{H}_\lambda, \lambda \in \Omega\}$ a measurable family of Hilbert spaces. For every $A \in \mathcal{A}$ define the *direct integral*

$$\mathfrak{H}(A) := \int_A \mathfrak{H}_\lambda d\mu(\lambda)$$

as the Hilbert space consisting of all families $\{|x_\lambda\rangle\}_{\lambda \in A}$ satisfying $\int_A \|x_\lambda\|_\lambda^2 d\mu(\lambda) < \infty$ endowed with the standard algebraic structure and with the scalar product $\langle \{x_\lambda\} | \{y_\lambda\} \rangle := \int_A \langle x_\lambda | y_\lambda \rangle_\lambda d\mu(\lambda)$. It is easy to see that $\mathfrak{H}(A)$ is complete, i.e., a Hilbert space in its own right [97]. Let A and B denote disjoint sets in $\mathcal{A}$ and define a unitary map from $\exp \mathfrak{H}(A) \otimes \exp \mathfrak{H}(B)$ onto $\exp \mathfrak{H}(A \cup B)$ by

$$U(A,B)(\exp u \otimes \exp v) := \exp(u \oplus v),$$

for all $u \in \mathfrak{H}(A), v \in \mathfrak{H}(B)$ and extended by linearity to all of $\exp \mathfrak{H}(A) \otimes \exp \mathfrak{H}(B)$. This shows that $\exp \mathfrak{H}(\Omega)$ is a continuous tensor product over Ω.

E

JOINT POSITION-MOMENTUM OBSERVABLES

La physique est tout, mon cher !

[from "Les Contes d'Hoffmann" by Jacques Offenbach/Jules Barbier]

E.1 GALILEAN SYSTEMS OF COVARIANCE IN NONRELATIVISTIC PHASE SPACE

The results in this section are well-known and are included in the present thesis only for illustration. A more detailed account can be found in the book by Davies [30] which is the main source for this section.

Consider the Hilbert space $L^2(\mathbb{R})$ together with the standard position observable Q which is a PV measure over the Borel sets of $\mathbb{R}$ defined by $\{Q(B)\varphi\}(x) := \chi_B(x)\varphi(x)$. Let f denote a probability density on $\mathbb{R}$, then

$$Q_f : \mathcal{B}(\mathbb{R}) \to \mathfrak{E}(L^2(\mathbb{R})), Q_f(B) := \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y)\chi_B(x-y)dy \right) Q(dx),$$

defines an observable on $\mathbb{R}$ which is interpreted as "unsharp" position observable. The integral converges in the weak operator topology. A similar construction is possible for the standard momentum observable P.

Now choose $\alpha \in L^2(\mathbb{R})$ with vanishing expectation value for the position observable Q and for the momentum observable P. Define $\alpha_{xy}(q) := e^{iyq}\alpha(q-x)$, then $\alpha_{xy} \in L^2(\mathbb{R})$ and the expectation values for Q and P in the state α_{xy} are x and y respectively. We define a POV measure on $\mathbb{R}^2$ by

$$A : \mathcal{B}(\mathbb{R}^2) \to \mathfrak{E}(L^2(\mathbb{R})), A(E) := \frac{1}{2\pi} \int_E |\alpha_{xy}\rangle\langle\alpha_{xy}| dxdy,$$

where the integral converges in the weak operator topology. This observable is interpreted as joint "unsharp" position-momentum observable. This interpretation is substantiated by the following observation. A induces in a natural way two POV measures on $\mathbb{R}$, namely, $A_1 : \mathcal{B}(\mathbb{R}) \to \mathfrak{E}(L^2(\mathbb{R})), A_1(B) := A(B \times \mathbb{R})$ and $A_2 : \mathcal{B}(\mathbb{R}) \to \mathfrak{E}(L^2(\mathbb{R})), A_2(B) := A(\mathbb{R} \times B)$. This POV measures are called the *marginal observables* of A and a straightforward computation shows that $A_1(B) = Q_{|\alpha|^2}(B)$ and $A_2(B) = P_{|\alpha|^2}(B)$, to wit, A_1 is an unsharp position observable and A_2 is an unsharp momentum observable. It can be shown that A, A_1 and A_2 respect the Heisenberg uncertainty relation. This has been discussed for instance in [30] and [23]. That A transforms appropriately under Galilean transformations has been shown in [23]. Hence, A is an example for a *Galilean system of covariance*.

E.2 LORENTZIAN SYSTEMS OF COVARIANCE IN RELATIVISTIC PHASE SPACE

In this section we describe the manifestly Lorentz covariant POV measures on relativistic phase space which have been constructed by Ali and Prugovečki [6], see also [144, 145].

The relativistic phase space of a classical particle with rest mass m and without spin is given by the seven-dimensional manifold

$$\Gamma_m^+ \equiv \mathcal{M} \times \mathfrak{V}_m^+,$$

where $\mathcal{M}$ denotes Minkowski space and where $\mathfrak{V}_m^+ := \{k \mid k^2 = k_\nu k^\nu = m^2c^2, k^0 > 0\}$ denotes the forward mass hyperboloid. Let σ denote an initial-data spacelike hypersurface of operational simultaneity and introduce $\Sigma_m^+ \equiv \sigma \times \mathfrak{V}_m^+$. Without loss of generality we assume that σ corresponds to $q_0 = 0$. On Σ_m^+ we introduce the measure element $d\Sigma_m^+(q,p) = d\mathbf{q}d\mathbf{p}$. This can be written in a covariant form (using an obvious notation) as $d\Sigma_m^+(q,p) = 2p^\nu d\sigma_\nu(q)\delta(p^2 - m^2c^2)d^4p$, where $d\sigma_\nu$ denotes the three-dimensional volume element on σ normal to σ. Now consider the Hilbert space $L^2(\Sigma_m^+)$ of all square integrable complex-valued functions $\varphi(q,p)$ on Σ_m^+ endowed with the scalar product

$$\langle\varphi_1|\varphi_2\rangle \equiv \int_{\Sigma_m^+} \varphi_1^*(q,p)\varphi_2(q,p)d\Sigma_m^+(q,p).$$

Using that σ corresponds to $q_0 = 0$, we introduce the time-evolution $\varphi(q^0) = \exp(-iP_0q^0)\varphi(0)$, where $P_0 = \sqrt{-\Delta_\mathbf{q} + m^2}$. [1]

Analogously we introduce the Hilbert space $L^2(\mathfrak{V}_m^+)$ consisting of all complex-valued square-integrable functions on the forward mass hyperboloid for mass m endowed with the scalar product

$$\langle\widetilde{\varphi}_1|\widetilde{\varphi}_2\rangle \equiv \int_{\mathfrak{V}_m^+} \widetilde{\varphi}_1^*(k)\widetilde{\varphi}_2(k)d\Omega_m(k),$$

where $d\Omega_m(k) \equiv \delta(k^2 - m^2c^2)d^4k$.

A Poincarè transformation (a,Λ) acts on $\mathbb{R}^4 \times \mathfrak{V}_m^+$ as

$$(a,\Lambda) : (q,p) \mapsto (a + \Lambda q, \Lambda p).$$

On $L^2(\Sigma_m^+)$ we introduce the following representation of the Poincarè group

$$U(a,\Lambda) : \varphi(q,p) \mapsto \varphi'(q,p) := \varphi(\Lambda^{-1}(q-a), \Lambda^{-1}p).$$

It is easily seen that this stipulation defines a unitary representation of the Poincarè group, i.e., satisfies $U(a',\Lambda')U(a,\Lambda) = U(a' + \Lambda'a, \Lambda'\Lambda)$. On $L^2(\mathfrak{V}_m^+)$ we introduce the following unitary representation of the Poincarè group

$$U(a,\Lambda) : \widetilde{\varphi}(k) \mapsto \widetilde{\varphi}'(k) := \exp(iak)\widetilde{\varphi}(\Lambda^{-1}k).$$

[1] Since we are in effect dealing only with solutions of the Klein-Gordon equation with positive energy, it is possible to make sense out of the square root of $-\Delta_\mathbf{q} + m^2$. This somewhat trivial point has been discussed at many places, for instance in an appendix of [150].

Now take a vector $|\eta\rangle \in L^2(\mathfrak{V}_m^+)$ which is invariant under spatial rotations, i.e., $U(0,R)|\eta\rangle = |\eta\rangle$ in an obvious notation. Then Ali and Prugovečki [6] have shown that the elements $|\eta_{q,p}\rangle := U(q,\Lambda_{p/m})|\eta\rangle$ considered as functions of (q,p) are elements of $L^2(\Sigma_m^+)$ and span a closed subspace of $L^2(\Sigma_m^+)$. To be precise, every $|\eta\rangle$ induces a isometric mapping from $L^2(\mathfrak{V}_m^+)$ to a closed subspace of $L^2(\Sigma_m^+)$ by $\widetilde{\varphi}(k) \mapsto \varphi(q,p) := \langle \eta_{q,p}|\widetilde{\varphi}\rangle$. We denote the closed subspace of $L^2(\Sigma_m^+)$ spanned by the $|\eta_{q,p}\rangle$ by $\mathbf{P}_\eta L^2(\Sigma_m^+)$.

Now a POV measure on the Borel subsets of $\mathcal{M} \times \mathfrak{V}_m^+$ with values in $\mathfrak{E}(\mathbf{P}_\eta L^2(\Sigma_m^+))$ can be defined by

$$E_\eta(B) := iC_\eta \int_B |\eta_{q,p}\rangle n^\nu \overleftrightarrow{\partial}_\nu \langle \eta_{q,p}| d^4q d\Omega_m(p),$$

for all Borel subsets B of $\mathcal{M} \times \mathfrak{V}_m^+$, where C_η denotes a normalization constant explicitly given by $\hbar C_\eta = (2\pi\hbar)^3 \int_{\mathfrak{V}_m^+} |\eta(k)|^2 d\Omega_m(k)$ and where n^ν is a unit vector normal to the hypersurfaces in $\mathcal{M}$ corresponding to constant q_0.

It is possible to show that E_η defines indeed a system of covariance, i.e., E_η satisfies

$$U(a,\Lambda)E_\eta(B)U^{-1}(a,\Lambda) = E_\eta(a + \Lambda B),$$

for all B. Accordingly, for every η the POV measure E_η can be interpreted as relativistic joint position-momentum observable for a single relativistic particle without spin. If the state φ of the particle belongs to the subspace $\mathbf{P}_\eta L^2(\Sigma_m^+)$ of $L^2(\Sigma_m^+)$, then the probability to find the particle in the subspace B of its phase space $\mathcal{M} \times \mathfrak{V}_m^+$ is given by $\langle \varphi|E_\eta(B)|\varphi\rangle$ which reduces to

$$\langle \varphi|E_\eta(B_o)|\varphi\rangle = \int_{B_o} |\varphi(q,p)|^2 d\Sigma_m^+(q,p),$$

whenever B_o is a subset of $\mathcal{M} \times \mathfrak{V}_m^+$ corresponding to *constant* q_0.

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Un catalogo egli è che ho fatt'io,
Osservate, leggete con me
[from "Don Giovanni" by Wolfgang Amadeus Mozart/Lorenzo da Ponte]

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