DESY 97-017

Lattice Renormalization Group Studies of the Two-dimensional $O(N)$ Symmetric Non-linear σ model

Dissertation
zur Erlangung des Doktorgrades
des Fachbereichs Physik
der Universität Hamburg

vorgelegt von
York Xylander
aus Cloppenburg

Hamburg 1997

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Gutachter der Disputation:	Prof. Dr. G. Mack Prof. Dr. J. Bartels
Datum der Disputation:	13.2.1997
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Abstract

In this work the two dimensional $O(N)$ symmetric non-linear σ model is studied with different blockspin Renormalization Group methods.

The idea of Part I is to compute the Renormalized Trajectory of the model in a renormalized coupling expansion by means of a scaling principle.

For the hierarchical approximation of the model the Renormalized Trajectory and the associated improved observables are determined to high orders of the running coupling. A numerical cross-check shows the range of validity of the method.

The Renormalized Trajectory of the full σ model is computed to first order in the running coupling. A Monte-Carlo simulation reveals that the improved action to this order does not have an improved scaling behavior.

In Part II the two dimensional $O(N)$ symmetric non-linear σ model is treated with the Mean Field Renormalization Group. A convenient parametrization of fluctuations on the sphere is presented and the fluctuation integral is performed in a saddle-point approximation. This gives rise to an effective action. A saddle-point equation for the background field is derived.

Zusammenfassung

In dieser Arbeit wird das zweidimensionale $O(N)$-invariante nichtlineare σ-Modell mit verschiedenen Blockspin Renormierungsgruppen Methoden untersucht.

Die Idee des ersten Teils besteht darin, die Renormierte Trajektorie des Modells in einer Störungsentwicklung in der renormierten Kopplung auf Grundlage eines Skalenprinzips zu bestimmen.

In der hierarchischen Approximation des Modells werden die Renormierte Trajektorie und die assoziierten verbesserten Observablen zu einer hohen Ordnung in der renormierten Kopplung bestimmt. Ein numerischer Test zeigt den Gültigkeitsbereich der Methode auf.

Die Renormierte Trajektorie des vollen σ-Modells wird zur ersten Ordnung in der renormierten Kopplung berechnet. Eine Monte-Carlo Simulation zeigt, daß die verbesserte Wirkung zu dieser Ordnung noch kein verbessertes Skalenverhalten aufweist.

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Introduction

A decade ago constructive field theorist like GAWEDZKI, KUPIAINEN, and BALABAN developed conceptual and computational tools to obtain rigourous results from block-spin renormalization group approaches. Their aim was to proove the existence of a continuum limit of lattice field theory.

This thesis is an attempt to link their concepts with today's ideas and to exploit their machinery with today's computer power in order

The good old two dimensional $O(N)$ symmetric non-linear σ model serves as a test and reference platform for the resulting new methods.

The thesis is split in two parts each of them coming with an introduction of its own. Hence I will be brief here.
Part I is devoted to the computation of the Renormalized Trajectory by C. WIECZERKOWSKI's scaling technique in renormalized perturbation theory.
Part II presents first attempts to apply G. MACK's approximative RG to the σ-model.
Chapter 1 contains an introduction to the notions of lattice field theory used in the subsequent chapters.

In continuation of earlier work done in our group [Spe94] most of the programs in the course of this thesis were written in `C++`. As a case study for the applicability of object-oriented approaches to high-end numerics the main points of interest were performance and reusabilty of the code. Another important software aspect of the work was the developing of convenient general purpose data analysis tools for the numerical practioner. This led to a monte-carlo analysis package `mcstat` and a data extraction tool `dat2tab`. The latter program reached the finals of the European Software Award. They can be obtained from `http::\\lienhard.desy.de\~xylander`.

Chapter 1

Lattice Field Theory

In this chapter a short introduction to lattice field theory is given. The relationship between the continuum limit, the Renormalization Group and Perfect Actions is clarified. (See [RF92],[WK74],[GK] for additional material)

1.1 Field Theory On The Lattice

Consider a D dimensional scalar continuum field theory with an Euclidean bare action. The theory is defined by the generating functional and its associated expectation values. These are formally given by functional integrals over the continuum fields.

To make sense out of these formal expressions one has to regularize the theory by introducing an infra-red (IR) and ultra-violet (UV) cutoff. The continuum theory is then defined with respect to some limit process that removes these regularizations.

Lattice Regularization

In the case of lattice quantum field theory the regularization is achieved by replacing the space-time continuum with a lattice Ω_a with lattice spacing a and linear extension $L = Na$. Periodic boundary conditions may be assumed.

The theory is then defined with respect to a bare lattice action $S_{a\,0}[\phi_a|\{\rho_{a\,0}\}]$. As a functional of lattice fields $\phi_a : \Omega_a \rightarrow \mathbb{R}$ it may be thought of being parametrized by some bare couplings $\{\rho_{a\,0}\}$. It can be obtained from the continuum action by replacing the continuum fields with lattice fields, continuum derivatives with lattice derivatives etc., but this process is by no means unique.

Under some assumptions [GJ87] the generating functional

$$Z_a[J_a] = \int \mathcal{D}\phi_a e^{-S_{a\,0}[\phi_a]+(J_a,\phi_a)} \tag{1.1}$$

and its expectation values

$$< \mathcal{O}_a[\phi_a] >= \frac{1}{Z_a[0]} \int \mathcal{D}\phi_a \mathcal{O}_a[\phi_a] e^{-S_{a\,0}[\phi_a]} \tag{1.2}$$

are now perfectly well defined mathematical objects.

Continuum Limit

A continuum limit $a \rightarrow 0$ of the lattice theory exists if it is possible to tune the bare couplings $\rho_{a\,0} = \rho_{a\,0}(a)$ while keeping some physical couplings fixed such that the GREEN-functions

$$\lim_{a \rightarrow \infty} < \prod_{i=1}^{n} \phi_a(z_{a\,i}) >_{\rho_{a\,0}} \tag{1.3}$$

remain finite.

In contrast to other regularizations these expressions can be evaluated non-perturbativly by means of Monte-Carlo simulations on the computer.

A computer can only handle numbers, i.e. dimensionless quantities. Therefore the above formulas have to be expressed in the language of unit lattice spin systems. This process is non-trivial because it might involve not only the canonical dimension of the quantities but also a priori unknown anomalous scaling dimensions. The natural length scale in the business is the lattice spacing a. E.g. the dimensionless action reads

$$S_0[\phi] = S_{a\,0}[a^{-\alpha}\phi(\cdot/a)] \tag{1.4}$$

Here $\phi : \Omega \rightarrow \mathbb{R}$ is a dimensionless lattice field living on a finite unit lattice Ω. The scaling dimension α of the field will be fixed later.

After feeding in values for the dimensionless couplings $\{\rho_0\}$ the computer returns numbers for the correlation functions

$$< \prod_{i=1}^{n} \phi(z_i) >_{\rho_0} \tag{1.5}$$

or the expectation values of other observables.

Fixing the Scale

To make contact with the physical world one now has to evaluate some expectation values and relate them with their experimental counterparts. This fixes the renormalized couplings.

A most prominent observable is the dimensionless correlation length ξ which is determined from the long-range decay of the two-point connected correlation function

$$< \phi(z)\phi(0) >_{\rho_0} \overset{|z| \gg 1}{\sim} \frac{1}{|z|^{\alpha}} \exp(-|z|/\xi). \tag{1.6}$$

It is a function of the bare couplings $\xi = \xi(\{\rho_0\})$. By relating the correlation length with the physical mass

$$m_{\text{phys}} = (\xi a)^{-1} \tag{1.7}$$

the scale a is expressed in physical units.

As can be seen from eq. (1.7) the limit $a \to 0$ with fixed physical mass m_{phys} can only be achieved if the correlation length of the associated statistical system diverges simultaneously. A statistical system with infinte correltaion length is called critical. Therefore the bare dimensionless couplings $\{\rho_0\}$ have to be tuned to take their critical values $\{\rho_0\} \to \{\rho_{\text{cr}}\}$. The manifold of critical actions will be called the critical surface.

In other words: To study the continuum limit of quantum lattice field theory leads to study the critical limit of a statistical system.

Numerically the approach to criticality means that larger and larger lattices have to be used to simulate the statistical system (The correlation length has to fit inside). Even worse, the method by which the Monte-Carlo algorithm generates new configurations becomes more and more inefficient while approaching criticality (*critical slowing down*).

1.2 Renormalization Group on the Lattice

The scale invariance of the critical statistical system is a result of fluctuations on all length scales. This makes the problem so extremely difficult to tackle. The Renormalization Group (RG) attacks it with a classical divide and conquer strategy. Instead of performing the critical functional integrals (1.1,1.2) in one large step they are devided into a cascade of simpler non-critical integrals.

The RG Recipe

The key ingredients of a dimensionless block-spin RG formalism are:

Block lattice: Starting from a base lattice Ω we introduce a blocked lattice $\bar{\Omega}$ by combining blocks of $s^D, s \in \mathbb{N}_+$ base points into single block points. The block lattice has a dimensionless lattice spacing of s.

Blockspin: The blockspin field $\bar{\phi}(\bar{z})$ is a lattice field living on the blocked lattice $\bar{z} \in \bar{\Omega}$. It is some kind of average of the base field $\phi(z)$ on the blocks $z \in \bar{z}$ performed by an averaging operator $\bar{\phi}(\bar{z}) = \bar{C}[\phi](\bar{z})$.

Rescaling: After the blocking step the coarse lattice $\bar{\Omega}$ is rescaled back to a unit lattice Ω'. In the thermodynamic limit of infinite lattices we have $\Omega = \Omega'$. Otherwise the volume has shrunk by s^{-D}. The rescaled blockspin field ϕ' is obtained from the non-rescaled one by virtue of a rescaling operator $\phi'(x) = (\mathcal{S}_{[1]}\bar{\phi})(x) = s^{-\alpha}\bar{\phi}(sx)$. The correct choice of the scaling dimension α is crucial to the entire procedure. In general it may not only involve the canonical dimension but also the anomalous dimension of the field. In this sense the latter can be considered

as a parameter of the RG. It is convenient to go directly from the base lattice to the rescaled block lattice by an rescaled averaging operator $C = \mathcal{S}\bar{C}$.

Weight Functional: Let $\mathsf{P}[\phi]$ be some kind of weight functional with

$$\int \mathcal{D}\phi' \mathsf{P}[C[\phi], \phi'] = 1. \tag{1.8}$$

As functional of the blockspin field it should be centered around those configurations close to $C[\phi]$. Examples will be given later. Additionally it should respect the symmetries of the action if the RG is supposed to preserve them.

Effective Action: By inserting eq. (1.8) into in the definition of the partition function and exchanging the sequence of integration we obtain

$$Z = \int \mathcal{D}\phi e^{-S[\phi]} = \int \mathcal{D}\phi' e^{-S'[\phi']} \tag{1.9}$$

with the effective action

$$e^{-(RS)[\phi']} := e^{-S'[\phi']} := \int \mathcal{D}\phi \mathsf{P}[C[\phi], \phi'] e^{-S[\phi]}. \tag{1.10}$$

Effective Observables: Let $\mathcal{O}[\phi]$ be an observable. The associated effective observable is given by

$$\mathcal{O}'[\phi'] = \frac{\int \mathcal{D}\phi \mathsf{P}[C[\phi], \phi'] \mathcal{O}[\phi] e^{-S[\phi])}}{\int \mathcal{D}\phi \mathsf{P}[C[\phi], \phi'] e^{-S[\phi])}} \tag{1.11}$$

and has the same expectation value with respect to S' as $\mathcal{O}$ has with respect to S

$$< \mathcal{O}[\phi] >_S = < \mathcal{O}'[\phi'] >_{S'} \tag{1.12}$$

Let us summarize: WILSON's RG relates two actions describing the same physics. The system is in equal means described by the base quantities S,$\mathcal{O}$ or by the blocked quantities S',$\mathcal{O}'$, the latter having a decreased number of degrees of freedom. The price for this is a more complicated action. Even when starting from a low-dimensional coupling manifold the RG transformation will most likely place the effective action away from this manifold. Of course not all of the generated couplings are equally important. The problem remains to decide which of them are important and which are not.

Improved Actions

The blocking step leaves long-ranged properties like the correlation length unchanged. Because of the rescaling step the dimensionless correlation length shrinks $\xi' = \xi/s$. By iterating the RG we therefore obtain a sequence of lattice actions $S_{[n]}$ with decreasing correlation lengths.

$$\xi_{[n]} = s^{-n} \xi_{[0]} \tag{1.13}$$

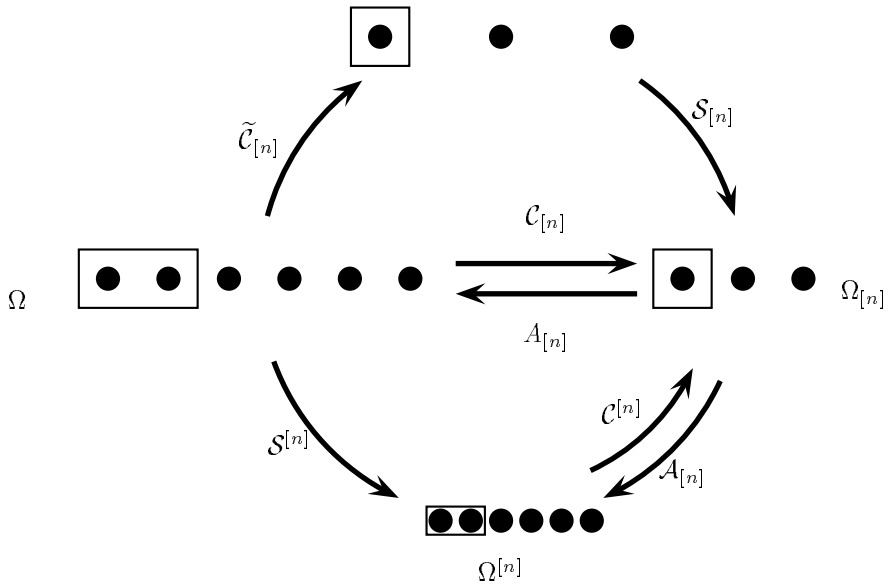

Figure 1.1: *Lattices and operators. See appendix A for the notation. The index* $_{[n]}$ *denotes the number of RG steps*

This feature can be exploited as follows: The larger the correlation length the more computational effort is required to produce a certain accurracy from a Monte-Carlo simulation. If we a) were able to compute an effective action $S_{[n]}$ from a bare action $S_{[0]}$ and b) had an efficient algorithm to simulate it we would have gained a lot: The physical predictions of both theories are by construction the same. But since the effective theory has a smaller correlation length it is easier to simulate.

Problem a) may even be solved by a simulation. Since the fluctuation integral is not critical it does not suffer from critical slowing down. This method is called Monte-Carlo Renormalization Group. The key problem in this field (and most other RG approaches) is to find an appropriate parametrization and reasonable truncations of the action. Problem b) is closely related to locality. The less localized the action is the more complicated is the program to simulate it. The propagators of the fluctuation fields are supposed to be extremely local. They should fall off within one block exponentially. The hope of WILSON therefore was that when starting with a local action the effective action would be local again.

Iteration Limit

Consider now the iteration limit

$$\lim_{n\to\infty} R^n S =: S_* \tag{1.14}$$

Because of eq. (1.13) the correlation length of S_* will be either 0 or ∞.

Such a limit will be invariant under the RG

$$RS_* = S_* \tag{1.15}$$

and is therefore a fixed-point (FP) of the RG. It turns out that these points encode the entire critical physics. Fixed points with zero correlation length are called infra-red fixed-points whereas those with diverging correlation length are termed ultra-violet fixed-points.

Linearized RG

It is highly instructive to study the linearized RG transformation at a fixed-point because it gives us some knowledge about the RG flow in the vincinity of the FP.

So let $\mathcal{L}_{S_*}$ denote the linearization of R at S_*. It turns out that $\mathcal{L}_{S_*}$ is actually the same transformation as the RG transformation eq. (1.11) for the effective observables evaluated at the fixed-point.

$$(\mathfrak{L}_{S_*}\mathcal{O})(\phi') = \frac{\int \mathcal{D}\phi \mathsf{P}[C[\phi], \phi']\mathcal{O}[\phi]e^{-S_*[\phi])}}{\int \mathcal{D}\phi \mathsf{P}[C[\phi], \phi']e^{-S_*[\phi])}} \tag{1.16}$$

Being a linear operator $\mathfrak{L}_{S_*}$ will have eigenobservables. They are classified according to their eigenvalues. Observables with eigenvalue larger than zero are called relevant. A perturbation in this direction is amplified by $\mathfrak{L}_{S_*}$ and hence by R. Those with eigenvalue less than zero are termed irrelevant. Perturbations in these directions are attracted to the FP. Eigenobservables with eigenvalue zero are called marginal. In order to understand their behaviour under R the linear approximation $\mathcal{L}_{S_*}$ is no longer sufficient. There is a close relationship between the eigenvalues and the critical exponents but we refrain from going into details here.

1.3 Renormalization, RG and Perfect Actions

Renormalization and RG

We now want to understand the relationship between the continuum limit and the RG.

Assume a RG topology with an UV fixed-point having one relevant direction and an attractive IR fixed-point. Assume further a manifold of bare actions parametrized by the couplings $\{\rho_0\}$ called canonical manifold. This manifold will generally have an intersection with the critical surface of the UV fixed-point. These actions have critical couplings $\{\rho_{\text{cr}}\}$ and are attracted towards the UV fixed-point under the RG. The actions away from the critical surface are not critical. If the RG is applied to them they are driven towards the UV fixed-point in the irrelevant directions but are attracted towards the IR fixed-point in the relevant direction. One can now imagine a one-dimensional manifold connecting the two fixed-points, which will be called the Renormalized Trajectory (RT). Its tangent at the UV fixed-point is given by the relevant direction. By construction all actions on the RT are mapped on the RT again. The RT is a stable manifold of the RG. The RG leaves an action on the RT invariant up to a flow of the relevant

couplings, in our case here one coupling. This coupling is called renormalized coupling ρ_{R} for reasons that will be explained later on.

When iterating the RG all non-critical actions are asymptotically driven towards the RT. This is the key to renormalizability and motivates its name. Start with an bare action on the canonical manifold given by bare couplings $\{\rho_{0[N_0]}\}$. After $N_0 \gg 1$ iterations of the RG it will be mapped on an effective action very close to the RT. By construction this action $S_{\text{RT}}(\rho_{\text{R}}) = R^{N_0} S_0(\rho_{0[N_0]})$ describes the same physics as the bare one. Now we turn the argumentation upside down and look for bare couplings $\{\rho_{0[N_0+N]}(\rho_{\text{R}})\}$ as function of the renormalized coupling such that the associated action is mapped on $S_{\text{RT}}(\rho_{\text{R}})$ after $N_0 + N$ RG steps. All three actions describe the same physics. But the two bare action have a different correlation length since $S_0(\rho_{0[N_0+N]})$ is related to the renormalized action by N additional RG steps.

$$\xi(\rho_{0[N_0+N]}(\rho_{\text{R}})) = s^N \xi(\rho_{0[N_0]}(\rho_{\text{R}})) \tag{1.17}$$

If we now perform the limit $N \to \infty$ the correlation length diverges and the bare couplings approach their critical values

$$\lim_{N\to\infty} \rho_{0[N_0+N]}(\rho_{\text{R}}) = \rho_{\text{cr}} \tag{1.18}$$

By construction even in this limit the bare action is equivalent to the renormalized action. The renormalized action is non-critical and leads to finite expectation values. Hence this is true for the critical bare action, too. But that is what renormalizabilty is all about.

The conclude: Theories on the RT describe continuum theories.

Perfect Actions

Now let us consider an asymptotically free theory, e.g. the two dimensional $O(N)$ symmetric non-linear σ model . The action for this theory may be written as $S[\phi] = \beta \mathfrak{H}[\phi]$ with coupling β and Hamiltonian $\mathfrak{H}[\phi]$. It turns out that the fixed-point action S_* ruling the model lies at $\beta = \infty$ and has only one relevant direction [1]

The Perfect Action (PA) of HASENFRATZ and NIEDERMAYER is an asymptotic approximation to the RT. It is defined by the flow of a classical RG transformation. In contrast to the full RG the classical RG does not perform the fluctuation integral but extremizes the action under the constraint of prescribed blockspin. Because of the asymptotic freedom $\beta = \infty$ the FP of the full transformation is the same as the classical one. Away from the FP this is no longer true. But at least for the two dimensional $O(N)$ symmetric non-linear σ model the RT seems to be parallel to the PA trajectory for quite some distance. Since actions on the RT describe continuum physics actions close to it should have excellent scaling properties. In the case of the Perfect Action of [HN94] it was possible to compute continuum quantities on a 3×3 lattice!

[1] Actually this direction is marginal in the linear approximation.[Wil]

Part I

The Renormalized Trajectory of the two dimensional $O(N)$ symmetric non-linear σ model

Chapter 2

Introduction

In this chapter the general philosophy of the method is described.

The main object of interest in quantum field theory on the lattice is the Renormalized Trajectory. The reason being that lattice actions on the RT directly describe continuum physics.

The Perfect Action of HASENFRATZ and NIEDERMAYER [HN94] for asymptotically free theories is an approximation to the true RT. It is obtained by solving a classical RG transformation at $\beta \to 0$ and then extrapolating to finite β. Although being only an asmyptotic form of the RT the PA for the two dimensional $O(N)$ symmetric non-linear σ model already has excellent scaling properties.

The work presented here is in some sense an improvement scheme in which the zeroth order approximation is a PA. It is based on the ideas of C. WIEZCERKOWSKI and is part of the *Running Coupling Expansion for Renormalized Trajectories from Renormalization Invariance* program [WX95a, WX95b, Wie96b, Wie96c, Wie96a].

As indicated by this lengthy name the concept is a combination of renormalized perturbation theory in a running coupling with the notion of scaling invariance.

Let the renormalized trajectory of the two dimensional $O(N)$ symmetric non-linear σ model be parametrized in terms of a local coordinate f. Recall that it is a curve in the space of $O(N)$ invariant actions which is stable under application of a RG transformation. In other words: The blockspin transformation leaves the action invariant up to a flow of the coupling

$$RS_{\text{RT}}[\phi|f] = S_{\text{RT}}[\phi|\beta(f)]. \tag{2.1}$$

The action is said to scale.

The complete dynamics of the renormalization group on the renormalized trajectory is encoded in the flow of the running coupling. We call $\beta(f)$ Callan-Symanzik function since it defines this discrete flow.

The essence of this work is: Eq. (2.1) can be solved perturbativly in an iterative scheme without reference to any bare action.

A blockspin transformation comes along with a linear mapping of observables which preserves their expectation values. This mapping is the linearized renormalization group

transformation. We shall consider this linear mapping $\mathcal{L}$ to the blockspin transformation over the renormalized trajectory. An observable will be called an eigenvector with eigenvalue $\epsilon(f)$ if it satisfies the equation

$$\mathcal{L}\mathcal{O}[\phi|f] = \epsilon(\beta(f))\mathcal{O}[\phi|\beta(f)]. \tag{2.2}$$

In accordance with the terminology for the action we will speak of this property as scaling.

The space of eigenvectors defines a moving frame in the tangent space over the renormalized trajectory. The set of eigenvalues will be called the spectrum of the theory. It is exactly calculable at the fixed point. Moving away from the fixed point on the renormalized trajectory it becomes perturbed.

In the vicinity of the ultraviolet fixed point it is therefore natural to perform a perturbation expansion in the running coupling to determine the spectrum from (2.2).

This entire program has been worked out for two dimensional $O(N)$ symmetric non-linear σ model in the so-called hierarchical approximation. The perturbative results have been cross-checked by numerical means with great success.

Additionally the setup for the program in the full model has been layed down. In contrast to the hierarchical situation it was not possible to find an analytical solution for the Perfect Action as starting point for the perturbative machinery. Instead the classical RG had to be treated perturbativly, too. This gave rise to a first order improved action. [1]. Software has been developed to check the scaling properties of improved actions. First tests with the given action were performed.

We have used a linear blockspin in this work. Therefore the fields have fluctuating length and are not living on the unit sphere. It is assumed that these general $O(N)$ invariant models are lying in the same universality class as the two dimensional $O(N)$ symmetric non-linear σ model and therefore share the same physics (See 4.6).

This part of the thesis is organized as follows:

In chapter 3 the properties of a linear blockspin transformation for scalar fields are discussed.

Chapter 4 is devoted to the implementation of above program for the two dimensional $O(N)$ symmetric non-linear σ model in the hierarchical approximation.

Finally, in chapter 5 the setup for the full model is described and first simulation results of the improved action are presented.

Parts of this work have been published in [WX95a], [WX95b].

[1] The term *improved* might be confusing. It is not meant here in the sense of SYMANZIK. Symanzik-improved actions converge *faster* towards the RT, i.e. the continuum limit. Our actions are scaling improved. The question of convergence speed although related has not been adressed. See 4.6.

Chapter 3

Renormalization Group (RG) on the lattice

In this section the basic concepts and formulas for a linear blockspin RG on the lattice are presented.

3.1 The RG-Transformation

The Averaging Operator

Let ϕ be a scalar field on a D-dimensional unit lattice Ω with action $S[\phi]$ (See appendix A for notation).

By means of the rescaled averaging operator $\mathcal{C} : \mathcal{H} \to \mathcal{H}'$ with

$$\mathcal{C}(x\,,z) = s^{-D-\alpha}\delta_{x,\left[\frac{z}{s}\right]} \tag{3.1}$$

we define the blockspin field $\phi' = \mathcal{C}\phi$. Here s denotes a scaling parameter $s \in \mathbb{N}_+$ and $\alpha = -\frac{D-2}{2}$ is the canonical dimension of ϕ. The adjoint operator is given by

$$\mathcal{C}^T(z,x) = \mathcal{C}(x,z) \tag{3.2}$$

Fields on the blocked lattice Ω' are interpolated by $\mathcal{C}^T$ onto block-constant fields on the base lattice Ω. (see fig. 1.1 for lattices)

The following relations hold:

$$\mathcal{C}\mathcal{C}^T \quad = \quad s^{\beta}\mathbb{I}' \tag{3.3}$$

with $\beta = -2\alpha - D \overset{(D=2)}{=} -2$.

The Transformation

Throughout this part we will use the following RG transformation R

$$e^{-(RS)[\phi']} = e^{-S'[\phi']} \sim \int \mathcal{D}\phi \; \delta_\kappa[\mathcal{C}\phi - \phi'] e^{-S[\phi]} \tag{3.4}$$

with Gaussian constraint

$$\delta_\kappa[\varphi] \sim \exp\left(-\frac{\kappa}{2}(\varphi,\varphi)\right). \tag{3.5}$$

By virtue of the weight functional $\mathsf{P}[\varphi] = \delta_\kappa[\varphi]$ the functional integral over ϕ is centered around those field configutations obeying $\mathcal{C}\phi = \phi'$. The smearing parameter κ can be used later on to tune the locality properties of the resulting effective action.

GK Formalism

K. GAWEDZKI and A. KUPIAINEN have introduced [GK], [GK80] a formalism[1] which turns out to be very convenient for perturbative explorations.

They exploit the fact that the flow of a quadratic action $S[\phi] = \frac{1}{2}(\phi, v^{-1}\phi)$ under R can be handled explicitly. One can easily see that the effective action S' is again quadratic but with an effective propagator v' given by

$$v' = \mathcal{C} v \mathcal{C}^T + \frac{1}{\kappa} := \mathsf{R} v. \tag{3.6}$$

Neglecting the last term this transformation has a very appealing interpretation: Propagation of blockfields on the blocked lattice is done by interpolating to the base lattice by means of $\mathcal{C}^T$ propagating then with the base propagator v and finally blocking back to the blocked lattice by virtue of $\mathcal{C}$.

Obviously any action $S[\phi] = T_v[\phi] + V[\phi]$ can be devided into a kinetic part $T_v[\phi] = \frac{1}{2}(\phi, v^{-1}\phi)$ with an arbitrary free propagator v and an interaction part. The freedom to choose the free propagator can be used to simplify the resulting equations or to optimize the perturbative expansion [Gri97].

The so-called background field is now defined by the requirement that it minimizes the free action part T_v subject to the weight P

$$T_v[\psi] + \frac{\kappa}{2}||\mathcal{C}\psi - \phi'||^2 = \text{extremal}. \tag{3.7}$$

The linear ansatz $\psi = A\phi'$ leads to the explicit solution

$$A = v\mathcal{C}^T v'^{-1}. \tag{3.8}$$

By splitting the base field $\phi = \psi + \zeta$ into the background field ψ and a fluctuation field ζ and making use of the convolution formula for Gaussian integrals (see Appendix C) the RG transformation R eq. (3.4) can be rewritten as

$$e^{-V'[\phi']} = e^{-(\mathcal{R}V)[\phi']} = \int d\mu_\Gamma[\zeta] e^{-V[A\phi'+\zeta]} \tag{3.9}$$

[1] Actually they treated the situation with $\kappa = 0$ where the weighting functional δ_κ becomes a δ-distribution.

with

$$\Gamma = v - Av'A^T. \tag{3.10}$$

The so-called interpolation operator $A : \mathcal{H}' \to \mathcal{H}$ interpolates blockspin fields to the base lattice. In contrast to the adjoint averaging operator with its constant interpolation A smears the fields more smoothly (see fig. 3.2). [2]

As mentioned in chapter 1 the locality of the fluctuation propagator Γ is essential for the entire RG philosophy. The GK fluctuation propagator is proven to have an exponential fall-off [GK]. See fig. 3.1.

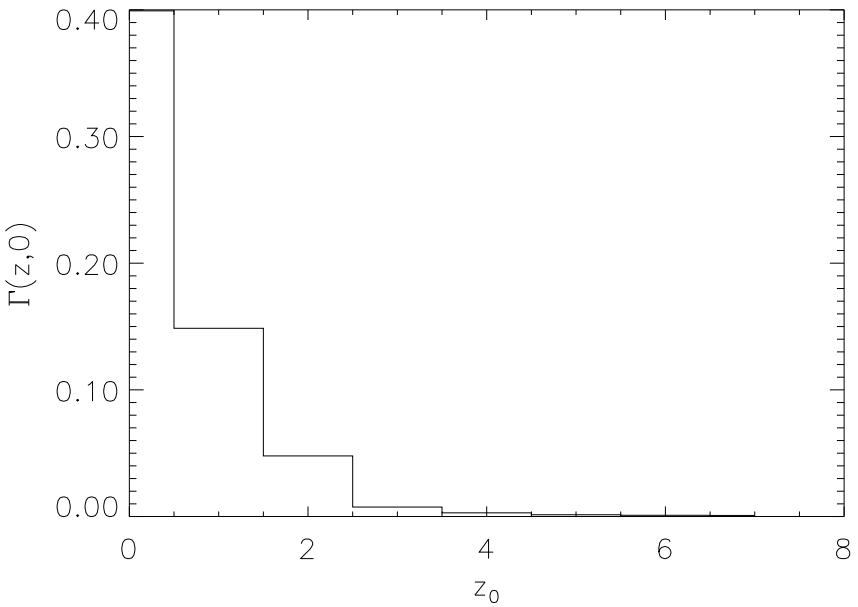

Figure 3.1: *Fall-off of the fluctuation propagtor* $\Gamma(z,0)$ *with* $z = (z_0, z_1)$. *Computation was done on a* 16×16 *lattice with scaling factor* $s = 4$. *Essentially the flucation propagtor is zero outside the block.*

What has been accomplished?. Instead of a RG flow of the action one now has a flow of the covariance v and the interaction V (often missleadingly called the Effective Potential) governed by RG transformations R and $\mathcal{R}$.

For later reference we mention the following useful relations

$$A = \kappa \Gamma \mathcal{C}^T \tag{3.11}$$

$$\Gamma = \left(v^{-1} + \kappa \mathcal{C}^T \mathcal{C}\right)^{-1} \tag{3.12}$$

$$v' = \left(\kappa \mathbb{I}' - \kappa^2 \mathcal{C} \Gamma \mathcal{C}^T\right)^{-1}. \tag{3.13}$$

[2] The averaging operator is closely related to wavelets. On a multigrid this allows an orthogonal decomposition of the field into its frequency components [Mac]

Iteration Limit versus Scaling Limit

The word *Group* in Renormalization Group is usually motivated by demanding that R fulfills the semi-group property

$$R(s)R(s) = R(s^2). \tag{3.14}$$

The Gaussian RG transformation R used here does not only depend [3] on the scaling factor s but additionally on the Gaussian parameter κ

$$R = R(s, \kappa). \tag{3.15}$$

A modified semi-group property holds

$$R(s, \kappa)R(s, \kappa) = R(s^2, \kappa_{[2]}(s, \kappa)) \tag{3.16}$$

with

$$\kappa_{[n]}(s, \kappa) = \frac{1 - s^\beta}{1 - s^{n\beta}}\kappa. \tag{3.17}$$

The continuum limit of a lattice field theory now can be approached by two methods. Either by an infinite iteration of RG steps with fixed scale $\lim_{n\to\infty} R^n(s, \kappa)$. Or alternatively one can consider the scaling limit of a single RG step $\lim_{n\to\infty} R(s^n, \kappa_{[n]}(s, \kappa))$ with appropriate tuning of the Gaussian parameter.

3.2 Quadratic Fixed-Points

Let us first focus on the flow of a covariance $v =: v_{[0]}$ under R. After n iterations we find

$$\begin{aligned} v_{[n+1]} &= \mathcal{C} v_{[n]} \mathcal{C}^T + \frac{1}{\kappa} \\ &= \mathcal{C}^2 v_{[n-1]} {\mathcal{C}^T}^2 + \left(1 + s^\beta\right) \frac{1}{\kappa} \\ &= \mathcal{C}^3 v_{[n-2]} {\mathcal{C}^T}^3 + \left(1 + s^\beta + s^{2\beta}\right) \frac{1}{\kappa} \\ &\vdots \\ &= \mathcal{C}^n v_{[0]} {\mathcal{C}^T}^n + \underbrace{\frac{1 - s^{n\beta}}{1 - s^\beta} \frac{1}{\kappa}}_{\kappa_{[n]}^{-1}} \end{aligned} \tag{3.18}$$

where we have used eq. (3.3).

[3] Actually the transformation depends on the rescaling dimension α, too. For simiplicity we are using here the canonical dimension of the field.

The first term can be written as

$$\begin{aligned}\left(\mathcal{C}^n v_{[0]} \mathcal{C}^{T\,n}\right)(x,x') &= s^{-2n(D+\alpha)} \sum_{y,y'} \delta_{x,[y/s^n]} v(y-y') \delta_{x',[y'/s^n]} \\ &= s^{-2nD} \sum_{\substack{z \in \square_n(x) \\ z' \in \square_n(x')}} s^{-2n\alpha} v\big(s^n z - s^n z'\big) \\ &= s^{-2nD} \sum_{\substack{z \in \square_n(x) \\ z' \in \square_n(x')}} s^{-2n\alpha} \int\limits_{p \in [0,2\pi]^D} v(p) e^{ip s^n (z-z')} \\ &= s^{-2nD} \sum_{\substack{z \in \square_n(x) \\ z' \in \square_n(x')}} \int\limits_{q \in [0,2\pi s^n]^D} s^{-n(2\alpha+D)} v(q/s^n) e^{iq(z-z')}.\end{aligned} \tag{3.19}$$

Assuming that the inverese propagator can be expanded as

$$v^{-1}(p) = C_0 + C_1 p^2 + C_2 p^4 \ldots \tag{3.20}$$

we get

$$\begin{aligned}s^{-n(2\alpha+D)} v(q/s^n) &= \frac{1}{C_0 s^{n(2\alpha+D)} + C_1 s^{n(2\alpha+D-2)} q^2 + C_2 s^{n(2\alpha+D-4)} q^4} \\ &= \frac{1}{C_0 s^{2n} + C_1 q^2 + C_2 s^{-2n} q^4}.\end{aligned} \tag{3.21}$$

In this form the iteration limit $n \to \infty$ can be taken. If the original propagator v has a mass term, i.e. $C_0 \neq 0$ then this mass acquires a factor s^2 in each RG step, eventually yielding

$$\lim_{n \to \infty} \left(\mathcal{C}^n v_{[0]} \mathcal{C}^{T\,n}\right)(x,x') = 0. \tag{3.22}$$

In this case the action flows towards a high-temperature fixed-point

$$\lim_{n \to \infty} v_{[n+1]}(x,x') = \frac{1}{1-s^\beta} \frac{1}{\kappa} \mathbb{I} = \frac{1}{\kappa_*} \mathbb{I} := v_{\text{ht}} = \mathsf{R} v_{\text{ht}}. \tag{3.23}$$

If on the other hand $v_{[0]}$ is a massless propagator [4] we end up in a massless fixed point [5]

$$\lim_{n \to \infty} v_{[n+1]}(x,x') = \int\limits_{\substack{\mathsf{z} \in \square(x) \\ \mathsf{z}' \in \square(x')}} (-\triangle)^{-1}(\mathsf{z},\mathsf{z}') + \frac{1}{\kappa_*}. \tag{3.24}$$

$\triangle(\mathsf{z},\mathsf{z}')$ is the continuum Laplacian and $\square(x)$ is the unit cube centered around x.

[4] The reader might be worried about the zero modes of the massless bare propagator. As shown [GK] one can introduce an IR regulator to take care of these. Since the RG transformation does not depend on the regulator in the scaling limit we will ignore this technical issue further on.

[5] Actually there is a line of fixed points parametrized by C_1. Here we have set $C_1 = 1$. This can always be achieved be means of a trivial wavefunction renormalization.

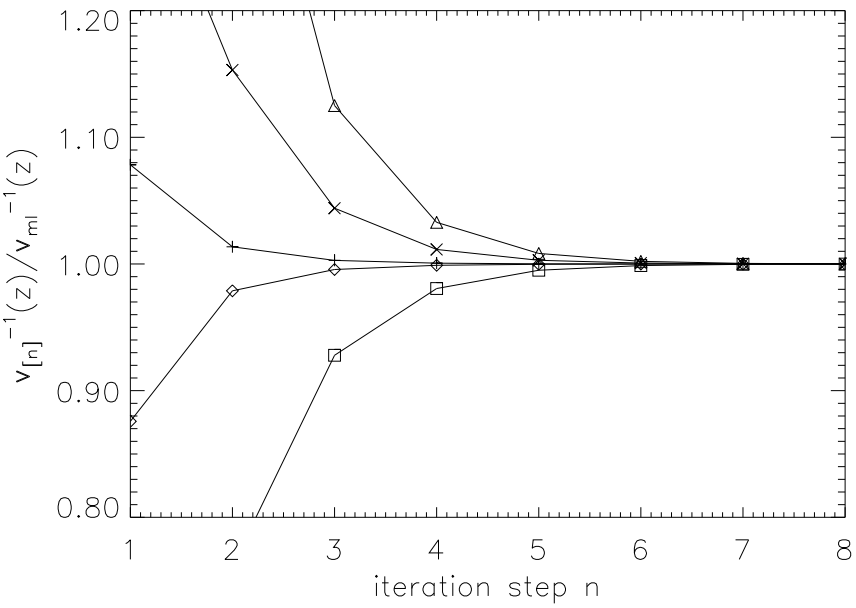

Figure 3.2: *Flow of the standard nearest neighbor lattice Laplacian* $-\Delta_{nn}(z) = {v_{[0]}}^{-1}(z)$ *towards* $v_{ml}^{-1}(z)$ *after* n *iteration steps. Computation was done on a* 32×32 *lattice with* $s = 2, \kappa = 8$ *by means of* WILSON*'s blow-up technique. The values for* $z = (0,1),(1,1),(2,0),(2,1),(2,2)$ *are drawn with* $+, *, \diamond, \triangle, \Box, \times$.

This formula closely resembles eq. (3.6) if one uses the averaging operator $\mathcal{C}^{[\infty]}$ (B.7) from the appendix B. Essentially its kernel is a translated characteristic function of the *continuum* unit cube. With these eq. (3.24) reads

$$v_{\text{ml}} = \mathcal{C}^{[\infty]} \, (-\triangle)^{-1} \, {\mathcal{C}^{[\infty]}}^T + \frac{1}{\kappa_*} \tag{3.25}$$

The quadratic FP v_{ml} is the inverse of the so-called Perfect Laplacian (see Appendix B.4). It is obtained by blocking the inverse continuum Laplacian onto the unit lattice. The spectrum of the Pefect Laplacian [BW74, HN94] is exact in the sense that it fulfills the continuum energy-momentum relation. This observation motivates the attribute *perfect*. In contrast to the standard lattice Laplacian it has not only nearest neighbor couplings yet the other couplings decay rapidly with the distance. See table 3.1. By tuning the Gaussian parameter κ one can modify this fall-off. It turns out that $\kappa \approx 8$ leads to the smallest interaction range. Please note that the limit (3.21) only exists because the rescaling dimension α has been chosen appropriately (see [BW74]). This demonstrates that the anomalous dimension of the field at the Gaussian fixed-point is zero.

$v_{\text{ml}}^{-1}(z)$	$z_1 = 0$	$z_1 = 1$	$z_1 = 2$
$z_0 = 0$	3.24027	-0.61802	-0.00199
$z_0 = 1$	-0.61802	-0.19033	-0.00067
$z_0 = 2$	-0.00199	-0.00067	-0.00162

Table 3.1: *Perfect Laplacian for* $\kappa = 8$. *All other couplings are smaller than* 10^{-4}.

3.3 ML and HT Picture

As mentioned before one is free to choose the propagator v in the split of the action $S[\phi] = T_v[\phi] + V[\phi]$. With the quadratic fixed-points $v_{\text{ml}}, v_{\text{ht}}$ we have two natural candidates. The RG transformation R for the action can therefore be expressed in two different ways $\mathcal{R}_{\text{ht}}, \mathcal{R}_{\text{ml}}$ for the potential. We will speak of the HT and the ML picture.

Later on it will be important to switch freely between the ML and HT picture. So let us start at the massless fixed-point v_{ml} and study the flow of a quadratic perturbation with kernel K_{ml}

$$V_{\text{ml}}[\phi] = \frac{1}{2}\left(\phi, K_{\text{ml}}\phi\right). \tag{3.26}$$

The transformation law for this kernel reads

$$K' = {A_{\text{ml}}}^T \left(K_{\text{ml}}^{-1} + \Gamma_{\text{ml}}\right)^{-1} A_{\text{ml}} = {A_{\text{ml}}}^T L_{\text{ml}} K_{\text{ml}} A_{\text{ml}} \tag{3.27}$$

with $L_{\text{ml}} = \left(1 + \Gamma_{\text{ml}} K_{\text{ml}}\right)^{-1}$.

What do we expect? Suppose that K has a non-vanishing zero mode. Therefore the total propagator $v = v_{\text{ml}} + K_{\text{ml}}^{-1}$ has a mass. According to eq. (3.22) it will then flow towards the high-temperature fixed-point v_{ht}. This means that the kernel K_{ml} flows to $K_{\text{ml}} \to K_{\text{ml}}^* := v_{\text{ht}}^{-1} - v_{\text{ml}}^{-1}$. Thus we have found the high-temperature fixed-point in terms of the transformation at the massless fixed-point. It is instructive to prove this directly from eq. (3.27) First note that

$$K_{\text{ml}}^* = v_{\text{ht}}^{-1} + \kappa \mathcal{C}^T \mathcal{C} - \left(v_{\text{ml}}^{-1} - \kappa \mathcal{C}^T \mathcal{C}\right) \overset{(3.12)}{=} \Gamma_{\text{ht}}^{-1} - \Gamma_{\text{ml}}^{-1} \tag{3.28}$$

from which we deduce

$$L_{\text{ml}}^* = \Gamma_{\text{ht}} \Gamma_{\text{ml}}^{-1}. \tag{3.29}$$

On the other hand

$$K_{\text{ml}}^* \overset{(3.13)}{=} \left(\kappa \mathbb{I}' - \kappa^2 \mathcal{C} \Gamma_{\text{ht}} \mathcal{C}^T\right) - \left(\kappa \mathbb{I}' - \kappa^2 \mathcal{C} \Gamma_{\text{ml}} \mathcal{C}^T\right) = -\kappa^2 \mathcal{C}\left[\Gamma_{\text{ht}} - \Gamma_{\text{ml}}\right] \mathcal{C}^T. \tag{3.30}$$

Equipped with these relations we verify that K_{ml}^* is indeed a fixed point solution of eq.

(3.27):

$$\begin{aligned} {A_{\text{ml}}}^T L^*_{\text{ml}} K^*_{\text{ml}} A_{\text{ml}} &\overset{(3.30,3.29)}{=} {A_{\text{ml}}}^T \left[\Gamma_{\text{ml}}^{-1} - \Gamma_{\text{ml}}^{-1}\Gamma_{\text{ht}}\Gamma_{\text{ml}}^{-1}\right] A_{\text{ml}} \\ &\overset{(3.8)}{=} \kappa^2 \mathcal{C}\Gamma_{\text{ml}} \left[\Gamma_{\text{ml}}^{-1} - \Gamma_{\text{ml}}^{-1}\Gamma_{\text{ht}}\Gamma_{\text{ml}}^{-1}\right] \Gamma_{\text{ml}}\mathcal{C}^T \\ &= -\kappa^2 \mathcal{C}\left[\Gamma_{\text{ht}} - \Gamma_{\text{ml}}\right]\mathcal{C}^T \overset{(3.30)}{=} K^*_{\text{ml}} \end{aligned} \tag{3.31}$$

We are now able to switch from the ML to the HT-perspective and vice versa. Define

$$V_{\text{ml}}[\phi] = \frac{1}{2}(\phi, K^*_{\text{ml}}\phi) + V_{\text{ht}}[\phi]. \tag{3.32}$$

By means of eq. (C.4) and using the abbrevation $\psi_{\text{ml}} = A_{\text{ml}}\phi'$ we find

$$e^{-{V'}_{\text{ml}}[\phi']} = e^{-\frac{1}{2}(\psi, L_{\text{ml}}K_{\text{ml}}\psi)} \int \mathrm{d}\mu_{(\Gamma^{-1}+K_{\text{ml}})^{-1}}[\zeta] e^{-V_{\text{ml}}[L_{\text{ml}}\psi+\zeta]}. \tag{3.33}$$

With $L^*_{\text{ml}}\psi_{\text{ml}} = L^*_{\text{ml}}A_{\text{ml}}\phi' = \Gamma_{\text{ht}}\Gamma_{\text{ml}}^{-1}\kappa\Gamma_{\text{ml}}\mathcal{C}^T\phi' = \kappa\Gamma_{\text{ht}}\mathcal{C}^T\phi' = A_{\text{ht}}\phi'$ the RG transformation with respect to v_{ht} reads as expected

$$e^{-{V'}_{\text{ht}}[\phi']} \sim \int d\mu_{\Gamma_{\text{ht}}}[\zeta] e^{-V_{\text{ht}}[A_{\text{ht}}\phi'+\zeta]}. \tag{3.34}$$

By interchanging the roles of HT and ML one yields the opposite relations.

3.4 Scaling Limit of the GAWEDZKI and KUPIAINEN Kernels

In section 3.2 we have encountered with eq. (3.25) a first example of a scaling limit. The lattice propagator v_{ml} can be written as a continuum propagator $(-\triangle)^{-1} + \frac{1}{\kappa_*}$ transported to the unit lattice by virtue of $\mathcal{C}^{[\infty]}$. By similar techniques the other kernels can also be expressed in this way.

Interpolation Operator

Let $*$ stand for ml or ht. Define a scale transformation

$$\left(\mathcal{S}_*\phi\right)(z) = s^{-\alpha_*}\phi(sz) \tag{3.35}$$

and rescaled interpolation operators

$$\mathcal{A}_*{}^{[m]} = \mathcal{S}_*^m A_*^m. \tag{3.36}$$

If the exponents α_* are chosen suitably then $\mathcal{A}_*{}^{[m]}$ has a scaling limit $\mathcal{A}_*{}^{[\infty]}$. For α_{ht} we find $\alpha_{\text{ht}} = -D - \alpha$ and for α_{ml} we get $\alpha_{\text{ml}} = \alpha$. An approximation of $\mathcal{A}_{\text{ml}}{}^{[\infty]}$ is shown in fig. 3.3.

Because of $\mathcal{C}^{[\infty]}\mathcal{A}_{\text{ml}}{}^{[\infty]} = \mathbb{I}$ the background field $\psi = \mathcal{C}^{[\infty]}\phi'$ fulfills $\mathcal{C}^{[\infty]}\psi = \phi'$. The background field is a continuum field with only one degree of freedom per unit volume.

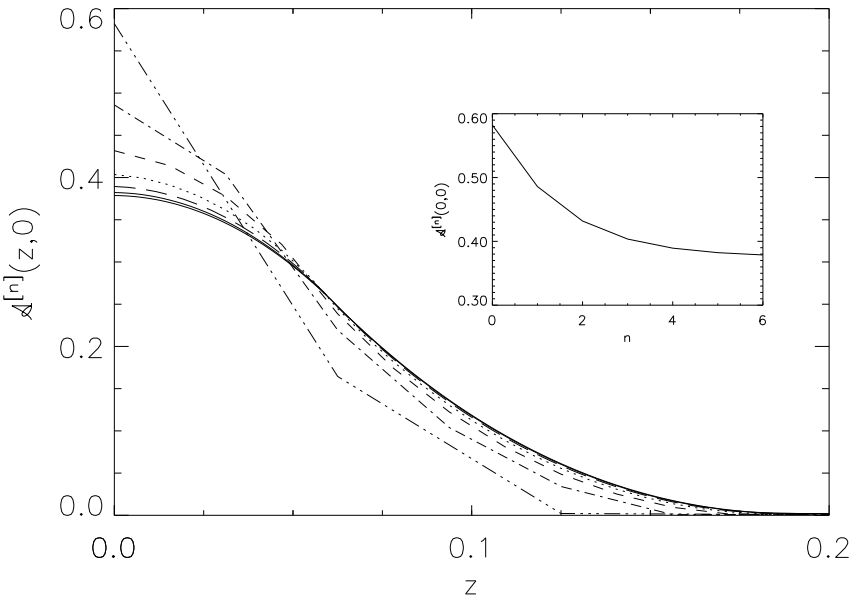

Figure 3.3: *Approach of the rescaled interpolation operator $\mathcal{A}_{ml}{}^{[n]}(\mathsf{z},0)$ to its scaling form for $n = 1,\ldots,8$ and $s = 2$. It was computed by blocking from quadratic lattices of sidelength s^{n+3} with scaling factor s^n making use of $A^n(s,\kappa) = A(s^n,\kappa_{[n]}(s,\kappa))$. The small picture shows the convergence of $\mathcal{A}_{ml}{}^{[n]}(0,0)$ as function of n.*

The K Kernel

We now want to find the scaling limit of K^*_{ml}. Starting point is the fixed point equation eq. (3.31) which can be rewritten as

$$K^*_{\text{ml}} = A_{\text{ht}}{}^T K^*_{\text{ml}} A_{\text{ml}}. \tag{3.37}$$

In coordinates this reads

$$\begin{aligned} K^*_{\text{ml}}(z_1,z_2) &= \int\limits_{x_1,x_2} A_{\text{ht}}{}^T(z_1,x_1) K^*_{\text{ml}}(x_1,x_2) A_{\text{ml}}(x_2,z_2) \qquad (3.38)\\ &= \int\limits_{\mathsf{z}_1,\mathsf{z}_2} s^{-\alpha_{\text{ht}}} A_{\text{ht}}{}^T(z_1,s\mathsf{z}_1) s^{-2D+\alpha_{\text{ht}}+\alpha_{\text{ml}}} K^*_{\text{ml}}(s\mathsf{z}_1,s\mathsf{z}_2) s^{-\alpha_{\text{ml}}} A_{\text{ml}}(s\mathsf{z}_2,z_2). \end{aligned}$$

Now, taking the scaling limit $s \to \infty$ one obtains

$$K^*_{\text{ml}}(z_1,z_2) = \int\limits_{\mathsf{z}_1,\mathsf{z}_2} \mathcal{A}_{\text{ht}}{}^{[\infty]T}(z_1,\mathsf{z}_1) \mathcal{K}^{*[\infty]}_{\text{ml}}(\mathsf{z}_1,\mathsf{z}_2) \mathcal{A}_{\text{ml}}{}^{[\infty]}(\mathsf{z}_2,z_2). \tag{3.39}$$

The rescaled kernel $\mathcal{K}_{\text{ml}}^{*[\infty]}$ turns out to be local

$$\begin{aligned}
\mathcal{K}_{\text{ml}}^{*[\infty]}(\mathsf{z}_1,\mathsf{z}_2) &= \lim_{s\to\infty} s^{-2D+\alpha_{\text{ht}}+\alpha_{\text{ml}}} K_{\text{ml}}^*(s\mathsf{z}_1, s\mathsf{z}_2) \\
&= \lim_{s\to\infty} s^{-2D+\alpha_{\text{ht}}+\alpha_{\text{ml}}} \int\limits_p K_{\text{ml}}^*(p) e^{ips(\mathsf{z}_1-\mathsf{z}_2)} \\
&= \lim_{s\to\infty} s^{-D+\alpha_{\text{ht}}+\alpha_{\text{ml}}} \int\limits_{\mathsf{p}} K_{\text{ml}}^*(\mathsf{p}/s) e^{i\mathsf{p}(\mathsf{z}_1-\mathsf{z}_2)} \\
&\overset{(D=2)}{=} \lim_{s\to\infty} \int\limits_{\mathsf{p}} K_{\text{ml}}^*(\mathsf{p}/s) e^{i\mathsf{p}(\mathsf{z}_1-\mathsf{z}_2)} \\
&\overset{(D=2)}{=} K_{\text{ml}}^*(\mathsf{p}=0)\delta(\mathsf{z}_1-\mathsf{z}_2). \qquad (3.40)
\end{aligned}$$

Since $K_{\text{ml}}^* = v_{\text{ht}}^{-1} - v_{\text{ml}}^{-1}$ and $v_{\text{ht}}^{-1}(\mathsf{p}=0) = \kappa_*, v_{\text{ml}}^{-1}(\mathsf{p}=0) = 0$ we ultimately get the scaling form

$$K_{\text{ml}}^*(z_1,z_2) = \kappa_* \int\limits_{\mathsf{z}} \mathcal{A}_{\text{ht}}{}^{[\infty]^T}(z_1,\mathsf{z}) \mathcal{A}_{\text{ml}}{}^{[\infty]}(\mathsf{z},z_2). \qquad (3.41)$$

3.5 The Linearized Transformation

We now want to study perturbations of the fixed points v_{ht} and v_{ml}. Let v_* stand for one of these. The linearized version of (3.9) applied to an observable $\mathcal{O}_{[0]}$ at $V_{[0]} = 0$ reads

$$\mathcal{O}_{[1]}[\phi_{[1]}] = (\mathcal{L}\mathcal{O}_{[0]})[\phi_{[1]}] = \int d\mu_{\Gamma_*}[\zeta] \mathcal{O}_{[0]}[A_*\phi_{[1]} + \zeta] \qquad (3.42)$$

Please note, that this is the same transformation (1.16) as for effective observables at a quadratic action.

Consider now the normal ordered observable

$$\mathcal{O}_{[0]}[\phi_{[0]}] = \; :e^{(\phi_{[0]},J)}:_{v_*} = e^{(\phi_{[0]},J)-\frac{1}{2}(J,v_*J)}. \qquad (3.43)$$

One readily computes

$$\begin{aligned}
\mathcal{O}_{[1]}[\phi_{[1]}] &= \int d\mu_{\Gamma_*}[\zeta] \exp\left\{(A_*\phi_{[1]}+\zeta, J) - \frac{1}{2}(J, v_*J)\right\} \\
&= \exp\left\{(A_*\phi_{[1]}, J) - \frac{1}{2}(J, [v_*-\Gamma_*]\, J)\right\} \\
&= \exp\left\{(\phi_{[1]}, A_*^T J) - \frac{1}{2}(A_*^T J, vA_*^T J)\right\} \\
&= \; :e^{(\phi_{[1]},A_*^T J)}:_{v_*} \qquad (3.44)
\end{aligned}$$

from which we deduce

$$: \phi_{[0]}(z_1) \cdots \phi_{[0]}(z_n) :_{v_*} \to \int\limits_{x_1, \ldots, x_n} A_*(z_1, x_1) \cdots A_*(z_n, x_n) : \phi_{[1]}(x_1) \cdots \phi_{[1]}(x_n) :_{v_*} . \tag{3.45}$$

We are seeking eigenobservables of $\mathcal{L}$. Up to a folding with an interpolation kernel the normal-ordered products are already invariant. Fortunately the rescaled interpolation kernels have a scaling limit ${\mathcal{A}_*}^{[\infty]}$. From their definition (3.36) we find

$${\mathcal{A}_*}^{[m]} A_* = \mathcal{S}_*^{-1} {\mathcal{A}_*}^{[m+1]} \tag{3.46}$$

and therefore

$${\mathcal{A}_*}^{[\infty]} A_* = \mathcal{S}_*^{-1} {\mathcal{A}_*}^{[\infty]}. \tag{3.47}$$

Now define observables

$$\mathcal{O}_*^{(n)}[\phi_{[0]}] := \int\limits_z : \psi_*(z)^n :_{v_*} \tag{3.48}$$

with a blockspin dependent continuum background field $\psi_*(\phi_{[0]}) = {\mathcal{A}_*}^{[\infty]} \phi_{[0]}$.

Because of (3.45) we obtain

$$\begin{aligned}
\left(\mathcal{L}\mathcal{O}_*^{(n)}\right)[\phi_{[0]}] &= \int\limits_{z \in \mathbb{R}^D} : \left({\mathcal{A}_*}^{[\infty]} A_* \phi_{[0]}\right)(z)^n :_{v_*} \\
&= \int\limits_{z \in \mathbb{R}^D} : \left(\mathcal{S}_*^{-1} {\mathcal{A}_*}^{[\infty]} \phi_{[0]}\right)(z)^n :_{v_*} \\
&= \int\limits_{z \in \mathbb{R}^D} s^{n\alpha_*} : \left({\mathcal{A}_*}^{[\infty]} \phi_{[0]}\right)(z/s)^n :_{v_*} \\
&= s^{n\alpha_* + D} \int\limits_{z \in \mathbb{R}^D} : \left({\mathcal{A}_*}^{[\infty]} \phi_{[0]}\right)(z)^n :_{v_*} \\
&= s^{n\alpha_* + D} \mathcal{O}_*^{(n)}[\phi_{[0]}].
\end{aligned} \tag{3.49}$$

Hence the observables $\mathcal{O}_*^{(n)}[\phi_{[0]}]$ are eigenobservables of $\mathcal{L}$ with eigenvalues $s^{n\alpha_* + D}$.

Chapter 4

The Hierarchical $O(N)$ model

In this chapter the RT of the hierarchical two dimensional $O(N)$ symmetric non-linear σ model is calculated in perturbation theory.

4.1 Hierarchical Approximation

There are many different perspectives to look at the hierarchical approximation. Here we will take the one promoted by C. WIECZERKOWSKI. Starting point is the RG transformation in the high-temperature picture where the high-temperature fixed point $v_{\text{ht}} = 1/\kappa_* \mathbb{I}$ is taken as free propagator.[1]

$$e^{-V'_{\text{ht}}[\phi']} = \int d\mu_{\Gamma_{\text{ht}}}[\zeta] e^{-V_{\text{ht}}[A_{\text{ht}}\phi' + \zeta]} . \tag{4.1}$$

Here we have introduced $V_{\text{ht}} = S - T_{\text{ht}}$. In the HT picture eqns. (3.8,3.10) read

$$A_{\text{ht}} = \mathcal{C}^T \tag{4.2}$$

$$\Gamma_{\text{ht}} = \frac{1}{\kappa_*} \left(\mathbb{I} - \mathcal{C}^T \mathcal{C} \right) . \tag{4.3}$$

The entire RG philosophy is based on the assumption that the fluctuation propagators are strongly decaying kernels. Only then the occuring fluctuation integral is non-critical and the decomposition of the full (critical) partition function into slices makes sense.

In the hierarchical approximation this assumption is taken to its very end and the fluctuation propagator is approximated by the pure local part [2] $\Gamma_{\text{ht}} = \frac{1}{\kappa_*} \mathbb{I}$.

[1] For the ease of notation in this section scalar fields are treated

[2] C. WIECZERKOWSKI has embedded the hierarchical RG into a more general framework with the latter as zeroth order approximation. In this quasi-hierarchical RG the non-locality $\mathcal{C}^T \mathcal{C}$ of eq. (4.3) is switched on continously by some parameter τ with $\tau = 0$ giving the hierarchical fluctuation propagator. The transformation for $\tau > 0$ then turns out to be a hierarchical transformation with a subsequent rescaling step. [Wie]

If one further assumes that $V_{\rm ht}$ is also ultra-local

$$V_{\rm ht}[\phi] \quad = \quad \int\limits_z \mathcal{V}(\phi(z)) \tag{4.4}$$

then the flucutation integral eq. (4.3) factorizes completely and the effective potential becomes ultra-local, too.

$$e^{-\mathcal{V}'_{\rm ht}(\hat{\phi}')} \quad = \quad \left[\int d\mu_{\gamma_{\rm ht}}(\hat{\zeta}) e^{-\mathcal{V}_{\rm ht}\left(s^{-\frac{D+2}{2}}\hat{\phi}'+\hat{\zeta}\right)}\right]^{s^D} . \tag{4.5}$$

This is the hierarchical RG transformation in the HT picture. We will mark local quantities with $\hat{}$ from now on.

Here we are only interested in the $D = 2$ case. In this special situation the ML picture turns out to be more convenient. Proceeding as in 3.3, eq. (4.5) will now be rewritten in the ML picture. Let $\hat{A}_{\rm ht} = s^{-\frac{D+2}{2}}$.

In the hierarchical version the quadratic fixed point $\frac{1}{2}\hat{K}^*_{\rm ht}\hat{\phi}^2$ of the equation eq. (4.5) reads

$$\hat{K}^*_{\rm ht} = s^D \hat{A}_{\rm ht}\hat{L}^*_{\rm ht}\hat{K}^*_{\rm ht}\hat{A}_{\rm ht} \tag{4.6}$$

Formally the only difference is the occurrence of a block volume factor. Using $\hat{L}^*_{\rm ht} = (1-\gamma_{\rm ht}\hat{K}^*_{\rm ht})^{-1}$ yields

$$\hat{K}^*_{\rm ht} = \frac{s^{-2}-1}{\gamma_{\rm ht}}. \tag{4.7}$$

Define now $\gamma^{-1}_{\rm ml} := \hat{K}^*_{\rm ht} + \gamma^{-1}_{\rm ht}$ which yields $\gamma_{\rm ht} = s^{-2}\gamma_{\rm ml}$. Then $\hat{L}^*_{\rm ht} = s^2$,$\hat{L}^*_{\rm ht}\hat{A}_{\rm ht} =: \hat{A}_{\rm ml} = 1$ and therefore we get

$$e^{-\mathcal{V}'_{\rm ml}(\hat{\phi}')} \quad = \quad \left[\int d\mu_{\gamma_{\rm ml}}(\hat{\zeta}) e^{-\mathcal{V}_{\rm ml}(\hat{\phi}'+\hat{\zeta})}\right]^{s^D} . \tag{4.8}$$

This now is the hierarchical RG transformation in the ML picture.

Please note that in eq. (4.5) both integer parameters s, D can be continued to real values. Additionally the value of $\gamma_{\rm ml}$ or $\gamma_{\rm ht}$ can be varied.

4.2 The RG Transformation

Let us consider the hierarchical blockspin transformation for N component fields

$$e^{-\hat{\mathcal{R}}\mathcal{V}'(\hat{\phi}')} = \mathcal{N}\left[\int \mathrm{d}\mu_\gamma(\hat{\zeta}) e^{-\mathcal{V}(\hat{\phi}'+\hat{\zeta})}\right]^2 . \tag{4.9}$$

in $D = 2$ dimensions with scale parameter $s = \sqrt{2}$. The subspace of $O(N)$ invariant interactions is stable under $\hat{\mathcal{R}}$ (4.9). If we restrict our attention to this subspace $O(N)$ invariance requires $\mathcal{V}(\hat{\phi})$ to be a function of the modulus $|\hat{\phi}|$. The normalization constant in (4.9) is conveniently chosen such that the interaction is always zero at its minimum. We will denote this minimum by r and call it the radius of the potential.

4.3 Perfect Action and the Renormalized Trajectory

To calculate the Perfect Action [HN94] for the hierarchical $O(N)$ model we write the potential in the form

$$\mathcal{V}(\hat{\phi}) = r^2 \tilde{\mathcal{V}} \left(\frac{|\hat{\phi}|}{r} - 1 \right) \tag{4.10}$$

with $r = \frac{1}{f}$. The hierarchical transformation then takes the form

$$\exp\left(-r'^2 \tilde{\mathcal{V}}'(|\hat{\phi}'| - 1)\right) = \left[\mathcal{N} \left(\frac{2\pi\gamma}{r^2} \right)^{-\frac{N}{2}} \int \mathrm{d}^N \hat{\zeta} \exp\left(-r^2 \left\{ \frac{\hat{\zeta}^2}{2\gamma} - \tilde{\mathcal{V}}(|\frac{1}{r}\hat{\phi}' + \hat{\zeta}| - 1) \right\} \right) \right]^2 \tag{4.11}$$

where the fluctuation field and the blockspin field have been rescaled by r. It turns out in perturbation theory that the effective radius is

$$r' = r - \frac{\gamma}{2}(N-1)\frac{1}{r} + O\left(\frac{1}{r^3}\right). \tag{4.12}$$

In the limit where r and hence also r' are sent to infinity the fluctuation integral can be evaluated by the saddlepoint method giving the FP-equation

$$\tilde{\mathcal{V}}^{\text{FP}}(|\hat{\phi}'| - 1) = 2 \inf_{\hat{\zeta} \in \mathbb{R}^N} \left(\frac{1}{2\gamma}\hat{\zeta}^2 + \tilde{\mathcal{V}}^{\text{FP}}(|\hat{\phi}' + \hat{\zeta}| - 1) \right). \tag{4.13}$$

This equation can be solved by the ansatz $c_2^{\text{FP}}(|\hat{\phi}| - r)^2$ with a single parameter c_2^{FP} yielding

$$c_2^{\text{FP}}(|\hat{\phi}'| - 1)^2 = 2 \inf_{\hat{\zeta} \in \mathbb{R}^N} \left(\frac{1}{2\gamma}\hat{\zeta}^2 + c_2^{\text{FP}}(|\hat{\phi}' + \hat{\zeta}| - 1)^2 \right). \tag{4.14}$$

Rexpressing this quadratic extremality condition as a Gaussian integral equation

$$\exp\left\{ -c_2^{\text{FP}}(|\hat{\phi}'| - 1)^2 \right\} = \left[\mathcal{N} \int \mathrm{d}\mu_\gamma(\hat{\zeta}) \exp\left\{ \frac{1}{2\gamma}\hat{\zeta}^2 + c_2^{\text{FP}}(|\hat{\phi}' + \hat{\zeta}| - 1)^2 \right\} \right]^2 \tag{4.15}$$

one recognizes the FP equation of the original transformation with the wellknown HT solution $c_2^{\text{FP}} = \hat{K}^*_{\text{ml}} = \frac{1}{2\gamma}$.

The Perfect Action approximation for the renormalized trajectory in this model is hence given by

$$\mathcal{V}_{\text{PA}}(\phi) = \frac{r^2}{2\gamma} \left(\frac{1}{r}|\hat{\phi}| - 1 \right)^2 = \frac{1}{2}(\hat{\phi}_r, \hat{K}^*_{\text{ml}} \hat{\phi}_r) \tag{4.16}$$

with $\hat{\phi}_r = |\hat{\phi}| - r$.

The right way to think of this formula is as a line of fixed points of the classical renormalization group transformation parametrized by r. Note that unlike [HN94] the action is not just multiplied by r^2. We have tested this approximation numerically as will be explained below.

A sufficiently general form of $\mathcal{V}_{\text{RT}}(\hat{\phi})$ for σ models close to the renormalized trajectory proves to be

$$\mathcal{V}_{\text{RT}}(\hat{\phi}) = \sum_{a \geq 2} P_a(f)(|\hat{\phi}| - \frac{1}{f})^a \tag{4.17}$$

with $P_2(f) = \frac{1}{2\gamma} + O(f^2)$ and $P_a(f) = O(f^a)$ for $a > 2$.

The running coupling is given by the inverse radius. In this setup $\hat{\phi}$ is not restricted to take values on the sphere with radius $\frac{1}{f}$, the reason being that our recursion does not preserve this condition. The normalization constant $\mathcal{N}$ is conveniently chosen such that $\mathcal{V}(\hat{\phi}) = 0$ for $|\hat{\phi}| = \frac{1}{f}$. We adopt this renormalization condition. When computing single renormalization group transformations we will speak of the previous action as the bare action and of the outcome as the effective or renormalized action.

Let us define the renormalized trajectory as the curve $\mathcal{V}_{\text{RT}}(\hat{\phi}|f)$ in the space of $O(N)$ invariant potentials, parametrized by the inverse radius $f = \frac{1}{r}$, with the following two properties:

1) $\mathcal{V}_{\text{RT}}(\hat{\phi}|f)$ is stable under the blockspin transformation $\hat{\mathcal{R}}$. It follows that there exists a function $\beta(f)$ such that

$$\mathcal{R}\mathcal{V}_{\text{RT}}(\hat{\phi}|f) = \mathcal{V}_{\text{RT}}(\hat{\phi}|\beta(f)). \tag{4.18}$$

In other words: A blockspin transformation acts on the renormalized trajectory by a transformation of the coordinate given by a β-function.

2) The asymptotic behavior of $\mathcal{V}_{\text{RT}}(\hat{\phi}|f)$ as the running coupling f goes to zero is given by

$$\begin{aligned} \mathcal{V}_{\text{RT}}(\hat{\phi}|f) &= \mathcal{V}^{(1)}_{\text{RT}}(\hat{\phi}|f) + O(f^2), \\ \mathcal{V}^{(1)}_{\text{RT}}(\hat{\phi}|f) &= \mathcal{V}_{\text{PA}}(\hat{\phi}|f). \end{aligned} \tag{4.19}$$

Up to corrections of second order in the running coupling the renormalized trajectory coincides with the Perfect Action.

4.4 Perturbation Theory

The perturbation expansion for the $O(N)$ model can be computed to high orders using computer algebra. Let us explain the method in a second order calculation for the Perfect Action. As bare potential we take

$$\mathcal{V}(\hat{\phi}) = \mathcal{V}_{\text{PA}}(\hat{\phi}) = \frac{1}{2\gamma} W(\hat{\phi}|f)^2 \tag{4.20}$$

with

$$W(\hat{\phi}|f) = |\hat{\phi}| - \frac{1}{f}. \tag{4.21}$$

The effective potential will be $O(N)$–invariant. Without loss of generality we can therefore take the blockspin to be given by $\hat{\phi}' = (r+\hat{\phi}'_r)\hat{e}$ with $\hat{e}$ being an N–component unit vector, say $(0,\ldots,0,1)^T$. The shift of $\hat{\phi}'_r$ serves to place us into the minimum of the bare potential. We then decompose orthogonally the fluctuation field with respect to the direction of $\hat{\phi}'$ into $\hat{\zeta} = \hat{\rho}\hat{e} + \hat{\xi}$. The one component variable $\hat{\rho}$ is the radial fluctuation field while the $N-1$ component variable $\hat{\xi}$ is the tangential fluctuation field.

Now the bare potential is expanded in powers of the coupling $f = \frac{1}{r}$. Up to second order it is given by

$$\begin{aligned}\mathcal{V}(\{r+\hat{\phi}'_r+\hat{\rho}\}e+\hat{\xi}) &= \frac{1}{2\gamma}(\hat{\phi}'_r+\hat{\rho})^2 + f\frac{1}{2\gamma}\hat{\xi}^2(\hat{\phi}'_r+\hat{\rho}) \\ &\quad -f^2\frac{1}{2\gamma}\hat{\xi}^2(\hat{\phi}'_r+\hat{\rho})^2 + \frac{1}{8\gamma}\hat{\xi}^4 \end{aligned}\tag{4.22}$$

One observes a zero order term which cannot be treated as perturbation. A closer look reveals that this term is precisely the HT-fixed point. Splitting the bare potential

$$\mathcal{V}(\{r+\hat{\phi}'_r+\hat{\rho}\}e+\hat{\xi}) = \frac{1}{2}(\hat{\phi}'_r+\hat{\rho},\hat{K}^*_{\text{ml}}\{\hat{\phi}'_r+\hat{\rho}\}) + \mathcal{V}^{(1)}(\hat{\phi}'_r+\hat{\rho},\hat{\xi}) \tag{4.23}$$

enables us to use eq. (C.4), yielding the RG equation

$$e^{-\mathcal{V}'(\{r+\hat{\phi}'_r\}e)} = e^{\frac{1}{2}(\hat{\phi}'_r,\hat{K}^*_{\text{ml}}\hat{\phi}'_r)}\left[\mathcal{N}\int \mathrm{d}\mu_{\gamma_{\text{ht}}}(\hat{\rho})\int \mathrm{d}\mu_{\gamma_{\text{ml}}}(\hat{\xi})e^{-\mathcal{V}^{(1)}(\hat{A}_{\text{ht}}\hat{\phi}'_r+\hat{\rho},\hat{\xi})}\right]^2. \tag{4.24}$$

with $\hat{A}_{\text{ht}} = \frac{1}{2}$.

In this mixed formulation the radial field is treated in the HT picture whereas the angular field is treated in the ML picture. The bare potential takes the form

$$\mathcal{V}^{(1)}(\hat{A}_{\text{ht}}\hat{\phi}'_r+\hat{\rho},\hat{\xi}) = f\frac{1}{2\gamma_{\text{ml}}}\hat{\xi}^2(\frac{1}{2}\hat{\phi}'_r+\hat{\rho}) - f^2\frac{1}{2\gamma_{\text{ml}}}\hat{\xi}^2(\frac{1}{2}\hat{\phi}'_r+\hat{\rho})^2 + \frac{1}{8\gamma_{\text{ml}}}\hat{\xi}^4 \tag{4.25}$$

At this point perturbation theory is applicable. Although the potential is non–polynomial to begin with only finitely many terms show up at finite order with a leading trilinear vertex. Note that the Perfect Action is recovered when fluctuations are completely neglected. The perturbation expansion is straight forward using the Gaussian correlations (see C.7)

$$\int \mathrm{d}\mu_{\gamma_{\text{ht}}}(\hat{\rho})\hat{\rho}^{2n} = \left(\frac{\gamma_{\text{ml}}}{2}\right)^n\prod_{m=0}^{n-1}(2m+1) \tag{4.26}$$

and

$$\int \mathrm{d}\mu_{\gamma_{\text{ml}}}(\hat{\xi})(\hat{\xi}^2)^n = \gamma_{\text{ml}}^n\prod_{m=0}^{n-1}(2m+N-1). \tag{4.27}$$

Computing the fluctuation integral to second order perturbation theory and neglecting field independent terms we obtain an effective potential of the form

$$\mathcal{V}'(\{r+\hat{\phi'}_r\}e) = \left(\frac{1}{2\gamma_{\rm ml}} - \frac{3}{8}(N-1)f^2\right)\hat{\phi'}_r^2 + \frac{1}{2}(N-1)f\hat{\phi'}_r + O(f^3). \tag{4.28}$$

We then determine the value δr of $\hat{\phi'}_r$ at which the effective potential attains its minimum and substitute $\hat{\phi'}_r = \hat{\phi'}_{r'} + \delta r$. The change of r is due to the linear term in $\hat{\phi'}_r$. The meaning of this variable is $\hat{\phi'}_{r'} = |\hat{\phi'}| - r'$ with $r' = r + \delta r$ denoting the renormalized radius.

To second order perturbation theory in f the change of the radius is $\delta r = -\frac{\gamma_{\rm ml}}{2}(N-1)f + O(f^3)$. From this we find a renormalized coupling $f' = \frac{1}{r'}$ of the form

$$f' = f + \frac{\gamma_{\rm ml}}{2}(N-1)f^3 + O(f^5). \tag{4.29}$$

(The vanishing of the f^4 term of the $\triangle\beta$ function follows from a fourth order calculation.) In particular we have confirmed that the model is perturbatively asymptotically free for $N > 1$. That is, when perturbation theory applies we find a flow where the coupling slowly grows and the radius slowly shrinks. The effective potential becomes

$$\mathcal{V}'(\hat{\phi'}_{r'}) = \left(\frac{1}{2\gamma_{\rm ml}} - \frac{3}{8}(N-1)f'^2\right)\hat{\phi'}_{r'}^2 + O(f'^3) \tag{4.30}$$

in terms of $\hat{\phi'}_{r'} = |\hat{\phi'}| - r'$ and f'. We also see that this action remains invariant in the sense of scaling to first order. Scaling violation shows up in a second order flow of the overall prefactor. In a zeroth improvement step they can be compensated for by changing the bare action into

$$\mathcal{V}(\hat{\phi}) = \left(\frac{1}{2\gamma_{\rm ml}} + c_2^{(2)}f'^2\right)\left(|\hat{\phi}| - \frac{1}{f}\right)^2. \tag{4.31}$$

The correct value of the improvement parameter is $c_2^{(2)} = \frac{3}{4}(N-1)$. The resulting action can then be seen to scale even to second order.

4.5 Improved Action

Let us now also remove the scaling violations of second order. As a bare potential we take the second order improved one which we write in the form

$$\mathcal{V}^{(2)}(\hat{\phi}) = P_2^{(2)}(f)\left(|\hat{\phi}| - \frac{1}{f}\right)^2 \tag{4.32}$$

with $P_2^{(2)}(f) = \left(\frac{1}{2\gamma_{\rm ml}} - \frac{3}{4}(N-1)f'^2\right)$ The effective potential computed to third order perturbation theory is given by

$$\mathcal{V}'^{(2)}(\hat{\phi}'_{r'}) = \left(\frac{1}{2\gamma_{\rm ml}} - \frac{3}{4}(N-1)f'^2\right)\hat{\phi}'^2_{r'} + \frac{7}{24}(N-1)f'^3\hat{\phi}'^3_{r'} + O(f'^4) \tag{4.33}$$

in terms of f'. What is new is a cubic term in $\hat{\phi}'_{r'} = |\hat{\phi}'| - r'$. As it will be generated anyway to third order it is natural to include it to this order in the bare action. In other words let us make the ansatz

$$\mathcal{V}^{(3)}(\hat{\phi}) = P_2^{(3)}(f)\left(|\hat{\phi}| - \frac{1}{f}\right)^2 + P_3^{(3)}(f)\left(|\hat{\phi}| - \frac{1}{f}\right)^3 \tag{4.34}$$

with $P_2^{(3)}(f) = P_2^{(2)}(f) + c_2^{(3)}f^3$ and $P_3^{(3)}(f) = c_3^{(3)}f^3$.

Here we have anticipated a possible cubic correction to $P_2(f)$ which in fact turns out not to exist. The ansatz contains two new improvement parameters. Computing again the effective potential to third order, the correct values of these are seen to be $c_2^{(3)} = 0$ and $c_3^{(3)} = \frac{7}{18}(N-1)$. This action reproduces itself up to fourth order corrections. The improvement scheme can now be iterated.

Suppose that we have found the potential

$$\mathcal{V}^{(n)}(\hat{\phi}) = \sum_{a=2}^{n} P_a^{(n)}(f)\left(|\hat{\phi}| - \frac{1}{f}\right)^a \tag{4.35}$$

which scales up to order n. For the potential of the next improvement level $n+1$ we make the ansatz

$$\mathcal{V}^{(n+1)}(\hat{\phi}) = \sum_{a=2}^{n+1} P_a^{(n+1)}(f)\left(|\hat{\phi}| - \frac{1}{f}\right)^a \tag{4.36}$$

containing polynomials $P_a^{(n+1)}(f) = \left[P_a^{(n)}(f) + c_a^{(n+1)}f^{n+1}\right]$ for $a \leq n$ and $P_{n+1}^{(n+1)}(f) = c_{n+1}^{(n+1)}f^{n+1}$. This general form is reproduced to this order. The corresponding effective potential is again of this form to order $n+1$

$$\mathcal{V}'^{(n+1)}(\hat{\phi}'_{r'}) = \sum_{a=2}^{n+1} P'^{(n+1)}_a(f')\hat{\phi}'^a_{r'} + O(f'^{n+2}) \tag{4.37}$$

containing polynomials $P'^{(n+1)}_a(f') = \left[P_a^{(n)}(f') + c'^{(n+1)}_a f'^{n+1}\right]$ for $a \leq n$ and $P'^{(n+1)}_{n+1}(f') = c'^{(n+1)}_{n+1}f'^{n+1}$.

The effective coefficients $c'^{(n+1)}_a$ depend linearly on their bare counterparts $c_a^{(n+1)}$. (To order $n+1$ they have no other choice.) In order that there be no scaling violation to order $n+1$ the polynomials $P_a^{(n+1)}$ and $P'^{(n+1)}_a$ have to be equal. From this we obtain a system of linear equations for the coefficients $c_a^{(n+1)}$. This system turns out to have a unique solution. With the general form

$$P_a^{(n)}(f) = \sum_{m=0}^{[\frac{n-a}{2}]} c_a^{(2m+a)}f^{2m+a} \tag{4.38}$$

we find up to fifth order

$$\begin{aligned}
c_2^{(2)} &= -\frac{3(N-1)}{4}, \\
c_2^{(4)} &= \left(\frac{9(N-1)^2}{8} - \frac{61(N-1)}{28}\right)\gamma_{\rm ml}, \\
c_3^{(3)} &= \frac{7(N-1)}{18}, \\
c_3^{(5)} &= \left(-\frac{5(N-1)^2}{8} + \frac{257(N-1)}{180}\right)\gamma_{\rm ml}, \\
c_4^{(4)} &= -\frac{15(N-1)}{56}, \\
c_5^{(5)} &= \frac{31(N-1)}{150},
\end{aligned} \tag{4.39}$$

We observe that the power series for $P_a(f)$ contains only even (odd) powers of f when a is even (odd). Furthermore we observe that the signs of the coefficients alternate. The complete series is not expected to converge due to instanton singularities. It would however be very interesting to apply the machinery of resummation methods to a high order approximation to the renormalized trajectory.

$$f' = f + \frac{(N-1)}{2}\gamma_{\rm ml} f^3 + \left(\frac{3(N-1)^2}{8} + \frac{13(N-1)}{12}\right)\gamma_{\rm ml}^2 f^5 + O(f^6) \tag{4.40}$$

For the sake of completeness we also include the recursion

$$\begin{aligned}
r' &= r - \frac{1}{2}\gamma_{\rm ml}(N-1)f - \left(\frac{1}{8}(N-1)^2 + \frac{13}{12}(N-1)\right)\gamma_{\rm ml} f^3 \\
&\quad - \left(\frac{1}{16}(N-1)^3 + \frac{13}{12}(N-1)^2 + \frac{47}{12}(N-1)\right)\gamma_{\rm ml}^3 f^5 + O(f^6)
\end{aligned} \tag{4.41}$$

for the effective radius.

4.6 Continuum Limit

Our definition of a renormalized trajectory does not refer to a continuum limit procedure. It is nevertheless identical with the continuum limit effective potential of models in the $O(N)$ universality class in the hierarchical renormalization scheme. To prove this we can perform the continuum limit using (4.41) as bare interaction. Define the bare coupling to be the n-fold preimage $\beta^{-n}(f)$ of a renormalized value f. By construction of the renormalized trajectory it follows that

$$\hat{\mathcal{R}}^n \mathcal{V}_{\rm RT}(\hat{\phi}|\beta^{-n}(f)) = \mathcal{V}_{\rm RT}(\hat{\phi}|f)) \tag{4.42}$$

for all numbers n of renormalization group steps. The continuum limit $n \to \infty$ is immediately performed since the right hand side of (4.42) is independent of n. All that is needed is an analysis of the recursion relation defined by the β-function or rather its inverse. This is a comparatively easy task. See for instance [GK86]. For $N > 1$ it follows that the bare coupling tends to zero as n goes to infinity. (One needs to take into account logarithmic corrections piled up by the term of third order in g^2, fifth order in f.) It follows that the perturbation expansion in the running coupling is valid. To make contact with the hierarchical real world one should also assign a scale, for instance in form of a lattice spacing a, to the point on the renormalized trajectory where the running coupling is given by the renormalized value f. The bare cutoff in (4.42) is then $s^{-n}a$ with s the block scale.

The sceptical reader may worry to what extent this construction is connected to the continuum limit of a bare theory defined by his favorite $O(N)$ invariant interaction. Consider for instance the standard interaction of the linear $O(N)$-model defined by

$$\mathcal{V}(\hat{\phi}|f) = \lambda f^2 \left(\hat{\phi}^2 - \frac{1}{f^2}\right)^2 . \tag{4.43}$$

Its continuum limit is constructed as the result of the infinite iteration

$$\lim_{n\to\infty} \hat{\mathcal{R}}^n \mathcal{V}(\hat{\phi}|f_{-n}(f)) = \mathcal{V}_{\text{cont}}(\hat{\phi}|f) . \tag{4.44}$$

Here the bare coupling $f_{-n}(f)$ is tuned such that the minimum of $\hat{\mathcal{R}}^n \mathcal{V}(\hat{\phi}|f_{-n}(f))$ is located at the renormalized radius $|\hat{\phi}| = \frac{1}{f}$. The continuum limit is universal as we have learnt from the work of Wilson [WK74]. Therefore the connection to the above definition is simply

$$\mathcal{V}_{\text{cont}}(\hat{\phi}|f) = \mathcal{V}_{\text{RT}}(\hat{\phi}|f) . \tag{4.45}$$

A rigorous proof of the existence of (4.44) and, inbetween the lines, also of (4.45) has been given by Gawedzki and Kupiainen [GK86]. More generally, the set of all bare interactions sharing (4.41) as their common continuum limit defines the universality class of the hierarchical $O(N)$-model. (4.43) is known to belong to this class and so is the original model with sharp constraint. So what we do, when we define the renormalized trajectory by the above two conditions, is to completely disentangle the admittingly also interesting questions if, how, and at what pace the continuum limit is reached by some particular bare model.

In practice we may not be able to compute (4.41) to all orders of perturbation theory in the running coupling for the models we are really interested in, for example the full nonlinear $O(N)$-model in terms of blockspin transformation on a unit lattice. In this situation the best one can do is to take the highest order approximation to (4.41) accessible as bare interaction. In the hierarchical model for instance already the second approximant

$$\mathcal{V}^{(2)}_{\text{RT}}(\hat{\phi}) = \left(\frac{1}{2\gamma_{\text{ml}}} + \frac{3}{4}(N-1)f'^2\right)\left(|\hat{\phi}| - \frac{1}{f}\right)^2 . \tag{4.46}$$

turns out to be an excellent starting point for a numerical study of the renormalized trajectory. Its main property is of course

$$\hat{\mathcal{R}}\mathcal{V}_{\text{RT}}^{(2)}(\hat{\phi}|f) = \mathcal{V}_{\text{RT}}^{(2)}(\hat{\phi}|\beta(f)) + O(\beta(f)^3). \tag{4.47}$$

It therefore coincides with the renormalized trajectory up to corrections of third order in f.

Let us finally mention that the third order approximant would be ideally suited for a rigorous construction along the lines of Gawedzki and Kupiainen [GK86] and of Pordt and Reisz [PR91]. The perturbative part is trivial since the action reproduces itself to third order. The construction therefore reduces to the proof of a stability bound which controls non perturbative corrections. This bound is already implicitly contained in the rigorous work on the $O(N)$-model with standard bare interaction (4.43).

4.7 Perfect Observables

Let us consider the hierarchical renormalization group transformation for local observables corresponding to (4.9) in the case of $D = 2$ dimensions with scale parameter $s = \sqrt{2}$. It is given by the linear transformation

$$\hat{\mathcal{L}}_{\mathcal{V}}\hat{\mathcal{O}}(\hat{\phi}') = \frac{\int \mathrm{d}\mu_\gamma(\hat{\zeta}) e^{-\mathcal{V}(\hat{\phi}'+\hat{\zeta})}\hat{\mathcal{O}}(\hat{\phi}'+\hat{\zeta})}{\int \mathrm{d}\mu_\gamma(\hat{\zeta}) e^{-\mathcal{V}(\hat{\phi}'+\hat{\zeta})}}. \tag{4.48}$$

The block volume is $s^D = 2$. (4.48) is the linearization of (4.9) divided by the block volume

$$\hat{\mathcal{L}}_{\mathcal{V}}\hat{\mathcal{O}}(\hat{\phi}') = \frac{1}{2}\frac{\partial}{\partial t}\hat{\mathcal{R}}(\mathcal{V} + t\hat{\mathcal{O}})_{|t=0}. \tag{4.49}$$

In other words the transformation of observables is the tangent map of the transformation of the potential.

By means of the transformations that led to eq. (4.24) one finds

$$\hat{\mathcal{L}}_{\mathcal{V}}\hat{\mathcal{O}}(\{\hat{A}_{\text{ht}}\hat{\phi}'_r + \hat{\rho}\}e) = \tag{4.50}$$
$$\frac{\int \mathrm{d}\mu_{\gamma_{\text{ht}}}(\hat{\rho}) \int \mathrm{d}\mu_{\gamma_{\text{ml}}}(\hat{\xi}) e^{-\mathcal{V}^{(1)}(\hat{A}_{\text{ht}}\hat{\phi}'_r+\hat{\rho},\hat{\xi})}\hat{\mathcal{O}}(\{r + \hat{A}_{\text{ht}}\hat{\phi}'_r + \hat{\rho}\}e + \hat{\xi})}{\int \mathrm{d}\mu_{\gamma_{\text{ht}}}(\hat{\rho}) \int \mathrm{d}\mu_{\gamma_{\text{ml}}}(\hat{\xi}) e^{-\mathcal{V}^{(1)}(\hat{A}_{\text{ht}}\hat{\phi}'_r+\hat{\rho},\hat{\xi})}}$$

At this point perturbation theory is again applicable. The expansion parameter is the inverse radius f.

An observable is called a moving eigenvector to the moving eigenvalue $\epsilon(f)$ if it satisfies the renormalization group equation

$$\hat{\mathcal{L}}_{\mathcal{V}}\hat{\mathcal{O}}(\hat{\phi}|f) = \epsilon(\beta(f))\hat{\mathcal{O}}(\hat{\phi}|\beta(f)). \tag{4.51}$$

Solutions to (4.51) will be called perfect observables. A parametrization well suited for perturbation theory is

$$\hat{\mathcal{O}}(\hat{\phi}|f) = \sum_{n=0}^{\infty} Q_n(f)\left(|\hat{\phi}| - \frac{1}{f}\right)^n. \tag{4.52}$$

Here the coefficients $Q_n(f)$ are taken to be power series in the running coupling f. (The coefficients are not expected to exhibit any singularity at zero coupling.) We will organize the solutions of (4.51) according to their zeroth order coefficients. The zeroth order observables are simply normal ordered monomials. Let us also perform the perturbation expansion for the observables of the form (4.52). Define

$$W(\hat{\phi}) = |\hat{\phi}| - \frac{1}{f}. \tag{4.53}$$

As in the potential we find a term of order zero in the running coupling. Separating it off we obtain

$$W(\{\frac{1}{f} + \frac{\hat{\phi'}_r}{2} + \hat{\rho})e + \hat{\xi}|f) = \frac{\hat{\phi'}_r}{2} + \hat{\rho} + W^{(1)}(\frac{\hat{\phi'}_r}{2} + \hat{\rho}, \hat{\xi}|f). \tag{4.54}$$

In the limit when the radius becomes infinite $\mathcal{V}^{(1)}$ vanishes. Since the zeroth order term does not depend on $\hat{\xi}$, the transformation (4.50) reduces in this case to the convolution with the radial Gaussian measure. This transformation is identical with that of a one-component scalar field in two dimensions at the high-temperature fixed-point. We are therefore immediately lead to normal ordered monomials.

Put $\gamma_* = \frac{\gamma_{\text{ht}}}{1-\hat{A}_{\text{ht}}^2} = \frac{2\gamma_{\text{ml}}}{3}$. From eq. (C.8) it then follows that

$$\int \mathrm{d}\mu_{\gamma_{\text{ht}}}(\hat{\rho}) : (\hat{A}_{\text{ht}}\hat{\phi'}_r + \hat{\rho})^n :^{\hat{\rho}}_{\gamma_*} = \hat{A}_{\text{ht}}^n : \hat{\phi} :^{\hat{\phi'}_r}_{\gamma_*} . \tag{4.55}$$

Therefore the eigenvectors are normal ordered monomials to zeroth order as expected. The zeroth order spectrum is $\epsilon_n = \frac{1}{2^n}$. Let us write the normal ordered monomials in the form

$$: \hat{\phi'}_r^{\,n} :_{\gamma_*} = \sum_{m=0}^{n} h_{n,m} \hat{\phi'}_r^{\,m} . \tag{4.56}$$

They are of course rescaled Hermite polynomials in the variable $\hat{\phi'}_r$. We then write the associated sequence of observables defined as solutions of the scaling equation (4.51) in the form

$$\hat{\mathcal{O}}_n(\hat{\phi}|f) = \sum_{m=0}^{\infty} Q_{n,m}(f) W(\hat{\phi}|f)^m . \tag{4.57}$$

The coefficients $Q_{n,m}(f)$ are given by the power series expansions

$$Q_{n,m}(f) = \sum_{l=0}^{\infty} d_{n,m}^{(l)} f^l \tag{4.58}$$

with zero order coefficients of the normal ordered form

$$Q_{n,m}^{(0)} = h_{n,m} . \tag{4.59}$$

That is, the observables are perturbations of normal ordered monomials. The perturbative form of the moving spectrum is

$$\epsilon_n(f) = \sum_{m=0}^{\infty} \epsilon_n^{(m)} f^m \tag{4.60}$$

with zero order coefficients

$$\epsilon_n^{(0)} = \frac{1}{2^n}. \tag{4.61}$$

This completes the setup for the improvement program for observables. Let us then turn to the question of how to compute the higher coefficients in the expansions (4.58) and (4.60) for the moving eigenvectors and observables. The strategy is an adaptation of the improvement program for the potential. Let us choose the first (nontrivial) observable $\hat{\mathcal{O}}_1(\hat{\phi}|f)$ as an example and perform a third order computation to some detail. The zeroth order argument provides us with the information that

$$\begin{aligned} \hat{\mathcal{O}}_1(\hat{\phi}|f) &= W(\hat{\phi}|f) + O(f), & (4.62)\\ \epsilon_1(f) &= \frac{1}{2} + O(f). & (4.63) \end{aligned}$$

Let us denote the zeroth order approximation by $\hat{\mathcal{O}}(\hat{\phi}|f) = W(\hat{\phi}|f)$. As we will see this observable already scales to second order. We immediately perform a perturbation expansion to third order for the effective observable of the zeroth order approximation. The result is

$$\begin{aligned} \hat{\mathcal{L}}_{\text{RT}}\hat{\mathcal{O}}(\hat{\phi}|f) &= \epsilon(f')\left(Q_0(f') + W(\hat{\phi}|f') + Q_2(f')W(\hat{\phi}|f')^2\right) + O((f')^4),\\ \epsilon(f') &= \frac{1}{2} + \frac{(N-1)\gamma_{\text{ml}}}{4}(f')^2,\\ Q_0(f') &= -\frac{5(N-1)\gamma_{\text{ml}}^2}{3}(f')^3,\\ Q_2(f') &= -\frac{(N-1)\gamma_{\text{ml}}}{6}(f')^3, \end{aligned} \tag{4.64}$$

where the old running coupling f is expressed in terms of the new running coupling $f' = \beta(f)$.

($\hat{\mathcal{L}}_{\text{RT}}$ is understood as an operator which is applied to the function $\hat{\mathcal{O}}(\hat{\phi}|f)$.) This change of coupling prepares in particular the ground for further iterations. Let us emphasize that not only the coefficients but also the coordinate functions depend on the running coupling and have to be adjusted. From this we conclude that the observable already scales to second order. The eigenvalue is therefore already correct to second order. To third order both a constant and a term quadratic in $W(\hat{\phi}|f)$ are generated.

Therefore the observable does not reproduce its dependence on the field to third order. To find the observable which scales to third order we make the ansatz

$$\hat{\mathcal{O}}(\hat{\phi}|f) = d_0 f^3 + W(\hat{\phi}|f) + d_2 f^3 W(\hat{\phi}|f)^2. \tag{4.65}$$

The ansatz involves two improvement parameters d_0 and d_2. To determine their value one again computes the effective observable starting from (4.65). The expansion gives

$$\begin{aligned}
\hat{\mathcal{L}}_{\rm RT}\hat{\mathcal{O}}(\hat{\phi}|f) &= \epsilon(f')\left(Q_0(f') + W(\hat{\phi}|f') + Q_2(f')W(\hat{\phi}|f')^2\right) + O((f')^4), \\
\epsilon(f') &= \frac{1}{2} + \frac{(N-1)\gamma_{\rm ml}}{4}(f')^2, \\
Q_0(f') &= \left(-\frac{5(N-1)\gamma_{\rm ml}^2}{3} + d_2\gamma_{\rm ml} + 2d_0\right)(f')^3, \\
Q_2(f') &= \left(-\frac{(N-1)\gamma_{\rm ml}}{6} + \frac{d_2}{2}\right)(f')^3.
\end{aligned} \tag{4.66}$$

The improvement parameters are then determined by the simple linear equations

$$\begin{aligned}
d_2 &= \frac{-(N-1)\gamma_{\rm ml}}{6} + \frac{d_2}{2}, \\
d_0 &= \frac{-5(N-1)\gamma_{\rm ml}^2}{3} + d_2\gamma_{\rm ml} + 2d_0\,.
\end{aligned} \tag{4.67}$$

The solution is

$$d_2 = \frac{-(N-1)\gamma_{\rm ml}}{3}, \quad d_0 = 2(N-1)\gamma_{\rm ml}^2. \tag{4.68}$$

To third order we therefore find

$$\hat{\mathcal{O}}_1(\hat{\phi}|f) = 2(N-1)\gamma_{\rm ml}^2 f^3 + W(\hat{\phi}|f) - \frac{(N-1)\gamma_{\rm ml}}{3}f^3 W(\hat{\phi}|f)^2 + O(f^4) \tag{4.69}$$

together with the eigenvalue

$$\epsilon_1(f) = \frac{1}{2} + \frac{(N-1)\gamma_{\rm ml}}{4}f^2 + O(f^4). \tag{4.70}$$

Note that the eigenvalue does not have a term of third order in the running coupling. This scheme is iterated in the obvious manner. Suppose that we have computed the observable to order t in the running coupling

$$\hat{\mathcal{O}}_1(\hat{\phi}|f) = \hat{\mathcal{O}}_1^{(t)}(\hat{\phi}|f) + O(f^{t+1}). \tag{4.71}$$

Moreover suppose that the order t improved observable is given by the general form

$$\begin{aligned}
\hat{\mathcal{O}}_1^{(t)}(\hat{\phi}|f) &= \sum_{n=0}^{t-1} Q_{1,n}^{(t)}(f)W(\hat{\phi}|f)^n, \\
Q_{1,0}^{(t)}(f) &= \sum_{r=0}^{t} d_{1,0}^{(r)} f^r, \\
Q_{1,1}^{(t)}(f) &= 1, \\
Q_{1,n}^{(t)}(f) &= \sum_{r=n+1}^{t} d_{1,n}^{(r)} f^r.
\end{aligned} \tag{4.72}$$

Then by induction it follows that the effective observable to order $t+1$ is again of this general form. It follows that to every order of perturbation theory only finitely many powers of the coordinate functions $W(\hat{\phi}|f)$ appear. The sums turn out to involve either even or odd powers in the running coupling respectively. We assume that $\hat{\mathcal{O}}_1^{(t)}(\hat{\phi}|f)$ scales to order t. That is,

$$\hat{\mathcal{L}}_{\text{RT}}\hat{\mathcal{O}}_1^{(t)}(\hat{\phi}|f) = \hat{\mathcal{O}}_1^{(t)}(\hat{\phi}|f') + O(f'^{t+1}). \tag{4.73}$$

Then we take an ansatz of the form (4.72) to order $t+1$ treating the coefficients of order $t+1$ in the running coupling as improvement parameters. We compute the effective observable. It depends linearly on the improvement parameters. To order $t+1$ it has no other choice. Then we claim invariance to obtain a linear system of equations for the improvement coefficients. This system turns out always to have a unique solution: the improved observable.

4.8 Hierarchical Fusion Rules

The outcome of our analysis is a system of observables $\hat{\mathcal{O}}_n(\hat{\phi}|f)$ on the renormalized trajectory parametrized by f. Under a hierarchical renormalization group transformation $\hat{\mathcal{O}}_n(\hat{\phi}|f)$ is multiplied by the moving eigenvalue $\epsilon_n(\beta(f))$ and the coordinate is changed to $\beta(f)$. This scheme is obviously iteratable which is the reason why we introduced it from the beginning. To compute general correlation functions we need one more ingredience which is the notion of hierarchical fusion rules. The general form of our observables in terms of $\hat{\phi'}_r = |\hat{\phi}| - r$ is

$$\begin{aligned}
\hat{\mathcal{O}}_n(\hat{\phi}|f) &= \sum_{r=0}^{\infty} \hat{\mathcal{O}}_n^{(r)}(\hat{\phi'}_r)\, f^r\,, \\
\hat{\mathcal{O}}_n^{(0)}(\hat{\phi'}_r) &= :\hat{\phi'}_r^{\,n}:, \\
\hat{\mathcal{O}}_n^{(1)}(\hat{\phi'}_r) &= 0, \\
\hat{\mathcal{O}}_n^{(2)}(\hat{\phi'}_r) &= \sum_{m=0}^{\left[\frac{n-2}{2}\right]} \hat{\mathcal{O}}^{(2)}_{n,n-2-2m} :\hat{\phi'}_r^{\,n-2-2m}:, \\
\hat{\mathcal{O}}_n^{(r)}(\hat{\phi'}_r) &= \sum_{m=0}^{\left[\frac{n+r-2}{2}\right]} \hat{\mathcal{O}}^{(r)}_{n,n+r-2-2m} :\hat{\phi'}_r^{\,n+r-2-2m}:\,.
\end{aligned} \tag{4.74}$$

The normal ordering covariance is $\gamma_* = \frac{2}{3}\gamma_{\text{ml}}$. Let us put $\gamma_{\text{ml}} = 1$ to simplify the notation. To zeroth order in f we rediscover normal ordered monomials in $\hat{\phi'}_r$. Their perturbations along the renormalized trajectory prove to have no first order terms in the running coupling f. To every order of perturbation theory in f we find only finitely many normal ordered powers of $\hat{\phi'}_r$. The highest power is $n+r-2$ for $r>2$. Asso-

ciated with this system of observables is a system of fusion rules defined by

$$\begin{aligned}\hat{\mathcal{O}}_n(\hat{\phi}|f)\hat{\mathcal{O}}_m(\hat{\phi}|f) &= \sum_{l=0}^{\infty} N_{n,m;l}(f)\hat{\mathcal{O}}_l(\hat{\phi}|f),\\ N_{n,m;l}(f) &= \sum_{r=0}^{\infty} N^{(r)}_{n,m;l}\, f^r\,. \end{aligned} \tag{4.75}$$

Furthermore from the fusion rules we obtain a symmetric bilinear form on the linear space of observables. It is defined by

$$\Big(\hat{\mathcal{O}}_n(\hat{\phi}|f), \hat{\mathcal{O}}_m(\hat{\phi}|f)\Big) = N_{n,m;0}(f). \tag{4.76}$$

The physical significance of this bilinear form is that in the thermodynamic limit only the overlap of an observable with the constant term is expected to survive. Since correlation functions are expected to decrease with distance, the spectrum consists of eigenvalues strictly smaller than one on the renormalized trajectory. To zeroth order of perturbation theory we recapture the fusion rules of normal ordered products

$$\begin{aligned} N^{(0)}_{n,m;l} &= \frac{n!m!}{n'!m'!l'!}\gamma_*^{l'},\\ n' &= \frac{1}{2}(l+m-n),\\ m' &= \frac{1}{2}(n+l-m),\\ l' &= \frac{1}{2}(n+m-l), \end{aligned} \tag{4.77}$$

for $|n-m| \le l \le n+m$ and $n+m-l \in 2\mathbb{Z}$, zero else. Furthermore to zeroth order the observables are orthogonal with respect to the bilinear form

$$N^{(0)}_{n,m;l} = \delta_{n,m}\, m!\, \gamma_*^m. \tag{4.78}$$

The simple pattern (4.77), (4.78) becomes perturbed as one moves away from the ultraviolet fixed point on the renormalized trajectory. The perturbation expansion for $N_{1,1;l}(f)$ to fifth order in f is for instance

$$\begin{aligned} N_{1,1;0}(f) &= \frac{2}{3} + \frac{5(N-1)}{9}\, f^2 - \left(\frac{20(N-1)^2}{27} - \frac{233(N-1)}{63}\right) f^4 + O(f^6),\\ N_{1,1;1}(f) &= -2(N-1)\, f^3 + \left(\frac{17(N-1)^2}{3} - 26(N-1)\right) f^5 + O(f^7),\\ N_{1,1;2}(f) &= 1 + O(f^6),\\ N_{1,1;3}(f) &= \frac{24(N-1)}{35}\, f^5 + O(f^7). \end{aligned} \tag{4.79}$$

The perturbation expansion for $N_{1,2;l}(f)$ to fifth order in f is given by

$$\begin{aligned}
N_{1,2;0}(f) &= \frac{40(N-1)^2}{21} f^5 + O(f^7), \\
N_{1,2;1}(f) &= \frac{4}{3} + \frac{10(N-1)}{9} f^2 - \left(\frac{40(N-1)^2}{27} - \frac{598(N-1)}{63} \right) f^4 + O(f^6), \\
N_{1,2;2}(f) &= -4(N-1) f^3 + \left(\frac{34(N-1)^2}{3} - \frac{404(N-1)}{7} \right) f^5 + O(f^7), \\
N_{1,2;3}(f) &= 1 - \frac{4(N-1)}{21} f^4 + O(f^6), \\
N_{1,2;4}(f) &= \frac{48(N-1)}{35} f^5 + O(f^7).
\end{aligned} \tag{4.80}$$

All other fusion rules are zero to fifth order. We observe that orthogonality is violated to fifth order for the first and second observable.

With the fusion rules we can compute correlation functions of our observables. Let us consider for example a general two-point function. It depends on the hierarchical distance k and the hierarchical lattice size k' of the system. The explicit formula is

$$\left\langle \hat{\mathcal{O}}_n(\hat{\phi}|f) \hat{\mathcal{O}}_m(\hat{\phi}|f) \right\rangle = \prod_{j=1}^{k} \left(\epsilon_n \left(\beta^j(f) \right) \epsilon_m \left(\beta^j(f) \right) \right) \sum_{l=0}^{\infty} N_{n,m;l} \left(\beta^k(f) \right) \prod_{i=1}^{k'-k} \epsilon_l \left(\beta^{k+i}(f) \right) \tag{4.81}$$

for the two point function on the point of the renormalized trajectory parametrized by f. This formula simply states that each observable is renormalized independently k-times. Each renormalization step produces a factor given by the eigenvalue of the corresponding observable at the location on the renormalized trajectory. After k steps the observables end up in the same block and are fused together there. The result of fusion is then renormalized $(k'-k)$-times to obtain the value of the two point function. The thermodynamic limit corresponds to $k' = \infty$. The formula (4.81) still holds in this limit provided one changes to a different parametrization of the renormalized trajectory at the point where the running coupling f diverges.

4.9 Numerical Results

When dealing with perturbation theory it is natural to question its validity. To tackle this problem we have determined our perfect observables and their corresponding eigenvalues numerically. The main technical task is to compute the transformations

$$\hat{\mathcal{R}} \mathcal{V}(\hat{\phi}') = \int \mathrm{d}\mu_\gamma(\hat{\zeta}) e^{-\mathcal{V}(\hat{\phi}' + \hat{\zeta})} \tag{4.82}$$

$$\hat{\mathcal{L}}_{\mathcal{V}} \hat{\mathcal{O}}(\hat{\phi}') = \frac{\int \mathrm{d}\mu_\gamma(\hat{\zeta}) e^{-\mathcal{V}(\hat{\phi}' + \hat{\zeta})} \hat{\mathcal{O}}(\hat{\phi}' + \hat{\zeta})}{\int \mathrm{d}\mu_\gamma(\hat{\zeta}) e^{-\mathcal{V}(\hat{\phi}' + \hat{\zeta})}}. \tag{4.83}$$

Both equations can be reduced to integrals of the type

$$\mathcal{I}_F(\hat{\phi}') = \int \mathrm{d}\mu_\gamma(\hat{\zeta}) F(|\hat{\phi}' + \hat{\zeta}|) \tag{4.84}$$

with certain scalar functions $F(\varphi)$. A shift $\hat{\zeta} \to \hat{\phi}' + \hat{\zeta}$ in the fluctuation field yields

$$\mathcal{I}_F(|\hat{\phi}'|) = (2\pi\gamma)^{N/2} \int \mathrm{d}^N\hat{\zeta} \exp\left(-\frac{1}{2\gamma_{\text{ml}}}[\hat{\zeta}^2 + \hat{\phi}'^2] + \frac{1}{\gamma_{\text{ml}}}\hat{\phi}'\hat{\zeta}\right) F(|\hat{\zeta}|). \tag{4.85}$$

By using polar coordinates and integrating out the polar angles we find for $N = 3$

$$\mathcal{I}_F(|\hat{\phi}'|) = \mathcal{N} \int_0^\infty \mathrm{d}R \exp\left(-\frac{1}{2\gamma_{\text{ml}}}[R^2 + |\hat{\phi}'|^2]\right) \sinh\left(\frac{|\hat{\phi}'|R}{\gamma}\right) \frac{RF(R)}{|\hat{\phi}'|}. \tag{4.86}$$

In this form the integral can be evaluated by standard numerical methods.

Equipped with integrators for (4.82) and (4.83) the strategy goes a follows: The first step is to determine $\mathcal{V}_{\text{RT}}$. For this purpose we start with the perturbatively improved action $\mathcal{V}_{\text{RT}}^{\text{pert}}(\hat{\phi}|r_0)$ at a given radius r_0 as bare potential and iterate the RG transformation (4.82). In each step the resulting potential is driven closer and closer towards the RT. After 10 steps we end up with a good approximation of $\mathcal{V}_{\text{RT}}(r)$ at some radius $r = r(r_0)$. In this iterative process the potential $\mathcal{V}(\varphi)$ is represented as a cubic spline with N_φ equidistant knots φ_i in the range $I_\varphi = [\varphi_{\min}, \varphi_{\max}]$. For each iteration one has to evaluate $\mathcal{V}' = \hat{\mathcal{R}}\mathcal{V}$ at these points. The interval I_φ must to be chosen in such a way that for the computation of $\mathcal{V}'(\varphi \in I_\varphi)$ contributions $\mathcal{V}(\varphi \notin I_\varphi)$ are numerically negligible in (4.82).

In the second step the eigenvalues and eigenoperators of the linearized RG transformation $\hat{\mathcal{L}}_{\text{RT}}$ at this very potential $\mathcal{V}_{\text{RT}}(r)$ are computed. The operator $\hat{\mathcal{L}}_{\text{RT}}$ acts on the infinite dimensional space of observables. Naturally a computer can only handle the restriction of $\hat{\mathcal{L}}_{\text{RT}}$ to a finite subspace. Our program uses the space spanned by the operators $B_m(\hat{\phi}|r) = (|\hat{\phi}| - r)^m$ with $m = 0, \ldots, M$. To obtain the representation matrix $L_{n,m}$ of $\hat{\mathcal{L}}_{\text{RT}}$ in this basis the image of every $B_m(\hat{\phi}|r)$ under $\hat{\mathcal{L}}_{\text{RT}}$ is numerically expanded by a finite difference method in terms of $B_n(\hat{\phi}|r')$

$$\hat{\mathcal{L}}_{\text{RT}} B_m(\phi|r) = \sum_{n=0}^{M} L_{n,m} B_n(\phi|r'). \tag{4.87}$$

Finally, the desired eigenvectors $\hat{\mathcal{O}}_m(r') = \sum_{n=0}^{M} Q_{n,m}(r') B_n(r')$ and their eigenvalues $\epsilon_n(r')$ are calculated from $L_{n,m}$. Fig. 4.6 shows the flow of the largest eigenvalues $\epsilon_m(r')$. As in the case of the potential the perturbative predictions are in excellent agreement with the numerical results down to a radius of about $r' \approx 4$. Then nonperturbative effects show up forcing the eigenvalues to become smaller. A similar behaviour can be found for the expansions coefficients $Q_{n,m}(r')$ except that deviations already show up at $r' \approx 10$.

At $r = r_{\text{cr}} \approx 2.04$ the effective radius $r'(r)$ vanishes and r ceases to be an appropriate parametrization of the RT but the eigenvalues and eigenvectors continue to flow against their fixpoint values at the HT-fixpoint.

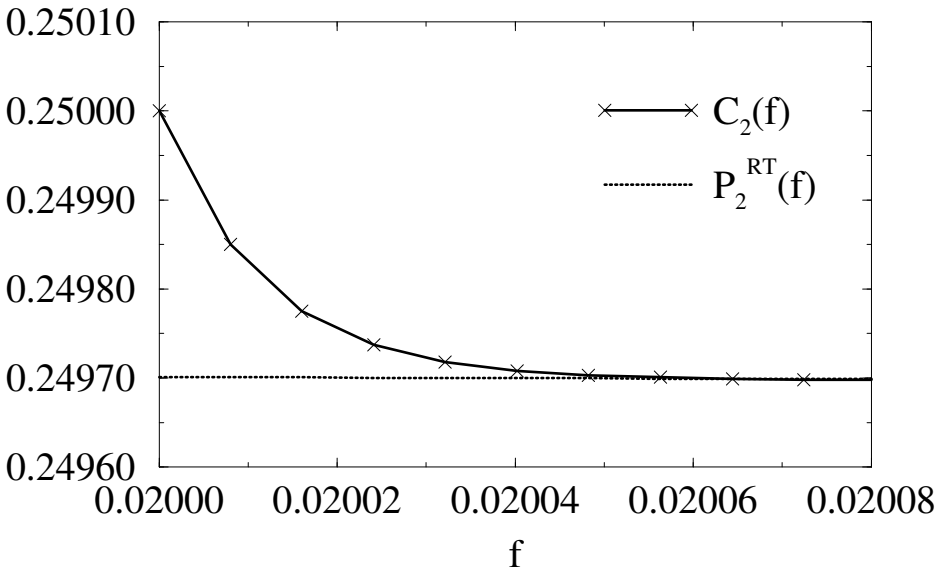

Figure 4.1: *Flow of the coupling $C_2(f)$ towards its renormalized value $P_2^{\mathrm{RT}}(f)$ under consecutive applications of the RG transformation. The calculation was performed with a bare radius of 50, $\gamma = 2$. The other couplings $C_a(f)$ with $2 < a \leq 6$ behave similarly.*

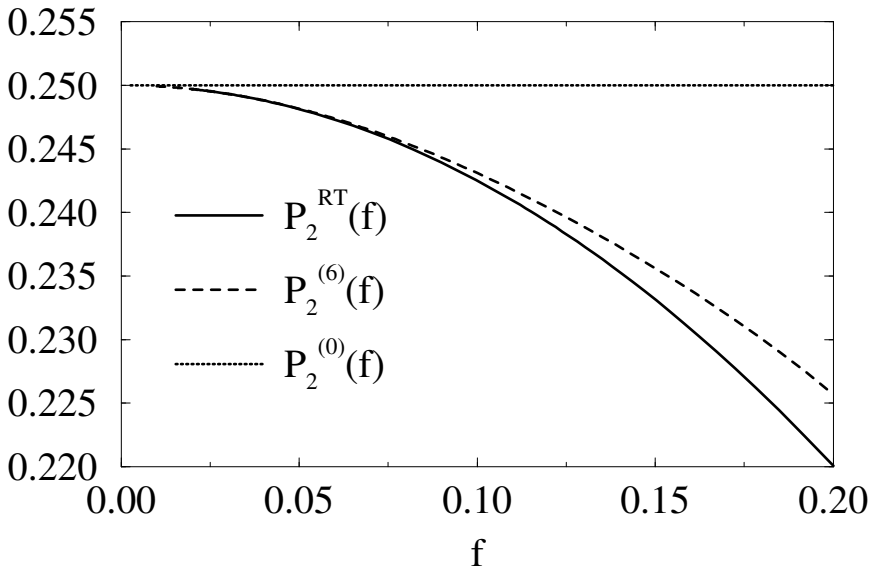

Figure 4.2: *Comparison of the perfect renormalized coupling $P_2^{\mathrm{RT}}(f)$ with its improved and unimproved approximations $P_2^{(6)}(f)$,$P_2^{(0)}(f)$*

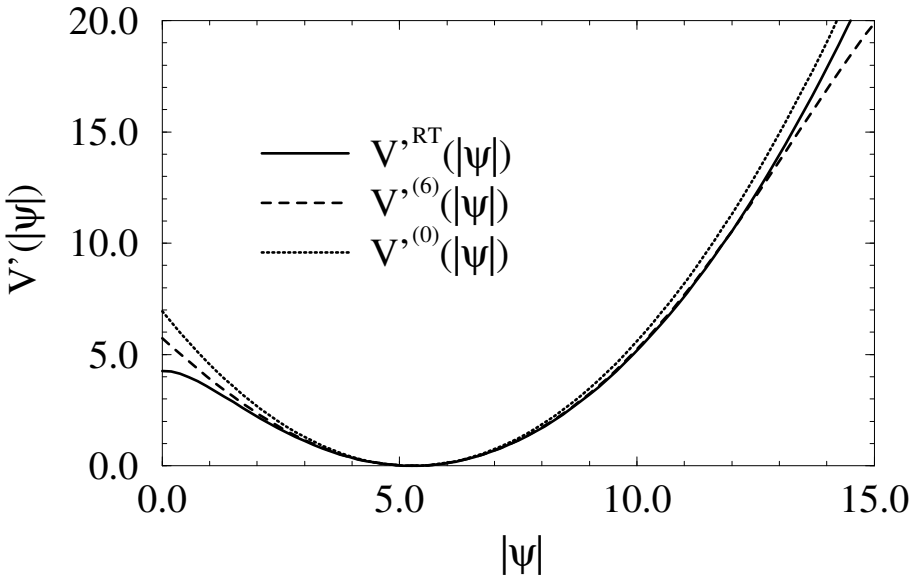

Figure 4.3: *Large field behavior of the effective potential.*

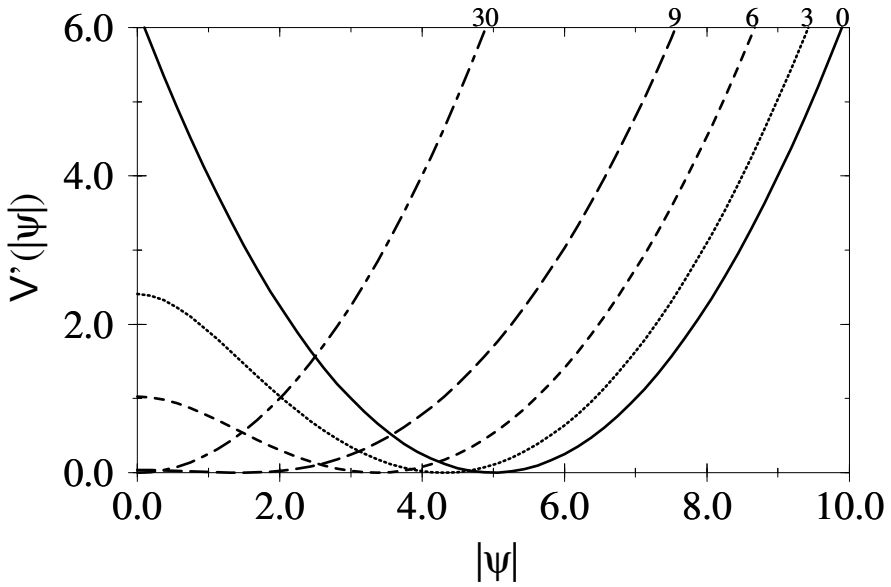

Figure 4.4: *Flow of the effective potential. The plot shows $V'(|\psi|)$ after n=0,3,6,9 and 30 RG steps starting with the unimproved Perfect Action approximation $V^{(0)}(|\psi|) = 1/(2\gamma)(|\psi| - r)^2$ at $r = 5, \gamma = 2$ and ending in the HT fixpoint $V_{HT}(|\psi|) = 1/(2\gamma)|\psi|^2$.*

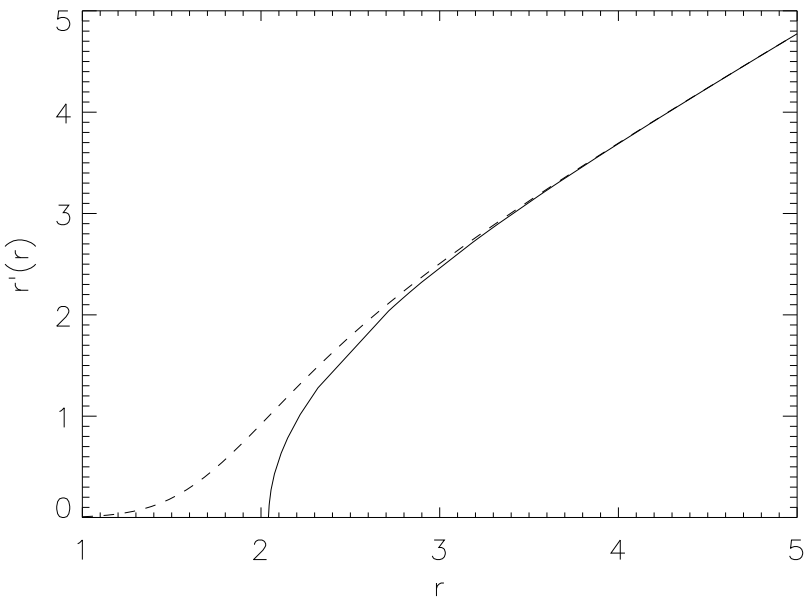

Figure 4.5: *Flow of the radius in the IR. Perturbative (dashed) and numerical (solid) result. At* $r_c \approx 2.04$ *the numerical effective radius vanishes with* $r' \approx |r - r_c|^{0.5}$.

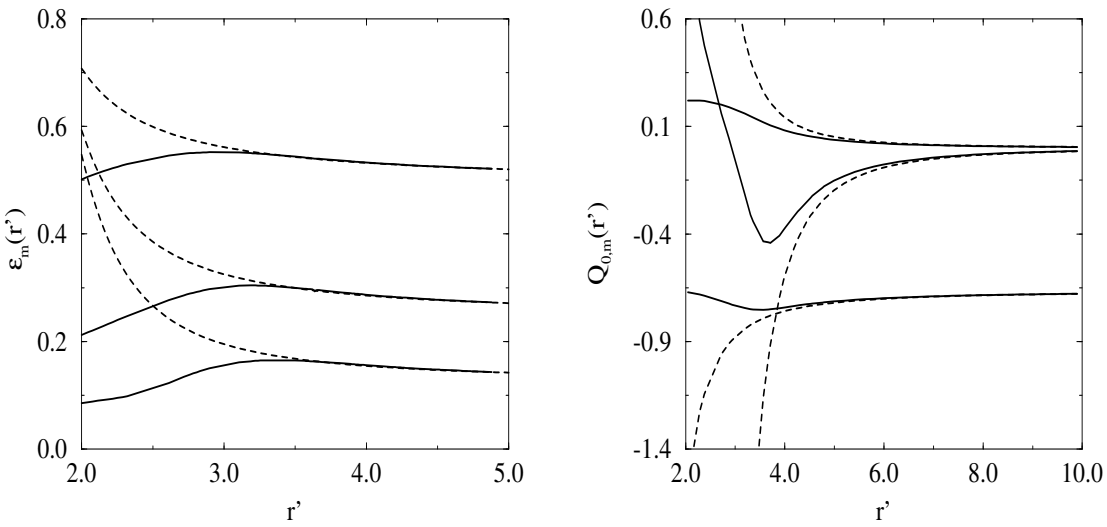

Figure 4.6: *Flow of the eigenvalues $\epsilon_m(r')$ and the expansion coefficients $Q_{n,m}(r')$ along the renormalized trajectory. From top to down $\epsilon_1, \epsilon_2, \epsilon_3$ and $Q_{0,1}, Q_{0,3}, Q_{0,2}$. Solid lines represent numerical data, dashed lines perturbative results. Calculations were done with $N_\varphi = 200, \varphi_{\min} = 0, \varphi_{\max} = 30, M = 10$.*

Chapter 5

The full model

In this chapter the RT for the full two dimensional $O(N)$ symmetric non-linear σ model is computed to first order perturbation theory.

5.1 The RG transformation

Now let ϕ be a N-dimensional field on Ω with an $O(N)$ invariant action $S[\phi]$. Let r denote the constant value of $|\phi|$ that minimizes $S[\phi]$.

$$\left.\frac{\delta S[\phi]}{\delta \phi(z)}\right|_{\phi(z)=r\,e_N} = 0 \tag{5.1}$$

As in the scalar case we define a Renormalization Group transformation $S' = \mathcal{R}S$ via

$$e^{-S'[\phi']} \sim \int \mathcal{D}\phi\, \delta_\kappa[\mathcal{C}\phi - \phi'] e^{-S[\phi]}. \tag{5.2}$$

Our ultimate goal is to find the renormalized trajectory of the non-linear $O(N)$-invariant σ-model in perturbation theory. This means that up to a flow of the coupling $f' = \beta(f)$ the action on the $S_{\text{RT}}[\phi|f]$ should be invariant with respect to $\mathcal{R}$:

$$\mathcal{R}S_{\text{RT}}[\phi|f] = S_{\text{RT}}[\phi|\beta(f)] \tag{5.3}$$

According to chapter 3 the RG transformation for the interaction $V[\phi]$ with respect to some bare propagator v reads

$$e^{-V'[\phi']} = e^{-(\mathcal{R}V)[\phi']} = \int d\mu_\Gamma[\zeta] e^{-V[A\phi'+\zeta]}. \tag{5.4}$$

By the same line of arguments as in the hierarchical case 4.3 the Perfect Action is obtained after a suitable rescaling from the fixed-point of the classical RG transformation

$$V_{\text{PA}}[\phi] = \mathcal{R}_{\text{cl}} V_{\text{PA}}[\phi] = \inf_{\zeta} \left\{ \frac{1}{2}(\zeta\,, \Gamma^{-1}\zeta) + V_{\text{PA}}[A\phi + \zeta] \right\}. \tag{5.5}$$

Unfortunately this equation can not be solved analytically in the full model. In [HN94] the analogous equation is treated by semi-numerical methods. This could be done here in principle, too. But we take a perturbative approach instead.

We will now perform a sequence of manipulations that finally give rise to an equivalent but for our purposes more convenient form of the RG transformation.

1. First we split up the field into components $\phi = \sigma + \pi$ with $\sigma = |\sigma| e_N$ and $\sigma\pi = 0$. Here e_N denotes the unit vector in the N direction leading to a blockspin independent split.

   The RG transformation then reads

   $$e^{-S'[\sigma',\pi']} \sim \int \mathcal{D}\sigma\mathcal{D}\pi\delta_\kappa[\mathcal{C}\sigma - \sigma']\delta_\kappa[\mathcal{C}\pi - \pi'] e^{-S[\sigma,\pi]}. \tag{5.6}$$

   The action $S[\sigma, \pi]$ is supposed to be invariant under

   $$\sigma \to \sigma - (\delta\omega\pi)\, e_N \qquad \pi \to \pi + \delta\omega\sigma. \tag{5.7}$$

2. Then we seperate off the massless kinetic fixed point term from the action and treat the potential at the shifted radial field $\sigma_r = \sigma - r$ with $r = re_N$.

   $$S[\sigma, \pi] = \frac{1}{2}(\sigma + \pi\,, v_{\text{ml}}^{-1}\,[\sigma + \pi]) + V_{\text{ml}}[\sigma_r\,, \pi] \tag{5.8}$$

   yielding the transformation

   $$e^{-V'_{\text{ml}}[\sigma'_r,\pi']} = \int d\mu_{\Gamma_{\text{ml}}}[\rho] d\mu_{\Gamma_{\text{ml}}}[\xi] \exp\{-V_{\text{ml}}[A_{\text{ml}}\sigma'_r + \rho, A_{\text{ml}}\pi' + \xi]\} \tag{5.9}$$

3. Now we treat the shifted radial field σ_r at the high-temperature fixed-point.

   $$V_{\text{ml}}[\sigma_r\,, \pi] = \frac{1}{2}(\sigma_r\,, K^*_{\text{ml}}\sigma_r) + V^{[1]}[\sigma_r\,, \pi] \tag{5.10}$$

   The interaction $V^{[1]}[\sigma_r\,, \pi]$ must be chosen in such a way that $V_{\text{ml}}[\sigma_r\,, \pi]$ is invariant under

   $$\sigma \to \sigma - (\delta\omega\pi)\, e_N \qquad \pi \to \pi + \delta\omega(\sigma_r + r). \tag{5.11}$$

   From (5.9) we get

   $$e^{-V'_{\text{ml}}[\sigma'_r,\pi']} = \int d\mu_{\Gamma_{\text{ml}}}[\rho] d\mu_{\Gamma_{\text{ml}}}(\xi)$$
   $$\exp\left\{ -\frac{1}{2}([A_{\text{ml}}\sigma'_r - r]\,, K^*_{\text{ml}}\,[A_{\text{ml}}\sigma' - r]) - V^{[1]}[A_{\text{ml}}\sigma'_r - r + \rho, A_{\text{ml}}\pi' + \xi] \right\} \tag{5.12}$$

Let $\psi_r = A_{\rm ml}\sigma' - r$. We now specialize to $D = 2$. Since in this case $\alpha_{\rm ml} = -\frac{D-2}{2} = 0$ we find $A_{\rm ml} r = s^{-\alpha_{\rm ml}} r = r$ and hence $\psi_r = A_{\rm ml}\sigma'_r$. As in section 3.3 we shift the radial fluctuation field $\rho \to (\mathsf{L}_* - \mathbb{I})\psi_r + \rho$ to obtain

$$\exp\{-V'_{\rm ml}[\sigma', \pi']\} = \exp\left\{-\frac{1}{2}(\sigma'_r, K^*_{\rm ml}\sigma'_r)\right\} \int d\mu_{\Gamma_{\rm ht}}[\rho] d\mu_{\Gamma_{\rm ml}}[\xi] \exp\left\{-V^{[1)}[A_{\rm ht}\sigma'_r + \rho, A_{\rm ml}\pi' + \xi]\right\}. \quad (5.13)$$

This transformation $\mathcal{R}$ is the starting point for our perturbative setup.

5.2 $O(N)$ Invariance

Before we start with perturbation theory let us explore what kind of restrictions the $O(N)$ invariance of the action $S[\sigma, \pi]$ imposes on

$$V_{\rm ml}[\sigma, \pi] = \sum_{i=0}^{\infty} V^{[i]}[\sigma, \pi] = \underbrace{\frac{1}{2}(\sigma_r, K^*_{\rm ml}\sigma_r)}_{V^{[0]}[\sigma,\pi]} + V^{[1)}[\sigma_r, \pi] \quad (5.14)$$

and therefore on $V^{[1)}[\sigma_r, \pi]$. From eq. (5.11) we deduce the following Ward identities for $V_{\rm ml}[\sigma, \pi]$

$$\left[-\left(\pi_a, \frac{\delta}{\delta\sigma_r}\right) + \left(\sigma_r + r, \frac{\delta}{\delta\pi_a}\right)\right] V_{\rm ml}[\sigma_r, \pi] = 0. \quad (5.15)$$

To order t this reads

$$\left[-\left(\pi_a, \frac{\delta}{\delta\sigma_r}\right) + \left(\sigma_r + r, \frac{\delta}{\delta\pi_a}\right)\right] V^{[t]}(\sigma_r, \pi) = -\left(1, \frac{\delta}{\delta\pi_a}\right) V^{[t+1]}[\sigma_r, \pi]. \quad (5.16)$$

Using $K^*_{\rm ml} = K^{*T}_{\rm ml}$ one finds for $t = 0$

$$-(\sigma_r, K^*_{\rm ml}\pi_a) = -\left(1, \frac{\delta}{\delta\pi_a}\right) V^{[1]}(\sigma_r, \pi). \quad (5.17)$$

This equation has the solution

$$V^{[1]}[\sigma_r, \pi] = \frac{1}{2}(\sigma_r, K^*_{\rm ml}\pi^2). \quad (5.18)$$

5.3 Perturbation Theory

Let us start with with zeroth order in $f = \frac{1}{r}$. From eq. (5.13) we immediately conclude that to this order the interaction

$$V^{(0)}_{\rm RT}[\sigma_r, \pi] = \frac{1}{2}(\sigma_r, K^*_{\rm ml}\sigma_r) \quad (5.19)$$

is RG invariant if we define

$$r' := r. \tag{5.20}$$

This is the same situation as in the hierarchical model (4): The radial field is at the HT fixed-point whereas the angular field resides at the ML fixed-point. Note that $V_{\text{RT}}^{(0)}$ is not $O(N)$ invariant but the corrections are of order $O(f)$.

As mentioned in the last section the first order improvement is dictated by $O(N)$-invariance giving

$$V_{\text{RT}}^{[1]}[\sigma_r\,,\pi] = \frac{1}{2}(\sigma_r\,, K_{\text{ml}}^*\pi^2) \tag{5.21}$$

A first order calculation of the effective interaction yields

$$\begin{aligned}
&\int d\mu_{\Gamma_{\text{ht}}}[\rho]d\mu_{\Gamma_{\text{ml}}}[\xi]\,\exp\left\{-V^{[1)}[A_{\text{ht}}{\sigma'}_r+\rho,\,A_{\text{ml}}\pi'+\xi]\right\}\\
&= \quad -\int d\mu_{\Gamma_{\text{ht}}}(\rho)d\mu_{\Gamma_{\text{ml}}}(\xi)\frac{f}{2}\left(\left\{A_{\text{ht}}{\sigma'}_r+\rho\right\}K_{\text{ml}}^*\left\{A_{\text{ml}}\pi'+\xi\right\}^2\right)\\
&= \quad -\frac{f}{2}\left(A_{\text{ht}}{\sigma'}_r\,, K_{\text{ml}}^*\left\{A_{\text{ml}}\pi'\right\}^2\right)-\frac{f}{2}(N-1)\int\limits_{z_1,z_2}(A_{\text{ht}}{\sigma'}_r)(z_1), K_{\text{ml}}^*(z_1-z_2)\Gamma_{\text{ml}}(z_2\,,z_2)
\end{aligned}$$

The linear term in ${\sigma'}_r$ results in a shift of the radius.

So the net effect of an RG transformation is that each ${\sigma'}_r$ acquires the convolution with an A_{ht} kernel and each π' acquires the convolution with an A_{ml} kernel.

To find the invariant form of the interaction we apply the same technique as in 3.4 and perform the scaling limit. After one RG step the interaction can be written as

$$\begin{aligned}
&\frac{f}{2}\int\limits_{\substack{z_1,z_2\\x_1,x_2,x_3}}{\sigma'}_r(x_1){A_{\text{ht}}}^T(x_1\,,z_1)K_{\text{ml}}^*(z_1\,,z_2)A_{\text{ml}}(z_2\,,x_2)\pi'(x_2)A_{\text{ml}}(z_2\,,x_3)\pi'(x_3) = \\
&\qquad \frac{f}{2}\int\limits_{\substack{z_1,z_2\\x_1,x_2,x_3}}{\sigma'}_r(x_1){\mathcal{A}_{\text{ht}}}^T(x_1\,,\mathsf{z}_1)s^{2D+\alpha_{\text{ht}}+2\alpha_{\text{ml}}}K_{\text{ml}}^*(s\mathsf{z}_1\,,s\mathsf{z}_2)\mathcal{A}_{\text{ml}}(\mathsf{z}_2\,,x_2)\mathcal{A}_{\text{ml}}(\mathsf{z}_2\,,x_3)\pi'(x_2)\pi'(x_3)
\end{aligned} \tag{5.22}$$

In $D=2$ we obtain

$$s^{2D+\alpha_{\text{ht}}+2\alpha_{\text{ml}}}K_{\text{ml}}^*(s\mathsf{z}_1\,,s\mathsf{z}_2)\overset{(D=2)}{=}s^2K_{\text{ml}}^*(s\mathsf{z}_1\,,s\mathsf{z}_2)=\mathcal{K}_{\text{ml}}^*(\mathsf{z}_1\,,\mathsf{z}_2) \tag{5.23}$$

This is exactly the same situation as in eq. (3.4) where we found

$$\lim_{s\to\infty}\mathcal{K}_{\text{ml}}^* = \kappa_*\mathbb{I}. \tag{5.24}$$

So we finally end up with the first order RT potential

$$V_{\text{RT}}^{(1)}[\sigma_r\,,\pi] = \frac{1}{2}\kappa_*(\mathcal{A}_{\text{ht}}{}^{(\infty)}\sigma_r\,,\mathcal{A}_{\text{ml}}{}^{(\infty)}\sigma_r)+\frac{f}{2}\kappa_*\left(\mathcal{A}_{\text{ht}}{}^{(\infty)}\sigma_r\,,\left\{\mathcal{A}_{\text{ml}}{}^{(\infty)}\pi\right\}^2\right). \tag{5.25}$$

Note that this potential is not $O(N)$ invariant to first order in f but the corrections to fix this are of order $O(f)^2$.

For numerical purposes it is convenient to write this in a manifest $O(N)$ invariant form. With the decomposition

$$\phi = \sigma + \pi = \sigma e + \pi = (r + \sigma_r)e + \pi \tag{5.26}$$

$$\tag{5.27}$$

we find

$$|\phi| = r + \sigma_r + \frac{f}{2}\pi^2 + O(f^2) \tag{5.28}$$

$$1 - (f|\phi|)^{-1} = f\sigma_r + \frac{f^2}{2}\pi^2 + O(f^2) \tag{5.29}$$

$$\left(1 - (f|\phi|)^{-1}\right)\phi = \left(\sigma_r + \frac{f}{2}\pi^2\right)e + f\sigma_r\left(\sigma_r + \pi\right) + O(f^2). \tag{5.30}$$

Therefore both

$$V_{\text{RT}}^{(1)}[\phi] = \frac{1}{2}\left(\{|\phi| - r\}\,, K_{\text{ml}}^{*}\{|\phi| - r\}\right) \tag{5.31}$$

and

$$V_{\text{RT}}^{(1)}[\phi] = \frac{1}{2}\left(\left\{\phi - r\frac{\phi}{|\phi|}\right\}\,, K_{\text{ml}}^{*}\left\{\phi - r\frac{\phi}{|\phi|}\right\}\right) \tag{5.32}$$

have the same first order expansion eq. (5.25).

5.4 Simulations

In this section the improved action is simulated. First the technical aspects of the simulations are explained. Then the scaling properties of the action are presented.

What has been accomplished so far? We have found a first order perturbative approximation to the renormalized trajectory which is RG invariant up to the flow in the coupling. In the hierarchical model this approximation already had very good scaling properties for sufficiently small values of the coupling f. We are therefore tempted to test the approximation for the full model at this early stage of perfection numerically by means of a Monte-Carlo simulation.

How can scaling be tested? The closer an action lies to the RT the better it describes continuum physics. HASENFRATZ and NIEDERMAYER gave an extremly impressive demonstration of their Perfect Action by simulating it on a 3×3 lattice. For a certain observable they obtained expectation values that were compatible with the continuum extrapolations gained from simulations with the standard action on much larger lattices.

It is by no means clear that all observables should exhibit such fantastic scaling properties. This would only be the case if the associated improved observables were used.

For the sake of comparability we have chosen to use the same observable as HASENFRATZ and NIEDERMAYER: the step scaling function of M. LÜSCHER,P. WEISZ and U. WOLFF [LWW91].

Let $g^2(L) = m(L)L$ be the running coupling where $m(L)$ denotes the massgap of the theory on a finite lattice with (spacial) length $L = Na$ for some value of the bare coupling β. The step scaling function $\Sigma(s, g^2, 1/N)$ relates the running coupling at length L with the running coupling at length $L' = sL$

$$g^2(sL) = \Sigma(s, g^2(L), a/L). \tag{5.33}$$

The continuum step scaling function $\sigma(s, g^2) = \lim_{a\to 0} \Sigma(s, g^2(L), a/L)$ is obtained by interpolating a sequence of measurements with fixed (physical) coupling and decreasing a/L.

Practically this means the following: Choose a lattice size N. Tune the bare coupling β until the measured running coupling takes a prescribed value. For historical reasons this is $g^2 = 1.0595$. Then simulate the theory at the same bare coupling but with $N' = 2N$ this yields $\Sigma(2, 1.0595, 1/N)$. Iterate the procedure with increasing lattice sizes and try to extrapolate to the continuum limit.

To test our improved action in this sense we need the following ingredients:

- An efficient Monte-Carlo updater for the action
- A method to measure the massgap
- A method to tune the bare coupling
- Analysis software to compute the errors.

These tasks are closely related and will be discussed now. It should be kept in mind that the software is supposed to serve as framework for future explorations of more complicated actions. This requires a certain level of generality and motivates the effort put into its development. As mentioned in the introduction the design of the programs was a case study for the usage of the programming language C++ in high-end numerics. The conclusions from this endeavour are presented in [SX96].

5.4.1 Monte Carlo Updater

We want to simulate a theory with $O(N)$-invariant action $S_{\text{RT}}[\phi]$.

$$S_{\text{RT}}[\phi] = \frac{1}{2}(\phi, \Box_{\text{ml}}\phi) - \frac{1}{2}(|\phi| - r, K^*_{\text{ml}}\{|\phi| - r\}) \tag{5.34}$$

After rescaling the field $\phi \rightarrow \frac{\phi}{r^2}$ and introducing

$$\beta = r^2 \tag{5.35}$$

this reads

$$S_{\text{RT}}[\phi] \quad = \quad \frac{\beta}{2}(\phi, \Box_{\text{ml}}\phi) - \frac{\beta}{2}(|\phi| - 1, K^*_{\text{ml}}\{|\phi| - 1\}). \tag{5.36}$$

The covariances $\Box_{\text{ml}} = v_{\text{ml}}^{-1}$ and $K^*_{\text{ml}} = v_{\text{nt}}^{-1} - v_{\text{ml}}^{-1}$ have more than nearest-neighbor interactions (see 3). This has to be taken into consideration when designing the updater.

Field Decomposition

It is convenient to decompose the Field $\phi = \rho\sigma$ into a radial field $\rho(z) \in \mathbb{R}_+$ and an angular field $\sigma(z) \in S_{N-1}$. This transformation of variables induces a change of the measure $\mathcal{D}\phi = \mathcal{D}\rho\rho^{N-1}\mathcal{D}\sigma$. Introducing

$$S_{\text{RT}}[\rho, \sigma] = S_{\text{RT}}[\rho\sigma] - (N-1)\int_z \ln \rho(z) \tag{5.37}$$

the partition function Z can be written as

$$Z \sim \int \mathcal{D}\phi e^{-S_{\text{RT}}[\phi]} \sim \int \mathcal{D}\rho\mathcal{D}\sigma e^{-S_{\text{RT}}[\rho,\sigma]}. \tag{5.38}$$

We are seeking a Monte-Carlo update procedure that generates configurations (ρ, σ) with the stationary distribution $\exp(-S_{\text{RT}}[\rho, \sigma])$. Let $\mathcal{P}(\rho', \sigma' \leftarrow \rho, \sigma)$ be the corresponding transition probability from an initial configuration (ρ, σ) to a final configuration (ρ', σ'). The stationarity condition reads

$$\exp(-S_{\text{RT}}[\rho', \sigma']) = \int \mathcal{D}\rho\mathcal{D}\sigma\mathcal{P}(\rho', \sigma' \leftarrow \rho, \sigma)\exp(-S_{\text{RT}}[\rho, \sigma]). \tag{5.39}$$

The MC procedure we used is obtained by consecutively applying a cluster update for the angular field and a Metropolis update for the radial field. Both of these algorithms fulfill detailed balance,i.e.

$$\mathcal{P}_{\mathrm{C}}(\rho,\sigma' \leftarrow \sigma)e^{-S_{\mathrm{RT}}[\rho,\sigma]} = \mathcal{P}_{\mathrm{C}}(\rho,\sigma \leftarrow \sigma')e^{-S_{\mathrm{RT}}[\rho,\sigma']} \tag{5.40}$$

$$\mathcal{P}_{\mathrm{M}}(\rho' \leftarrow \rho,\sigma)e^{-S_{\mathrm{RT}}[\rho,\sigma]} = \mathcal{P}_{\mathrm{M}}(\rho \leftarrow \rho',\sigma)e^{-S_{\mathrm{RT}}[\rho',\sigma]} \tag{5.41}$$

but the composition

$$\mathcal{P}(\rho',\sigma' \leftarrow \rho,\sigma) = \mathcal{P}_{\mathrm{M}}(\rho' \leftarrow \rho,\sigma')\mathcal{P}_{\mathrm{C}}(\rho,\sigma' \leftarrow \sigma) \tag{5.42}$$

generally does not. Nevertheless $\mathcal{P}$ leeds to the wanted stationary distribution as one can see by inserting eq. (5.42) into the rhs. of eq. (5.39) and making use of eqns. (5.40,5.41) and the normalization of $\mathcal{P}_{\mathrm{C}}, \mathcal{P}_{\mathrm{M}}$:

$$\int \mathcal{D}\sigma' \mathcal{P}_{\mathrm{C}}(\rho,\sigma' \leftarrow \sigma) = 1 \tag{5.43}$$

$$\int \mathcal{D}\rho' \mathcal{P}_{\mathrm{M}}(\rho' \leftarrow \rho,\sigma) = 1. \tag{5.44}$$

The same is true for

$$\mathcal{P}(\rho',\sigma' \leftarrow \rho,\sigma) = \mathcal{P}_{\mathrm{M}}^{m_{\mathrm{M}}}(\rho' \leftarrow \rho,\sigma')\mathcal{P}_{\mathrm{C}}^{m_{\mathrm{C}}}(\rho,\sigma' \leftarrow \sigma) \tag{5.45}$$

with $m_{\mathrm{C}}, m_{\mathrm{M}} \in \mathbb{N}_+$.

Cluster Algorithm

For the updating of the anguler field σ a modified WOLFF-Cluster-Algorithm ([Wol89]) was used. The modification is due to the more-than-nearest-neighbor-coupling of the action. For this reason not only bonds to the nearest-neighbors but to all coupled neighbors had to be considered for activation. Consider a spin $\phi(z)$ that has been flipped. The bond to a coupled site z' is then activated with the probability

$$p(\phi(z),\phi(z')) = 1 - \exp\left\{\min\left[0, -2\beta\Box_{\mathrm{ml}}(z,z')(R\phi(z))(R\phi(z')\right]\right\}. \tag{5.46}$$

Here R denotes the randomly chosen reflection plane for the one-cluster.

A recursive implementation for the algorithm was used. For analysis purposes the program keeps track of the value of the action. Since the computation of this value from scratch after every update is extremely costly only the differences due to an update were taken into account.

One has to be very careful when using a recursive algorithm in this situation, because the bond-energy that is computed at some stage may be changed in the course of the recursion.

Radial Metropolis Updater

As mentioned above the radial field ρ was updated with a standard 4-hit Metropolis procedure. Consider a local update $\phi \rightarrow \phi'$ with $\phi'(z) = \phi(z)$ for $z \neq z(0)$. For the change of the action one finds

$$\begin{aligned}
\Delta S[z_0|\phi' \leftarrow \phi] = & \\
& \frac{\beta}{2}\left[|\phi'(z_0)| - |\phi(z_0)|\right]\left(\Box_{\text{ml}}(0) - K^*_{\text{ml}}(0)\right) \\
+ \quad & \beta \int\limits_{z_1 \neq z_0} \left[\phi'(z_0) - \phi(z_0)\right] \Box_{\text{ml}}(z_0 - z_1)\phi(z_1) \\
- \quad & \frac{\beta}{2}\left(|\phi'(z_0)| - |\phi(z_0)|\right)\left[2K^*_{\text{ml}}(0) - 2\int\limits_{z_1 \neq z_0} K^*_{\text{ml}}(z_0 - z_1)\left\{|\phi(z_1)| - 1\right\}\right]
\end{aligned} \tag{5.47}$$

By inserting $\phi = \rho\sigma$ one easily obtains the change of the action for a pure radial update $\rho \rightarrow \rho'$. Eq. (5.47) for a full update was used to check the combined Cluster-Metropolis algorithm.

The ratio between Metropolis sweeps and Cluster updates was tuned manually in such a way that the number of local updates was approximately the same[1].

Local Randomization

Within the local updating routine random trial values for the field $\phi(z)$ are needed.

Random angular fields $\sigma(z)$ are drawn from a uniform distribution on the unit surface $\mathcal{S}_{N-1}$ with the usual method ([WPF92],[Kal90]): Generate N equidistributed random numbers $\varphi_i \in [0, 1)$ until $\varphi^2 \leq 1$. Then set $\sigma(z) = \varphi/||\varphi||$.

A trial radius ρ' is produced with reference to its old value ρ by

$$\rho' = |\rho - \tau/2 + \tau x| \tag{5.48}$$

with $x \in [0, 1)$ being a uniform (pseudo-) random number. The parameter τ can be tuned to optimize the accept/reject ratio of the local updater.

Interaction Covariances

The Perfect Laplacian $\Box_{\text{ml}}$ was computed in momentum space by means of eq. (B.18) and then transformed into position space with a FFT. Since eq. (B.18) is only valid on an infinite lattice, small corrections had to be made in order to keep the propagator massless. After initialization of the propagator with the infinite lattice values it was normalized such that $\int_z \Box_{\text{ml}}(z) = 0$ and $\int_z \Box_{\text{ml}}(z)z^2 = -4$ (see [HN94]). Both $\Box_{\text{ml}}$ and K^*_{ml} couple to more than just nearest-neigbors, actually their support is the entire lattice. Since they fall off rather fast (see (B)), the program allows to vary the range of interaction taken into account. Numerically 2 levels seem to be enough.

[1] On a lattice with M sites the Metropolis algorithm makes M local updates per sweep. If the average cluster size was C then M/C cluster updates were made.

5.4.2 Measurements

In this section we are specializing to $D = 2$. Lattice points $z \in \Omega$ are written as $z = (t, x)$. The temporal extent is denoted by T and the spacial extent by L.

Correlation Function

Let

$$\Phi(t) = \sum_{x=0}^{L-1} \phi(t, x) \tag{5.49}$$

be the timeslice field associated with the lattice field $\phi(z)$. $\Phi(t)$ is the partial fourier transform of $\phi(z)$ at zero spacial momentum.

The connected zero momentum timeslice correlation function is defined as

$$C(\tau) := \langle \Phi(t_0)\Phi(t_0 + \tau) \rangle_{\mathrm{c}}. \tag{5.50}$$

Massgap

We want to measure the massgap which is determined by the long range properties of C.

For $T = L = \infty$ the mass gap is given by

$$C(\tau) \sim e^{-m|\tau|}. \tag{5.51}$$

To determine the mass gap on a finite lattice with periodic boundary conditions both in temporal and spacial direction the following methods were used:

- By fitting $C(\tau)$ self-consistently to

  $$a_0 + a_1 \left[e^{-m\tau} + e^{-m(T-\tau)} \right] \tag{5.52}$$

  in the range $\tau = 2m^{-1} \ldots T/2$ [Wol89].

- In the effective mass method described in [MM94] the mass gap is computed from the ratios of the correlation function $r_{ij} = \frac{C(\tau_i)}{C(\tau_j)}$ at a fixed timeslice triple τ_1, τ_2, τ_3 by means of

  $$(r_{13} - 1)(x^{t_2} + x^{-t_2} - x^{t_3} - x^{-t_3}) = (r_{23} - 1)(x^{t_1} + x^{-t_1} - x^{t_3} - x^{-t_3}). \tag{5.53}$$

  Here we have used the notation $m = -\log x$ and $t_i = \frac{T}{2} - \tau_i$.

For both methods the measured observable was

$$D(\tau) = \frac{1}{T} \sum_{t=0}^{T-1} \Phi(\tau + t)\Phi(t) \tag{5.54}$$

with $\tau = 0, \ldots, T-1$. The two methods have the disadvantage that the decay of the correlation function is poluted by higher energy states of the transfer matrix. This problem is circumvented on a lattice with free boundary conditions in temporal direction (Assuming $T > 3L$).

- In this situation the method of [LWW91] is used. The massgap is there determined by

$$e^{-m(L)} = \frac{D(L)}{D(L)}. \tag{5.55}$$

Here the observable is

$$D(\tau) = \frac{1}{T-2L-\tau} \sum_{t=L}^{T-1-L-\tau} \Phi(\tau+t)\Phi(t). \tag{5.56}$$

Improved Estimators

In the case of the cluster updater the statistical error of the observables can be reduced by an order of magnitude when using improved cluster estimators [Wol89]

Let $\mathcal{O}(\phi)$ be an observable with average

$$< \mathcal{O}(\phi) >= \frac{1}{Z} \int \mathcal{D}\phi \mathcal{O}(\phi) e^{-S_{\mathrm{RT}}[\phi]}. \tag{5.57}$$

Then

$$\mathcal{O}_{\mathrm{C}}(\rho, \sigma) = \int \mathcal{D}\sigma' \mathcal{O}(\rho\sigma') \mathcal{P}_{\mathrm{C}}(\rho, \sigma' \leftarrow \sigma) \tag{5.58}$$

has the same average:

$$\begin{aligned}
< \mathcal{O}_{\mathrm{C}}(\rho, \sigma) > &= \frac{1}{Z} \int \mathcal{D}\rho \mathcal{D}\sigma \mathcal{O}_{\mathrm{C}}(\rho, \sigma) e^{-S_{\mathrm{RT}}[\rho, \sigma]} \\
&= \frac{1}{Z} \int \mathcal{D}\rho \mathcal{D}\sigma \int \mathcal{D}\sigma' \mathcal{O}(\rho\sigma') \mathcal{P}_{\mathrm{C}}(\rho, \sigma' \leftarrow \sigma) e^{-S_{\mathrm{RT}}[\rho, \sigma]} \\
&= \frac{1}{Z} \int \mathcal{D}\rho \mathcal{D}\sigma \int \mathcal{D}\sigma' \mathcal{O}(\rho\sigma') \mathcal{P}_{\mathrm{C}}(\rho, \sigma \leftarrow \sigma') e^{-S_{\mathrm{RT}}[\rho, \sigma']} \\
&= \frac{1}{Z} \int \mathcal{D}\rho \mathcal{D}\sigma' \mathcal{O}(\rho\sigma') e^{-S_{\mathrm{RT}}[\rho, \sigma']} \\
&= < \mathcal{O}(\phi) > .
\end{aligned} \tag{5.59}$$

Following the arguments of Wolff one finds for the two-point correlation function $\mathcal{O}(\phi) = \phi(z_1)\phi(z_2)$

$$\mathcal{O}_{\mathrm{C}}(\phi) = N \frac{|\Omega|}{|\mathrm{C}|} \Theta_{\mathrm{C}}(z_1) \Theta_{\mathrm{C}}(z_2) [R\phi(z_1)][R\phi(z_2)]. \tag{5.60}$$

β-shift

As described in 5.4 one has to tune the bare coupling β such that the running coupling takes some specific value. This can be done as follows [LWW91]: Because of

$$< \mathcal{O} >_{\beta_0+\Delta\beta} = \frac{< \mathcal{O} \exp(-\Delta S) >_{\beta_0}}{< \exp(-\Delta S) >_{\beta_0}} \tag{5.61}$$

the expectation value of $\mathcal{O}$ at $\beta = \beta_0 + \Delta\beta$ can be obtained by measuring at β_0. Here $\Delta S = \frac{\Delta\beta}{\beta_0} S$. As shown in [LWW91] an improved cluster estimator for

$$\mathcal{O}_{\Delta\beta}[\phi] = \phi(z_1)\phi(z_2) \exp(-\Delta S[\phi]) \tag{5.62}$$

is given by

$$\begin{aligned} \mathcal{O}_{\Delta\beta,\mathrm{C}} &= N \frac{|\Omega|}{|\mathrm{C}|} \Theta_{\mathrm{C}}(z_1)\Theta_{\mathrm{C}}(z_2)[R\phi(z_1)][R\phi(z_2)] \frac{1}{2} \left(e^{-\Delta S[\phi]} + e^{-\Delta S[\phi']} \right) \\ &\quad + N \frac{|\Omega|}{|\mathrm{C}|} \Theta_{\mathrm{C}}(z_1)(1-\Theta_{\mathrm{C}}(z_2))[R\phi(z_1)][R\phi(z_2)] \frac{1}{2} \left(e^{-\Delta S[\phi]} - e^{-\Delta S[\phi']} \right) \end{aligned} \tag{5.63}$$

Here ϕ' denotes the spin-flipped cluster configuration. For each measurement the $S[\phi]$ and $S[\phi']$ are written out.

Error Analysis

Because of the β-shift requirement we decided to split the simulation and the analysis totally. Therefore the simulation program did not perform any binning or averaging but wrote huge amounts (typically 100MB) of raw data on disk. The data were organized in rows of $D[0], D[1], \ldots, D[T/2-1], S$ for unimproved and $DC[0], DCC[1], \ldots, DCC[T/2-1], DCC[T/2-1], S, S'$ for improved correlation functions. In the latter case $DC[t]$ and $DCC[t]$ refer to sums over the cluster and sums over the complement of the cluster as given by eq. (5.63).

We have developed an interactive analysis package based on `IDL, Interactive Data Language` to perform the statistical analysis. Here a brief list of features:

- Computation of averages and errors of primary quantities by binning, integrated autocorrelation time, exponential autocorrelation time.
- Computation of averages and errors of secondary quantities with Jackknife and Bootstrap method. The function by which the secondary quantity is calculated may be chosen at runtime. This makes it easy to use e.g. different massgap definitions.
- Automatic handling of β-shifts. Some secondary functions not only take observable values but also their errors as argument (e.g. for fitting). Because of eq. (5.61) even shifted primary quantities are actually secondary ones. Therefore a Jackknife within the Jackknife is performed to compute the correct statistical errors for the subsequent application of the secondary function.
- Compact visualization of the relevant statistical information.

5.4.3 Numerical Tests

Simulation program and analysis program are rather complicated pieces of software. Before using them for production an extensive testing phase was run. In the following ϕ-Metropolis refers to a simulation where the full field ϕ was updated by a 4-hit Metropolis procedure. ρ-Metropolis + σ-Cluster means, that only the radial field ρ was updated with Metropolis, whereas for the angular part σ the cluster algorithm was used.

The Gaussian Model

For a pure Gaussian $S[\phi] = \frac{\beta}{2}(\phi, v^{-1}\phi)$ all expectation values can be computed analytically. As covariance we used the $\kappa = 8$ Perfect Laplacian $v^{-1} = v_{\text{ml}}^{-1} + M^2$. with additional mass $M = 1$ The two-point function yields

$$< \phi(z_1)\phi(z_2) >= N v(z_1 - z_2). \tag{5.64}$$

Tab. 5.1 shows the results. The different methods are in excellent agreement with the

Method	$v(0,1)$	$v(1,1)$
Exact	0.180188	0.107227
ϕ-Metropolis,$\beta = 1.0$	0.1801(3)	0.1074(3)
ϕ-Metropolis,$\beta = 1.1 \rightarrow 1.0$	0.1803(5)	0.1071(5)
ρ-Metropolis + σ-Cluster,$\beta = 1.0$	0.180(1)	0.108(1)
ρ-Metropolis + σ-Cluster,$\beta = 1.1 \rightarrow 1.0$	0.1803(5)	0.1074(5)
ρ-Metropolis + σ-Cluster, improved, $\beta = 1.0$	0.1812(8)	0.1076(7)

Table 5.1: *Results for the two-point function of a Gaussian action. Simulations were performed on a 8×8 lattice with periodic boundary conditions at $M = 1.0$. The average cluster size was 2.6. Too small for the cluster algorithm to outrun the 4-hit Metropolis.*

analytic prediction.

Mass Gap Determination

For a test of the mass gap determination we have recomputed values for the standard action.

In [Wol89] the massgap for the standard model on a periodic lattice at $\beta = 1.4$ were cited. The results in tab. 5.2 are in good agreement with the literature value. The effective mass method seems to be less stable than the massgap by self-consistent fitting.

In [LWW91] the massgap on a lattice with free boundary conditions in temporal direction were presented. These results could be reproduced as tab. 5.3 shows.

Judging from these numbers the software appears to work correctly.

Method	m
[Wol89]	0.1449(7)
mass fit	0.143(5)
eff. mass	0.143(9)

Table 5.2: *Comparison of the methods to determine the massgap with* WOLFF *for periodic lattice* $T = 64, L = 64, \beta = 1.4$. *Simulation was done with a cluster updater and improved observables*

Method	$m(5)$
[LWW91]	0.2119(1)
non-improved	0.214(3)
improved	0.2113(5)

Table 5.3: *Comparison with* LÜSCHER ET AL *for free temporal and periodic spacial boundary*$T = 64, L = 5, \beta = 1.5699$. *The Simulation was performed with a cluster updater and the massgap was determined with the ratio method.*

5.4.4 Numerical Scaling Results

After these lengthy preliminaries we are now able to perform the scaling test of HASENFRATZ and NIEDERMAYER for our improved action.

- The first task was to tune the bare coupling β such that $Lm(L) = 1.0598$. We choose $L = 5$ and measured the massgap. Fig. 5.1 shows that the massgap value $m(5) = 0.2119$ is compatible within our accuracy for bare couplings β_0 in the range $[1.8460, 1.8475]$.
- In a second step we performed at $\beta_0 = 1.84665$ a simulation on a lattice with $L = 10$. In the above β_0-range this yields values $\Sigma(2, 1.0595, 1/5) \in [1.364, 1.378]$ for the step scaling function.

These numbers have to be compared with standard results.
The usual standard action gives [LWW91]

$$\Sigma(2, 1.0595, 1/5) = 1.2905(10) \tag{5.65}$$

the Perfect Action [HN94] yields

$$\Sigma(2, 1.0595, 1/5) = 1.264(1). \tag{5.66}$$

The value for the continuum extrapolation of the step-scaling function is cited in [LWW91] as

$$\Sigma(2, 1.0595, 0 = 1.264(2). \tag{5.67}$$

In words: Our action is further away from the continuum limit than the standard action.

This is not totally surprising. Looking at eq. (5.22) we observe that the perturbation theory predicts asymptotic freedom for $N > 1$. Eq. (5.22) means that the running coupling does not change to first order: $f' = f + O(f)$.

Hence higher orders will have to be taken into account to improve the scaling properties. This work is in progress.

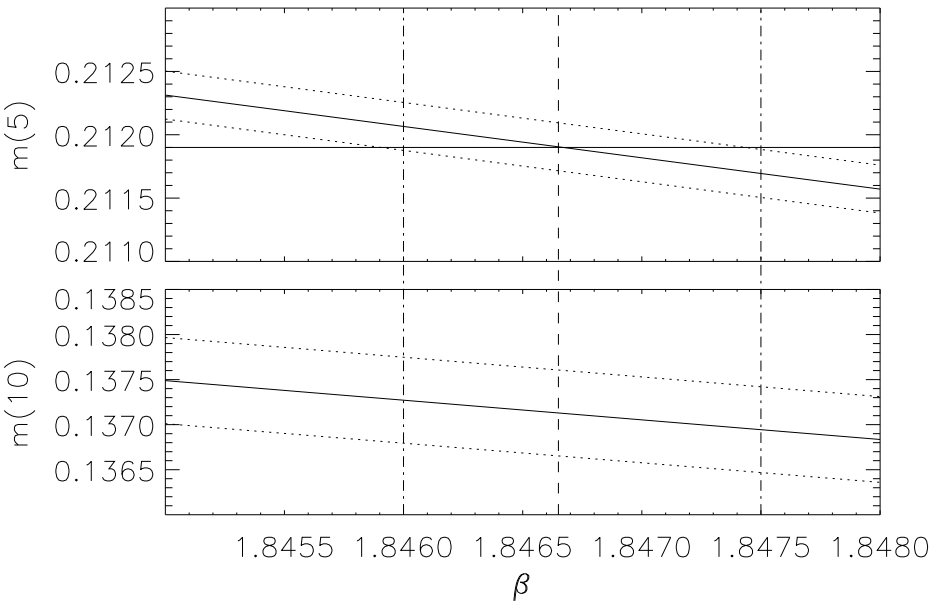

Figure 5.1: *β-shifted results for the massgap on $L = 5$ and $L = 10$. Simulation was done for $T = 32$ with free boundary conditions in temporal direction. The updates were performed with the mixed ρ-Metropolis+σ-Cluster algorithm at $\beta = 1.84665$. Simulation runs with the other updaters and the other massgap definitions gave similar results but with less accuracy.*

5.5 Conclusion

This part of the thesis is based on the idea to compute a renormalized trajectory in terms of a scaling analysis of a single renormalization group step [Wie96c]. It was worked out for the two dimensional $O(N)$ symmetric non-linear σ model in a running coupling expansion.

For the hierarchical approximation of the model it was possible to determine the renormalized trajectory and the associated improved observables to high orders of the

running coupling. By numerical means the range of validity of the method was determined to reach down to $f = 0.25$.

The setup for the full model has been fixed and the renormalized trajectory was computed to first order. A Monte-Carlo simulation revealed that the *improved* action scales even worse than the standard action. This negative result does not come totally unexpected. The absence of a wavefunction renormalization to this order leads to a wrong $N - 1$ asyomtotic freedom prediction. In a second order calculation with a continuous momentum cutoff RG of the POLCHINSKI-type [Pol84] C. WIECZERKOWSI has proven that the correct asymptotic $N - 2$ behaviour can be restored when an appropriate wavefunction renormalization is performed. From the latter and this work it is in principle clear what has to be done to get a nicely scaling lattice action. Unfortunately this involves the computation of complicated sums over lattice kernels. The software developed in course of this thesis will be the foundation for this future work. Despite the technical difficulties, the merrits of a lattice calculation should not be underestimated. The possibility to check the validity of the method by hard numerical numbers distinguishes the blockspin renormalization group from other RG approaches.

Part II

Mean Field Renormalization Group (MFRG) study of the two dimensional $O(N)$ symmetric non-linear σ model

Chapter 6

Introduction

From the conceptual point of view the RG is a beautiful tool to explore quantum field theory and critical phenomena. But whenever one tries to realize the RG in a computational scheme one has to deal with a potentially infinite dimensional space of couplings. This inevitably requires a parametrization and - because of finite computer resources - a truncation of the effective actions.[1] Usually this is done ad hoc by neglecting terms that are irrelevant in a perturbative sense. This is not really justified. The RT is a low-dimensional manifold parametrized by the renormalized (relevant) couplings. This means that when the renormalized couplings are given all the infinitely many non-relevant couplings are already *determined*. This is what renormalizability means. It does not mean that the non-relevant couplings are *zero*!

In [GMXP96] a new method for the calculation of RG flows on the lattice was presented. The key feature being a parametrization of the action that is preserved under an approximative RG transformation. The approximations have a limited range of validity. One advantage of the method is that it signals the breakdown of its own applicability. This is not be underestimated. Hereby even the non-applicablity of the scheme to a model is a physically relevant result.

In chapter 7 the essentials of the method for the scalar case are briefly summarized. Chapter 8 then contains first attempts to apply the method to the two dimensional $O(N)$ symmetric non-linear σ model .

The (computational) link to part I is the appearance of the GK kernels providing us with a framework to perform functional integrations over high-frequency fields. A comparison between the results of a MFRG and pure GK perturbation theory as used in part I may be found in [Gri97].

[1] As mentioned by HASENFRATZ and NIEDERMAYER in [HN94] this problem prevented the field of improved actions to become a numerical success for years. Their Perfect Action for the two dimensional $O(N)$ symmetric non-linear σ model requires the solution of the FP equation of a classical RG transformation which was performed semi-analytical. As stated in [HN94] the big success was only possible because they were able to parametrize the action in a suitable way

Chapter 7

MFRG for scalar fields

In this chapter a short presentation of the MFRG method for scalar fields is given.

The approximative RG scheme presented [GMXP96] is based on two approximations: Starting from a lattice action given by a certain parametrization the fluctuation integral is performed in a saddle-point approximation. It turns out that the RG transformation repects the parametrization. The result is a functional recursion relation for the action. By means of an additional localization approximation one obtains a simplified relation that can be solved by the computer. For details the reader is refered to [Gri97]. Here only those aspects are elucidated that are needed for the subsequent chapter. For pedagogical reasons we will use dimensionfull quantities in this part. Therefore the lattice Ω^i has a lattice spacing a_i etc.

7.1 Saddle Point Approximation

In contrast to part I a fixed-blockspin RG will be used now

$$e^{-S^{i+1}[\Phi]} = \int D\phi\, \delta(\mathcal{C}^i\phi - \Phi) e^{-S^i[\phi]}. \tag{7.1}$$

It can be obtained from eq. (3.4) in the limit $\kappa \to \infty$. This limit is an essential requirement for the method to work.

The index i denotes that the action $S^i[\phi]$ is obtained after i RG steps from a starting action S^0.

The background field ψ in the GK-formalism of chapter 3 was defined via eq. (3.7) as the field that minimized the *free* action subject to the (smeared) Gaussian constraint. This definition led to the linear relation $\psi = A\phi'$ between background and blockspin field. Additionally it induced a split of the base field into the high- and low-frequeny components $\phi = \psi + \zeta$.

Here the mean field Ψ^i is determined as the field that minimizes the *full* action

$$S^i[\Psi^i] = min$$

subject to the constraint

$$\mathcal{C}^i \Psi^i = \Phi. \tag{7.2}$$

This way of proceeding was proposed by T. BALABAN in his work on gauge theories [TB]. It has the advantage over the GK-scheme that it preserves manifest gauge invariance.

The functional relation between the background field and the blockspin field is in general no longer linear.

But with the method of Lagrange one obtains a nonlinear equation for $\Psi^i[\Phi]$

$$\begin{aligned} S^{i\prime}[\Psi^i] &= \mathcal{C}^{iT} \lambda^i, & (7.3)\\ \mathcal{C}^i \Psi^i &= \Phi & (7.4) \end{aligned}$$

where $\lambda^i[\Phi](x)$ are Lagrange multipliers, $x \in \Omega^{i+1}$. The prime denotes a derivative with respect to ϕ. Again the field is split

$$\phi(z) = \Psi^i[\Phi](z) + \zeta^i(z)$$

into background field $\Psi^i[\Phi]$ and fluctuation field ζ^i. But since we are now at the real minimum the expansion of the action around $\Psi^i[\Phi]$ has no linear term in ζ^i. Making a saddle-point approximation we can therefore compute the effective action analytically. The result is

$$S^{i+1}[\Phi] = S^i[\Psi^i[\Phi]] - \frac{1}{2}\mathrm{tr} \ln \Gamma^i[\Psi^i[\Phi]]. \tag{7.5}$$

This is the parametrization of the action mentioned above. In words: The action $S^{i+1}[\Phi]$ depends on the field Φ through a functional $\Psi^i[\Phi]$. To find $\Psi^i[\Phi]$ is the same crucial step as in the case of HASENFRATZ and NIEDERMAYER.

The fluctuation propagator is given by[1]

$$\Gamma^i[\Psi^i] = S^{i\prime\prime-1}[\Psi^i] - S^{i\prime\prime-1}[\Psi^i]\mathcal{C}^{iT}(\mathcal{C}^i S^{i\prime\prime-1}[\Psi^i]\mathcal{C}^{iT})^{-1}\mathcal{C}^i S^{i\prime\prime-1}[\Psi^i]. \tag{7.6}$$

and is the pseudo-inverse of $S^{i\prime\prime-1}[\Psi^i]$ on the space $\mathcal{H}^i_{\mathcal{C}}$ of lattice fields ζ^i with vanishing blockaverage $\mathcal{C}^i\zeta^i = 0$[2].

For $W^i[\phi] = -S^{i\prime\prime}[\phi]$ and $-S^{i\prime}[\phi]$ a recursion relation can be derived. They are depicted graphically in fig. 7.2,7.3 with the notation of fig. 7.1. This recursion relation refers to the derivative of the background field with respect to the blockspin field. Fortunatly a convenient formula for this quantity can be found:

$$\Psi^i{}_{,x}[\Phi](z) \equiv \frac{\delta}{\delta\Phi(x)}\Psi^i[\Phi] = (1 - \Gamma^i S^{i\prime\prime})\mathcal{C}^{iT}(z,x). \tag{7.7}$$

[1] This equation is to be compared with eq. (3.10). Replace $S^{i\prime\prime-1}$ by $T^{\prime\prime-1} = v$ and make use of $v' = \mathcal{C}v\mathcal{C}^T, A = v\mathcal{C}^T v'^{-1}$.

[2] These additional relations are the advantage of the fixed-blockspin method

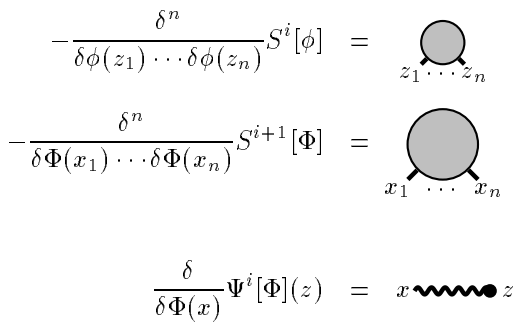

Figure 7.1: *Graphical notation for the derivatives of the action and the background field*

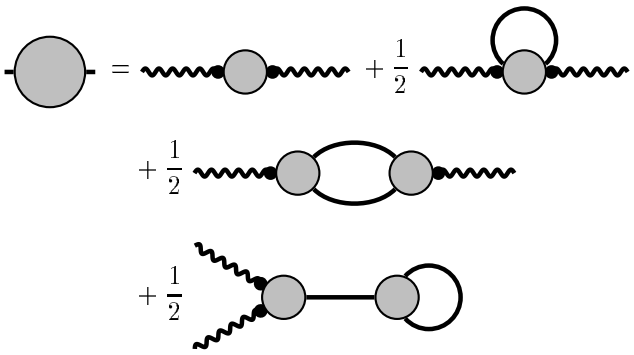

Figure 7.2: *Recursion relation for* $W^i = -S^{i\prime\prime}$

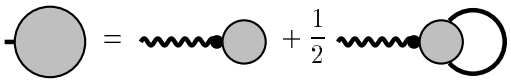

Figure 7.3: Recursion relation for $-S^{i+1\prime}$ in the mean-field approximation

7.2 Localization Approximation

We will now present a line of thought that introduces and justifies the localization approximation

1. Consider the action $S^{i+1}[\Phi]$ at level $^{i+1}$. It will contain a kinetic term that makes configurations with large derivatives very unlikely. We may therefore assume that Φ ist not very rough on scale a_{i+1}.

2. The background field $\Psi^i[\Phi]$ interpolates from the coarse lattice Ω^{i+1} to the finer lattice Ω^i. It is determined by extremizing the action $S^i[\phi]$. By the same argument as above the kinetic term in S^i will suppress high-frequency fluctuations and render the background field a smooth function on scale a_i.[3]. For big enough scaling factor $s = a_{i+1}/a_i$ we may therefore consider $\Psi^i[\Phi]$ as approximatly constant over distances a_i.

3. This leads to the essence behind the localization approximation: Since the action $S^{i+1}[\Phi]$ depends on the field Φ only through the background field $\Psi^i[\Phi]$ and since the latter behaves smoothly we claim: To compute $S^{i+1}[\Phi]$ we only have to evaluate $S^i[\phi]$ for nearly constant fields. More precisely one will need to compute the first and second derivative of S^i for constant fields.

4. As mentioned in chapter 1 the basic assumption of the RG is that it preserves locality. Hence $S^{i\prime\prime}[\phi](zw)$ should have the locality properties of a Laplacian: It is zero if z and w are seperated by more than a lattice spacing a_i. Because of argument 3 this means that $S^{i\prime\prime}[\Psi^i[\Phi]](zw)$ can only depend on $\Psi^i[\Phi]$ at these points. Hence it is justified to make the following hermitean localization approximation for W^i

$$-S^{i\prime\prime}[\Psi^i](z,w) = W^i[\Psi^i](z,w) \approx \frac{1}{2}\Big(\overline{W}^i(z,w|\Psi^i(z)) + \overline{W}^i(z,w|\Psi^i(w))\Big). \tag{7.8}$$

   The kernels $\overline{W}^i$ are supposed to behave like a Laplacian, too.

5. It is possible to compute S^{i+1} up to a constant. given $S^{i\prime}[\phi = \text{const}]$ and $S^{i\prime\prime}$. Hence it is sufficient to have recursion relations for these two.

6. The recursion relation for $W^{i+1}(x,y)$ (fig. 7.2) involves three different quantities:

   - Being the inverse of $S^{i\prime\prime}[\Psi^i]$ on the space of fields with $\mathcal{C}^i\zeta = 0$ the fluctuation propagator is expected to fall-off within one block lattice spacing a_{i+1}.
   - The kernels $W^i(z,w)$ have according to argument 4 a range of one lattice spacing a_i.
   - The derivative of the background field $\Psi^{i\prime}$ being determined by the latter two quantities has an exponential decay with decay length a_{i+1}.

[3] The interpolation operators A from part one may serve as example for this behaviour. See fig. 3.3

This shows that $W^{i+1}[\Phi]$ has similar locality properties on the scale of the new lattice spacing a_{i+1} as W^i had on the scale a_i and gives rise to a recursion relation for the localized kernels $\overline{W}^i$ [GMXP96]. These relations are obtained from the recursion relations fig. 7.2, 7.3.

In other words: We have found a localized parametrization of the action that is preserved under the flow of the RG in our approximation.

The assumptions that lead to the approximation may break down. A striking feature of the method is that such a breakdown can be monitored [Gri97]. Hence even a failure of the method can provide physical insight.

Chapter 8

MFRG for the two dimensional $O(N)$ symmetric non-linear σ model

In this chapter first attempts to apply the MFRG method to the two dimensional $O(N)$ symmetric non-linear σ model are presented.

Let us go through the RG recipe from chapter 1.2 and try to collect the ingredients.

8.1 Choice of the Blockspin

The first decision to be made is the choice of the blockspin.

In contrast to part I we will here use a non-linear averaging operator.

$$\phi'(x) = C[\phi](x) := \frac{(\mathcal{C}\phi)(x)}{|(\mathcal{C}\phi)(x)|} \tag{8.1}$$

which preserves the unit length of the fields.

This definition is equivalent to demanding that the blockspin field $\phi'(x)$ solves the equation

$$F[\phi'|\phi] := \mathcal{C}\phi - \phi'(\phi' \cdot \mathcal{C}\phi) = \mathcal{P}^{\phi'}\mathcal{C}\phi = 0. \tag{8.2}$$

where $\mathcal{P}^{\phi'} = \mathbb{I} - \phi' \otimes \phi'$ denotes the projector on fields perpendicular to ϕ'.

In words: The blockspin field $\phi' \in \mathcal{S}^{N-1}$ is determined in such a way that the average of the components of ϕ perpendicular to ϕ' vanishes. Because of $\phi' \cdot F[\phi'|\phi] = 0$ this gives $N-1$ independent conditions. See appendix A for our index notation.

8.2 Background Field

Before going into a detailed calculation in local coordinates let us explore the consequences of the above blockspin choice for the determination of the background field.

As in chapter 7 the background field ψ is defined with respect to an extremality condition

$$S[\psi] = \text{extr.}, \qquad \text{subject to} \quad C[\psi] = \phi' \quad \Leftrightarrow \quad \mathcal{C}\mathcal{P}^{\phi'}\psi = 0 \tag{8.3}$$

but additionally it has to fulfill the constraint $\psi^2 = 1$. This requires the introduction of a second Lagrange multiplier field $\mu(z)$ besides λ. From

$$W[\psi, \mu, \lambda|\phi'] := S[\psi] + (\mu, \{\psi^2 - 1\}) + (\mathcal{C}^T\mathcal{P}^{\phi'}\lambda, \psi) = \quad \text{extr.} \tag{8.4}$$

we deduce

$$S'[\psi] + 2\mu\psi + \mathcal{P}^{\phi'}\mathcal{C}^T\lambda = 0\,. \tag{8.5}$$

By multiplying this eq. with ψ the Lagrange multiplier μ can be eliminated eventually yielding

$$\mathcal{P}^{\psi}\left[S'[\psi] + \mathcal{C}^T\mathcal{P}^{\phi'}\lambda\right] = 0 \tag{8.6}$$

This equation substitutes for eq. (7.3) in the scalar case. The appearance of the blockspin dependent projectors $\mathcal{P}^{\psi}$, $\mathcal{P}^{\phi'}$ makes the situation far more difficult.

8.3 Weight Functional

We will now introduce the blockspin into the partition function by means of the weight functional $\mathsf{P}[C[\phi], \phi'] = \hat{\delta}[C[\phi], \phi']$ obeying

$$\int \hat{\mathcal{D}}\phi'\hat{\delta}[C[\phi], \phi'] = 1. \tag{8.7}$$

Here $\hat{\mathcal{D}}$ denotes the $O(N)$-invariant functional measure over $\mathcal{S}^{N-1}$ and $\hat{\delta}$ is the spherical delta-functional.

Consider a local parametrization $\phi = (\phi_0, \ldots, \phi_{N-1}) = (\phi_0, \vec{\phi})$ of the half-sphere with local coordinates $\phi_a\,, a = 1, \ldots, N-1$ and $\phi_0 = \sqrt{1 - \vec{\phi}^2}$. In these coordinates eq. (8.7) reads

$$\int \hat{\mathcal{D}}\phi'\hat{\delta}[C[\phi]|\phi'] = \int \left(\prod_{a=1}^{N-1} \mathcal{D}{\phi'}_a\right) \left|\det \frac{\delta F_b}{\delta {\phi'}_c}\right| \left(\prod_{d=1}^{N-1} \delta[F_d[\phi'|\phi]]\right) = 1. \tag{8.8}$$

where δ is the usual cartesian delta functional for the d-th component. Please note that the determinant is taken from a $N-1 \times N-1$ matrix not a $N \times N$ matrix. [1]

[1] The determinant is as in the FADDEEV-POPOV-trick a generalization of

$$\int_x \delta(f(x)) = \int_x \frac{1}{|f'(x)|}\delta(x - x_0)$$

where $f(x_0) = 0$.

Because of $O(N)$-invariance we are free to choose the coordinate system such that $e_0 = C[\phi]$. With this choice one finds

$$F_d[\phi'|\phi] = -{\phi'}_d {\phi'}_0 \mathcal{C}\phi_0 \tag{8.9}$$

and hence

$$\frac{\delta F_b}{\delta {\phi'}_c} = \mathcal{C}\phi_0 \left(-{\phi'}_0 \delta_{b,c} + \frac{{\phi'}_b {\phi'}_c}{{\phi'}_0} \right) \tag{8.10}$$

The determinant of this expression can be written manifestly covariant as

$$\left| \det \frac{\delta F_b}{\delta {\phi'}_c} \right| = |\phi' \mathcal{C}\phi|^{N-1}. \tag{8.11}$$

We finally yield the expression

$$\int \mathcal{D}^{N-1}\vec{\phi'} |\phi' \mathcal{C}\phi|^{N-1} \delta^{N-1}[\mathcal{P}^{\phi'} \mathcal{C}\phi] = 1 \tag{8.12}$$

with $\mathcal{D}^{N-1}\vec{\phi'} := \prod_{d=1}^{N-1} \mathcal{D}{\phi'}_d$, $\delta^{N-1} := \prod_{d=1}^{N-1} \delta_d$.

8.4 Effective Action

Inserting eq. (8.12) into the partition functional and changing the order of integration as usual, one obtains the effective action $S'[\phi']$. In a local coordinate system with $e_0(x) = \phi'(x)$ it is defined as

$$e^{-S'[\phi']} = \int \mathcal{D}^{N-1}\vec{\phi} \delta^{N-1}[\mathcal{P}^{\phi'} \mathcal{C}\phi] e^{-E[\phi,\phi']}. \tag{8.13}$$

The exponent functional is given by

$$E[\phi,\phi'] = S[\phi] + (N-1) \int_x \phi'(x)(\mathcal{C}\phi)(x) - \frac{1}{2} \int_z \ln(1-\vec{\phi}^2) \tag{8.14}$$

where the last term comes from the $\mathcal{S}^{N-1}$-measure. This fluctuation integral will now be treated in the saddle-point approximation with the background field ψ extremizing $E[\phi,\phi']$ subject to the constraint $(\mathcal{P}^{\phi'} \mathcal{C}\phi)_a = 0$. The constraint $\psi^2 = 1$ is automatically fulfilled in the local coordinate system.

8.5 Split of the Field

Suppose we have succeeded to determine the background field. The next step is then to introduce a local split of the basefield into background and fluctuation field.

$$\phi(z) = G(\psi(z), \zeta(z)) \tag{8.15}$$

Its purpose is to decouple the low-frequency components of the field from the high-frequency components. Apart from this there are some technical restrictions on G:

- When inserting the split into the weight functional this should result into a *linear* constraint on ζ. Only then we are able to make use of the GK formalism to transform the functional integral eq. (8.13) into a Gaussian fluctuation integral with fluctuation propagator Γ. A look at the blockspin definition reveals that this requires $\phi'([z])\zeta(z) = 0$, i.e. the fluctuation fields are from the tangentspace $\mathcal{T}_{\phi'([z])}\mathcal{S}^{N-1}$ to $\mathcal{S}^{N-1}$ at the blockspin field.
- We want to expand the action around the background field $\psi(z)$. This naturally leads to fluctuation fields being from the tangentspace $\mathcal{T}_{\psi(z)}\mathcal{S}^{N-1}$ to $\mathcal{S}^{N-1}$. This is the $N-1$ dimensional linear space tangent to the background field ψ.

To fulfill both requirements simultaneously we will use fluctuation fields perpendicular to ϕ' but will have to transform the derivatives.

Consider the following two coordinate systems[2] $\{e^\alpha(z)\}, \{e'^\alpha(z)\}$ with

$$e^0(z) = \phi'(z) \tag{8.16}$$
$$e'^0(z) = \psi(z) \tag{8.17}$$

and arbitrary e^a, e'^a.

The split of the field in these systems reads

$$\phi = \sqrt{1-\vec{\zeta}^2}e^0 + \zeta_a e^a = \sqrt{1-\vec{\xi'}^2}e'^0 + \xi'_a e'^a. \tag{8.18}$$

Additionally a shift of ζ_a is performed. We write $\zeta_a(z) = \xi_a(z) + \psi_a(z)$ with $\psi_a(z) = (\psi(z)\cdot e^a(z))$. The two coordinate vectors $\vec{\xi}$ and $\vec{\xi'}$ are related to each other by a transfomation $\vec{\xi'} = \vec{\Xi}'[\vec{\xi}]$ with $\vec{\Xi}'[0] = 0$. [3] Following above strategy the exponent functional will be re-expressed in terms of the fluctuation coordinates $\vec{\xi'}$ and is then expanded with respect to the fluctuation coordinates $\vec{\xi}$:

$$\begin{aligned} E[\vec{\Xi}'[\vec{\xi}]|\psi] &= E[0|\psi] \\ &+ \left(\frac{\delta E[\vec{\Xi}'[\vec{\xi}]|\psi]}{\delta\vec{\xi'}}\Big|_0 \frac{\delta\vec{\Xi}'[\vec{\xi}]}{\delta\vec{\xi}}\Big|_0, \vec{\xi}\right) \\ &+ \frac{1}{2}\left(\vec{\xi}, \left\{\frac{\delta\vec{\Xi}'[\vec{\xi}]}{\delta\vec{\xi}}\Big|_0 \frac{\delta^2 E[\vec{\Xi}'[\vec{\xi}]|\psi]}{\delta\vec{\xi'}\delta\vec{\xi'}}\Big|_0 \frac{\delta\vec{\Xi}'[\vec{\xi}]}{\delta\vec{\xi}}\Big|_0\right\}\vec{\xi}\right) \\ &+ \frac{1}{2}\left(\vec{\xi}, \left\{\frac{\delta E[\vec{\Xi}'[\vec{\xi}]|\psi]}{\delta\vec{\xi'}}\Big|_0 \frac{\delta^2\vec{\Xi}'[\vec{\xi}]}{\delta\vec{\xi}\delta\vec{\xi}}\Big|_0\right\}\vec{\xi}\right) \end{aligned} \tag{8.19}$$

$$= E[0|\psi] + (E^{;}[0|\psi], \vec{\xi}) + \frac{1}{2}(\vec{\xi}, E^{;;}[0|\psi]\vec{\xi}) \tag{8.20}$$

Here $;$ denotes a derivative with respect to $\vec{\xi'}$.

[2] This parametrization is in the spirit of the one used by POLYAKOV in his famous proof of asymptotic freedom of the two dimensional $O(N)$ symmetric non-linear σ model [Pol75]

[3] This follows from $\sqrt{1-(\vec{\xi}+\vec{\psi})^2}\phi' + (\xi_a+\psi_a)e^a = \sqrt{1-\vec{\Xi}'[\vec{\xi}]^2}\psi + \Xi'_a[\vec{\xi}]e'^a$

The background field is defined by requiring that the term linear in $\vec{\xi}$ vanishes subject to the constraint that $\mathcal{P}^{\phi'}\mathcal{C}\psi = 0$ or in coordinates $\mathcal{C}\psi_a = 0$.

$$\int_{z'} \frac{\delta E[\vec{\Xi}'[\vec{\xi}]|\psi]}{\delta \xi'_b(z')}\Big|_0 \frac{\delta \Xi'_b[\vec{\xi}](z')}{\delta \xi_a(z)}\Big|_0 = \mathcal{C}^T \lambda_a(z). \tag{8.21}$$

Let

$$Q^T_{ab}(z,x) := \int_{z'} \frac{\delta \Xi'^{-1}_b[\vec{\xi}'](z)}{\delta \xi'_a(z')}\Big|_0 \mathcal{C}^T(z',x) \tag{8.22}$$

then the saddle-point condition reads

$$\frac{\delta E[\vec{\Xi}'[\vec{\xi}]|\psi]}{\delta \xi'_a(z)}\Big|_0 = \left(Q^T\lambda\right)_a(z) \tag{8.23}$$

This is as close as one can get to the scalar case. The geometrical constraint is implemented by the operator

$$Q^T(z,x) : \mathcal{T}_{\phi'(x)}\mathcal{S}^{N-1} \to \mathcal{T}_{\psi(z)}\mathcal{S}^{N-1} \tag{8.24}$$

Since $\zeta_a e^a$ is perpendicular to ϕ' and because of the constraint $\mathcal{C}\psi_d = 0$ the delta functional in the $\vec{\xi}$ coordinates becomes

$$\delta^{N-1}[\mathcal{P}^{\phi'}\mathcal{C}\phi] = \prod_{d=1}^{N-1} \delta[\mathcal{C}\xi_d]. \tag{8.25}$$

This situation is now suited for the application of a $N-1$ dimensional GK-formalism with fluctuation propagator

$$\Gamma[\psi] = E^{\shortparallel -1}[0|\psi] - E^{\shortparallel -1}[0|\psi]\mathcal{C}^{i^T}(\mathcal{C}E^{\shortparallel -1}[0|\psi]\mathcal{C}^T)^{-1}\mathcal{C}E^{\shortparallel -1}[0|\psi] \tag{8.26}$$

yielding the effective action in the saddle-point approximation

$$S'[\phi'] = E[\psi[\phi']] - \frac{1}{2}\mathrm{tr}\ln\Gamma[\psi[\phi']]. \tag{8.27}$$

8.6 Conclusion

The evaluation of eq. (8.27) is still in progress. The points to be attacked next are:

- With eqns. (8.27), (8.26), (8.23) we have at least formally been able to perform the saddle-point approximation. In the scalar case it was possible to derive an extremely useful equation for the derivative of the background field. In the present case this is obstructed because of the blockspin dependent projectors in the saddle-point eq. (8.23). Alternative equations will have to be derived.

- The derivatives $E^{[}[0|\psi]$ have to be computed in coordinates. For this purpose a nice trick from POLYAKOV is useful [Pol75]: He introduced a gauge field

$$\mathsf{A}_{\mu}^{\prime\alpha\,\beta} = e^{\prime\,\alpha}(z+\mu)\left(\partial_{\mu} e^{\prime\,\beta}\right)(z) \tag{8.28}$$

and a covariant derivative

$$\mathsf{D}_{\mu}^{\prime\alpha\,\beta} = \partial_{\mu}\delta^{\alpha\,\beta} + \mathsf{A}_{\mu}^{\prime\alpha\,\beta} \tag{8.29}$$

such that e.g. the bare action can be rewritten as

$$S[\phi] = S[\vec{\xi}'|\psi] = \frac{\beta}{2}\int_{z} (\mathsf{D}'_{\mu}\xi')^2(z). \tag{8.30}$$

In the above formula ξ'_0 is as usual an abbreviation for $\sqrt{1-\vec{\xi'}^2}$. From these expressions it is easy to compute the derivatives.

- Motivated by the studies of other models [Gri97],[JW97] it might by necessary to perform a self-consistent normalordering in advance of the saddle-point approximation to avoid instabilities.

- A convenient localization approximation has to be made. This is essential to end up with manageable equations.

In his famous work [Pol75] POLYAKOV used a similar RG technique to prove the asymptotic freedom of the model. The key difference is that he used sharp momentum cutoffs yielding a clean separation of the high- and low-frequency field modes. Although the technical problems introduced by the lattice are severe the prospect to be able to simulate the effective action and hence be able to check the validity of the assumptions justifies further efforts.

Appendix A

Notational preliminaries

In this appendix our notation is fixed

Let the base lattice

$$\Omega_{[0]} = \left(\mathbb{Z}/N_{[0]}\mathbb{Z}\right)^D \tag{A.1}$$

be a finite D dimensional hyper-cubic lattice with unit lattice spacing and extension $N_{[0]}$ in every dimension. Position-space fields $\phi_{[0]}(z)$ are supposed to meet periodic boundary conditions. Therefore the corresponding momentum-space fields $\phi_{[0]}(p)$ are periodically defined on the lattice.

$$\tilde{\Omega}_{[0]} = \left(\frac{2\pi}{N_{[0]}}\mathbb{Z}/N_{[0]}\mathbb{Z}\right)^D \tag{A.2}$$

with lattice spacing $\frac{2\pi}{N_{[0]}}$ and length 2π. Introducing the symbols

$$\int_z := \sum_{z\in\Omega_{[0]}} \qquad \int_p := \frac{1}{{N_{[0]}}^D} \sum_{p\in\tilde{\Omega}_{[0]}} \tag{A.3}$$

we retrieve the familiar equations

$$\phi_{[0]}(z) = \int\limits_p \phi_{[0]}(p) e^{+ipz} \qquad \phi_{[0]}(p) = \int\limits_z \phi_{[0]}(z) e^{-ipz} \tag{A.4}$$

$$\delta_{[0]}(z) = \int\limits_p e^{ipz} =: \mathbb{I}_{[0]}(z) \qquad \delta_{[0]}(p) = \int\limits_z e^{-ipz} =: \mathbb{I}_{[0]}(p) \tag{A.5}$$

$$\phi_{[0]}(z) = \int\limits_{z'} \phi_{[0]}(z')\delta(z-z') \qquad \phi_{[0]}(p) = \int\limits_{p'} \phi_{[0]}(p')\delta(p-p') \tag{A.6}$$

$$\left(\phi_{[0]}, \phi_{[0]}\right) = \int\limits_z |\phi_{[0]}(z)|^2 \qquad \left(\phi_{[0]}, \phi_{[0]}\right) = \int\limits_p |\phi_{[0]}(p)|^2. \tag{A.7}$$

The Hilbertspace associated with the above inner products is denoted as $\mathcal{H}(\Omega_{[0]})$ or $\mathcal{H}_{[0]}$. The functional measure on $\mathcal{H}(\Omega_{[0]})$ is defined as

$$\mathcal{D}\phi_{[0]} := \prod_z \mathrm{d}\phi_{[0]}(z). \tag{A.8}$$

For the block lattice $\Omega_{[1]}$ and its momentum-space counterpart $\tilde{\Omega}_{[1]}$ analog expressions can be written down simply by changing the index $_{[0]}$ into $_{[1]}$ and using $N_{[1]} = \frac{N_{[0]}}{s}$ with a blocking factor $s \in \mathbb{N}$. The objects after n blocking steps are indexed as $_{[n]}$. Sometimes the index $_{[0]}$ for base quantities will omitted and the blocked quantities are then denoted with a prime $'$.

It is convenient to define the following two lattices

$$\Lambda = (\mathbb{Z}/s\mathbb{Z})^D \qquad \tilde{\Lambda} = \left(\frac{2\pi}{s}\mathbb{Z}/s\mathbb{Z}\right)^D. \tag{A.9}$$

Then any $z \in \Omega_{[0]}$ or $p \in \tilde{\Omega}_{[0]}$ can be decomposed into

$$z = sx + w \qquad p = \frac{q}{s} + l \tag{A.10}$$

with $w \in \Lambda, x \in \Omega_{[1]}, q \in \tilde{\Omega}_{[1]}, l \in \tilde{\Lambda}$. Defining

$$\int\limits_w := \sum_{w\in\Lambda} \qquad \int\limits_l := \frac{1}{s^D} \sum_{l\in\tilde{\Lambda}} \tag{A.11}$$

we can split the integrals

$$\int\limits_z = \int\limits_w \int\limits_x \qquad \int\limits_p = \int\limits_q \int\limits_l. \tag{A.12}$$

Additionally to the covariant (scale) objects $_{[n]}$ we introduce their contravariant counterparts.

For example

$$\Omega^{[n]} = \left(s^{-n}\mathbb{Z}/N_{[n]}\mathbb{Z}\right)^D \tag{A.13}$$

is a lattice with lattice spacing s^{-n}. The associated Hilbertspace of lattice fields is denoted by $\mathcal{H}^{[n]}$. In the limit $n \to \infty$ we identify $\Omega^{[\infty]}$ with $\mathbb{R}^D$.

The transition between both worlds is performed by scale transformations $\mathcal{S}^{[n]} : \mathcal{H}_{[0]} \to \mathcal{H}^{[n]}$ being defined as

$$\mathcal{S}^{[n]}(z, \mathsf{z}) \quad := \quad s^{-n\alpha}\delta_{[0]}(z - s^n\mathsf{z}). \tag{A.14}$$

Due to our definitions of integrals and delta-functions this means

$$\mathcal{S}^{[n]^T}(\mathsf{z}, z) \quad := \quad s^{-n(\alpha+D)}\delta^{[n]}(z - s^n\mathsf{z}) \tag{A.15}$$

$$\mathcal{S}^{[n]^{-1}}(z, \mathsf{z}) \quad := \quad s^{-n\alpha}\delta_{[n]}(\frac{z}{s^n} - \mathsf{z}). \tag{A.16}$$

Up to now only scalar fields ϕ have been introduced. When refering to a N component field this is written as $\phi = (\phi_0, \ldots, \phi_{N-1})$.

In order to confuse the reader we will heavily make use of the following implicit conventions

$$\alpha, \beta, \text{ etc} \quad \in \quad \{0, \ldots, N-1\} \tag{A.17}$$

$$a, b, \text{ etc} \quad \in \quad \{1, \ldots, N-1\} \tag{A.18}$$

$$A, B, \text{ etc} \quad \in \quad \{2, \ldots, N-1\} \tag{A.19}$$

$$z \quad \in \quad \Omega_{[0]} \tag{A.20}$$

$$x \quad \in \quad \Omega_{[1]} \tag{A.21}$$

$$\mathsf{z} \quad \in \quad \Omega^{[n]} \tag{A.22}$$

We used the following notation for pertubative orders. A quantity $Q^{(n)}$ is improved up to order n. $Q^{[n]}$ contains only terms that are exactly of order n whereas $Q^{(n)}$ has no terms of order less than n.

Appendix B

GK-Kernels

In this appendix the momentum space representations for the GAWEDZKI *and* KUPIAINEN *kernels are derived.*

In joint work with M. GRIESSL a C++ -class library for the GK kernels has been written. It provides structural classes like `Lattice`, `Layer`, `SymKernel` together with appropriate access methods and application classes like the kernels. The wide adjustability of parameters (lattice dimension, extension etc.) and options (dimension-full/less, Delta- or Gauss-blockspin) makes it a very convenient and flexible tool for lattice RG calculations. The kernels are computed in momentum space by the following formulas and then are shuffled to position space by a FFT.

B.1 The Averaging Operator C

The averaging operator $\mathcal{C}$ and its adjoint $\mathcal{C}^T$ are defined in position-space via

$$\mathcal{C}(x,z) := s^{-\alpha-D}\delta_{x,\left[\frac{z}{s}\right]} =: \mathcal{C}^T(z,x) \tag{B.1}$$

with $\left[\frac{z}{s}\right]$ denoting the $\mathbb{Z}^D$ part of $\frac{z}{s}$. Its fourier representation reads

$$\mathcal{C}(x,z) = \int_p \mathcal{C}(p)e^{ip(sx-z)} \qquad \mathcal{C}(p) := \int_z \mathcal{C}(0,z)e^{ipz} \tag{B.2}$$

with

$$\mathcal{C}(p) = s^{-\alpha-D}\prod_{\mu=1}^{D}\left[\frac{\sin(p_\mu s/2)}{\sin(p_\mu/2)}e^{-ip_\mu(s-1)/2}\right]. \tag{B.3}$$

Note that

$$\mathcal{C}(x,z) = \mathcal{C}(x-x', z-sx') \qquad \forall x' \in \Omega_{[1]} \tag{B.4}$$

$$\mathcal{C}\mathcal{C}^T = s^{-2\alpha-D}\,\mathbb{I}_{[1]} \tag{B.5}$$

$$\mathcal{C}(l) = s^{-\alpha}\delta^D_{l,\mathbf{0}}. \tag{B.6}$$

Let $\mathcal{C}_{[n]}(s) := \mathcal{C}(s^n)$. We now introduce a contravariant averaging operator (see A for notation) $\mathcal{C}^{[n]} : \mathcal{H}^{[0]} \to \mathcal{H}_{[n]}$ by means of

$$\mathcal{C}^{[n]}(x,\mathsf{z}) := \left(\mathcal{C}_{[n]}\mathcal{S}^{[n]\,-1}\right)(x,\mathsf{z}) = \delta_{[n]}(x-[\mathsf{z}]) = \delta_{x,[\mathsf{z}]}. \tag{B.7}$$

In the infinite volume case one can take the limit $n \to \infty$. $\mathcal{C}^{[\infty]}$ turns out to be the characteristic function of the unit cube at x

$$\mathcal{C}^{[\infty]}(x,\mathsf{z}) := \begin{cases} 1 & \text{for} \quad \mathsf{z}-x \in [0,1]^D \\ 0 & \text{otherwise} \end{cases}. \tag{B.8}$$

Its Fourier-transform reads

$$\mathcal{C}^{[\infty]}(\mathsf{p}) \quad = \quad \prod_{\mu=1}^{D}\left[\frac{\sin(\frac{\mathsf{p}_\mu}{2})}{\frac{\mathsf{p}_\mu}{2}} e^{+i\frac{\mathsf{p}_\mu}{2}}\right]. \tag{B.9}$$

Later on we will need $\mathcal{C}^{[\infty]}(\mathsf{p})$ with $\mathsf{p} = p+k$. Refering to the identities $\sin(\frac{p_\mu+k_\mu}{2}) = (-1)^{\frac{k_\mu}{2\pi}}\sin(\frac{p_\mu}{2}) = e^{i\frac{k_\mu}{2}}\sin(\frac{p_\mu}{2})$ one finds in this case

$$\mathcal{C}^{[\infty]}(p+k) = \prod_{\mu=1}^{D}\left[\frac{\sin(\frac{p_\mu}{2})}{\frac{p_\mu+k_\mu}{2}}\right] e^{+i\frac{p_\mu}{2}}. \tag{B.10}$$

B.2 The Base-lattice Propagator $v_{[0]}$

The base-lattice propagator $v_{[0]}$ is formally defined as the inverse of the negative base-lattice Laplacian

$$\Box_{[0]}(z_1,z_2) = -\sum_{\mu=1}^{D}\left[\delta_{[0]}(z_1-z_2+\mu) + \delta_{[0]}(z_1-z_2-\mu) - 2\delta_{[0]}(z_1-z_2)\right] \tag{B.11}$$

$$\Box_{[0]}(p) = \hat{p}^2 \tag{B.12}$$

with $\hat{p_\mu} = 2\sin\left(\frac{p_\mu}{2}\right)$.

B.3 The Block-lattice Propagator $v_{[1]}$

The block-lattice propagator $v_{[1]}$ is defined as

$$v_{[1]} := \mathcal{C}v_{[0]}\mathcal{C}^T + \frac{1}{\kappa}\mathbb{I}_{[1]}. \tag{B.13}$$

This leads to the Fourier representation

$$v_{[1]}(x_1, x_2) = \int_q v_{[1]}(q) e^{iq(x_1 - x_2)} \qquad v_{[1]}(q) = \int_x v_{[1]}(x, 0) e^{-iqx} \tag{B.14}$$

with

$$v_{[1]}(q) = \int_l \mathcal{C}(\frac{q}{s} + l) v_{[0]}(\frac{q}{s} + l) \mathcal{C}^*(\frac{q}{s} + l) + \frac{1}{\kappa}. \tag{B.15}$$

Note that

$$v_{[1]}(q = 0) = s^{-2\alpha - D} v_{[0]}(0) + \frac{1}{\kappa} \tag{B.16}$$

$$v_{[1]}(q) = v_{[1]}(q + sl) \qquad \forall l \in \tilde{\Lambda}. \tag{B.17}$$

B.4 Perfect Laplacian

The Perfect Laplacian can be obtained by two methods. Either by numerical iteration of the blocking procedure or directly by its fourier representation.

The first method is instructive for itself and was implemented by BELL and WILSON [BW74]. Using a computer one is obviously limited to finite lattices. It is not feasible nor necessary to start from a say $(2^{10})^D$ lattice and to iterate down to $(2^4)^D$ lattice. Instead WILSON suggested to blow-up the lattice after each RG step by inserting zeros in the middle of the matrix representing the propagator-kernel (fig. B.1). Then it is possible to perform an arbitrary number of iteration steps on moderately sized lattices (e.g. $(32)^D$). Beware: Since the insertion has to respect the lattice symmetry one has to take care of even/odd lattice extensions. We will now derive the fourier representation

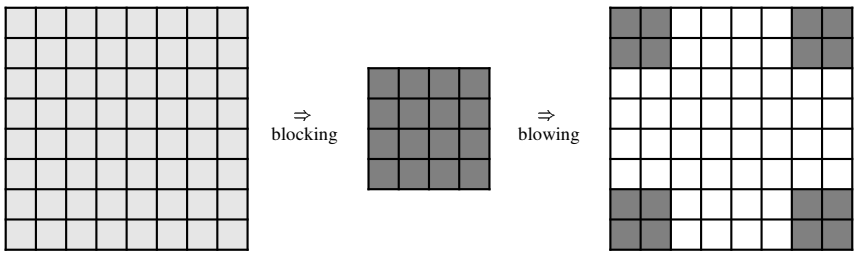

Figure B.1: WILSON*'s blow-up technique*

of the Perfect Laplacian. With eq. (B.10) it is easy to compute the fourier transform of

the GK-propagator

$$\begin{aligned} v_{\mathrm{GK}}(\mathsf{z}_1-\mathsf{z}_2) &= \int\limits_{\mathsf{z}_1,\mathsf{z}_2} \mathcal{C}^{[\infty]}(\mathsf{z}_1,\mathsf{z}_1)\,(-\Delta)^{-1}\,(\mathsf{z}_1-\mathsf{z}_2)\mathcal{C}^{[\infty]\,T}(\mathsf{z}_2,\mathsf{z}_2) \\ &= \int\limits_p \int\limits_k \frac{\mathcal{C}^{[\infty]}(p+k)\mathcal{C}^{[\infty]\,T}(p+k)}{(p+k)^2} e^{ip(\mathsf{z}_1-\mathsf{z}_2)} \\ &= \int\limits_p \int\limits_k \frac{1}{(p+k)^2} \prod_{\mu=1}^{D} \frac{\sin(\frac{p_\mu+k_\mu}{2})}{\frac{p_\mu+k_\mu}{2}} \frac{\sin(\frac{p_\mu+k_\mu}{2})}{\frac{p_\mu+k_\mu}{2}} e^{ip(\mathsf{z}_1-\mathsf{z}_2)} \\ &= \int\limits_p \int\limits_k \frac{1}{(p+k)^2} \prod_{\mu=1}^{D} \frac{\sin^2(\frac{p_\mu}{2})}{\left(\frac{p_\mu+k_\mu}{2}\right)^2} e^{ip(\mathsf{z}_1-\mathsf{z}_2)}. \end{aligned} \tag{B.18}$$

Here we have made use of $k_\mu = 2\pi\hat{k}_\mu$ and therefore $\exp(ik\mathsf{z}) = 1$. Note that $v_{\mathrm{GK}}^{-1}(p=0) = 0$.

B.5 The Interpolation Operator A

Being defined as

$$A(z,x) = \int\limits_{z',x'} v_{[0]}(z,z')\mathcal{C}^*(z',x')v_{[1]}{}^{-1}(x',x) \tag{B.19}$$

the fourier representation of the interpolation operator reads

$$A(z,x) = \int\limits_p A(p)e^{+ip(z-sx)} \qquad A(p) = \int\limits_z A(z,0)e^{-ipz} \tag{B.20}$$

with

$$A(p) = v_{[0]}(p)\mathcal{C}^*(p)v_{[1]}{}^{-1}(sp). \tag{B.21}$$

From this we find

$$A(l) = \frac{1}{s^{-\alpha-D} + \frac{s^\alpha}{\kappa\, v_{[0]}(0)}}\delta_{l,0} \tag{B.22}$$

$$s^{-\alpha-D}\int\limits_z A(z,x) = 1 - \frac{1}{\kappa v_{[0]}(p=0)s^{-2\alpha-D}+1} \tag{B.23}$$

$$\int\limits_x A(z,x) = \frac{1}{s^{-\alpha-D} + \frac{1}{\kappa\, v_{[0]}(0)}}. \tag{B.24}$$

B.6 The Fluctuation Propagator Γ

The fluctuation propagator Γ is defined as

$$\Gamma = v_{[0]} - A v_{[1]} A^T . \tag{B.25}$$

This yields the fourier representation

$$\Gamma(z_1, z_2) = \int\limits_{l_1, l_2} \int\limits_{q_1} \Gamma(l_1, q_1, l_2) e^{i(\frac{q_1}{s}+l_1)z_1 - i(\frac{q_1}{s}+l_2)z_2} \tag{B.26}$$

with

$$\Gamma(l_1, q_1, l_2) = v_{[0]}(\frac{q_1}{s} + l_1) s^D \delta_{l_1, l_2} - A(\frac{q_1}{s} + l_1) v_{[1]}(q_1) A^T(\frac{q_1}{s} + l_2). \tag{B.27}$$

We deduce

$$\Gamma(l_1, 0, l_2) = \begin{cases} v_{[0]}(l_1) s^D \delta_{l_1, l_2} & l_1 \neq 0 \\ \frac{\frac{1}{\kappa} s^D}{s^{-2\alpha - D} + \frac{1}{\kappa v_{[0]}(0)}} & l_1 = 0 \end{cases} . \tag{B.28}$$

The fluctuation propagator is invariant under block-translations:

$$\Gamma(z_1, z_2) = \Gamma(z_1 + x, z_2 + x) \quad \text{with} \quad x \in \Lambda. \tag{B.29}$$

Without loss of generality it is therefore sufficient to consider only $\Gamma(z, z')$ with $z' = w' \in \Lambda$ which will be denoted by $\Gamma(z, w')$.

Sometimes it is more convenient to write this as

$$\Gamma(w, x, w') := \Gamma(w + x, w'). \tag{B.30}$$

For computational purposes it is important to know how many independent values of $\Gamma(z, w')$ have to be calculated in order to determine it's behavior completely. At first sight (B.30) there are $(N_{[1]} s N_{[1]})^D$ values, which is e.g. 531441 for $N_{[0]} = 9, s = 3, D = 4$. This number can be reduced drastically by taking the lattice symmetry of the blocking procedure into account which is given by the averaging operator $\mathcal{C}(x - z)$. The center of symmetry C lies in the middle of the first block.[1] All lattice symmetries S can be reduced to reflections at planes through C. To make this more clear let us rearrange the base-lattice $\Omega_{[0]}$:

1. Shift the block-center C into the origin $z = 0$ of the lattice.

$$\hat{z}_\mu \mapsto \begin{cases} \hat{z}_\mu - \frac{s-1}{2} & : \quad s \text{ odd} \\ 1 + (\hat{z}_\mu - \frac{s}{2})2 & : \quad s \text{ even} \end{cases} \tag{B.31}$$

[1] If s is even this point does not belong to the block-lattice $\Omega_{[1]}$.

2. Center the lattice by virtue of periodic boundary conditions

$$\hat{z}_\mu \mapsto \begin{cases} \hat{z}_\mu - N_{[0]} & : \quad \hat{z}_\mu > \frac{N_{[0]}-1}{2} & \text{and } N_{[0]} \text{ odd, } s \text{ odd} \\ & : \quad \hat{z}_\mu > \frac{N_{[0]}}{2} & \text{and } N_{[0]} \text{ even, } s \text{ odd} \\ \hat{z}_\mu - 2N_{[0]} & : \quad \hat{z}_\mu > N_{[0]} - 1 & \text{and } N_{[0]} \text{ even, } s \text{ even} \end{cases}$$

The case $N_{[0]}$ odd, s odd can not occur since $N_{[0]}$ must be a multiple of s.

Figure B.2 shows the symmetries for two-dimensional lattices. The generalization to higher dimensions is obvious.

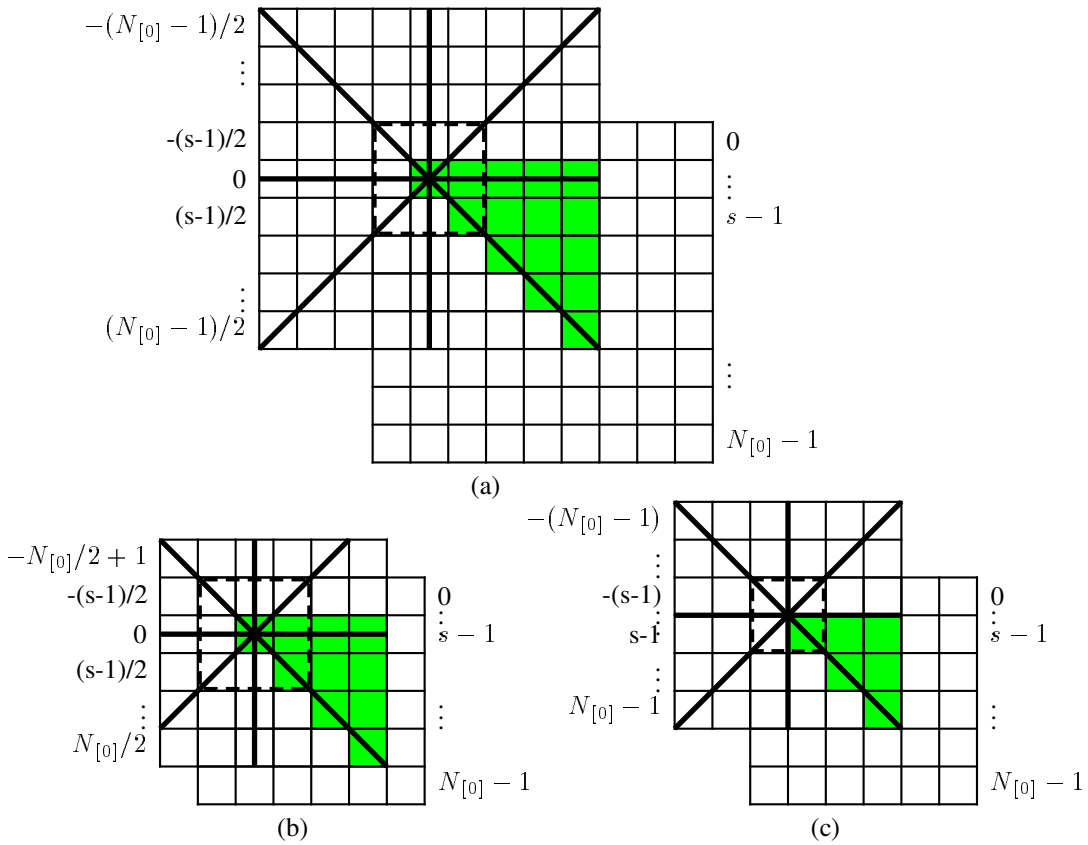

Figure B.2: *Symmetries of the blocking-procedure for a two-dimensional lattice $N_{[0]}$. The first block is framed with a dashed line. Symmetry-axes are drawn as solid lines and the wedge is colored grey. (a) $N_{[0]}$ odd, s odd (b) $N_{[0]}$ even, s odd (c) $N_{[0]}$ even, s even.*

We see that any arbitrary $z \in \Omega_{[0]}$ can be transformed into a reduced variable $\bar{z} = Sz \in \text{wedge}(D, s, N_{[0]})$ with

$$\text{wedge}(D, s, N_{[0]}) := \left\{ z \in \Omega_{[0]} \mid 0 \leq \hat{z}_D \leq \ldots \leq \hat{z}_1 \right\} \tag{B.32}$$

by means of an appropriate symmetry transformation $S = S(z)$.

From lattice symmetry follows

$$\Gamma(z,w) = \Gamma(Sz, Sw) = \Gamma(\bar{z}, Sw). \tag{B.33}$$

In general Sw does not lie in wedge$(D, s, N_{[0]})$. Thus we deduce that $\Gamma(z,w)$ has no more than $\sharp\text{wedge}(D, s, N_{[0]}) \times N_{[1]}$ independent values.

To compute this estimate we make use of the following recursion relation:

$$\sharp\text{wedge}(D > 1, s, N_{[0]}) = \sum_{n=1}^{\eta(s,N_{[0]})} \sharp\text{wedge}(D-1, s, n) \tag{B.34}$$

$$\sharp\text{wedge}(D = 0, s, n) = 1 \tag{B.35}$$

$$\eta(s, N_{[0]}) := \begin{cases} \frac{N_{[0]}-1}{2} + 1 & : \text{ for } s \text{ odd, } N_{[0]} \text{ odd} \\ \frac{N_{[0]}}{2} + 1 & : \text{ for } s \text{ odd, } N_{[0]} \text{ even} \\ \frac{N_{[0]}}{2} & : \text{ for } s \text{ even, } N_{[0]} \text{ even} \end{cases}$$

In the above case $D = 4, N_{[0]} = 9, s = 3$ one finds $\sharp\text{wedge}(4,3,9) = 70$ so the number of Γ-values to be calculated reduces to 5670! This estimate is good enough for our applications although there is still symmetry left unused in the w-sector. A brute-force application of all symmetry operations shows that the exact number of independent values in our case amounts to 2380.

Appendix C

Gaussian measures

In this chapter a few properties of Gaussian Measures are collected (See [GJ87]).

Any real, positive, symmeteric linear operator Γ on $\mathcal{H}(\Omega)$ will be called a covariance operator.

The Gaussian measure with mean zero for the N-component field ζ is defined as

$$\mathrm{d}\mu_\Gamma[\zeta] := \det(2\pi\Gamma)^{-\frac{N}{2}}\mathcal{D}\zeta e^{\frac{1}{2}(\zeta,\Gamma^{-1}\zeta)}. \tag{C.1}$$

It is characterized by its generating functional

$$\int \mathrm{d}\mu_\Gamma[\zeta]e^{(J,\zeta)} \quad = \quad \exp\left\{\frac{1}{2}(J,\Gamma J)\right\}. \tag{C.2}$$

For covariance operators $\Gamma, \Gamma_1, \Gamma_2$ with $\Gamma = \Gamma_1 + \Gamma_2$ the convolution theorem holds

$$\int \mathrm{d}\mu_\Gamma[\zeta]F[\zeta] \quad = \quad \int \mathrm{d}\mu_{\Gamma_1}[\zeta_1]\int \mathrm{d}\mu_{\Gamma_2}[\zeta_2]F[\zeta_1+\zeta_2]. \tag{C.3}$$

By making use of eq. (C.1) one obtains the extremely useful relation

$$\begin{aligned}\int \mathrm{d}\mu_\Gamma[\zeta]e^{-\frac{1}{2}(\psi+\zeta,K\{\psi+\zeta\})}F[\zeta] = \\ \mathcal{N}\int \mathrm{d}\mu_{(\Gamma^{-1}+K)^{-1}}[\zeta]e^{-\frac{1}{2}(\psi,LK\psi)}F[L\psi+\zeta]\end{aligned} \tag{C.4}$$

with $L = (1+\Gamma K)^{-1}$.

Normalordering with respect to a covariance Γ is defined by

$$:e^{(\zeta,J)}:_\Gamma = e^{(\zeta,J)-\frac{1}{2}(J,\Gamma J)}. \tag{C.5}$$

For the ultra-local case with

$$\mathrm{d}\mu_\gamma(\hat{\zeta}) = (2\pi\gamma)^{\frac{-N}{2}}\mathrm{d}^N\hat{\zeta}e^{-\frac{1}{2\gamma}\hat{\zeta}^2} \tag{C.6}$$

we mention the Wick formula

$$\int \mathrm{d}\mu_\gamma(\hat{\zeta})(\hat{\zeta}^2)^n \quad = \quad \gamma^n \prod_{m=0}^{n-1}(2m+N). \tag{C.7}$$

For $N = 1$ we find

$$\int \mathrm{d}\mu_{\gamma_1}(\hat{\zeta}) : (\hat{A}\hat{\phi} + \hat{\zeta})^n :_{\gamma_2}^{\hat{\zeta}} = \hat{A}^n : \hat{\phi} :_{\frac{\gamma_2 - \gamma_1}{\hat{A}^2}}^{\hat{\phi}} \tag{C.8}$$

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Acknowledgment

Throughout the last three years I had the great pleasure to enjoy the friendship of some very special people. Without their help this thesis would definitely not exist:

First of all I thank GERHARD MACK for sharing his insights about life, the universe and everything else with me. It is an honour to be a member of his universal doctor family.

I would like to thank CHRISTIAN WIECZERKOWSKI for giving me the opportunity to collaborate with him. His incredible energy and constant support were extremely encouraging.

MAX GRIESSL has suffered most from my impatience, bad moods and attempts to speak Bavarian. Working with him in contrast was highly enjoyable.

MARCUS, UTE, MARKUS, MARTIN, BERNHARD. Danke!

The hospitality of I. Inst. für Theor. Physik in Münster is highly recommended, especially KLAUS SPANDEREN's generous chocolate donations during long night shifts.

Thanks to JOACHIM HEIN, KARL JANSEN, KLAUS PINN and MARTIN LÜSCHER for their simulational guidance.

MARTIN MEIER-SCHELLERSHEIM, CHRISTIAN DOLLE AND JAN WÜRTHNER tried their best to correct my tzpos. Don't blame them ...

Support from the **Deutsche Forschungs Gemeinschaft** and the **German Israeli Foundation** is gratefully acknowledged.

Finally, I would like to thank all those who made the time in the notorious *Hobbykeller* so memorable.