Self-consistent Calculation of Real Space Renormalization Group Flows and Effective Potentials

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Abstract

We present a self-consistent method for the calculation of renormalization group flows in lattice field theory. The truncation of the action is done in a way that the flow preserves a simple parameterization of the action. Higher powers of derivatives of the field are neglected in the actions, but no polynomial approximation in the field is made. In each step, the integration over the fluctuation field (high frequency components of the field) is performed by a saddle point method where the saddle point depends on the block-spin. A numerical implementation of the method on the lattice allows us to compare the resulting flow with other techniques.

Because this method may not be accurate enough for certain models we discuss an improvement of the method where the saddle point approximation is preceded by self-consistent normal ordering.

As an alternative we determine the renormalization group flow by a perturbative expansion of the effective potential. The expansion is given in orders of the fluctuation propagator.

In the last part numerical results are presented and compared. Also we determine the critical line with the help of the self-consistent saddle point approximation.

In this paper we treat scalar field theories as an example. The basic limitations of the methods are also discussed. They come from a possible breakdown of stability which may occur e.g. when a composite block-spin would be needed.

Zusammenfassung

In dieser Arbeit wird eine selbst-konsistente Methode zur Berechnung von Renormierungsgruppen-Flüssen von Gitterfeldtheorien vorgestellt. Die Wirkung wird derart vereinfacht, daäs der Fluß eine einfache Parametrisierung erhält. Höhere Ableitungen des Feldes werden in der Wirkung vernachlässigt, aber es wird keine polynomiale Approximation bezüglich des Feldes angewandt. In jedem Schritt wird die Integration über das Fluktuationsfeld (hochfrequente Anteile des Feldes) durch eine Sattelpunktsmethode ausgeführt, wobei der Sattelpunkt vom Blockspin abhängt. Eine numerische Umsetzung der Methode auf dem Gitter erlaubt es uns den resultierenden Fluß mit anderen Methoden zu vergleichen.

Da es für bestimmte Modelle sein kann, daß diese Methode nicht genau genug ist, diskutieren wir eine Verbesserung. Dabei wird der Sattelpunktsmethode eine selbstkonsistente Normalordnung vorgelagert.

Als eine Alternative bestimmen wir den Renormierungsgruppenfluß durch eine Störungsentwicklung des effektiven Potentials. Die Entwicklung wird durch die Ordnung des Fluktuationspropagators bestimmt.

Im letzten Teil werden numerische Resultate vorgestellt und verglichen. Außerdem bestimmen wir die kritische Linie mit der Hilfe der selbstkonsistenten Sattelpunktsmethode.

In dieser Arbeit behandeln wir skalare Feldtheorien als ein Beispiel. Die grundlegenden Gültigkeitsgrenzen der Methoden werden auch diskutiert. Sie werden durch mögliche Instabilitäten verursacht, die auftreten könnnen, wenn z.B. ein zusammengesetztes Blockspinfeld nötig wäre.

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1 Introduction

Effective field theories describe quantum field theories at low energies. They arise from general quantum field theories which describe also high energies by somehow integrating out high energy *fluctuations*. This reduces the degrees of freedom which makes the treatment of these theories more manageable. That is one important reason why effective field theories are used. For low energies effective field theories describe the same physics as the original field theories. The effect of the fluctuations which influence also the low energy physics are taken into account by more complicated interactions in the effective action.

In general the calculation of the effective action, which is part of a renormalization group transformation, can not be done analytically in exact form. Approximations have to be applied. First we treat the D-dimensional space as a finite D-dimensional (euclidian) lattice. On the finite lattice numerical calculations with the computer are possible. Second we approximate the integration over the fluctuations which is still a ∞-dimensional problem.

In section 2 we give a short overview of the self-consistent approximation developed in this paper. In section 3 the first method is developed, which we call *saddle point approximation*. The next section presents an algorithm which implements the saddle point approximation. Because in general it is not accurate enough we improved it with a preceded normal ordering which is described in section 5. Section 6 lists some reasons how the method described so far can fail.

Section 7 describes an alternative way of calculating the flow of the effective potential. It uses perturbation theory and is based on a polynomial approximation of the potential. The effective potential is expanded up to third order in the fluctuation propagator.

We used numerical calculations on the lattice to compare these approximations of the calculation of the effective potential. The results of the numerical studies are summarized in section 8. The saddle point approximation proves to be a useful method with good accuracy whereas the perturbative expansion is not well suited for the iterative calculation of the flow. It fails for large values of the field. A comparison between the result the self-consistent saddle point approximation, perturbation theory and Monte Carlo data for the effective potentials is shown in figure 14. Also in this section we determined the critical line by means of the saddle point approximation. The hope is to find a fixed point with the help of the critical line. Because the method is limited to m_0^2 which are not too strong negative only some part of the critical line can be studied.

The conclusions summarize this paper.

Sections 2, 3 and 5 have been a joint work with G. Mack, G. Palma and Y. Xylander and have already been published in [8].

2 Renormalization group flow on the lattice

Suppose we start from the Euclidean action $S^0[\phi^0]$ of a field theory which lives on some lattice Λ^0 or on the continuum. It depends on a field ϕ^0 on Λ^0. We wish to compute a sequence of effective actions $S^i[\phi^i]$ which live on lattices Λ^i of increasing lattice spacing a_i, i=0,1,2,... .

In principle the sequence of actions is defined once we fix a block-spin definition. The block-spin definition determines a field $\Phi = \phi^{i+1}$ on the coarser lattice Λ^{i+1} as some kind of average

$$\Phi = \mathcal{C}^i \phi$$

of the field $\phi = \phi^i$ on the lattice Λ^i. The actions are defined recursively by

$$Z^i[\Phi] = e^{-S^{i+1}[\Phi]} = \int D\phi\, \delta(\mathcal{C}^i\phi - \Phi) e^{-S^i[\phi]}. \tag{1}$$

For given Φ, $Z^i[\Phi]$ is the partition function of an *auxiliary theory* in which only those variables are integrated out which are to be interpreted later as high frequency modes of the field.

We will also need certain expectation values of this auxiliary theory. If $A = A[\phi]$ is an observable, we set

$$\langle A\rangle^i = Z^i[\Phi]^{-1} \int D\phi A[\phi] \delta(\mathcal{C}^i\phi - \Phi) e^{-S^i[\phi]}. \tag{2}$$

These expectation values depend on Φ, but we neglect to indicate this dependence explicitly. In the calculations, two auxiliary Φ-dependent quantities will play an important role, the *background field* $\Psi^i[\Phi]$ and the *fluctuation propagator* $\Gamma^i[\Psi^i]$. The fluctuation propagator is considered to depend on the block-spin Φ through the background field $\Psi^i = \Psi^i[\Phi]$. Both auxiliary quantities are defined as expectation values of the auxiliary theory.

$$\Psi^i(z) \equiv \Psi^i[\Phi](z) = \langle\phi(z)\rangle^i, \tag{3}$$

$$\Gamma^i[\Psi^i](z,w) = \langle\phi(z)\phi(w)\rangle^i - \langle\phi(z)\rangle^i\langle\phi(w)\rangle^i. \tag{4}$$

These definitions differ from those of Gawedzki and Kupiainen, cp. [4, 13]. In their work, expectation values of a free theory were used. The present approach is more in the spirit of Balaban's work on gauge theories, where expansions of the full action around its minimum were used for reasons of gauge invariance [2].

The validity of the method requires that the auxiliary theory has an infrared cutoff of order a_{i+1}^{-1}. Therefore the fluctuation propagator must decay exponentially with decay length no larger than about a_{i+1}. When this condition is violated, it signals a bad choice of the block-spin definition.

We are not able to calculate the functional integrals (1,2) exactly. Therefore approximations are necessary. The calculation is done in a self-consistent way. Both the background field and the fluctuation propagator are used in the calculation. Conditions will be imposed on them which ensure that equations (3) and (4) are fulfilled within the accuracy of the approximation. We discuss in subsection 3.4 how a truncation of the effective action to a manageable form can be justified. No expansion in powers of fields is involved.

It follows from the definition (4) that the fluctuation propagator satisfies the constraints

$$\mathcal{C}^i\Gamma^i[\Psi^i] = 0 = \Gamma^i[\Psi^i]\mathcal{C}^{i*}. \tag{5}$$

The simplest choice of block-spin for a theory of scalar fields is as follows. One identifies the sites x of the coarser lattice Λ^{i+1} with disjoint hypercubes in the lattice Λ^i, and one chooses the block-spin as the average of the original field on these hypercubes

$$\Phi(x) = \int_z \mathcal{C}^i(x,z)\phi(z) = \operatorname*{av}_{z\in x}\phi(z), \qquad \text{where} \qquad \int_z \equiv (a_i)^D \sum_{z\in\Lambda^i}.$$

One can try to improve on the locality properties of the effective actions by modifying the block-spin procedure [9].

We may imagine starting from a lattice of finite volume. After a finite number of steps we arrive at a lattice Λ^N which consists of only a single site. The field Φ on this lattice is some average of an average ... of an average of the original field ϕ^0. Let us interpret it as magnetization. The action $S^N[\Phi]$ will give us the constraint effective potential – i.e. the free energy – as a function of the magnetization.

We will make the following assumptions on the choice of block-spin in this paper. We assume that $\mathcal{C}^i$ is a linear map, and that $\mathcal{C}^i\mathcal{C}^{i*}$ is invertible. (For gauge theories, linearity will have to be given up.) We assume also that $\mathcal{C}^{i*}$ interpolates constant fields to constant fields, and we impose the following normalization condition

$$\int_{x\in\Lambda^{i+1}} \mathcal{C}^i\phi(x) = \int_{z\in\Lambda^i}\phi(z)$$

for constant fields. For the above mentioned choice of block-spin, the equation is an identity for arbitrary fields ϕ.

Let $\mathcal{H}^i$ be the Hilbert space of square summable functions on Λ^i, and let $\mathcal{H}^i_{\mathcal{C}}$ be the subspace of functions ζ^i with vanishing block average $\mathcal{C}^i\zeta^i = 0$. Invertibility of $\mathcal{C}^i\mathcal{C}^{i*}$ ensures the existence of the orthogonal decomposition

$$\mathcal{H}^i = \widetilde{\mathcal{C}}^{i*}\mathcal{H}^{i+1} \oplus \mathcal{H}^i_{\mathcal{C}},$$

where $\widetilde{\mathcal{C}}^i = (\mathcal{C}^i\mathcal{C}^{i*})^{-1}\mathcal{C}^i$. This is true because every $\phi \in \mathcal{H}^i$ can be decomposed as

$$\phi = \widetilde{\mathcal{C}}^{i*}\Phi + \zeta^i$$

with $\Phi = \mathcal{C}^i\phi \in \mathcal{H}^{i+1}$ and $\zeta^i = \phi - \widetilde{\mathcal{C}}^{i*}\Phi$. The first summand is obviously in $\widetilde{\mathcal{C}}^{i*}\mathcal{H}^{i+1}$ and the second is in $\mathcal{H}^i_{\mathcal{C}}$ because it vanishes when we apply $\mathcal{C}^i$: $\mathcal{C}^i\zeta^i = \mathcal{C}^i\phi - \mathcal{C}^i\widetilde{\mathcal{C}}^{i*}\Phi = \Phi - \Phi = 0$. Also the orthogonal decomposition is true because the scalar product vanishes: $(\zeta^i, \widetilde{\mathcal{C}}^{i*}\Phi) = ((\mathcal{C}^i\mathcal{C}^{i*})^{-1}\mathcal{C}^i\zeta^i, \Phi) = 0$.

Notation: We will use letters $z, w...$ for sites in Λ^i and $x, y...$ for sites in Λ^{i+1}.

3 The saddle point approximation

3.1 Parameterization of the action

We wish to compute the action $S^{i+1}[\Phi]$ which depends on a field Φ on the lattice Λ^{i+1} from the action $S^i[\phi]$ on the finer lattice Λ^i. We will make some approximations to perform the calculation. We present a parameterization of the actions which will be preserved by the approximate renormalization group flow.

The action $S^{i+1}[\Phi]$ will depend on the field Φ through a functional $\Psi^i[\Phi]$. The field Ψ^i lives on the lattice Λ^i. It is called the background field.

The recursion formula will have the form

$$S^{i+1}[\Phi] = S^i[\Psi^i] - \frac{1}{2}\mathrm{tr}\ln\Gamma^i[\Psi^i], \tag{6}$$

$$\Psi^i = \Psi^i[\Phi]. \tag{7}$$

Approximations will be made such that

1. The parameterization of the action preserves its form
2. The fluctuation propagator, the effective action and its first two field derivatives, and the background field $\Psi^i[\Phi]$ and its derivative $\Psi^{i\prime}[\Phi]$ with respect to the field Φ will enter into the recursion relations, but they need only be calculated for constant fields. This computational problem is fit for a PC.

We will use a dual notation for derivatives with respect to fields

$$S^{i\prime}[\phi](z) \equiv S^i{}_{,z}[\phi] = \frac{\delta}{\delta\phi(z)}S^i[\phi]\ ,$$

and similarly for second derivatives $S^i{}_{,zw}$.

The background field Ψ^i will be determined as a functional of Φ by the saddle point condition which involves the previous action

$$S^i[\Psi^i] = min \ \text{ subject to } \ \mathcal{C}^i\Psi^i = \Phi\ .$$

The fluctuation field propagator Γ^i lives on lattice Λ^i. It is a selfadjoint integral operator with kernel

$$\Gamma^i[\phi](z,w)\ , \quad z,w \in \Lambda^i$$

and is defined as pseudoinverse of $S^{i\prime\prime}[\phi]$ for all ϕ in the sense that

$$\Gamma^i[\phi]S^{i\prime\prime}[\phi]\Gamma^i[\phi] = \Gamma^i[\phi]. \tag{8}$$

In the case where the field ϕ is equal to the background field $\Psi^i[\Phi]$, the fluctuation field propagator is positive definite on $\mathcal{H}^i_{\mathcal{C}}$. It satisfies the constraints (5), viz.

$$\mathcal{C}^i\Gamma^i[\Psi^i] = 0 = \Gamma^i[\Psi^i]\mathcal{C}^{i*}. \tag{9}$$

Therefore the space $\mathcal{H}^i$ is mapped into $\mathcal{H}^i_{\mathcal{C}}$ by Γ^i, and Γ^i vanishes on $\widetilde{\mathcal{C}}^{i*}\mathcal{H}^{i+1}$. The trace tr which appears in the action (6) is to be understood as a trace over $\mathcal{H}^i_{\mathcal{C}}$.

We will explain later on how the background field and the fluctuation propagator can be evaluated by a self consistent approximation which involves neglect of higher order terms in gradients of fields.

It is the crux of any real space renormalization group method to find truncated forms of the effective actions which can be parameterized in a manageable form.

Often this is done in an ad hoc fashion which throws away some pieces which are irrelevant in a *perturbative* sense - higher powers of fields, for instance. This is not

really justified because existing irrelevant terms get suppressed in the next RG-step. But new irrelevant terms are created by the marginal and relevant ones at the same time. As a consequence, there is a kind of equilibrium, so that (along the renormalized trajectory) the irrelevant terms are determined by the relevant and marginal ones. They are not necessarily very small and they influence the flow of the marginal and relevant coupling constants. Throwing them away after each renormalization group step introduces therefore a systematic error which accumulates.

In principle there is a better way. One could determine the irrelevant pieces as a function of the marginal and relevant ones by solving a fixed point equation. But until now this is practical only in simplified (hierarchical) models [16].

The motivation for our method of truncation will be described below. We will argue that our approximation becomes more and more accurate the larger the scaling factor (ratio of lattice spacings) $s = a_{i+1}/a_i$.

On the other hand, the saddle point approximation becomes exact in the limit $s \mapsto 1$ (when the nature of the cutoff permits such a limit). It becomes less accurate with increasing scaling factor because the phase space for high frequency modes increases.

The truncation consists of a local approximation to the second derivative $S^{i\prime\prime} = -W$ of the action. For more precise notation, set

$$-S^i{}_{,zw}[\phi] = W^i[\phi](z,w) \tag{10}$$

$$\left(W^i[\phi]f\right)(z) = \int_w W[\phi](z,w)f(w) \tag{11}$$

We will approximate W^i by a function $\overline{W}^i(z,w|\xi)$ which depends only on the value ξ of the field at one site of the lattice Λ^i. If $f \in \mathcal{H}^i$ is a function on Λ^i then we approximate

$$\left(W^i[\phi]f\right)(z) = \frac{1}{2}\int_w \left[\overline{W}^i(z,w|\phi(z)) + \overline{W}^i(z,w|\phi(w))\right] f(w)\ . \tag{12}$$

$\overline{W}^i$ can be determined from the knowledge of W^i for *constant* fields ϕ. Symmetrization is performed to maintain hermiticity of W.

To understand the meaning of the approximation, suppose that S^i is the action of a ϕ^4-theory. Then

$$\left(S^{i\prime\prime}f\right)(z) = \left[-\Delta + m^2 + \frac{1}{2}\lambda\phi(z)^2\right] f(z). \tag{13}$$

We see that our approximation is exact in this case because the field dependent part of $S^{i\prime\prime}$ only involves the field ϕ at a single point z.

In particular, the approximation is exact for free field theories. Since the saddle point approximation is also exact for free fields, our method is exact for free field theory.

A general action can be decomposed into a potential and a term whose first derivative with respect to the field vanishes for constant fields. We call it a generalized kinetic term.

$$S^i[\phi] = \text{generalized kinetic term} + V^i[\phi]\ . \tag{14}$$

A field independent contribution to V^i is of no interest. Therefore it suffices to know the derivative $V^{i\prime}$. It is uniquely determined if we know the derivative $S^{i\prime}$ of the action for *constant* fields.

Our method consists in deriving recursion relations for V^i and W^i i.e. for the first and second derivatives $S^{i\prime}$ and $W^i = -S^{i\prime\prime}$ of the action evaluated at *constant* fields.

Moreover we wish to obtain the effective actions as a function of the block-spin and not only as a function of the background field. Therefore we will need information on the block-spin dependence of the background field. It turns out that the use of the recursion formulae requires knowledge of the derivative of the background field with respect to the block-spin. There is a formula for this:

$$\Psi^i{}_{,x}[\Phi](z) \equiv \frac{\delta}{\delta\Phi(x)}\Psi^i[\Phi] = (1-\Gamma^i S^{i\prime\prime})\tilde{\mathcal{C}}^{i*}(z,x). \tag{15}$$

For a block-spin $\overline{\Phi}$ which lives on a lattice made of a single site, translation invariance implies that there exists a saddle point $\Psi^i = \Psi^i[\overline{\Phi}]$ of the action which is also constant. Because of the constraint $\mathcal{C}^i\Psi^i = \Phi$ and our normalization conventions it follows that

$$\Psi^i[\overline{\Phi}] = \widetilde{\mathcal{C}}^{i*}\overline{\Phi}\ . \tag{16}$$

So we know the background field in this case, assuming the saddle point is the minimum. If it is not, this signals a breakdown of stability. We will come back to such possibilities. They are familiar from the Maxwell construction in thermodynamics.

The argument can be extended to show that to any block-spin $\overline{\Phi}$ which is constant on Λ^{i+1} there exists a saddle point (16) which is constant, assuming that translational invariance is not spontaneously broken. If the invariance under translations by block lattice vectors is not spontaneously broken in the auxiliary theory with expectation values (2), then the appropriate saddle point Ψ will be invariant under block lattice translations. Therefore it is equal to the constraint minimum of the action on a single block with periodic boundary conditions. Now we can appeal to translation invariance of the action under translations on the fine lattice Λ^i again to conclude that there exists a constant saddle point unless translational symmetry is spontaneously broken. It may happen that it is spontaneously broken or the saddle point is not a minimum. Again, this signals a breakdown of stability.

In view of its interpretation as a propagator for the high frequency modes of the field which are integrated out in a renormalization group step, the kernel $\Gamma^i[\phi](z,w)$ of the fluctuation field propagator must decay exponentially with distance $|z-w|$ with decay length at most one block lattice spacing a_{i+1} (see [1]). This is a condition which limits the range of applicability of the present method. It could be monitored during the computation of the renormalization group flow. When it is violated, this will typically entail a violation of the locality properties of the next action S^{i+1} as well. Locality requires that $S^i{}_{,zw}$ decays exponentially with the distance between z and w with decay length no larger than one lattice spacing a_i.

When this is violated it is typically a sign that the choice of the block-spin definition is not appropriate. This can happen for several reasons, see section 6.

3.2 The detailed procedure of saddle point approximation

Let S^i be a generic action. We now want to evaluate the effective action S^{i+1}

$$e^{-S^{i+1}[\Phi]} = \int D\phi\,\delta(\mathcal{C}^i\phi - \Phi)e^{-S^i[\phi]} \tag{17}$$

in a saddle point approximation. For this purpose we split the field

$$\phi(z) = \Psi^i[\Phi](z) + \zeta^i(z) \quad (z \in \Lambda^i).$$

into the background field $\Psi^i[\Phi]$ and a fluctuation field ζ^i. The background field has also the meaning of a mean field, because in the Gaussian approximation that we will use for the action S^i eq. (3) holds.

The mean field Ψ^i is determined as a functional of the block-spin field Φ by a saddle point condition which involves the previous action

$$S^i[\Psi^i] = min \ \text{ subject to } \ \mathcal{C}^i\Psi^i = \Phi \ .$$

The result is a nonlinear equation for $\Psi^i[\Phi]$

$$S^{i\prime}[\Psi^i] = \mathcal{C}^{i*}\lambda^i, \tag{18}$$
$$\mathcal{C}^i\Psi^i = \Phi \ . \tag{19}$$

$\lambda^i[\Phi](x)$ are Lagrange multipliers, $x \in \Lambda^{i+1}$.

Expanding the action around the mean field to second order in ζ^i yields

$$S^i[\Psi^i + \zeta^i] = S^i[\Psi^i] + \int_z S^i{}_{,z}[\Psi^i]\zeta^i(z) + \frac{1}{2}\int_{z,w} S^i{}_{,zw}[\Psi^i]\zeta^i(z)\zeta^i(w). \tag{20}$$

The linear term in the expansion vanishes

$$(S^{i\prime}, \zeta^i) = (\mathcal{C}^{i*}\lambda^i, \zeta^i) = (\lambda^i, \mathcal{C}^i\zeta^i) = 0,$$

since due to (19) $\mathcal{C}^i\zeta^i = 0$. Therefore we get

$$e^{-S^{i+1}[\Phi]} \approx \int_{\mathcal{H}^i} D\zeta^i \delta(\mathcal{C}\zeta^i) e^{-S^i[\Psi^i[\Phi]] - \frac{1}{2}\zeta^i S^{i\prime\prime}[\Psi^i[\Phi]]\zeta^i} \tag{21}$$
$$= e^{-S^i[\Psi^i[\Phi]]} \int_{\mathcal{H}^i_{\mathcal{C}}} D\zeta^i e^{-\frac{1}{2}\zeta^i \Gamma^i[\Psi^i[\Phi]]^{-1}\zeta^i} \tag{22}$$

Note that $S^{i\prime\prime}$ is a linear operator on $\mathcal{H}^i$. In appendix A.1 we show that Γ^i can be written as

$$\Gamma^i = S^{i\prime\prime -1} - S^{i\prime\prime -1}\mathcal{C}^{i*}(\mathcal{C}^i S^{i\prime\prime -1}\mathcal{C}^{i*})^{-1}\mathcal{C}^i S^{i\prime\prime -1} \ . \tag{23}$$

Because of $\mathcal{C}^i\Gamma^i = 0 = \Gamma^i\mathcal{C}^{i*}$ the fluctuation propagator vanishes on $\mathcal{C}^{i*}\mathcal{H}^{i+1}$ and is a linear operator $\mathcal{H}^i \to \mathcal{H}^i_{\mathcal{C}}$. Therefore we can use the quantity $(\Gamma^i)^{-1} = S^{i\prime\prime}\Big|_{\mathcal{H}^i_{\mathcal{C}}}$ in (22). The extension of Γ^i to all of $\mathcal{H}^i$ is a pseudoinverse of $S^{i\prime\prime}$ as described by (8).

The remaining fluctuation integral is then Gaussian with covariance $\Gamma^i[\Psi^i[\Phi]]$ and may easily be performed. The result is

$$S^{i+1}[\Phi] = S^i[\Psi^i[\Phi]] - \frac{1}{2}\mathrm{tr}\ln\Gamma^i[\Psi^i[\Phi]], \tag{24}$$

The trace tr is to be understood as a trace over $\mathcal{H}^i_{\mathcal{C}}$.

If one uses in (21)

$$\delta(\phi) = \lim_{\kappa\to\infty} e^{-\frac{\kappa}{2}\int_z \phi(z)^2} \ , \tag{25}$$

one obtains the following alternative expression for the fluctuation propagator.

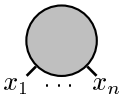
$$\Gamma^i[\Psi^i[\Phi]] = \lim_{\kappa\to\infty} \left(S^{i\prime\prime}[\Psi^i[\Phi]] + \kappa\mathcal{C}^{i*}\mathcal{C}^i\right)^{-1} \ . \tag{26}$$

3.3 Recursion relations

By iterating the saddle point approximation we obtain S^{i+2} from S^{i+1}:

$$S^{i+2}[\varphi] = S^{i+1}[\Psi^{i+1}[\varphi]] - \frac{1}{2}\mathrm{tr}\ln\Gamma^{i+1}[\Psi^{i+1}[\varphi]].$$

$$-\frac{\delta^n}{\delta\phi(z_1)\cdots\delta\phi(z_n)}S^i[\phi] \quad = \quad$$

$$-\frac{\delta^n}{\delta\Phi(x_1)\cdots\delta\Phi(x_n)}S^{i+1}[\Phi] \quad = \quad$$ 

$$\frac{\delta}{\delta\Phi(x)}\Psi^i[\Phi](z) \quad = \quad$$

Figure 1: Definition of graphical notation for the derivatives of the action and of the background field

Now we need Γ^{i+1}. It is defined in terms of $S^{i+1\prime\prime}$ by an equation similar to (23). Therefore we try to get a recursion relation for

$$W^i[\phi] = -S^{i\prime\prime}[\phi]\,.$$

In appendix A.5 the following recursion relation will be deduced by differentiating eq. (6). (We neglect to write functional $\Psi^i[\Phi]$-dependencies on the right hand side.)

$$\begin{aligned}
&W^{i+1}[\Phi](x,y) = \\
&\int_{z,w\in\Lambda^i} W^i(z,w)\Psi^i{}_{,y}(z)\Psi^i{}_{,x}(w) \\
&+\ \frac{1}{2}\int_{z,w,z_i\in\Lambda^i} \Gamma^i(z_1,z_2)W^i{}_{,z}(z_2,z_3)\Gamma^i(z_3,z_4)W^i{}_{,w}(z_4,z_1)\Psi^i{}_{,x}(z)\Psi^i{}_{,y}(w) \\
&+\ \frac{1}{2}\int_{z,w,z_1,z_2\in\Lambda^i} \Gamma^i(z_1,z_2)W^i{}_{,zw}(z_2,z_1)\Psi^i{}_{,x}(z)\Psi^i{}_{,y}(w) \\
&+\ \frac{1}{2}\int_{z,w,z_j\in\Lambda^i} \Gamma^i(z_1,z_2)W^i{}_{,z}(z_2,z_1)\Gamma^i(z,z_3)W^i{}_{,w}(z_3,z_4)\Psi^i{}_{,y}(w)\Psi^i{}_{,x}(z_4).
\end{aligned} \tag{27}$$

This functional relation is an exact consequence of eq. (6), no localization approximation has been made yet.

We see that we need an equation for the derivative $\Psi^{i\prime}$ of the background field. In appendix A.2 it will be shown that

$$\Psi^{i\prime} = (1-\Gamma^i S^{i\prime\prime})\widetilde{\mathcal{C}}^{i*}. \tag{28}$$

The recursion relation is shown in graphical form in figure 2. The graphical notation is described in figure 1.

We see that the recursion relation for $S^{i\prime\prime}$ contains a 1-particle reducible term. This is an artifact of the simple saddle point approximation which will disappear in the improved version of section 5.

As described in section 3.1 the action S^i can be split into a potential term $V^i[\phi]$ and a generalized kinetic term whose first field derivative vanishes for constant fields; the potential term is determined by the derivative $S^{i\prime}$ evaluated at constant field.

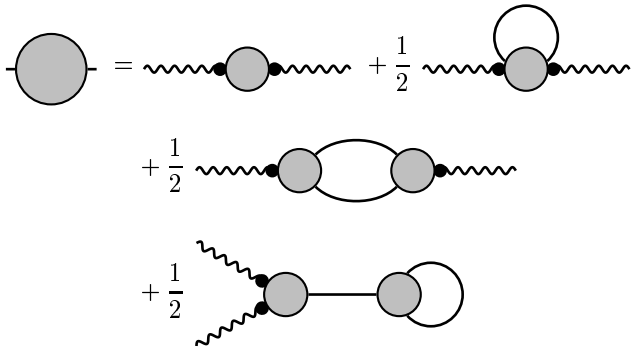


Figure 2: Recursion relation for $W^i = -S^{i\prime\prime}$ in saddle point approximation

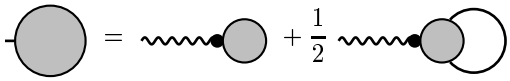


Figure 3: Recursion relation for $-S^{i+1\prime}$ in saddle point approximation

We give the recursion relation for the first derivative $S^{i\prime}$ in a general form. Using (see appendix A.3)

$$\Gamma^i{}_{,z} = \Gamma^i W^i{}_{,z} \Gamma^i \,. \tag{29}$$

one gets by differentiation of (24)

$$S^{i+1}{}_{,x}[\Phi] = \int\limits_z \Psi^i{}_{,x}[\Phi](z) \Big\{ S^i{}_{,z}[\Psi^i[\Phi]] - \frac{1}{2} \mathrm{tr} \Gamma^i[\Psi^i[\Phi]] W^i{}_{,z}[\Psi^i[\Phi]] \Big\} \,. \tag{30}$$

The graphical illustration is shown in figure 3.

For constant block-spin field the derivative of the generalized kinetic term vanishes. Therefore the recursion relation for the potential term has the same form as (30):

$$V^{i+1}{}_{,x}[\Phi] = \int\limits_z \Psi^i{}_{,x}[\Phi](z) \Big\{ V^i{}_{,z}[\Psi^i[\Phi]] - \frac{1}{2} \mathrm{tr} \Gamma^i[\Psi^i[\Phi]] W^i{}_{,z}[\Psi^i[\Phi]] \Big\} \,. \tag{31}$$

3.4 Justification of the localization approximation; application to constant fields

Let us summarize what we have achieved so far. Given the functionals V^i and W^i we can compute the fluctuation propagator Γ^i via equation (23) and the derivative of the background field $\Psi^{i\prime}$ via equation (28). This enables us to calculate W^{i+1} and V^{i+1} by means of recursion relations (27) and (31). At least this can be done in principle. Because of the functional nature of these equations the calculation is too complicated for numerical purposes. We will arrive at a significant simplification by arguing that we may focus attention on constant fields.

Fields Φ with very large derivatives have very small probability because of the kinetic term in the action. We may therefore assume that Φ does not have such violent behavior.

$\Psi^i[\Phi](z)$ is an interpolation of the field Φ from the coarser lattice Λ^{i+1} to the lattice Λ^i. The interpolation is determined by a minimality condition on the action. Because of the kinetic term in the action, we expect that the result of the interpolation is a smooth function on Λ^i. This means that we may consider $\Psi^i[\Phi]$ as nearly constant over distances a_i, assuming that the scaling factor $s = a_{i+1}/a_i$ is big enough

By locality $S^i{}_{,zw}[\Psi^i[\Phi]]$ is very nearly zero if z is not within about one lattice spacing of w, and it depends only on the field within a neighborhood of about one lattice spacing a_i of z and w. Therefore we may approximate $W^i[\Psi^i]$ by a hermitian operator $\overline{W}^i$ which involves only the value $\xi = \Psi^i(z)$ of the field at one point (see also (12)),

$$W^i[\Psi^i](z,w) \approx \frac{1}{2}\Big(\overline{W}^i(z,w|\Psi^i(z)) + \overline{W}^i(z,w|\Psi^i(w))\Big), \tag{32}$$

This justifies the localization approximation for W^i. $\overline{W}^i$ has locality properties similar to a Laplacian.

To determine $W^{i+1}[\Phi]$ we need $\Gamma^i[\phi]$ for $\phi = \Psi^i[\Phi]$. $\Gamma^i(x,y)$ depends on the field ϕ on a domain of diameter about a_{i+1}. Unless Φ is constant, the field ϕ need not be nearly constant over such a big domain. But $\Gamma^i[\phi]$ is determined by $S^{i\prime\prime}[\phi] = -W^i$ and we can use the localization approximation for this latter quantity in order to determine Γ^i for general field Φ if we need it.

Looking now at the recursion relation for $W^{i+1}(x,y)$ we see that three different quantities are involved:

- The fluctuation propagator has an exponential decay of one block lattice spacing a_{i+1} (see [1] and figure 4).
- The operator $W^i(z,w)$ and its derivatives are very local by assumption and have therefore a range of one lattice spacing a_i.
- The derivative of the background field $\Psi^{i\prime}$ is also determined by these quantities and has exponential decay with decay length a_{i+1}.

Combining these arguments we see that $W^{i+1}[\Phi]$ has similar locality properties on the scale of the new lattice spacing a_{i+1} as W^i had on the scale a_i.

Therefore we will be entitled to make similar approximations on the next level as before.

Now we can perform functional derivatives of W^i for almost constant fields:

$$\begin{aligned} W^i{}_{,z}[\phi](z_1,z_2) \approx \frac{1}{2}\Big(&\overline{W}^{i\prime}(z_1,z_2|\phi(z_1))\delta(z-z_1) \\ &+ \overline{W}^{i\prime}(z_1,z_2|\phi(z_2))\delta(z-z_2)\Big) \end{aligned} \tag{33}$$

$$\begin{aligned} W^i{}_{,zw}[\phi](z_1,z_2) \approx \frac{1}{2}\Big(&\overline{W}^{i\prime\prime}(z_1,z_2|\phi(z_1))\delta(z-z_1)\delta(w-z_1) \\ &+ \overline{W}^{i\prime\prime}(z_1,z_2|\phi(z_2))\delta(z-z_2)\delta(w-z_2)\Big)\,, \end{aligned} \tag{34}$$

where $'$ denotes the derivative of $\overline{W}^i(z_1,z_2|\xi)$ with respect to the single variable ξ. These formulas are valid when smeared with test functions in variables z and w which are nearly constant over distances a_i.

Setting the block-spin field constant the fluctuation propagator becomes a function of the field value. By virtue of (28) the derivative of the background field can be written as a simple derivative:

$$\Psi^i{}_{,x}[\,\overline{\Phi}\,](z) \equiv \Psi^{i\prime}(z,x|\overline{\Phi}) = \int_{z_1}\Big(\delta(z,z_1) + \int_w \Gamma^i(z,w|\overline{\Phi})\overline{W}^i(w,z_1|\overline{\Phi})\Big)\widetilde{\mathcal{C}}^{i*}(z_1,x)\,. \tag{35}$$

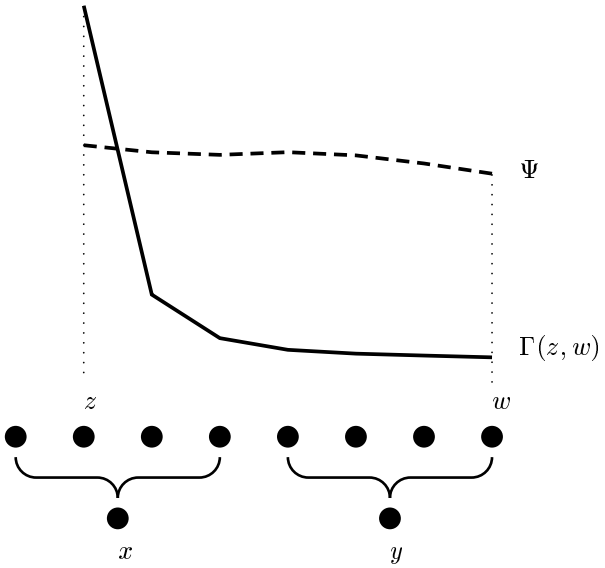


Figure 4: The decay of the fluctuation propagator.

For constant block-spin fields, inserting (33) and (34) into the recursion relation (27) for W gives therefore a recursion relation for $\overline{W}^i$:

$$\begin{aligned}
\overline{W}^{i+1}(x,y|\overline{\Phi}) &= \int_{z,w\in\Lambda^i} \overline{W}^i(z,w|\overline{\Phi})\Psi^{i\prime}(z,y|\overline{\Phi})\Psi^{i\prime}(w,x|\overline{\Phi}) \\
&+ \frac{1}{8}\int_{z_j\in\Lambda^i} \Gamma^i(z_1,z_2|\overline{\Phi})\overline{W}^{i\prime}(z_2,z_3|\overline{\Phi})\Gamma^i(z_3,z_4|\overline{\Phi})\overline{W}^{i\prime}(z_4,z_1|\overline{\Phi}) \\
&\qquad \times\Big\{\Psi^{i\prime}(z_2,x|\overline{\Phi})+\Psi^{i\prime}(z_3,x|\overline{\Phi})\Big\}\Big\{\Psi^{i\prime}(z_4,y|\overline{\Phi})+\Psi^{i\prime}(z_1,y|\overline{\Phi})\Big\} \\
&+ \frac{1}{4}\int_{z_1,z_2\in\Lambda^i} \Gamma^i(z_1,z_2|\overline{\Phi})\overline{W}^{i\prime\prime}(z_2,z_1|\overline{\Phi}) \\
&\qquad \times\Big\{\Psi^{i\prime}(z_1,x|\overline{\Phi})\Psi^{i\prime}(z_1,y|\overline{\Phi})+\Psi^{i\prime}(z_2,x|\overline{\Phi})\Psi^{i\prime}(z_2,y|\overline{\Phi})\Big\} \\
&+ \frac{1}{8}\int_{z_j\in\Lambda^i} \Gamma^i(z_1,z_2|\overline{\Phi})\overline{W}^{i\prime}(z_2,z_1|\overline{\Phi})\overline{W}^{i\prime}(z_3,z_4|\overline{\Phi})\Psi^{i\prime}(z_4,x|\overline{\Phi}) \\
&\qquad \times\Big\{\Gamma^i(z_2,z_3|\overline{\Phi})+\Gamma^i(z_1,z_3|\overline{\Phi})\Big\}\Big\{\Psi^{i\prime}(z_3,y|\overline{\Phi})+\Psi^{i\prime}(z_4,y|\overline{\Phi})\Big\}.
\end{aligned} \tag{36}$$

For constant fields we have translation symmetry. Therefore the derivative of the potential must be independent of the coordinate:

$$V^i{}_{,z}[\overline{\phi}] = \overline{V}^{i\prime}(\overline{\phi}) = \text{const.} \tag{37}$$

To obtain the recursion relation of the effective potential we insert the local approximation for W^i:

$$\begin{aligned}
\overline{V}^{i+1\prime}(\overline{\Phi}) &= \int_z \Psi^i{}_{,x}(z|\overline{\Phi})\overline{V}^{i\prime}(\overline{\Phi}) \\
&\quad - \frac{1}{4}\int_{z_1,z_2} \Gamma^i(z_1,z_2|\overline{\Phi})\overline{W}^{i\prime}(z_2,z_1|\overline{\Phi})\Big(\Psi^i{}_{,x}(z_1|\overline{\Phi})+\Psi^i{}_{,x}(z_2|\overline{\Phi})\Big).
\end{aligned} \tag{38}$$

We simplify this. By assumption of section 2

$$\int_{x\in\Lambda^{i+1}} \mathcal{C}^i(x,z) = 1\,.$$

Differentiating the constraint (19)

$$\int_z \mathcal{C}^i(x',z)\,\Psi^i{}_{,x}(z) = \delta(x'-x)\,, \tag{39}$$

and integrating over x' we see that

$$\int_{z\in\Lambda^i} \Psi^i{}_{,x}[\Phi](z) = 1\,.$$

This leads to the simpler recursion relation

$$\begin{aligned}\overline{V}^{i+1\prime}(\overline{\Phi}) &= \overline{V}^{i\prime}(\overline{\Phi}) \\ &\quad - \frac{1}{4}\int_{z_1,z_2} \Gamma^i(z_1,z_2|\overline{\Phi})\overline{W}^{i\prime}(z_2,z_1|\overline{\Phi})\Big(\Psi^i{}_{,x}(z_1|\overline{\Phi}) + \Psi^i{}_{,x}(z_2|\overline{\Phi})\Big)\,.\end{aligned} \tag{40}$$

One can use this to determine the constraint effective potential, i.e. the potential on the last lattice Λ^N which consists of one point only.

4 Numerical treatment

4.1 Getting started

We want to calculate the constraint effective potential on the lattice numerically. We consider an example. For starting action we take the standard φ^4-theory in two dimensions,

$$S^0(\phi^0) = -\int_{z,w\in\Lambda^0} \frac{1}{2}\phi^0(z)\Delta(z,w)\phi^0(w) \; + \int_{z\in\Lambda^0} \Big(\frac{1}{2}m_0^2\phi^0(z)^2 + \frac{\lambda_0}{4!}\phi^0(z)^4\Big). \tag{41}$$

To calculate $\overline{W}^1[\Phi]$ and $\overline{V}^1[\Phi]$ we need $\overline{W}^0[\phi^0]$, $\overline{W}^{0\prime}[\phi^0]$, $\overline{W}^{0\prime\prime}[\phi^0]$, $\overline{V}^{0\prime}[\phi^0]$ and $\Gamma^0[\phi^0]$ for constant fields $\phi^0 = \overline{\phi}$. We write W^0 (see also (13)) in a symmetric form:

$$\begin{aligned} W^0(z_1,z_2|\phi^0) &= \Delta(z_1,z_2) - (m_0^2 + \frac{\lambda_0}{2}\phi^0(z_1)^2)\delta(z_1-z_2) &(42)\\ &= \Delta(z_1,z_2) - \Big(m_0^2 + \frac{\lambda_0}{4}(\phi^0(z_1)^2 + \phi^0(z_2)^2)\Big)\delta(z_1-z_2) &(43)\end{aligned}$$

Therefore we have for $\overline{W}^0$

$$\overline{W}^0(z_1,z_2|\xi) = \Delta(z_1,z_2) - \Big(m_0^2 + \frac{\lambda_0}{2}\xi^2\Big)\delta(z_1-z_2) \tag{44}$$

which is not an approximation because equation (32) is *exact* in this case. Now it follows for the constant field $\overline{\phi}$

$$\begin{aligned} \overline{W}^0(z_1,z_2|\overline{\phi}) &= \Delta(z_1,z_2) - \Big(m_0^2 + \frac{\lambda_0}{2}\overline{\phi}^2\Big)\delta(z_1-z_2) &(45)\\ \overline{W}^{0\prime}(z_1,z_2|\overline{\phi}) &= -\lambda\overline{\phi}\delta(z_1-z_2) &(46)\\ \overline{W}^{0\prime\prime}(z_1,z_2|\overline{\phi}) &= -\lambda\delta(z_1-z_2) &(47)\end{aligned}$$

Similarly, the potential

$$\overline{V}^{0\prime}(\overline{\phi}) \;=\; V^0_{,z}[\phi^0]\Big|_{\phi^0=\overline{\phi}} \;=\; m_0^2\overline{\phi} + \frac{\lambda}{3!}\overline{\phi}^3 \tag{48}$$

4.2 Relations in momentum space

We have fixed the starting point. Next we want to calculate the recursion relations for $\overline{W}^i$ and $\overline{V}^i$. To reduce the amount of work we switch to momentum space. Because we insert constant fields we have translation symmetry. Therefore the Fourier transformation looks quite simple. For the notation see appendix B.

$$\begin{aligned} \overline{W}^i(z_1,z_2|\overline{\phi}) &= \int_p \overline{W}^i(p|\overline{\phi})e^{ip(z_1-z_2)} &(49)\\ \Psi^i{}_{,x}(z|\overline{\phi}) &= \int_p \Psi^i(p|\overline{\phi})e^{ip(z-x)} &(50)\\ \Gamma^i(z_1,z_2|\overline{\phi}) &= \int_{l,q,l'} \Gamma^i(l,q,l'|\overline{\phi})e^{i((q+l)z_1-(q+l')z_2)} &(51)\end{aligned}$$

and similar for the derivatives, which are now simple derivatives with respect to $\overline{\phi}$. Γ^i has only translation symmetry on the block lattice. Therefore it is not diagonal in momentum space (see appendix B).

Transforming the recursion relations into momentum space leads to

$$\begin{aligned}
\overline{W}^{i+1}(q|\overline{\phi}) &= \int_l \Psi^{i\prime *}(q+l|\overline{\phi})\overline{W}^i(q+l|\overline{\phi})\Psi^{i\prime}(q+l|\overline{\phi}) \\
&+ \frac{1}{8}\int_{q',l_1,\dots,l_4} \Gamma^i(l_1,q',l_2|\overline{\phi})\Gamma^i(l_3,q+q',l_4|\overline{\phi})\Psi^{i\prime *}(q-l_2+l_3|\overline{\phi})\Psi^{i\prime}(q-l_1+l_4|\overline{\phi}) \\
&\quad \times\Big\{\overline{W}^{i\prime}(q+q'+l_3|\overline{\phi})+\overline{W}^{i\prime}(q'+l_2|\overline{\phi})\Big\}\Big\{\overline{W}^{i\prime}(q'+l_1|\overline{\phi})+\overline{W}^{i\prime}(q+q'+l_4|\overline{\phi})\Big\} \\
&+ \frac{1}{4}\int_{q',l_1,l_2,l_3} \Gamma^i(l_1,q',l_2|\overline{\phi})\Psi^{i\prime *}(q-l_2+l_3|\overline{\phi})\Psi^{i\prime}(q-l_1+l_3|\overline{\phi}) \\
&\quad \times\Big\{\overline{W}^{i\prime\prime}(q'+l_1|\overline{\phi})+\overline{W}^{i\prime\prime}(q'+l_2|\overline{\phi})\Big\} \qquad (52) \\
&+ \frac{1}{8}\int_{q',l_1,\dots,l_4} \Gamma^i(l_1,q',l_2|\overline{\phi})\Gamma^i(-l_1+l_2,0,l_4|\overline{\phi})\Psi^{i\prime *}(q+l_3|\overline{\phi})\Psi^{i\prime}(q+l_3+l_4|\overline{\phi}) \\
&\quad \times\Big\{\overline{W}^{i\prime}(q'+l_1|\overline{\phi})+\overline{W}^{i\prime}(q'+l_2|\overline{\phi})\Big\}\Big\{\overline{W}^{i\prime *}(q+l_3|\overline{\phi})+\overline{W}^{i\prime}(l_4|\overline{\phi})\Big\}
\end{aligned}$$

and

$$\begin{aligned}
\overline{V}^{i+1\prime}(\overline{\phi}) &= \overline{V}^{i\prime}(\overline{\phi}) - \frac{1}{4}\int_{q,l,l'} \Gamma^i(l,q,l'|\overline{\phi})\Psi^{i\prime}(-l+l'|\overline{\phi}) \\
&\quad \times\Big\{\overline{W}^{i\prime}(q+l|\overline{\phi})+\overline{W}^{i\prime}(q+l'|\overline{\phi})\Big\}. \qquad (53)
\end{aligned}$$

We need also the formula (28) in momentum space:

$$\Psi^{i\prime}(q+l|\overline{\phi}) = \mathcal{C}^{i*}(q+l) + \int_{l'} \Gamma^i(l,q,l'|\overline{\phi})\overline{W}^i(q+l'|\overline{\phi})\mathcal{C}^{i*}(q+l') \qquad (54)$$

4.3 Algorithm

We are now able to calculate all quantities needed. The following algorithm gives an overview of the procedure:

0. $i=0$: Given the starting action fix the values of $\overline{W}^0(p|\overline{\phi}_j)$ and $\overline{V}^{0\prime}(\overline{\phi}_j)$ for a number of constant field values $\overline{\phi}_1,\dots,\overline{\phi}_n$ and for all momenta p.
1. Calculate $\overline{W}^{i\prime}(p|\overline{\phi}_j)$ and $\overline{W}^{i\prime\prime}(p|\overline{\phi}_j)$ by numerical differentiation.
2. Determine $\Gamma^i(l,q,l'|\overline{\phi}_j)$ with the help of the explicit formula (23).
3. Calculate $\Psi^{i\prime}(q+l|\overline{\phi}_j)$ by (54).
4. Insert all into recursion relations for $\overline{W}^{i+1}(q|\overline{\phi}_j)$ and $\overline{V}^{i+1\prime}(\overline{\phi}_j)$.
5. Increment i and goto 1 until $i=N$.

We want to see numerically how this procedure works to produce the correct – derivative of the – effective potential. The numerical results are compared with Monte-Carlo calculations and perturbative expansion. This will be summarized in section 8.

5 Improved saddle point method with normal ordering

5.1 Advanced method with normal ordering

The localization approximation becomes more and more accurate the larger the scaling factor (ratio of lattice spacings) $s = a_{i+1}/a_i$. On the other hand, the saddle point approximation becomes exact in the limit $s \mapsto 1$ (when the nature of the cutoff permits such a limit). It becomes less accurate with increasing scaling factor because the phase space for high frequency modes increases. This is one reason why we choose to consider a fairly sophisticated improvement of the saddle point method.

The self consistent improvement of the saddle point integration consists of two steps. In the first step, the old action $S^i[\phi] = S^i[\Psi^i + \zeta^i]$ is self-consistently normal ordered in the fluctuation field ζ^i; at the same time the background field $\Psi^i = \Psi^i[\Phi]$ is determined. Symbolically

$$S^i[\phi] =: T^i[\Psi^i, \zeta^i] : \tag{55}$$

The precise definition and properties of the normal ordered form T^i of the action S^i will be described presently. In the second step, the fluctuation field integration is performed, neglecting normal products of higher order than second in the fluctuation field. The integral is again Gaussian, and the result reads

$$S^{i+1}[\Phi] = T^i[\Psi^i, 0] - \frac{1}{2}\mathrm{tr}\ln\Gamma^i[\Psi^i], \tag{56}$$

The trace is again over the space $\mathcal{H}^i_{\mathcal{C}}$ of functions ζ^i on Λ^i with vanishing block average, $\mathcal{C}^i\zeta^i = 0$.

The method of self-consistent normal ordering is very old - see [17]; sometimes it is called the Feynman Bogoliubov method. Therefore we will refer to our method here as Feynman Bogoliubov approximation. The definition and properties of the normal ordered action T^i are as follows.

Let $d\mu_\Gamma$ be the normalized Gaussian measure (free field measure) with covariance (propagator) Γ. Given the fluctuation propagator Γ^i, the normal ordered amplitude T^i is defined as

$$T^i[\Psi^i, \zeta^i] = \int d\mu_{\Gamma^i[\Psi^i]}(\xi) S[\Psi^i + \zeta^i + \xi] \tag{57}$$

$$= \exp\left(\frac{1}{2}\frac{\delta}{\delta\xi}\Gamma^i[\Psi^i]\frac{\delta}{\delta\xi}\right) S[\Psi^i + \zeta^i + \xi]\Big|_{\xi\,=\,0} \tag{58}$$

The second formula is valid when S^i is differentiable. The first formula is more general, it can be used to define a normal product expansion also for discontinuous actions S^i (cp. the section 5 in [8]).

We use the following notation for functionals with two arguments:

$$\frac{\delta}{\delta\Psi^i(z)}A[\Psi^i, \zeta^i] = A_{,z}[\Psi^i, \zeta^i], \qquad \frac{\delta}{\delta\zeta^i(z)}A[\Psi^i, \zeta^i] = A_{;z}[\Psi^i, \zeta^i].$$

The notation is a little sloppy. Because $\mathcal{C}^i\zeta^i = 0$, the second expression only makes sense after smearing with a test function f which obeys $\mathcal{C}^i f = 0$, viz.

$$\int_z f(z)\frac{\delta}{\delta\zeta^i(z)}A[\Psi^i, \zeta^i] = \frac{d}{d\tau}A[\Psi^i, \zeta^i + \tau f]\Big|_{\tau\,=\,0}.$$

$$-\frac{\delta^n}{\delta\xi^i(z_1)\cdots\delta\xi^i(z_n)}T^i[\Psi^i,\xi^i]\Big|_{\xi^i=0} \quad = \quad \overset{z_1\cdots z_n}{\text{(N)}}$$

Figure 5: Graphical notation for normal ordered amplitudes

The background field $\Psi^i = \Psi^i[\Phi]$ will be determined by the condition that the expansion of the action T^i in powers of the fluctuation field has no linear term:

$$T^i[\Psi^i,\zeta^i] = T^i[\Psi^i,0] + \frac{1}{2}\zeta^i T^i{}_{;;}[\Psi^i,0]\zeta^i + ... \qquad \text{for } \mathcal{C}^i\zeta^i = 0. \tag{59}$$

That is the saddle point condition

$$\int_z \zeta^i(z)T^i{}_{;z}[\Psi^i] = 0 \quad \text{if } \mathcal{C}^i\zeta^i = 0 \tag{60}$$

which is equivalent to

$$T^i{}_{;}[\Psi^i] = \mathcal{C}^{i*}\lambda^i, \tag{61}$$

$$\mathcal{C}^i\Psi^i = \Phi\ . \tag{62}$$

$\lambda^i[\Phi](x)$ are Lagrange multipliers, $x \in \Lambda^{i+1}$.

Now we specify the fluctuation propagator Γ^i as pseudo inverse of $T^i{}_{;;}$ so that

$$\Gamma^i[\Psi^i]T^i{}_{;;}[\Psi^i]\Gamma^i[\Psi^i] = \Gamma^i[\Psi^i]. \tag{63}$$

$$\mathcal{C}^i\Gamma^i[\Psi^i] = \Gamma^i[\Psi^i]\mathcal{C}^{i*} = 0. \tag{64}$$

As in the saddle point approximation the fluctuation propagator satisfies the constraints (5), is positive definite on $\mathcal{H}^i_{\mathcal{C}}$, and equal to the inverse of $T^i{}_{;;}$ on $\mathcal{H}^i_{\mathcal{C}}$. But $T^i{}_{;;}$ depends itself on the propagator Γ^i. Therefore (63) leads to a gap equation for Γ^i.

One sees that the background field Ψ^i is the minimum of the normal ordered form T^i of the action S^i on the surface with prescribed block-spin $\Phi = \mathcal{C}^i\phi$, and Γ^i is the inverse of the Hessian $T^i{}_{;;}$ at this minimum; it depends on Ψ^i. (The Hessian is a quadratic form on the tangent space $\mathcal{H}^i_{\mathcal{C}}$ to the surface.)

The determination of the fluctuation field propagator Γ^i involves solving a gap equation. The gap equation is obtained by differentiating eq. (58) twice at $\zeta^i = 0$.

$$T^i{}_{;;} = e^{\left(\frac{1}{2}\frac{\delta}{\delta\xi}\Gamma^i[\Psi]\frac{\delta}{\delta\xi}\right)}S^{i\prime\prime}[\Psi^i+\xi]\Big|\xi = 0 \tag{65}$$

$$\Gamma^i[\Psi^i] = T^i{}_{;;}[\Psi^i,0]^{-1} \text{ on } \mathcal{H}^i_{\mathcal{C}}. \tag{66}$$

The gap equation is shown in graphical form in figure 7. Figure 5 explains the graphical notation. Further derivatives can be applied and we obtain a generalization of eq. (65) as shown in figure 6.

This completes the discussion of the improved saddle point approximation. Eq. (56) is still a formidable functional recursion formula. We need to make it practical by our second approximation of localization. The localization approximation has to be made for the second derivative $T^i{}_{;;}$ and not for $S^{i\prime\prime}$. In addition to the formula for the second derivative $S^{i\prime\prime}$ there is a formula which expresses the first derivative $S^{i\prime}$ of the ordinary action in terms of the first derivative $T^i{}_{;}$ of the previous normal ordered action; this furnishes the potential up to an additive constant.

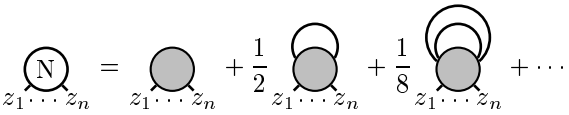


Figure 6: Definition of the normal ordered amplitudes. Further legs generated by Ψ-derivatives will be marked by primes

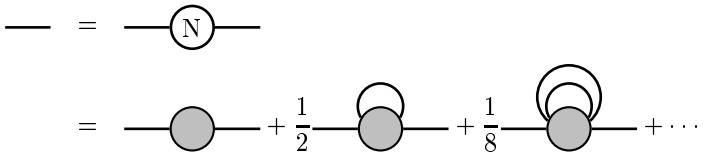


Figure 7: Gap equation for the fluctuation propagator Γ^i in Feynman-Bogoliubov approximation

We have a similar formula for the derivative of the background field:

$$\Psi^i{}_{,x}(z) = (1 - \Gamma_B{}^i T^i{}_{;,})\tilde{\mathcal{C}}^{i*}(z,x). \tag{67}$$

where $\Gamma_B{}^i$ is defined as inverse of $T^i{}_{;,}$ on $\mathcal{H}^i_{\mathcal{C}}$. In contrast to Γ^i it is defined as inverse of the mixed derivative $T^i{}_{;,}$ which involves derivatives with respect to the fluctuation field and the background field.

5.2 Gaussian integration

We wish to evaluate the integral (1) for S^{i+1}, using the quadratic approximation (55,59) for S^i, viz.

$$S^i[\Psi^i + \zeta^i] = T^i[\Psi^i, 0] + \frac{1}{2} : \zeta^i T^i{}_{;;}[\Psi^i, 0]\zeta^i :$$

We undo the normal ordering again. Dropping arguments $\xi = 0$ we obtain

$$S^i[\Psi^i + \zeta^i] = T^i[\Psi^i] + \frac{1}{2}\zeta^i T^i{}_{;;}[\Psi^i]\zeta^i - \frac{1}{2}\mathrm{tr} T^i{}_{;;}[\Psi^i]\Gamma^i[\Psi^i]. \tag{68}$$

The trace is over $\mathcal{H}^i_{\mathcal{C}}$. By arguments similar to appendix A.1 the fluctuation propagator has the explicit form

$$\Gamma^i = T^i{}_{;;}{}^{-1} - T^i{}_{;;}{}^{-1}\mathcal{C}^{i*}(\mathcal{C}^i T^i{}_{;;}{}^{-1}\mathcal{C}^{i*})^{-1}\mathcal{C}^i T^i{}_{;;}{}^{-1}. \tag{69}$$

As in the saddle point approximation there is an alternative formula for the fluctuation propagator

$$\Gamma^i[\Psi^i[\Phi]] = \lim_{\kappa\to\infty}\left(T^i{}_{;;}[\Psi^i[\Phi]] + \kappa\mathcal{C}^{i*}\mathcal{C}^i\right)^{-1}. \tag{70}$$

We argue that the last term in (68) is a constant independent of Ψ^i. Multiplying $T^i{}_{;;}$ from the right of (69) gives

$$\Gamma^i T^i{}_{;;} = 1 - T^i{}_{;;}{}^{-1}\mathcal{C}^{i*}(\mathcal{C}^i T^i{}_{;;}{}^{-1}\mathcal{C}^{i*})^{-1}\mathcal{C}^i. \tag{71}$$

We take the trace of (71)

$$\begin{aligned}\mathrm{tr}(\Gamma^i T^i{}_{;;}) &= \mathrm{tr}1 - \mathrm{tr}\Big(T^i{}_{;;}{}^{-1}\mathcal{C}^{i*}(\mathcal{C}^i T^i{}_{;;}{}^{-1}\mathcal{C}^{i*})^{-1}\mathcal{C}^i\Big) \\ &= \mathrm{tr}\ \delta_z(0) - \mathrm{tr}\ \delta_x(0) \\ &= \sum_z 1 - \sum_x 1 = \text{const.} \qquad \text{q.e.d.}\end{aligned}$$

Now we can again evaluate the functional integral (1) in Gaussian approximation with the result

$$S^{i+1}[\Phi] = T^i[\Psi^i[\Phi], 0] - \frac{1}{2}\mathrm{tr}\ln\Gamma^i[\Psi^i[\Phi]]. \tag{72}$$

5.3 Relations between derivatives of the normal ordered action

We derive a relation between the Ψ^i- and ξ-derivative by using the following well known change of covariance lemma for Gaussian measures [5]. If the covariance Γ_s depends parametrically on a variable s, then

$$\frac{d}{ds}\int d\mu_{\Gamma_s}(\phi)A[\phi] = \frac{1}{2}\int d\mu_{\Gamma_s}(\phi)\frac{\delta}{\delta\phi}\dot{\Gamma}_s\frac{\delta}{\delta\phi}A[\phi]\,, \tag{73}$$

where $\dot{\Gamma}_s = d\Gamma_s/ds$.

We denote Ψ^i-derivatives by commas "," and ξ-derivatives by semicolons ";" as before. Applying the change of covariance lemma to the definition (57) of the normal ordered amplitude we obtain the relation

$$T^i{}_{,z}[\Psi^i] = T^i{}_{;z}[\Psi^i] + \frac{1}{2}\mathrm{tr}\Gamma_{,z}[\Psi^i]T^i{}_{;;}[\Psi^i]. \tag{74}$$

After another ξ-derivative we get

$$T^i{}_{,z\,;w}[\Psi^i] = T^i{}_{;zw}[\Psi^i] + \frac{1}{2}\mathrm{tr}\Gamma_{,z}[\Psi^i]T^i{}_{;;;w}[\Psi^i]. \tag{75}$$

5.4 Recursion relation for the first and second derivatives of the action

We differentiate the recursion relation (72) to obtain a formula for the first derivative of the action S^{i+1}. The derivative of the tr ln is computed in the same way as before. The result reads

$$S^{i+1}{}_{,x} = \int_z \Psi^i{}_{,x}(z)\left(T^i{}_{,z}[\Psi^i] - \frac{1}{2}\mathrm{tr}\Gamma^i{}_{,z}[\Psi^i]T^i{}_{;;}\right). \tag{76}$$

We re-express the Ψ^i-derivative in the first term using eq. (74). A cancelation occurs and we obtain the final form of the recursion relation for the first derivative of the action.

$$S^{i+1}{}_{,x} = \int_z \Psi^i{}_{,x}(z)T^i{}_{;z}[\Psi^i]. \tag{77}$$

Next we turn to the second derivative. We differentiate eq. (77) once more. Since $T^i{}_{;z}$ is the ξ-derivative of the normal ordered action we can use eq. (74) again to evaluate the derivative of $T^i{}_{;z}$ with the help of

$$\Gamma^i{}_{,z}[\Psi^i] = -\Gamma^i[\Psi^i]T^i{}_{;;\,,z}[\Psi^i]\Gamma^i[\Psi^i]. \tag{78}$$

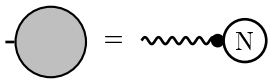


Figure 8: Recursion relation for $S^{i+1\prime}$ in the Feynman Bogoliubov approximation

Differentiating the factor $\Psi^i{}_{,x}(z)$ produces a term

$$\int_z \Psi^i{}_{,xy}(z) T_{;z}. \tag{79}$$

Differentiating the constraint (62) twice shows that $\Psi^i{}_{,xy} \in \mathcal{H}^i_{\mathcal{C}}$ because

$$\int_z \mathcal{C}^i(x',z)\Psi^i{}_{,x}(z) = \delta(x'-x)\,, \tag{80}$$

implies

$$\int_z \mathcal{C}^i(x',z)\Psi^i{}_{,xy}(z) = 0\,. \tag{81}$$

The saddle point condition (60) implies therefore that expression (79) vanishes. As a result we obtain the recursion formula for the second derivative of the action

$$\begin{aligned} S^{i+1}{}_{,xy} &= \int_{z,w} \Psi^i{}_{,x}(z) T^i{}_{;z\,,w} \Psi^i{}_{,y}(w) && (82)\\ &= \int_{z,w} \Psi^i{}_{,x}(z) \left(T^i{}_{;zw} - \frac{1}{2}\,\mathrm{tr}(\Gamma^i T^i{}_{;;\,;z} \Gamma^i T^i{}_{;;\,,w}) \right) \Psi^i{}_{,y}(w) && (83) \end{aligned}$$

We neglected to indicate arguments Ψ^i of propagators and normal ordered action. We see that there is no 1-particle reducible contribution.

A subtle feature of this formula is the appearance of two different vertices in the last term; one involves a Ψ^i-derivative and the other a ξ-derivative to create the external leg. This feature is familiar from Schwinger Dyson equations, cp. section 5 of [8]. Neglecting the difference would only neglect a two loop correction, though.

The recursion relations in Feynman Bogoliubov approximation are shown in figures 8 and 9. They involve propagators which are to be determined as solutions of the gap equation (65), shown graphically in figure 7.

5.5 Derivative of the background field

The recursion relations involve the derivative $\Psi^i{}_{,x}$ of the background field. Differentiating the saddle point condition (60) we obtain

$$\int_{z,w} \zeta^i(w) \Psi^i{}_{,x}(z) T^i{}_{;w,z}[\Psi^i] = 0 \tag{84}$$

We claim that this equation and the constraint (80) are solved by

$$\Psi^i{}_{,x}(z) = \tilde{\mathcal{C}}^{i*}(z,x) - \int_{w,u} \Gamma_B{}^i(z,w) T^i{}_{;w\,,u} \tilde{\mathcal{C}}^{i*}(u,x), \tag{85}$$

where $\tilde{\mathcal{C}}^{i*} = \mathcal{C}^{i*}(\mathcal{C}^i\mathcal{C}^{i*})^{-1}$ as before and $\Gamma_B{}^i$ is defined as the inverse of $T^i{}_{;,}$ on $\mathcal{H}^i_{\mathcal{C}}$. The proof proceeds as in the saddle point section.

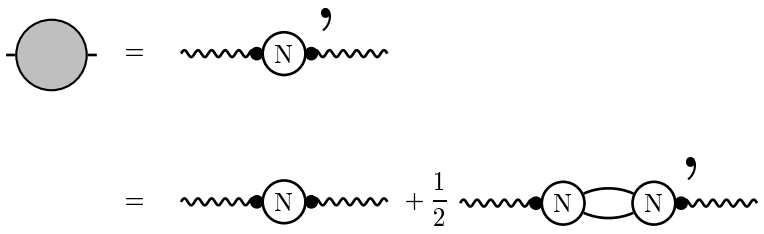


Figure 9: Recursion relations for the $S^{i+1\prime\prime}$ in the Feynman Bogoliubov approximation. The second formula is obtained from the first with the help of eq. (75).

5.6 Discussion of the localization approximation

There are new aspects of the localization approximation which need to be discussed. The consideration of this section assumed that the starting amplitude S^i is accurately known. Because of this, it is appropriate to regard the recursion relations as recursion relations for the normal ordered amplitudes T^i, and to make the localization approximation for these amplitudes, *not* for S^i. The localization approximation for T^i is justified by appealing to the same arguments as before: In computing S^{i+1} we only need $T^i{}_{;;}$ for smooth fields.

Thus we may regard one step in the recursion relations as composed of three parts

1. Localization approximation on T^i for smooth fields .
2. Computation of S^{i+1} from T^i for smooth fields
3. computation of T^{i+1} from S^{i+1}

When proceeding in this way, the result for S^{i+1} must be substituted into the (gap) equations which determine the normal ordering. In general there result graphs which contain propagators from two successive length scales i and $i+1$. This complication does not arise when only 1-loop graphs are retained.

There is one further technical complication. In contrast with $S^{i\prime\prime}$, $T^i{}_{;;}$ is not necessarily exactly invariant under arbitrary translations by lattice vectors in Λ^i but only under block lattice translations. Therefore it it is not easy to calculate its pseudoinverse (the fluctuation field propagator) exactly. One can proceed as follows, however.

Because $T^i{}_{;;}$ and Γ^i are determined by a gap equation one can start with some guess for $T^i{}_{;;}$, e.g. with the pseudoinverse of $S^{i\prime\prime}$ which was used in the saddle point approximation. Then one iterates the gap equation.

The gap equations are here considered for reasons of accuracy and not because there are infrared problems. Quite on the contrary, the propagators must have exponential decay. Therefore the iteration can be expected to converge very fast.

6 How locality can fail

Things can go badly wrong when locality of the effective action fails due to a bad choice of block-spin. It is therefore essential to monitor whether the method can be expected to give reliable results by monitoring the range

$$\operatorname*{av}_{z\in 0} \frac{\int_w |(z-w)W(z,w|\Psi)|}{\int_w |W(z,w|\Psi)|} \tag{86}$$

of the interaction for each value of the constant field which enters into the computation of the desired quantity.

Failures of locality can occur for several reasons which require different remedies.

One sees from the recursion relation for the derivatives of the action that good locality properties can only be expected if the fluctuation propagator $\Gamma^i(z,w|\Psi)$ decays with distance $(z-w)$ with decay length no more than one block lattice spacing .

For a ϕ^4-action

$$\Gamma = \lim_{\kappa\mapsto\infty} \left(-\Delta + m^2 + \frac{1}{2}\lambda\phi(z)^2 + \kappa \mathcal{C}^{i\,*}\mathcal{C}^i \right)^{-1}. \tag{87}$$

Following Balaban [1], one can estimate the decay properties by the lowest eigenvalue of the operator in brackets with Δ replaced by the Laplacian Δ^N with Neumann boundary conditions on the block boundary; basically this eigenvalue should be strictly positive and of order at least one in units where $a_{i+1} = 1$. We are interested chiefly in (nearly) constant ϕ; in this case one can also find the decay by Fourier analysis cp. ref. [4]. The $\kappa\mathcal{C}^{i\,*}\mathcal{C}^i$-term effectively eliminates the zero mode of Δ^N (constants) and the next eigenvalue of Δ^N has the desired magnitude. Therefore things go wrong when $m^2 + \frac{1}{2}\lambda\phi(z)^2$ becomes too strongly negative. This can happen when $\phi \approx 0$ and m^2 is too strongly negative - i.e. near the top of a Mexican hat which is too high, cp. figure 10.

When m^2 is negative enough, something still more terrible happens. Numerical work revealed that the minimal action for constant block-spin zero is reached for background fields which have periodic domain walls of alternating slope. In other words, the auxiliary theory of section 2 shows spontaneous breaking of translation symmetry by block lattice vectors. With respect to the approximated fluctuation integral (21) the mean field of the saddle point approximation is equal to the expectation value $\langle\phi\rangle$. This expectation value can be calculated easily with Monte Carlo for the exact fluctuation integral (17) which is believed to give an idea for the mean field. An example is shown in figure 11. We see an oscillating mean field which shows spontaneous breaking of symmetry in one direction.

The above estimate of eigenvalues shows that the instability will occur for smaller and smaller values of m_0^2 and the larger the blocking factor is in a single RG-step. In figure 11, a blocking factor 16 is used.

At intermediate RG-steps one could try to remedy this by choosing block-spins of fixed length - i.e. integrating out the modulus of the field - to obtain nonlinear σ-model type effective actions.

But anyway we cannot calculate the dependence of the constraint effective potential on the magnetization in the unstable region (figure 10) because the assumption of no spontaneous breaking of translational symmetry is physically wrong there.

Nonlocalities can appear in another more interesting way which is relevant for studies of a dynamical Higgs mechanism, in particular *superconductivity.* The remedy in this case is the introduction of a composite Bose field as a block-spin. In superconductivity it represents Cooper pairs. This mechanism was studied in some detail by Grabowski [7].

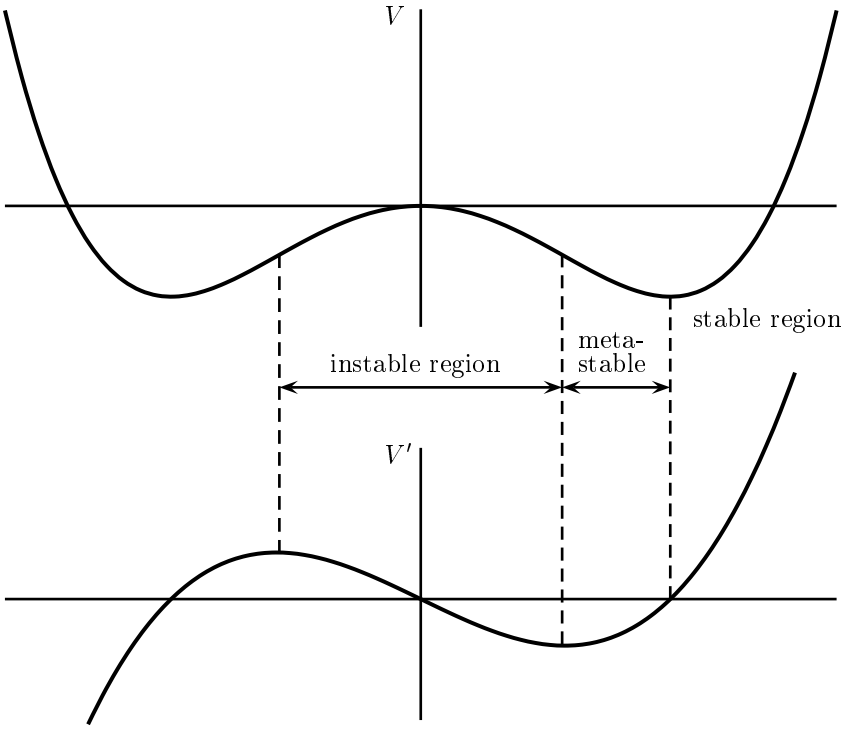


Figure 10: Potential and its derivative. The middle region is unstable and unphysical. It is separated from the stable region by a meta-stable region. Our method cannot compute the constraint effective potential in the unstable region because we assumed that translation invariance is not spontaneously broken

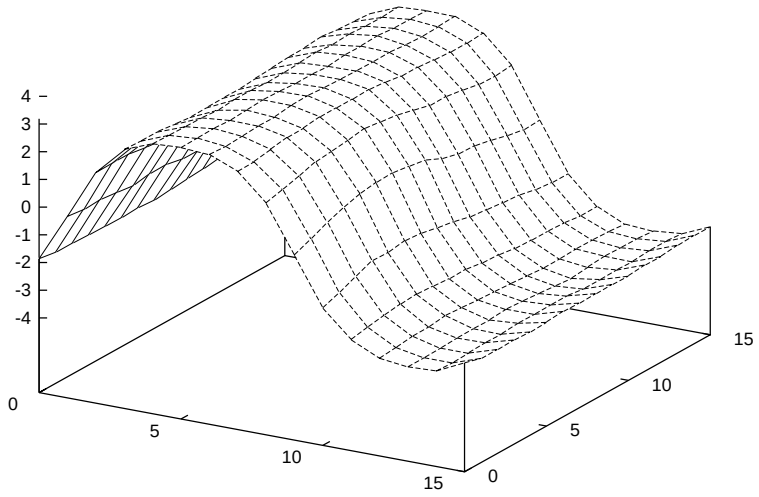


Figure 11: An example where spontaneous breakdown of translation invariance occurs in the fluctuation field integral. The expectation value of ϕ of one Monte Carlo run is plotted. This was calculated on a 16×16 lattice with blocking factor 16, $m_0^2 = -0.8$, $\lambda_0^{(4)} = 0.4$, number of equilibration sweeps is 200, nr of sweeps for measurement is 2000, $\Phi = 0.1$.

7 Perturbative calculation of the effective potential

7.1 Definition of the effective potential

As an alternative we calculated the effective potential perturbatively, using the Gawedzki-Kupianen formalism [4]. The action $S^i[\phi]$ is split into a kinetic term $\frac{1}{2}\phi(v^i)^{-1}\phi$ and a potential $V^i[\phi]$. v^i is the free propagator. Again the notion of fluctuation field ζ^i with $\mathcal{C}^i\zeta^i = 0$ is used and we have a background field Ψ^i:

$$\phi = \Psi^i + \zeta^i \,. \tag{88}$$

The background field is an interpolation of the block-spin field:

$$\Psi^i = \mathcal{A}^i\Phi \,.$$

In the Gawedzki-Kupianen formalism the interpolation operator $\mathcal{A}^i$ is a linear operator. It is defined as

$$\mathcal{A}^i = v^i\mathcal{C}^{i*}(v^{i+1})^{-1} \tag{89}$$

The background field minimizes the kinetic term in the action under the subsidiary condition $\mathcal{C}^i\Psi^i = \Phi$. Here v^{i+1} is the free propagator for the block-spin field:

$$v^{i+1} = \mathcal{C}^i v^i \mathcal{C}^{i*} \tag{90}$$

Inserting the split of the field ϕ (88) into (17) we get for the fluctuation integral

$$e^{-S^{i+1}[\Phi]} = e^{-\frac{1}{2}\Phi(v^{i+1})^{-1}\Phi - V^{i+1}[\Phi]} = e^{-\frac{1}{2}\Phi(v^{i+1})^{-1}\Phi}\int_{\mathcal{H}^i_{\mathcal{C}}} D\zeta^i e^{-\frac{1}{2}\zeta^i\Gamma^{i-1}\zeta^i - V^i[\Psi^i+\zeta^i]} \tag{91}$$

where the fluctuation propagator is defined as[1]

$$\Gamma^i = v^i - \mathcal{A}^i v^{i+1}\mathcal{A}^{i*} \,.$$

Because the kinetic part $\Phi(v^{i+1})^{-1}\Phi$ is already fixed by definition we focus on the effective potential:

$$e^{-V^{i+1}[\Phi]} = \int \mathrm{d}\mu_{\Gamma^i}(\zeta^i)\, e^{-V^i[\mathcal{A}^i\Phi+\zeta^i]} \tag{92}$$

For brevity $V^{i+1}[\Phi]$ is also written as $V^{i+1}[\psi]$ where $\psi \equiv \Psi^i = \mathcal{A}^i\Phi$.[2]

We start with ϕ^4-potential:

$$V^0[\phi^0] = \int \mathrm{d}z \Big(\frac{1}{2}m_0^2\phi^0(z)^2 + \frac{\lambda_0^{(4)}}{4!}\phi^0(z)^4\Big) \tag{93}$$

The mass term is put into the potential, i.e. the kinetic term has no mass term (v_0 is the propagator):

$$\frac{1}{2}\int \mathrm{d}z_1\mathrm{d}z_2\phi^0(z_1)v_0^{-1}(z_1,z_2)\phi^0(z_2) = -\frac{1}{2}\int \mathrm{d}z\phi^0(z)\Delta_z\phi^0(z) \tag{94}$$

Of course it is also possible to split the mass term and put one part into the inverse propagator and the other part into the interaction. .

[1] Inserting (89) and (90) one sees the same formula for Γ^i as in (23). For free fields v^i and $S^{i\prime\prime-1}$ are identical.

[2] To confuse the reader we use another symbol for the background field. But this leads to more readable formulas below.

7.2 Perturbative expansion of V^{i+1}

Now we do perturbation theory in powers of Γ^i.

In [11] the following nice formula is proven which leads to a convenient calculation of the perturbative expansion (it is also a version of Wicks theorem): [3]

$$e^{-V^{i+1}[\psi]} = e^{\frac{1}{2}\frac{\delta}{\delta\psi}\Gamma^i\frac{\delta}{\delta\psi}} e^{-V^i[\psi]} = e^A e^B \tag{95}$$

with

$$A = \frac{1}{2}\frac{\delta}{\delta\psi}\Gamma^i\frac{\delta}{\delta\psi} = \frac{1}{2}\int \mathrm{d}z_1\mathrm{d}z_2 \frac{\delta}{\delta\psi(z_1)}\Gamma^i(z_1,z_2)\frac{\delta}{\delta\psi(z_2)} \tag{96}$$

$$B = -V^i[\psi] \stackrel{\text{for } i=0}{=} -\int \mathrm{d}z \left(\frac{1}{2}m_0^2\psi(z)^2 + \frac{\lambda_0^{(4)}}{4!}\psi(z)^4\right). \tag{97}$$

We want to calculate $V^{i+1}[\psi]$ up to order two in Γ^i. Because one has to take the logarithm of $e^A e^B$ only connected parts have to be considered. A is of order one in Γ^i. Therefore e^A needs to be expanded up to order two.

$$\begin{aligned} e^A e^B &= 1 + B + AB + \frac{1}{2}AB^2 + \frac{1}{6}AB^3 + \frac{1}{2}A^2B + \frac{1}{4}A^2B^2 + \frac{1}{12}A^2B^3 \\ &\quad + O({\Gamma^i}^3) + \text{nonconnected} \end{aligned}$$

Some derivatives:

$$\frac{\delta}{\delta\psi(z)}B^2 = 2B\frac{\delta B}{\delta\psi(z)} \tag{98}$$

$$\frac{\delta^2}{\delta\psi(z)\delta\psi(w)}B^2 = 2\frac{\delta B}{\delta\psi(z)}\frac{\delta B}{\delta\psi(w)} + 2B\frac{\delta^2 B}{\delta\psi(z)\delta\psi(w)} \tag{99}$$

Using (99) gives

$$AB^2 = \int \mathrm{d}z\mathrm{d}w \frac{\delta B}{\delta\psi(z)}\Gamma^i(z,w)\frac{\delta B}{\delta\psi(w)} + 2B\cdot AB = \frac{\delta B}{\delta\psi}\Gamma^i\frac{\delta B}{\delta\psi} + 2B\cdot AB$$

The second term is nonconnected. The same procedure gives

$$\begin{aligned} A^2B^2 &= \int_{u,v,w,z} \Gamma^i(z,w)\frac{\delta^3 B}{\delta\psi(z)\delta\psi(w)\delta\psi(v)}\Gamma^i(v,u)\frac{\delta B}{\delta\psi(u)} \\ &\quad + \int_{u,v,w,z} \frac{\delta^2 B}{\delta\psi(z)\delta\psi(v)}\Gamma^i(z,w)\Gamma^i(v,u)\frac{\delta^2 B}{\delta\psi(w)\delta\psi(u)} \\ &\quad + 2\int_{z,w} \frac{\delta B}{\delta\psi(z)}\Gamma^i(z,w)\frac{\delta AB}{\delta\psi(w)} + \text{nonconnected} \\ &= 2\int_{u,v,w,z} \Gamma^i(z,w)\frac{\delta^3 B}{\delta\psi(z)\delta\psi(w)\delta\psi(v)}\Gamma^i(v,u)\frac{\delta B}{\delta\psi(u)} \\ &\quad + \int_{u,v,w,z} \frac{\delta^2 B}{\delta\psi(z)\delta\psi(v)}\Gamma^i(z,w)\Gamma^i(v,u)\frac{\delta^2 B}{\delta\psi(w)\delta\psi(u)} \\ &\quad + \text{nonconnected} \end{aligned}$$

[3] Not every field can be written as $\mathcal{A}^i\Phi$. Therefore $\frac{\delta}{\delta\psi}$ is not well defined. To be correct one should take the derivative with respect to some general field:

$$e^{-V^{i+1}[\psi]} = e^{\frac{1}{2}\frac{\delta}{\delta\zeta}\Gamma^i\frac{\delta}{\delta\zeta}} e^{-V^i[\psi+\zeta]}\Big|_{\zeta=0}$$

But the results are not influenced. This was also mentioned in [11], section 5.2.

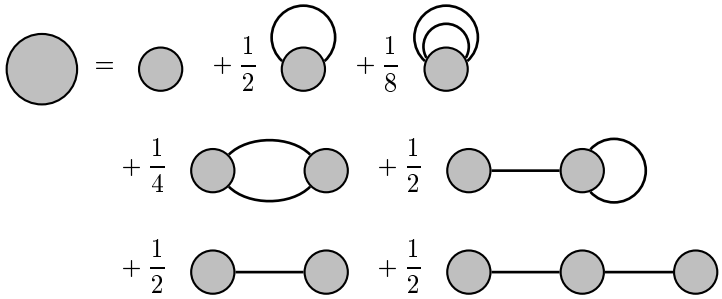


Figure 12: Expansion of $-V^{i+1}$ to second order of Γ^i. We use a similar graphical notation as in figure 1. Here the vertices represent $-V^i$ instead of $-S^i$.

and

$$A^2B^3 = 6\int_{u,v,w,z} \frac{\delta B}{\delta\psi(z)}\Gamma^i(z,w)\frac{\delta^2 B}{\delta\psi(w)\delta\psi(v)}\Gamma^i(v,u)\frac{\delta B}{\delta\psi(u)} + \text{nonconnected}$$

AB^3 and all terms AB^n, A^2B^n for $n > 3$ are nonconnected.
Collecting all terms gives

$$\begin{aligned}
-V^{i+1}[\psi] =& B + \frac{1}{2}\int_{w,z}\Gamma^i(z,w)\frac{\delta^2 B}{\delta\psi(z)\delta\psi(w)} \\
&+ \frac{1}{8}\int_{u,v,w,z}\Gamma^i(z,w)\Gamma^i(v,u)\frac{\delta^4 B}{\delta\psi(z)\delta\psi(w)\delta\psi(v)\delta\psi(u)} \\
&+ \frac{1}{2}\int_{w,z}\frac{\delta B}{\delta\psi(z)}\Gamma^i(z,w)\frac{\delta B}{\delta\psi(w)} \\
&+ \frac{1}{2}\int_{u,v,w,z}\Gamma^i(z,w)\frac{\delta^3 B}{\delta\psi(z)\delta\psi(w)\delta\psi(v)}\Gamma^i(v,u)\frac{\delta B}{\delta\psi(u)} \\
&+ \frac{1}{4}\int_{u,v,w,z}\frac{\delta^2 B}{\delta\psi(z)\delta\psi(v)}\Gamma^i(z,w)\Gamma^i(v,u)\frac{\delta^2 B}{\delta\psi(w)\delta\psi(u)} \\
&+ \frac{1}{2}\int_{u,v,w,z}\frac{\delta B}{\delta\psi(z)}\Gamma^i(z,w)\frac{\delta^2 B}{\delta\psi(w)\delta\psi(v)}\Gamma^i(v,u)\frac{\delta B}{\delta\psi(u)} \\
&+ O({\Gamma^i}^3)
\end{aligned} \tag{100}$$

The graphical representation is shown in figure 12.

Until now this can be applied to every potential $V = V^i$. Different V's which could be used are for instance

$$V(\phi) = \int \mathrm{d}z\Big(\frac{1}{2}m^2\phi(z)^2 + \sum_{n\geq 4,\text{even}}^{N}\frac{\lambda^{(n)}}{n!}\phi(z)^n\Big) \tag{101}$$

or

$$V(\phi) = \int \mathrm{d}z_1\mathrm{d}z_2\frac{1}{2}\phi(z_1)K(z_1,z_2)\phi(z_2) + \sum_{n\geq 4,\text{even}}^{N}\int \mathrm{d}z\frac{\lambda^{(n)}}{n!}\phi(z)^n \tag{102}$$

where $K(z_1, z_2)$ is a symmetric kernel, which represents a kinetic part put into the potential.

For our starting potential (93) we have the following derivatives ($i = 0$):

$$\frac{\delta}{\delta\psi(z)}B = -m_i^2\psi(z) - \frac{\lambda_i^{(4)}}{3!}\psi(z)^3 \tag{103}$$

$$\frac{\delta^2}{\delta\psi(z)\delta\psi(w)}B = (-m_i^2 - \frac{\lambda_i^{(4)}}{2}\psi(z)^2)\delta(z-w) \tag{104}$$

$$\frac{\delta^3}{\delta\psi(z)\delta\psi(w)\delta\psi(v)}B = -\lambda_i^{(4)}\psi(z)\delta(z-w)\delta(z-v) \tag{105}$$

$$\frac{\delta^4}{\delta\psi(z)\delta\psi(w)\delta\psi(v)\delta\psi(u)}B = -\lambda_i^{(4)}\delta(z-w)\delta(z-v)\delta(z-u) \tag{106}$$

Inserting into (100) gives the effective potential up to second order in Γ^i. The notation of the second column will be described later on:

$$V^{i+1}[\psi] = \tag{107}$$

$$\frac{1}{2}m_i^2\int \mathrm{d}z\psi(z)^2 \qquad \to \frac{1}{2}g_{2,1}^{i+1}R_{2,1}^{i+1}(x_1,x_2)$$

$$+\frac{\lambda_i^{(4)}}{4!}\int \mathrm{d}z\psi(z)^4 \qquad \to \frac{1}{4!}g_{4,1}^{i+1}R_{4,1}^{i+1}(x_1,\dots,x_4)$$

$$-\frac{1}{2}m_i^4\int_{z_1,z_2}\psi(z_1)\Gamma^i(z_1,z_2)\psi(z_2) \qquad \to \frac{1}{2}g_{2,2}^{i+1}R_{2,2}^{i+1}(x_1,x_2)$$

$$+\frac{\lambda_i^{(4)}}{4}\int_z\psi(z)^2\Gamma^i(z,z) \qquad \to \frac{1}{2}g_{2,3}^{i+1}R_{2,3}^{i+1}(x_1,x_2)$$

$$-\frac{1}{2}m_i^2\lambda_i^{(4)}\int_{z_1,z_2}\psi(z_1)\Gamma^i(z_1,z_1)\Gamma^i(z_1,z_2)\psi(z_2) \qquad \to \frac{1}{2}g_{2,5}^{i+1}R_{2,5}^{i+1}(x_1,x_2)$$

$$-\frac{1}{4}m_i^2\lambda_i^{(4)}\int_{z_1,z_2}\psi(z_1)^2\Gamma^i(z_1,z_2)^2 \qquad \to \frac{1}{2}g_{2,6}^{i+1}R_{2,6}^{i+1}(x_1,x_2)$$

$$+\frac{1}{2}m_i^6\int_{z_1,z_2,z_3}\psi(z_1)\Gamma^i(z_1,z_2)\Gamma^i(z_2,z_3)\psi(z_3) \qquad \to \frac{1}{2}g_{2,4}^{i+1}R_{2,4}^{i+1}(x_1,x_2)$$

$$-\frac{1}{3!}m_i^2\lambda_i^{(4)}\int_{z_1,z_2}\psi(z_1)\Gamma^i(z_1,z_2)\psi(z_2)^3 \qquad \to \frac{1}{4!}g_{4,2}^{i+1}R_{4,2}^{i+1}(x_1,\dots,x_4)$$

$$-\frac{1}{2}\frac{(\lambda_i^{(4)})^2}{3!}\int_{z_1,z_2}\psi(z_1)\Gamma^i(z_1,z_1)\Gamma^i(z_1,z_2)\psi(z_2)^3 \qquad \to \frac{1}{4!}g_{4,5}^{i+1}R_{4,5}^{i+1}(x_1,\dots,x_4)$$

$$+\frac{1}{3!}m_i^4\lambda_i^{(4)}\int_{z_1,z_2,z_3}\psi(z_1)\Gamma^i(z_1,z_2)\Gamma^i(z_2,z_3)\psi(z_3)^3 \qquad \to \frac{1}{4!}g_{4,4}^{i+1}R_{4,4}^{i+1}(x_1,\dots,x_4)$$

$$-\frac{(\lambda_i^{(4)})^2}{2^4}\int_{z_1,z_2}\psi(z_1)^2\Gamma^i(z_1,z_2)^2\psi(z_2)^2 \qquad \to \frac{1}{4!}g_{4,6,1}^{i+1}R_{4,6,1}^{i+1}(x_1,\dots,x_4)$$

$$+\frac{1}{4}m_i^4\lambda_i^{(4)}\int_{z_1,z_2,z_3}\psi(z_1)\Gamma^i(z_1,z_2)\psi(z_2)^2\Gamma^i(z_2,z_3)\psi(z_3) \qquad \to \frac{1}{4!}g_{4,8}^{i+1}R_{4,8}^{i+1}(x_1,\dots,x_4)$$

$$-\frac{(\lambda_i^{(4)})^2}{2\cdot 3!^2}\int_{z_1,z_2}\psi(z_1)^3\Gamma^i(z_1,z_2)\psi(z_2)^3 \qquad \to \frac{1}{6!}g_{6,2}^{i+1}R_{6,2}^{i+1}(x_1,\dots,x_6)$$

$$+\frac{m_i^2(\lambda_i^{(4)})^2}{2\cdot 3!^2}\int_{z_1,z_2,z_3}\psi(z_1)^3\Gamma^i(z_1,z_2)\Gamma^i(z_2,z_3)\psi(z_3)^3 \qquad \to \frac{1}{6!}g_{6,4}^{i+1}R_{6,4}^{i+1}(x_1,\dots,x_6)$$

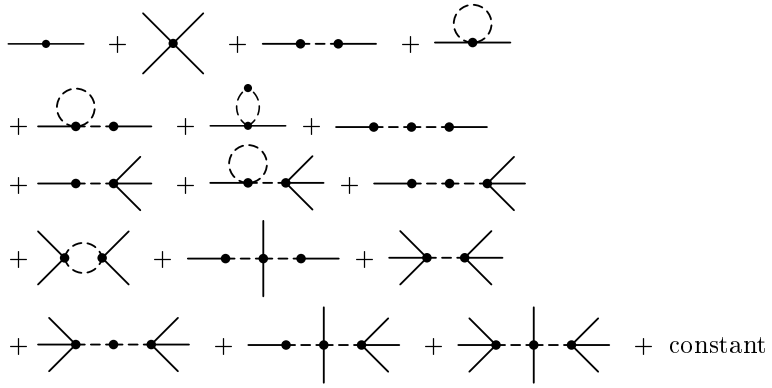
$$+\frac{m_i^2}{12}(\lambda_i^{(4)})^2\int_{z_1,z_2,z_3}\psi(z_1)\Gamma^i(z_1,z_2)\psi(z_2)^2\Gamma^i(z_2,z_3)\psi(z_3)^3\to\frac{1}{6!}g_{6,8}^{i+1}R_{6,8}^{i+1}(x_1,\dots,x_6)$$

$$+\frac{(\lambda_i^{(4)})^3}{4\cdot 3!^2}\int_{z_1,z_2,z_3}\psi(z_1)^3\Gamma^i(z_1,z_2)\psi(z_2)^2\Gamma^i(z_2,z_3)\psi(z_3)^3\quad\to\frac{1}{8!}g_{8,8}^{i+1}R_{8,8}^{i+1}(x_1,\dots,x_8)$$

$$+\frac{1}{2}m_i^2\int_z\Gamma^i(z,z)-\frac{1}{4}m_i^4\int_{z_1,z_2}\Gamma^i(z_1,z_2)\Gamma^i(z_2,z_1)+\frac{1}{8}\lambda_i^{(4)}\int_z\Gamma^i(z,z)^2$$

or graphically

V^{i+1} =



Now ψ has to be replaced by $\mathcal{A}^i\Phi$. Every term in $V^{i+1}[\Phi]$ has the form

$$\frac{1}{n!}g_{n,j,k}^{i+1}\int_{x_1,\dots,x_n}R_{n,j,k}^{i+1}(x_1,\dots,x_n)\Phi(x_1)\cdots\Phi(x_n)\,.$$

Look in (107) at the second column to see the correspondence. Equal j describe the same topology of the internal Γ^i-lines. k numbers different terms with the same j. [4]

$$\begin{aligned}V^{i+1}[\Phi] &= \sum_{n\geq 2,\text{even}}\int_{x_1,\dots,x_n}\sum_{j,k}\frac{g_{n,j,k}^{i+1}}{n!}R_{n,j,k}^{i+1}(x_1,\dots,x_n)\Phi(x_1)\cdots\Phi(x_n)\\ &=: \sum_{n\geq 2,\text{even}}\int_{x_1,\dots,x_n}\frac{1}{n!}R_n^{i+1}(x_1,\dots,x_n)\Phi(x_1)\cdots\Phi(x_n)\end{aligned}\qquad(108)$$

E.g. the third term in $V^{i+1}[\Phi]$ looks now

$$g_{2,2}^{i+1}=-m_i^4\,,\qquad R_{2,2}^{i+1}(x_1,x_2)=\int_{z_1,z_2}\mathcal{A}^i(z_1,x_1)\Gamma^i(z_1,z_2)\mathcal{A}^i(z_2,x_2)$$

For $n=2$ one can write $R_2^{i+1}(x_1,x_2)\equiv R_2^{i+1}(x_1-x_2)$ because of translational invariance. $\mathcal{A}^i$ and Γ^i are translational invariant with respect to the block lattice. Therefore by redefinition of the integration variables z_i one can move x_2 to the left:

$$R_{2,2}^{i+1}(x_1,x_2)=\int_{z_1,z_2}\mathcal{A}^i(z_1,x_1-x_2)\Gamma^i(z_1,z_2)\mathcal{A}^i(z_2,0)\equiv R_{2,2}^{i+1}(x_1-x_2)$$

[4] We will omit k if there is only one term in our approximation which is mostly the case. For $R_{4,6,1}^{i+1}$ we will get a similar graph $R_{4,6,2}^{i+1}$ in section 7.5.

7.3 Local part of V^{i+1}

Because $\mathcal{A}^i$ and Γ^i are quite local with respect to the block lattice spacing one expects a good approximation by the local part. [5] Only the zeroth order of the Taylor expansion of $\Phi(x_i)$ around $\Phi(x)$ is used:

$$\begin{aligned}\Phi(x_i) &= \Phi(x) + [\Phi(x_i) - \Phi(x)] \\ &= \Phi(x) + \nabla\Phi(x) + \frac{1}{2}\nabla^2\Phi(x) + \cdots\end{aligned}$$

This means all $\Phi(x_i)$ are taken at the same point x:

$$\overline{V}^{i+1}[\Phi] = \sum_{n\geq 2,\text{even}} \int_{x,x_2,\ldots,x_n} \frac{1}{n!} R_n^{i+1}(x, x_2, \ldots, x_n)\Phi(x)^n$$

Then all $R^{i+1}_{n,j,k}(x_1,\ldots,x_n)$ reduce to $\widehat{R}^{i+1}_{n,j,k}(q_1 = 0,\ldots,q_n = 0)$, where $\widehat{R}^{i+1}_{n,j,k}(q_1,\ldots,q_n)$ is the fourier transformed $R^{i+1}_{n,j,k}(x_1,\ldots,x_n)$. The proof is simple.
Because of translation symmetry we can write

$$R^{i+1}_{n,j,k}(x_1,\ldots,x_n) \equiv R^{i+1}_{n,j,k}(x_2 - x_1,\ldots,x_n - x_1).$$

Together we have

$$\begin{aligned}\int_{x_1,\ldots,x_n} & R^{i+1}_{n,j,k}(x_1, x_2, \ldots, x_n)\Phi(x_1)^n \\ &= \int_{x_1,\ldots,x_n} R^{i+1}_{n,j,k}(x_2 - x_1, \ldots, x_n - x_1)\Phi(x_1)^n \\ &= \int_{x_2',\ldots,x_n'} R^{i+1}_{n,j,k}(x_2', \ldots, x_n') \int_{x_1} \Phi(x_1)^n \\ &= \widehat{R}^{i+1}_{n,j,k}(q_2 = 0,\ldots,q_n = 0) \int_{x_1} \Phi(x_1)^n \\ &\equiv \widehat{R}^{i+1}_{n,j,k}(q_1 = 0,\ldots,q_n = 0) \int_{x_1} \Phi(x_1)^n .\end{aligned}$$

For practical reasons it is easier to simplify $R^{i+1}_{n,j,k}$ with the help of $\int_x \mathcal{A}^i(z,x) = 1$ (see [4]); e.g.

$$\begin{aligned}\int_{x_1,\ldots,x_4} & R^{i+1}_{4,6,1}(x_1, x_2, x_3, x_4)\Phi(x_1)^4 \\ &= \int_{x_1,\ldots,x_4} \int_{z_1,z_2} \mathcal{A}^i(z_1, x_1)\mathcal{A}^i(z_1, x_2)\Gamma^i(z_1, z_2)^2\mathcal{A}^i(z_2, x_3)\mathcal{A}^i(z_2, x_4)\Phi(x_1)^4 \\ &= \int_x \int_{z_1,z_2} \mathcal{A}^i(z_1, x)\Gamma^i(z_1, z_2)^2\Phi(x)^4\end{aligned}$$

One can split z_1 into $z_1 = w - y + x$ and redefine $z_2 = z - y + x$, where w is out of the first block and y out of the block lattice:

$$\begin{aligned}\int_{x_1,\ldots,x_4} R^{i+1}_{4,6,1}(x_1, x_2, x_3, x_4)\Phi(x_1)^4 &= \int_{w\in 0,y} \mathcal{A}^i(w, y) \int_z \Gamma^i(w, z)^2 \int_x \Phi(x)^4 \\ &= \int_{w\in 0,z} \Gamma^i(w, z)^2 \int_x \Phi(x)^4 \\ &= \widehat{R}^{i+1}_{4,6,1}(0) \int_x \Phi(x)^4\end{aligned}$$

[5] This is equivalent to calculating the effective potential only for constant fields.

Because of $\mathcal{C}^i\Gamma^i = 0$ [6] many terms of the local part vanish. E.g.

$$\begin{aligned}
\widehat{R}^{i+1}_{2,2}(0) &= \int_x \int_{z_1,z_2} \mathcal{A}^i(z_1,x)\Gamma^i(z_1,z_2)\mathcal{A}^i(z_2,0) \\
&= \int_{z_1,z_2} \Gamma^i(z_1,z_2)\mathcal{A}^i(z_2,0) \\
&= \int_y \int_{w\in 0} \Gamma^i(y+w,z_2)\mathcal{A}^i(z_2,0) \\
&\sim \int_y \int_z \mathcal{C}^i(y,z)\Gamma^i(z,z_2)\mathcal{A}^i(z_2,0) \\
&= 0
\end{aligned}$$

We split the integral $\int_{z_1}$ into two integrals $\int_y \int_{w\in 0}$. The first is a integral over the block lattice and the second over the first block in the base lattice. The later can be considered as proportional to the average operator with our choice of $\mathcal{C}^i$ (as average over one block).
All terms vanish where at least one external line is connected to at most one internal line, which is represented by Γ^i.
In order two of Γ^i only five constants are nonzero:

$$\widehat{R}^{i+1}_{2,1}(0) = \int_x \int_z \mathcal{A}^i(z,x)\mathcal{A}^i(z,0) = 1$$

$$\widehat{R}^{i+1}_{4,1}(0) = \int_{x_1,x_2,x_3} \int_z \mathcal{A}^i(z,x_1)\mathcal{A}^i(z,x_2)\mathcal{A}^i(z,x_3)\mathcal{A}^i(z,0) = 1$$

$$\widehat{R}^{i+1}_{2,3}(0) = \int_x \int_z \mathcal{A}^i(z,x)\Gamma^i(z,z)\mathcal{A}^i(z,0) = \int_{w\in 0} \Gamma^i(w,w) \tag{109}$$

$$\widehat{R}^{i+1}_{2,6}(0) = \int_x \int_{z_1,z_2} \mathcal{A}^i(z_1,x)\mathcal{A}^i(z_1,0)\Gamma^i(z_1,z_2)^2 = \int_{w\in 0,z} \Gamma^i(w,z)^2 \tag{110}$$

$$\begin{aligned}
\widehat{R}^{i+1}_{4,6,1}(0) &= \int_{x_1,x_2,x_3} \int_{z_1,z_2} \mathcal{A}^i(z_1,x_1)\mathcal{A}^i(z_1,x_2)\Gamma^i(z_1,z_2)^2 \cdot \\
&\qquad \cdot \mathcal{A}^i(z_2,x_3)\mathcal{A}^i(z_2,0) = \int_{w\in 0,z} \Gamma^i(w,z)^2
\end{aligned}$$

where w runs over the first block and z over the base lattice. The last two have identical values.
Writing the local part of $V^{i+1}[\Phi]$ in the form

$$\overline{V}^{i+1}[\Phi] = \frac{1}{2}m^2_{i+1}\int_x \Phi(x)^2 + \sum_{n\geq 4,\text{even}} \frac{\lambda^{(n)}_{i+1}}{n!}\int_x \Phi(x)^n$$

one gets for the effective coupling constants in order two of Γ^i after one step

$$m^2_{i+1} = m^2_i + \frac{\lambda^{(4)}_i}{2}\widehat{R}^{i+1}_{2,3}(0) - \frac{1}{2}m^2_i\lambda^{(4)}_i\widehat{R}^{i+1}_{2,6}(0) \tag{111}$$

$$\lambda^{(4)}_{i+1} = \lambda^{(4)}_i - \frac{3}{2}(\lambda^{(4)}_i)^2\widehat{R}^{i+1}_{4,6,1}(0) \tag{112}$$

$$\lambda^{(2n)}_{i+1} = 0 \qquad \forall\ n > 2 \tag{113}$$

[6] This is only valid for certain block-spin definitions. E.g. for Gauß block-spin we have $\mathcal{C}^i\Gamma^i = \frac{1}{\kappa}\mathcal{A}^{i*}$. κ is the same parameter as in (25), but stays finite.

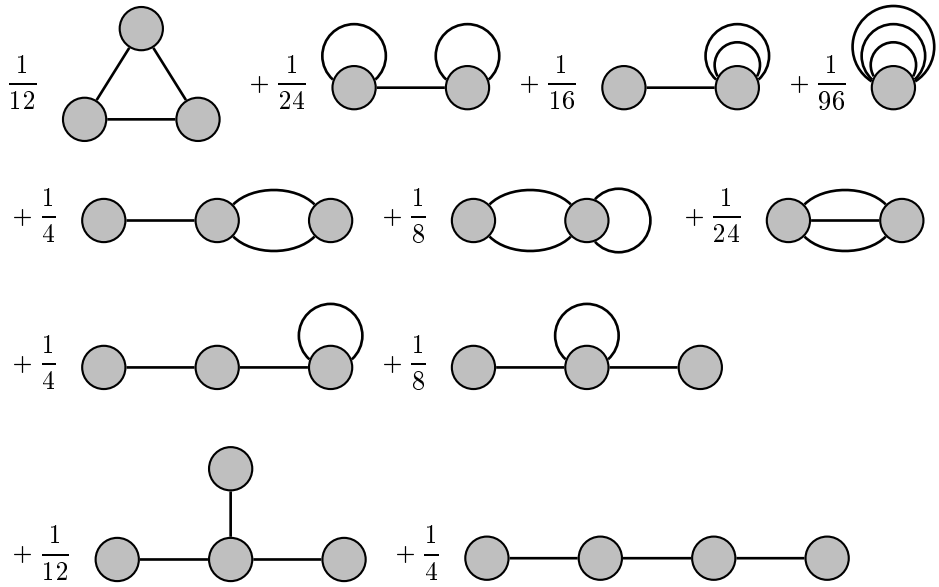


Figure 13: Contributions to the effective potential to third order of the fluctuation propagator.

7.4 Higher order

We see in equation (113) that no higher effective coupling constants are generated by the approximated renormalization group transformation. But in general the *full* renormalization group generates all coupling constants of type $\lambda_{i+1}^{(2n)}$ (see e.g. [12]). Already after one step they should occur. Therefore the accuracy of our approximation in second order of Γ^i is bad. It is bad at least for large values of the field. For small values of the field or for small couplings it may be sufficient.

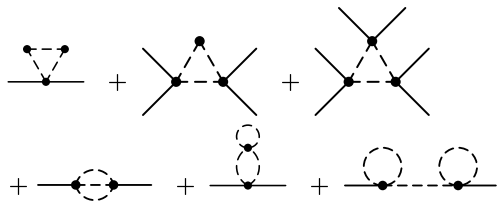
The next order leads to a lot of new terms, which are shown in a general form

in figure 13. Only six terms do not vanish for the local part:

$$
\begin{aligned}
V^{i+1}_{3rd\ order}[\psi] &= \frac{1}{8}m_i^4\lambda_i^{(4)}\int_{z_1,z_2,z_3}\psi(z_1)^2\Gamma^i(z_1,z_2)\Gamma^i(z_2,z_3)\Gamma^i(z_3,z_1)\\
&+\frac{1}{16}m_i^2(\lambda_i^{(4)})^2\int_{z_1,z_2,z_3}\psi(z_2)^2\Gamma^i(z_1,z_2)\psi(z_2)^2\Gamma^i(z_2,z_3)\Gamma^i(z_3,z_1)\\
&+\frac{1}{96}(\lambda_i^{(4)})^3\int_{z_1,z_2,z_3}\psi(z_1)^2\Gamma^i(z_1,z_2)\psi(z_2)^2\Gamma^i(z_2,z_3)\psi(z_3)^2\Gamma^i(z_3,z_1)\\
&-\frac{1}{24}(\lambda_i^{(4)})^2\int_{z_1,z_2}\psi(z_1)\Gamma^i(z_1,z_2)^3\psi(z_2)\\
&-\frac{1}{16}(\lambda_i^{(4)})^2\int_{z_1,z_2}\psi(z_1)^2\Gamma^i(z_1,z_2)^2\Gamma^i(z_2,z_2)\\
&-\frac{1}{16}(\lambda_i^{(4)})^2\int_{z_1,z_2}\psi(z_1)\Gamma^i(z_1,z_1)\Gamma^i(z_1,z_2)\Gamma^i(z_2,z_2)\psi(z_2)\\
&+\text{const.}+\dots\\
&\equiv \frac{1}{2}g^{i+1}_{2,9}\int_{x_1,x_2}R^{i+1}_{2,9}(x_1,x_2)+\frac{1}{4!}g^{i+1}_{4,9,1}\int_{x_1,\dots,x_4}R^{i+1}_{4,9,1}(x_1,\dots,x_4)\\
&+\frac{1}{6!}g^{i+1}_{6,9}\int_{x_1,\dots,x_6}R^{i+1}_{6,9}(x_1,\dots,x_6)+\frac{1}{2}g^{i+1}_{2,10}\int_{x_1,x_2}R^{i+1}_{2,10}(x_1,x_2)\\
&+\frac{1}{2}g^{i+1}_{2,11,1}\int_{x_1,x_2}R^{i+1}_{2,11,1}(x_1,x_2)+\frac{1}{2}g^{i+1}_{2,12}\int_{x_1,x_2}R^{i+1}_{2,12}(x_1,x_2)+\dots
\end{aligned}
$$

or graphically

$V^{i+1}_{3rd\ order}$ =



We used a maple program to determine the third order as described in appendix C. [7]

Again we focus on the local part:

$$
\begin{aligned}
\widehat{R}^{i+1}_{2,9}(0) &= \int_{w\in 0,z_1,z_2}\Gamma^i(w,z_1)\Gamma^i(z_1,z_2)\Gamma^i(z_2,w) \;=\; \widehat{R}^{i+1}_{4,9,1}(0) \;=\; \widehat{R}^{i+1}_{6,9,1}(0)\\
\widehat{R}^{i+1}_{2,10}(0) &= \int_{w\in 0,z}\Gamma^i(w,z)^3\\
\widehat{R}^{i+1}_{2,11,1}(0) &= \int_{w\in 0,z}\Gamma^i(w,z)^2\Gamma^i(z,z)\\
\widehat{R}^{i+1}_{2,12}(0) &= \int_{w\in 0,z}\Gamma^i(w,w)\Gamma^i(w,z)\Gamma^i(z,z)
\end{aligned}
$$

These expressions lead to more accurate effective coupling constants as modification

[7] Of course we could have used the program to calculate the second order. But we wanted to illustrate the detailed calculations which are simulated by the program.

of (111), (112) and (113):

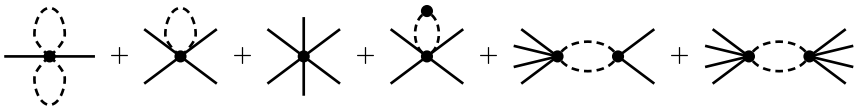
$$\begin{aligned}
m_{i+1}^2 &= m_i^2 + \frac{\lambda_i^{(4)}}{2}\widehat{R}_{2,3}^{i+1}(0) - \frac{1}{2}m_i^2\lambda_i^{(4)}\widehat{R}_{2,6}^{i+1}(0) + \frac{1}{4}m_i^4\lambda_i^{(4)}\widehat{R}_{2,9}^{i+1}(0) \\
&\quad -\frac{1}{12}(\lambda_i^{(4)})^2\widehat{R}_{2,10}^{i+1}(0) - \frac{1}{8}(\lambda_i^{(4)})^2\widehat{R}_{2,11,1}^{i+1}(0) - \frac{1}{8}(\lambda_i^{(4)})^2\widehat{R}_{2,12}^{i+1}(0) \\
\lambda_{i+1}^{(4)} &= \lambda_i^{(4)} - \frac{3}{2}(\lambda_i^{(4)})^2\widehat{R}_{4,6,1}^{i+1}(0) + \frac{3}{2}m_i^2(\lambda_i^{(4)})^2\widehat{R}_{4,9,1}^{i+1}(0) \\
\lambda_{i+1}^{(6)} &= \frac{15}{2}(\lambda_i^{(4)})^3\widehat{R}_{6,9,1}^{i+1}(0) \\
\lambda_{i+1}^{(2n)} &= 0 \qquad \forall\, n > 3
\end{aligned}$$

7.5 One more coupling constant

In the previous subsection we increased the order of the perturbative expansion to get one more nonvanishing coupling constant $\lambda_{i+1}^{(6)}$. We want to iterate the renormalization group transformation. Therefore for the next step we start with an potential of type (101) with $N = 6$. The perturbative expansion of certain order gives new terms with the new coupling constant $\lambda_i^{(6)}$. We used again the maple program of appendix C. The nonvanishing terms of the local part up to second order are

$$\begin{aligned}
&\frac{1}{16}\lambda_i^{(6)}\int_z \psi(z)^2\Gamma^i(z,z)^2 + \frac{1}{48}\lambda_i^{(6)}\int_z \psi(z)^4\Gamma^i(z,z) + \frac{1}{6!}\lambda_i^{(6)}\int_z \psi(z)^6 \\
&-\frac{1}{48}m_i^2\lambda_i^{(6)}\int_{z_1,z_2}\psi(z_1)^4\Gamma^i(z_1,z_2)^2 - \frac{1}{96}\lambda_i^{(4)}\lambda_i^{(6)}\int_{z_1,z_2}\psi(z_1)^2\Gamma^i(z_1,z_2)^2\psi(z_2)^4 \\
&-\frac{1}{2304}(\lambda_i^{(6)})^2\int_{z_1,z_2}\psi(z_1)^4\Gamma^i(z_1,z_2)^2\psi(z_2)^4 \\
\equiv\ &\frac{1}{2}g_{2,13}^{i+1}\int_{x_1,x_2} R_{2,13}^{i+1}(x_1,x_2) + \frac{1}{4!}g_{4,3}^{i+1}\int_{x_1,\dots,x_4} R_{4,3}^{i+1}(x_1,\dots,x_4) \\
&+\frac{1}{6!}g_{6,1}^{i+1}\int_{x_1,\dots,x_6} R_{6,1}^{i+1}(x_1,\dots,x_6) + \frac{1}{4!}g_{4,6,2}^{i+1}\int_{x_1,\dots,x_4} R_{4,6,2}^{i+1}(x_1,\dots,x_4) \\
&+\frac{1}{6!}g_{6,6}^{i+1}\int_{x_1,\dots,x_6} R_{6,6}^{i+1}(x_1,\dots,x_6) + \frac{1}{8!}g_{8,6}^{i+1}\int_{x_1,\dots,x_8} R_{8,6}^{i+1}(x_1,\dots,x_8)
\end{aligned}$$

or graphically



The contributions to the local part are:

$$\begin{aligned}
\widehat{R}_{2,13}^{i+1}(0) &= \int_{w\in 0}\Gamma^i(w,w)^2 \\
\widehat{R}_{4,3}^{i+1}(0) &= \int_{w\in 0}\Gamma^i(w,w) \\
\widehat{R}_{4,6,2}^{i+1}(0) &= \int_{w\in 0,z}\Gamma^i(w,z)^2 \;=\; \widehat{R}_{6,6}^{i+1}(0) \;=\; \widehat{R}_{8,6}^{i+1}(0) \\
\widehat{R}_{6,1}^{i+1}(0) &= 1
\end{aligned}$$

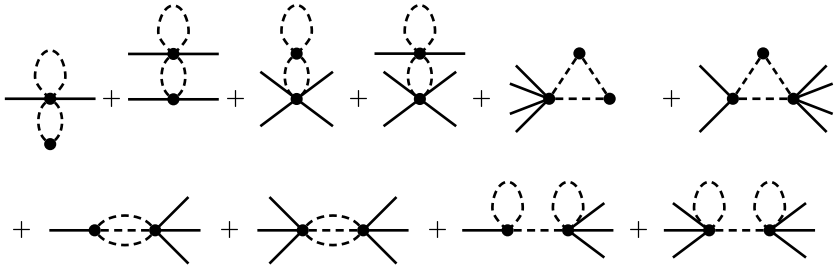
Except of $\widehat{R}^{i+1}_{2,13}(0)$ there is no need to calculate new values, because the simplified expressions on the right hand side occured already in (109) and (110). Only the numerical factors $g^{i+1}_{n,j,k}$ differ.

If we want to iterate many times we have to truncate now. The term $\widehat{R}^{i+1}_{8,6}(0)$ will be neglected for the next step. Otherwise we would get new terms by the perturbative expansion again and again. One has to balance the expense with the effort of using all nonvanishing coupling constants. Also we assume that coupling constants $\lambda^{(n)}_i$ are small which correspond to large powers of the field (if we do not iterate too often).

For the third order in Γ^i we get

$$
\begin{aligned}
&-\frac{1}{16}m_i^2\lambda_i^{(6)}\int_{z_1,z_2}\psi(z_1)^2\Gamma^i(z_1,z_1)\Gamma^i(z_1,z_2)^2\\
&-\frac{1}{32}\lambda_i^{(4)}\lambda_i^{(6)}\int_{z_1,z_2}\psi(z_1)^2\Gamma^i(z_1,z_1)\Gamma^i(z_1,z_2)^2\psi(z_2)^2\\
&-\frac{1}{192}\lambda_i^{(4)}\lambda_i^{(6)}\int_{z_1,z_2}\Gamma^i(z_1,z_1)\Gamma^i(z_1,z_2)^2\psi(z_2)^4\\
&-\frac{1}{384}(\lambda_i^{(6)})^2\int_{z_1,z_2}\psi(z_1)^2\Gamma^i(z_1,z_1)\Gamma^i(z_1,z_2)^2\psi(z_2)^4\\
&+\frac{1}{96}m_i^4\lambda_i^{(6)}\int_{z_1,z_2,z_3}\psi(z_1)^4\Gamma^i(z_1,z_2)\Gamma^i(z_2,z_3)\Gamma^i(z_3,z_1)\\
&+\frac{1}{96}m_i^2\lambda_i^{(4)}\lambda_i^{(6)}\int_{z_1,z_2,z_3}\psi(z_1)^2\Gamma^i(z_1,z_2)\psi(z_2)^4\Gamma^i(z_2,z_3)\Gamma^i(z_3,z_1)\\
&-\frac{1}{72}\lambda_i^{(4)}\lambda_i^{(6)}\int_{z_1,z_2}\psi(z_1)\Gamma^i(z_1,z_2)^3\psi(z_2)^3\\
&-\frac{1}{864}(\lambda_i^{(6)})^2\int_{z_1,z_2}\psi(z_1)^3\Gamma^i(z_1,z_2)^3\psi(z_2)^3\\
&-\frac{1}{48}\lambda_i^{(4)}\lambda_i^{(6)}\int_{z_1,z_2}\psi(z_1)\Gamma^i(z_1,z_1)\Gamma^i(z_1,z_2)\Gamma^i(z_2,z_2)\psi(z_2)^3\\
&-\frac{1}{576}(\lambda_i^{(6)})^2\int_{z_1,z_2}\psi(z_1)^3\Gamma^i(z_1,z_1)\Gamma^i(z_1,z_2)\Gamma^i(z_2,z_2)\psi(z_2)^3\\
\equiv\ &\frac{1}{2}g^{i+1}_{2,11,2}\int_{x_1,x_2}R^{i+1}_{2,11,2}(x_1,x_2)+\frac{1}{4!}g^{i+1}_{4,11,1}\int_{x_1,\dots,x_4}R^{i+1}_{4,11,1}(x_1,\dots,x_4)\\
&+\frac{1}{4!}g^{i+1}_{4,11,2}\int_{x_1,\dots,x_4}R^{i+1}_{4,11,2}(x_1,\dots,x_4)+\frac{1}{6!}g^{i+1}_{6,11}\int_{x_1,\dots,x_6}R^{i+1}_{6,11}(x_1,\dots,x_6)\\
&+\frac{1}{4!}g^{i+1}_{4,9,2}\int_{x_1,\dots,x_4}R^{i+1}_{4,9,2}(x_1,\dots,x_4)+\frac{1}{6!}g^{i+1}_{6,9,2}\int_{x_1,\dots,x_6}R^{i+1}_{6,9,2}(x_1,\dots,x_6)\\
&+\frac{1}{4!}g^{i+1}_{4,10}\int_{x_1,\dots,x_4}R^{i+1}_{4,10}(x_1,\dots,x_4)+\frac{1}{6!}g^{i+1}_{6,10}\int_{x_1,\dots,x_6}R^{i+1}_{6,10}(x_1,\dots,x_6)\\
&+\frac{1}{4!}g^{i+1}_{4,12}\int_{x_1,\dots,x_4}R^{i+1}_{4,12}(x_1,\dots,x_4)+\frac{1}{6!}g^{i+1}_{6,12}\int_{x_1,\dots,x_6}R^{i+1}_{6,12}(x_1,\dots,x_6)
\end{aligned}
$$

or graphically



The contributions to the local part are:

$$\begin{aligned}
\widehat{R}^{i+1}_{2,11,2}(0) &= \int_{w\in 0,z} \Gamma^i(w,z)^2\Gamma^i(w,w) \;=\; \widehat{R}^{i+1}_{4,11,1}(0) \;=\; \widehat{R}^{i+1}_{4,11,2}(0) \;=\; \widehat{R}^{i+1}_{6,11}(0) \\
\widehat{R}^{i+1}_{4,9,2}(0) &= \int_{w\in 0,z_1,z_2} \Gamma^i(w,z_1)\Gamma^i(z_1,z_2)\Gamma^i(z_2,w) \;=\; \widehat{R}^{i+1}_{6,9,2}(0) \\
\widehat{R}^{i+1}_{4,10}(0) &= \int_{w\in 0,z} \Gamma^i(w,z)^3 \;=\; \widehat{R}^{i+1}_{6,10}(0) \\
\widehat{R}^{i+1}_{4,12}(0) &= \int_{w\in 0,z} \Gamma^i(w,w)\Gamma^i(w,z)\Gamma^i(z,z) \;=\; \widehat{R}^{i+1}_{6,12}(0)
\end{aligned}$$

We have already neglected terms which contribute to effective couplings $\lambda^{(8)}_{i+1}$ or higher.

The values of the third order in the previous section can be reused again.

7.6 Resulting recursion equations

We put all terms together and get the resulting recursion equations for the coupling constants:

$$\begin{aligned}
m^2_{i+1} &= m^2_i + \frac{\lambda^{(4)}_i}{2}\widehat{R}^{i+1}_{2,3}(0) - \frac{1}{2}m^2_i\lambda^{(4)}_i\widehat{R}^{i+1}_{2,6}(0) + \frac{1}{4}m^4_i\lambda^{(4)}_i\widehat{R}^{i+1}_{2,9}(0) \\
&\quad -\frac{1}{12}(\lambda^{(4)}_i)^2\widehat{R}^{i+1}_{2,10}(0) - \frac{1}{8}(\lambda^{(4)}_i)^2\widehat{R}^{i+1}_{2,11,1}(0) - \frac{1}{8}(\lambda^{(4)}_i)^2\widehat{R}^{i+1}_{2,12}(0) \\
&\quad +\frac{1}{8}\lambda^{(6)}_i\widehat{R}^{i+1}_{2,13}(0) - \frac{1}{8}m^2_i\lambda^{(6)}_i\widehat{R}^{i+1}_{2,11,2}(0) \\
\lambda^{(4)}_{i+1} &= \lambda^{(4)}_i - \frac{3}{2}(\lambda^{(4)}_i)^2\widehat{R}^{i+1}_{4,6,1}(0) + \frac{3}{2}m^2_i(\lambda^{(4)}_i)^2\widehat{R}^{i+1}_{4,9,1}(0) + \frac{1}{2}\lambda^{(6)}_i\widehat{R}^{i+1}_{4,3}(0) \\
&\quad -\frac{1}{2}m^2_i\lambda^{(6)}_i\widehat{R}^{i+1}_{4,6,2}(0) - \frac{3}{4}\lambda^{(4)}_i\lambda^{(6)}_i\widehat{R}^{i+1}_{4,11,1}(0) - \frac{1}{8}\lambda^{(4)}_i\lambda^{(6)}_i\widehat{R}^{i+1}_{4,11,2}(0) \\
&\quad +\frac{1}{4}m^4_i\lambda^{(6)}_i\widehat{R}^{i+1}_{4,9,2}(0) - \frac{1}{3}\lambda^{(4)}_i\lambda^{(6)}_i\widehat{R}^{i+1}_{4,10}(0) - \frac{1}{2}\lambda^{(4)}_i\lambda^{(6)}_i\widehat{R}^{i+1}_{4,12}(0) \\
\lambda^{(6)}_{i+1} &= \lambda^{(6)}_i + \frac{15}{2}(\lambda^{(4)}_i)^3\widehat{R}^{i+1}_{6,9,1}(0) - \frac{15}{2}\lambda^{(4)}_i\lambda^{(6)}_i\widehat{R}^{i+1}_{6,6}(0) - \frac{15}{8}(\lambda^{(6)}_i)^2\widehat{R}^{i+1}_{6,11}(0) \\
&\quad +\frac{15}{2}m^2_i\lambda^{(4)}_i\lambda^{(6)}_i\widehat{R}^{i+1}_{6,9,2}(0) - \frac{5}{6}(\lambda^{(6)}_i)^2\widehat{R}^{i+1}_{6,10}(0) - \frac{5}{4}(\lambda^{(6)}_i)^2\widehat{R}^{i+1}_{6,12}(0) \\
\lambda^{(2n)}_{i+1} &= 0 \qquad \forall\; n > 3
\end{aligned}$$

Our approximation contains all terms up to third order (in the fluctuation propagator) of the perturbative expansion of the effective potential. Higher coupling constants than $\lambda^{(6)}_i$ are not taken into account for the next step. We concentrate

on the local part. The starting potential is a ϕ^4-potential and all further effective potentials of the renormalization group flow are ϕ^6-potentials.

In spite of the big number of terms which occur in the recursion equations above there are only seven values which need to be calculated for one iteration step. Namely

$$\begin{aligned} &\widehat{R}^{i+1}_{2,3}(0), \qquad \widehat{R}^{i+1}_{2,6}(0), \qquad \widehat{R}^{i+1}_{2,9}(0), \qquad \widehat{R}^{i+1}_{2,10}(0), \\ &\widehat{R}^{i+1}_{2,11,1}(0), \qquad \widehat{R}^{i+1}_{2,12}(0), \qquad \widehat{R}^{i+1}_{2,13}(0). \end{aligned} \tag{114}$$

All other non-vanishing values $\widehat{R}^{i+1}_{n,j,k}(0)$ have equal values to one of these seven. As mentioned earlier for fixed j it holds

$$\widehat{R}^{i+1}_{n',j,k'}(0) = \widehat{R}^{i+1}_{n,j,k}(0) \quad \text{for all } n', k'.$$

Actually we calculated the values (114) in momentum space similar to section 4. There is less work to be done because the involved operators are diagonal in momentum space (the fluctuation propagator is only diagonal on the dual block lattice, see appendix B). For completeness we write down the momentum space representation of (114):

$$\widehat{R}^{i+1}_{2,3}(0) = \int_{l,q} \Gamma^i(l,q,l)$$

$$\widehat{R}^{i+1}_{2,6}(0) = \int_{l,q,l'} \Gamma^i(l,q,l')^2$$

$$\widehat{R}^{i+1}_{2,9}(0) = \int_{q,l_1,l_2,l_3} \Gamma^i(l_1,q,l_2)\Gamma^i(l_2,q,l_3)\Gamma^i(l_3,q,l_1)$$

$$\widehat{R}^{i+1}_{2,10}(0) = \int_{q_1,q_2,l_1,l_1',l_2,l_2'} \Gamma^i(l_1,q_1,l_1')\Gamma^i(l_2,q_2,l_2')\Gamma^i(-l_1-l_2,-q_1-q_2,-l_1'-l_2')$$

$$\widehat{R}^{i+1}_{2,11}(0) = \int_{q_1,q_2,l_1,l_1',l_2,l_2'} \Gamma^i(l_1,q_1,l_1')\Gamma^i(l_2,q_2,l_2')\Gamma^i(-l_1,-q_1,l_2-l_1'-l_2')$$

$$\widehat{R}^{i+1}_{2,12}(0) = \int_{q_1,q_2,l_1,l_1',l_2,l_2'} \Gamma^i(l_1,q_1,l_1')\Gamma^i(l_2,q_2,l_2')\Gamma^i(l_1'-l_1,0,l_2-l_2')$$

$$\widehat{R}^{i+1}_{2,13}(0) = \int_{q_1,q_2,l_1,l_1',l_2} \Gamma^i(l_1,q_1,l_1')\Gamma^i(l_2,q_2,l_1+l_2-l_1')$$

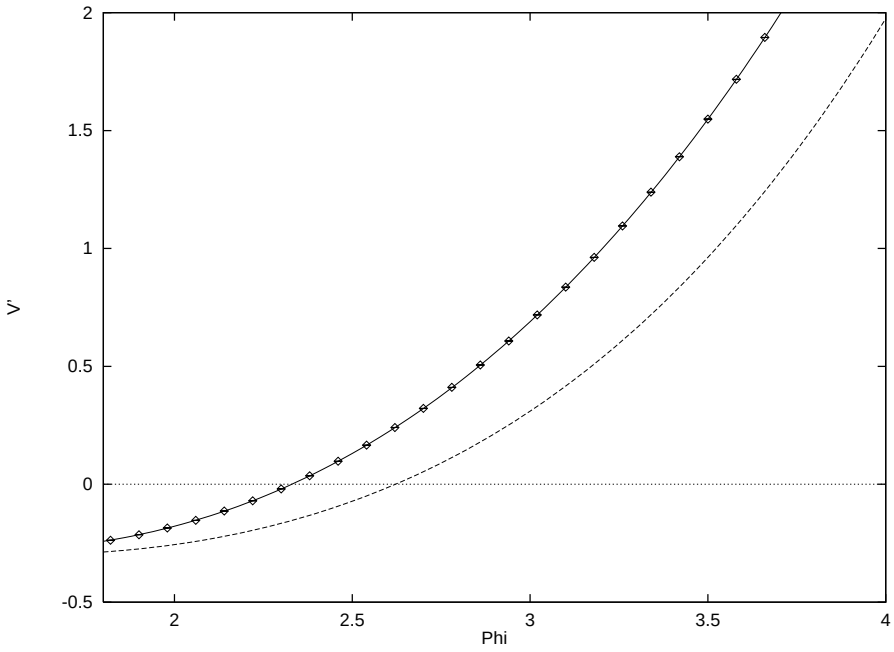


Figure 14: The derivative of the constraint effective potential is plotted against the block-spin Φ. The solid line is the saddle point approximation and the dashed line the perturbative calculation. The open squares represent the Monte Carlo results. This was calculated on a 16×16 lattice with blocking factor 2, $m_0^2 = -0.4$ and $\lambda_0^{(4)} = 0.4$. Note that the minimum of the potential is where $V' = 0$ (see figure 10). Our method does not permit to calculate the potential in the instable region where $V^{i\prime\prime} < 0$ (cp. figure 10 too).

8 Results

All numerical calculations have been done with the *perfect* laplacian described in [9] (1994), which is believed to bring the renormalization flow nearer to the renormalized trajectory.

8.1 Comparison of the two methods with Monte Carlo

The two methods *perturbative calculation* and *saddle point approximation* differ in their accuracy.

The perturbative calculation is a very good approximation for small field values Φ and bad for the outer region. It is a method to determine coupling constants which are coefficients of a Taylor expansion around a certain field value Φ. Here we expanded around $\Phi = 0$. Therefore it is natural that it is a good approximation around the expansion point, i.e. small field values. For large field values one needs higher and higher coupling constants $\lambda_i^{(n)}$ to improve the accuracy of a truncated Taylor expansion (polynomial approximation).

The second is a non-polynomial approximation and there is no global expansion point. It is accurate for all field values and has the correct asymptotic for large field values.

As a result we get the following picture:

- For small field values the perturbative calculation is comparable in accuracy to the saddle point approximation.

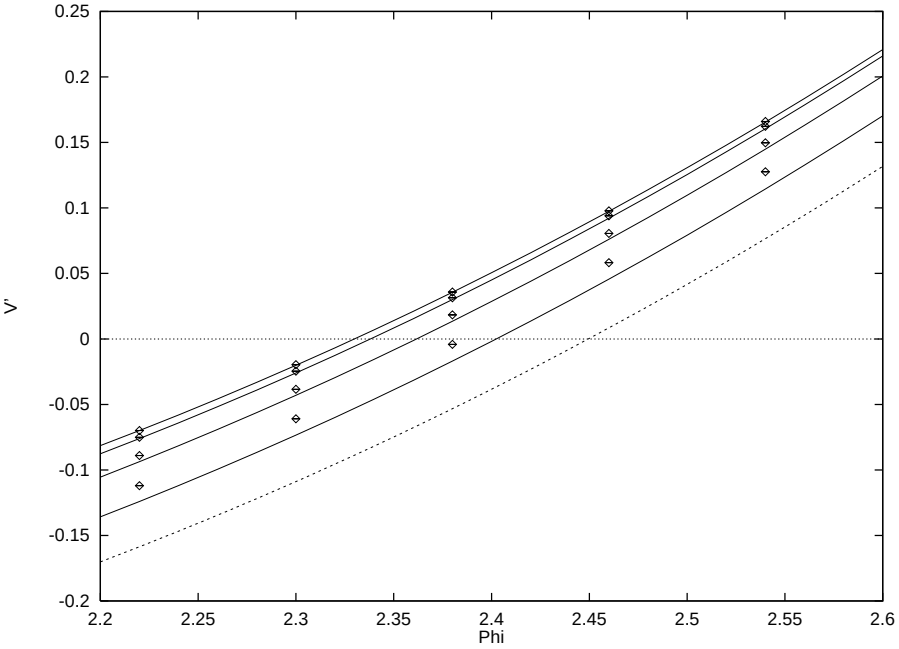


Figure 15: The flow of the derivative of the effective potential is plotted against the block-spin Φ in the vicinity of the minimum of the potential. The lines represent $\overline{V}^{i\prime}$ of the saddle point approximation for $i = 1, 2, 3, 4$ from the bottom and the open squares the Monte Carlo results for $i = 1, 2, 3, 4$ from the botton too (the determination of the flow is described in appendix D.3). The dotted line is the starting potential. This was calculated on a 16×16 lattice with blocking factor 2, $m_0^2 = -0.4$ and $\lambda_0^{(4)} = 0.4$. The extension of the lattice is getting smaller after every blocking step by factor 2. On the other hand the lattice spacing grows by factor 2, so that the lattice volume stays constant. No rescaling to the unit lattice is performed.

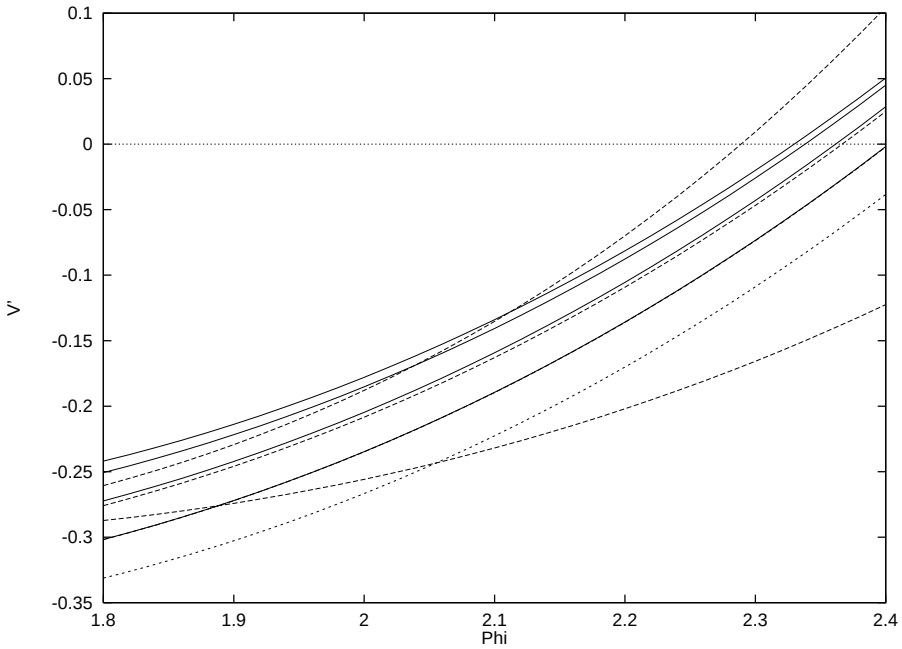


Figure 16: The flow of the derivative of the effective potential is plotted against the block-spin Φ in the vicinity of the minimum of the potential. The various methods are plotted. The solid lines represent $\overline{V}^{i\prime}$ of the saddle point approximation for $i = 1, 2, 3, 4$ from the bottom. The dotted line at the bottom is the original potential. The dashed lines represent the perturbative calculated potentials also for $i = 1, 2, 3, 4$ from the bottom. The line which crosses all lines at the bottom corresponds to $i = 4$. The lines for $i = 1$ coincide. This was calculated on a 16×16 lattice with blocking factor 2, $m_0^2 = -0.4$ and $\lambda_0^{(4)} = 0.4$.

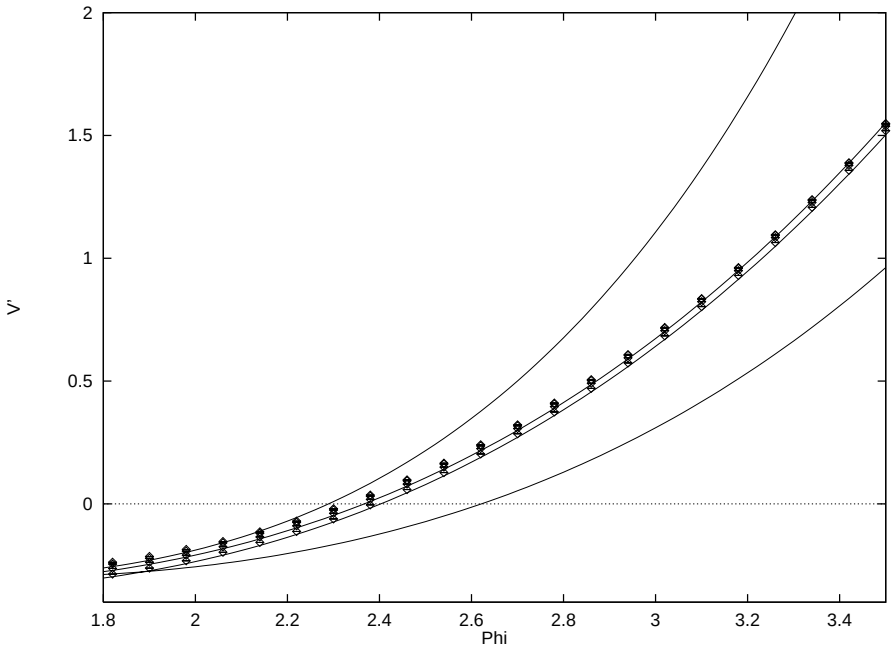


Figure 17: The flow of the derivative of the effective potential is plotted against the block-spin Φ for large field values. The open squares represent the Monte Carlo results for $i = 1, 2, 3, 4$ and the lines the perturbative calculated potentials for $i = 1, 2, 3, 4$. The line at the bottom corresponds to $i = 4$. The lines for $i = 1, 2$ are nearby the correct Monte Carlo points. This was calculated on a 16×16 lattice with blocking factor 2, $m_0^2 = -0.4$ and $\lambda_0^{(4)} = 0.4$.

- For larger field values the perturbative calculation goes rapidly wrong, while the saddle point approximation remains accurate and has the correct asymptotic (see figure 14).

  Only for the first two iterations the perturbative calculation is a good approximation, because only a few higher couplings are generated and therefore the neglection of higher couplings is less dramatic (see figure 17). This advantage of the saddle point approximation is of particular interest in theories with spontaneous symmetry breaking, where the field acquires a nonvanishing vacuum expectation value $\langle\phi\rangle$. One is mainly interested in $\phi \approx \langle\phi\rangle$.

- For the first iterations the potentials of both methods have a significant distance to the Monte Carlo potentials, whereas for the last iterations the distance decreases fast for the saddle point approximation. The perturbative calculation goes rapidly wrong as stated above. See figures 15 and 16.

- The calculation of coupling constants at the expansion point $\Phi = 0$ is not an appropriate technique to calculate the renormalization group flow for many iterations. This is true both for the perturbative calculation and the Monte Carlo method (see appendix D).

One can observe finite size effects too. Starting on lattices with different sizes we see in figure 18 that the first two iterations are identical. The next iteration differs slightly from each other. The last two iterations for the larger lattice lead to negative couplings and therefore to an unrealistic unstable effective potential.

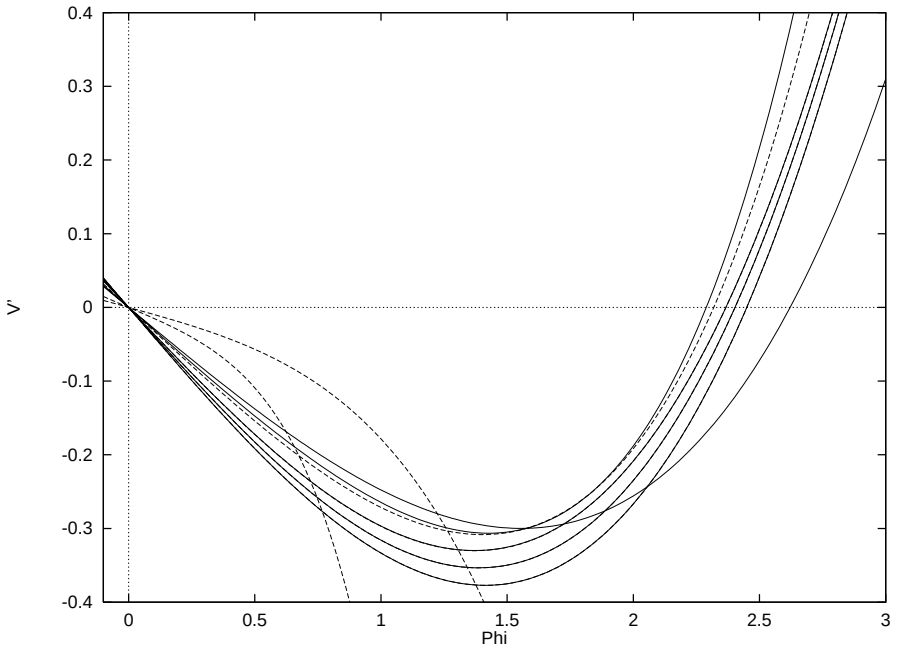


Figure 18: The flow of the derivative of the effective potential is plotted against the block-spin Φ, calculated with perturbative expansion. The solid lines represent V^i for $i = 0, 1, 2, 3, 4$ from the bottom starting on a 16×16 lattice with blocking factor 2. For the dashed lines the calculation started on a 32×32 lattice with the same blocking factor. For $i = 0, 1, 2$ the lines coincide (therefore the dashed lines are not visible). For $i = 3$ the dashed line seems to *fit* better into the flow. For both calculations the last two iterations do not seem to be trustworthy. The original coupling values are $m_0^2 = -0.4$ and $\lambda_0^{(4)} = 0.4$.

- The perturbative calculation should not be performed on lattices smaller than 8×8.

Also we can see finite size effects for the saddle point approximation if we start from larger lattices. In figure 19 the flow for two different lattices is compared. For the first two iterations the flows coincide, but the third iterations differ. In the physical region the difference is small, but in the unphysical region the difference is quite large. Looking at small values of Φ the effective potential calculated on the larger lattice looks more trustworthy because there is no change in the sign of the curvature.

8.2 Critical line

A short introduction to renormalization group on the lattice is given in [15]. It covers most of the statements given here.

To study critical phenomena one has to rescale the effective potential. This is described in appendix E for the canonical dimension. Because the renormalization group is a semi group it is sufficient to rescale the resulting flow of the effective potential at the end. Several iterations can be interpreted as one step with larger blocking factor.

In order to find a fixed point the field has to be rescaled with a certain dimension which is the canonical dimension plus the so called *anomalous* dimension. The anomalous dimension is a priori not known. Suppose we know the critical

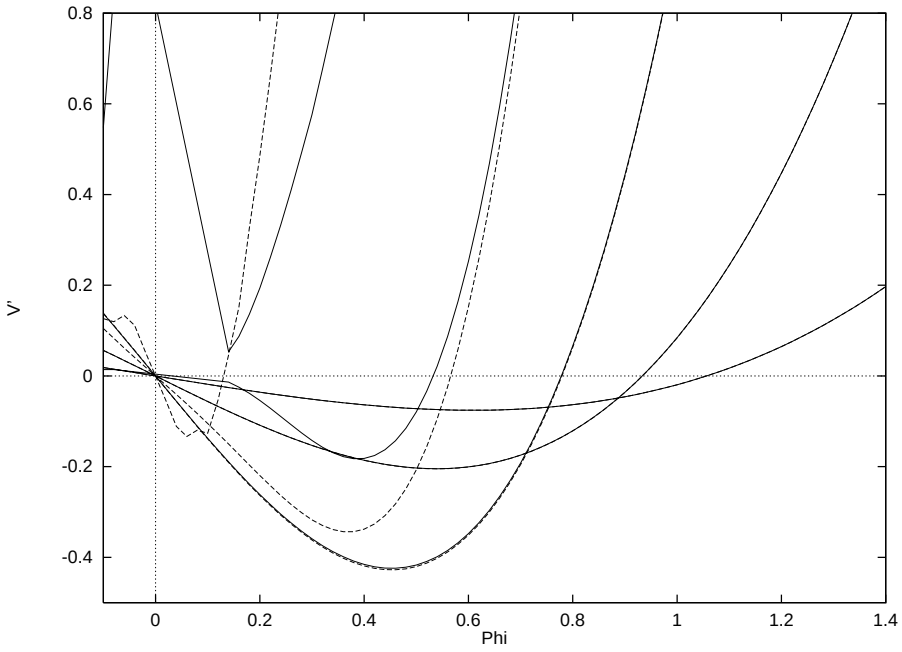


Figure 19: With the canonical dimension rescaled flow of the derivative of the effective potential is plotted against the block-spin Φ, calculated with saddle point approximation. The solid lines represent V^i for $i = 0, 1, 2, 3, 4$ from the bottom (looking at $\Phi = 1$) starting on a 16×16 lattice with blocking factor 2. For the dashed lines the calculation started on a 32×32 lattice with the same blocking factor. For $i = 0, 1, 2$ the lines coincide (therefore the dashed line are not visible). For $i = 3$ the two flows differ mostly in the unphysical region. For $i = 4$ both lines are unstable. The original coupling values are $m_0^2 = -0.186$ and $\lambda_0^{(4)} = 1$.

point. Then e.g. the minimum of the potential stays at the same field value under renormalization group transformations. Therefore a fixed minimum is a necessary condition for finding a fixed point.

The renormalization group transformation gives us the freedom to choose the kind of the renormalization condition. Here we require that the minimum of the potential stays at the same value of $\overline{\phi}$ (in the broken phase). This leads to a wave function renormalization after every step additional to the canonical rescaling.

Applied to our flow of the derivative of the effective potential we rescale first V^i according to its canonical dimension D_ϕ as described in appendix E:

$$V^{i\,\prime}_{R,z}[\Phi] = s^{i(D+D_\phi)} V^{i\prime}_{,s^i z}[s^{iD_\phi}\Phi(\frac{\cdot}{s^i})] \tag{115}$$

or for constant fields $\Phi = \overline{\phi}$

$$\overline{V}^i_R{}'(\overline{\phi}) = s^{i(D+D_\phi)} \overline{V}^{i\prime}(s^{iD_\phi}\overline{\phi}) \tag{116}$$

Here it is assumed that the blocking factor s is constant for all iterations, i.e. $a_i = s^i a_0$ (s^i is the i-th power of s).

If $\overline{\phi}^0_{min}$ is the value for the minimum of the starting potential and $\overline{\phi}^i_{min}$ the value after the i-th iteration then the anomalous dimension D_a is defined as

$$s^{iD_a} = \frac{\overline{\phi}^i_{min}}{\overline{\phi}^0_{min}}.$$

Then the resulting effective potential, which can be used for the study of critical phenomena, is

$$\overline{V}^i_{RC}{}'(\overline{\phi}) = s^{iD_a}\overline{V}^i_R{}'\left(s^{iD_a}\overline{\phi}\right) \tag{117}$$

We have applied it to the saddle point approximation and tried to find the critical line.

The two-dimensional ϕ^4-theory lies in the same universality class as the two-dimensional Ising model (see e.g. [10]). With zero external magnetic field the Ising model has one relevant field which corresponds to the mass term in the ϕ^4-theory. Around the fixed point one linearizes the renormalization group and gets the eigenvalues which determine the scaling behavior under renormalization group transformations. Transferring this picture in the space of all parameters which characterize a certain parameterization of the potentials we have one direction at the fixed point which is repulsive and all other directions are attractive. I.e. relevant parameters become larger and irrelevant parameters become smaller during the flow. The *critical surface* is defined as the surface in the space of all actions where the relevant fields vanish (see [14]).

Now we look for potentials of ϕ^4-type which lie on the critical surface. I.e. we search for the crossing line between the critical surface and the plane of bare parameters which is described by setting all parameters except of m_0^2 and $\lambda_0^{(4)}$ to zero.

For fixed $\lambda_0^{(4)}$ we looked for m_0^2 where after three iterations the derivative of the effective potential coincides at the best in the neighborhood of the minimum with the previous iteration. One expects to be on the renormalized trajectory after three iterations. One example is shown in figure 20. The resulting critical line is shown in figure 21.

We give a short description of the idea behind this procedure. The critical line splits our plane described above in two areas. In one of these the flow runs from a double well structure to a single well, where the effective potential has a unique

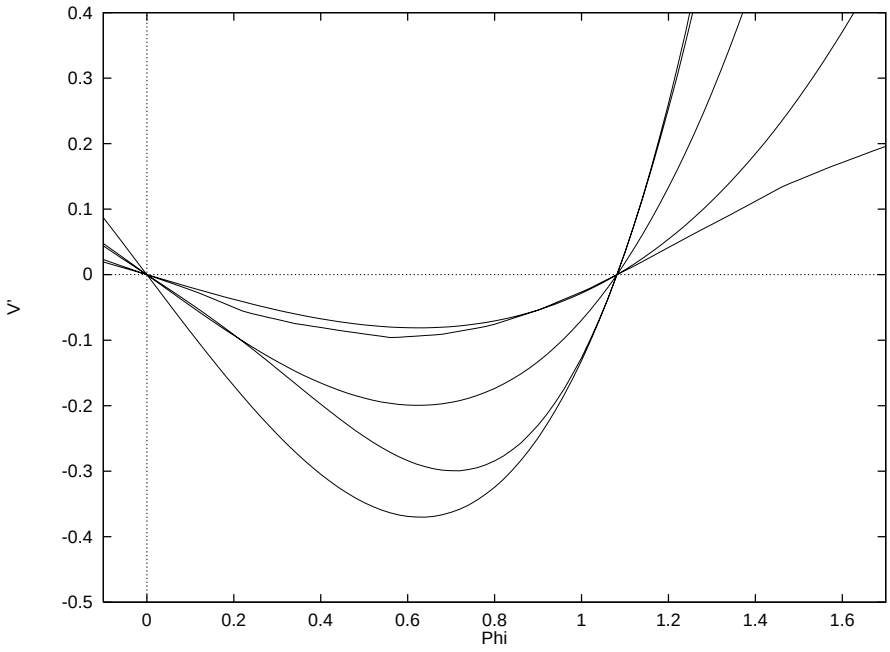


Figure 20: The rescaled derivative of the constraint effective potential is plotted against the block-spin $\overline{\phi}$. The minimum of the potential is fixed at the same value of $\overline{\phi}$. The lines from the bottom are $i = 2, 3, 1, 4, 0$. This was calculated on a 16×16 lattice with blocking factor 2, $m_0^2 = -0.186$ and $\lambda_0^{(4)} = 1$. Note that the minimum of the potential is where $V' = 0$ (see figure 10.)

minimum at $\Phi = 0$. In the other area the height of the mexican hat becomes larger and larger, i.e. the effective mass m_i^2 runs to $-\infty$. If we could start exactly on the critical line the flow of the effective potential would run on the critical surface attracted by a fixed point. Because we use an approximation we are not able to determine exactly the critical line. If we start just below or above the critical surface the flow is attracted by the fixed point in its irrelevant directions, whereas the relevant direction pushes the flow away from the surface. The closer we start at the surface the closer the flow reaches the fixed point. If the flow is stable long enough we will arrive near the fixed point. But in general we have too less number of iterations where we get a stable flow. Therefore we search for some point on the critical line which is somehow nearest to the fixed point. The hope is that the number of iterations for a stable flow are sufficient to come close to the fixed point.

Nevertheless we could not find any fixed point. It seems to have a strong negative coupling m_0^2. Because the saddle point approximation breaks down for too negative m_0^2 we can not study the area around the fixed point. Or we start to far away from the fixed point so that three to four iterations are not sufficient to lead the flow near the fixed point.

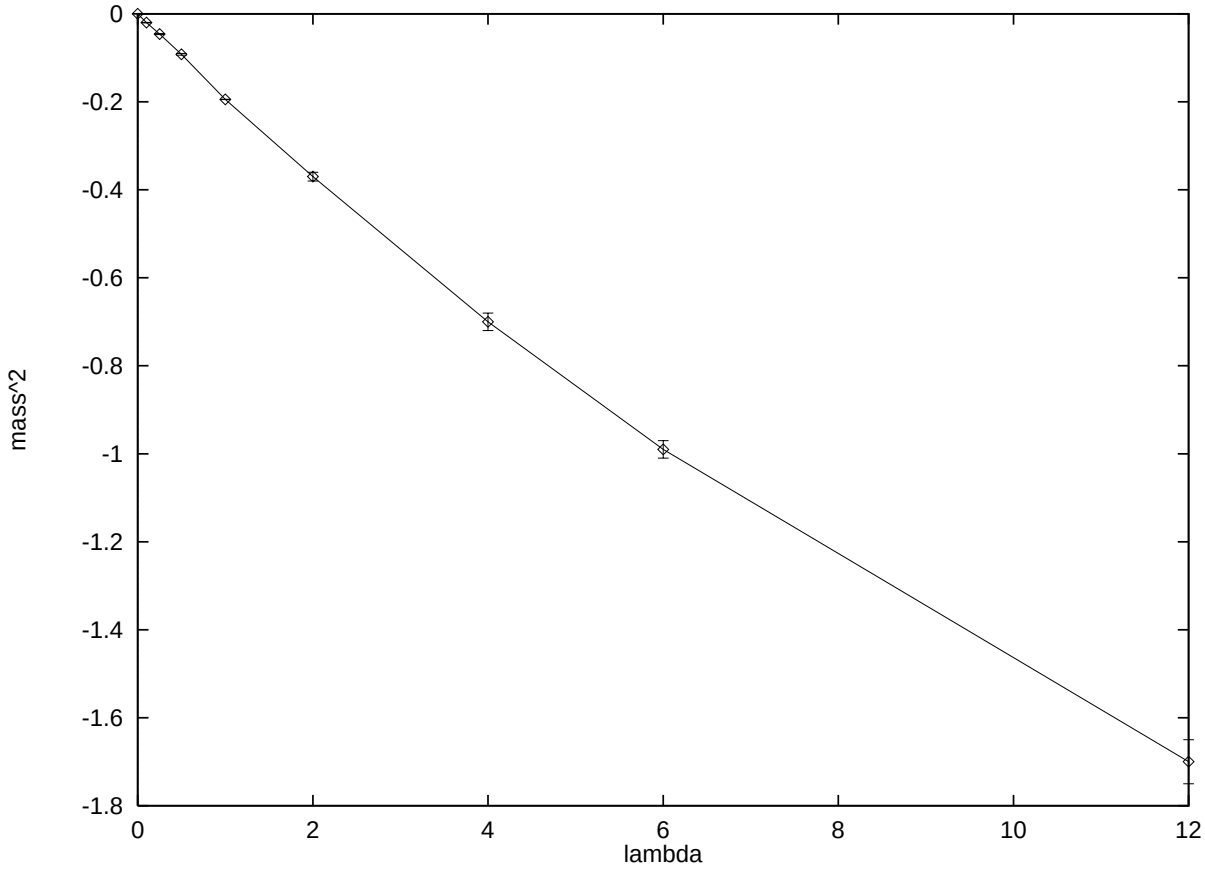


Figure 21: The critical line is plotted. The errorbars desribe the uncertainty of m_0^2 where we could decide whether the pair $(\lambda_0^{(4)}, m_0^2)$ lies above or below the critical line.

9 Conclusion

In this paper it was shown that the saddle point approximation is a useful approximation to calculate the flow of the effective potential under real space renormalization group transformations. The approximation is not a polynomial approximation and suffers therefore not so strongly from truncating the action than the perturbative calculation. It can be applied to the whole physical and meta-stable region. We started with the well known saddle point approximation for the calculation of the effective potential which is equivalent to a 1-loop approximation. This procedure can be iterated and preserves a simple parameterization of the action. As an addition we applied a localization approximation which made the calculation of the flow manageable.

The numerical results showed a good agreement in the calculation of the constraint effective potential with Monte Carlo results. Nevertheless there are differences. The first iterations have significant distance to the Monte Carlo flow, but the distance decreases fast for the next iterations.

We introduced an improved saddle point approximation which is preceded by normal ordering. It takes more terms of the recursion equation into account before the localization approximation is applied. On the other side there occurs no tadpole like graph in the recursion equation for $S^{i+1\prime\prime}$. This is quite of interest for theories where these graphs make problems as e.g. the sine gordon model.

For comparison we determined the flow of the effective potential by means of perturbation theory. Here we expanded in powers of the fluctuation propagator, which is slightly different from the usual expansion in powers of the quartic coupling λ_i (see e.g. [11]). Also in contrast to the literature we iterated the perturbation theory for the ϕ^4-theory instead of using the effective potential after one step for Monte Carlo simulations.

The advantage of the perturbative calculation of the effective potential is that one needs to calculate the kernels $\widehat{R}(0)$ only once. Then they can be used for every value of the coupling constants (where perturbative calculation makes sense). It is a good approximation for small values of the coupling constants. For a given order in Γ^i of the expansion the truncation of higher couplings makes it not usable for more than two iterations (with $N = 6$). Also for large field values it goes rapidly wrong, which is due to the truncation of a Taylor expansion. One could try to use more coupling constants. Of course also non-perturbative effects are not taken into account.

The numerical implementation of the improved saddle point method with normal ordering was not studied. Because $S^{i\prime\prime}$, $T^i{}_{;;}$ are not diagonal in momentum space the numerical effort is much larger. Derivations of the normal ordered form T^i of S^i are determined by iterations of a gap equation. This raises also the effort. Nevertheless it is believed that it can improve the accuracy of numerical treatments.

Also we tried to determine the critical line with the rescaled flow calculated by the saddle point approximation. The critical line separates the space of the coupling constants in two parts: The first part contains the couplings of actions which run from the broken phase into the symmetric phase (or stay in the symmetric phase). The second parameterizes actions which become actions with an infinite effective mass. A fixed point could not be found. For a too negative m_0^2 we got too strong instabilities which made it impossible to determine the critical line for more negative m_0^2.

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A Formulas for saddle point approximation

A.1 Form of Γ

We are looking for the propagator which corresponds to the quadratic term $\frac{1}{2}\zeta^i S^{i\prime\prime}[\Psi^i[\Phi]]\zeta^i$ in (21). The fluctuation fields ζ^i obey the constraint $\mathcal{C}^i\zeta^i = 0$. Let[8]

$$\Gamma^i = S^{i\prime\prime -1} - S^{i\prime\prime -1}\mathcal{C}^{i*}(\mathcal{C}^i S^{i\prime\prime -1}\mathcal{C}^{i*})^{-1}\mathcal{C}^i S^{i\prime\prime -1}\,. \tag{118}$$

It obeys the constraint $\mathcal{C}^i\Gamma^i = 0$. Therefore $\Gamma^i : \mathcal{H}^i \to \mathcal{H}^i_{\mathcal{C}}$. We now want to show that

$$(\Gamma^i)^{-1} = S^{i\prime\prime}$$

This means that

$$\Gamma^i S^{i\prime\prime}\Big|_{\mathcal{H}^i_{\mathcal{C}}} = 1_{\mathcal{H}^i_{\mathcal{C}}} = S^{i\prime\prime}\Gamma^i\Big|_{\mathcal{H}^i_{\mathcal{C}}}\,.$$

The first equation is equivalent to

$$\Gamma^i S^{i\prime\prime}\zeta^i = \zeta^i \qquad \forall\ \zeta^i \in \mathcal{H}^i_{\mathcal{C}}\,.$$

Applying $S^{i\prime\prime}\zeta^i$ to (118) from the right confirms this.
The second equation is true because it is the adjoint of the first.

A.2 Proof of the relation for the background field

Differentiating the defining equation (18) for the background field by the chain rule, we obtain

$$S^{i\prime\prime}[\Psi^i]\Psi^{i\prime} = \mathcal{C}^{i*}\lambda^{i\prime} \tag{119}$$

for some $\lambda^{i\prime} \in \mathcal{H}^{i+1}$. Multiplying (119) with $S^{i\prime\prime -1}$

$$\Psi^{i\prime} = S^{i\prime\prime -1}\mathcal{C}^{i*}\lambda^{i\prime} \tag{120}$$

and $\mathcal{C}^i$ gives

$$\mathcal{C}^i\Psi^{i\prime} = \mathcal{C}^i S^{i\prime\prime -1}\mathcal{C}^{i*}\lambda^{i\prime}$$

By virtue of $\mathcal{C}^i\Psi^{i\prime} = 1$ it follows

$$1 = (\mathcal{C}^i S^{i\prime\prime -1}\mathcal{C}^{i*})\lambda^{i\prime}$$

Therefore

$$(\mathcal{C}^i S^{i\prime\prime -1}\mathcal{C}^{i*})^{-1} = \lambda^{i\prime} \tag{121}$$

We have now one possible form of $\Psi^{i\prime}$:

$$\Psi^{i\prime} = S^{i\prime\prime -1}\mathcal{C}^{i*}(\mathcal{C}^i S^{i\prime\prime -1}\mathcal{C}^{i*})^{-1} \tag{122}$$

Multiplying Γ^i in (118) with $S^{i\prime\prime}$ from right and using (121) gives

$$\begin{aligned}\Gamma^i S^{i\prime\prime} &= 1 - S^{i\prime\prime -1}\mathcal{C}^{i*}(\mathcal{C}^i S^{i\prime\prime -1}\mathcal{C}^{i*})^{-1}\mathcal{C}^i \\ &= 1 - S^{i\prime\prime -1}\mathcal{C}^{i*}\lambda^{i\prime}\mathcal{C}^i\end{aligned}$$

[8] $S^{i\prime\prime}$ has in general no zero modes. If there are some zero modes (e.g. for $\phi = 0$) then they can be eliminated with the help of projection operators before building the inverse $S^{i\prime\prime -1}$. This is well known (see [4]).

Inserting (120) leads to

$$\Gamma^i S^{i\prime\prime} = 1 - \Psi^{i\prime} \mathcal{C}^i. \tag{123}$$

By multiplying $\mathcal{C}^{i*}$ from the right we get the result

$$\Psi^{i\prime} = (1 - \Gamma^i S^{i\prime\prime}) \widetilde{\mathcal{C}}^{i*}. \tag{124}$$

A.3 Derivative of Γ^i

Multiplying (123) with Γ^i from the right and using $\mathcal{C}^i \Gamma^i = 0$ gives

$$\Gamma^i S^{i\prime\prime} \Gamma^i = \Gamma^i.$$

Differentiation leads to

$$\Gamma^{i\prime} = \Gamma^{i\prime} S^{i\prime\prime} \Gamma^i + \Gamma^i S^{i\prime\prime\prime} \Gamma^i + \Gamma^i S^{i\prime\prime} \Gamma^{i\prime}\,.$$

$S^{i\prime\prime\prime}\Gamma^i$ and $\Gamma^i S^{i\prime\prime}$ can be replaced now by (123). Using $\mathcal{C}^i \Gamma^{i\prime} = 0$ leads to the result

$$\Gamma^{i\prime} = -\Gamma^i S^{i\prime\prime\prime} \Gamma^i$$

or expressed with W^i

$$\Gamma^i{}_{,z} = \Gamma^i W^i{}_{,z} \Gamma^i\,. \tag{125}$$

A.4 Determination of the background field

An equation for $\Psi^i{}_{,x}[\Phi](z)$ was already derived in A.2. It involves quantities which are obtained together with the propagator. Expressed in terms of W^i it looks

$$\Psi^{i\prime} = (1 + \Gamma^i W^i) \widetilde{\mathcal{C}}^{i*}.$$

To determine the recursion relation of W^i we need also the second derivative. Using the formula for the derivative of the fluctuation propagator (125) we find

$$\Psi^i{}_{,xy}[\Phi](z) = \int\limits_{w,z_1,z_2 \in \Lambda^i} \Gamma^i(z,z_1)\, W^i{}_{,w}(z_1,z_2)\, \Psi^i{}_{,y}[\Phi](w)\, \Psi^i{}_{,x}[\Phi](z_2). \tag{126}$$

A.5 Recursion relation for W^i

To get W^{i+1} out of (24) we need the second derivative of S^{i+1}.

Consider first

$$\begin{aligned}\frac{\delta}{\delta\Phi(x)} \mathrm{tr} \ln \Gamma^i[\Psi^i[\Phi]] &= \mathrm{tr}\, \Gamma^i[\Psi^i[\Phi]] \frac{\delta}{\delta\Phi(x)} W^i[\Psi^i[\Phi]] \\ &= \int_z \mathrm{tr}\Big(\Gamma^i[\Psi^i[\Phi]]\, W^i{}_{,z}[\Psi^i[\Phi]]\Big)\, \Psi^i{}_{,x}[\Phi](z)\end{aligned}$$

where we have used (125) and $W^i{}_{,z}[\Psi^i]$ or $\Psi^i{}_{,x}[\Phi]$ denote the functional derivatives

$$\frac{\delta}{\delta\Psi^i(z)} W^i[\Psi^i]\,, \qquad\qquad \frac{\delta}{\delta\Phi(x)} \Psi^i[\Phi]\,.$$

We differentiate once more to obtain the second derivative and omit the functional argument $\Psi^i[\Phi]$:

$$\begin{aligned}
&\frac{\delta^2}{\delta\Phi(x)\delta\Phi(y)}\mathrm{tr}\ln\Gamma^i[\Psi^i[\Phi]] \\
&= \int\limits_{z,w\in\Lambda^i} \mathrm{tr}\Big(\Gamma^i\, W^i{}_{,z}\Gamma^i\, W^i{}_{,w} + \Gamma^i W^i{}_{,zw}\Big)\, \Psi^i{}_{,x}[\Phi](z)\Psi^i{}_{,y}[\Phi](w) \\
&\quad + \int\limits_{z\in\Lambda^i} \mathrm{tr}\Big(\Gamma^i W^i{}_{,z}\Big)\, \Psi^i{}_{,xy}[\Phi](z) \\
&= \int\limits_{z,w,z_i\in\Lambda^i} \Gamma^i(z_1,z_2)W^i{}_{,z}(z_2,z_3)\Gamma^i(z_3,z_4)W^i{}_{,w}(z_4,z_1) \\
&\qquad\qquad \Psi^i{}_{,x}[\Phi](z)\Psi^i{}_{,y}[\Phi](w) \\
&\quad + \int\limits_{z,w,z_1,z_2\in\Lambda^i} \Gamma^i(z_1,z_2)W^i{}_{,zw}(z_2,z_1)\,\Psi^i{}_{,x}[\Phi](z)\Psi^i{}_{,y}[\Phi](w) \\
&\quad + \int\limits_{z,z_1,z_2\in\Lambda^i} \Gamma^i(z_1,z_2)W^i{}_{,z}(z_2,z_1)\,\Psi^i{}_{,xy}[\Phi](z).
\end{aligned} \tag{127}$$

Now we consider $S^i[\Psi^i[\Phi]]$. By the chain rule

$$\frac{\delta^2}{\delta\Phi(x)\delta\Phi(y)}S^i[\Psi^i[\Phi]] = \Psi^i{}_{,x}[\Phi]^*\, S^{i\prime\prime}[\Psi^i]\, \Psi^i{}_{,y}[\Phi] + S^{i\prime}[\Psi^i]\, \Psi^i{}_{,xy}[\Phi]. \tag{128}$$

By the saddle point condition, $S^{i\prime}[\Psi^i] = \mathcal{C}^{i*}\lambda^i$. Therefore the last term looks in coordinates

$$\int\limits_{z\in\Lambda^i} \frac{\delta^2}{\delta\Phi(x)\delta\Phi(y)}\Psi^i[\Phi](z) \int\limits_{x_1\in\Lambda^{i+1}} \mathcal{C}^{i*}(z,x_1)\lambda^i(x_1)\,.$$

Inserting (126) shows that this term vanishes because of $\Gamma^i\mathcal{C}^{i*} = 0$. Therefore

$$W^{i+1}[\Phi](x,y) = \Psi^i{}_{,x}[\Phi]^*\, W^i[\Psi^i[\Phi]]\, \Psi^i{}_{,y}[\Phi] + \frac{1}{2}\frac{\delta^2}{\delta\Phi(x)\delta\Phi(y)}\mathrm{tr}\ln\Gamma^i[\Psi^i[\Phi]]\,.$$

Using (127) and (126) leads to the result (27).

B Notations for the lattice

Let Λ^i be a D-dimenional position space lattice with lattice spacing a_i and length $L_i = N_i a_i$. The block factor between two successive lattices shall be denoted by $s = a_{i+1}/a_i$.

The block lattice Λ^{i+1} is defined as a lattice with the same length, but with lattice spacing a_{i+1} which is s times larger than a_i. Therefore the number of points in one direction is reduced: $N_{i+1} = N_i/s$. All lattice points of Λ^i which are situated in a cube of size s^D correspond to one block lattice point; the lattice Λ^i is splitted into disjoint *blocks*. See figure 22 for an example in two dimensions.

The correspondence can be represented by a split of coordinates $z = x + w$, where $z \in \Lambda^i$, $x \in \Lambda^{i+1}$ and

$$w \in \Lambda^{i,i+1} := \{z \in \Lambda^i : 0 \le z_j < s; 0 \le j < D\}\,.$$

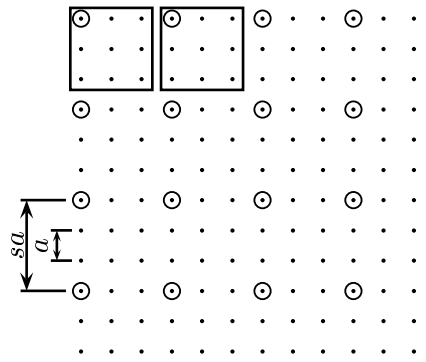

Figure 22: Starting from a 12×12 lattice with lattice spacing a the block lattice has lattice spacing $3a$ and is represented by the circles. The lattice points corresponding to one block lattice point are surrounded by a frame.

The integral over the lattice is defined as a sum with volume factor:

$$\int\limits_{z \in \Lambda^i} := (a_i)^D \sum_{z \in \Lambda^i} .$$

The split of coordinates leads to a split of the integral:

$$\int\limits_z = \int\limits_x \int\limits_w$$

with the definitions

$$\int\limits_x := (a_{i+1})^D \sum_{x \in \Lambda^{i+1}} \quad , \quad \int\limits_w := \frac{1}{s^D} \sum_{w \in \Lambda^{i,i+1}} \tag{129}$$

The corresponding dual lattice $\tilde{\Lambda}^i$ has a lattice spacing $\tilde{a}_i = 2\pi/L_i$ and length $\tilde{L}_i = 2\pi/a_i$. It is convenient to introduce an additonal momentum space lattice $\tilde{\Lambda}^{i,i+1}$ with spacing $\tilde{a}_{i,i+1} = \tilde{L}_i/s$ and length $\tilde{L}_{i,i+1} = s\tilde{L}_i$. Then any $p \in \tilde{\Lambda}^i$ can be decomposed into $p = q + l$ with $q \in \tilde{\Lambda}^{i+1}$ and $l \in \tilde{\Lambda}^{i,i+1}$. The definitions

$$\int\limits_p := \frac{\tilde{a}_i^D}{(2\pi)^D} \sum_{p \in \tilde{\Lambda}^i} \quad , \quad \int\limits_q := \frac{\tilde{a}_{i+1}^D}{(2\pi)^D} \sum_{q \in \tilde{\Lambda}^{i+1}} \quad , \quad \int\limits_l := \sum_{l \in \tilde{\Lambda}^{i,i+1}} \tag{130}$$

imply the split

$$\int\limits_p = \int\limits_q \int\limits_l . \tag{131}$$

We now want to derive the momentum space representation (51) of the fluctuation propagator. First note that any linear operator $L : \mathcal{H}^i \to \mathcal{H}^i$ with $L(z, z') = L(z - z')$ can be written as

$$L(z - z') = \int\limits_p L(p) e^{ip(z-z')}. \tag{132}$$

Examples are $\overline{W}^i(z,z'|\overline{\phi})$ in the saddle point approximation and the laplacian $\Delta^i(z,z')$. In the latter case one gets

$$\Delta^i(p) = -\hat{p}^2 \qquad , \qquad \hat{p}_\mu = \frac{2}{a_i}\sin\left(\frac{p_\mu a_i}{2}\right). \tag{133}$$

The representation (132) also holds for the averaging operator $\mathcal{C}^i(x,z)$ since $x \in \mathcal{H}^{i+1} \subset \mathcal{H}^i$ with

$$\mathcal{C}^i(p) = \prod_{\mu=1}^{D}\left(\frac{\sin(p_\mu s a_i/2)}{s\sin(p_\mu a_i/2)}e^{+ip_\mu(s-1)a_i/2}\right) \tag{134}$$

The fluctuation propagator Γ^i on the other hand is only invariant under translations $\Gamma^i(z,z') = \Gamma^i(z-x,z'-x)$ with $x \in \Lambda^{i+1}$ and therefore has the more complicated Fourier representation

$$\Gamma^i(z,z') = \int\limits_{l,q,l'} \Gamma^i(l,q,l')e^{i(q+l)z - i(q+l')z'} \tag{135}$$

with

$$\begin{aligned}
\Gamma^i(l,q,l') &= v^i(q+l)\delta_{l,l'} \\
&\quad -v^i(q+l)\mathcal{C}^{i*}(q+l)u^{i^{-1}}(q)\mathcal{C}^i(q+l')v^i(q+l') && (136)\\
u^i(q) &= \int\limits_l \mathcal{C}^i(q+l)v^i(q+l)\mathcal{C}^{i*}(q+l) && (137)\\
v^i(p) &:= -\overline{W}^i(\overline{\phi})^{-1}(p) && (138)
\end{aligned}$$

C Maple program for perturbative expansion

We used a maple program to calculate the perturbative expansion of the effective potential in third order of the fluctuation propagator.

According to equation (95) in section 7 the program calculates symbolically $e^A e^B$ upto a certain order in Γ^i. A and B are defined as in equations (96) and (97).

`Expansion(`*`order`*`)` is the main procedure which calculates the perturbative expansion upto the order *order*. The result is expressed in a general form similiar to (100). B and its derivatives stay in the resulting expression. [9] The derivatives are denoted by `B[i,z.j]`. The first index represents the number of derivatives and the second a corresponding coordinate. We start with local potentials. That means that more than one derivatives $\delta/\delta\psi(z)$ generate delta-functions (see e.g. (103) to (106)). Therefore it is sufficient to mark only one coordinate which is the argument of one of the derivatives $\delta/\delta\psi(z_j)$. The other coordinates have the same value due to the delta-functions. Internally the procedure `dBtermReplace( ... )` replaces symbolic derivatives $\delta B/\delta\psi(z_j)$ by `B[1,z.j]` and δ`B[i,z.k]`$/\delta\psi(z_j)$ by `B[i+1,z.j]` for $i > 1$. Also the delta-functions are taken into account. In the latter case all coordinates z_k are replaced by z_j.

`applyA(`*`expr,psi,varIndex1,varIndex2,maxIndex`*`)` applies A to *expr*. First it takes the derivative of *expr* with respect to ψ. Then it calls `dBreplace( ... )` (which calls `dBtermReplace( ... )` for every term) to replace the symbolic derivatives and to set the associated coordinate $z_{varIndex1}$. Second it takes the derivative

[9] It is not really the general form, because the delta-functions of higher derivatives are already evaluated (see below).

of the result again, multiplies with $\Gamma(z_{varIndex1}, z_{varIndex2})$ and calls `dBreplace( ... )` again. Now the coordinate is set to $z_{varIndex2}$.

`Expansion(`*`order`*`)` starts with e^B and calls `applyA( ... )` according to the Taylor expansion of e^A. E.g. `Expansion(3)` calls `applyA( ... )` three times and divides by 3!. The argument *maxIndex* of `applyA( ... )` is set to $2*$*order*, because there are maximal $2*$*order* different coordinates possible (every A has two different coordinates). The exponential factor e^B is reproduced by differentiation. We get rid of it by dividing through e^B. Then the procedure calls `Sort( ... )` which sorts the indices of the coordinates to fulfill the following scheme:

$$B^{(1)}(z_0), \ldots, B^{(1)}(z_i), B^{(2)}(z_{i+1}), \ldots, B^{(2)}(z_j), B^{(3)}(z_{j+1}), \ldots$$

This helps to sum up terms of the same type, i.e. which have the same topology but different numerical factors. Also nonconnected terms are deleted.

Of course we want the expansion for a certain starting potential $V[\psi]$. In the procedure `Replace( ... )` every `B[i,z.j]` is replaced by the explicit derivative of $-V$ expressed by powers of $\psi(z_j)$ and the coupling constants. The global variable `maxOrder` decides which concrete potential should be used. `maxOrder=4` (`maxOrder=6`) means that the potential (101) with $N = 4$ ($N = 6$) is used. This variable is also used to indicate which is the highest derivative of B which does not vanish.

As a result we get a lot of terms. But almost all terms have symmetries with respect to renaming of indices, i.e. many terms are of equal type (they have the same topology). The program does not recognize them because the indices of the coordinates are different. Therefore the procedure `permSort( ... )` permutes all indices and tries to sum up terms of the same type. This can be very time consuming, especially for higher orders.

To compare with (107) one has to integrate over all coordinates which occur in the result. To see examples of usage have a look at the end of the listing.

```
# Maple-programm to calculate the perturbative expansion of the
#      effective Potential Veff[Psi]
# author:  Max Griessl
#
#   relevant variables to set:
#      maxOrder   highest derivative of V which is nonzero
#      order      desired order of Gamma
#   relevant procedures:
#      Expansion(order)   calculates all terms of the desired order
#                         in a general form with B and its
#                         derivatives
#      Replace(expression,psi,2*order)   Replaces B^(i)(psi(z.j))
#                         by corresponding derivatives of V (the
#                         content has to be adapted to V;
#                         here maxOrder=4 means V has nonzero
#                         couplings m^2, lambda^(4)  and maxOrder=6
#                         means additional lambda^(6))
#      permSort(expression,psi,2*order)  the indices are permuted
#                         and terms of the same type are added
#
# have a look at the end of the listing how they are used

with(combinat,permute);

#
```

```
# Replace derivatives of B and of corresponding coordinates
# in ‘expr‘
#
dBreplace:=proc(expr,psi,varIndex,maxIndex)
local i,termList,nrList;
#
# splitting up ‘expr‘ into terms
nrList:='nrList';
termList:=ListOfTerms(expr,nrList);
#
# every term will be treated
for i from 1 to nrList do
   termList:=subsop(i=dBtermReplace(termList[i],psi,varIndex,
                   maxIndex),termList);
od;
#
# all terms will be connected again to one expression. Zero-terms
# will be deleted.
convert(termList,‘+‘);
end;

#
# Replace derivatives of B and of corresponding coordinates
# in ‘expr‘, which is only one term
#
dBtermReplace:=proc(expr,psi,varIndex,maxIndex)
local j,i,temp,temp1,exprt;
global maxOrder;
#
# first derivatives of order one of B are replaced. No
# delta-functions are created, i.e. the coordinates are assigned
# to the desired index 'varIndex'
#
exprt:=subs(diff(B(psi),psi)=B[1,z.varIndex](psi),expr);
for j from 1 to maxIndex do
#
# every derivative of B^(j)(psi) is replaced by B^(j+1)(psi)
       temp:=exprt;
       for i from 1 to maxOrder-1 do
              temp1:=subs(diff(B[i,z.j](psi),psi)=
                           B[i+1,z.varIndex](psi),temp);
              temp:=temp1;
       od;
#
# if the expression has changed the corresponding delta-function
# has to be applied
       if temp<>exprt then
              exprt:=subs(z.j=z.varIndex,temp);
       fi;
#
# if the term has derivatives of type B^(maxOrder+1)(psi) the whole
# term is zero
       temp:=subs(diff(B[maxOrder,z.j](psi),psi)=0,exprt);
       if temp<>exprt then
```

```
              RETURN(0);
       fi;
od;
RETURN(exprt);
end;

#
# applies A to `expr` once
# A has internal coordinates `varIndex1` and `varIndex2`
# A=1/2*d/dpsi(varIndex1)Gamma(varIndex1,varIndex2)d/dpsi(varIndex2)
#
applyA:=proc(expr,psi,varIndex1,varIndex2,maxIndex)
local temp;
       temp:=dBreplace(diff(expr,psi),psi,varIndex1,maxIndex);
       temp:=expand(diff(temp,psi)*Gamma[z.varIndex1,z.varIndex2]
                    / 2);
       RETURN(dBreplace(temp,psi,varIndex2,maxIndex));
end;

#
# sorts the indices of the coordinates, i.e.
# B^(1)(psi(z_0)),...,B^(1)(psi(z_i)),B^(2)(psi(z_i+1)),...
#   ...B^(4)(psi(z_n))
# deletes nonconnected terms
#
Sort:=proc(expr,psi,maxIndex)
local i,j,k,termList,nrList,temp1,temp2,currenti,connectedIndex,
      gam,permList;
global maxOrder;
#
# every term of the expression 'expr' is treated on its own
#
nrList:='nrList';
termList:=ListOfTerms(expr,nrList);
for i from 1 to nrList do
   temp1:=termList[i];
   currenti:=0;
#
# first count number of Gammas and store their indices in gam[]
# 'currenti' holds the number of Gammas
   for j from 1 to maxIndex do
      for k from 1 to maxIndex do
        temp2:=subs(Gamma[z.j,z.k]=1,temp1);
        if temp2<>temp1 then
           currenti:=currenti+1;
           gam[currenti,1]:=j;
           gam[currenti,2]:=k;
        fi
      od
   od;
#
# second look for the number of Gammas which are connected
# the number is stored in 'connectedIndex' (no multiple counting!)
   connectedIndex:=0;
```

```
   for j from 1 to currenti do
      for k from j+1 to currenti do
            if gam[j,1]=gam[k,1] or gam[j,1]=gam[k,2] \
               or gam[j,2]=gam[k,1] or gam[j,2]=gam[k,2] then
           connectedIndex:=connectedIndex+1;
        fi;
      od
   od;
#
# all Gammas in the term has to be connected, otherwise it will
# be deleted (only connected terms contribute to Veff)
# there must be at least currenti-1 connections to be connected
   if connectedIndex>=currenti-1 then
#
# the coordinates are sorted like above
# the coordinates have had values out of  x1...xmaxIndex
# now they start with x0
      currenti:=0;
      for j from 1 to maxOrder do
        for k from 1 to maxIndex do
           temp2:=subs(B[j,z.k]=B[j,z.currenti],temp1);
           if temp2<>temp1 then
              temp1:=subs(z.k=z.currenti,temp2);
              currenti:=currenti+1;
              temp2:=subs(z.currenti=z.k,temp1);
              temp1:=temp2;
           fi
         od
      od;
      termList:=subsop(i=temp1,termList);
   else
#
# nonconnected terms are deleted
      termList:=subsop(i=0,termList);
   fi
od;
#
# connect all terms together
RETURN(convert(termList,`+`));
end;


#
# splitting up 'expr' in distinct terms
#
ListOfTerms:=proc(expr,nrList)
local i,termList;
termList:=convert(expr+1,list);
nrList:=nops(termList)-1;
#
# if 'expr' is allready one term 'convert(expr,list)' would give
# the factors of 'expr'
# therefore '1' is added to get really a list
# now the term equal to '1' is deleted again
```

```
i:=1;
while termList[i]<>1 do i:=i+1; od;
RETURN(subsop(i=NULL,termList));
end;

#
# all permutations of indices are used to find terms of equal type
# which can be simplified to one term
#
permSort:=proc(expr,psi,maxIndex)
local i,j,k,ii,jj,termList,nrList,temp1,temp2,temp3,permList,perm,nr,
      tempList,pl,nrPerm,currenti;
#
# move indices from 0..maxIndex-1 to 1..maxIndex
# this makes it possible to collect all indices starting from 0 and
# with differences one
temp1:=expr;
for k from maxIndex by -1 to 1 do
       temp1:=subs(z.(k-1)=z.k,temp1);
od;
#
# every term is treated on its own
#
nrList:='nrList';
termList:=ListOfTerms(temp1,nrList);
#
# rename indices of z_i; they start with i=0 and have differences 1
#
for j from 1 to nrList do
  temp1:=termList[j];
  currenti:=0;
  for k from 1 to maxIndex do
       temp2:=subs(z.k=z.currenti,temp1);
       if temp2<>temp1 then
               currenti:=currenti+1;
               temp1:=subs(z.currenti=z.k,temp2);
       fi
  od;
  termList:=subsop(j=temp1,termList);
od;
#
# generate all permutations from 1 to 'maxIndex'
permList:=permute(maxIndex);
nrPerm:=nops(permList);
#
# run through all terms except of the first; it is used
# as reference
for j from 2 to nrList do
   temp1:=termList[j];
   #
   # run through all permutations
   for pl from 1 to nrPerm do
      perm:=permList[pl];
      temp2:=temp1;
```

```
  #
  # apply permutation to indices of the selected term
  # y.i is used as temporary copy
  for i from 1 to maxIndex do
     temp2:=subs(z.(i-1) = y.i,temp2);
  od;
  for i from 1 to maxIndex do
     temp2:=subs(y.i=z.(perm[i]-1),temp2);
  od;
  #
  # run through all possible pairs of indices
  for ii from 0 to maxIndex-1 do
     for jj from ii+1 to maxIndex-1 do
        #
        # just swapping the coordinates of Gamma
        # can help to simplify; the symmetry of
        # Gamma allows swapping of the coordinates
# because the original terms are sorted it is sufficient
# to swap only unsorted coordinates
        temp2:=subs(Gamma[z.jj,z.ii]=Gamma[z.ii,z.jj],temp2);
        #
        # if the permutation of the indices or the swap
        # changes temp2 then we try to find a similar
        # term
        if temp2 <> temp1 then
           #
           # run through all previous terms
           for i from 1 to j-1 do
              if termList[i]<>0 then
                 #
                 # try to add both terms
                 temp3:=expand(termList[i]+temp2);
                 #
                 # find out how many terms are in the
                 # result
                 nr:='nr';
                 tempList:=ListOfTerms(temp3,nr);
                 #
                 # did the addition reduce the number
                 # of terms?
                 if nr = 1 then
                    #
                    # assign the sum to the i-th term and
                    # delete the j-th term
                    termList:=subsop(i=temp3,j=0,termList);
                    #
                    # setting i=j exits the loop
                    i:=j;
                    #
                    # exit pl-loop
                    pl:=nrPerm+1;
                    #
                    # exit ii- and jj-loops
                    jj:=maxIndex; ii:=maxIndex;
```

```
                    fi;
                 fi;
              od;
           fi;
        od;
     od;
  od;
od;
#
# put all terms together again
RETURN(convert(termList,`+`));
end;


#
# calculates the effective potential expressed with B^(i)(psi) of
# `order`-th order in Gamma
#
Expansion:=proc(order)
local i,j,temp,b;
b:=psi->exp(B(psi));
temp:=b(psi);
for i from 1 to order do
   temp:=expand(applyA(temp,psi,2*i-1,2*i,2*i)/i!);
od;
temp:=expand(temp/b(psi));
temp:=Sort(temp,psi,2*order);
#
# the symmetry of Gamma is used
for i from 0 to 2*order-1 do
   for j from i+1 to 2*order-1 do
      temp:=subs(Gamma[z.j,z.i]=Gamma[z.i,z.j],temp);
   od;
od;
RETURN(-temp);
end;

#
# Replaces B^(i)(psi(z.j)) by corresponding derivatives of V
#
Replace:=proc(expr,psi,maxIndex)
local i,temp;
temp:=expr;
for i from 0 to maxIndex-1 do
   if maxOrder = 4 then
      temp:=subs({B[1,z.i](psi)=-(m^2*psi(z.i)
                                  +lambda/3!*psi(z.i)^3),
                  B[2,z.i](psi)=-(m^2+lambda/2*psi(z.i)^2),
                  B[3,z.i](psi)=-lambda*psi(z.i),
                  B[4,z.i](psi)=-lambda},temp);
   elif maxOrder = 6 then
      temp:=subs({B[1,z.i](psi)=-(m^2*psi(z.i)+lambda/3!*psi(z.i)^3
                                  +lambda[6]/5!*psi(z.i)^5),
                  B[2,z.i](psi)=-(m^2+lambda/2*psi(z.i)^2
```

```
                                +lambda[6]/4!*psi(z.i)^4),
                    B[3,z.i](psi)=-lambda*psi(z.i)-lambda[6]/3!
                                  *psi(z.i)^3,
                    B[4,z.i](psi)=-lambda-lambda[6]/2!*psi(z.i)^2,
                    B[5,z.i](psi)=-lambda[6]*psi(z.i),
                    B[6,z.i](psi)=-lambda[6]},temp);
   fi;
od;
RETURN(expand(temp));
end;


### main

# to get a reusable output which can be inserted again into maple
interface(prettyprint=0);

# global variable; describes the highest derivative of V which
# is nonzero
maxOrder := 4;

# to calculate the effective potential upto second order use this
# line
calcOrder:=2;
Veff:=1/2*m^2*psi(z)^2+lambda/4!*psi(z)^4+Expansion(1)
      + Expansion(calcOrder);
Veff:=permSort(Replace(Veff,psi,2*calcOrder),psi,2*calcOrder);


# to calculate the third order
#calcOrder:=3;
#Veff:=Expansion(calcOrder);

# to simplify the result of the third order
#Veff:=permSort(Replace(Veff,psi,2*calcOrder),psi,2*calcOrder);
```

D Monte Carlo calculation of the effective potential

To compare the various methods we use as a standard method Monte Carlo simulations (see e.g. [3]). It has the advantage that it gives us the *almost correct* values (under certain conditions) together with statistical errors. [10] The error can be reduced by just running the simulation longer. The disadvantage is the long time to get accurate results (otherwise there would be no need to develop other methods).

D.1 Determination of the derivative of the effective potential

We set $\Phi = \text{const.}$ similar to the other methods described in this paper. With $\mathcal{A}^i$ as defined in section 7 the backgrounfd field ψ is constant too and has the same value as Φ. Using a sloppy notation in this case we write for ψ also Φ. We take the

[10] In principle the results become exact in the limit of infinite running time.

derivative of

$$V^{i+1}[\Phi = \text{const}] = -\ln \int \mathrm{d}\mu_{\Gamma^i}(\zeta^i)\, e^{-V^i[\Phi+\zeta^i]}$$

with respect to the constant field Φ:

$$\frac{\mathrm{d}V^{i+1}}{\mathrm{d}\Phi}[\Phi] = \frac{\int \mathrm{d}\mu_{\Gamma^i}(\zeta^i)\, \frac{\mathrm{d}V^i}{\mathrm{d}\Phi}[\Phi+\zeta^i] e^{-V^i[\Phi+\zeta^i]}}{\int \mathrm{d}\mu_{\Gamma^i}(\zeta^i)\, e^{-V^i[\Phi+\zeta^i]}} = \left\langle \frac{\mathrm{d}V^i}{\mathrm{d}\Phi}[\Phi+\cdot] \right\rangle \tag{139}$$

The result is an expectation value of the derivative of the potential which is an adequate form for Monte Carlo simulations.

We determine the expectation value for various constant field values Φ by Monte Carlo simulations (Metropolis algorithm) together with its statistical error and compare it with the saddle point approximation and the perturbative calculation in section 8.

D.2 Determination of effective coupling constants

We set $\Phi = \text{const.}$ similar to the other methods described in this paper. Then nonlocal parts of $V^{i+1}[\Phi]$ vanish and the effective potential can be described as a power series similar to (101) with $N = \infty$. Taking derivatives with respect to the constant field Φ at $\Phi = 0$ we get the effective couplings:

$$m^2_{i+1} = \frac{1}{\Omega} \frac{\mathrm{d}^2 V^{i+1}[\Phi]}{\mathrm{d}\Phi^2}\bigg|\Phi = 0 \tag{140}$$

$$\lambda^{(4)}_{i+1} = \frac{1}{\Omega} \frac{\mathrm{d}^4 V^{i+1}[\Phi]}{\mathrm{d}\Phi^4}\bigg|\Phi = 0 \tag{141}$$

$$\lambda^{(6)}_{i+1} = \frac{1}{\Omega} \frac{\mathrm{d}^6 V^{i+1}[\Phi]}{\mathrm{d}\Phi^6}\bigg|\Phi = 0 \tag{142}$$

where Ω is the block lattice volume $\Omega = \int_x 1$.

First some derivatives:

$$V^{i+1}[\Phi = \text{const}] = -\ln \int \mathrm{d}\mu_{\Gamma^i}(\zeta^i)\, e^{-V[\Phi+\xi]}$$

$$\frac{\mathrm{d}V^{i+1}}{\mathrm{d}\Phi} = \frac{\int \mathrm{d}\mu_{\Gamma^i}(\zeta^i)\, \frac{\mathrm{d}V}{\mathrm{d}\Phi}[\Phi+\xi] e^{-V[\Phi+\xi]}}{\int \mathrm{d}\mu_{\Gamma^i}(\zeta^i)\, e^{-V[\Phi+\xi]}} = \left\langle \frac{\mathrm{d}V}{\mathrm{d}\Phi}[\Phi+\cdot] \right\rangle =: \langle V^1 \rangle \equiv M_1$$

The following useful formula holds for an arbitrary functional $A[\Phi]$, which can be seen easily:

$$\frac{\mathrm{d}}{\mathrm{d}\Phi} \langle A \rangle = \left\langle \frac{\mathrm{d}A}{\mathrm{d}\Phi} - AV^1 \right\rangle + \langle A \rangle \langle V^1 \rangle$$

Then

$$\begin{aligned}
\frac{\mathrm{d}^2 V^{i+1}}{\mathrm{d}\Phi^2} &= M_2 + M_1^2 \\
\frac{\mathrm{d}^4 V^{i+1}}{\mathrm{d}\Phi^4} &= M_4 + 4M_3M_1 + 12M_2M_1^2 + 3M_2^2 + 6M_1^4 \\
\frac{\mathrm{d}^6 V^{i+1}}{\mathrm{d}\Phi^6} &= M_6 + 6M_5M_1 + 30M_4M_1^2 + 15M_4M_2 + 120M_3M_1^3 + 120M_3M_2M_1 \\
&\quad + 10M_3^2 + 360M_2M_1^4 + 270M_2^2M_1^2 + 30M_2^3 + 120M_1^6
\end{aligned}$$

where

$$
\begin{aligned}
M_2 &= \left\langle V^2 - V^{1\,2} \right\rangle \\
M_3 &= \left\langle V^3 - 3V^2V^1 + V^{1\,3} \right\rangle \\
M_4 &= \left\langle V^4 - 4V^3V^1 - 3V^{2\,2} + 6V^2V^{1\,2} - V^{1\,4} \right\rangle \\
M_5 &= \Big\langle V^5 - 5V^4V^1 - 10V^3V^2 + 10V^3V^{1\,2} + 15V^{2\,2}V^1 \\
&\quad -10V^2V^{1\,3} + V^{1\,5} \Big\rangle \\
M_6 &= \Big\langle V^6 - 6V^5V^1 - 15V^4V^2 + 15V^4V^{1\,2} + 60V^3V^2V^1 \\
&\quad -10V^{3\,2} + 15V^{2\,3} - 20V^3V^{1\,3} - 45V^{2\,2}V^{1\,2} + 15V^2V^{1\,4} - V^{1\,6} \Big\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
V^1 &= \frac{\mathrm{d}V}{\mathrm{d}\Phi}[\Phi+\xi] = \int_z \Big(m_0^2(\Phi+\xi(z)) + \sum_{n\geq 4,\text{even}}^{N} \frac{\lambda_0^{(n)}}{(n-1)!}(\Phi+\xi(z))^{(n-1)} \Big) \\
V^2 &= \frac{\mathrm{d}^2V}{\mathrm{d}\Phi^2}[\Phi+\xi] = \int_z \Big(m_0^2 + \sum_{n\geq 4,\text{even}}^{N} \frac{\lambda_0^{(n)}}{(n-2)!}(\Phi+\xi(z))^{(n-2)} \Big) \\
V^k &= \frac{\mathrm{d}^kV}{\mathrm{d}\Phi^k}[\Phi+\xi] = \sum_{n\geq 4,\ \text{even},\ \mathrm{n-k}\geq 0}^{N} \frac{\lambda_0^{(n)}}{(n-k)!}(\Phi+\xi(z))^{(n-k)} \qquad \forall\ k>2
\end{aligned}
$$

(The same is valid for (101) and (102)).

Now it is clear what has to be done. First one determines the expectation values M_i at $\Phi = 0$ by Monte Carlo and do statistical analysis for every M_i to get the statistical errors. This can be done in one Monte Carlo run simultanously because the operator A in $\langle A \rangle$ does not influence the sweep-procedure.

Second one combines the results to the coupling constants according to (140), (141) and (142).

D.3 Comparison of both variants

The Monte Carlo method has also different properties depending on the expectation values one measures. If one calculates the coupling constants the resulting effective potential will be accurate for small field values, but not for large field values. As stated in section 8 one has to truncate the Taylor expansion and only a finite number of coupling constants are used. It has the advantage that only one Monte Carlo run is necessary to get all chosen constants simultaneously as described above. On the other side it needs a large number of sweeps to get accurate results because the expectation values used are quite complicate.

Calculating the (derivative of the) effective potential directly with Monte Carlo gives correct and accurate values for large field values. But for small field values a simple Metropolis algorithm can not be applied, because ergodicity is not fulfilled. One gets domain walls which make it improbable that the markov chain runs through all possible configurations. One needs for every field value Φ one Monte Carlo simulation on its own with less number of sweeps than above.

The flow of the effective potential for these two variants is shown in figure 23. The flow was calculated by increasing the blocking factor s in powers of 2 starting from the same lattice. We see in figure 23 that for larger blocking factors the accuracy of the coupling constants or of V^1 for small field values Φ decreases. This is due to a decrease of the acceptance rate for larger blocks (see [6]).

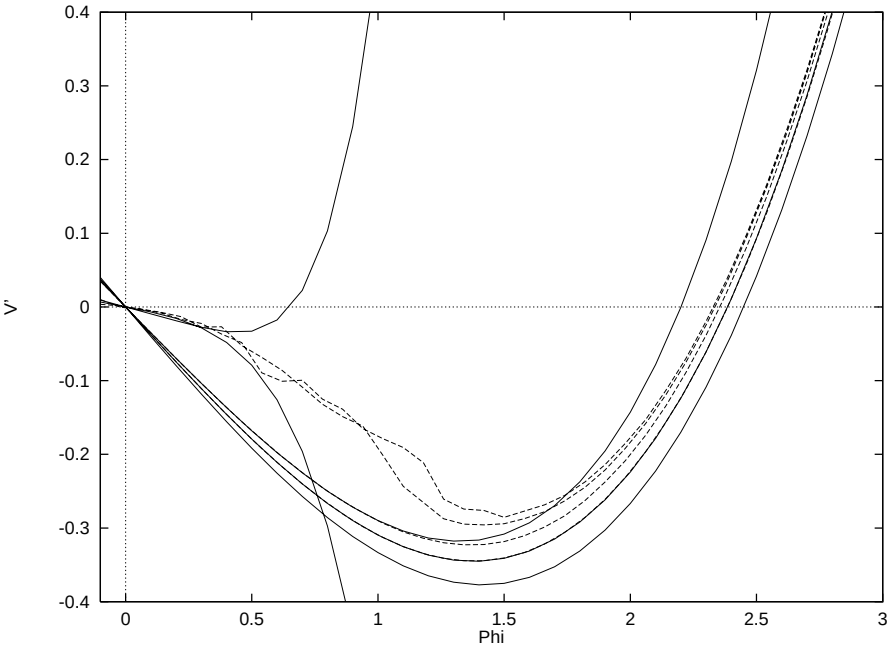

Figure 23: The flow of the derivative of the effective potential is plotted against the block-spin Φ. Two variants of Monte Carlo calculation were used. The solid lines represent the polynomial approximation and the dashed lines the measurement of the expectation value $\langle V^1 \rangle$. This was calculated on a 16×16 lattice with blocking factors 2,4,8 and 16, $m_0^2 = -0.4$ and $\lambda_0^{(4)} = 0.4$. The number of sweeps are 10000 for $\langle V^1 \rangle$ and 10000 for the first line and 100000 for the remaining lines of the polynomial approximation.

E Rescaling of the effective action

E.1 Dimensionless expressions

The well known starting point is setting all physical constants to the dimensionless value one. The action S has the same dimension as the Planck constant $\hbar$. Therefore by setting $\hbar = 1$ the action becomes dimensionless: $[S] = 1$.

Similar by setting the velocity of light equal to one $c = 1$ leads to $[L] = [T]$, i.e. the unit of length is equal to the unit of time. Also this implies $[M] = [L]^{-1}$.

Looking at the free scalar theory $S[\phi] = \int \mathrm{d}^D x \, \frac{1}{2}\phi(x)(-\Delta + m^2)\phi(x)$ we see that the canonical dimension of the scalar field is $[L]^{-\frac{D-2}{2}} =: [L]^{D_\phi}$.

Writing the action as a Taylor expansion

$$\begin{aligned} S[\phi] = \text{const.} + \int \mathrm{d}^D x \, \phi(x) \frac{\delta}{\delta\phi(x)} S[\phi] \\ + \int \mathrm{d}^D x \, \mathrm{d}^D y \, \phi(x) \frac{\delta^2}{\delta\phi(x)\phi(y)} S[\phi]\phi(y) + \dots \end{aligned} \tag{143}$$

we see that the first derivative has dimension $[L]^{-(D+D_\phi)}$ and similar the second derivative has dimension $[L]^{-2(D+D_\phi)}$. In general the n-th derivative has dimension $[L]^{-n(D+D_\phi)} = [L]^{-n\frac{D+2}{2}}$.

Now we construct dimensionless quantities on the lattice. All quantities are divided by a power of the lattice spacing to the corresponding dimension:

$$\widetilde{\phi}(\hat{z}) := \frac{\phi(a\hat{z})}{a^{D_\phi}} \tag{144}$$

$$\widetilde{S}^{(n)}(\hat{z}_1, \dots, \hat{z}_n|\widetilde{\phi}) := a^{n(D+D_\phi)} \, S^{(n)}\Big(a\hat{z}_1, \dots, a\hat{z}_n | a^{D_\phi} \widetilde{\phi}(\frac{\cdot}{a})\Big) \tag{145}$$

Here ϕ is a field on the infinite lattice $\Lambda = (a\mathbb{Z})^D$ with lattice spacing a. $\widetilde{\phi}$ is a field on the infinite lattice $\widetilde{\Lambda} = \mathbb{Z}^D$ with dimensionless lattice spacing 1. $\hat{z}, \hat{z}_i \in \widetilde{\Lambda}$.

Doing one RG-step we get the effective action $S^1[\Phi]$ as a functional of the block-spin field Φ. The latter is a field on the infinite lattice $\Lambda^1 = (sa\mathbb{Z})^D$ with lattice spacing sa, with s the blocking factor. Similar to the effective action we get effective expressions for the derivatives $S^{1\,(n)}$.

Again we can define dimensionless quantities by dividing with appropriate powers of the effective lattice spacing sa ($\hat{x}, \hat{x}_i \in \Lambda^1$):

$$\widetilde{\Phi}(\hat{x}) := \frac{\Phi(sa\hat{x})}{(sa)^{D_\phi}} \tag{146}$$

$$\begin{aligned} \widetilde{S}^{1\,(n)}(\hat{x}_1, \dots, \hat{x}_n|\widetilde{\Phi}) := \\ (sa)^{n(D+D_\phi)} \, S^{1\,(n)}\Big(sa\hat{x}_1, \dots, sa\hat{x}_n | (sa)^{D_\phi} \widetilde{\Phi}(\frac{\cdot}{sa})\Big) \end{aligned} \tag{147}$$

E.2 Rescaling procedure

We want to map the effective quantities back onto the lattice Λ. This is necessary to do unlimited RG-steps on a finite lattice by a computer.

This can be done directly with the help of the dimensionless quantities. The quantities in (146,147) live on the unit lattice. To define quantities on the lattice with lattice spacing a we just multiply the dimensionless quantities by the appropriate powers of a ($z, z_i \in \Lambda$):

$$\phi(z) := a^{D_\phi} \, \widetilde{\Phi}(\frac{z}{a}) \tag{148}$$

$$S_R^{1\,(n)}(z_1, \dots, z_n|\phi) := a^{-n(D+D_\phi)} \, \widetilde{S}^{1\,(n)}\Big(\frac{z_1}{a}, \dots, \frac{z_n}{a} | a^{-D_\phi}\phi(\cdot a)\Big) \tag{149}$$

Inserting (146,147) we see how the quantities on Λ depend on the effective quantities on Λ^1:

$$\begin{aligned} \phi(z) &= s^{-D_\phi}\, \Phi(sz) && (150) \\ S_R^{1\,(n)}(z_1,\ldots,z_n|\phi) &= s^{n(D+D_\phi)}\, S^{1\,(n)}\Big(sz_1,\ldots,sz_n|s^{D_\phi}\phi(\tfrac{\cdot}{s})\Big) && (151) \end{aligned}$$

Defining the rescaling operator $R : \mathcal{H}(\Lambda^1) \to \mathcal{H}(\Lambda)$ as

$$(R\Phi)(z) := s^{-D_\phi}\, \Phi(sz) \tag{152}$$

we see that the rescaled effective action can be defined as

$$S_R^1[\phi] = S^1[R^{-1}\phi] \tag{153}$$

and for the derivatives one can write

$$S_R^{1\,(n)}[\phi] = s^{n(D+2D_\phi)} \underbrace{(R \otimes \cdots \otimes R)}_{n-\text{times}} \circ S^{1\,(n)}[R^{-1}\phi] \tag{154}$$

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