Construction of Kink Sectors for Two-Dimensional Quantum Field Theory Models (An Algebraic Approach)

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Abstract

Several two-dimensional quantum field theory models have more than one vacuum state. Familiar examples are the Sine-Gordon and the ϕ_2^4-model. It is known that in these models there are also states, called kink states, which interpolate different vacua. A general construction scheme for kink states in the framework of algebraic quantum field theory is developed in a previous paper. However, for the application of this method, the crucial condition is the split property for wedge algebras in the vacuum representations of the considered models. It is believed that the vacuum representations of $P(\phi)_2$-models fulfill this condition, but a rigorous proof is only known for the massive free scalar field. Therefore, we investigate in a construction of kink states which can directly be applied to a large class of quantum field theory models, by making use of the properties of the dynamics of a $P(\phi)_2$ and Yukawa$_2$ models.

1 Introduction

Studying $1+1$-dimensional quantum field theories from an axiomatic point of view shows that kink sectors naturally appear in the theory of superselection sectors [20, 21, 51]. This paper is concerned with the construction of kink sectors for concrete quantum field theory models, like $P(\phi)_2$ and Yukawa$_2$ models.[1]

Our subsequent analysis is placed into the framework of algebraic quantum field theory which has been turned out to be a successful formalism to describe physical concepts like observables, states , superselection sectors (charges) and statistics. These notions can appropriately be described by mathematical concepts like C*-algebras, positive linear functionals and equivalence classes of representations. For the convenience of the reader, we shall state the relevant definitions and assumptions here.

Let $\mathcal{O} \subset \mathbb{R}^{1,s}$ be a region in space-time. We denote by $\mathfrak{A}(\mathcal{O})$ the algebra generated by all observables which can be measured within $\mathcal{O}$. For technical reasons we always suppose that $\mathfrak{A}(\mathcal{O})$ is a C*-algebra and $\mathcal{O}$ is a double cone, i.e. a bounded and causally complete region. Motivated by physical principles, we make the following assumptions:

(1) The assignment

$$\mathfrak{A} : \mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$$

is an isotonous net of C*-algebras, i.e. if $\mathcal{O}_1$ is contained in $\mathcal{O}_2$, then $\mathfrak{A}(\mathcal{O}_1)$ is a C*-sub-algebra of $\mathfrak{A}(\mathcal{O}_2)$. The isotony encodes the fact that each observable which can be measured within $\mathcal{O}$ can also be measured in every larger region. Furthermore, the C*-inductive limit

$$C^*(\mathfrak{A})$$

of the net $\mathfrak{A}$ can be constructed since the set of double cones is directed. We refer to [50] for this notion.

(2) Two local operations which take place in space-like separated regions should not influence each other. The *principle of locality* is formulated as follows: If the regions $\mathcal{O}_1$ and $\mathcal{O}_2$ are space-like separated, then the elements of $\mathfrak{A}(\mathcal{O}_1)$ commute with those of $\mathfrak{A}(\mathcal{O}_2)$.

(3) Each operator a which is localized in a region $\mathcal{O}$ should have an equivalent counterpart which is localized in the translated region $\mathcal{O} + x$.

[1] Parts are extracted from the PhD thesis [54].

The *principle of translation symmetry* is encoded by the existence of an automorphism group $\{\alpha_x; x \in \mathbb{R}^{1,s}\}$ which acts on the C*-algebra $C^*(\mathfrak{A})$ such that α_x maps $\mathfrak{A}(\mathcal{O})$ onto $\mathfrak{A}(\mathcal{O}+x)$.

A net of C*-algebras which fulfills the conditions *(1)* to *(3)* is called a *translationally covariant Haag-Kastler net*.

In order to discuss particle-like concepts, we select an appropriate class $\mathfrak{S}$ of normalized positive linear functionals, called states, of $C^*(\mathfrak{A})$. We require that the states $\omega \in \mathfrak{S}$ fulfill the conditions:

(1) There exists a strongly continuous unitary representation of the translation group $U : x \mapsto U(x)$ on the GNS[2]-Hilbert space $\mathcal{H}$ which implements the translations in the GNS-representation π, i.e.

$$\pi(\alpha_x a) = U(x)\pi(a)U(-x)$$

for each $a \in C^*(\mathfrak{A})$.

(2) The stability of a physical system is encoded in the spectrum condition (positivity of the energy), i.e. the spectrum (of the generator) of $U(x)$ is contained in the closed forward light cone.

These conditions are also known as the *Borchers criterion*. States which satisfy the Borchers criterion and which are, in addition, translationally invariant are called *vacuum states*.

Kinks already appear in classical field theories and the typical systems in which they occur are 1+1-dimensional. Familiar examples are the Sine-Gordon and the ϕ^4_2-model. We briefly describe the latter:

The Lagrangian density of the model is given by

$$\mathfrak{L}(\phi, x) = \frac{1}{2}\,\partial_\mu\phi(x)\partial^\mu\phi(x) - U(\phi(x))$$

where the potential U is given by

$$U(z) := \lambda/2\,(z^2 - a)^2 \quad .$$

The energy of a classical field configuration ϕ is

$$E(\phi) = \int \mathrm{d}\mathbf{x}\left(\frac{1}{2}(\partial_0\phi(0,\mathbf{x}))^2 + \frac{1}{2}(\partial_1\phi(0,\mathbf{x}))^2 + U(\phi(0,\mathbf{x}))\right) \quad .$$

[2] Given a state $\omega \in \mathfrak{S}$, we obtain via GNS-construction a Hilbert space $\mathcal{H}$, a *-representation π of $C^*(\mathfrak{A})$ on $\mathcal{H}$ and a vector $\Omega \in \mathcal{H}$ such that $\langle\Omega, \pi(a)\Omega\rangle = \omega(a)$ for each $a \in C^*(\mathfrak{A})$. The triple $(\mathcal{H}, \pi, \Omega)$ is called the GNS-triple of ω.

With the choice of U, given above, the absolute minimum value of U is zero and thus the energy functional $E : \phi \mapsto E(\phi)$ is positive.

There are two configurations $\phi_\pm$ with zero energy $E(\phi_\pm) = 0$:

$$\phi_\pm : (t, \mathbf{x}) \mapsto \pm\, a \quad .$$

These configurations are invariant under space-time translations and represent the vacua of the classical system.

There are two further configurations $\phi_s, \phi_{\bar{s}}$ which are stationary points of the energy functional E. They are given by

$$\phi_s : (t, \mathbf{x}) \mapsto a\ \tanh(\ \sqrt{\lambda} a \mathbf{x}\ ) \quad \text{and} \quad \phi_{\bar{s}} : (t, \mathbf{x}) \mapsto -a\ \tanh(\ \sqrt{\lambda} a \mathbf{x}\ ) \ .$$

These configurations represent the kinks of the classical system which *interpolate* the vacua $\phi_\pm$. Indeed, we have for the kink ϕ_s

$$\lim_{\mathbf{x} \to \pm\infty} \phi_s(t, \mathbf{x}) = \phi_\pm(t, \mathbf{x}) = \pm a \quad . \tag{1}$$

The configuration $\phi_{\bar{s}}$, which interpolates the vacua $\phi_\pm$ in the opposite direction, represents the anti-kink of ϕ_s. Both of them have the same energy, namely

$$E(\phi_s) = E(\phi_{\bar{s}}) = \frac{4}{3}\sqrt{\lambda} a^3 \quad .$$

From the classical example above, we see that the crucial properties of a kink are to interpolate vacuum configurations as well as to be a configuration of finite energy. Motivated by these properties, in quantum field theory a *kink state* ω is defined as follows:

The interpolation property: For each observable a, the limits

$$\lim_{\mathbf{x} \to \pm\infty} \omega(\alpha_{(t,\mathbf{x})}(a)) = \omega_\pm(a) \tag{2}$$

exist and $\omega_\pm$ are vacuum states. Note that equation (2) is the quantum version of the interpolation property (1).

Positivity of the energy: ω fulfills the Borchers criterion.

In the literature the concept of *kink* as described above is often called *soliton* (see [26, 27]) or more seldom *lump* (see [11]). In the subsequent, we shall use the word kink.

In [52], a construction scheme for kink states has been developed which is based on general principles. In order to make the comprehension of the subsequent sections easier we shall state the main ideas here. The construction of an interpolating kink state is based on a simple physical idea: Let $\mathfrak{A}$ be a Haag-Kastler net of W*-algebras in $1+1$-dimensions. Each double cone $\mathcal{O}$ splits our system into two infinitely extended laboratories, namely the laboratory which belongs to the left space-like complement $\mathcal{O}_{LL}$, and the laboratory $\mathcal{O}_{RR}$ which belongs the right space-like complement $\mathcal{O}_{RR}$. In order to prepare an interpolating kink state, we wish to prepare one vacuum state ω_1 in the left laboratory $\mathcal{O}_{LL}$, and another vacuum state ω_2 in the right laboratory $\mathcal{O}_{RR}$. This can only be done if the preparation of ω_1 does not disturb the preparation procedure of ω_2. In other words, the physical operations which take place in the laboratory on the left side $\mathcal{O}_{LL}$ should be statistically independent of those which take place in $\mathcal{O}_{RR}$.

Therefore, we require that there exists a vacuum representation π_0 such that the W*-tensor product

$$\mathfrak{A}_{\pi_0}(\mathcal{O}_{LL}) \,\overline{\otimes}\, \mathfrak{A}_{\pi_0}(\mathcal{O}_{RR})$$

is unitarily isomorphic to the von Neumann algebra

$$\mathfrak{A}_{\pi_0}(\mathcal{O}_{LL}) \vee \mathfrak{A}_{\pi_0}(\mathcal{O}_{RR})$$

where $\mathfrak{A}_{\pi_0}$ is the net in the vacuum representation π_0. [3] This condition is equivalent to the existence of a type I factor $\mathcal{N}$ which sits between $\mathfrak{A}_{\pi_0}(\mathcal{O}_{RR})$ and $\mathfrak{A}_{\pi_0}(\mathcal{O}_R)$:

$$\mathfrak{A}_{\pi_0}(\mathcal{O}_{RR}) \subset \mathcal{N} \subset \mathfrak{A}_{\pi_0}(\mathcal{O}_R) \ .$$

Here $\mathcal{O}_R$ is the space-like complement of $\mathcal{O}_{LL}$. In other words, the inclusion

$$\mathfrak{A}_{\pi_0}(\mathcal{O}_{RR}) \subset \mathfrak{A}_{\pi_0}(\mathcal{O}_R) \tag{3}$$

is *split*.

A detailed investigation of standard split inclusions of W*-algebras has been carried out by S. Doplicher and R. Longo [18]. We also refer to the results of D. Buchholz [8], C. D'Antoni and R. Longo [13] and C. D'Antoni and K. Fredenhagen [12].

Let ω_1 and ω_2 be two inequivalent vacuum states whose restrictions to each local algebra $\mathfrak{A}(\mathcal{O})$ are normal. Using the isomorphy

$$\mathfrak{A}_{\pi_0}(\mathcal{O}_{LL}) \,\overline{\otimes}\, \mathfrak{A}_{\pi_0}(\mathcal{O}_{RR}) \cong \mathfrak{A}_{\pi_0}(\mathcal{O}_{LL}) \vee \mathfrak{A}_{\pi_0}(\mathcal{O}_{RR})$$

[3] for an unbounded region $\mathcal{U}$, $\mathfrak{A}_{\pi_0}(\mathcal{U})$ denotes the von Neumann algebra which is generated by all local algebras $\mathfrak{A}_{\pi_0}(\mathcal{O})$ with $\mathcal{O} \subset \mathcal{U}$.

we conclude that the map

$$ab \mapsto \omega_1(a)\omega_2(b) \ \text{, } a \text{ is localized in } \mathcal{O}_{LL} \text{ and } b \text{ is localized in } \mathcal{O}_{RR},$$

defines a state of the algebra $C^*(\mathfrak{A}, \mathcal{O}_{LL} \cup \mathcal{O}_{RR})$ which, by the Hahn-Banach theorem, can be extended to a state ω of the C*-algebra of all observables. The state ω interpolates the vacua ω_1 and ω_2 correctly, but for an explicit construction of an interpolating state which satisfies the *Borchers criterion*, some technical difficulties have to be overcome.

The condition that the inclusion (3) is split is sufficient to develop a general construction scheme for interpolating kink states. We shall give a brief description of it here.

Step 1: We consider the W*-tensor product of the net $\mathfrak{A}$ with itself:

$$\mathfrak{A} \,\overline{\otimes}\, \mathfrak{A} : \mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}) \,\overline{\otimes}\, \mathfrak{A}(\mathcal{O})$$

The map α_F which is given by interchanging the tensor factors,

$$\alpha_F : a_1 \otimes a_2 \mapsto a_2 \otimes a_1$$

is called the *flip automorphism*. Since the inclusion (3) is split, the flip automorphism is unitarily implemented on $\mathfrak{A}_{\pi_0} \,\overline{\otimes}\, \mathfrak{A}_{\pi_0}(\mathcal{O}_{RR})$ by a unitary operator θ which is contained in $\mathfrak{A}_{\pi_0} \,\overline{\otimes}\, \mathfrak{A}_{\pi_0}(\mathcal{O}_R)$ [13]. The adjoint action of θ induces an automorphism

$$\beta := (\pi_0 \otimes \pi_0)^{-1} \circ \mathrm{Ad}(\theta) \circ \pi_0 \otimes \pi_0$$

which maps local algebras into local algebras. For each observable a which is localized in the left space-like complement of $\mathcal{O}$ we have $\beta(a) = a$, and for each observable b which is localized in the right space-like complement of $\mathcal{O}$ we have $\beta(b) = \alpha_F(b)$. Note that β may depend on the choice of the vacuum representation π_0.

Step 2: It is obvious that the state

$$\omega := \omega_1 \otimes \omega_2 \circ \beta|_{C^*(\mathfrak{A}) \otimes \mathbf{1}}$$

interpolates ω_1 and ω_2. Let π_1 and π_2 be the GNS-representations of ω_1 and ω_2 respectively. Then the GNS-representation

$$\pi = \pi_1 \otimes \pi_2 \circ \beta|_{C^*(\mathfrak{A} \otimes \mathbf{1})}$$

of ω is translationally covariant because the automorphism

$$\alpha_x \circ \beta \circ \alpha_{-x} \circ \beta$$

is implemented by a cocycle $\gamma(x)$ of *local* operators in $C^*(\mathfrak{A})$. The positivity of the energy can be proven by showing the additivity of the energy-momentum spectrum for automorphisms like β. This together implies that ω is an interpolating kink state.

In comparison to [26, 27], our construction scheme has the following advantages:

- $\oplus$ It is independent of specific details of the considered model because the split property (3), which is the crucial condition for applying the construction scheme, can be motivated by general principles.
- $\oplus$ It can be applied to pairs of vacuum sectors which are not related by a symmetry transformation.

Unfortunately, there is one disadvantage which is the price we have to pay for using a model independent analysis.

- $\ominus$ The split property for wedge algebras (3) has to be proven for the vacuum states of the model under consideration if we want to apply our construction scheme to it. It is believed that the vacuum states of the $P(\phi)_2$- and Yukawa$_2$ models fulfill this condition, but a rigorous proof is only known for the massive free Bose and Fermi field [12, 8, 57].

In the present paper, we investigate an alternative construction of kink states which can directly be applied to models. It is convenient to formulate our setup in the time slice formulation of a quantum field theory. The time slice-formulation has two main aspects. First, the *Cauchy data* with respect to a given space-like plane Σ which describes the boundary conditions at time $t = 0$. Second, the *dynamic* which describes the time evolution of the quantum fields.

The *Cauchy data* of a quantum field theory are given by a net of v.Neumann-algebras

$$\mathfrak{M} := \{\mathfrak{M}(\mathcal{I}) \subset \mathfrak{B}(\mathcal{H}_0);\ \mathcal{I} \text{ is open and bounded interval in } \Sigma\}$$

represented on a Hilbert-space $\mathcal{H}_0$. This net has to satisfy the following conditions:

(1) The net is isotonous, i.e. if $\mathcal{I}_1 \subset \mathcal{I}_2$, then $\mathfrak{M}(\mathcal{I}_1) \subset \mathfrak{M}(\mathcal{I}_2)$.

(2) The net is local, i.e. if $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$, then $\mathfrak{M}(\mathcal{I}_1) \subset \mathfrak{M}(\mathcal{I}_2)'$.

(3) There exists a unitary and strongly continuous representation

$$U : \mathbf{x} \in \mathbb{R} \mapsto U(\mathbf{x}) \in \mathcal{U}(\mathcal{H}_0)$$

of the spatial translations in $\Sigma \cong \mathbb{R}$, such that $\alpha_{\mathbf{x}} := \mathrm{Ad}(U(\mathbf{x}))$ maps $\mathfrak{M}(\mathcal{I})$ onto $\mathfrak{M}(\mathcal{I} + \mathbf{x})$.

A one-parameter group of automorphisms $\alpha = \{\alpha_t \in \mathrm{Aut}(\mathfrak{M}); t \in \mathbb{R}\}$ ($\mathrm{Aut}(\mathfrak{M})$ denotes the automorphisms of $C^*(\mathfrak{M})$) is called a *dynamics* of the net $\mathfrak{M}$ if the following conditions are fulfilled:

(a) The automorphism group α has propagation speed $\mathrm{ps}(\alpha) \leq 1$, where $\mathrm{ps}(\alpha)$ is defined by:

$$\mathrm{ps}(\alpha) := \inf\{\beta' | \alpha_t \mathfrak{M}(\mathcal{I}) \subset \mathfrak{M}(\mathcal{I}_{\beta'|t|}); \forall t, \mathcal{I}\} \ .$$

Here $\mathcal{I}_s := \mathcal{I} + (-s, s)$ denotes the interval, enlarged by $s > 0$.

(b) The automorphisms $\{\alpha_t \in \mathrm{Aut}(\mathfrak{M}); t \in \mathbb{R}\}$ commute with the automorphism group of spatial translations $\{\alpha_{\mathbf{x}} \in \mathrm{Aut}(\mathfrak{M}); \mathbf{x} \in \mathbb{R}\}$, i.e.:

$$\alpha_t \circ \alpha_{\mathbf{x}} = \alpha_{\mathbf{x}} \circ \alpha_t \quad ; \quad \forall \mathbf{x}, t \ .$$

The set of all dynamics of $\mathfrak{M}$ is denoted by $\mathrm{dyn}(\mathfrak{M})$.

For our purposes it is crucial to distinguish carefully the C*-inductive limit $C^*(\mathfrak{M})$ of the net $\mathfrak{M}$ and the corresponding C*- and W*-algebras, which belong to an unbounded region $\mathcal{J} \subset \Sigma$. They are denoted by

$$C^*(\mathfrak{M}, \mathcal{J}) := \overline{\bigcup_{\mathcal{I} \subset \mathcal{J}} \mathfrak{M}(\mathcal{I})}^{||\cdot||} \quad \text{and} \quad \mathfrak{M}(\mathcal{J}) := \bigvee_{\mathcal{I} \subset \mathcal{J}} \mathfrak{M}(\mathcal{I}) \quad \text{respectively.}$$

The Cauchy data of the $P(\phi)_2$- and the Yukawa$_2$ model are given by the nets of the corresponding free fields at time $t = 0$. For these Cauchy data it can be proven that the inclusion

$$\mathfrak{M}(\mathcal{I}_{RR}) \subset \mathfrak{M}(\mathcal{I}_R)$$

is split [8, 57, 52].

Let us briefly explain how kink states can be constructed if the following conditions are assumed:

(i) The dynamics of the model satisfies an appropriate *extendibility condition* which we shall explain later.

(ii) The vacuum states are local Fock states which is automatically satisfied for $P(\phi)_2$ and Yukawa$_2$ models [32, 56].

Step 1': We consider the twofold net

$$\mathfrak{M}\,\overline{\otimes}\,\mathfrak{M} : \mathcal{I} \mapsto \mathfrak{M}(\mathcal{I})\,\overline{\otimes}\,\mathfrak{M}(\mathcal{I}) \ .$$

Like in *Step 1* of our previous construction scheme, the split property implies that on $\mathfrak{M}(\mathcal{I}_{RR})\,\overline{\otimes}\,\mathfrak{M}(\mathcal{I}_{RR})$, the flip automorphism is implemented by a unitary operator $\theta_{\mathcal{I}}$ [13]. The adjoint action of $\theta_{\mathcal{I}}$ is an automorphism $\beta^{\mathcal{I}}$ which has the following properties:

(i) The automorphism $\beta^{\mathcal{I}}$ acts trivially on observables which are localized in the left complement of $\mathcal{I}$ and it acts like the flip on observables which are localized in the right complement of $\mathcal{I}$.

(ii) The automorphism $\beta^{\mathcal{I}}$ maps local algebras into local algebras.

Note that the automorphism $\beta^{\mathcal{I}}$ does not depend on the dynamics α.

Step 2': Let ω_1, ω_2 be two vacuum states with respect to a given dynamics α. The state

$$\omega := \omega_1 \otimes \omega_2 \circ \beta^{\mathcal{I}}|_{C^*(\mathfrak{M})\otimes\mathbb{C}\mathbf{1}}$$

interpolates the vacua ω_1 and ω_2. Moreover, it is covariant under spatial translations since for each $\mathbf{x}$ the operator

$$\gamma(0,\mathbf{x}) = (\alpha_{\mathbf{x}} \otimes \alpha_{\mathbf{x}})(\theta_{\mathcal{I}})\theta_{\mathcal{I}}$$

is localized in a sufficiently large bounded interval. Indeed, the unitary operators

$$U(0,\mathbf{x}) := (U_1(0,\mathbf{x}) \otimes U_2(0,\mathbf{x}))\,(\pi_1 \otimes \pi_2)(\gamma(0,-\mathbf{x}))$$

implement the spatial translations in the GNS-representation of ω where U_1 and U_2 implement the translations in the GNS-representations π_1, π_2 of ω_1 and ω_2 respectively.

Step 3': It remains to be proven that ω is translationally covariant with respect to the dynamics α. For this purpose, we wish to construct a cocycle $\gamma(0,t)$ such that the operators

$$U(t,0) := (U_1(t,0) \otimes U_2(t,0))\,(\pi_1 \otimes \pi_2)(\gamma(-t,0))$$

implement the dynamics α in the GNS-representation of ω. The operator

$$\gamma(t,0) := (\alpha_t \otimes \alpha_t)(\theta_{\mathcal{I}})\theta_{\mathcal{I}}$$

is a formal solution. Unfortunately, the flip implementer $\theta_{\mathcal{I}}$ is *not* contained in any local algebra and the term $(\alpha_t \otimes \alpha_t)(\theta_{\mathcal{I}})$ has no mathematical meaning unless α is the free dynamics. However, it can be given a meaning in some cases. We shall see that for an interacting dynamics there exists a suitable cocycle of the operators $\gamma(t,0)$ such that $\gamma(t,0)$ is localized in a bounded interval whose size depends linearly on $|t|$.

In order to formulate a sufficient condition for the existence of $\gamma(t,0)$, we construct an extension of the net $\mathfrak{M} \,\overline{\otimes}\, \mathfrak{M}$. We define $\hat{\mathfrak{M}}(\mathcal{I})$ to be the von Neumann algebra which is generated by $\mathfrak{M}(\mathcal{I}) \,\overline{\otimes}\, \mathfrak{M}(\mathcal{I})$ and the operator $\theta_{\mathcal{I}}$. The net

$$\hat{\mathfrak{M}} : \mathcal{I} \mapsto \hat{\mathfrak{M}}(\mathcal{I})$$

is an extension of $\mathfrak{M} \,\overline{\otimes}\, \mathfrak{M}$ which does not fulfill locality. This is due to the non-trivial implementation properties of $\theta_{\mathcal{I}}$. We shall call a dynamics α *extendible* if there exists a dynamics $\hat{\alpha}$ of $\hat{\mathfrak{M}}$ which is an extension of $\alpha \otimes \alpha$. Indeed,

$$t \mapsto \gamma(t,0) := \hat{\alpha}_t(\theta_{\mathcal{I}})\theta_{\mathcal{I}}$$

is a cocycle which has the desired properties. Finally, we conclude like in Step 3 of our previous construction scheme that the state

$$\omega := \omega_1 \otimes \omega_2 \circ \beta^{\mathcal{I}}|_{C^*(\mathfrak{M})\otimes\mathbb{C}\mathbf{1}}$$

is a kink state where ω_1, ω_2 are vacuum states with respect to the dynamics α.

Since the extendibility condition is rather technical one might bother that it is only fulfilled for few exceptional cases. Fortunately, this is not true. There is a large class of quantum field theory models whose dynamics are extendible. We shall prove that the extendibility holds for the following models:

(i) $P(\phi)_2$-models.

(ii) Yukawa$_2$ models.

(iii) Special types of Wess-Zumino models.

Note that a Dirac spinor field contributes to the field content of the Yukawa$_2$ and Wess-Zumino models, and the nets of Cauchy data fulfill *twisted duality* instead of Haag duality [57]. According to recent results which have been established by M. Müger [47], our results remain true for these cases also.

Wess-Zumino models have been studied in several papers. We refer to the work of A. Jaffe, A. Lesniewski, J. Weitsman and S. Janowsky [40, 43,

44, 41, 42]. It has been proven in [41] that some Wess-Zumino models possess more than one vacuum sector. An application of our construction scheme proves the existence of kink states for these models.

2 Preliminaries

In the first part (Section 2.1) of this preliminary section, we briefly describe how to construct a Haag-Kastler net from a given net of Cauchy data and a given dynamics. Examples for physical states with respect to an interacting dynamics are given in the second part (Section 2.2).

2.1 From Cauchy Data to Haag-Kastler Nets

We denote by $U(\mathfrak{M})$ the group of unitary operators in $C^*(\mathfrak{M})$. Let $\mathfrak{G}(\mathbb{R},\mathfrak{M})$ be the group which is generated by the set

$$\{(t,u)|\ t\in\mathbb{R} \text{ and } u\in U(\mathfrak{M})\ \}$$

modulo the following relations:

(1) For each $u_1,u_2\in U(\mathfrak{M})$ and for each $t_1,t_2,t\in\mathbb{R}$, we require:

$$(t,u_1)(t,u_2)=(t,u_1u_2) \text{ and } (t,1)=1$$

(2) For $u_1\in\mathfrak{M}(\mathcal{I}_1)$ and $u_2\in\mathfrak{M}(\mathcal{I}_2)$ with $\mathcal{I}_1\subset(\mathcal{I}_2+[-|t|,|t|])^c$ we require for each $t_1\in\mathbb{R}$:

$$(t_1+t,u_1)(t_1,u_2)=(t_1,u_2)(t_1+t,u_1)$$

We conclude from relation *(1)* that (t,u) is the inverse of (t,u^*). Furthermore, a localization region in $\mathbb{R}\times\Sigma$ can be assigned to each element in $\mathfrak{G}(\mathbb{R},\mathfrak{M})$. A generator (t,u), $u\in\mathfrak{M}(\mathcal{I})$ is localized in $\mathcal{O}\subset\mathbb{R}\times\Sigma$ if $\{t\}\times\mathcal{I}\subset\mathcal{O}$. The subgroup of $\mathfrak{G}(\mathbb{R},\mathfrak{M})$ which is generated by elements which are localized in the double cone $\mathcal{O}$, is denoted by $\mathfrak{G}(\mathcal{O})$.

We easily observe that relation *(2)* implies that group elements commute if they are localized in space-like separated regions.

The translation group in $\mathbb{R}^2$ is naturally represented by group-automorphisms of $\mathfrak{G}(\mathbb{R},\mathfrak{M})$. They are defined by the prescription

$$\beta_{(t,\mathrm{x})}(t_1,u):=(t+t_1,\alpha_{\mathrm{x}}u)\quad .$$

Thus the subgroup $\mathfrak{G}(\mathcal{O})$ is mapped onto $\mathfrak{G}(\mathcal{O}+(t,\mathrm{x}))$ by $\beta_{(t,\mathrm{x})}$.

To construct the universal Haag-Kastler net, we build the group C*-algebra $\mathfrak{B}(\mathcal{O})$ with respect to $\mathfrak{G}(\mathcal{O})$. For convenience, we shall describe the construction of $\mathfrak{B}(\mathcal{O})$ briefly.

In the first step we build the *-algebra $\mathfrak{B}_0(\mathcal{O})$ which is generated by all complex valued functions a on $\mathfrak{G}(\mathcal{O})$, such that

$$a(u) = 0 \quad \text{for almost each } u \in \mathfrak{G}(\mathcal{O})\,.$$

We write such a function symbolically as a formal sum, i.e.

$$a = \sum_u a(u)\; u$$

The product and the *-relation is given as follows:

$$ab = \sum_u a(u)\; u \cdot \sum_{u'} b(u')\; u' = \sum_{u'} \Big(\sum_u a(u) b(u^{-1}u') \Big) u'$$

$$a^* = \sum_u \bar{a}(u^{-1})\; u$$

It is well known, that the algebra $\mathfrak{B}_0(\mathcal{O})$ has a C*-norm which is given by

$$||a|| := \sup_\pi ||\pi(a)||_\pi$$

where the supremum is taken over each Hilbert space representation π of $\mathfrak{B}_0(\mathcal{O})$. Finally, we define $\mathfrak{B}(\mathcal{O})$ as the closure of $\mathfrak{B}_0(\mathcal{O})$ with respect to the norm above.

The C*-algebra which is generated by all local algebras $\mathfrak{B}(\mathcal{O})$ is denoted by $C^*(\mathfrak{B})$. By construction, the group isomorphisms $\beta_{(t,\mathrm{x})}$ induce a representation of the translation group by automorphisms of $C^*(\mathfrak{B})$.

Observation: The net of C*-algebras

$$\mathfrak{B} := \{\mathfrak{B}(\mathcal{O}) | \mathcal{O} \text{ is a bounded double cone in } \mathbb{R}^2 \}$$

is a translationally covariant Haag-Kastler net.

We have to mention that the universal net $\mathfrak{B}$ is not Lorentz covariant. The universal properties of the net $\mathfrak{B}$ are stated in the following Proposition:

Proposition 2.1 : *Each dynamics* $\alpha \in \mathrm{dyn}(\mathfrak{M})$ *induces a C*-homomorphism*

$$\iota_\alpha : C^*(\mathfrak{B}) \to C^*(\mathfrak{M})$$

such that

$$\iota_\alpha \circ \beta_{(t,\mathrm{x})} = \alpha_{(t,\mathrm{x})} \circ \iota_\alpha \quad ,$$

for each $(t, \mathrm{x}) \in \mathbb{R}^2$. *In particular,*

$$\mathfrak{A}_\alpha : \mathcal{O} \mapsto \mathfrak{A}_\alpha(\mathcal{O}) := \iota_\alpha(\mathfrak{B}(\mathcal{O}))''$$

is a translationally covariant Haag-Kastler net.

Proof. Given a dynamics α of $\mathfrak{M}$. We conclude from $\mathrm{ps}(\alpha) \leq 1$ that the prescription

$$(t, u) \mapsto \alpha_t u$$

defines a C*-homomorphism

$$\iota_\alpha : C^*(\mathfrak{B}) \to C^*(\mathfrak{M}) \ .$$

In particular, ι_α is a representation of $C^*(\mathfrak{B})$ on the Hilbert space $\mathcal{H}_0$. This statement can be obtained by using the relations, listed below.

(a)

$$\iota_\alpha((t, u_1)(t, u_2)) = \alpha_t u_1 \alpha_t u_2 = \alpha_t(u_1 u_2) = \iota_\alpha(t, u_1 u_2)$$

(b) If (t_1, u_1) and $(t_1 + t, u_2)$ are localized in space-like separated regions, then we obtain from $\mathrm{ps}(\alpha) \leq 1$:

$$[\iota_\alpha(t_1, u_1), \iota_\alpha(t_1 + t, u_2)] = \alpha_{t_1}[u_1, \alpha_t u_2] = 0$$

(c)

$$\iota_\alpha(\beta_{(t,\mathrm{x})}(t_1, u)) = \iota_\alpha(t + t_1, \alpha_{\mathrm{x}} u) = \alpha_{(t,\mathrm{x})} \alpha_{t_1} u$$

□

In general we expect that for a given dynamics α the representation ι_α is not faithful. Hence each dynamics defines a two-sided ideal

$$J(\alpha) := \iota_\alpha^{-1}(0) \in C^*(\mathfrak{B})$$

in $C^*(\mathfrak{B})$ which we call the *dynamical ideal* with respect to α and the quotient C*-algebras

$$\mathfrak{B}(\mathcal{O})/J(\alpha) \cong \mathfrak{A}_\alpha(\mathcal{O})$$

may depend on the dynamics α. Indeed, if $\mathcal{O}$ is a double cone whose base is *not contained* in Σ, then for different dynamics α_1, α_2 the algebras $\mathfrak{A}_{\alpha_1}(\mathcal{O})$ and $\mathfrak{A}_{\alpha_2}(\mathcal{O})$ are different. On the other hand, if the base of $\mathcal{O}$ is contained in Σ, then we conclude from the fact that the dynamics α has finite propagation speed and from Proposition 2.1:

Corollary 2.2 : *If $\mathcal{I} \subset \Sigma$ is the base of the double cone $\mathcal{O}$, then the algebra $\mathfrak{A}_\alpha(\mathcal{O})$ is independent of α. In particular, the C*-algebra*

$$C^*(\mathfrak{M}) = \overline{\bigcup_{\mathcal{I}} \mathfrak{M}(\mathcal{I})}^{||\cdot||} = \overline{\bigcup_{\mathcal{O}} \mathfrak{A}_\alpha(\mathcal{O})}^{||\cdot||}$$

is the C-inductive limit of the net $\mathfrak{A}_\alpha$.*

From the discussion above, we see that two dynamics with the same dynamical ideal induces the same quantum field theory.

2.2 Examples for Physical States

Let us consider the set $\mathfrak{S}$ of all *locally normal states* on $C^*(\mathfrak{M})$, i.e. for each state $\omega \in \mathfrak{S}$ and for each bounded interval $\mathcal{I}$, the restriction

$$\omega|_{\mathfrak{M}(\mathcal{I})}$$

is a normal state on $\mathfrak{M}(\mathcal{I})$.

As mentioned in the introduction, we are interested in states with vacuum and particle-like properties, i.e. states which satisfies the *Borchers criterion* (See the Introduction for this notion).

Notation: Given a dynamics $\alpha \in \mathrm{dyn}(\mathfrak{M})$. We denote the corresponding set of all locally normal states which satisfies the Borchers criterion by $\mathfrak{S}(\alpha)$ and analogously the set of all vacuum states by $\mathfrak{S}_0(\alpha)$. Moreover, we write for the set of vacuum sectors

$$\mathrm{sec}_0(\alpha) := \{[\omega] | \omega \in \mathfrak{S}_0(\alpha)\} \tag{4}$$

where $[\omega]$ denotes the unitary equivalence class of the the GNS-representation of ω.

Examples: Examples for vacuum states are the vacua of the $P(\phi)_2$-models [32, 33]. The interacting part of the cutoff Hamiltonian is given by a Wick polynomial of the time zero field ϕ_0, i.e.

$$H_1(\mathcal{I}) = H_1(\chi_{\mathcal{I}}) =: P(\phi_0) : (\chi_{\mathcal{I}})$$

where $\chi_{\mathcal{I}}$ is a test function with $\chi_{\mathcal{I}}(\mathbf{x}) = 1$ for $\mathbf{x} \in \mathcal{I}$ and $\chi_{\mathcal{I}}(\mathbf{y}) = 0$ on the complement of a slightly lager region $\hat{\mathcal{I}} \supset \mathcal{I}$. It is well known that $H_1(\mathcal{I})$ is a self-adjoint operator, which has a joint core with the free Hamiltonian h_0, and is affiliated with $\mathfrak{M}(\hat{\mathcal{I}})$. The operator $h_1(\mathcal{I})$ induces a automorphism group $\alpha_{\mathcal{I}}$ which is given by

$$\alpha_{\mathcal{I},t}(a) := e^{iH_1(\mathcal{I})t} a e^{-iH_1(\mathcal{I})t} \quad .$$

Consider the inclusion of intervals $\mathcal{I}_0 \subset \mathcal{I}_1 \subset \mathcal{I}_2$. Then we have for each $a \in \mathfrak{M}(\mathcal{I}_0)$:

$$\alpha_{\mathcal{I}_1,t}(a) = \alpha_{\mathcal{I}_2,t}(a)$$

Hence, there exists a one-parameter automorphism group $\{\alpha_{1,t} \in \mathrm{Aut}(\mathfrak{M}); t \in \mathbb{R}\}$ such that $\alpha_{1,t}$ acts on $a \in \mathfrak{M}(\mathcal{I})$ as follows:

$$\alpha_{1,t}(a) = \alpha_{\mathcal{I},t}(a) \quad ; \quad \forall t \in \mathbb{R}$$

The automorphism group $\{\alpha_{1,t} \in \mathrm{Aut}(\mathfrak{M}); t \in \mathbb{R}\}$ is a dynamics of $\mathfrak{M}$ with zero propagation speed, i.e. $\mathrm{ps}(\alpha_1) = 0$.

Since $H_1(\mathcal{I})$ has a joint core with the free Hamiltonian H_0, we are able to define the Trotter product of the automorphism groups α_0 and α_1 which is given for each local operator $a \in \mathfrak{M}(\mathcal{I})$ by

$$\alpha_t(a) := (\alpha_0 \times \alpha_1)_t(a) = s - \lim_{n \to \infty} (\alpha_{0,t/n} \circ \alpha_{1,t/n})^n(a) \quad .$$

The limit is taken in the strong operator topology. Furthermore,the propagation speed is sub-additive with respect to the Trotter product [32], i.e.

$$\mathrm{ps}(\alpha_0 \times \alpha_1) \leq \mathrm{ps}(\alpha_0) + \mathrm{ps}(\alpha_1)$$

and we conclude that $\alpha \in \mathrm{dyn}(\mathfrak{M})$ is a dynamics of $\mathfrak{M}$. We call the dynamics α *interacting*.

It is shown by Glimm and Jaffe [32] that there exist vacuum states ω with respect to the interacting dynamics α. We have to mention, that there is *no* vector ψ in Fock space $\mathcal{H}_0$, such that the state

$$a \mapsto \langle \psi, a\psi \rangle$$

is a vacuum state with respect to an interacting dynamics α, but there is a sequence of vectors (Ω_n) in $\mathcal{H}_0$ such that the weak* limit

$$\omega = w^* - \lim_n \langle \Omega_n, \cdot \Omega_n \rangle$$

is a vacuum state with respect to the dynamics α.

3 On the Existence of Kink States

The main theorem (Theorem 3.2) of this paper is formulated in the first part (Section 3.1) of the present section. In order to prepare the proof of Theorem 3.2, we need some technical preliminaries which are given in Section 3.2. In the last part (Section 3.3), we prove a criterion for the existence of kink states (the extendibility of the dynamics) which turns out to be satisfied by the $P(\phi)_2$ and Yukawa$_2$ models.

3.1 The Main Result

We now reformulate the definition (see Introduction) of a kink state within the time-slice formulation.

Definiton 3.1 : Let $\alpha \in \mathrm{dyn}(\mathfrak{M})$ be a dynamics of $\mathfrak{M}$. A state ω of $\mathfrak{M}$ is called a *kink state*, interpolating vacuum states $\omega_1, \omega_2 \in \mathfrak{S}_0(\alpha)$ if

(a) ω satisfies the Borchers criterion

(b) and there exists a bounded interval $\mathcal{I}$, such that ω fulfills the relations:

$$\pi|_{C^*(\mathfrak{M},\mathcal{I}_{LL})} \cong \pi_1|_{C^*(\mathfrak{M},\mathcal{I}_{LL})} \quad \text{and} \quad \pi|_{C^*(\mathfrak{M},\mathcal{I}_{RR})} \cong \pi_2|_{C^*(\mathfrak{M},\mathcal{I}_{RR})}$$

where the symbol $\cong$ means unitarily equivalent and $(\mathcal{H}, \pi, \Omega)$, $(\mathcal{H}_j, \pi_j, \Omega_j)$ are the GNS-triples of the states $\omega \in \mathfrak{S}(\alpha)$ and $\omega_j \in \mathfrak{S}_0(\alpha); j = 1, 2$ respectively.

The set of all kink states which interpolate ω_1 and ω_2 is denoted by $\mathfrak{S}(\alpha|\omega_1, \omega_2)$.

As already mentioned in the Introduction, a criterion for the existence of an interpolating kink state, can be obtained by looking at the construction method of [52]. In our context, we have to select a class of dynamics which are equipped with *good properties*. Such a selection criterion is developed in section 3. We shall show that each dynamics of a $P(\phi)_2$-model satisfies this criterion which leads to the following result:

Theorem 3.2 : *If $\alpha \in \mathrm{dyn}(\mathfrak{M})$ is a dynamics of a model with $P(\phi)_2$ plus Yukawa$_2$ interaction, then for each pair of vacuum states $\omega_1, \omega_2 \in \mathfrak{S}_0(\alpha)$ there exists an interpolating kink state $\omega \in \mathfrak{S}(\alpha|\omega_1, \omega_2)$.*

We shall prepare the proof of Theorem 3.2 during the subsequent sections.

3.2 Technical Preliminaries

Definiton 3.3 : Let $\mathfrak{M}$ be a net of Cauchy data. We denote by $G(\mathfrak{M})$ the group of unitary operators $u \in \mathfrak{B}(\mathcal{H}_0)$ whose adjoint actions $\chi_u := \mathrm{Ad}(u)$ commute with the spatial translations, i.e.:

$$\chi_u \circ \alpha_{\mathbf{x}} = \alpha_{\mathbf{x}} \circ \chi_u \quad .$$

Let $\alpha \in \mathrm{dyn}(\mathfrak{M})$ be a dynamics of the net $\mathfrak{M}$. Then we define the following sub-group of $G(\mathfrak{M})$:

$$G(\alpha, \mathfrak{M}) := \{u \in G(\mathfrak{M}) | \chi_u \circ \alpha_t = \alpha_t \circ \chi_u \text{ for each } t \in \mathbb{R}.\}$$

Remark: Each operator $u \in G(\alpha, \mathfrak{M})$ induces a symmetry of the Haag-Kastler net $\mathfrak{A}_\alpha$.

We make the following assumptions for the net of Cauchy data $\mathfrak{M}$:

Assumption:

(a) The net $\mathfrak{M}$ fulfills duality, i.e.

$$\mathfrak{M}(\mathcal{I})' = \mathfrak{M}(\mathcal{I}_{LL}) \vee \mathfrak{M}(\mathcal{I}_{RR}) \tag{5}$$

(b) There exists a dynamics α_0 and a normalized vector Ω_0 in $\mathcal{H}_0$, such that

$$\omega_0 = \langle \Omega_0, (\cdot)\Omega_0 \rangle$$

is a vacuum state with respect to the dynamics α_0.

(c) For each bounded interval $\mathcal{I}$, the inclusion

$$(\mathfrak{M}(\mathcal{I}_{RR}), \mathfrak{M}(\mathcal{I}_R))$$

is split.

According to our assumption, we conclude form the Theorem of Reeh and Schlieder that Ω_0 is a standard vector for the inclusion $(\mathfrak{M}(\mathcal{I}_{RR}), \mathfrak{M}(\mathcal{I}_R))$ which implies that

$$\Lambda(\mathcal{I}) := (\mathfrak{M}(\mathcal{I}_{RR}), \mathfrak{M}(\mathcal{I}_R), \Omega_0) \tag{6}$$

is a standard split inclusion for each interval $\mathcal{I}$. and hence (see [18]) there exists a unitary operator

$$w_{\mathcal{I}} : \mathcal{H}_0 \otimes \mathcal{H}_0 \to \mathcal{H}_0$$

such that for $a \in \mathfrak{M}(\mathcal{I}_{LL})$ and $b \in \mathfrak{M}(\mathcal{I}_{RR})$ we have:

$$w_{\mathcal{I}}(a \otimes b)w_{\mathcal{I}}^* = ab$$

Thus there is an interpolating type I factor $\mathcal{N} \cong \mathfrak{B}(\mathcal{H}_0)$, i.e.

$$\mathfrak{M}(\mathcal{I}_{RR}) \subset \mathcal{N} \subset \mathfrak{M}(\mathcal{I}_R)$$

which is given by

$$\mathcal{N} := w_{\mathcal{I}}(\mathbf{1} \otimes \mathfrak{B}(\mathcal{H}_0))w_{\mathcal{I}}^* \quad .$$

Hence we obtain an embedding of $\mathfrak{B}(\mathcal{H}_0)$ into the algebra $\mathfrak{M}(\mathcal{I}_R)$:

$$\Psi_{\mathcal{I}} : F \in \mathfrak{B}(\mathcal{H}_0) \mapsto w_{\mathcal{I}}(\mathbf{1} \otimes F)w_{\mathcal{I}}^* \in \mathfrak{M}(\mathbf{x}, \infty)$$

This embedding is called the *universal localizing map*.

Remark: We shall make a few remarks on the assumptions given above.

(i) The results, which we shall establish in the following, remain to be correct if the net of Cauchy data fulfills *twisted duality* instead of duality [47, 57].

(ii) For the application of our analysis to quantum field theory models, like $P(\phi)_2$- or Yukawa$_2$ models, we can choose as Cauchy data tensor products of the time-zero algebras of the massive free Bose or Fermi field. The time-zero algebras of the massive free Bose field fulfill the assumptions *(a)* and *(b)* and it has been shown [53, Appendix] (compare also [8]) that *(c)* is also fulfilled. Replacing duality by twisted duality, the assumptions *(a)* to *(c)* hold for the massive free Fermi field, too [57].

(iii) The state ω_0 plays the role of a free massive vacuum state, called the bare vacuum.

Proposition 3.4 : *Let $u \in G(\mathfrak{M})$ be an operator and let $\mathcal{I}$ be a bounded interval. Then there exists a canonical automorphism $\chi_u^{\mathcal{I}}$ with the properties:*

(1) The relations

$$\chi_u^{\mathcal{I}}|_{C^*(\mathfrak{M},\mathcal{I}_{LL})} = \mathrm{id}_{C^*(\mathfrak{M},\mathcal{I}_{LL})} \quad \textit{and} \quad \chi_u^{\mathcal{I}}|_{C^*(\mathfrak{M},\mathcal{I}_{RR})} = \chi_u|_{C^*(\mathfrak{M},\mathcal{I}_{RR})} \quad (7)$$

hold.

(2) There exists a strongly continuous map $\gamma^1_{(u,\mathcal{I})} : \Sigma \to C^*(\mathfrak{M})$ such that:

(i)

$$\mathrm{Ad}(\gamma^1_{(u,\mathcal{I})}(\mathbf{x})) = \alpha_{\mathbf{x}} \circ \chi_u^{\mathcal{I}} \circ \alpha_{-\mathbf{x}} \circ (\chi_u^{\mathcal{I}})^{-1} \ .$$

(ii) The cocycle condition is fulfilled:

$$\gamma^1_{(u,\mathcal{I})}(\mathbf{x}+\mathbf{y}) = \alpha_{\mathbf{x}}(\gamma^1_{(u,\mathcal{I})}(\mathbf{y}))\gamma^1_{(u,\mathcal{I})}(\mathbf{x}) \ .$$

Proof.

(1) In the same manner as in [52], we show that

$$\mathrm{Ad}(\Psi_{\mathcal{I}}(\mathbf{1} \otimes u))(\mathfrak{M}(\hat{\mathcal{I}})) \subset \mathfrak{M}(\hat{\mathcal{I}})$$

if the interval $\hat{\mathcal{I}}$ contains $\mathcal{I}$. This implies that

$$\chi_u^{\mathcal{I}} := \mathrm{Ad}(\Psi_{\mathcal{I}}(\mathbf{1} \otimes u))$$

is a well defined automorphism of $C^*(\mathfrak{M})$. By using the properties of the universal localizing map $\Psi_{\mathcal{I}}$, we conclude that $\chi_u^{\mathcal{I}}$ fulfills equation (7).

(2) By a straight forward generalization of the proof of [52, Proposition 4.2], we conclude that the statement *(2)* holds where $\gamma^1_{(u,\mathcal{I})}(\mathbf{x})$ is given by:

$$\gamma^1_{(u,\mathcal{I})}(\mathbf{x}) = \Psi_{\mathcal{I}+\mathbf{x}}(\mathbf{1} \otimes u)\Psi_{\mathcal{I}}(\mathbf{1} \otimes u^*) \ .$$

□

Let ω be a vacuum state with respect to the dynamics α and let $u \in G(\alpha, \mathfrak{M})$, then the state

$$\omega_u^{\mathcal{I}} := \omega \circ \chi_u^{\mathcal{I}}$$

seem to be good a good candidate for an interpolating kink state. Indeed, it follows from the construction of $\chi_u^{\mathcal{I}}$ that

$$\omega_u^{\mathcal{I}}|_{C^*(\mathfrak{M},\mathcal{I}_{RR})} = \omega \circ \chi_u|_{C^*(\mathfrak{M},\mathcal{I}_{RR})}$$

$$\omega_u^{\mathcal{I}}|_{C^*(\mathfrak{M},\mathcal{I}_{LL})} = \omega|_{C^*(\mathfrak{M},\mathcal{I}_{LL})} \ .$$

Hence $\omega_u^{\mathcal{I}}$ interpolates ω and $\omega \circ \chi_u$. To decide whether $\omega_u^{\mathcal{I}}$ is a positive energy state, we investigate in the subsequent section, how $\chi_u^{\mathcal{I}}$ is transformed under the action of a dynamics α.

3.3 When Does a Theory Possess Kink States?

Let α be a dynamics and $G \subset G(\alpha, \mathfrak{M})$ be a *finite* subgroup. By using the universal localizing map $\Psi_{\mathcal{I}}$, we obtain for each bounded interval $\mathcal{I}$ a unitary representation of G

$$U_{\mathcal{I}} : G \ni g \mapsto U_{\mathcal{I}}(g) := \Psi_{\mathcal{I}}(\mathbf{1} \otimes g) \in \mathfrak{M}(\mathcal{I}_R) \ .$$

In the previous section it has been shown that $U_{\mathcal{I}}(g)$ implements an automorphism $\chi_g^{\mathcal{I}}$ which is covariant under spatial translations (Proposition 3.4). For a dynamics $\alpha \in \mathrm{dyn}(\mathfrak{M})$, we wish to construct a cocycle $\gamma_{(g,\mathcal{I})}$ in order to show that $\chi_g^{\mathcal{I}}$ is an interpolating automorphism. The *formal* operator

$$\gamma_{(g,\mathcal{I})}(t, \mathbf{x}) := \alpha_{(t,\mathbf{x})}(U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^*$$

seems to be a useful Ansatz since it formally implements the automorphism

$$\alpha_{(t,\mathbf{x})} \circ \chi_g^{\mathcal{I}} \circ \alpha_{(-t,-\mathbf{x})} \circ (\chi_g^{\mathcal{I}})^{-1} \ .$$

Unfortunately, the operators $U_{\mathcal{I}}(g)$ are *not* contained in $C^*(\mathfrak{M})$ and the term $\alpha_{(t,\mathbf{x})}(U_{\mathcal{I}}(g))$ has no well defined mathematical meaning. To get a well defined solution for $\gamma_{(g,\mathcal{I})}$, we construct an extension of the net $\mathfrak{M}$ which contains the operators $U_{\mathcal{I}}(g)$ (compare also [47]).

Definiton 3.5 : Let $G \subset G(\mathfrak{M})$ be a compact sub-group. The net $\mathfrak{M} \rtimes G$ is defined by the assignment

$$\mathfrak{M} \rtimes G : \mathcal{I} \mapsto (\mathfrak{M} \rtimes G)(\mathcal{I}) := \mathfrak{M}(\mathcal{I}) \vee U_{\mathcal{I}}(G)'' \ .$$

Proposition 3.6 : *Let $\mathcal{I}$ be a bounded interval, then the map*

$$\pi^{\mathcal{I}} : \mathfrak{M}(\mathcal{I}) \rtimes G \ni a \cdot g \mapsto a\, U_{\mathcal{I}}(g) \in \mathfrak{M}(\mathcal{I}) \vee U_{\mathcal{I}}(G)''$$

is a faithful representation of the crossed product $\mathfrak{M}(\mathcal{I}) \rtimes G$.

Proof. First, we easily observe that $\pi^{\mathcal{I}}$ is a well defined representation of $\mathfrak{M}(\mathcal{I}) \rtimes G$. According to [39, Theorem 2.2, Corollary 2.3], we conclude that the crossed product $\mathfrak{M}(\mathcal{I}) \rtimes G$ is isomorphic to the von Neumann algebra $\mathfrak{M}(\mathcal{I}) \vee U_{\mathcal{I}}(G)''$ and $\pi^{\mathcal{I}}$ is a W*-isomorphism. □

Definiton 3.7 : A one parameter automorphism group α, which satisfies the conditions, listed below, is called a G-dynamics of the extended net $\mathfrak{M} \rtimes G$.

(a) α is a dynamics of the net $\mathfrak{M} \rtimes G$ (See Introduction).

(b) The automorphisms α_t commute with the automorphisms χ_g, i.e.

$$\alpha_t \circ \chi_g = \chi_g \circ \alpha_t \ ; \text{ for each } t \in \mathbb{R} \text{ and for each } g \in G.$$

The set of all G-dynamics of $\mathfrak{M} \rtimes G$ is denoted by $\mathrm{dyn}_G(\mathfrak{M} \rtimes G)$.

Proposition 3.8 : *Let $\alpha \in \mathrm{dyn}_G(\mathfrak{M} \rtimes G)$ be a G-dynamics and $\mathcal{I}$ be a bounded interval. Then the operator*

$$\gamma^0_{(g,\mathcal{I})}(t) := \alpha_t(U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^*$$

is contained in $\mathfrak{M}(\mathcal{I}_{|t|})$ where $\mathcal{I}_{|t|}$ denotes the enlarged interval $\mathcal{I} + (-|t|, |t|)$ and the operator

$$\gamma_{(g,\mathcal{I})}(t, \mathbf{x}) := \alpha_{(t,\mathbf{x})}(U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^*$$

fulfills the cocycle condition:

$$\gamma_{(g,\mathcal{I})}(t + t', \mathbf{x} + \mathbf{x}') = \alpha_{(t,\mathbf{x})}(\gamma_{(g,\mathcal{I})}(t', \mathbf{x}'))\gamma_{(g,\mathcal{I})}(t, \mathbf{x}) \ .$$

Proof. For $a \in C^*(\mathfrak{M}, \mathcal{I}_{|t|,RR})$, the operator $\alpha_{-t}(a)$ is contained in $C^*(\mathfrak{M}, \mathcal{I}_{RR})$ which implies

$$\begin{aligned}
a\ \alpha_t(U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^* &= \alpha_t(\alpha_{-t}(a)U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^* \\
&= \alpha_t(U_{\mathcal{I}}(g)\chi_g\alpha_{-t}(a))U_{\mathcal{I}}(g)^* \\
&= \alpha_t(U_{\mathcal{I}}(g)\alpha_{-t}\chi_g(a))U_{\mathcal{I}}(g)^* \\
&= \alpha_t(U_{\mathcal{I}}(g))\chi_g(a)U_{\mathcal{I}}(g)^* \\
&= \alpha_t(U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^*\ a
\end{aligned}$$

and we conclude:

$$\alpha_t(U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^* \in C^*(\mathfrak{M}, \mathcal{I}_{|t|,RR})' = \mathfrak{M}(\mathcal{I}_{|t|,L})$$

By a similar argument, $\alpha_t(U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^*$ is contained in $\mathfrak{M}(\mathcal{I}_{|t|,R})$ and we conclude from duality that it is contained in $\mathfrak{M}(\mathcal{I}_{|t|})$. The cocycle condition for $\gamma_{(g,\mathcal{I})}$ is obviously fulfilled and the proposition follows. □

Definiton 3.9 : Let $\alpha \in \mathrm{dyn}(\mathfrak{M})$ be a dynamics and $G \subset G(\mathfrak{M})$ be a compact subgroup. We shall call α *G-extendible* if there exists a G-dynamics $\hat{\alpha}$ of the extended net $\mathfrak{M} \rtimes G$, such that

$$\hat{\alpha}_t|_{C^*(\mathfrak{M})} = \alpha_t$$

for each $t \in \mathbb{R}$.

We are now prepared to prove one of our key results:

Theorem 3.10 : *Let $\alpha \in \mathrm{dyn}(\mathfrak{M})$ be a G-extendible dynamics and let $\chi_g^{\mathcal{I}}$ be the automorphism which can be constructed by Proposition 3.4. Then for each vacuum state ω with respect to α the state*

$$\omega_g^{\mathcal{I}} := \omega \circ \chi_g^{\mathcal{I}}$$

is a kink state which interpolates ω and $\omega \circ \chi_g$.

Proof. As postulated, there exists an extension $\hat{\alpha} \in \mathrm{dyn}_G(\mathfrak{M} \rtimes G)$ of α. We show that for each $g \in G$ the operator

$$\gamma^0_{(g,\mathcal{I})}(t) := \hat{\alpha}_t(U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^*$$

implements the automorphism

$$\alpha_t \circ \chi_g^{\mathcal{I}} \circ \alpha_{-t} \circ (\chi_g^{\mathcal{I}})^{-1}$$

on $C^*(\mathfrak{M})$. Indeed, we have for each $a \in C^*(\mathfrak{M})$:

$$\begin{aligned}
\mathrm{Ad}(\gamma^0_{(g,\mathcal{I})}(t))a &= \hat{\alpha}_t(U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^*\, a\, U_{\mathcal{I}}(g)\hat{\alpha}_t(U_{\mathcal{I}}(g))^* \\
&= \hat{\alpha}_t(U_{\mathcal{I}}(g))\,(\chi_g^{\mathcal{I}})^{-1}(a)\,\hat{\alpha}_t(U_{\mathcal{I}}(g))^* \\
&= \hat{\alpha}_t\Big(U_{\mathcal{I}}(g)\alpha_{-t}\Big((\chi_g^{\mathcal{I}})^{-1}(a)\Big)U_{\mathcal{I}}(g)^*\Big) \\
&= \alpha_t\Big(U_{\mathcal{I}}(g)\alpha_{-t}\Big((\chi_g^{\mathcal{I}})^{-1}(a)\Big)U_{\mathcal{I}}(g)^*\Big) \\
&= \alpha_t \circ \chi_g^{\mathcal{I}} \circ \alpha_{-t} \circ (\chi_g^{\mathcal{I}})^{-1}(a)
\end{aligned}$$

Finally we conclude from Proposition 3.8 that

$$\gamma_{(g,\mathcal{I})}(t,\mathbf{x}) := \alpha_{(t,\mathbf{x})}(U_{\mathcal{I}}(g))U_{\mathcal{I}}(g)^*$$

is a cocycle where $\gamma_{(g,\mathcal{I})}(t,\mathbf{x})$ is localized in a sufficiently large bounded interval. By a straight forward generalization of the proof of [52, Proposition 5.5] we conclude that $\omega_u^{\mathcal{I}}$ is a positive energy state which implies the result. □

The dynamics α of $P(\phi)_2$- and Yukawa$_2$ models are locally implementable by unitary operators. More precisely, for each bounded interval $\mathcal{I}$ and for each positive number $\tau > 0$, there exists a unitary operator $u(\mathcal{I},\tau|t)$ with the properties:

(1) If $|t_1|, |t_2|, |t_1 + t_2| < \tau$, then we have

$$u(\mathcal{I},\tau|t_1 + t_2) = u(\mathcal{I},\tau|t_1)u(\mathcal{I},\tau|t_2) \ .$$

(2) For $|t| < \tau$, the operator $u(\mathcal{I},\tau|t)$ implements α_t on $\mathfrak{M}(\mathcal{I})$, i.e.:

$$\alpha_t(a) = u(\mathcal{I},\tau|t)\ a\ u(\mathcal{I},\tau|t)^* \ ; \ \text{for each } a \in \mathfrak{M}(\mathcal{I}). \tag{8}$$

Let $G \subset G(\alpha,\mathfrak{M})$ be a compact sub-group. In order to show that α is G-extendible, it is sufficient to prove that the operators

$$u(\mathcal{I}_1,\tau|t)U_{\mathcal{I}}(g)u(\mathcal{I}_1,\tau|t)^* \ ,$$

which are the obvious candidates for $\hat{\alpha}(U_I(g))$, are independent of $\mathcal{I}_1$ for $\mathcal{I}_1 \supset \mathcal{I}$ and $|t| \leq \tau$.

Lemma 3.11 : *If for each $\mathcal{I} \subset \mathcal{I}_1$, for each $\tau < \tau_1$ and for each $g \in G$ the equation*

$$u(\mathcal{I},\tau|t)U_{\mathcal{I}}(g)u(\mathcal{I},\tau|t)^* = u(\mathcal{I}_1,\tau_1|t)U_{\mathcal{I}}(g)u(\mathcal{I}_1,\tau_1|t)^* \tag{9}$$

holds, then the dynamics α is G-extendible. Here $u(\mathcal{I},\tau|t)$ are unitary operators which fulfill equation (8).

Proof. Let $(\mathcal{I}_n,\tau_n)_{n\in\mathbb{N}}$ be a sequence, such that $\lim_n \mathcal{I}_n = \mathbb{R}$ and $\lim_n \tau_n = \infty$. We conclude from our assumption (equation (9)) that the uniform limit

$$\hat{\alpha}_t(a) := \lim_{n\to\infty} \mathrm{Ad}(u(\mathcal{I}_n,\tau_n|t))(a)$$

exists. Thus $\hat{\alpha} : t \mapsto \hat{\alpha}_t$ is a well defined one-parameter automorphism group, extending the dynamics α. It remains to be proven that $\hat{\alpha}$ has propagation speed $\mathrm{ps}(\hat{\alpha}) \leq 1$. Since $\hat{\alpha}$ is an extension of α and $\mathrm{ps}(\alpha) \leq 1$, we conclude for each $a \in C^*(\mathfrak{M}, \mathcal{I}_{|t|,RR})$ and for each $b \in C^*(\mathfrak{M}, \mathcal{I}_{|t|,LL})$:

$$\begin{aligned} ab\ \hat{\alpha}_t(U_{\mathcal{I}}(g)) &= \hat{\alpha}_t(\alpha_{-t}(a)\alpha_{-t}(b)U_{\mathcal{I}}(g)) \\ &= \hat{\alpha}_t(U_{\mathcal{I}}(g)\alpha_{-t}\chi_g(a)\alpha_{-t}(b)) \\ &= \hat{\alpha}_t(U_{\mathcal{I}}(g))\ \chi_g(a)b \end{aligned}$$

Thus the operator $\hat{\alpha}_t(U_{\mathcal{I}}(g))$ is contained in $\mathfrak{M}(\mathcal{I}_{|t|,R})$ and implements χ_g on
$\mathfrak{M}(\mathcal{I}_{|t|,RR})$. This finally implies:

$$\hat{\alpha}_t(U_{\mathcal{I}}(g))U_{\mathcal{I}_{|t|}}(g)^* \in \mathfrak{M}(\mathcal{I}_{|t|})$$

and the lemma follows. □

Let us consider the twofold W*-tensor product of the net of Cauchy data, i.e.:

$$\mathfrak{M} \,\overline{\otimes}\, \mathfrak{M} : \mathcal{I} \mapsto \mathfrak{M}(\mathcal{I}) \,\overline{\otimes}\, \mathfrak{M}(\mathcal{I})$$

Observation:

(i) If the net $\mathfrak{M}$ fulfills the conditions *(a)* to *(c)* of the previous section, then the net $\mathfrak{M} \,\overline{\otimes}\, \mathfrak{M}$ fulfills them, too.

(ii) Let $\alpha \in \mathrm{dyn}(\mathfrak{M})$ be a dynamics of $\mathfrak{M}$, then $\alpha^{\otimes 2}$ is a dynamics of $\mathfrak{M} \,\overline{\otimes}\, \mathfrak{M}$. Note that the flip operator u_F, which is given by

$$u_F : \mathcal{H}_0 \otimes \mathcal{H}_0 \to \mathcal{H}_0 \otimes \mathcal{H}_0 \ ; \ \psi_1 \otimes \psi_2 \mapsto \psi_2 \otimes \psi_1$$

is contained in $G(\alpha^{\otimes 2}, \mathfrak{M} \,\overline{\otimes}\, \mathfrak{M})$. Hence u_F induces an embedding of $\mathbb{Z}_2$ into $G(\alpha^{\otimes 2}, \mathfrak{M} \,\overline{\otimes}\, \mathfrak{M})$.

(iii) According to Definition 3.5, we can construct a non-local extension

$$\hat{\mathfrak{M}} := (\mathfrak{M} \,\overline{\otimes}\, \mathfrak{M}) \rtimes \mathbb{Z}_2$$

of the twofold net $\mathfrak{M} \,\overline{\otimes}\, \mathfrak{M}$. Let $\Psi_{\mathcal{I}}$ be the universal localizing map of the standard split inclusion

$$\Lambda(\mathcal{I}) \otimes \Lambda(\mathcal{I}) = (\mathfrak{M}(\mathcal{I}_{RR}) \,\overline{\otimes}^{\,2}, \mathfrak{M}(\mathcal{I}_R) \,\overline{\otimes}^{\,2}, \Omega_0 \otimes \Omega_0)$$

and define $\theta_{\mathcal{I}} := \Psi_{\mathcal{I}}(1 \otimes u_F)$. Then the algebra $\hat{\mathfrak{M}}(\mathcal{I})$ is simply given by

$$\hat{\mathfrak{M}}(\mathcal{I}) = ((\mathfrak{M} \,\overline{\otimes}\, \mathfrak{M}) \rtimes \mathbb{Z}_2)(\mathcal{I}) = (\mathfrak{M}(\mathcal{I}) \,\overline{\otimes}\, \mathfrak{M}(\mathcal{I})) \vee \{\theta_{\mathcal{I}}\}'' \quad .$$

(iv) By Proposition 3.4, there exists a canonical automorphism

$$\beta^{\mathcal{I}} := \mathrm{Ad}(\theta_{\mathcal{I}}) \ , \tag{10}$$

associated with the pair $(u_F, \mathcal{I})$.

Notation: Let α be a dynamics of $\mathfrak{M}$. In the sequel, we shall call α *extendible* if $\alpha^{\otimes 2}$ is $\mathbb{Z}_2$-extendible.

We conclude this section by the following corollary which can be derived by a direct application of Theorem 3.10:

Corollary 3.12 : *Let $\alpha \in \mathrm{dyn}(\mathfrak{M})$ be an extendible dynamics, then for each pair of vacuum states $\omega_1, \omega_2 \in \mathfrak{S}_0(\mathfrak{A}_\alpha)$, the state*

$$\omega = \mu_{\beta^{\mathcal{I}}}(\omega_1 \otimes \omega_2)$$

is a kink state.

4 Application to Quantum Field Theory Models

We show that a sufficient condition for the existence of interpolating automorphisms, i.e. the extendibility of the dynamics, is satisfied for the $P(\phi)_2$, the Yukawa$_2$ and special types of Wess-Zumino models.

4.1 Kink States in $P(\phi)_2$-Models

We shall show that the dynamics of $P(\phi)_2$-models are extendible. As described in Section 2.2 the dynamics of a $P(\phi)_2$-model consists of two parts.

(1) The first part is given by the free dynamics α_0, with propagation speed $\mathrm{ps}(\alpha_0) = 1$,

$$\alpha_{0,t}(a) = e^{iH_0 t} a e^{-iH_0 t}$$

where $(H_0, D(H_0))$ is the free Hamiltonian which is a self-adjoint operator on the domain $D(H_0) \subset \mathcal{H}_0$.

(2) The second part is a dynamics α_1 with propagation speed $\mathrm{ps}(\alpha_1) = 0$, i.e. $\alpha_{1,t}$ maps each local algebra $\mathfrak{M}(\mathcal{I})$ onto itself. The interaction part of the full Hamiltonian is given by a Wick polynomial of the time-zero field ϕ:

$$H_1(\mathcal{I}) = H_1(\chi_{\mathcal{I}}) = \int \mathrm{d}\mathbf{x} \; : P(\phi(\mathbf{x})) : \; \chi_{\mathcal{I}}(\mathbf{x})$$

where $\chi_{\mathcal{I}}$ is a smooth test function which is one on $\mathcal{I}$ and zero on the complement of a slightly lager region $\hat{\mathcal{I}} \supset \mathcal{I}$. The unitary operator $\exp(itH_1(\mathcal{I}))$ implements the dynamics α_1 locally, i.e. for each $a \in \mathfrak{M}(\mathcal{I})$ we have:

$$\alpha_{1,t}(a) := e^{iH_1(\mathcal{I})t} a e^{-iH_1(\mathcal{I})t} \quad .$$

Definiton 4.1 : An operator valued distribution $v : S(\mathbb{R}) \to L(\mathcal{H}_0)$ is called an *ultra local interaction*, if the following conditions are fulfilled:

(1) For each real valued test function $f \in S(\mathbb{R})$, $v(f)$ is self-adjoint and has a common core with H_0.

(2) Let $f \in S(\mathbb{R})$ be a real valued test function with support in a bounded interval $\mathcal{I}$, then the spectral projections of $v(f)$ are contained in $\mathfrak{M}(\mathcal{I})$.

(3) For each pair of test functions $f_1, f_2 \in S(\mathbb{R})$, the spectral projections of $v(f_1)$ commute with the spectral projections of $v(f_2)$.

Remark: It has been proven in [32], that the Wick polynomials of the time zero fields are ultra local interactions. Furthermore, each ultra local interaction v induces a dynamics $\alpha^v \in \mathrm{dyn}(\mathfrak{M})$ with propagation speed $\mathrm{ps}(\alpha^v) = 0$. Let $\mathcal{I}$ be a bounded interval and let $\chi_{\mathcal{I}} \in S(\mathbb{R})$ be a positive test function with $\chi_{\mathcal{I}}(\mathbf{x}) = 1$ for each $\mathbf{x} \in \mathcal{I}$. Indeed, by an application of J. Glimm's and A. Jaffe's analysis [32], we conclude that the automorphisms

$$\alpha^v_t : \mathfrak{M}(\mathcal{I}) \to \mathfrak{M}(\mathcal{I}) \; ; \; a \mapsto \mathrm{Ad}(\, \exp(itv(\chi_{\mathcal{I}}))\,)a$$

define a dynamics with zero propagation speed. In the sequel, we shall call a dynamics α^v *ultra local* if it is induced by an ultra local interaction v.

In order to prove that a dynamics α, which is given by the Trotter product

$$\alpha = \alpha_0 \times \alpha^v$$

of a free and an ultra local dynamics, is extendible, we show that each part of the dynamics can be extended separately.

Since the free part of the dynamics can be extended to the algebra $\mathfrak{B}(\mathcal{H}_0)$ of all bounded operators on the Fock space $\mathcal{H}_0$, it is obvious that α_0 is extendible. Thus it remains to be proven the following:

Lemma 4.2 : *Each ultra local dynamics $\alpha^v \in \mathrm{dyn}(\mathfrak{M})$ is extendible.*

Proof. Let us consider any ultra local interaction v. For each test function $f \in S(\mathbb{R})$, we introduce the unitary operator

$$u(f|t) := e^{itv(f)} \otimes e^{itv(f)} \quad .$$

Let $\mathcal{I}$ be a bounded interval and denote by $\mathcal{I}_\epsilon$, $\epsilon > 0$, the enlarged interval $\mathcal{I} + (-\epsilon, \epsilon)$. We choose test functions $\chi^{(\mathcal{I},\epsilon)} \in S(\mathbb{R})$ such that

$$\chi^{(\mathcal{I},\epsilon)}(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in \mathcal{I} \\ 0 & \mathbf{x} \in \mathcal{I}_\epsilon^c = \mathcal{I}_\epsilon \backslash \mathbb{R} \end{cases} \quad .$$

For an interval $\hat{\mathcal{I}} \supset \mathcal{I}_\epsilon$, the region $\hat{\mathcal{I}}_\epsilon \backslash \mathcal{I}_\epsilon$ consists of two connected components $(\hat{\mathcal{I}}_\epsilon \backslash \mathcal{I}_\epsilon)_\pm$ and there exist test functions $\chi^\pm \in S(\mathbb{R})$ with

$$\begin{aligned} &\mathrm{supp}(\chi^-) \subset (\hat{\mathcal{I}}_\epsilon \backslash \mathcal{I}_\epsilon)_- \subset \mathcal{I}_{LL} \\ &\mathrm{supp}(\chi^+) \subset (\hat{\mathcal{I}}_\epsilon \backslash \mathcal{I}_\epsilon)_+ \subset \mathcal{I}_{RR} \\ &\chi^{(\hat{\mathcal{I}},\epsilon)} - \chi^{(\mathcal{I},\epsilon)} = \chi^+ + \chi^- \quad . \end{aligned}$$

Let us write

$$u(\mathcal{I},\epsilon|t) := u(\chi^{(\mathcal{I},\epsilon)}|t) \ \text{ and } \ u_\pm(t) := u(\chi^\pm|t).$$

Since we have $[u(f_1|t), u(f_2|t)] = 0$ for any pair of test functions $f_1, f_2 \in S(\mathbb{R})$, we obtain for each $\epsilon > 0$ and for $\mathcal{I}_\epsilon \subset \hat{\mathcal{I}}$:

$$u(\hat{\mathcal{I}},\epsilon|t) = u(\mathcal{I},\epsilon|t) u_-(t) u_+(t) \tag{11}$$

If we make use of the fact that $u_+(t)$ is α_F-invariant and localized in $\mathcal{I}_{RR}$, we conclude that $\theta_\mathcal{I}$ and $u_\pm(t)$ commute. Thus we obtain

$$\mathrm{Ad}(u(\hat{\mathcal{I}},\epsilon|t))\theta_\mathcal{I} = \mathrm{Ad}(u(\mathcal{I},\epsilon|t))\theta_\mathcal{I} \tag{12}$$

which depends only of the localization interval $\mathcal{I}$ since $\epsilon > 0$ can be chosen arbitrarily small. According to Lemma 3.11, the automorphisms

$$\hat{\alpha}_t^v : \hat{\mathfrak{M}}(\mathcal{I}) \ni a \mapsto \mathrm{Ad}(u(\mathcal{I},\epsilon|t))a \in \hat{\mathfrak{M}}(\mathcal{I})$$

define a dynamics of $\hat{\mathfrak{M}}$ whose restriction to $\mathfrak{M} \,\overline{\otimes}\, \mathfrak{M}$ is $\alpha^v \otimes \alpha^v$. Thus α^v is extendible. □

If $\hat{\alpha}_0$ denotes the natural extension of the free dynamics $\alpha_0^{\otimes 2}$ to $\hat{\mathfrak{M}}$ and let $\hat{\alpha}^v$ be the extension of the ultra local dynamics $\alpha^v \otimes \alpha^v$ then, by using the Trotter product, we conclude that the dynamics

$$\hat{\alpha} := \hat{\alpha}_0 \times \hat{\alpha}^v$$

is an extension of the dynamics $(\alpha_0 \times \alpha^v)^{\otimes 2}$ to $\hat{\mathfrak{M}}$. This leads to the following result:

Proposition 4.3 : *Each dynamics of a $P(\phi)_2$-model is extendible.*

Proof. The statement follows from Lemma 4.2 and from the fact that each dynamics of a $P(\phi)_2$-model is a Trotter product of the free dynamics α_0 and an ultra local dynamics α_1. □

The existence of interpolating kink states in $P(\phi)_2$-models is an immediate consequence of Proposition 4.3.

Corollary 4.4 : *Let $\alpha \in \mathrm{dyn}(\mathfrak{M})$ be a dynamics of a $P(\phi)_2$-model. Then for each pair of vacuum states $\omega_1, \omega_2 \in \mathfrak{S}_0(\alpha, \mathfrak{M})$ there exists an interpolating kink state $\omega \in \mathfrak{S}(\omega_1, \omega_2)$.*

Proof. By Proposition 4.3 each dynamics of a $P(\phi)_2$-model is extendible and we can apply Corollary 3.12 which implies the result. □

4.2 The Dynamics of the Yukawa$_2$ Model

Since the dynamics of a Yukawa$_2$-like model can not be written as a Trotter product which consists of a free and an ultra local dynamics, it is a bit more complicated to show that these dynamics are extendible. We briefly summarize here the construction of the Yukawa$_2$ dynamics which has been carried out by J. Glimm and A. Jaffe [32]. We also refer to the work of R. Schrader [55, 56].

Let $\mathfrak{M}_s$ and $\mathfrak{M}_a$ be the nets of Cauchy data for the free Bose and Fermi field, represented on the Fock spaces $\mathcal{H}_s$ and $\mathcal{H}_a$ respectively. The Cauchy data of the Yukawa$_2$ model are given by the W*-tensor product $\mathfrak{M} := \mathfrak{M}_s \,\overline{\otimes}\, \mathfrak{M}_a$ of the nets $\mathfrak{M}_s$ and $\mathfrak{M}_a$. Moreover, we set $\mathcal{H}_0 := \mathcal{H}_s \otimes \mathcal{H}_a$.

Step 1: In the first step, a Hamiltonian, which is regularized by an UV-cutoff $c_0 > 0$ and an IR-cutoff $c_1 > 1$, $c_0 << c_1$, is constructed. For this purpose, one chooses test functions $\delta_{c_0}, \chi_{c_1} \in S(\mathbb{R})$ with the properties:

(a)

$$\mathrm{supp}(\delta_{c_0}) \subset (-c_0, c_0) \quad \text{and} \quad \int \mathrm{d}\mathbf{x}\, \delta_{c_0}(\mathbf{x}) = 1$$

(b)

$$\mathrm{supp}(\chi_{c_1}) \subset (-c_1 - 1, c_1 + 1) \quad \text{and} \quad \chi_{c_1}(\mathbf{x}) = 1 \quad \text{for each } \mathbf{x} \in (-c_1, c_1).$$

The UV-regularized fields are given by

$$\phi(c_0, \mathbf{x}) := (\phi * \delta_{c_0})(\mathbf{x}) \quad \text{and} \quad \psi(c_0, \mathbf{x}) := (\psi * \delta_{c_0})(\mathbf{x}) \tag{13}$$

where ϕ is a massive free Bose field and ψ a free Dirac spinor field at $t = 0$. The fields, defined by equation (13), act on $\mathcal{H}_0$ via the operators

$$\Phi(c_0, \mathbf{x}) := \phi(c_0, \mathbf{x}) \otimes 1\!\!1_{\mathcal{H}_a} \quad \text{and} \quad \Psi(c_0, \mathbf{x}) := 1\!\!1_{\mathcal{H}_s} \otimes \psi(c_0, \mathbf{x}) \ .$$

The regularized Hamiltonian $H(c_0, c_1)$ can be written as a sum of three parts:

(1) The free Hamiltonian H_0 which is given by

$$H_0 = H_{0,s} \otimes 1\!\!1_{\mathcal{H}_a} + 1\!\!1_{\mathcal{H}_s} \otimes H_{0,a}$$

where $H_{0,s}$ and $H_{0,a}$ are the free Hamilton operators of the Bose and the Fermi field respectively.

(2) The regularized Yukawa interaction term:

$$H_Y(c_0, c_1) = \int \mathrm{d}\mathbf{x}\, \chi_{c_1}(\mathbf{x})\, \Phi(c_0, \mathbf{x}) \ : \bar{\Psi}(c_0, \mathbf{x}) \Psi(c_0, \mathbf{x}) :$$

(3) The counterterms:

$$H_C(c_0, c_1) = \sum_{n=0}^{N} z_n(c_0) \int \mathrm{d}\mathbf{x}\, \chi_{c_1}(\mathbf{x}) \ : \Phi(\mathbf{x})^n :$$

where $z_n(c_0)$ are suitable renormalization constants.

The following statement has been established by J.Glimm and A.Jaffe [32, 34]:

Theorem 4.5 : *The counterterms $H_C(c_0, c_1)$ can be chosen in such a way that*

(1) the cutoff Hamiltonian $H(c_0, c_1) = (H_0 + H_Y(c_0, c_1) + H_C(c_0, c_1))^{**}$ is a positive and self adjoint operator with domain $D(H_0)$.

(2) The uniform limit

$$R(c_1, \zeta) = \lim_{c_0 \to 0} (H(c_0, c_1) - \zeta)^{-1}$$

is the resolvent of a self adjoint operator $H(c_1)$.

(3) $H(c_1)$ is the limit of $H(c_0, c_1)$ in the strong graph topology.

Notation: In the sequel, we shall use the following notation:

$$u(c_0, c_1, t) := \exp(itH(c_0, c_1)) \quad \text{and} \quad u(c_1, t) := \exp(itH(c_1)) \, .$$

Remark: The aim is to show that $H(c_1)$ induces a dynamics of $\mathfrak{M}$, given locally by the equation

$$\alpha_t|_{\mathfrak{M}(\mathcal{I})} = \mathrm{Ad}(\, u(c_1, t)\,) \quad \text{for } \mathcal{I}_{|t|} := \mathcal{I} + (-|t|, |t|) \subset (-c_1, c_1).$$

However, in comparison to the $P(\phi)_2$-models, there are some more technical difficulties which have to be overcome.

(i) The Hamiltonian $H(c_1)$ is only defined as a limit of the Hamiltonians $H(c_0, c_1)$ and it has no mathematical meaning when written as a sum

$$H_0 + H_Y(c_1) + H_C(c_1) \quad .$$

Thus the construction scheme for a dynamics, as it has been used for $P(\phi)_2$-models, does not apply.

(ii) On the other hand, one might try to apply $P(\phi)_2$-like methods to the Hamiltonian $H(c_0, c_1)$, for which the UV-cutoff is not removed. For this purpose, one wishes to write $H(c_0, c_1)$ as a sum $H(c_0, c_1) = H_1(c_0, c_1) + H_2(c_0, c_1)$ where $H_1(c_0, c_1)$ induces a dynamics α_1 with propagation speed $\mathrm{ps}(\alpha_1) \leq 1$ and $H_2(c_0, c_1)$ induces a dynamics α_2 with propagation speed $\mathrm{ps}(\alpha_2) = 0$.

The difficulty with writing such a decomposition for $H(c_0, c_1)$ arises from the fact that the Yukawa interaction term $H_Y(c_0, c_1)$ induces an automorphism group with infinite propagation speed.

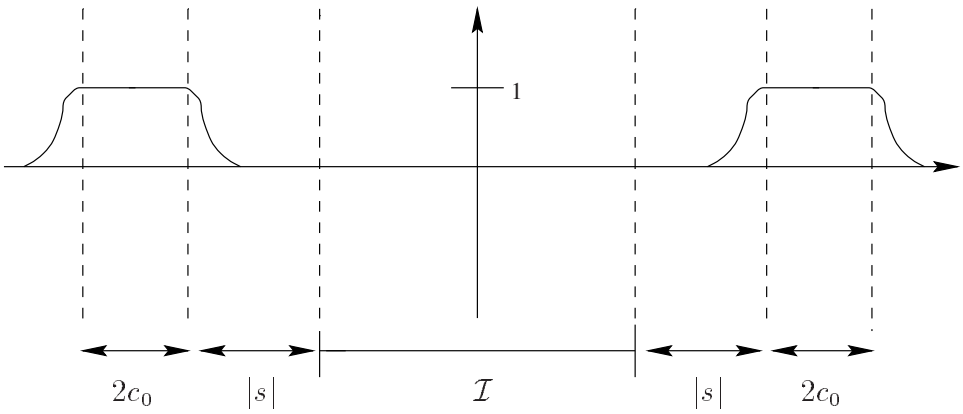

Figure 1: The figure shows the graph of the function $\chi_{(\mathcal{I},s,c_0)}$.

Step 2: In the next step, one introduces test functions $\chi_{(\mathcal{I},s,c_0)}$ (see Figure 1), depending on a bounded interval $\mathcal{I}$, a real number $s > 0$ and the UV-cutoff c_0, fulfilling the conditions

$$\begin{aligned} &\mathrm{supp}(\chi_{(\mathcal{I},s,c_0)}) \subset \mathcal{I}_{2c_0+|s|+\epsilon}\backslash\mathcal{I}_{|s|-\epsilon} \quad \text{and} \\ &\chi_{(\mathcal{I},s,c_0)}(\mathbf{x}) = 1 \quad \text{if } \mathbf{x} \in \mathcal{I}_{2c_0+|s|}\backslash\mathcal{I}_{|s|}. \end{aligned} \tag{14}$$

Here $\epsilon << c_0$ is any sufficiently small positive number. The Hamiltonian $H(c_0, c_1)$ is replaced by the operator

$$H(\mathcal{I}, s, c_0, c_1) := H_0 + H_C(c_0, c_1) + H_Y(\mathcal{I}, s, c_0, c_1) \tag{15}$$

depending additionally on $\mathcal{I}$ and s, where $H_Y(\mathcal{I}, s, c_0, c_1)$ is given by

$$H_Y(\mathcal{I}, s, c_0, c_1) := \int \mathrm{d}\mathbf{x}\, \Phi(c_0, \mathbf{x}) \; :\bar{\Psi}(c_0, \mathbf{x})\Psi(c_0, \mathbf{x}): \; (\; \chi_{c_1}(\mathbf{x}) - \chi_{(\mathcal{I},s,c_0)}(\mathbf{x})\;) \quad .$$

In order to construct from these data a c_1-independent approximation of the dynamics which maps $\mathfrak{M}(\mathcal{I})$ onto $\mathfrak{M}(\mathcal{I}_{|t|})$, one defines the unitary operators

$$w(\mathcal{I}, c_0, c_1, t) := \prod_{j=1}^{n} \exp\left(i\frac{t}{n} H(\,\mathcal{I}, (n-j)n^{-1}t, c_0, c_1\,)\right)$$

where n is equal to the integral part of $|c_0^{-1}t|$. The lemma, given below, has been established in [32].

Lemma 4.6 : [32, Lemma 9.1.2] *The adjoint action of* $w(\mathcal{I}, c_0, c_1, t)$ *induces an automorphism*

$$\alpha_t^{(\mathcal{I},c_0)} := \mathrm{Ad}(w(\mathcal{I}, c_0, c_1, t)) : \mathfrak{M}(\mathcal{I}) \to \mathfrak{M}(\mathcal{I}_{|t|})$$

which is independent of c_1.

Step 3: For technical reasons, to control convergence as c_0 tends to zero, the length of time propagation is scaled, and one defines for $\lambda \in [0, 1]$ the c_1-independent automorphism

$$\alpha_t^{(\mathcal{I},c_0,\lambda)} := \mathrm{Ad}(w(\mathcal{I}, c_0, c_1, \lambda, t)) : \mathfrak{M}(\mathcal{I}) \to \mathfrak{M}(\mathcal{I}_{|t|})$$

where $w(\mathcal{I}, c_0, c_1, \lambda, t)$ is given by

$$w(\mathcal{I}, c_0, c_1, \lambda, t) := \prod_{j=1}^{n} \exp\left(i \frac{\lambda \cdot t}{n} H(\mathcal{I}, (n-j)n^{-1}t, c_0, c_1) \right) \ .$$

The final approximation is given by averaging over λ:

$$\alpha_t^{(\mathcal{I},c_0,\ell)}(a) := \int \mathrm{d}\lambda \ f_\ell(\lambda)\, \alpha_t^{(\mathcal{I},c_0,\lambda)}(a)$$

where f_ℓ is a positive continuous function such that

$$\int \mathrm{d}\lambda \ f_\ell(\lambda) = 1 \quad \text{and } \mathrm{supp}(f_\ell) \subset [1-\ell, 1], \ell \leq 1.$$

Finally, J. Glimm and A. Jaffe have established the result:

Theorem 4.7 : [32, Theorem 9.1.3] *There exists a function* $c : \ell \mapsto c_\ell$ *with* $\lim_{\ell \to 0} c_\ell = 0$ *such that*

$$\alpha_t^Y(a) := w - \lim_{\ell \to 0} \alpha_t^{(\mathcal{I},c_\ell,\ell)}(a) = u(c_1, t)\ a\ u(c_1, t)^* \tag{16}$$

for each $a \in \mathfrak{M}(\mathcal{I})$ *and for each sufficiently large* c_1.

4.3 Kink States in Models with Yukawa$_2$ Interaction

We shall use an analogous strategy as above (step 1- step 3) in order to show that the dynamics α^Y, which is given due to Theorem 4.7 is extendible.

Theorem 4.8 : *The dynamics* α^Y *of the Yukawa*$_2$ *model is extendible.*

Let us prepare the proof. First, we give a few comments on the notation to be used.

Notation:

(a) In the sequel, we write $\hat{w}(\cdots) = w(\cdots)^{\otimes 2}$ and $\hat{u}(\cdots) = u(\cdots)^{\otimes 2}$ for the corresponding quantities of the twofold theory. As in step 3 above, we also define the automorphism

$$\hat{\alpha}_t^{(\mathcal{I},c_0,\lambda)} := \mathrm{Ad}(\hat{w}(\mathcal{I},c_0,c_1,\lambda,t))$$

and the average

$$\hat{\alpha}_t^{(\mathcal{I},c_0,\ell)}(a) = \int \mathrm{d}\lambda\; f_\ell(\lambda)\, \hat{\alpha}_t^{(\mathcal{I},c_0,\lambda)}(a) \quad .$$

(b) Let ω_0 be the vacuum state with respect to the free dynamics which is induced by H_0. We denote by $\Psi_{\mathcal{I}}$ the universal localizing map of the standard split inclusion $\Lambda(\mathcal{I})\otimes\Lambda(\mathcal{I})$ and we define $\theta_{\mathcal{I}} := \Psi_{\mathcal{I}}(\mathbf{1}\otimes u_F)$.

Lemma 4.9 : *The adjoint action of $\hat{w}(\mathcal{I},c_0,c_1,t)$ induces an automorphism*

$$\hat{\alpha}_t^{(\mathcal{I},c_0)} : \hat{\mathfrak{M}}(\mathcal{I}) \to \hat{\mathfrak{M}}(\mathcal{I}_{|t|})$$

which is independent of c_1.

Proof. By Lemma 4.6, it is sufficient to prove that

$$\mathrm{Ad}(\hat{w}(\mathcal{I},c_0,c_1,t))\theta_{\mathcal{I}}$$

is c_1-independent. Indeed, following the arguments in the proof of Proposition 4.3, we conclude that

$$\theta'_{\mathcal{I}} := \exp(i\tau H(\mathcal{I},s,c_0,c_1))^{\otimes 2}\; \theta_{\mathcal{I}}\; \exp(-i\tau H(\mathcal{I},s,c_0,c_1))^{\otimes 2}$$

is c_1-independent for $|\tau| \leq c_0$ and that $\theta'_{\mathcal{I}}$ is contained in $\hat{\mathfrak{M}}(\mathcal{I}_{|s|+|\tau|})$. Composing n such maps, we obtain the lemma. □

In complete analogy to Theorem 4.7 we have:

Lemma 4.10 :

$$\hat{\alpha}^Y(a) := w - \lim_{\ell\to 0} \hat{\alpha}_t^{(\mathcal{I},c_\ell,\ell)}(a) = \hat{u}(c_1,t)\, a\, \hat{u}(c_1,t)^*$$

For each $a \in \hat{\mathfrak{M}}(\mathcal{I})$ and for each sufficiently large c_1.

Proof. By Theorem 4.7, we conclude that the lemma holds for each $a \in \mathfrak{M}(\mathcal{I}) \overline{\otimes} \mathfrak{M}(\mathcal{I})$. Hence it remains to be proven that

$$w - \lim_{\ell \to 0} \hat{\alpha}_t^{(\mathcal{I}, c_\ell, \ell)}(\theta_\mathcal{I}) = \hat{u}(c_1, t)\, \theta_\mathcal{I}\, \hat{u}(c_1, t)^* \quad .$$

The Corollary 9.1.9 of [32] states:

$$w - \lim_{\ell \to 0} \int \mathrm{d}\lambda \, (\, \hat{w}(\mathcal{I}, c_\ell, c_1, \lambda, t) \, - \, \hat{u}(c_\ell, c_1, \lambda t)\,) f_\ell(\lambda) = 0 \quad .$$

We define

$$\theta_\mathcal{I}(\ell, t) := \hat{\alpha}_t^{(\mathcal{I}, c_\ell, \ell)}(\theta_\mathcal{I}) \quad \text{and} \quad \bar{\theta}_\mathcal{I}(\ell, t) := \int \mathrm{d}\lambda \, f_\ell(\lambda)\, \mathrm{Ad}(\, \hat{u}(c_\ell, c_1, \lambda t)\,)\theta_\mathcal{I} \quad .$$

The Schwarz's inequality implies for each $\psi \in \mathcal{H}_0 \otimes \mathcal{H}_0$:

$$|\langle \psi, \theta_\mathcal{I}(\ell, t) - \bar{\theta}_\mathcal{I}(\ell, t)\psi \rangle|$$
$$\leq 2||\psi|| \cdot \left(\int \mathrm{d}\lambda \, f_\ell(\lambda)\, ||(\, \hat{w}(\mathcal{I}, c_\ell, c_1, \lambda, t) \, - \, \hat{u}(c_\ell, c_1, \lambda t)\,)\psi||^2 \right)^{1/2}$$

Since $||(v-u)\psi||^2 = 2 \cdot \mathrm{Re}(\langle (v-u)\psi, u\psi \rangle)$, we obtain:

$$|\langle \psi, \theta_\mathcal{I}(\ell, t) - \bar{\theta}_\mathcal{I}(\ell, t)\psi \rangle| \;\leq\; 4||\psi|| \cdot \left(\int \mathrm{d}\lambda \, f_\ell(\lambda) \right.$$
$$\left. \mathrm{Re} \left\langle \, (\hat{w}(\mathcal{I}, c_\ell, c_1, \lambda, t) \, - \, \hat{u}(c_\ell, c_1, \lambda t)\,)\psi, \hat{u}(c_\ell, c_1, \lambda t)\psi \right\rangle \right)^{1/2}$$

which proves the lemma. □

Proof of Theorem 4.8: We conclude from Lemma 4.10 and Lemma 3.11 that the automorphism group $\hat{\alpha}^Y$ is a dynamics of the extended net $\hat{\mathfrak{M}}$ whose restriction to $\mathfrak{M} \overline{\otimes} \mathfrak{M}$ is $\alpha^Y \otimes \alpha^Y$. Thus α^Y is extendible. □

Remark: According to [56], each dynamics α^{Y+P} of a quantum field theory model with Yukawa$_2$ plus $P(\phi)_2$ boson self-interaction is extendible.

Finally, we conclude from Theorem 4.8:

Corollary 4.11 : *Let α^{Y+P} be a dynamics of a quantum field theory model with Yukawa$_2$ plus $P(\phi)_2$ boson self-interaction. For each pair ω_1, ω_2 of vacuum states with respect to α^{Y+P}, there exists a kink state ω in $\mathfrak{S}(\alpha^{Y+P}|\omega_1, \omega_2)$.*

4.4 Wess-Zumino Models

One interesting class of quantum field theory models which possess more than one vacuum sector are the $N = 2$ Wess-Zumino models in two-dimensional space-time. Their properties have been studied in several papers [40, 43, 44, 41, 42] and we summarize the main results which are established there in order to setup our subsequent analysis.

The field content of these models with a finite volume cutoff $c > 0$ consists of one *complex* Bose field ϕ_c and one Dirac spinor field ψ_c, acting as operator valued distributions on the Fock spaces

$$\begin{aligned}\mathcal{H}_a(c) &:= \bigoplus_{n=0}^{\infty} L_2(T_c, \mathbb{C}^2)^{\otimes_a} \\ \mathcal{H}_s(c) &:= \bigoplus_{n=0}^{\infty} L_2(T_c, \mathbb{C})^{\otimes_s}\end{aligned}$$

where a, s stands for symmetrization or anti-symmetrization of the tensor product and $L_2(T_c, \mathbb{C}^k)$ $(k = 1, 2)$ denotes the Hilbert space of $\mathbb{C}^k$-valued and square integrable functions, living on the circle T_c of length c. The net of Cauchy data for the finite volume theory is given by

$$\mathfrak{M}_c : (-c, c) \supset \mathcal{I} \mapsto \mathfrak{M}_c(\mathcal{I}) = \mathfrak{M}_{c,s}(\mathcal{I}) \,\overline{\otimes}\, \mathfrak{M}_{c,a}(\mathcal{I})$$

where the nets $\mathfrak{M}_{c,s}$ and $\mathfrak{M}_{c,a}$ are defined by the assignments:

$$\mathfrak{M}_{c,s} : (-c, c) \supset \mathcal{I} \mapsto \mathfrak{M}_{c,s}(\mathcal{I}) := \left\{ e^{i(\phi_c(f_1) + \pi_c(f_2))} \;\middle|\; \mathrm{supp}(f_j) \subset \mathcal{I} \right\}''$$

$$\mathfrak{M}_{c,a} : (-c, c) \supset \mathcal{I} \mapsto \mathfrak{M}_{c,a}(\mathcal{I}) := \left\{ \psi_c(f_1), \bar{\psi}_c(f_2) \;\middle|\; \mathrm{supp}(f_j) \subset \mathcal{I} \right\}''$$

where π_c is the canonically conjugate of ϕ_c.

Let $\mathfrak{M} := \mathfrak{M}_{c=\infty}$ be the net of Cauchy data in the infinite volume limit, then the map

$$\iota_c : \begin{pmatrix} \phi(f_{11}) & \pi(f_{12}) \\ \psi(f_{21}) & \bar{\psi}(f_{22}) \end{pmatrix} \mapsto \begin{pmatrix} \phi_c(f_{11}) & \pi_c(f_{12}) \\ \psi_c(f_{21}) & \bar{\psi}_c(f_{22}) \end{pmatrix} \;\; ; \;\; \mathrm{supp}(f_{ij}) \subset (-c, c)$$

is a W*-isomorphism which identifies the nets $\mathfrak{M}$ and $\mathfrak{M}_c$ for those regions $\mathcal{I}$ which are contained in $(-c, c)$.

The interaction part of the *formal* Hamiltonian consists of two parts.

(a) A $P(\phi)_2$-like part:

$$H_P(v,c) = \int_{T_c} \mathrm{d}\mathbf{x} \ : |v'(\Phi_c)|^2 : \ - \ : |\Phi_c|^2 :$$

(b) A Yukawa$_2$-like part:

$$H_Y(v,c) := \int_{T_c} \mathrm{d}\mathbf{x} \ : \bar{\Psi}_c \begin{pmatrix} v''(\Phi_c) - 1 & 0 \\ 0 & v''(\Phi_c)^* - 1 \end{pmatrix} \Psi_c :$$

where v is a polynomial of degree $\deg(v) = n$, called *superpotential*, and the fields Φ_c and Ψ_c are given by

$$\Phi_c := \phi_c \otimes 1_{\mathcal{H}_a(c)} \quad \text{and} \quad \Psi_c := 1_{\mathcal{H}_s(c)} \otimes \psi_c \ .$$

According to the results of [40, 42, 43, 44], it has been shown that, there is a self-adjoint Fredholm operator $Q(v,c)$, called *supersymmetry generator*, on
$\mathcal{H}_0(c) := \mathcal{H}_s(c) \otimes \mathcal{H}_a(c)$. The *Fredholm index* of $Q(v,c)$

$$\mathrm{ind}(Q(v,c)) = \dim \ker(Q(v,c)) - \dim \mathrm{coker}(Q(v,c))$$

has been computed in [43]. The result is

$$|\mathrm{ind}(Q(v,c))| = \deg(v) - 1 \ .$$

The space $\mathcal{H}_0(c)$ may be decomposed $\mathcal{H}_0(c) = \mathcal{H}_+(c) \oplus \mathcal{H}_-(c)$ into the eigenspaces of the fermion parity operator $\Gamma := (-1)^{N_a}$, where N_a is the fermion number operator. With respect to this decomposition, the operator $Q(v,c)$ has the form

$$Q(v,c) = \begin{pmatrix} 0 & Q_+(v,c) \\ Q_-(v,c) & 0 \end{pmatrix} \ .$$

The full Hamiltonian of the finite volume model is given by

$$H(v,c) = Q(v,c)^2$$

which implies:

$$\dim \ker(H(v,c)) = |\dim \ker(Q_+(v,c)) - \dim \ker(Q_-(v,c))| = \deg(v) - 1$$

The Hamiltonian $H(v,c)$ induces a dynamics $\alpha^{(v,c)}$ of the finite volume model and we conclude from the results of [40]:

Theorem 4.12 : [40, Theorem 1] *There exists at least* $\deg(v) - 1$ *vacuum sectors with respect to the dynamics* $\alpha^v := \alpha^{(v,c=\infty)}$ *of the model in the infinite volume limit.*

4.5 Kink States in Wess-Zumino Models

In order to prove the existence of kink sectors, we now apply the results which have been established in Section 4.1 and Section 4.3 to $N = 2$ Wess-Zumino Models.

The case $\deg(v) = 3$*:* Let us have a closer look at the simplest non-trivial case $\deg(v) = 3$. We let

$$v'(z) = \lambda_2 z^2 + \lambda_1 z + \lambda_0 \quad .$$

As in the previous sections (equation (13)), we introduce the UV-regularized fields:

$$\Phi(c_0, \mathbf{x}) := (\Phi * \delta_{c_0})(\mathbf{x}) \quad \text{and} \quad \Psi(c_0, \mathbf{x}) := (\Psi * \delta_{c_0})(\mathbf{x})$$

where δ_{c_0} is a smooth test function with support in $(-c_0, c_0)$. We obtain for the $P(\phi)_2$-like part of the regularized interaction Hamiltonian

$$H_P(v; c_0, c_1) = \int \mathrm{d}\mathbf{x} \chi_{c_1}(\mathbf{x}) \Big[$$

$$: |\lambda_2 \Phi(c_0, \mathbf{x})^2 + \lambda_1 \Phi(c_0, \mathbf{x}) + \lambda_0|^2 : - : |\Phi(c_0, \mathbf{x})|^2 : \Big]$$

and for the Yukawa$_2$-like part:

$$H_Y(v; c_0, c_1) = \int \mathrm{d}\mathbf{x} \chi_{c_1}(\mathbf{x}) \Big[$$

$$: \bar{\Psi}(c_0, \mathbf{x}) \begin{pmatrix} 2\lambda_2 \Phi(c_0, \mathbf{x}) + \lambda_1 - 1 & 0 \\ 0 & 2\bar{\lambda}_2 \Phi(c_0, \mathbf{x})^* + \bar{\lambda}_1 - 1 \end{pmatrix} \Psi(c_0, \mathbf{x}) : \Big]$$

Using the same techniques as in Section 4.1 and Section 4.3, we obtain the corollary (see also Corollary 4.11):

Corollary 4.13 : *Let v be a superpotential of degree $\deg(v) = 3$. Then the following statements are true:*

(1) The dynamics $\alpha^v \in \mathrm{dyn}(\mathfrak{M})$ of the model in the infinite volume limit is extendible.

(2) There exists two different vacuum sectors $e_1, e_2 \in \mathrm{sec}_0(\alpha^v, \mathfrak{M})$ and two different kink sectors $\theta \in \mathrm{sec}(e_1, e_2)$, $\bar{\theta} \in \mathrm{sec}(e_2, e_1)$.

The case $\deg(v) > 3$*:* We close this section by discussing the remaining case.

In order to show the extendibility of $\alpha^v \in \mathrm{dyn}(\mathfrak{M})$, we can try to proceed in the same manner as for the case $\deg(v) = 3$. According to Section 4.3 (*Step 2* and *Step 3*), we construct an approximation

$$\mathfrak{M}(\mathcal{I}) \,\overline{\otimes}\, \mathfrak{M}(\mathcal{I}) \ni a \mapsto \hat{\alpha}_t^{(v;\mathcal{I},c_0,\ell)}(a) := \int \mathrm{d}\lambda \; f_\ell(\lambda)\, \hat{\alpha}_t^{(v;\mathcal{I},c_0,\lambda)}(a)$$

of the dynamics $\alpha^v \otimes \alpha^v$ of the twofold theory. Provided that the corresponding result of Lemma 4.9 is true for the case $\deg(v) > 3$ also, the linear maps $\hat{\alpha}_t^{(v;\mathcal{I},c_0,\ell)}$ can be extended to the algebra $\hat{\mathfrak{M}}(\mathcal{I})$.

For the generalization of Theorem 4.8, it seems that the most difficult part is to show that there exists a function $c : \ell \mapsto c_\ell$ with $\lim_{\ell\to 0} c_\ell = 0$ such that

$$\alpha_t^v(a) := w - \lim_{\ell\to 0} \alpha_t^{(v;\mathcal{I},c_\ell,\ell)}(a) \;. \tag{17}$$

The regularized Yukawa-like part of the Hamilton density contains terms of the form

$$:\Psi^{(i)}(c_0,\mathbf{x})\Psi^{(j)}(c_0,\mathbf{x}): \; :\Phi(c_0,\mathbf{x})^k:$$

$$i,j \in \{0,1\},\, i \neq j \text{ and } k \leq \deg(v) - 2,$$

where $\Psi^{(j)}$ denotes the j-component of the Dirac spinor field Ψ. Since there are contributions with $k > 1$, the proof of Theorem 4.7 does not directly apply.

Provided that for each superpotential v the dynamics α^v is extendible, we conclude that for each pair of vacuum sectors $e_1, e_2 \in \mathrm{sec}_0(\alpha^v, \mathfrak{M})$ there exists a kink state $\omega \in \mathfrak{S}(e_1, e_2)$. Then the model possesses at least $\deg(v)(\deg(v)-1)$ different non-trivial kink sectors.

5 Conclusion and Outlook

In the present paper, a construction scheme for kink sectors has been developed which can be applied to a large class of quantum field theory models. Most of the techniques which are used, except those in the proof of the extendibility of the dynamics, concern operator algebraic methods. They are model independent in the sense that they can be derived from first principles. There are still some interesting open problems and we shall make a few remarks on them here.

Some further remarks on kink states: Let us consider a quantum field theory model $(P(\phi)_2,\, Y_2)$, possessing vacua $\omega_1\,, \omega_2$ which are related by a symmetry χ. According to Theorem 3.10, there exists an automorphism $\chi^{\mathcal{I}}$ which induces a kink state $\omega = \omega_1 \circ \chi^{\mathcal{I}}$. Note that ω is a pure state in this case.

Alternatively, we obtain a kink state $\hat{\omega}$ by passing to the two folded tensor product of the theory with itself first and then by restricting the α_F-interpolating automorphism $\beta^{\mathcal{I}}$, whose existence follows also from Theorem 3.10, to the first tensor factor, i.e.:

$$\hat{\omega} = \omega_1 \otimes \omega_2 \circ \beta_{C^*(\mathfrak{A})\otimes\mathbf{1}} \quad .$$

Provided the split property for wedge algebras holds for the interacting vacua [52], then, by applying a recent result of M. Müger [48], we conclude that $[\hat{\omega}]$ is nothing else but the infinite multiple of $[\omega]$.

The problem of reducibility: The problem of reducibility arises if the vacua under consideration are not related by a symmetry since then our construction scheme leads to kink representations of the form

$$\pi = \pi_1 \otimes \pi_2 \circ \beta|_{C^*(\mathfrak{A})\otimes\mathbf{1}}$$

where β is an automorphism and $\pi_1\,, \pi_2$ are vacuum representations. The representation π is not irreducible and whether π can be decomposed into irreducible sub-representations is still an open problem. Some of our results [53, Theorem 4.4.3] suggest that π is, in non exceptional cases, an infinite multiple of one irreducible component.

Kink sectors in $d > 1+1$ dimensions: It would be desirable to apply our program to quantum field theories in higher dimensions. Let us suppose a theory, given by a net of W*-algebras $\mathfrak{A}$, possesses two locally normal vacuum states $\omega_1\,, \omega_2$.

As a sensible generalization of a kink states to $d > 1+1$, we propose to consider locally normal states ω which fulfill the interpolation condition:

$$\omega|_{C^*(\mathfrak{A},S_1)} = \omega_1|_{C^*(\mathfrak{A},S_1)} \quad \text{and} \quad \omega|_{C^*(\mathfrak{A},S_2')} = \omega_2|_{C^*(\mathfrak{A},S_2')} \tag{18}$$

where S_1, S_2, $S_1 \subset S_2$, are space-like cones. The state ω describes the coexistence of two phases which are separated by the *phase boundary* $\partial S := S_1' \cap S_2$.

Let us assume duality for space-like cones in the vacuum representations under consideration. Furthermore, we assert that the inclusion

$$\Lambda = (\mathfrak{A}_{\pi_1}(S_1), \mathfrak{A}_{\pi_1}(S_2), \Omega_1)$$

is standard split. Here $(\mathcal{H}_1, \pi_1, \Omega_1)$ is a GNS-triple with respect to ω_1.

Unfortunately, for $d > 1+1$ the phase boundary ∂S is not compact and therefore our construction scheme can not directly be generalized to higher dimensions.

In order to overcome this difficulties, we consider a sequence of standard split inclusions

$$\Lambda_n := (\mathfrak{A}_{\pi_1}(\mathcal{O}_{1n}), \mathfrak{A}_{\pi_1}(\mathcal{O}_{2n}), \Omega_1)$$

where $\mathcal{O}_{1n} \subset\subset \mathcal{O}_{2n}$ are bounded double cones such that $\mathcal{O}_{jn}$ tends to S_j for $n \to \infty$.

As in the $1+1$-dimensional case we pass now to the two folded tensor product of the theory with itself. Denote by $\Psi_{\Lambda_n \otimes \Lambda_n}$ the universal localizing map with respect to the inclusion $\Lambda_n \otimes \Lambda_n$. Since the operators

$$\theta_n := \Psi_{\Lambda_n \otimes \Lambda_n}(1 \otimes u_F)$$

are localized in a bounded region, we may define the following automorphisms of $C^*(\mathfrak{A})$:

$$\beta_n := (\pi_1 \otimes \pi_1)^{-1} \circ \beta_n \circ (\pi_1 \otimes \pi_1) \ .$$

We obtain a sequence of states $\{\omega_n, n \in \mathbb{N}\}$ where ω_n is given by:

$$\omega_n := \omega_1 \otimes \omega_2 \circ \beta_n|_{C^*(\mathfrak{A})\otimes \mathbf{1}} \ .$$

For large n the states ω_n have *almost* the correct interpolation property, namely for each pair of local observables a, b where a is localized S_2' and b is localized in S_1, there exists a sufficiently large N such that

$$\omega_n(a) = \omega_1(a) \quad \text{and} \quad \omega_n(b) = \omega_2(b)$$

for each $n > N$. Note that each state ω_n fulfills the Borchers criterion since ω_n belongs to the vacuum sector $[\omega_1]$.

In order to obtain generalized kink states, we propose to investigate weak*-limit points of the sequence $\{\omega_n, n \in \mathbb{N}\}$. Note that each weak*-limit ω_ι point of the sequence $\{\omega_n, n \in \mathbb{N}\}$ fulfills the interpolation condition (18). It remains to be proven that the weak*-limit points are locally normal.

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