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A non-abelian square root of abelian vertex operators

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Abstract. Kadanoff's "correlations along a line" in the critical two-dimensional Ising model [1] are reconsidered. They are the analytical aspect of a representation of abelian chiral vertex operators as quadratic polynomials, in the sense of operator valued distributions, in non-abelian exchange fields. This basic result has interesting applications to conformal coset models. It also gives a new explanation for the remarkable relation between the "doubled" critical Ising model and the free massless Dirac theory [2]. As a consequence, analogous properties as for the Ising model order/disorder fields with respect both to doubling and to restriction along a line are established for the two-dimensional local fields with chiral SU(2) symmetry at level 2.

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1. Introduction and outline

Wightman distributions can be pointwise multiplied, unlike general distributions, and the product is again a Wightman distribution. This fact [3, Chap. V] is due to the spectral properties, i.e., support properties in momentum space, of Wightman fields. The new Wightman field resulting from this prescription is known as the p-product of the factor fields.

The converse is in general not true: the square root of a Wightman distribution is in general not a Wightman distribution. There is, however, at least one famous non-trivial example, due to Schroer and Truong [2]. The exponentials of the potential for the axial current of a free Dirac field in two dimensions can be defined (with an appropriate renormalization) and give rise to local Wightman fields. The sine and the cosine, at a specific value of the coefficient in the exponential, turn out to be the p-squares of the order and disorder fields of the Ising model. This result establishes a deep field theoretical relation between the "doubled" Ising model and the free Dirac theory. Analytically it means that the vacuum expectation values of the latter can be computed by taking square roots of v.e.v.'s of the former; the double-valuedness of the square root gives rise to the mutual non-locality of the order and disorder fields.

The purpose of this paper is the discussion of another instance of nontrivial p-product factorization and some of its consequences. The basic formula (3.4) represents a chiral abelian vertex operator (cf. Sect. 2.1) as a p-quadratic expression in terms of chiral non-abelian exchange fields (Sect. 2.2); the latter are the well-known chiral constituents of the order and disorder fields of the critical Ising field theory [5].

The p-product of chiral abelian vertex operators is another abelian vertex operators, with a quadratic addition law for the charges. Conversely, the p-factorization of abelian vertex operators into abelian vertex operators of smaller charges is an almost trivial fact (cf. Sect. 2.1 below). In contrast, the new feature of our basic result is the *non-abelian* nature of the constituents, reflected in their non-abelian braiding and fusion properties.

The p-product is well-defined also for non-local fields, as long as they satisfy the remaining Wightman axioms. We shall refer to such fields as non-local Wightman fields. Indeed, our basic result (3.4) is in terms of non-local (chiral) fields, while some of its applications pertain to local fields in two dimensions.

Viewing Wightman fields as operator valued distributions, the p-product is the pointwise multiplication as a distribution, and the tensor product as an operator. The p-product, denoted by $\Phi \odot \Psi$, of two Wightman fields Φ and Ψ on Hilbert spaces H_Φ and H_Ψ therefore lives on $H_\Phi \otimes H_\Psi$ (its cyclic subspace may be smaller). Symbolically we write

$$(\Phi \odot \Psi)(f) = \int d^d x f(x)\ \Phi(x) \otimes \Psi(x). \tag{1.1}$$

The precise statement is that the tensor product of Wightman distributions extends to singular smearing functions of the form $f(x)\delta^d(x-y)$ where f is a smooth test function.

The first input for the basic result (cf. Sect. 3) is the old observation in the critical Ising model [1] due to Kadanoff (for the magnetization σ alone) and Kadanoff and Ceva (for mixed correlations involving also the disorder parameter μ) that upon restriction along a line in two-dimensional space, the correlations of σ and μ satisfy certain "Γ-selection rules" and simplify considerably. Their analytical form becomes that of the correlations of appropriately assigned electric charges in a one-dimensional Coulomb gas.

Translated to the associated quantum field theory in 1+1 dimensions, this observation is tantamount to a simple identification between the order and disorder fields along a time-like axis and chiral abelian vertex operators E_α of charge $\alpha = \pm\frac{1}{2}$ and scaling dimension $h = \frac{1}{8}$. Explicitly, the Kadanoff-Ceva formulae read

$$\begin{cases} \sigma|_{x=0} \cong E_{+\frac{1}{2}} P_+ + E_{-\frac{1}{2}} P_- \\ \mu|_{x=0} \cong E_{-\frac{1}{2}} P_+ + E_{+\frac{1}{2}} P_- \end{cases} \tag{1.2a}$$

$$\Leftrightarrow \qquad E_{\pm\frac{1}{2}} \cong \sigma|_{x=0} P_\pm + \mu|_{x=0} P_\mp, \tag{1.2b}$$

where $P_\pm$ are the projections onto the subspaces of integer and half-integer charge. (For the precise notations see Sect. 2.1 below.)

The general theory says that the restriction of a Wightman field in d dimensions to a time-like hyperplane (such as $x^1 = 0$) yields another Wightman field in one dimension less [4]. For $d = 1 + 1$, the resulting field is a chiral field; the "positive" direction (pertinent to the spectrum condition) is inherited from the forward time-like direction.

In order to prevent confusion, we note that the commutation relations obtained by restricting a 1+1-dimensional field to the line $x = 0$ of course test the time-like commutation relations

of that field. There is no conflict between σ and μ being local fields and their representations (1.2a) in terms of non-local vertex operators.

The second input is the well-known fact that the (local, but not mutually local) 1+1-dimensional order and disorder fields σ and μ possess a bilinear factorization into chiral exchange fields a and b along with their adjoints [5]. The latter, often also called non-abelian vertex operators referring to the non-abelian superselection structure of the underlying chiral observables (here: the stress-energy tensor T with central charge $c = \frac{1}{2}$), are the primary fields of scaling dimension $h = \frac{1}{16}$ for the latter [6], making different transitions between the three sectors of T. We shall call a, b, a^+ and b^+ the "elementary fields" in the sequel. They are non-local Wightman fields. (Still, they are relatively local with respect to the chiral observables; in this sense, all non-local fields throughout this article are not as non-local as could be. Genuinely non-local fields without any organizing local observables might be far more intricate objects.)

We have to distinguish the factorization in two independent light-cone coordinates $u = t+x$, $v = t-x$, as mentioned here, from p-factorization described before. Let Φ and Ψ be two chiral (non-local) Wightman fields; then the $\times$-product

$$(\Phi \times \Psi)(f) = \int dt\, dx f(t,x)\ \Phi(t+x) \otimes \Psi(t-x) \tag{1.3}$$

defines a 1+1-dimensional (a priori also non-local) Wightman field. The $\times$-product is just the ordinary tensor product of operator valued distributions (except for the passage to light-cone coordinates), and is therefore the tensor product both as a distribution and as an operator.

The bilinear factorization of the order and disorder fields is given in eqs. (3.1) below. For the moment, the gross structure

$$(\sigma \quad \text{and} \quad \mu) \sim (\text{linear}) \times (\text{linear}) \tag{1.4}$$

is sufficient. (Here and in the sequel expressions like "linear" or "quadratic" mean "linear or p-quadratic polynomial in the elementary fields".)

The point is now that the $\times$-product, when restricted along a time-like axis (such as $x = 0$), turns into the p-product; symbolically:

$$(\Phi \times \Psi)(t,0) = (\Phi \odot \Psi)(t). \tag{1.5}$$

Thus, restricting a 1+1-dimensional field which is obtained as a $\times$-product of chiral fields produces the p-product of the chiral fields.

We apply this observation to the decomposition (1.4) and insert it into the Kadanoff-Ceva result (1.2b). This yields a representation of the chiral abelian vertex operators of charge $\pm\frac{1}{2}$ (scaling dimension $h = \frac{1}{8}$) as a quadratic p-polynomial in the (non-local and non-abelian) elementary fields of scaling dimension $h = \frac{1}{16}$

$$E_{\pm\frac{1}{2}} \sim (\text{quadratic in the elementary fields}). \tag{1.6}$$

The exact formula, eq. (3.4), is the basic result of the present note. While it is quite surprising as it stands, it becomes even more interesting by its consequences. Namely, we apply it in several ways by exhibiting various familiar fields of various chiral and two-dimensional models as homogeneous p- and $\times$-product combinations in the elementary fields a and b; we thereby reproduce several well-known observations from a unifying point of view, as well as produce several new results.

1. In Sect. 4.1 we give another proof for the massless case of the previously mentioned observation by Schroer and Truong [2] in the "doubled" Ising model. It relates the p-squares of the order and disorder fields to the axial potential of the massless Dirac theory. The latter has the form $\varphi_R(t-x)-\varphi_L(t+x)$, so its exponentials are $\times$-product of (oppositely charged) chiral abelian vertex operators. The identifications given in [2] are the sine and cosine formulae

$$\sigma_D \equiv \sigma \odot \sigma \cong \frac{1}{\sqrt{2}i}(E_{-\frac{1}{2}} \times E_{\frac{1}{2}} - E_{\frac{1}{2}} \times E_{-\frac{1}{2}}), \tag{1.7a}$$

$$\mu_D \equiv \mu \odot \mu \cong \frac{1}{\sqrt{2}}(E_{-\frac{1}{2}} \times E_{\frac{1}{2}} + E_{\frac{1}{2}} \times E_{-\frac{1}{2}}). \tag{1.7b}$$

Since σ and μ are $\times$-bilinear in the elementary fields, their p-squares are $\times$-products of p-quadratic expressions in the elementary fields. The latter arrange in such a way that the basic result turns them into abelian chiral vertex operators, thus reproducing eqs. (1.7) (up to a unitary similarity transformation, and only on the joint cyclic Hilbert space of the fields under consideration).

2. We apply our methods to conformal coset models [7]. In these models, the primary fields with respect to a current algebra are factorized into primary fields with respect to some current subalgebra and primary fields with respect to the corresponding coset algebra ("branching"). This separation of degrees of freedom is another example of p-factorization.

In particular, we consider the U(2) currents of a doublet of complex chiral fermions (the primary fields), and its subalgebra of SU(2) currents at level 1 (Sect. 2.3.). The separation of the SU(2) and U(1) (= coset) degrees of freedom of the complex fermion doublet [8] is an instance of the trivial abelian p-factorization of abelian vertex operators as mentioned before. It gives rise to the pseudo-real doublet of SU(2)-primary fields represented in terms of chiral abelian vertex operators with scaling dimension $h = \frac{1}{4}$.

In the next step (Sect. 4.2.), the latter are p-factorized further into two abelian vertex operators with scaling dimension $h = \frac{1}{8}$, which in turn are p-quadratic in the elementary fields by the basic result. The resulting p-quartic expression for the level 1 primary fields, when written in the form (linear) $\odot$ (cubic), is precisely the coset model factorization with respect to the coset stress-energy tensor (with central charge $c = \frac{1}{2}$) and the level 2 currents, when the level 2 theory is embedded into the tensor product of two level 1 theories [7].

The primary chiral exchange fields of scaling dimension $h = \frac{3}{16}$ at level 2 are pseudo-real isospin doublets of non-abelian exchange fields A_i and B_i along with their adjoints, making transitions between the three sectors of the SU(2) current algebra. The previous discussion

yields a p-cubic representation in terms of the elementary fields

$$A_i \quad \text{and} \quad B_i \sim (\text{cubic in the elementary fields}). \tag{1.8}$$

The exact formulae are eqs. (4.12) below. This representation explains the observation that the solutions to the Knizhnik-Zamolodchikov equation [9] for isospin $\frac{1}{2}$ at level 2 are cubic expressions in the $h = \frac{1}{16}$ conformal blocks of the $c = \frac{1}{2}$ minimal model [6].

3. In Sect. 4.3 we consider the defining matrix field g_{ij} of the Wess-Zumino-Witten (WZW) model (= generalized Thirring model) [10] with $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry. It is $\times$-bilinear in the primary chiral exchange field doublet (A_i and B_i and their adjoints) of the level 2 SU(2) current algebra, just as σ and μ factorize into the chiral a and b fields.

According to our previous result (1.8), the chiral exchange field doublets are p-cubic in the elementary fields, and hence g_{ij} are of the form (cubic) $\times$ (cubic). It is possible to split off a p-quadratic part in the form of abelian vertex operators $E_{\pm\frac{1}{2}}$ in either chiral factor, and the remaining p-factor (linear) $\times$ (linear) can be identified with σ and μ for the diagonal and off-diagonal matrix elements. The gross structure is

$$g_{\mathrm{diag}} \sim \sigma \odot (E_{\pm\frac{1}{2}} \times E_{\mp\frac{1}{2}}), \tag{1.9a}$$

$$g_{\mathrm{offd}} \sim \mu \odot (E_{\pm\frac{1}{2}} \times E_{\pm\frac{1}{2}}). \tag{1.9b}$$

The trace of g involves σ as well as a field which we already know how to identify with $\sigma \odot \sigma$ by eq. (1.7a); hence

$$\mathrm{Tr}\, g \cong \sqrt{2}\, (\sigma \odot \sigma \odot \sigma). \tag{1.10}$$

Next, we restrict g_{ij} to a time-like axis and obtain a p-polynomial of order six in the elementary fields. By eqs. (1.2a) and (1.9), these polynomials turn into cubic p-products of vertex operators $E_{\pm\frac{1}{2}}$. The resulting representation (4.19) of $g_{ij}|_{x=0}$ by abelian chiral vertex operators with scaling dimension $\frac{3}{8}$ generalizes Kadanoff's result (1.2a) to the WZW model.

4. In Sect. 4.4 we consider p-quadratic expressions in the 1+1-dimensional matrix field g_{ij}, notably the p-determinant. In terms of the elementary fields, this determinant field is of the form (order six) $\times$ (order six). Again arranging p-quadratic polynomials in either chiral factor to yield abelian vertex operators according to the basic result, and performing some astute similarity transformations, the p-determinant field is finally found to be

$$\mathrm{Det}_{\odot}\, g \cong \frac{1}{\sqrt{2}i}(E_{-\beta} \times E_{\beta} - E_{\beta} \times E_{-\beta}) \tag{1.11}$$

with charge $\beta = \frac{1}{2}\sqrt{3}$ (rather than $\frac{1}{2}$ in eq. (1.7a)). This generalizes the result of Schroer and Truong [2] (in the massless case) to the WZW model.

A technical comment might be in order. Throughout the paper, we prove identities between various fields only "on the cyclic Hilbert space, and up to a unitary similarity transformation which preserves the vacuum vector". This means, of course, no weakening of the statement,

since the vacuum expectation values which determine the fields are never affected. A system of fields must of course be subjected to the same transformation. One should, however, keep in mind other fields which are possibly present, and which might or might not be transformed at the same time. Notably, the property of being primary with respect to some chiral stress-energy tensor or current algebra is defined by specific commutation relations with these chiral observables, and will be preserved only if the latter are. This is, e.g., guaranteed if the unitary operator which implements the similarity transformation is a constant on each irreducible representation space of the chiral observables. Otherwise, the latter would have to be transformed along with the primary fields. Even if the state of affairs is not explicitly mentioned in every single instance, it is contextually evident everywhere in the paper by inspection of the specific form of the similarity transformation at hand.

The present work contains the results of the diploma thesis of the first author (K.F.) [21].

2. Preparation of the ground

2.1. Abelian vertex operators

Unlike the massless free scalar field in 1+1 dimensions, its exponentials exist on a positive definite Hilbert space [3,12,13,14]. Mandelstam [15] has introduced such exponentials depending on only one chiral coordinate when he bosonized the massless Thirring model (at vanishing coupling constant) in terms of the sine-Gordon model (at $\beta^2 = 4\pi$). These fields, denoted by $E_\alpha = {:}\exp i\alpha\varphi{:}$ in the sequel, are now known as (abelian) chiral vertex operators. The "triple ordering" refers to the creation and annihilation parts of the auxiliary chiral scalar field and involves a renormalization parameter μ of the dimension of mass which is understood in the limit $\mu \searrow 0$. For rather streamlined treatments see, e.g., [13] and [16].

The complex free chiral fermion field ψ can be identified with a vertex operator of unit charge $\alpha = 1$. Its current $j = :\psi\psi^*: = \partial\varphi$ generates a U(1) symmetry with charge operator $Q = \frac{1}{2\pi}\int j(x)dx$. By virtue of

$$QE_\alpha = E_\alpha(Q + \alpha), \tag{2.1}$$

the U(1) symmetry extends to the chiral vertex operators of arbitrary charge α.

Chiral vertex operators of general charge are non-local chiral Wightman fields, satisfying anyonic commutation relations

$$E_\alpha(x)E_\beta(y) = \mathrm{e}^{\pm i\pi\alpha\beta}E_\beta(y)E_\alpha(x) \quad \text{if} \quad x \neq y \tag{2.2}$$

with the sign $\pm = \mathrm{sign}\,(x-y)$ in the exponent. They are conformally covariant fields with scaling dimension $h = \frac{1}{2}\alpha^2$ with respect to the stress-energy tensor $\frac{i}{4\pi}:\psi\overset{\leftrightarrow}{\partial}\psi^*: = \frac{1}{4\pi}:j^2:$ of the complex fermion (with central charge $c = 1$).

The vacuum expectation values of chiral vertex operators satisfy charge conservation and are given by

$$\langle\Omega, E_{\alpha_1}(x_1)\dots E_{\alpha_n}(x_n)\Omega\rangle = \prod_{i<j}\Delta(x_i - x_j)^{-\alpha_i\alpha_j} \qquad \text{provided} \quad \sum\alpha_i = 0\ . \tag{2.3}$$

The distributions

$$\Delta(x-y)^{2h} = \left[\frac{-i}{x-y-i\epsilon}\right]^{2h} = \frac{1}{\Gamma(2h)}\int_0^\infty dk k^{2h-1} \mathrm{e}^{-ik(x-y)} \tag{2.4}$$

contributing to these v.e.v.'s have spectral support on the positive momenta.

Abelian chiral vertex operators with multi-component charges can be defined as exponentials of several independent massless free fields, e.g.,

$$E_{(\alpha,\beta)} = :\exp i(\alpha\varphi_1 + \beta\varphi_2): \cong E_\alpha \odot E_\beta \ . \tag{2.5}$$

Their vacuum expectation values are given by the same formula (2.3), replacing only products of charges by scalar products of multi-component charges. This replacement results in the factorization of the total v.e.v.'s into the v.e.v.'s for the charge components and therefore justifies the equivalence in eq. (2.5).

On the other hand, it implies an obvious $\mathrm{O}(N)$ symmetry for the vertex operators with N-component charges, corresponding to rotations and reflections in charge space. Therefore on its own cyclic Hilbert space, the field (2.5) is also equivalent to $E_{\sqrt{\alpha^2+\beta^2}}$.

2.2. Exchange fields

Chiral exchange fields arise by a spectral decomposition with respect to global conformal transformations of 1+1-dimensional fields with conformal symmetry [17] and subsequent chiral factorization. They arise also as point-like limits of bounded intertwining operators in the framework of algebraic quantum field theory [18,19], or as intertwiners between modules of chiral symmetry algebras such as Virasoro or affine Kac-Moody algebras [20]. They are the fundamental charge-carrying entities which make transitions between the various superselection sectors of a theory of local chiral observables. As such, they are defined on a "source" Hilbert space and take values in a "range" Hilbert space which are both irreducible sectors of the observables, and are relatively local with respect to the latter.

In specific models where the chiral observables are generated by the stress-energy tensor or by a current algebra, it is convenient to distinguish "primary" exchange fields by their specific commutation relations with the observables. These imply certain partial differential equations, reflecting the symmetry of the model, for the vacuum expectation values of primary exchange fields whose solutions are known as "conformal blocks" [6,9]. The commutation relations of chiral exchange fields with each other are given by a non-abelian representation of the braid group. Abelian vertex operators (restricted to a charge sector) are the exchange fields with respect to an abelian current algebra.

The primary chiral exchange fields of the Ising model were first treated in detail in [5]. In this model, the chiral observables are generated by the stress-energy tensor with central charge $c = \frac{1}{2}$. There are three superselection sectors, which carry representations of the conformal group with primary scaling dimensions 0, $\frac{1}{16}$ and $\frac{1}{2}$, respectively.

The primary field for the trivial sector 0 is just the unit operator. The primary field for the sector $\frac{1}{2}$ is a real free fermion. There are three exchange fields corresponding to this primary field, which are given by the free fermion field in the Neveu-Schwarz representation restricted to the subspaces of even and odd Fermi number, and the free Fermi field in the Ramond representation. The former two are each other's adjoints $\psi_{\rm NS} : \mathcal{H}_0 \to \mathcal{H}_{\frac{1}{2}}$ and $\psi^+_{\rm NS} : \mathcal{H}_{\frac{1}{2}} \to \mathcal{H}_0$, and the latter is a selfadjoint field $\psi_{\rm R} : \mathcal{H}_{\frac{1}{16}} \to \mathcal{H}_{\frac{1}{16}}$.

The primary exchange fields for the sector $\frac{1}{16}$ are the chiral constituents of the order and disorder parameter of the 1+1-dimensional Ising model. There are two "raising" operators a and b along with their adjoints:

$$a : \mathcal{H}_0 \to \mathcal{H}_{\frac{1}{16}} \quad \text{and its adjoint} \quad a^+ : \mathcal{H}_{\frac{1}{16}} \to \mathcal{H}_0 \tag{2.6a}$$

$$b : \mathcal{H}_{\frac{1}{16}} \to \mathcal{H}_{\frac{1}{2}} \quad \text{and its adjoint} \quad b^+ : \mathcal{H}_{\frac{1}{2}} \to \mathcal{H}_{\frac{1}{16}} \tag{2.6b}$$

The commutation relations of these chiral exchange fields are given by a non-abelian representation of the braid group of the Hecke type [5].

We have chosen the normalization such that the two-point function is $\langle \Omega, a^+(x)a(y)\Omega\rangle = \Delta(x-y)^{\frac{1}{8}}$. The only possible four-point functions are given by

$$\langle \Omega, a^+(x_1)a(x_2)a^+(x_3)a(x_4)\Omega\rangle = V^{\frac{1}{8}} \cdot f(x) \tag{2.7a}$$

$$\langle \Omega, a^+(x_1)b^+(x_2)b(x_3)a(x_4)\Omega\rangle = V^{\frac{1}{8}} \cdot g(x) \tag{2.7b}$$

where the first factor with

$$V(x_1,x_2,x_3,x_4) = \frac{\Delta(x_1-x_2)\Delta(x_3-x_4)\Delta(x_1-x_4)\Delta(x_2-x_3)}{\Delta(x_1-x_3)\Delta(x_2-x_4)} \tag{2.8}$$

is the four-point function of abelian vertex operators with alternating charges and the proper scaling dimension, and the "conformal block" functions

$$f(x) = \sqrt{\tfrac{1}{2}(1+\sqrt{1-x})} \quad \text{and} \quad g(x) = \sqrt{\tfrac{1}{2}(1-\sqrt{1-x})} \tag{2.9}$$

depend only on the conformally invariant cross ratio $x = \frac{\Delta(x_1-x_3)\Delta(x_2-x_4)}{\Delta(x_1-x_2)\Delta(x_3-x_4)}$.

The higher $2n$-point functions were derived in closed form in [5].

2.3. The SU(2) primary fields at level 1 in terms of vertex operators

A doublet of complex free chiral fermions $\psi_\pm$ gives rise to a theory with U(2) symmetry. Its stress-energy tensor with $c = 2$ is given by $T = \frac{i}{4\pi}(:\psi_+ \overleftrightarrow{\partial} \psi^*_+: + :\psi_- \overleftrightarrow{\partial} \psi^*_-:)$, and the currents are

$$j^A = :\psi_i \tau^A_{ij} \psi^*_j: \qquad (A = 0,1,2,3) \tag{2.10}$$

where $\tau^0 = \mathbb{1}$ and τ^a $(a = 1,2,3)$ are the Pauli matrices.

The component j^0 generates the abelian U(1) current algebra with stress-energy tensor $T_U = \frac{1}{8\pi} :(j^0)^2:$ $(c = 1)$, and j^a generate the SU(2) current algebra at level 1 with stress-energy tensor T_S given by the Sugawara formula quadratic in the currents $(c = 1)$, such that $T = T_U + T_S$. The U(1) and the SU(2) currents commute with each other, and both transform the fermion doublet in the obvious way.

The U(1) and the SU(2) degrees of freedom of the fermion doublet can be separated as follows in a very explicit manner [8]. The crucial point is the fact that chiral charged fermions can be represented by abelian vertex operators of unit charge.

The doublet of charged chiral fermions can be represented by a pair of abelian vertex operators with orthogonal unit charge vectors $\alpha_\pm$. Since the latter constitute a pair of commuting fermions, a Klein transformation is needed to make them anti-commute. Rather than the standard choice (multiplying, e.g., E_{α_+} by the exponential $\exp(i\pi\alpha_- Q)$ of the charge operator of the other component), we prefer a more symmetric one and multiply both components by $\exp(i\pi\frac{\alpha_+ - \alpha_-}{2} Q)$. Both choices are unitarily equivalent by conjugation with the exponential of some quadratic polynomial of the charge operators.

We choose $\alpha_\pm = \frac{1}{\sqrt{2}} \cdot (1, \pm 1)$ and apply the p-factorization of multi-component vertex operators (eq. (2.5)). This gives rise to the simple p-factorization

$$\psi_\pm = \phi \odot \phi_\pm \tag{2.11}$$

where

$$\phi = E_\alpha \tag{2.12}$$

is an ordinary abelian vertex operator of charge $\alpha = \frac{1}{\sqrt{2}}$ carrying the U(1) symmetry, and

$$\phi_\pm = E_{\pm\alpha}\ \mathrm{e}^{i\pi\alpha Q} \tag{2.13}$$

is a pseudoreal doublet carrying the SU(2) symmetry. Pseudoreality means

$$\phi_i^* = i\varepsilon_{ij}\ \phi_j C = -i\varepsilon_{ij}\ C\phi_j \tag{2.14}$$

where $\varepsilon_{+-} = 1$ and $C = \mathrm{e}^{2\pi i\alpha Q} = C^{-1}$ is a Casimir operator which equals $+1$ (-1) on all states of integer (half-integer) isospin and thus commutes with the SU(2) symmetry.

In this representation, the factorization of the current algebra becomes manifest: the U(1) current is $j^0 = \sqrt{2}\ \partial\varphi \otimes 1\!\!1$, and the SU(2) currents are found in the Frenkel-Kac [21] representation $j^3 = 1\!\!1 \otimes \sqrt{2}\ \partial\varphi$ and $j^\pm = 1\!\!1 \otimes E_{\pm\sqrt{2}}$.

Both the U(1) field and the SU(2) doublet are primary fields w.r.t. the respective current algebra, and have scaling dimension $h = \frac{1}{4}$. They both satisfy anyonic commutation relations according to eq. (2.2). The two-point functions are $\langle\Omega, \phi^*\phi\Omega\rangle = \Delta^{\frac{1}{2}}$, or equivalently $\langle\Omega, \phi_i\phi_j\Omega\rangle = -i\varepsilon_{ij}\Delta^{\frac{1}{2}}$.

The cyclic Hilbert space of the pseudoreal doublet contains all charges which are integer multiples of $\alpha = \frac{1}{\sqrt{2}}$. The primary exchange fields which interpolate the two sectors of the SU(2) current algebra at level 1 (integer and half-integer isospin), are obtained by restricting $\phi_\pm$ to the subspaces of even or odd charge in units of α, or equivalently of integer or half-integer isospin.

2.4. Goddard-Kent-Olive (GKO) coset construction

Consider two current algebras $\mathcal{A}_i$ for some simple Lie group G at level k_i generated by the currents j_i^a, and with stress-energy tensor T_i given by the affine Sugawara formula. The diagonal currents

$$J^a = j_1^a \otimes \mathbb{1} + \mathbb{1} \otimes j_2^a \tag{2.15}$$

generate a current algebra $\mathcal{A}$ at level $k_1 + k_2$ within $\mathcal{A}_1 \otimes \mathcal{A}_2$. This current algebra has its own stress-energy tensor T given by the Sugawara formula in terms of J^a which has the same commutation relations with J^a as the total stress-energy tensor $T_1 \otimes \mathbb{1} + \mathbb{1} \otimes T_2$. The difference T_{coset} is a stress-energy tensor which commutes with the current algebra $\mathcal{A}$. It belongs (possibly along with other operators) to the coset model $\mathcal{C}$ of fields in $\mathcal{A}_1 \otimes \mathcal{A}_2$ commuting with $\mathcal{A}$.

Embedding a primary exchange field at level k_1 by the prescription $\phi \otimes \mathbb{1}$ into the Hilbert space of $\mathcal{A}_1 \otimes \mathcal{A}_2$, yields operators which have primary commutation relations with both the diagonal currents J^a and the coset stress-energy tensor. Therefore, according to the same scheme of separation of degrees of freedom as in Sect. 2.3, there is a factorization of the form

$$\phi \otimes \mathbb{1} \cong \sum \varphi_c \odot \varphi \tag{2.16}$$

where φ_c are primary exchange fields w.r.t. the coset theory and φ are primary exchange fields w.r.t. the current algebra at level $k_1 + k_2$. The quantum numbers of the exchange fields on the right hand side are determined by the branching rules of representations of $\mathcal{A}_1 \otimes \mathcal{A}_2$ in restriction to $\mathcal{C} \otimes \mathcal{A}$; in particular, φ carries the G quantum numbers, and the scaling dimensions are additive.

The symbol $\odot$ on the right hand side of the GKO decomposition (2.16) refers to the tensor product of representation spaces of $\mathcal{C} \otimes \mathcal{A}$, while in contrast, on the left hand side $\otimes$ refers to the tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2$. The passage is made by a reorganization of the Hilbert space which, in a special case, will become explicit along the way in Sect. 4.2.

3. The basic result

Chiral exchange fields can be combined in various ways to yield local 1+1-dimensional fields. A systematic way to do so was described in [22] in terms of the algebraic theory of superselection sectors. (Helas, the assignment of σ and μ was interchanged in [22], and no care was taken to construct local fields which are closed under conjugation; in order to repair that defect, the self-adjoint local fields below differ from the result in [22] by a non-unitary similarity transformation. One can actually adjust the correct formulae "by hand" from the known commutation relations and four-point functions of the elementary a and b fields [5] by imposing the correct commutation relations [2] and four-point functions [5] for σ and μ.)

The upshot is the following. The local field σ interpolates between the "diagonal" charge sectors $\mathcal{H}_{ss} \equiv \mathcal{H}_s \otimes \mathcal{H}_s$ where $\mathcal{H}_s$ ($s = 0, \frac{1}{16}, \frac{1}{2}$) are the three superselection sectors of the

chiral observables. It is given by the standard formula $\sigma = a \times a + b \times b + (\text{h.c.})$. The field μ, however, carries an "excess $\mathbb{Z}_2$ charge" which makes transitions to the "non-diagonal" sectors $\mathcal{H}_{0\frac{1}{2}}$, $\mathcal{H}_{\frac{1}{2}0}$ and another copy of $\mathcal{H}_{\frac{1}{16}\frac{1}{16}}$. In order to describe both fields simultaneously, σ has of course to be defined also on the non-diagonal sectors. There is some ambiguity how to define these fields which consists in unitary similarity transformations which commute with the chiral observables in order not to spoil the primary commutation relations. These unitaries are given by a complex phase on each of the four simple sectors, and a unitary 2×2 matrix which mixes the two copies of $\mathcal{H}_{\frac{1}{16}\frac{1}{16}}$.

Our representation of choice is the following.

$$\sigma = u \otimes (a \times a + b^+ \times b^+) + v \otimes (a \times b^+ + b^+ \times a) + (\text{h.c.}), \tag{3.1a}$$

$$\mu = v \otimes (a \times a - b^+ \times b^+) + iu \otimes (a \times b^+ - b^+ \times a) + (\text{h.c.}). \tag{3.1b}$$

We introduced two orthogonal unit vectors u and v in $\mathbb{C}^2$ in order to distinguish the two copies of $\mathcal{H}_{\frac{1}{16}\frac{1}{16}}$ within $\mathcal{H}_{\frac{1}{16}\frac{1}{16}} \oplus \mathcal{H}_{\frac{1}{16}\frac{1}{16}} = \mathbb{C}^2 \otimes \mathcal{H}_{\frac{1}{16}\frac{1}{16}}$; they are multiplied like rectangular matrices. Obviously, σ preserves both $\mathcal{H}_{00} \oplus (u \otimes \mathcal{H}_{\frac{1}{16}\frac{1}{16}}) \oplus \mathcal{H}_{\frac{1}{2}\frac{1}{2}}$ and $\mathcal{H}_{0\frac{1}{2}} \oplus (v \otimes \mathcal{H}_{\frac{1}{16}\frac{1}{16}}) \oplus \mathcal{H}_{\frac{1}{2}0}$, while μ alternates between these spaces. Conversely, μ preserves the analogous spaces with u and v interchanged, while σ alternates between them. The following diagram summarizes the situation: σ interpolates according to the horizontal arrows, while μ interpolates according to the diagonal arrows:

$$\begin{array}{ccccc} (00) & \longleftrightarrow & u \otimes (\frac{1}{16}\frac{1}{16}) & \longleftrightarrow & (\frac{1}{2}\frac{1}{2}) \\ & \nwarrow\!\!\searrow & & \nwarrow\!\!\searrow & \\ (0\frac{1}{2}) & \longleftrightarrow & v \otimes (\frac{1}{16}\frac{1}{16}) & \longleftrightarrow & (\frac{1}{2}0) \end{array} \tag{3.2}$$

Every single term in (3.1) corresponds to an arrow in (3.2).

Our normalization is such that the two-point functions are $\langle \Omega, \sigma\sigma\Omega \rangle = \langle \Omega, \mu\mu\Omega \rangle = (\Delta_L \Delta_R)^{\frac{1}{8}}$ in terms of the left- and right-moving light-cone coordinates. All mixed four-point functions [5], e.g., $\langle \Omega, \mu\mu\sigma\sigma\Omega \rangle = (V_L V_R)^{\frac{1}{8}}(f_L f_R - g_L g_R)$, can be read off eqs. (2.7) and (3.1).

Each of the fields σ and μ satisfies local commutativity at space-like distance with itself, but they are not mutually local. Instead, they satisfy the "dual" commutation relations [2]

$$\sigma(x)\mu(y) = \pm\mu(y)\sigma(x) \quad \text{if} \quad (x-y)^2 < 0 \tag{3.3}$$

with the sign $\pm = \text{sign}\,(x^1 - y^1)$.

Let us consider now the Kadanoff-Ceva formulae (1.2). Evaluating the fields σ and μ at $x = 0$ means replacing the $\times$-product in (3.1) by the p-product $\odot$. The projections P_+ and P_- project onto the subspaces generated from the vacuum by an even resp. odd number of applications of the fields σ and μ, i.e., P_+ projects onto $\mathcal{H}_+ = \bigoplus_{s,t=0,\frac{1}{2}} \mathcal{H}_{st}$ while P_- projects onto $\mathcal{H}_- = \mathbb{C}^2 \otimes \mathcal{H}_{\frac{1}{16}\frac{1}{16}}$.

Thus inserting (3.1) into (1.2b) yields the basic result

$$\begin{aligned} E_{+\frac{1}{2}} = {} & u \otimes (a \odot a + b^+ \odot b^+) - iu^+ \otimes (a^+ \odot b - b \odot a^+) \\ & + v \otimes (a \odot b^+ + b^+ \odot a) + v^+ \otimes (a^+ \odot a^+ - b \odot b), \end{aligned} \tag{3.4}$$

and $E_{-\frac{1}{2}} = E^*_{+\frac{1}{2}}$.

Like all formulae below, eq. (3.4) can easily be checked in all mixed four-point functions of $E_{\pm\frac{1}{2}}$, using eqs. (2.3) and (2.7). The important point is, however, its validity on the entire joint cyclic Hilbert space of these fields.

The charge conjugation invariance $E_{+\frac{1}{2}} \leftrightarrow E_{-\frac{1}{2}}$ is manifest in this representation. It is implemented by the involutive unitary operator $\Pi \cdot (P_{00} + iP_{0\frac{1}{2}} - iP_{\frac{1}{2}0} - P_{\frac{1}{2}\frac{1}{2}} + M \otimes P_{\frac{1}{16}\frac{1}{16}})$ involving the discrete "flip" operator $\Pi : \mathcal{H}_{st} \to \mathcal{H}_{ts}$ which interchanges the two tensor factors, and a matrix M in the multiplicity space of $\mathcal{H}_{\frac{1}{16}\frac{1}{16}}$ which interchanges u and v.

Eq. (3.4) shows a $\mathbb{Z}_4$ charge structure as follows. We redisplay the diagram (3.2) with only the arrows pertaining to $E_{+\frac{1}{2}}$:

$$\begin{array}{ccccc} (00) & \longrightarrow & u \otimes (\frac{1}{16}\frac{1}{16}) & \longleftarrow & (\frac{1}{2}\frac{1}{2}) \\ & \nwarrow\!\!\searrow & & \swarrow\!\!\nearrow & \\ (0\frac{1}{2}) & \longrightarrow & v \otimes (\frac{1}{16}\frac{1}{16}) & \longleftarrow & (\frac{1}{2}0) \end{array} \tag{3.5}$$

It becomes apparent that the subspaces H_i given by

$$\begin{aligned} H_0 &= \mathcal{H}_{00} \oplus \mathcal{H}_{\frac{1}{2}\frac{1}{2}} & \qquad H_1 &= u \otimes \mathcal{H}_{\frac{1}{16}\frac{1}{16}} \\ H_2 &= \mathcal{H}_{0\frac{1}{2}} \oplus \mathcal{H}_{\frac{1}{2}0} & \qquad H_3 &= v \otimes \mathcal{H}_{\frac{1}{16}\frac{1}{16}} \end{aligned} \tag{3.6}$$

are the subspaces of charge $(i \bmod 4)\frac{1}{2}$. The four terms in eq. (3.4) correspond in turn to $P_{i+1}E_{+\frac{1}{2}}P_i = E_{+\frac{1}{2}}P_i$, $i = 0, 1, 2, 3$.

In terms of these subspaces we have the interpolation diagrams

$$\sigma : \begin{cases} H_0 \leftrightarrow H_1 \\ \\ H_3 \leftrightarrow H_2 \end{cases} \qquad \mu : \begin{cases} H_0 & H_1 \\ \updownarrow & \updownarrow \\ H_3 & H_2 \end{cases} \qquad E_{\frac{1}{2}} : \begin{cases} H_0 \to H_1 \\ \uparrow \quad \downarrow \\ H_3 \leftarrow H_2 \end{cases} \tag{3.7}$$

The first two diagrams pertain to the 1+1-dimensional theory. The $\mathbb{Z}_4$ charge equals the Kadanoff-Ceva Γ-charge which is conserved "off the line" only mod 4. In this picture, the real chiral Ising fermions ψ_L and ψ_R arising in the operator product expansions of $\mu\sigma$ and $\sigma\mu$ carry two units of $\mathbb{Z}_4$ charge.

4. Applications

4.1. The doubled Ising model

In contrast to the order and disorder fields $\varphi = \sigma, \mu$ with their dual commutation relations (3.3), the "doubled" fields $\varphi_D = \varphi \odot \varphi$ satisfy local commutativity also with each other. The asserted identifications (1.7) which we are going to establish are equivalent to

$$\mu_D + i\sigma_D \cong \sqrt{2} \cdot E_{-\frac{1}{2}} \times E_{\frac{1}{2}} \tag{4.1}$$

where the fields on the right hand side are also local fields made from anyonic chiral constituents [16].

We compute $E_{-\frac{1}{2}} \times E_{\frac{1}{2}}$ from the basic result (3.4). Its cyclic Hilbert space involves only the conjugate charge sectors $H_{-i} \otimes H_i$. For the vacuum expectation values, we may therefore omit all terms which are defined on $H_j \otimes H_i$, $j \neq -i$. The remaining terms are

$$\begin{aligned} E_{-\frac{1}{2}} \times E_{\frac{1}{2}} \cong\ & Y \otimes (aa - b^+b^+) \times (aa + b^+b^+) \\ & + iX \otimes (ab^+ - b^+a) \times (ab^+ + b^+a) \\ & + X^+ \otimes (a^+a^+ + bb) \times (a^+a^+ - bb) \\ & - iY^+ \otimes (a^+b + ba^+) \times (a^+b - ba^+) \end{aligned} \tag{4.2}$$

on the cyclic subspace. We omitted the $\odot$ symbols, and put $X = u \otimes v$, $Y = v \otimes u$.

On the other hand, we compute the doubled fields from eqs. (3.1). Each of the fields σ and μ, according to the scheme (3.7), takes each of the subspaces H_i into a specific H_j. Hence repeated application of the doubled fields connects the vacuum sector $H_0 \otimes H_0$ only with the diagonal sectors $H_i \otimes H_i$, and the latter again exhaust the joint cyclic Hilbert space. Thus we obtain

$$\begin{aligned} \mu_D + i\sigma_D \cong\ & (V + iU) \otimes (aa \times aa + b^+b^+ \times b^+b^+ + iaa \times b^+b^+ + ib^+b^+ \times aa) \\ & - (V - iU) \otimes (ab^+ \times ab^+ + b^+a \times b^+a - iab^+ \times b^+a - ib^+a \times ab^+) \\ & + (V^+ + iU^+) \otimes (a^+a^+ \times a^+a^+ + bb \times bb + ia^+a^+ \times bb + ibb \times a^+a^+) \\ & - (V^+ - iU^+) \otimes (a^+b \times a^+b + ba^+ \times ba^+ - ia^+b \times ba^+ - iba^+ \times a^+b) \end{aligned} \tag{4.3}$$

on the joint cyclic subspace. We put $U = u \otimes u$, $V = v \otimes v$ and arranged the factors as $(A \times B) \odot (C \times D) \cong (A \odot C) \times (B \odot D)$. Note that due to the different factor ordering, the individual subspaces $H_j \otimes H_i$ in eq. (4.2) are different from $H_j \otimes H_i$ in eq. (4.3).

Eq. (4.3) is transformed into $\sqrt{2}$ times eq. (4.2) by conjugation with a unitary operator which preserves the vacuum vector. The operator which does the job equals (in an obvious notation) $(P_{00\times00} - P_{\frac{1}{2}\frac{1}{2}\times\frac{1}{2}\frac{1}{2}} + P_{0\frac{1}{2}\times\frac{1}{2}0} - P_{\frac{1}{2}0\times0\frac{1}{2}}) + i(P_{00\times\frac{1}{2}\frac{1}{2}} - P_{\frac{1}{2}\frac{1}{2}\times00} + P_{0\frac{1}{2}\times0\frac{1}{2}} - P_{\frac{1}{2}0\times\frac{1}{2}0}) + M \otimes P_{\frac{1}{16}\frac{1}{16}\times\frac{1}{16}\frac{1}{16}}$, with a unitary matrix M in the multiplicity space of $\mathcal{H}_{\frac{1}{16}\frac{1}{16}} \otimes \mathcal{H}_{\frac{1}{16}\frac{1}{16}}$ taking $V + iU \mapsto \sqrt{2}\,Y$ and $V - iU \mapsto \sqrt{2}\,X$. This similarity transformation commutes with the chiral observables of the doubled Ising field theory (two stress-energy tensors with central charge $c = \frac{1}{2}$ on each light-cone), and therefore preserves the structure of the full model.

It might be objected that one obtains apparently different results when one restricts the doubled fields according to eqs. (1.7) to the line $x = 0$, and when one doubles the restricted fields according to eqs. (1.2a); namely

$$\begin{aligned} \sigma_D|_{x=0} &= \tfrac{1}{\sqrt{2}\,i}(E_\alpha - E_{-\alpha}) \\ \mu_D|_{x=0} &= \tfrac{1}{\sqrt{2}}(E_\alpha + E_{-\alpha}) \end{aligned} \qquad \text{vs.} \qquad \begin{aligned} (\sigma|_{x=0})_D &= E_\alpha P_+ + E_{-\alpha} P_- \\ (\mu|_{x=0})_D &= E_{-\alpha} P_+ + E_\alpha P_- \end{aligned} \tag{4.4}$$

where $P_\pm$ now project on the spaces of even resp. odd charge in units of $\alpha = \frac{1}{\sqrt{2}}$. In fact, the latter form can be transformed into the former by a chain of similarity transformations

exploiting SU(2) invariance. It is implemented by the unitary operator $\mathcal{V}\mathrm{e}^{-i\pi(\alpha Q)^2}\mathcal{U}\mathrm{e}^{i\pi(\alpha Q)^2}$ where first $\mathrm{e}^{i\pi(\alpha Q)^2}$ transforms the abelian vertex operators into the pseudoreal SU(2) doublet $\phi_\pm$ along with the SU(2)-invariant Casimir operator $C = P_+ - P_-$ (cf. Sect. 2.3), next the SU(2) transformation $\mathcal{U}$ takes $\phi_\pm \mapsto \sqrt{\frac{\mp i}{2}}(\phi_+ \mp \phi_-)$, then $\mathrm{e}^{-i\pi(\alpha Q)^2}$ reproduces the vertex operators, and finally $\mathcal{V} = P_+ + \sqrt{-i}P_-$ (which commutes with the other three operators) removes the charge dependence. We refrain from an explicit demonstration, since a similar transformation will be discussed in detail later (Sect. 4.4).

4.2. The SU(2) primary fields at level 2

We consider the GKO coset $(\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_1)/\mathrm{SU}(2)_2$. The coset algebra is the chiral stress-energy tensor with $c = \frac{1}{2}$ whose primary exchange fields were described in Sect. 2.2.

The level 2 current algebra has also three sectors $\mathcal{K}_I$ distinguished by the isospin $I = 0, \frac{1}{2}, 1$ of the ground states of the conformal Hamiltonian. The primary exchange fields of isospin $I = \frac{1}{2}$ are four SU(2) doublets which make transitions in analogy to eqs. (2.6):

$$A_i : \mathcal{K}_0 \to \mathcal{K}_{\frac{1}{2}} \quad \text{and} \quad A_i^+ : \mathcal{K}_{\frac{1}{2}} \to \mathcal{K}_0 \tag{4.5a}$$

$$B_i : \mathcal{K}_{\frac{1}{2}} \to \mathcal{K}_1 \quad \text{and} \quad B_i^+ : \mathcal{K}_1 \to \mathcal{K}_{\frac{1}{2}} \tag{4.5b}$$

The fields A_i^+ and B_i^+ are not the adjoints of A_i and B_i, since the latter transform in the conjugate representation. We may, however, choose the normalizations such that

$$X_i^* = -\varepsilon_{ij} X_j^+, \qquad (X_i^+)^* = \varepsilon_{ij} X_j \qquad (X = A, B, \quad i, j = \pm). \tag{4.6}$$

Like the elementary fields, the chiral exchange fields of the SU(2) level 2 current algebra satisfy braid group commutation relations given by another non-abelian Hecke type representation.

The two-point function is $\langle \Omega, A_i^+ A_j \Omega \rangle = \varepsilon_{ij} \Delta^{\frac{3}{8}}$. The only non-vanishing four-point functions of these exchange fields have the structure $\langle \Omega, A_i^+ A_j A_k^+ A_l \Omega \rangle = V^{\frac{3}{8}} \cdot F_{ijkl}(x)$ and $\langle \Omega, A_i^+ B_j^+ B_k A_l \Omega \rangle = V^{\frac{3}{8}} \cdot G_{ijkl}(x)$, with V and x as in Sect. 2.2. The SU(2) tensors F and G are the two independent "conformal block" solutions to the Knizhnik-Zamolodchikov differential equation

$$\left[(k+2)\partial_x - \frac{C_{12}}{x} + \frac{C_{23}}{1-x} \right] W(x) = 0 \tag{4.7}$$

where $k = 2$ is the level and $C_{mn} = (\vec{I}_m + \vec{I}_n)^2$ are the quadratic SU(2) Casimir operators on the product representations due to the m-th and n-th field entry. The solutions

$$\begin{aligned} F_{ijkl} &= \ \ \varepsilon_{ij}\varepsilon_{kl} \cdot f(f^2 + g^2) \ - \ \varepsilon_{ik}\varepsilon_{jl} \cdot 2fg^2 \\ G_{ijkl} &= -\varepsilon_{ij}\varepsilon_{kl} \cdot g(f^2 + g^2) \ + \ \varepsilon_{ik}\varepsilon_{jl} \cdot 2f^2 g \end{aligned} \tag{4.8}$$

involve the same functions $f(x)$ and $g(x)$ as in eq. (2.9).

Our intention is to give an explanation for the cubic structure of these vacuum expectation values in terms of the v.e.v.'s of the elementary fields. For this purpose, we recall how the level 2 primary exchange fields can be obtained by the GKO coset construction from the level 1 primaries $\phi_\pm \otimes \mathbb{1}$.

According to the general discussion in Sect. 2.4., the branching rules (in particular, the SU(2) quantum numbers and scaling dimensions) determine the possible terms in the coset factorization of $\phi_\pm \otimes \mathbb{1}$. The result is

$$\phi_i \otimes \mathbb{1} = a \odot A_i - ia^+ \odot A_i^+ + b \odot B_i + ib^+ \odot B_i^+ . \tag{4.9}$$

Here the absolute coefficients 1 in front of the first and third term are chosen by a normalization freedom, and the relative coefficients $\pm i$ in front of the adjoint terms are determined by the pseudoreality property (2.14) of ϕ_i along with eq. (4.6). Eq. (4.9) can be cross-checked by inserting the four-point functions (2.7) and (4.8) and comparing with the four-point functions of $\phi_\pm$ which are computed with the help of eqs. (2.13) and (2.3).

Now comes the crucial step. According to eq. (2.13), $\phi_\pm$ have a representation in terms of vertex operators of charge $\pm\alpha$ with $\alpha^2 = \frac{1}{2}$. According to eq. (2.5) with the choice $\alpha = (\frac{1}{2}, \frac{1}{2})$, the latter can be written in the form $E_{\pm\frac{1}{2}} \odot E_{\pm\frac{1}{2}}$, and according to eq. (3.4), the vertex operators of charge $\pm\frac{1}{2}$ can be written as p-quadratic polynomials in the elementary exchange fields. The operator $\mathrm{e}^{i\pi\alpha Q}$ in (2.13) takes values i^p in the sectors of charge $p\alpha$, and the charge quantum numbers p mod 4 in the representation (3.4) have been described in eqs. (3.5) and (3.6). Since in the cyclic Hilbert space of $E_{\pm\frac{1}{2}} \odot E_{\pm\frac{1}{2}}$ only these "diagonal" charges occur, the mixed terms can be omitted from the p-square of (3.4). The result is

$$\begin{aligned} \phi_+ \otimes \mathbb{1} \cong\ & [u \otimes (a \odot a + b^+ \odot b^+)]^{\odot 2} + i[-iu^+ \otimes (a^+ \odot b - b \odot a^+)]^{\odot 2} \\ & -[v \otimes (a \odot b^+ + b^+ \odot a)]^{\odot 2} - i[v^+ \otimes (a^+ \odot a^+ - b \odot b)]^{\odot 2} . \end{aligned} \tag{4.10}$$

We reorder this expression by separating the first p-factor (and at the same time make it more lucid by omitting the $\otimes$ and $\odot$ signs):

$$\begin{aligned} \phi_+ \otimes \mathbb{1} \cong\ & a \odot [Ua(aa + b^+b^+) - Vb^+(ab^+ + b^+a)] \\ & +a^+ \odot [-iU^+b(a^+b - ba^+) - iV^+a^+(a^+a^+ - bb)] \\ & +b \odot [iU^+a^+(a^+b - ba^+) + iV^+b(a^+a^+ - bb)] \\ & +b^+ \odot [Ub^+(aa + b^+b^+) - Va(ab^+ + b^+a)] \end{aligned} \tag{4.11}$$

where $U = u \otimes u$, $V = v \otimes v$ are just another pair of orthogonal unit vectors pointing to two of the four copies of $\mathcal{H}^{\otimes 2}_{\frac{1}{16}\frac{1}{16}}$ in the representation space of $E_{\pm\frac{1}{2}} \odot E_{\pm\frac{1}{2}}$.

Of course, also the embedded field $1 \otimes \phi_\pm$ has an equivalent representation in terms of exchange fields of the coset and level 2 chiral observables. It does, however, not live on the same cyclic Hilbert space $(\mathcal{H}_0 \otimes \mathcal{K}_0) \oplus (\mathcal{H}_{\frac{1}{16}} \otimes \mathcal{K}_{\frac{1}{2}}) \oplus (\mathcal{H}_{\frac{1}{2}} \otimes \mathcal{K}_1)$ as $\phi_\pm \otimes 1$. E.g., the field $\phi_+ \odot \phi_+$ belongs to a triplet of the diagonal SU(2) and has scaling dimension $\frac{1}{2}$. These quantum numbers are found in the sectors $\mathcal{H}_0 \otimes \mathcal{K}_1$ and $\mathcal{H}_{\frac{1}{2}} \otimes \mathcal{K}_0$. In fact, the joint cyclic

Hilbert space contains apart from the mentioned five sectors also a second copy of $\mathcal{H}_{\frac{1}{16}} \otimes \mathcal{K}_{\frac{1}{2}}$, where the two embedded doublets have similar interpolation properties as σ and μ in (3.2). We refrain from computing these formulae.

By comparing (4.11) with (4.9), we read off the p-cubic representations of the level 2 primary exchange doublet:

$$\begin{aligned} A_+ &= Ua(aa + b^+b^+) - Vb^+(ab^+ + b^+a) \\ A_+^+ &= U^+b(a^+b - ba^+) + V^+a^+(a^+a^+ - bb) \\ B_+ &= iU^+a^+(a^+b - ba^+) + iV^+b(a^+a^+ - bb) \\ B_+^+ &= -iUb^+(aa + b^+b^+) + iVa(ab^+ + b^+a) \end{aligned} \tag{4.12}$$

while the $-$ components are determined by (4.6).

This interpretation of (4.11) means that we choose the embedding in such a way that the coset stress-energy tensor (with $c = \frac{1}{2}$) acts on the Hilbert spaces belonging to the first p-factor. Then the Sugawara stress-energy tensor of the level 2 current algebra (with $c = \frac{3}{2}$) acts on the triple tensor products of Hilbert spaces belonging to the remaining three p-factors in the form of three commuting $c = \frac{1}{2}$ "stress-energy tensors". The latter, of course, do not individually generate the conformal transformations of any level 2 fields, nor are the level 2 currents defined on the individual factor Hilbert spaces; the true stress-energy tensor with $c = \frac{3}{2}$ is the sum of the three $c = \frac{1}{2}$ fields.

A non-trivial cross-check of our result (4.12) is that it reproduces the four-point functions (4.8) (using eq. (2.7)).

Finally, for later use, we display another representation which amounts to viewing the last two p-factors in (4.12) together with the multiplicity vectors as contributions to the abelian vertex operators from which they originate (cf. eq. (3.4)). In this form,

$$\begin{aligned} A_+ &= a \odot E_+P_0 - b^+ \odot E_+P_2 \\ A_+^+ &= ib \odot E_+P_1 + a^+ \odot E_+P_3 \\ B_+ &= -a^+ \odot E_+P_1 + ib \odot E_+P_3 \\ B_+^+ &= -ib^+ \odot E_+P_0 + ia \odot E_+P_2 \end{aligned} \tag{4.13}$$

where $E_+ \equiv E_{+\frac{1}{2}}$ and P_i project onto the subspaces (3.6). We observe that each of the three sectors between which the elementary fields in these expressions interpolate arises in two orthogonal copies, distinguished by different abelian charge projections in the second p-factor. Again, the $-$ components are determined by (4.6).

One might proceed and compute also the primary isospin 1 triplet of real fermions, e.g., from the operator product expansion of the fields (4.12). Inserting the o.p.e.'s for the elementary fields, it is evident that one will obtain the three permutations of $\psi \odot \mathbb{1} \odot \mathbb{1}$ where ψ is the real Ising chiral fermion, along with some Klein transformations which make the three components anti-commute. Since nothing exciting happens, we do not present the explicit calculations.

4.3. The 1+1-dimensional Wess-Zumino-Witten (WZW) field at level 2

The 1+1-dimensional fundamental field of the WZW model is a 2×2 matrix with $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry (by left and right matrix multiplication). It is a local field which carries the quantum numbers of the non-local primary exchange doublet with respect to both chiral current algebras.

At level 2, it can be represented in the form [22]

$$g_{ij} = (A_i \times A_k + A_i^+ \times A_k^+ + B_i \times B_k + B_i^+ \times B_k^+)\ \varepsilon_{kj} \tag{4.14}$$

where the ε is necessary because the right action of SU(2) is in the conjugate representation. This form parallels the representation (3.1a) of the order parameter on its own cyclic subspace (we do not consider the analogue of the disorder parameter, which would enlarge the joint cyclic Hilbert space and require a second copy of $\mathcal{K}_{\frac{1}{2}\frac{1}{2}}$ as in eqs. (3.1b).). Due to eq. (4.2), one has

$$g_{ij}^* = \varepsilon_{ik}\varepsilon_{jl}\ g_{kl}. \tag{4.15}$$

We insert the expressions (4.13) for the primary exchange fields into the definition (4.14). As before, we first consider the interpolation scheme within the joint cyclic subspace. Every component g_{ij} is a sum of contributions of the form $X_{ij}^{kl} \odot (E \times E)P_{kl}$ where $(E \times E)P_{kl}$ is the $\times$-product of the abelian vertex operators from eq. (4.13) on the charge sector $H_k \otimes H_l$ according to eq. (3.7), the accompanying $\times$-product of the elementary fields from eq. (4.13) being called X_{ij}^{kl}. As it turns out, on the cyclic Hilbert space every such operator X_{++}^{kl} (resp. X_{+-}^{kl}) can be identified with a contribution to σ (resp. μ) as in eqs. (3.1), restricted to a Γ-charge sector H_i according to eq. (3.6). E.g., the component g_{+-} involves the combination $X_{00} = a \times a - b^+ \times b^+$ which appears in the representation (3.1a) of μ on H_0. For this identification, no multiplicity vectors u, v are needed to keep the two copies H_1 and H_3 of $\mathcal{H}_{\frac{1}{16}\frac{1}{16}}$ apart, since the accompanying abelian charge projection operators P_{kl} already have the same effect.

The result of these considerations is the following interpolation diagram in which the horizontal arrows represent the action of the component g_{+-} which involves μ and $E_{+\frac{1}{2}} \times E_{+\frac{1}{2}}$ while the diagonal arrows represent g_{++} which involves σ and $E_{+\frac{1}{2}} \times E_{-\frac{1}{2}}$ (the other two matrix elements interpolate according to the reversed arrows according to eq. (4.15)):

$$\begin{array}{ccccccccc} 0 \odot 00 & \longrightarrow & 3 \odot 11 & \longrightarrow & 0 \odot 22 & \longrightarrow & 3 \odot 33 & \longrightarrow & 0 \odot 00 \\ & \searrow\!\!\nearrow & & \searrow\!\!\nearrow & & \searrow\!\!\nearrow & & \searrow\!\!\nearrow & \\ 2 \odot 02 & \longrightarrow & 1 \odot 13 & \longrightarrow & 2 \odot 20 & \longrightarrow & 1 \odot 31 & \longrightarrow & 2 \odot 02 \end{array} \tag{4.16}$$

The vertices $i \odot kl$ of this diagram are labelled by the Γ-charge sectors H_i between which σ and μ interpolate, and the abelian charge sectors $H_k \otimes H_l$ between which the vertex operators $E_{\pm\frac{1}{2}}$ interpolate, according to eqs. (3.7).

The explicit computation of the coefficients yields

$$\begin{aligned} g_{++} &\cong \sigma \odot (E_{+\frac{1}{2}} \times E_{-\frac{1}{2}})(-P_{00} + iP_{11} - iP_{22} - P_{33} + iP_{02} - iP_{13} + P_{20} + P_{31}) \\ g_{+-} &\cong \mu \odot (E_{+\frac{1}{2}} \times E_{+\frac{1}{2}})(P_{00} + P_{11} - P_{22} + P_{33} + iP_{02} + P_{13} - iP_{20} - P_{31}). \end{aligned} \tag{4.17}$$

The charge dependences in eqs. (4.17) can be removed to a large extent, but not entirely, by a unitary similarity transformation of the form $\mathbb{1} \odot \sum_{ij} \omega_{ij} P_{ij}$ with suitable complex phases ω_{ij} on the charge sectors $H_i \otimes H_j$. For later use, we display two convenient simplifications of eqs. (4.17):

$$\begin{aligned} g_{++} &\cong \sigma \odot (E_{+\frac{1}{2}} \times E_{-\frac{1}{2}}) \\ g_{+-} &\cong \mu \odot (E_{+\frac{1}{2}} \times E_{+\frac{1}{2}} \mathrm{e}^{-i\pi Q}) \end{aligned} \tag{4.18a}$$

or

$$\begin{aligned} g_{++} &\cong \sigma \odot (E_{+\frac{1}{2}} \mathrm{e}^{-i\pi Q} \times E_{-\frac{1}{2}} \mathrm{e}^{-i\pi Q}) \\ g_{+-} &\cong \mu \odot (E_{+\frac{1}{2}} \mathrm{e}^{-i\pi Q} \times E_{+\frac{1}{2}}) \end{aligned} \tag{4.18b}$$

The other matrix elements are given by eq. (4.15), i.e., $g_{--} = g_{++}^*$ and $g_{-+} = -g_{+-}^*$.

It is amusing to observe how the Klein transformations $\mathrm{e}^{-i\pi Q}$ in these expressions "mimic" the same dual commutation relations for the abelian vertex operator fields as for the order and disorder fields (eq. (3.3)), and hence serve to cancel the corresponding signs such that the p-products are indeed local. It is also possible to transfer the Klein transformation in eq. (4.18) from the vertex operators to the order and disorder fields, changing their dual commutation relations into anyonic ones.

In the representation (4.18a), the trace field $\mathrm{Tr}\, g$ looks particularly simple, and with eq. (1.7a) turns into

$$\mathrm{Tr}\, g \cong \sqrt{2}\, (\sigma \odot \mu \odot \mu) \cong \sqrt{2}\, \sigma \odot \sigma \odot \sigma. \tag{4.19}$$

The latter equivalence holds since the cosine $\mu \odot \mu$ differs from the sine $\sigma \odot \sigma$ by a U(1) transformation, cf. Sect. 2.1.

On the line $x = 0$, the fields g_{ij} can be represented as abelian vertex operators, e.g., inserting (1.2a) into (4.18a)

$$\begin{aligned} g_{++}|_{x=0} &\cong E_\beta (P_0 + P_2) + E_\gamma (P_1 + P_3) \\ g_{+-}|_{x=0} &\cong E_\delta (P_0 - P_2) + i E_\varepsilon (P_1 - P_3) \end{aligned} \tag{4.20}$$

where $\beta = \frac{1}{2}(1,1,-1)$, $\gamma = \frac{1}{2}(-1,1,-1)$, $\delta = \frac{1}{2}(-1,1,1)$, $\varepsilon = \frac{1}{2}(1,1,1)$ (with $\beta + \delta = \gamma + \varepsilon$), and P_i project on the subspaces of charge $(\gamma - \delta)Q = (i \bmod 4)\frac{1}{2}$, or equivalently on the subspaces of right-handed WZW isospin component $j_R^3 = (i \bmod 4)\frac{1}{2}$. These formulae are another example of "abelianization" of correlations along the line, just like Kadanoff's result for the Ising model. Note, however, the non-trivial angles between the charge vectors.

It is interesting to verify the unitarity of the matrix-valued field g_{ij} in these representations, and to compute the associated chiral SU(2) currents. Of course, one has to consider operator quadratic expressions of the form $g_{ij} g_{kj}^*$ or $g_{ij} \partial g_{kj}^*$ which have to be renormalized. For conformal fields of chiral scaling dimensions $(h, \bar{h})$ the proper way of doing so is a point-split regularization and renormalization by $\Delta_L^{-2h} \Delta_R^{-2\bar{h}}$ in order to single out the leading contribution to the operator product expansion. This indeed reproduces unitarity (using $\circ\!\!\circ \sigma\sigma \circ\!\!\circ = \circ\!\!\circ \mu\mu \circ\!\!\circ = \circ\!\!\circ E_{\pm\frac{1}{2}} E_{\mp\frac{1}{2}} \circ\!\!\circ = \mathbb{1}$; actually one finds that rather $g/\sqrt{2}$ is unitary since we have normalized every matrix element such that its two-point function $\langle \Omega, g_{ij}^* g_{ij} \Omega \rangle$ is

$(\Delta_L\Delta_R)^{\frac{3}{8}}$), and gives the level 2 currents $J^a_L = i\mathrm{Tr}\,\tau^a \,{}^\circ_\circ\partial_+ g g^*{}^\circ_\circ \cong J^a \times \mathbb{1}$ in the form

$$\begin{aligned} J^3 &\cong -\mathbb{1} \odot 2\partial\varphi \cong (\alpha_1+\alpha_2)\partial\varphi \\ J^+ &\cong (E_{+1}+E_{-1}) \odot E_{-1} \cong E_{\alpha_1}+E_{\alpha_2}, \end{aligned} \tag{4.21}$$

where $\alpha_1 = (1,-1)$, $\alpha_2 = (-1,-1)$, and similar for the right-moving currents $J^a_R = i\mathrm{Tr}\,\tau^a \,{}^\circ_\circ\partial_- g^* g^\circ_\circ \cong \mathbb{1} \times J^a$. In order to obtain this result, we have used the relevant leading contributions to the operator product expansions $E_{\pm\frac{1}{2}}(x)E_{\pm\frac{1}{2}}(y) \approx \Delta(x-y)^{-\frac{1}{4}}E_{\pm 1} + \cdots$ and $E_{\pm\frac{1}{2}}E_{\mp\frac{1}{2}} \approx \Delta^{\frac{1}{4}}(\mathbb{1} \pm \frac{i}{2}(x-y)\partial\varphi + \cdots)$, as well as $\sigma\sigma \approx \mu\mu \approx (\Delta_L\Delta_R)^{\frac{1}{8}}\mathbb{1} + \cdots$ and $\mu\sigma \approx i\sigma\mu \approx \sqrt{i}\Delta_L^{-\frac{3}{8}}\Delta_R^{\frac{1}{8}}\psi_L + \cdots$ where the left-moving real chiral Ising fermion ψ_L can in turn be represented in the form $\frac{1}{2}(E_{+1}+E_{-1}) \times \mathbb{1}$, up to another Klein transformation which makes ψ_L and ψ_R anti-commute. The form (4.21) complies with eq. (2.14) with two Frenkel-Kac representations at level 1, if one changes the basis to $\alpha_1 = (\sqrt{2},0)$, $\alpha_2 = (0,\sqrt{2})$.

4.4. The determinant field

Every p-quadratic expression in either the diagonal or the off-diagonal matrix elements g_{ij}, eqs. (4.18), can be expressed in terms of abelian vertex operators alone, since the doubled non-abelian factors μ and σ can. This observation generalizes the Schroer-Truong formulae (1.7).

A particularly simple representation of the sine form (1.11) can be obtained for the p-determinant field (which is invariant under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$)

$$\mathrm{Det}_\odot\, g := \tfrac{1}{2}\,\varepsilon_{ik}\varepsilon_{jl}\, g_{ij} \odot g_{kl} = \tfrac{1}{2}(g_{11}\odot g_{22} + g_{22}\odot g_{11} - g_{12}\odot g_{21} - g_{21}\odot g_{12})\,. \tag{4.22}$$

In order to compute the p-determinant, we take the liberty to use the representations (4.18a) and (4.18b) for the first and second p-factor, respectively. As for the abelian vertex operators in eqs. (4.18), we observe that they always combine in (4.22) in the form $E_{\pm\frac{1}{2}} \odot E_{\mp\frac{1}{2}} \cong E_{\pm\alpha}$ where $\alpha = \frac{1}{\sqrt{2}}$. Consequently, all charges in the cyclic Hilbert space are integer multiples of α, and the charge operators involved may be replaced by $\mathrm{e}^{i\pi Q} \odot \mathbb{1} \cong \mathrm{e}^{i\pi\alpha Q}$ and $\mathbb{1} \odot \mathrm{e}^{i\pi Q} \cong \mathrm{e}^{-i\pi\alpha Q} = \mathrm{e}^{i\pi\alpha Q}\cdot C$ where $C = \mathrm{e}^{2\pi i\alpha Q} = C^{-1}$ is the Casimir operator encountered previously in Sect. 2.3.

The crucial point now is that an enhanced symmetry emerges on the vertex operators of charge $\pm\alpha$ which was not present on the vertex operators of charge $\pm\frac{1}{2}$ in the representation (4.13), i.e., before taking the p-square (and which spoils the original (squared) level 2 affine chiral SU(2) symmetries generated by (4.21)). Namely, all terms contributing to (4.22) involve $E_{\pm\alpha}\mathrm{e}^{i\pi\alpha Q} \times E_{\pm\alpha}\mathrm{e}^{i\pi\alpha Q}$, i.e., the components of the pseudoreal SU(2) level 1 doublet $\phi_\pm$ as in eq. (2.13), possibly along with some Casimir operators. Explicitly, we have

$$\begin{aligned} \mathrm{Det}_\odot\, g &\cong \frac{1}{2}\big[\sigma_D \odot (\phi_+ C \times \phi_- C + \phi_- \times \phi_+) - i\mu_D \odot (\phi_+ C \times \phi_+ C + \phi_- \times \phi_-)\big] \\ &\cong \frac{1}{2\sqrt{2}i}\Big[(E_{-\frac{1}{2}} \times E_{+\frac{1}{2}}) \odot (\phi_+ + \phi_-) \times (\phi_+ + \phi_-) \\ &\qquad + (E_{+\frac{1}{2}} \times E_{-\frac{1}{2}}) \odot (\phi_+ - \phi_-) \times (\phi_+ - \phi_-)\Big]\,. \end{aligned} \tag{4.23}$$

For the second line, we observed that on the cyclic Hilbert space $C \otimes C$ equals $\mathbb{1}$, and we evaluated σ_D and μ_D according to eqs. (1.7).

The new level 1 chiral SU(2) symmetries permit to rotate the components $(\phi_+ \pm \phi_-)$ into $\sqrt{2}\phi_\mp$ in the left chiral factor and into $\pm\sqrt{2}\phi_\pm$ in the right chiral factor. This unitary similarity transformation turns the determinant field into

$$\mathrm{Det}_\odot\, g \cong \frac{1}{\sqrt{2}i}\left[(E_{-\frac{1}{2}} \times E_{+\frac{1}{2}}) \odot (\phi_- \times \phi_+) - (E_{+\frac{1}{2}} \times E_{-\frac{1}{2}}) \odot (\phi_+ \times \phi_-)\right]. \tag{4.24}$$

After this transformation, the cyclic Hilbert space is seen to carry opposite charges in the two chiral factors. We may therefore insert $\mathbb{1} = \mathbb{1} \odot (\mathrm{e}^{i\pi\alpha Q} \otimes \mathrm{e}^{i\pi\alpha Q})$ and re-substitute the vertex operators $E_{\pm\alpha}$ for the doublet $\phi_\pm$. This gives finally

$$\mathrm{Det}_\odot\, g \cong \frac{1}{\sqrt{2}i}(E_{-\beta} \times E_\beta - E_\beta \times E_{-\beta}) \tag{4.25}$$

with $\beta = \frac{1}{2}(1, \sqrt{2})$ or, after an O(2) rotation in charge space, $\beta = \frac{1}{2}\sqrt{3}$. The non-triviality of this formula can be esteemed upon the intricate exercise to test it only at the four-point level.

5. Conclusions

We have established a sort of "Quantum Field Lego" with chiral non-abelian exchange fields which can be recombined in a large variety of ways to produce several local and non-local, chiral and 1+1-dimensional Wightman fields belonging to different models. It is the operator formulation of some remarkable findings in the critical Ising model by Kadanoff and Ceva and by Schroer and Truong, and predicts, among other things, generalizations thereof in the context of a WZW (= generalized Thirring) model.

A prominent feature of the basic decomposition (3.4) is that it produces non-abelian exchange fields, satisfying commutation relations according to a non-abelian representation of the braid group and possessing a non-abelian fusion structure, by factorization of abelian vertex operators with anyonic commutation relations and additive charge structure.

We have seen along the way (Sects. 2.3 and 4.3) how the generators of the chiral symmetry can be explicitly computed in our operator representations and are found, of course, in complete agreement with well-known results.

Throughout the paper, we have mentioned only en passant the issue of braid group commutation relations. Let us just add that p-products go along with the tensor products of representations of the braid group; notably the basic result singles out an abelian subrepresentation within the square of the Hecke type representations associated with the elementary fields.

Let us consider the question whether one may expect parallel findings in other minimal or coset models. Since abelian vertex operators are the primary fields for stress-energy tensors

with integer central charge c, one may expect representations of abelian vertex operators as p-products of non-abelian ones whenever a stress-energy tensor with integer central charge occurs in the product of the chiral observables. In the case of the doubled Ising model, the latter is just the sum of the two stress-energy tensors with $c = \frac{1}{2}$, and in the case of the determinant field, it is the coset stress-energy tensor with $c = 1$ for the coupling of two level 2 current algebras into the level 4 current algebra. Admitting sufficiently high p-powers, any model with rational central charge presumably provides another set of non-abelian roots of abelian vertex operators. On the other hand, the "Kadanoff mechanism" of quadratic abelianization upon restriction to a time-like axis, which in the present model we consider to be the reason for the possibility of several independent recombinations of the same elementary fields, can only work when the two opposite chiral central charges add up to an integer. This property of course distinguishes the Ising model with $c = \frac{1}{2}$ and the SU(2) model at level 2 with $c = \frac{3}{2}$ studied in this paper from other minimal or WZW models.

An immediate further application, not treated in this note, is the construction of the primary field ϕ_{21} with dimension $\frac{7}{16}$ of the minimal model with $c = \frac{7}{10}$ which arises in the coset construction as the singlet contribution to the p-product of the SU(2) primary doublets at level 1 (eq. (2.13)) and at level 2 (eq. (4.12)), commuting with the level 3 currents.

Although the basic result involves chiral and non-local fields, several of our applications pertain to ordinary local Wightman fields in 1+1-dimensional Minkowski space-time.

We want to draw the attention to a point of general importance which is illustrated by eq. (4.25). By definition, the determinant field is p-quadratic in the four mutually local fields g_{ij}. On the other hand, the right hand side of eq. (4.25), by elementary identities using the representation (1.7), equals one half of

$$\sigma_D \odot \sigma_D \odot \sigma_D - \sigma_D \odot \mu_D \odot \mu_D - \mu_D \odot \sigma_D \odot \mu_D - \mu_D \odot \mu_D \odot \sigma_D \ . \tag{5.1}$$

(This formula would not be guessed from eqs. (4.18) according to which g_{ij} are not p-cubic in σ and μ.) The determinant field is thus at the same time p-cubic in local fields. Yet, a common sixth-order factorization into local fields cannot be perceived (at least not in terms of our elementary fields). This indicates, with due precaution, that something like a "prime decomposition" with respect to the p-product does not exist within the class of *local* Wightman fields.

We consider our work as an exercise with non-canonical fields from which several lessons on the general theory of Wightman fields can be drawn. We want to refute the possible impression that the construction of p-products is just an artificial trick to construct new Wightman fields without a proper physical meaning from old ones. On the contrary, all the product fields in this article were previously known and considered in their own model context. The new aspect is the passage between different models enabled by p-products.

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