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# INTEGRABILITY, COORDINATE SYSTEMS, AND SEPARATION OF VARIABLES IN PATH INTEGRALS

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ABSTRACT

In this contribution I summarize the achievements of separation of variables in integrable quantum systems from the point of view of path integrals. This includes the free motion on homogeneous spaces, and motion subject to a potential force, and I would like to propose systematic investigations of parametric coordinate systems on homogeneous spaces.

The problem of separation of variables in physical systems has been studied for a long time. First, the question arises, is a system separable, i.e., is it integrable (which is not the same)? Many efforts have been put into the investigation to construct separable systems. For the usual physical systems this means to consider potential or magnetic forces, breaking the homogeneity of the free motion, which still allow complete separation of variables in some coordinate system. In three-dimensional space this means that there must be at least three functionally independent constants of motion, representing the separation constants. If there are less than three constants of motion, in a quantum system they are called observables, the physical system is not entirely separable, a property which leads usually to some sort of chaoticity, e.g.[16].

If there are more than three observables in a systems, such a system is called superintegrable. Generally, in $d$ dimensions a system is called maximally superintegrable if there are $2d-1$ constants of motion. The best known examples are the isotropic oscillator and the Coulomb potential in spaces of constant curvature, respectively. This property has the consequence that separability in more that one coordinate system is possible. For instance, in three dimensions the oscillator has five constants of motion and is separable in eight (out of eleven) coordinate systems, and the Coulomb potential has also five constants of motion (including the famous Pauli-Lenz-Runge vector) and is separable in four coordinate systems.

Whereas for a particular physical system one is most concerned with separability, in homogeneous spaces one wants to know the total number of separable coordinate systems for the free Schrödinger (Helmholtz-) equation, or equivalently, how many inequivalent sets of commuting observables can be constructed.

Obviously, these two topics attracted a considerable amount of attention. For the most important two- and three-dimensional spaces of constant curvature, i.e., Euclidean space, the sphere and the hyperboloid, this question was settled by Olevskiĭ [28] in a classical article. Fischer et al. [6] studied spherical functions for generalized rotation groups. Later on, Kalnins and Miller studied in a series of articles higher dimensional spaces, for instance in [21] the $\mathrm{O}(2,2)$ hyperboloid, in [22] four-dimensional flat space (Euclidean and Minkowski space); systematically, these studied have been summarized by Kalnins [17] who gave an account how to construct separable coordinate systems in spaces of constant curvature in higher dimensions.

Concerning potential problems, Eisenhardt [5] investigated the kind of potentials where the Schrödinger equation is soluble, and Smorodinski, Winternitz et al. [26] started a systematic study of integrable potentials in three-dimensional Euclidean space.

From the point of view of path integrals one has to deal with two difficulties: First, the problem is question must be separated, second, expansion formulas must be found in order to complete the path integration. Whereas it is straightforward to formulate a separation formula for separable coordinate systems [8, 9, 14], the finding of expansion formulas is quite another matter. By an expansion formula we

mean, for instance, the expansion of plane waves into circular waves according to

$$\mathrm{e}^{z\cos\psi} = \sum_{\nu\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\nu\psi} I_\nu(z) \ , \tag{1}$$

which (and its higher dimensional generalization) lies at the very basis for path integrals in polar coordinates [15, 30]. Fortunately, more expansion formulas have been found and have given rise to set up a set of Path Integral Identities which may be called Basis Path Integrals. They are

1. *The Gaussian Path Integral.* This includes the path integral for the harmonic oscillator, as well as for the general quadratic Lagrangian.

2. *The Besselian Path Integral.* This is the path integral for the (generalized) radial harmonic oscillator.

3. *The Legendrian Path Integrals.* These two path integrals correspond to the path integral solution of the Pöschl-Teller, respectively modified Pöschl-Teller potential, respectively the Rosen-Morse oscillator [2, 7, 25].

In [8] I have tried to summarize the possibilities of achieving path integral solutions in homogeneous spaces according to the lines of the Basic Path Integrals, the results are listed in table 1. As it turned out, the above listed Basic Path Integrals are far from complete in doing the job. I could find several other expansion formulas and path integral identities connected with elliptic and spheroidal coordinates. These expansion formulas were used together with interbases expansion relations according to

$$|\mathbf{k}>= \int dE_{\mathbf{l}}\, C_{\mathbf{l},\mathbf{k}} |\mathbf{l}> \quad . \tag{2}$$

Here $|\mathbf{k}>$ stands for a basis of eigenfunctions of the relevant Hamiltonian in the coordinate space representation $\mathbf{k}$, and $\int dE_{\mathbf{l}}$ is the spectral-expansion with respect to the coordinate space representation $\mathbf{l}$ with coefficients $C_{\mathbf{l},\mathbf{k}}$ which can be discrete, continuous or both. For example, in two-dimensional Euclidean space we consider the expansion for an arbitrary $\alpha$ ($h = pd/2$, with $d$ the parameter in the elliptic coordinate system $(\mu,\nu)$, and $p$ the momentum, [27, p.185])

$$\begin{aligned}\exp\left[\mathrm{i}p(x\cos\alpha + y\sin\alpha)\right] &= 2\sum_{n=0}^{\infty} \mathrm{i}^n \mathrm{ce}_n(\alpha;h^2)\mathrm{Mc}_n^{(1)}(\mu;h)\mathrm{ce}_n(\nu;h^2) \\ &+2\sum_{n=1}^{\infty} \mathrm{i}^{-n}\mathrm{se}_n(\alpha;h^2)\mathrm{Ms}_n^{(1)}(\mu;h)\mathrm{se}_n(\nu;h^2) \ .\end{aligned} \tag{3}$$

$\mathrm{ce}_n, \mathrm{se}_n$, $\mathrm{Mc}_n^{(1)}, \mathrm{Ms}_n^{(1)}$ are periodic and non-periodic Mathieu functions, which are even and odd, respectively. Equation (3) represents the expansion of plane waves into elliptical waves, similarly as the expansion (1) into circular waves.

The corresponding attempt to review path integrals for separable potentials were performed in [11]–[14]. However, there the emphasize was concentrated on superintegrable systems in spaces of constant curvature, i.e., in two- and three-dimensional

Table 1: Path Integration on Homogeneous Spaces

| Homogeneous Space | Number of Coordinate Systems | Number of Systems in which Path Integration is possible |
|---|---|---|
| Two-Dimensional Pseudo-Euclidean Space | 10 | 10 |
| Three-Dimensional Pseudo-Euclidean Space | 54 | 32 |
| Four-Dimensional Pseudo-Euclidean Space | 261 | 182 |
| Two-Dimensional Euclidean Space | 4 | 4 |
| Three-Dimensional Euclidean Space | 11 | 9 |
| Four-Dimensional Euclidean Space | 42 | 35 |
| Two-Dimensional Sphere | 2 | 2 |
| Three-Dimensional Sphere | 6 | 5 |
| Two-Dimensional Pseudosphere | 9 | 9 |
| Three-Dimensional Pseudosphere | 34 | 24 |
| O(2, 2)-Hyperboloid | 72 | 33 |

Euclidean space and on the two- and three-dimensional sphere and hyperboloid. Finally, in [9] a Table of Path Integrals will be accomplished which attempts to summarize the achievements of solving Feynman path integrals in general, i.e., this will include not only the numerous applications of the Basic Path Integrals, like general quadratic Lagrangians with electric and magnetic fields, Coulomb potentials, monopole systems and path integral formulations for group spaces, but will also contain path integral formulations for boundary conditions, point interactions, coherent states, fermions, supersymmetric quantum mechanics, and some field theory formulæ, respectively the generating functionals.

All the known solutions of the Basic Path Integrals are related to the quantum motion on the $SU(2)$–sphere and SU(1, 1)–hyperboloid, respectively their symmetry group [20]. The quantum motion can have a discrete, or a continuous spectrum, or

both, and the relevant spectrum we need emerging from the spectrum of the group manifold SU(1,1) is of the form (by $m$ I denote the mass of a test particle, $p > 0$ is its momentum)

$$E_{\sigma,j_0} = -\frac{\hbar^2}{2m}[j_0^2 + \sigma(\sigma+2)] \ , \qquad \begin{array}{l} j_0 = 0, \sigma = -1 + ip \ , \\ j_0 = 2n \ (n \in \mathbb{N}), \sigma = -1 \ , \end{array} \tag{4}$$

for the continuous, respectively discrete spectrum. In this respect most of the usual solvable problems, free motion and potential problems, are covered within this scheme, including the four Basic Path Integrals.

Looking at the number of coordinate system representations on the SU(1,1)–hyperboloid, respectively the O(2,2)–hyperboloid [21] one realizes that all the known solutions in terms of cartesian, spherical or parabolic coordinates come from just a few special coordinate space representations of the corresponding matrix elements of the group. This means that the power of the majority of the remaining coordinate space representations has not been exploited yet, with the exceptions of elliptic and spheroidal coordinates in flat space [8], and elliptic and elliptic–cylindrical coordinates on the sphere [10]. The main difficulty one encounters in the investigation of the more complicated, i.e., parametric coordinate systems, is that very little seems to be known about the corresponding theory of special functions in terms of these coordinates. These special functions are usually defined by means of recurrence relations. These recurrence relations usually cannot be resolved in terms of well-defined power series as it is the case for the (confluent) hypergeometric functions.

Concerning path integrals, one is interested in expansion theorems like (1) or (3). The former corresponds to an expansion connecting two solutions of the Schrödinger equation which contains no free parameter, the second connecting two solutions of the Schrödinger equation which contains one parameter, which defines the semi-axis of the ellipse. Usually, such expansion theorems can be obtained by considering the overlap functions between the wave-functions of two coordinate systems, i.e., one considers an interbases expansion theorem (2). A considerable effort has been done to investigate such interbases expansion for potential problems connecting the well-known spherical and parabolic coordinate systems, e.g. [19] (for a more comprehensive bibliography, e.g. [8, 14]), because the corresponding solutions of the Schrödinger equation take on a well-known form. The difficulties arise if one considers the relevant expansions of say spherical coordinates, and a parametric coordinate system, say, spheroidal coordinates. Here it is often necessary to construct the solutions of the Schrödinger equation in terms of the parametric coordinate systems by means of the interbases expansion coefficients from an already known solution. In this procedure one uses the corresponding observables of the parametric coordinate system, and this approach guarantees at the same time that the so constructed solutions are properly normalized; e.g., the result of [10], where this issue was addressed for the spherical and cylindrical coordinate systems on the sphere $S^{(3)}$ to construct the solution of the Schrödinger equation of the two one-parametric ellipso-cylindrical coordinate systems.

The question of taking into account two-parametric coordinate systems is even more difficult, and in comparison to one-parametric systems even less is known. For instance, the investigations of the harmonic oscillator in ellipsoidal coordinates [18], and the quantum motion on the three-dimensional sphere in terms of ellipsoidal coordinates [1].

The notion of the so-called "hidden symmetry" or "dynamical symmetry" in an ordinary potential problem stems from the following observation: One considers the quantum motion on a group manifold, say on the two-dimensional hyperboloid $\Lambda^{(2)}$: $\mathbf{u}^2 = u_0^2 - u_1^2 - u_2^2 = 1$. There exists nine coordinate systems on $\Lambda^{(2)}$ which separates the Schrödinger, respectively the Helmholtz equation. Choosing one coordinate system, say the spherical coordinate system, and separating off the circular variable $\varphi \in [0, 2\pi)$ one is left with the Schrödinger equation for the potential problem $V(\tau) = \frac{\hbar^2}{2m}(l^2 - 1/4)/\sinh^2\tau$, $\tau > 0$, $(l \in \mathbb{Z})$. Taking hyperbolic coordinates, and separating off the continuous variable $\tau_2 \in \mathbb{R}$ one is left with the Schrödinger equation for the potential $V(\tau_1) = \frac{\hbar^2}{2m}(k^2 - 1/4)/\cosh^2\tau_2$, $\tau_2 \in \mathbb{R}$, $(k \in \mathbb{R})$. Taking horicyclic coordinates and separating off the continuous variable $x \in \mathbb{R}$ one is left with the Schrödinger equation for the Liouville potential $V(\tau) = \frac{\hbar^2}{2m}(k^2 - 1/4)\mathrm{e}^{2\varrho}$, $\varrho \in \mathbb{R}$, $(k \in \mathbb{R})$. Therefore we see that the dynamical symmetry "hidden" in a well-known potential problem corresponds in each case to the same quantum motion on the hyperboloid, however realized in different coordinate space representations, and the symmetry is the rotational symmetry of the hyperboloid $\Lambda^{(2)} \subset \mathrm{SO}(2,1)/\mathrm{O}(2)$. Similar, the Pöschl-Teller potential is related to the quantum motion on SU(2), and the modified Pöschl-Teller potential is related to SU(1,1). Actually, every path integral which is solvable can be reformulated in terms of a quadratic–form–Hamiltonian, respectively Lagrangian, i.e., by means of a "hidden" or "dynamical" symmetry. [1] This is the essence of the Duistermaat-Heckman Theorem [3]. In these coordinate system representations for generalized rotation groups (no free parameters) one obtains after separating off the "trivial" motion of subgroups coordinates *one dimensional potential problems.*

On the other hand, one-parametric coordinate system produce after separating off the subgroup coordinate *two-dimensional potentials.* For instance, in the prolate-spheroidal systems in $\mathbb{R}^3$ one obtains after separating off the circular variable $\varphi \in [0, 2\pi)$ the potential ([8], $l \in \mathbb{Z}, k \in \mathbb{R}, \mu > 0, \nu \in (0, \pi), d > 0$)

$$V(\mu,\nu) = \frac{\hbar^2}{2m}k^2d^2(\sinh^2\mu + \sin^2\nu) + (l^2 - \tfrac{1}{4})\frac{\hbar^2}{2md^2}\left(\frac{1}{\sinh^2\mu} + \frac{1}{\sin^2\nu}\right) , \qquad (5)$$

and similarly for oblate spheroidal coordinates. Therefore it is a general feature that pure subgroup coordinates (like spherical coordinates) generate by means of the separation procedure one-dimensional potential problems with the corresponding

[1] In the case of the Coulomb problem, this is the O(4)-group for the discrete spectrum and the O(3,1)-group for the continuous spectrum.

"hidden symmetry" of the motion on the group space, generic one-parametric coordinate systems generate two-dimensional potential problems with the corresponding "hidden symmetry" of the motion on the group space, etc. [2] The couplings in the potentials which emerge and may have originally taken on only rational values, can be analytically continued to any number, provided the problem at hand remains well-defined. [3] The structure of the coordinate systems on the $\mathrm{O}(2,2)$ hyperboloid is thus as follows:

1. $\mathrm{O}(2,2) \supset E(1,1)$: 10 coordinate systems.
2. $\mathrm{O}(2,2) \supset \mathrm{O}(2,1)$: 9 coordinate systems.
3. $\mathrm{O}(2,2) \supset \mathrm{O}(1,2)$: 9 coordinate systems.
4. 19 semi–split coordinate systems which are one-parametric.
5. 22 non–split coordinate systems which are two-parametric.
6. 3 non–orthogonal coordinate systems.

The majority of all coordinate systems come from *non-subgroup coordinate systems* about next to nothing is known, let alone harmonic analysis in terms of the corresponding eigenfunctions representations in terms of the bound states wave–functions and the scattering solutions.

As an instructive example, let us consider the prolate-spheroidal coordinate system in four-dimensional Euclidean space, where we separate off the circular variables and perform a space-time transformation [8, 9, 15, 24]

$$\int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathcal{D}\mu(t) \int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathcal{D}\nu(t) \frac{d^4}{4} (\sinh^2\mu + \sin^2\nu) \sinh 2\mu \sin 2\nu \prod_{i=1,2} \int\limits_{\varphi_i(t')=\varphi_i'}^{\varphi_i(t'')=\varphi_i''} \mathcal{D}\varphi_i(t)$$

$$\times \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} d^2 \Big( (\sinh^2\mu + \sin^2\nu)(\dot\mu^2 + \dot\nu^2) + \sinh^2\mu \sin^2\nu \dot\varphi_1^2 + \cosh^2\mu \cos^2\nu \dot\varphi_2^2 \Big) \right.\right.$$

$$\left.\left. + \frac{\hbar^2}{2md^2(\sinh^2\mu + \sin^2\nu)} \left( \frac{1}{\sin^2\nu \cos^2\nu} + \frac{1}{\sinh^2\mu \cosh^2\mu} \right) \right] dt \right\}$$

$$= \left( \tfrac{d^2}{4} \sinh 2\mu' \sinh 2\mu'' \sin 2\nu' \sin 2\nu' \sin 2\nu'' \right)^{-1/2}$$

[2] Note that parabolic coordinates are non-parametric generic coordinates which generate a two-dimensional radial harmonic oscillator potential, of which the path integral is of the Besselian type, and therefore leads to already known results. The range of known special functions, for instance the Bessel and Whittaker functions, the Legendre and hypergeometric functions, or the spheroidal functions allows only a sufficient number of indices to cover the two and three-dimensional wave-functions.

[3] For instance, in Besselian and Legendrian Path Integrals the angular momentum number $l$ may become purely imaginary, or complex with $\Re(l) > -1/2$.

$$\times \sum_{m_{1,2}\in\mathbb{Z}} \frac{\mathrm{e}^{\mathrm{i}m_1(\varphi_1''-\varphi_1')+\mathrm{i}m_2(\varphi_2''-\varphi_2')}}{4\pi^2} \int_{\mathbb{R}} \frac{dE}{2\pi\,\mathrm{i}} \mathrm{e}^{-\mathrm{i}ET/\hbar} \int_0^\infty ds''$$
$$\times \int_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathcal{D}\mu(t) \int_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathcal{D}\nu(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}(\dot\mu^2+\dot\nu^2)+Ed^2(\sinh^2\mu+\sin^2\nu)\right.\right.$$
$$\left.\left.-\frac{\hbar^2}{2m}\left(\frac{m_1^2-\frac{1}{4}}{\sin^2\nu}+\frac{m_2^2-\frac{1}{4}}{\cos^2\nu}+\frac{m_1^2-\frac{1}{4}}{\sinh^2\mu}-\frac{m_2^2-\frac{1}{4}}{\cosh^2\mu}\right)\right]ds\right\}\ . \tag{6}$$

It is obvious that the corresponding path integral solution in terms of the wavefunction expansion is clearly a generalization of the three-dimensional case. Whereas in three dimensions the spheroidal wavefunctions yield for $d = 0$ Legendre functions, the spheroidal wavefunctions in four dimensions have (modified) Pöschl-Teller wavefunctions as their degenerations. I did not find explicit representations of the corresponding wavefunctions in the literature. However, we can propose heuristically such wavefunctions by taking into account the theory of [27]. Looking at the prolate spheroidal coordinate system, we see that in the limit $d \to 0$ the spherical system must emerge with the cylindrical coordinate system on $S^{(3)}$ as the proper subsystem. Therefore it follows that the wavefunctions in the variable $\nu$ must be generalization of the Pöschl-Teller wavefunctions $\Phi_n^{(\alpha,\beta)}(z)$, and the wavefunctions in the variable $\mu$ must be again generalizations of modified Bessel functions $I_\nu(z)$. The proper quantum numbers which must be taken into account are $l \in \mathbb{N}_0$ and $m_1, m_2 \in \mathbb{Z}$. Hence we propose spheroidal wavefunctions $\mathrm{ps}_l^{(m_1,m_2)}(\cos\nu; p^2d^2)$ and $S_l^{(m_1,m_2),(1)}(\cosh\mu; pd)$ together with the following limiting correspondence ($\gamma = pd$)

$$\mathrm{ps}_l^{(m_1,m_2)}(\cos\nu;\gamma^2) \quad \stackrel{\gamma\to 0}{\simeq} \quad (\sin\nu)^{|m_1|}(\cos\nu)^{|m_1|}P_l^{(m_1,m_2)}(\cos 2\nu)\ , \tag{7}$$
$$S_l^{(m_1,m_2),(1)}(\cosh\mu;\gamma) \quad \stackrel{\gamma\to 0}{\simeq} \quad \frac{2\pi}{pr}J_{l+1}(pr)\ . \tag{8}$$

We propose consequently the interbasis expansion (up to phase factors and redefinition of functions)

$$\exp\left[\mathrm{i}pd(\sinh\mu\sin\vartheta\sin\nu\cos\alpha+\cosh\mu\cos\vartheta\cos\nu\cos\beta)\right]$$
$$=\sum_{l=0}^\infty \sum_{m_1,m_2\in\mathbb{Z}} 2(|m_1|!+|m_2|!+2l+1)\frac{l!\Gamma(|m_1|+|m_2|+l+1)}{\Gamma(|m_1|+l+1)\Gamma(|m_2|+l+1)}$$
$$\times \mathrm{e}^{\mathrm{i}m_1\alpha+\mathrm{i}m_2\beta}S_l^{(m_1,m_2),(1)}(\cosh\mu;\gamma)\mathrm{ps}_l^{(m_1,m_2)}(\cos\vartheta;\gamma^2)\mathrm{ps}_l^{(m_1,m_2)}(\cos\nu;\gamma^2)\ . \tag{9}$$

By means of this expansion the path integration can be done with the conjectural result of the path integral (6)

$$K(\mu'',\mu',\nu'',\nu',\varphi_1'',\varphi_1',\varphi_2'',\varphi_2';T)=\sum_{l=0}^\infty \sum_{m_1,m_2\in\mathbb{Z}} \frac{\mathrm{e}^{\mathrm{i}m_1(\varphi_1''-\varphi_1')+\mathrm{i}m_2(\varphi_2''-\varphi_2')}}{4\pi^2}$$
$$\times 2(|m_1|!+|m_2|!+2l+1)\frac{l!\Gamma(|m_1|+|m_2|+l+1)}{\Gamma(|m_1|+l+1)\Gamma(|m_2|+l+1)}$$

$$\times \frac{2}{\pi} \int_0^\infty p^3 dp\, S_l^{(m_1,m_2),(1)}(\cosh\mu''; pd) S_l^{(m_1,m_2),(1)\,*}(\cosh\mu'; pd)$$
$$\times \mathrm{ps}_l^{(m_1,m_2)\,*}(\cos\nu'; p^2d^2)\mathrm{ps}_l^{(m_1,m_2)}(\cos\nu''; p^2d^2)\mathrm{e}^{-\mathrm{i}\hbar p^2 T/2m} \quad . \tag{10}$$

A proper definition of these functions is involved. For instance, the function $\mathrm{ps}_\mu^{(m_1,m_2)}(z;\gamma^2)$ must be expanded in terms of $(1-z^2)^{|m_1|/2}(z)^{|m_1|}P_\mu^{(m_1,m_2)}(2z^2-1)$ with expansion coefficients $a_{\mu,\kappa}^{m_1,m_2}(\gamma^2)$, $\kappa \in \mathbb{Z}$. These coefficients then satisfy some recurrence relations which characterize the spheroidal functions. The case of oblate spheroidal coordinates is similar. A detailed study of these mathematical issues is beyond the scope of this overview and is not discussed here any further, c.f. [14]. Of course, similar considerations can also be made for path integral formulations in four-dimensional pseudo-Euclidean space.

Therefore new solutions, expansion theorems, and overlap functions, i.e., inter-bases expansions, for parametric coordinate systems on the SU(1,1)-, respectively the O(2,2)-hyperboloid would provide the possibility to study in a more systematic way the following issues:

1. New special functions will provide new identities between (old and new) special functions.

2. Expansion theorems of plane waves into parametric waves. The relations between other wave-expansions then can be obtained by the corresponding overlap functions.

3. The solution of the Schrödinger equation in parametric coordinate systems in terms of parametric special functions, in flat space and spaces of (positive and negative) constant curvature alike in terms of Lamé and Heun-functions and polynomials [23, 29].

4. The investigation of old potential problems in new settings.

5. The analysis of physical systems possessing symmetry properties which can be expressed in parametric coordinates as, e.g., in the two-center problem.

6. The investigation of unfamiliar perturbations for, e.g., the Coulomb potential or the harmonic oscillator.

7. Finding new Basic Path Integrals which go beyond the standard solutions.

Exploiting the full power of the group theoretical methods, i.e., symmetry methods in the spirit of A.Yu.Smorodinsky, would therefore enlarge our knowledge and possibilities of harmonic analysis on hyperboloids, special functions, separation of variables, and, of course, of path integrals.

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