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A local (perturbative) construction of observables in gauge theories: the example of QED

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Abstract. - Interacting fields can be constructed as formal power series in the framework of causal perturbation theory. The local field algebra $\tilde{\mathcal{F}}(\mathcal{O})$ is obtained without performing the adiabatic limit; the (usually bad) infrared behavior plays no role. To construct the observables in gauge theories we use the Kugo-Ojima formalism; we define the BRST-transformation $\tilde{s}$ as a graded derivation on the algebra of interacting fields and use the implementation of $\tilde{s}$ by the Kugo-Ojima operator Q_{int}. Since our treatment is local, the operator Q_{int} differs from the corresponding operator Q of the free theory. We prove that the Hilbert space structure present in the free case is stable under perturbations. All assumptions are shown to be satisfied in QED.

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1. Introduction

The quantization of gauge theories is a longstanding problem of theoretical physics. Since the works of Tomonaga, Schwinger, Feynman and Dyson in the late fourties the problem is solved for QED from a pragmatic point of view: the predictions (e.g. on the magnetic moment of the electron) are in perfect agreement with experiment. In the sixties and seventies the quantization of nonabelian gauge theories was developed by Feynman [F], Faddeev-Popov [FP], t'Hooft, Becchi-Rouet-Stora [BRS], Kugo-Ojima [KO] and others. Weinberg and Salam proposed to base the theory of electroweak interactions on a spontaneously broken gauge model, which has survived the last thirty years.

The ultraviolet divergences appearing in quantum field theory can be removed by various renormalization methods. An elegant method is causal perturbation theory which was developed by Epstein and Glaser [EG] on the basis of ideas due to Stückelberg and Bogoliubov and Shirkov [BS]. However, the *infrared problem* is only partially solved. One aspect is that charged particles cannot be eigenstates of the mass operator (they have to be "infraparticles" [Sch, Bu2]). Another aspect of the infrared problem are the divergences which appear in the adiabatic limit $g \to$ const. of the S-matrix, where g is a space-time dependent coupling 'constant'. In QED these divergences are logarithmic and cancel in the cross section. (This is proven at least at low orders of the perturbation series [S].) Moreover, Blanchard and Seneor [BlSe] proved that the adiabatic limit of Green's and Wightman functions exists for QED. But in nonabelian gauge theories the divergences are worse. Perturbation theory seems to be unable to describe the long distance properties of these models ("confinement"). There is the hope that these two aspects of the infrared problem are directly connected. (See e.g. Scharf in [S], sect. 3.12, where the existence and uniqueness of the adiabatic limit of the S-matrix is proposed to be a good criterion for a selection of physical states.)

The main message of the present paper is that a *local* construction of the observables in gauge theories is possible without performing the adiabatic limit. Hence the infrared divergences do not occur in the construction of the model. They rather appear on the level of the long distance properties of the theory. We hope that due to its local character, our construction can be generalized to curved space-times, continuing the program of [BFK,BF].

The quantization of gauge fields in a renormalizable gauge requires an indefinite metric space. Afterwards, one has to prove that the Wightman distributions of gauge invariant fields fulfil the condition of positivity. In [DHKS1,S] the free Kugo-Ojima charge Q which implements the BRST-transformation of free fields is used to select the physical Hilbert space. But the commutator of Q with the interacting gauge invariant fields vanishes only up to a divergence. One may expect that in the adiabatic limit the positivity is satified, but a proof in the nonabelian case seems to be rather hard. Here we adopt another point of view which avoids the discusion of the adiabatic limit. Our way out is to work with the interacting Kugo-Ojima charge $Q_{\rm int}$ [KO], which implements the BRST-transformation [BRS] of the interacting fields. By means of the Ward identities we prove that the commutator of $Q_{\rm int}$ with the gauge invariant interacting fields of QED is in fact zero. The infrared divergences which remained an open problem in the Kugo-Ojima formalism are absent in our treatment. However, since $Q_{\rm int} \neq Q$ before the adiabatic limit, the construction of the physical Hilbert space cannot be done on the level of free fields. We show that the physical Hilbert space of the interacting model is obtained as a deformation of the physical Hilbert space of the free theory. Here we adopt ideas from deformation quantization as developed by Bordemann

and Waldmann [BW].

The paper is organized as follows. In the next section we study the interacting fields in the framework of causal perturbation theory [BS,EG]. They are formal power series of (unbounded) operators in the Fock space of free fields. We point out that up to unitary equivalence the interacting fields depend on the interaction Lagrangian only *locally*. In section 3 we specialize to QED and compute commutators of interacting fields to all orders. Thereby we essentially use the Ward identities, which are proven in appendix B.

In section 4 we turn to the problems specific to gauge theories, the elimination of the unphysical fields and the mentioned positivity. We give a general *local* construction of the *observables* in gauge theories and of the *physical Hilbert space* in which the observables are faithfully represented. We prove that this structure is stable if we replace the free fields by the interacting ones.

This general construction relies on some assumptions. They are verified in the case of QED in section 5. The main problem is the construction of the interacting Kugo-Ojima charge $Q_{\rm int}$. To avoid a volume divergence we embed the algebra of interacting fields on an arbitrary finitely extended region into the corresponding algebra over a spacetime with compact spatial sections. This does not change the algebraic relations. We expect that also the corresponding Hilbert space representations of the local algebras of observables are unitarily equivalent, but this remains to be proven. The technical details of the free quantum gauge fields on the spatially compactified Minkowski space are written down in appendix A.

2. Perturbative Construction of Interacting Fields

In the framework of causal perturbation theory [BS,EG,St3,S,BF], interacting fields can be constructed as formal power series of operator valued distributions on a dense invariant domain $\mathcal{D}$ in the Fock space of incoming fields. The interacting fields $A_{\rm int\,\mathcal{L}}(x)$ (A is a Wick polynomial of incoming fields) depend on an interaction Lagrangian $\mathcal{L}$ which is a Wick polynomial of incoming fields with test functions $g \in \mathcal{D}(\mathbf{R}^4)$ as coefficients so that the interaction is switched on only within a finitely extended region of spacetime.

The crucial observation is that the dependence of the interacting fields on the interaction Lagrangian is *local*, in the sense that given a causally complete finitely extended open spacetime region $\mathcal{O}$, Lagrangians $\mathcal{L}_1$ and $\mathcal{L}_2$ which differ only within a closed region which does not intersect the closure of $\mathcal{O}$, lead to unitarily equivalent fields within $\mathcal{O}$, i.e. there exists a unitary formal power series V of operators on $\mathcal{D}$ such that

$$V A_{\rm int\,\mathcal{L}_1}(x) V^{-1} = A_{\rm int\,\mathcal{L}_2}(x), \qquad \forall x \in \mathcal{O}, \tag{2.1}$$

and V does not depend on A [BF]. This property (2.1) is a direct consequence of causality, which we are now going to explain.

Building blocks for the construction of interacting fields are the time ordered products $T(A_1(x_1)...A_n(x_n))$ of Wick polynomials of free fields. They are multilinear (with $\mathcal{C}^\infty$ functions as coefficients) symmetrical operator valued distributions on the dense domain $\mathcal{D}$ which satisfy the causal factorization property

$$\textbf{(Causality)} \quad T(A_1(x_1)...A_n(x_n)) = T(A_1(x_1)...A_k(x_k))T(A_{k+1}(x_{k+1})...A_n(x_n)) \tag{2.2}$$

if $x_j \notin \bar{V}_+ + x_i$, $i = 1, ..., k$, $j = k+1, ..., n$, where $\bar{V}_+$ is the closed forward light cone in Minkowski space.

Causality (2.2) and symmetry determine the time ordered products on the set of pairwise different points. Moreover, if the time ordered products of less than n factors are everywhere defined, the time ordered product of n factors is uniquely determined up to the total diagonal $x_1 = ... = x_n$. Thus renormalization amounts to an extension, for every n, of time ordered products to the total diagonal. This extension is always possible, and it can be done such that the conditions of Poincare covariance (w.r.t. some unitary positive energy representation U of the Poincare group $\mathcal{P}_+^\uparrow$)

$$\textbf{(N1)} \quad \mathrm{Ad}\, U(L)(T(A_1(x_1)...A_n(x_n))) = T(\mathrm{Ad}\, U(L)(A_1(x_1))...\mathrm{Ad}\, U(L)(A_n(x_n))), \quad L \in \mathcal{P}_+^\uparrow \tag{2.3}$$

and of unitarity hold,[1]

$$\textbf{(N2)} \qquad T(A_1(x_1)...A_n(x_n))^+ = \sum_{P \in \mathrm{Part}\,\{1,...,n\}} (-1)^{|P|+n} \prod_{p \in P} T(A_i(x_i)^+,\, i \in p). \tag{2.4}$$

($^+$ means the adjoint on $\mathcal{D}$, $(\phi, B(f)\psi) = (B^+(\bar{f})\phi, \psi), \quad \phi, \psi \in \mathcal{D}$.) The generating functional for the time ordered products is the S-matrix $S(\mathcal{L})$, $\mathcal{L} = \sum_{i=1}^N g_i A_i$, $g_i \in \mathcal{D}(\mathbf{R}^4)$

$$S(\mathcal{L}) = \sum_{n=0}^\infty \frac{i^n}{n!} \int d^4x_1 ... d^4x_n\, T(\mathcal{L}(x_1)...\mathcal{L}(x_n)), \tag{2.5}$$

i.e.

$$T(A_{i_1}(x_1)...A_{i_n}(x_n)) = \frac{\delta^n}{i^n \delta g_{i_1}(x_1)...\delta g_{i_n}(x_n)} S(\mathcal{L})|_{g_1=...=g_N=0}. \tag{2.6}$$

Finally, the interacting field $A_{\mathrm{int}\,\mathcal{L}}$ corresponding to the Wick polynomial A of the free fields, is defined by [BS,EG,DKS1]

$$A_{\mathrm{int}\,\mathcal{L}}(x) = \frac{\delta}{i\delta h(x)} S(\mathcal{L})^{-1} S(\mathcal{L} + hA)|_{h=0}. \tag{2.7}$$

By inserting (2.5) into (2.7) one obtains the perturbative expansion of the interacting fields

$$A_{\mathrm{int}\,\mathcal{L}}(x) = A(x) + \sum_{n=1}^\infty \frac{i^n}{n!} \int d^4x_1 ... d^4x_n\, R(\mathcal{L}(x_1)...\mathcal{L}(x_n); A(x)), \tag{2.8}$$

with the 'totally retarded products'

$$R(A_1(x_1)...A_n(x_n); A(x)) \stackrel{\mathrm{def}}{=} \sum_{I \subset \{1,...,n\}} (-1)^{|I|} \bar{T}(A_i(x_i),\, i \in I) T(A_j(x_j),\, j \in I^c, A(x)), \tag{2.9}$$

where $I^c \stackrel{\mathrm{def}}{=} \{1, ..., n\} \setminus I$ and $\bar{T}$ denotes the 'antichronological product'. The corresponding generating functional is the inverse S-matrix

$$S(\mathcal{L})^{-1} = \sum_{n=0}^\infty \frac{(-i)^n}{n!} \int d^4x_1 ... d^4x_n\, \bar{T}(\mathcal{L}(x_1)...\mathcal{L}(x_n)),$$

[1] We work throughout with the conventions of [EG], not with the ones of [S].

and the antichronological products can be obtained from the time ordered products by the usual inversion of a formal power series, namely the r.h.s. of (2.4).

The arbitrariness in the extensions of time ordered products to coinciding points can be further restricted. Let $\phi_1, ..., \phi_N$ be the free fields in terms of which the model is defined, which satisfy the linear field equation

$$\sum_j D_{ij}\phi_j = 0, \tag{2.10}$$

(where D_{ij} is a matrix whose entries are differential operators such that D is a relativistic invariant hyperbolic differential operator with a unique solution of the Cauchy problem) and with C-number commutators

$$[\phi_j(x), \phi_k(y)] = i\Delta_{jk}(x-y), \qquad \Delta_{jk} = \Delta^{\rm ret}_{jk} - \Delta^{\rm av}_{jk} \tag{2.11}$$

($\Delta^{\rm ret,\,av}_{jk}$ retarded resp. advanced Green's function of D_{ij}, i.e. supp $\Delta^{\rm ret,\,av}_{jk} \subset \bar{V}_{+,-}$). We define to every Wick polynomial A the sub Wick polynomials $\frac{\partial A}{\partial \phi_j}$ by [2]

$$[A(x), \phi_k(y)] = i\sum_j \frac{\partial A}{\partial \phi_j}(x)\Delta_{jk}(x-y). \tag{2.12}$$

We then require

$$\textbf{(N3)} \quad [T(A_1(x_1)...A_n(x_n)), \phi_j(x)] = i\sum_{k=1}^{n}\sum_l T(A_1(x_1)...\frac{\partial A_k}{\partial \phi_l}(x_k)...A_n(x_n))\Delta_{lj}(x_k - x) \tag{2.13}$$

and (cf. [St2])

$$\textbf{(N4)} \quad \sum_j D^x_{ij} T(A_1(x_1)...A_n(x_n)\phi_j(x)) = i\sum_{k=1}^{n} T(A_1(x_1)...\frac{\partial A_k}{\partial \phi_i}(x_k)...A_n(x_n))\delta(x_k - x). \tag{2.14}$$

The first condition means that the time ordered product is determined up to a C-number by the time ordered products of sub Wick polynomials, whereas the second condition defines uniquely the time ordered products with additional free field factors once it is given away from the diagonal. It translates into the differential equation

$$\textbf{(N4}') \quad \sum_j D^x_{ij} R(A_1(x_1)...A_n(x_n); \phi_j(x)) = i\sum_{k=1}^{n} R(A_1(x_1)...\hat{k}...A_n(x_n); \frac{\partial A_k}{\partial \phi_i}(x_k))\delta(x_k - x) \tag{2.15}$$

[2] If A contains *derivated* free fields the definition (2.12) is replaced by

$$[A(x), \phi_k(y)] = i\int d^4z \sum_j \frac{\delta A(x)}{\delta \phi_j(z)}\Delta_{jk}(z-y)$$

and similar modifications appear in the following formulas.

for the totally retarded products (2.9), where the hat means the omission of the corresponding factor. By means of (2.8) we see that the requirement (2.14) implies the field equation for the interacting field $\phi_{\mathrm{int}\,j,\mathcal{L}}$,

$$\sum_j D_{ij}\phi_{\mathrm{int}\,j,\mathcal{L}}(x) = -\Big(\frac{\partial\mathcal{L}}{\partial\phi_i}\Big)_{\mathrm{int}\,\mathcal{L}}(x). \tag{2.16}$$

The remaining arbitrariness is the freedom in the extension of the expectation values $\omega(T(A_1(x_1)...A_n(x_n)))$, where ω is some state (e.g. the vacuum), to the diagonal. This freedom consists in adding a distribution with support on the diagonal. Its form is restricted by covariance and by the requirement that the degree of the singularity at the diagonal, measured in terms of Steinmann' scaling degree [Ste,BF], may not be increased by the extension.

The requirements (2.13) and (2.14) are purely algebraic normalization conditions for the time ordered products resp. the interacting fields. They are independent of the choice of some state and, hence, are well suited for the generalization to curved spacetimes.

For later purpose we are going to list some properties of the totally retarded products (2.9). By means of causality (2.2) one easily finds that they have totally retarded support,

$$\mathrm{supp}\, R(A_1(x_1)...A_n(x_n);A(x)) \subset \{(x_1,...x_n;x) \mid x_i \in x+\bar{V}_-,\ \forall i=1,...n\}. \tag{2.17}$$

This means that the interacting fields $A_{\mathrm{int}\,\mathcal{L}}(x)$ (2.7-8) depend only on the interaction in the past of x, i.e. solely on $\mathcal{L}|_{x+\bar{V}_-}$.

The following lemma describes the structure of the totally retarded products with a free field factor.

Lemma 1: Let $(\phi_j)_j$ be free fields and the normalizations fulfil **(N4)**. Then

$$\textbf{(A)}\quad R(A_1(x_1)...A_n(x_n);\phi_i(x)) = i\sum_{k=1}^{n}\sum_l \Delta^{\mathrm{ret}}_{il}(x-x_k)R(A_1(x_1)...\hat{k}...A_n(x_n);\frac{\partial A_k}{\partial\phi_l}(x_k)),$$

$$\textbf{(B)}\quad R(A_1(x_1)...A_{n-1}(x_{n-1})\phi_j(y);A(x)) = i\sum_{m=1}^{n-1}\sum_h \Delta^{\mathrm{av}}_{jh}(y-x_m)R(A_1(x_1)...\frac{\partial A_m}{\partial\phi_h}(x_m)...$$

$$...A_{n-1}(x_{n-1});A(x)) + i\sum_h \Delta^{\mathrm{av}}_{jh}(y-x)R(A_1(x_1)...A_{n-1}(x_{n-1});\frac{\partial A}{\partial\phi_h}(x))$$

Thereby note

$$\textbf{(C)}\quad R(A_1(x_1)...A_n(x_n);A(x)) = 0 \quad \text{if } n\geq 1 \text{ and some } A_j\ (1\leq j\leq n) \text{ or } A \text{ is a C}-\text{number}.$$

Proof: The last statement **(C)** is easily obtained from the definition (2.9) and, if A is the C-number, by taking $\sum_{I\subset\{1,...,n\}}\bar{T}(A_i(x_i),\, i\in I)T(A_j(x_j),\, j\in I^c)=0$ into account. Alternatively one can argue by means of **(N1)** and the translation invariance of C-number fields that the non-validity of **(C)** would contradict the support property (2.17).

(A) Due to

$$D_{ij}\Delta^{\mathrm{ret,\,av}}_{jk}(x) = \delta_{ik}\delta(x), \tag{2.18}$$

the expression **(A)** is a solution of the hyperbolic differential equation (2.15). Moreover, it is the only solution which fulfills the support property (2.17).

To prove **(B)** we note that **(N4)** implies

$$D^y_{ij} R(A_1(x_1)...A_{n-1}(x_{n-1})\phi_j(y); A(x)) = i\sum_{m=1}^{n-1}\delta(y-x_m)R(A_1(x_1)...\frac{\partial A_m}{\partial\phi_i}(x_m)...$$

$$...A_{n-1}(x_{n-1}); A(x)) + i\delta(y-x)R(A_1(x_1)...A_{n-1}(x_{n-1}); \frac{\partial A}{\partial\phi_i}(x)), \tag{2.19}$$

analogously to (2.15). Again there is only one solution of (2.19) which respects (2.17), namely **(B)**. □

In the next section we will compute commutators of interacting fields by means of

Proposition 2: The (anti-) commutator of two interacting fields can be written in the form

$$[A^1_{\mathrm{int}\,\mathcal{L}}(x), A^2_{\mathrm{int}\,\mathcal{L}}(y)]_\mp = -\sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^4y_1...d^4y_n$$

$$\Big\{R(\mathcal{L}(y_1)...\mathcal{L}(y_n)A^1(x); A^2(y)) \mp R(\mathcal{L}(y_1)...\mathcal{L}(y_n)A^2(y); A^1(x))\Big\}. \tag{2.20}$$

The anticommutator appears only if A^1 and A^2 have either both an odd number of spinor or both an odd number of ghost fields.[3]

Proof: Due to $S(\mathcal{L}+hA)^{-1}S(\mathcal{L}+hA) = 1$ we can write (2.7) in an alternative way

$$A_{\mathrm{int}\,\mathcal{L}}(x) = -\frac{\delta}{i\delta h(x)}S(\mathcal{L}+hA)^{-1}S(\mathcal{L})|_{h=0}. \tag{2.21}$$

and, hence, we get

$$A^1_{\mathrm{int}\,\mathcal{L}}(x)A^2_{\mathrm{int}\,\mathcal{L}}(y) = \frac{\delta^2}{\delta h_1(x)\delta h_2(y)}S(\mathcal{L}+h_1A^1)^{-1}S(\mathcal{L}+h_2A^2)|_{h_1=0=h_2}. \tag{2.22}$$

Next we note that the first term on the r.h.s. of (2.20) is equal to

$$-\frac{\delta^2}{i^2\delta h_1(x)\delta h_2(y)}S(\mathcal{L}+h_1A^1)^{-1}S(\mathcal{L}+h_1A^1+h_2A^2)|_{h_1=0=h_2}. \tag{2.23}$$

Therefore, the assertion (2.20) is equivalent to

$$[A^1_{\mathrm{int}\,\mathcal{L}}(x), A^2_{\mathrm{int}\,\mathcal{L}}(y)] = \frac{\delta^2}{\delta h_1(x)\delta h_2(y)}[S(\mathcal{L}+h_1A^1)^{-1}S(\mathcal{L}+h_1A^1+h_2A^2)-$$

$$-S(\mathcal{L}+h_2A^2)^{-1}S(\mathcal{L}+h_1A^1+h_2A^2)]|_{h_1=0=h_2}, \tag{2.24}$$

which holds obviously true by (2.22). If A^i is fermionic the corresponding test function h_i is Grassmann valued (i.e. an anticommuting C-number, see e.g. [S], appendix D) and, hence, the commutators possibly turn into anticommutators. □

By means of the support property (2.17) we immediately see from (2.20) that the interacting fields are local

$$[A^1_{\mathrm{int}\,\mathcal{L}}(x), A^2_{\mathrm{int}\,\mathcal{L}}(y)]_\mp = 0 \qquad \text{if} \qquad (x-y)^2 < 0. \tag{2.25}$$

Of course this can also be proven in a more direct way. Note that Proposition 2 also provides a decomposition of the commutator into a retarded (i.e. $(x-y)\in\bar{V}_+$) and advanced part ($(x-y)\in\bar{V}_-$).

[3] We work with the convention that the Fock space of the (free) incoming fields is the tensor product of the photon, spinor and ghost Fock spaces (see (A.15)). Hence a free spinor fields *commutes* with a free ghost field.

3. Commutators of Interacting Fields in QED

In QED the interaction is given by

$$\mathcal{L}(x) = g(x) : \overline{\psi}(x)\gamma_\mu A^\mu(x)\psi(x) :, \qquad g \in \mathcal{D}(\mathbf{R}^4), \tag{3.1}$$

where A^μ is the free photon field and $\overline{\psi}$, ψ are the free spinor fields. In addition, we introduce a pair u, $\tilde{u}$ of free, anticommuting ghost fields

$$\square u = 0 = \square \tilde{u}, \quad \{u(x), u(y)\} = 0, \quad \{\tilde{u}(x), \tilde{u}(y)\} = 0, \quad \{u(x), \tilde{u}(y)\} = -iD(x-y), \tag{3.2}$$

where D is the massless Pauli-Jordan distribution. In QED the ghost fields do not couple and could therefore be eliminated. We are, however, interested in a formulation of the gauge structure which can be generalized to the nonabelian case, where the ghosts seem to be indispensable [F,FP,DHKS1].

The only non-vanishing anticommutator of the spinor fields is

$$\{\psi(x), \overline{\psi}(y)\} = -iS(x-y) = -i(i\gamma^\mu \partial_\mu + m)\Delta(x-y) \tag{3.3}$$

and Δ denotes the massive Pauli-Jordan distribution. The quantization of the photon field A^μ is done in the Feynman gauge with the commutator

$$[A^\mu(x), A^\nu(y)] = ig^{\mu\nu} D(x-y). \tag{3.4}$$

The $*$-operation is introduced by $A^{\mu\,*} = A^\mu$,

$$u^* \stackrel{\text{def}}{=} u, \qquad \tilde{u}^* \stackrel{\text{def}}{=} -\tilde{u}. \tag{3.5}$$

and $\psi(x)^* = \overline{\psi}(x)\gamma_0$. Unitarity **(N2)** (with respect to the *-operation) implies for the interacting fields

$$A^\mu_{\text{int}\,\mathcal{L}}(x)^* = A^\mu_{\text{int}\,\mathcal{L}}(x), \qquad j^\mu_{\text{int}\,\mathcal{L}}(x)^* = j^\mu_{\text{int}\,\mathcal{L}}(x), \qquad \psi_{\text{int}\,\mathcal{L}}(x)^*\gamma_0 = \overline{\psi}_{\text{int}\,\mathcal{L}}(x), \tag{3.6}$$

where g is assumed to be real-valued and j^μ is the matter current

$$j^\mu(x) \stackrel{\text{def}}{=} : \overline{\psi}(x)\gamma^\mu\psi(x) : . \tag{3.7}$$

In QED the field equations (2.16) read

$$\square A^\mu_{\text{int}\,\mathcal{L}}(x) = -g(x) j^\mu_{\text{int}\,\mathcal{L}}(x), \tag{3.8}$$

$$(i\gamma_\mu \partial^\mu - m)\psi_{\text{int}\,\mathcal{L}}(x) = -g(x)(\gamma_\mu A^\mu \psi)_{\text{int}\,\mathcal{L}}(x) \tag{3.9}$$

and similar for $\overline{\psi}_{\text{int}\,\mathcal{L}}$.

An important restriction of the normalizations (i.e. the freedom in the extension to the total diagonal) comes from the *Ward identities*

$$\textbf{(N5)} \quad \partial^y_\mu T\Big(j^\mu(y) A_1(x_1)...A_n(x_n)\Big) = i\sum_{j=1}^n \delta(y-x_j) T\Big(A_1(x_1)...(\theta A_j)(x_j)...A_n(x_n)\Big), \tag{3.10}$$

where A_j is a sub Wick monom of $\mathcal{L}$ (3.1) and $(\theta A_j) \stackrel{\text{def}}{=} \frac{d}{d\alpha}|_{\alpha=0} A_{j\,\alpha}$ is the infinitesimal action of the global $U(1)$ transformation

$$\psi \to \psi_\alpha \stackrel{\text{def}}{=} e^{i\alpha}\psi, \qquad \overline{\psi} \to \overline{\psi}_\alpha \stackrel{\text{def}}{=} e^{-i\alpha}\overline{\psi}, \qquad A^\mu \to A^\mu, \qquad u \to u, \qquad \tilde{u} \to \tilde{u}, \qquad \alpha \in \mathbf{R}. \tag{3.11}$$

We prove in appendix B that all Ward identities **(N5)** can be fulfilled and are compatible with the other above mentioned normalization conditions **(N1)**, **(N2)**, **(N3)** and **(N4)**. The subset of Ward identities involving only factors j and $\mathcal{L}$

$$\partial_\mu^y T\Big(j^\mu(y)\mathcal{L}(x_1)...\mathcal{L}(x_k)j^{\mu_1}(x_{k+1})...j^{\mu_{n-k}}(x_n)\Big) = 0 \tag{3.12}$$

is equivalent to the free (perturbative) operator gauge invariance of QED ([S], [DHKS2] and section 5.1). (3.10) implies the same Ward identities for the antichronological products $\bar{T}$ up to a global factor (-1) on the r.h.s.. With that one easily derives from (3.10) the Ward identities for the totally retarded products

$$\partial_\mu^y R\Big(A_1(x_1)...A_{n-1}(x_{n-1})j^\mu(y); A(x)\Big) = i\delta(y-x)R\Big(A_1(x_1)...A_{n-1}(x_{n-1}); (\theta A)(x)\Big) +$$

$$+ i\sum_{k=1}^{n-1}\delta(y-x_k)R\Big(A_1(x_1)...(\theta A_k)(x_k)...A_{n-1}(x_{n-1}); A(x)\Big)$$

and

$$\partial_\mu^y R\Big(A_1(x_1)...A_n(x_n); j^\mu(y)\Big) = i\delta(y-x_k)R\Big(A_1(x_1)...\hat{k}...A_n(x_n); (\theta A_k)(x_k)\Big),$$

where $\hat{k}$ means that $A_k(x_k)$ is omitted. Especially we obtain

$$\partial_\mu^x R(\mathcal{L}(x_1)...\mathcal{L}(x_n); j^\mu(x)) = 0, \qquad \text{i.e.} \qquad \partial_\mu j^\mu_{\text{int}\,\mathcal{L}}(x) = 0. \tag{3.13}$$

Hence $j^\mu_{\text{int}\,\mathcal{L}}$ is a conserved current.

The Ward identities **(N5)** also imply that the corresponding charge operator implements the infinitesimal $U(1)$-action θ on the interacting fields, i.e.

$$[j^0_{\text{int}\,\mathcal{L}}(f), A_{\text{int}\,\mathcal{L}}(x)] = i(\theta A)_{\text{int}\,\mathcal{L}}(x),$$

where A is a sub Wick monom of $\mathcal{L}$ (3.1), and for the test function $f \in \mathcal{D}(\mathbf{R}^4)$ we assume that there exists $h \in \mathcal{D}(\mathbf{R})$ such that

$$f(y) = h(y_0) \quad \forall y = (y_0, \mathbf{y}) \quad \text{in a neighbourhood of} \quad x + (\bar{V}_+ \cup \bar{V}_-) \quad \text{and} \quad \int dy_0\, h(y_0) = 1.$$

To prove this we first note $[j^0_{\text{int}\,\mathcal{L}}(f), A_{\text{int}\,\mathcal{L}}(x)] = \int d^4y\, h(y_0)[j^0_{\text{int}\,\mathcal{L}}(y), A_{\text{int}\,\mathcal{L}}(x)]$ by the support property (2.25) of the commutator, and by means of Proposition 2 this is equal to

$$-\sum_{n=0}^{\infty}\frac{i^n}{n!}\int dy_1...dy_n\, g(y_1)...g(y_n)\int dy\,\Big\{\Big([h(y_0)-h(y_0-a)]+h(y_0-a)\Big)\cdot$$

$$\cdot R(\mathcal{L}(y_1)...\mathcal{L}(y_n)j^0(y); A(x)) - \Big([h(y_0)-h(y_0-b)]+h(y_0-b)\Big)R(\mathcal{L}(y_1)...\mathcal{L}(y_n)A(x); j^0(y))\Big\}.$$

Due to the support property (2.17) of the R-products we can choose a and b such that the contributions from $h(y_0-a)$ and $h(y_0-b)$ vanish. Setting $k(y)\equiv k_0(y_0)\stackrel{\rm def}{=}\int_{-\infty}^{y_0}dz\,[h(z)-h(z-a)]$, the $[h(y_0)-h(y_0-a)]$-term is equal to

$$\ldots\int dy\,k(y)\partial_\mu^y R(\mathcal{L}(y_1)\ldots\mathcal{L}(y_n)j^\mu(y);A(x))=ik(x)(\theta A)_{{\rm int}\,\mathcal{L}}(x),$$

where we have inserted a Ward identity. For the $[h(y_0)-h(y_0-b)]$-term we obtain $i(1-k(x))(\theta A)_{{\rm int}\,\mathcal{L}}(x)$ by a similar procedure. So the sum of all terms is in fact $i(\theta A)_{{\rm int}\,\mathcal{L}}(x)$. □

By means of the Ward identity (3.13) and Lemma 1, **(A)** we obtain

$$\partial_\mu A^\mu_{{\rm int}\,\mathcal{L}}(x)=\partial_\mu A^\mu(x)+\sum_{n=1}^{\infty}\frac{i^{n+1}}{n!}\int d^4x_1\ldots d^4x_n\cdot$$

$$\cdot\sum_{l=1}^{n}g(x_1)\ldots\partial_\mu g(x_l)\ldots g(x_n)D^{\rm ret}(x-x_l)R(\mathcal{L}^0(x_1)\ldots\hat{l}\ldots\mathcal{L}^0(x_n);j^\mu(x_l)),\qquad(3.14)$$

where

$$\mathcal{L}^0(x)\stackrel{\rm def}{=}:\overline{\psi}(x)\gamma_\mu A^\mu(x)\psi(x):,\qquad\text{i.e.}\qquad\mathcal{L}(x)=g(x)\mathcal{L}^0(x).\qquad(3.15)$$

Thus in the formal partial adiabatic limit $g|_{x+\bar{V}_-}\to$ const. the interacting field $\partial_\mu A^\mu_{{\rm int}\,\mathcal{L}}(x)$ agrees with the corresponding free one.

Let

$$\mathcal{O}_{x,y}\stackrel{\rm def}{=}(x+V_+)\cap(y+V_-).\qquad(3.16)$$

We will study the algebra $\tilde{\mathcal{F}}(\mathcal{O})$ generated by

$$\{A_{{\rm int}\,\mathcal{L}}(f)=\int d^4x\,A_{{\rm int}\,\mathcal{L}}(x)f(x)|f\in\mathcal{D}(\mathcal{O}),A=A^\mu,\psi,\overline{\psi},j^\mu\ldots\},\qquad(3.17)$$

where $\mathcal{O}$ is the double cone

$$\mathcal{O}\stackrel{\rm def}{=}\mathcal{O}_{(-r,\mathbf{0}),(r,\mathbf{0})},\qquad(r>0)\qquad(3.18)$$

which is centered at the origin, and the test function $g\in\mathcal{D}(\mathbf{R}^4)$ in (3.1), (3.15) is assumed to fulfil

$$g(x)=e=\text{const.},\qquad\forall x\in\mathcal{O}.\qquad(3.19)$$

We are now going to compute some commutators of $\partial_\mu A^\mu_{{\rm int}\,\mathcal{L}}(x)$ with elements of $\tilde{\mathcal{F}}(\mathcal{O})$.

Proposition 3: The following relations hold for the interacting fields at points $x,y\in\mathcal{O}$.

(1) $$[\partial_\mu A^\mu_{{\rm int}\,\mathcal{L}}(x),A^\nu_{{\rm int}\,\mathcal{L}}(y)]=i\partial^\nu D(x-y),$$

(2) $$[\partial_\mu A^\mu_{{\rm int}\,\mathcal{L}}(x),\partial_\nu A^\nu_{{\rm int}\,\mathcal{L}}(y)]=0,$$

(3) $$[\partial_\mu A^\mu_{{\rm int}\,\mathcal{L}}(x),\psi_{{\rm int}\,\mathcal{L}}(y)]=D(x-y)e\psi_{{\rm int}\,\mathcal{L}}(y),$$

(4) $$[\partial_\mu A^\mu_{{\rm int}\,\mathcal{L}}(x),\overline{\psi}_{{\rm int}\,\mathcal{L}}(y)]=-D(x-y)e\overline{\psi}_{{\rm int}\,\mathcal{L}}(y),$$

$$\textbf{(5)} \qquad [\partial_\mu A^\mu_{\text{int}\,\mathcal{L}}(x), \mathcal{L}^0_{\text{int}\,\mathcal{L}}(y)] = i(\partial_\mu D)(x-y) j^\mu_{\text{int}\,\mathcal{L}}(y).$$

Proof: Since $\partial_\mu A^\mu_{\text{int}\,\mathcal{L}}$ is a solution of the wave equation for $x \in \mathcal{O}$, it is sufficient to check these relations for the advanced or retarded part of the commutator.

(1) and (2): Inserting Lemma 1 **(B)** and **(C)** into Proposition 2 we obtain

$$[A^\mu_{\text{int}\,\mathcal{L}}(x), A^\nu_{\text{int}\,\mathcal{L}}(y)]_{\text{av}} = -ig^{\mu\nu} D^{\text{av}}(x-y) - \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4z_1 \ldots d^4z_n \, g(z_1)\ldots g(z_n)$$

$$i \sum_{m=1}^{n} D^{\text{av}}(x-z_m) R(\mathcal{L}^0(z_1)\ldots j^\mu(z_m)\ldots\mathcal{L}^0(z_n); A^\nu(y)). \tag{3.20}$$

From the support properties of the retarded products (2.17) we see that the integration over z_m is confined to the double cone

$$z_m \in \mathcal{O}_{x,y} \subset \mathcal{O}. \tag{3.21}$$

Let us now consider $[\partial_\mu A^\mu_{\text{int}\,\mathcal{L}}(x), A^\nu_{\text{int}\,\mathcal{L}}(y)]$. We want to show that the divergence ∂^x_μ of all terms in (3.20) with index $n \geq 1$ vanish. In fact, the divergence ∂^x_μ can be written as $-\partial^{z_m}_\mu D^{\text{av}}(x-z_m)$. So formally, after a partial integration in z_m we get two terms: (i) A divergence of $R(\mathcal{L}^0\ldots j\ldots\mathcal{L}^0; A)$ with respect to a j-vertex, which vanishes due to the Ward identities **(N5)**. (ii) A term $\sim \partial_\mu g(z_m)$ which vanishes since g is constant within $\mathcal{O}$ (3.19,3.21). The argument can easily be made rigorous by smearing with test functions. Hence

$$[\partial_\mu A^\mu_{\text{int}\,\mathcal{L}}(x), A^\nu_{\text{int}\,\mathcal{L}}(y)]_{\text{av}} = -i\partial^\nu D^{\text{av}}(x-y), \qquad \forall x, y \in \mathcal{O}, \tag{3.22}$$

which implies **(1)**. Formula **(2)** follows since D is a solution of the wave equation.

(3): To prove **(3)** we proceed analogously to (3.20) and obtain

$$[\partial_\mu A^\mu_{\text{int}\,\mathcal{L}}(x), \psi_{\text{int}\,\mathcal{L}}(y)]_{\text{av}} = -\sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4z_1 \ldots d^4z_n \, g(z_1)\ldots g(z_n) \cdot$$

$$\cdot i \sum_{m=1}^{n} R(\mathcal{L}^0(z_1)\ldots j^\mu(z_m)\ldots\mathcal{L}^0(z_n); \psi(y)) \partial^x_\mu D^{\text{av}}(x-z_m). \tag{3.23}$$

We integrate by parts with respect to z_m and insert the Ward identity

$$\partial^{z_m}_\mu R\big(\mathcal{L}^0(z_1)\ldots j^\mu(z_m)\ldots\mathcal{L}^0(z_n); \psi(y)\big) =$$

$$= -\delta(z_m - y) R\big(\mathcal{L}^0(z_1)\ldots\hat{m}\ldots\mathcal{L}^0(z_n); \psi(y)\big). \tag{3.24}$$

Because of $z_m \in \mathcal{O}$ (3.21) the terms $\sim \partial g(z_m)$ vanish and we end up with

$$[\partial_\mu A^\mu_{\text{int}\,\mathcal{L}}(x), \psi_{\text{int}\,\mathcal{L}}(y)]_{\text{av}} =$$

$$= -D^{\text{av}}(x-y) g(y) \psi_{\text{int}\,\mathcal{L}}(y) \qquad \text{for} \qquad x, y \in \mathcal{O}, \tag{3.25}$$

This proves **(3)**.

(4): The relation for the conjugate spinor follows by applying the *-operation.

(5): By means of Proposition 2 we get

$$[\partial_\mu A^\mu_{\mathrm{int}\,\mathcal{L}}(x), \mathcal{L}^0_{\mathrm{int}\,\mathcal{L}}(y)]_{\mathrm{ret}} =$$

$$\sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4z_1 ... d^4z_n \, g(z_1)...g(z_n) \partial^x_\mu R(\mathcal{L}^0(z_1)...\mathcal{L}^0(z_n)\mathcal{L}^0(y); A^\mu(x)). \tag{3.26}$$

From part **(A)** of Lemma 1 this is equal to

$$\sum_{n=0}^{\infty} \frac{i^{n+1}}{n!} \int dz_1 ... dz_n \, g(z_1)...g(z_n) \{\partial_\mu D^{\mathrm{ret}}(x-y) R((\mathcal{L}^0(z_1)...\mathcal{L}^0(z_n); j^\mu(y)) +$$

$$+ \sum_{m=1}^{n} \partial_\mu D^{\mathrm{ret}}(x-z_m) R(\mathcal{L}^0(z_1)...\hat{m}...\mathcal{L}^0(z_n)\mathcal{L}^0(y); j^\mu(z_m))\}. \tag{3.27}$$

By a partial integration with respect to z_m and the same reasoning as above, we see that the second term does not contribute, hence we find

$$[\partial_\mu A^\mu_{\mathrm{int}\,\mathcal{L}}(x), \mathcal{L}^0_{\mathrm{int}\,\mathcal{L}}(y)]_{\mathrm{ret}} = i(\partial_\mu D^{\mathrm{ret}})(x-y) j^\mu_{\mathrm{int}\,\mathcal{L}}(y), \tag{3.28}$$

which finally implies **(5)**. □

4. Connection of observable algebras and field algebras in perturbative gauge theories

The observation (2.1) applies to the algebra $\tilde{\mathcal{F}}(\mathcal{O})$ (3.17) with a unitary formal power series V. We conclude that the net of algebras of local interacting fields $\tilde{\mathcal{F}}(\mathcal{O}_1)$, $\mathcal{O}_1 \subset \mathcal{O}$, up to unitary equivalence is uniquely determined by $g|_{\mathcal{O}}$. Since $\mathcal{O}$ is arbitrary, the full net of local algebras can be constructed without performing the adiabatic limit $g \to$ constant.

This general procedure can be applied also to *gauge theories*. But there the (local) algebras of interacting fields contain unphysical fields like vector potentials and ghosts. Therefore the question arises whether there is a local construction of the algebra of observables and the physical Hilbert space, i.e. a (pre)Hilbert space on which the observables can be faithfully represented.

4.1 Local construction of observables in gauge theories

Let $\mathcal{O} \to \mathcal{F}(\mathcal{O})$ be a net of $\mathbf{Z}_2$-graded *-algebras. In addition, we are given a graded derivation s on $\cup_{\mathcal{O}} \mathcal{F}(\mathcal{O}) =: \mathcal{F}$ with $s^2 = 0$, $s(\mathcal{F}(\mathcal{O})) \subset \mathcal{F}(\mathcal{O})$ and $s(F^*) = -(-1)^{\delta(F)} s(F)^*$, where $(-1)^{\delta(F)}$ is the $\mathbf{Z}_2$-gradiation of $F \in \mathcal{F}$.

The observables should be s-invariant. Therefore, we consider the kernel of s, $\mathcal{A}_0 := s^{-1}(0) \subset \mathcal{F}$, and $\mathcal{A}_0(\mathcal{O}) := \mathcal{A}_0 \cap \mathcal{F}(\mathcal{O})$. $\mathcal{A}_0$ is a *-algebra:

$$A, B \in \mathcal{A}_0 \Longrightarrow s(AB) = s(A)B + (-1)^{\delta(A)} A s(B) = 0, \qquad \text{i.e.} \quad AB \in \mathcal{A}_0. \tag{4.1}$$

We set $\mathcal{A}_{00} := s(\mathcal{F})$. Due to $s^2 = 0$ it is a subspace of $\mathcal{A}_0$; it is even a 2-sided ideal in $\mathcal{A}_0$

$$s(F)A = s(FA) - (-1)^{\delta(F)}Fs(A) = s(FA) \in \mathcal{A}_{00} \tag{4.2}$$

and similarly $As(F) \in \mathcal{A}_{00}$ if $A \in \mathcal{A}_0$. The algebra of observables is defined as the quotient

$$\mathcal{A} \stackrel{\text{def}}{=} \frac{\mathcal{A}_0}{\mathcal{A}_{00}} \tag{4.3}$$

and

$$\mathcal{O} \to \mathcal{A}(\mathcal{O}) \stackrel{\text{def}}{=} \frac{\mathcal{A}_0 \cap \mathcal{F}(\mathcal{O})}{\mathcal{A}_{00} \cap \mathcal{F}(\mathcal{O})} \tag{4.4}$$

is the net of algebras of local observables.

4.2 Representation of the observables in the physical pre Hilbert space

We now ask: under which conditions does $\mathcal{A}$ have a nontrivial representation by operators on a pre Hilbert space, such that

$$(A^*\phi, \psi) = (\phi, A\psi), \qquad \forall A \in \mathcal{A}. \tag{4.5}$$

For this purpose we assume that $\mathcal{F}$ has a faithful representation on an inner product space $(\mathcal{K}, <.,.>)$ such that

$$< F^*\phi, \psi > = < \phi, F\psi >, \qquad \forall F \in \mathcal{F},$$

and s is implemented by an operator Q on $\mathcal{K}$, i.e.

$$s(F) = QF - (-1)^{\delta(F)}FQ, \tag{4.6}$$

such that

$$< Q\phi, \psi > = < \phi, Q\psi > \qquad \text{and} \qquad Q^2 = 0. \tag{4.7}$$

The assumptions (4.7) are made in order to fulfil $s(F^*) = -(-1)^{\delta(F)}s(F)^*$ and $s^2 = 0$. Note that if the inner product on $\mathcal{K}$ is positive definite, we find $< Q\phi, Q\phi > = < \phi, Q^2\phi > = 0$, hence $Q = 0$ and thus also $s = 0$. Hence for nontrivial s the inner product must necessarily be indefinite.

Since the physical states should be s-invariant, we consider the kernel of Q: $\mathcal{K}_0 \stackrel{\text{def}}{=} \operatorname{Ke} Q$. Let $\mathcal{K}_{00}$ be the range of Q. Because of $Q^2 = 0$ we have $\mathcal{K}_{00} \subset \mathcal{K}_0$. We assume:

$$\begin{aligned} &(\textbf{Positivity}) \qquad \text{(i)} \quad < \phi, \phi > \geq 0 \qquad \forall \phi \in \mathcal{K}_0, \\ &\text{and} \qquad \text{(ii)} \quad \phi \in \mathcal{K}_0 \quad \wedge \quad < \phi, \phi > = 0 \quad \Longrightarrow \quad \phi \in \mathcal{K}_{00}. \end{aligned} \tag{4.8}$$

Then

$$\mathcal{H} \stackrel{\text{def}}{=} \frac{\mathcal{K}_0}{\mathcal{K}_{00}}, \qquad < [\phi_1], [\phi_2] >_{\mathcal{H}} \stackrel{\text{def}}{=} < \psi_1, \psi_2 >_{\mathcal{K}}, \quad \psi_j \in [\phi_j] := \phi_j + \mathcal{K}_{00} \tag{4.9}$$

is a pre Hilbert space. (Due to (4.7) the definition of $< [\phi_1], [\phi_2] >_{\mathcal{H}}$ is independent of the choice of the representatives $\psi_j \in [\phi_j]$, $j = 1, 2$.)

Now we assure that

$$\pi([A])[\phi] \stackrel{\text{def}}{=} [A\phi] \tag{4.10}$$

is a well defined representation on $\mathcal{H}$ (where $A \in \mathcal{A}_0$, $\phi \in \mathcal{K}_0$, $[A] := A + \mathcal{A}_{00}$). Namely, let $A + s(B)$, $A \in \mathcal{A}_0$, $B \in \mathcal{F}$, be a representative of $[A] \in \mathcal{A}$ in $\mathcal{F}$, and let $\phi + Q\psi$, $\phi \in \mathcal{K}_0$, $\psi \in \mathcal{K}$ be a representative of $[\phi] \in \mathcal{H}$ in $\mathcal{K}$. We have to show that $A\phi \in \mathcal{K}_0$ and $(A + s(B))(\phi + Q\psi) - A\phi \in \mathcal{K}_{00} = Q\mathcal{K}$. But $QA\phi = s(A)\phi + (-1)^{\delta(A)}AQ\phi = 0$, and

$$\begin{aligned} s(B)\phi + (A + s(B))Q\psi &= (QB - (-1)^{\delta(B)}BQ)(\phi + Q\psi) - (-1)^{\delta(A)}(s(A)\psi - QA\psi) = \\ &= QB(\phi + Q\psi) + (-1)^{\delta(A)}QA\psi \in Q\mathcal{K}. \end{aligned}$$

4.3 Stability under deformations

It is gratifying that the described structure is *stable under deformations*, e.g. by turning on the interaction. Let $\mathcal{K}$ be fixed and replace $F \in \mathcal{F}$ by a formal power series $\tilde{F} = \sum_n g^n F_n$ with $F_0 = F$ and $F_n \in \mathcal{F}$, $\delta(F_n) = \text{const}$. In the same way replace s and Q by formal power series $\tilde{s} = \sum_n g^n s_n$ (each s_n is a graded derivation), $\tilde{Q} = \sum_n g^n Q_n$, $Q_n \in L(\mathcal{K})$, with $s_0 = s$, $Q_0 = Q$ and

$$\tilde{s}^2 = 0, \quad \tilde{Q}^2 = 0, \quad <\tilde{Q}\phi, \psi> = <\phi, \tilde{Q}\psi> \quad \text{and} \quad \tilde{s}(\tilde{F}) = \tilde{Q}\tilde{F} - (-1)^{\delta(\tilde{F})}\tilde{F}\tilde{Q}. \tag{4.11}$$

We can then define $\tilde{\mathcal{A}} \stackrel{\text{def}}{=} \frac{\text{Ke}\,\tilde{s}}{\text{Ra}\,\tilde{s}}$. $\mathcal{K}_0$ and $\mathcal{K}_{00}$ have to be replaced by formal power series $\tilde{\mathcal{K}}_0 := \text{Ke}\,\tilde{Q}$ and $\tilde{\mathcal{K}}_{00} := \text{Ra}\,\tilde{Q}$ with coefficients in $\mathcal{K}$. Due to the above result, the algebra $\tilde{\mathcal{A}}$ has a natural representation on $\tilde{\mathcal{H}} \stackrel{\text{def}}{=} \frac{\tilde{\mathcal{K}}_0}{\tilde{\mathcal{K}}_{00}}$. The inner product on $\mathcal{K}$ induces an inner product on $\tilde{\mathcal{H}}$ which assumes values in the formal power series over $\mathbf{C}$. We adopt the point of view that a formal power series $\tilde{b} = \sum_n g^n b_n$, $b_n \in \mathbf{C}$ is *positive* if there is another formal power series $\tilde{c} = \sum_n g^n c_n$, $c_n \in \mathbf{C}$ with $\tilde{c}^* \tilde{c} = \tilde{b}$, i.e. $b_n = \sum_{k=0}^n \bar{c}_k c_{n-k}$. This is equivalent to the condition [4]

$$b_n \in \mathbf{R} \qquad \forall n \in \mathbf{N}_0, \tag{4.12}$$

and

$$\exists k \in \mathbf{N}_0 \cup \{\infty\} \qquad \text{such that} \qquad b_l = 0 \quad \forall l < 2k,$$
$$\text{and} \qquad b_{2k} > 0 \qquad \text{in the case} \quad k < \infty. \tag{4.13}$$

We now show that the assumptions concerning the positivity of the inner product are automatically fulfilled for the deformed theory, if they hold true in the undeformed model.

Theorem 4: Let the positivity assumption (4.8) be fulfilled in zeroth order. Then
(i) $<\tilde{\phi}, \tilde{\phi}> \geq 0 \qquad \forall \tilde{\phi} \in \tilde{\mathcal{K}}_0$,
(ii) $\tilde{\phi} \in \tilde{\mathcal{K}}_0 \quad \wedge \quad <\tilde{\phi}, \tilde{\phi}> = 0 \quad \Longrightarrow \quad \tilde{\phi} \in \tilde{\mathcal{K}}_{00}$.
(iii) For every $\phi \in \mathcal{K}_0$ there exists a power series $\tilde{\phi} \in \tilde{\mathcal{K}}_0$ with $(\tilde{\phi})_0 = \phi$.
(iv) Let π and $\tilde{\pi}$ be the representations (4.10) of $\mathcal{A}$, $\tilde{\mathcal{A}}$ on $\mathcal{H}$, $\tilde{\mathcal{H}}$ respectively. Then $\tilde{\pi}(\tilde{A}) \neq 0$ if $\pi(A_0) \neq 0$.

Proof of Theorem 4, (i) and (ii): Let $\tilde{\phi} \in \tilde{\mathcal{K}}_0$ and $b_n = \sum_{k=0}^n <\phi_k, \phi_{n-k}>$. b_n clearly is real. $\tilde{Q}\tilde{\phi} = 0$ implies $Q_0\phi_0 = 0$, hence $\phi_0 \in \mathcal{K}_0$ and $b_0 \geq 0$. If $b_0 > 0$ (i) follows. If $b_0 = 0$ we know that there is some $\psi_0 \in \mathcal{K}$ with $\phi_0 = Q_0\psi_0$. Let $\psi_k^{(0)} := \psi_0 \delta_{k,0}$ and $\tilde{\psi}^{(0)} := \sum_k g^k \psi_k^{(0)} = \psi_0$. Then

$$\tilde{\eta}^{(0)} := \tilde{\phi} - \tilde{Q}\tilde{\psi}^{(0)}, \tag{4.14}$$

is a formal power series with vanishing term of zeroth order. We now proceed by induction and assume that $b_0 = b_1 = ... = b_{2n} = 0$ and there is some formal power series $\tilde{\psi}^{(n)} = \sum_k g^k \psi_k^{(n)}$ with coefficients in $\mathcal{K}$ such that

$$\tilde{\eta}^{(n)} := \sum_k g^k \eta_k^{(n)} = \tilde{\phi} - \tilde{Q}\tilde{\psi}^{(n)} \tag{4.15}$$

[4] Bordemann and Waldmann [BW] work with a weaker definition of positivity in the case of a formal Laurent series with real coefficients: they only require that the smallest non-vanishing coefficient is positive, it does not need to be an even coefficient.

vanishes up to order n. Then

$$b_{2n+1} =< \tilde{\eta}^{(n)} + \tilde{Q}\tilde{\psi}^{(n)}, \tilde{\eta}^{(n)} + \tilde{Q}\tilde{\psi}^{(n)} >_{2n+1} =< \tilde{\eta}^{(n)}, \tilde{\eta}^{(n)} >_{2n+1} = 0 \tag{4.16}$$

and $b_{2n+2} =< \eta^{(n)}_{n+1}, \eta^{(n)}_{n+1} >$. Since $\tilde{Q}\tilde{\eta}^{(n)} = 0$ we get $Q_0\eta^{(n)}_{n+1} = 0$, i.e. $\eta^{(n)}_{n+1} \in \mathcal{K}_0$ and $b_{2n+2} \geq 0$. If $b_{2n+2} > 0$ we obtain (i), otherwise $\exists \psi_{n+1} \in \mathcal{K}$ with $\eta^{(n)}_{n+1} = Q_0\psi_{n+1}$, and we can define

$$\psi^{(n+1)}_k := \psi^{(n)}_k + \delta_{n+1,k}\psi_{n+1}. \tag{4.17}$$

One easily verifies

$$(\tilde{\phi} - \tilde{Q}\tilde{\psi}^{(n+1)})_k = 0, \qquad \forall k = 0, 1, ..., n+1. \tag{4.18}$$

Either the induction stops at some n or we find a $\tilde{\psi}$ with $\tilde{\phi} = \tilde{Q}\tilde{\psi}$, i.e. $\tilde{\phi} \in \tilde{\mathcal{K}}_{00}$.

To prove (iii) we again proceed by induction and assume that there exists a power series $\tilde{\phi}^{(n)}$ such that $\tilde{Q}\tilde{\phi}^{(n)}$ vanishes up to order n. This is certainly true for $n = 0$. Then $0 = (\tilde{Q}^2\tilde{\phi}^{(n)})_{n+1} = Q_0(\tilde{Q}\tilde{\phi}^{(n)})_{n+1}$, hence $(\tilde{Q}\tilde{\phi}^{(n)})_{n+1} \in \mathcal{K}_0$. Moreover $0 = < \tilde{Q}\tilde{\phi}^{(n)}, \tilde{Q}\tilde{\phi}^{(n)} >_{2n+2} =< (\tilde{Q}\tilde{\phi}^{(n)})_{n+1}, (\tilde{Q}\tilde{\phi}^{(n)})_{n+1} >$, thus $(\tilde{Q}\tilde{\phi}^{(n)})_{n+1} \in \mathcal{K}_{00}$ and there exists a $\phi_{n+1} \in \mathcal{K}$ with $(\tilde{Q}\tilde{\phi}^{(n)})_{n+1} + Q_0\phi_{n+1} = 0$. We then set $(\tilde{\phi}^{(n+1)})_k := (\tilde{\phi}^{(n)})_k + \delta_{n+1,k}\phi_{n+1}$ and find that $\tilde{Q}\tilde{\phi}^{(n+1)}$ vanishes up to order $n+1$.

$$\tilde{\phi} := \lim_{n\to\infty} \tilde{\phi}^{(n)} \in \tilde{\mathcal{K}}_0 \tag{4.19}$$

is then the wanted formal power series.

It remains to prove (iv). $\tilde{\pi}(\tilde{A}) = 0$ means $\tilde{A} = \sum_k g^k A_k \in \mathrm{Ke}\,\tilde{s}$ with $\tilde{A}\tilde{\phi} \in \tilde{\mathcal{K}}_{00}, \forall \tilde{\phi} \in \tilde{\mathcal{K}}_0$. By means of (iii) this implies in zeroth order $A_0\phi_0 \in \mathcal{K}_{00}$, $\forall \phi_0 \in \mathcal{K}_0$, i.e. $\pi(A_0) = 0$. □

Note that $\phi \to \tilde{\phi}$ is non-unique and this holds true also for the induced relation between $\mathcal{H}$ and $\tilde{\mathcal{H}}$.

The unit $\tilde{1}$ in an algebra of formal power series is $\tilde{1} = (1, 0, 0, ...) = 1g^0$, and $\tilde{a} = \sum_{k=0}^{\infty} a_k g^k$ is invertible iff a_0 is invertible. We denote by $\tilde{\mathbf{C}}$ the formal power series over $\mathbf{C}$ and consider $\tilde{\mathcal{K}} \stackrel{\mathrm{def}}{=} \{\tilde{\phi} = \sum_n \phi_n g^n | \phi_n \in \mathcal{K}\}$ and $\tilde{\mathcal{F}} \stackrel{\mathrm{def}}{=} \{\tilde{F} = \sum_n F_n g^n | F_n \in \mathcal{F}\}$ as $\tilde{\mathbf{C}}$-modules. This is possible because the usual multiplication of power series yields maps

$$\tilde{\mathbf{C}} \times \tilde{\mathcal{F}} \to \tilde{\mathcal{F}} : (\tilde{a}, \tilde{A}) \to \tilde{a}\tilde{A} = \tilde{A}\tilde{a}, \qquad \tilde{\mathbf{C}} \times \tilde{\mathcal{K}} \to \tilde{\mathcal{K}} : (\tilde{a}, \tilde{\phi}) \to \tilde{a}\tilde{\phi} = \tilde{\phi}\tilde{a},$$

which fulfil the relations

$$\tilde{A}(\tilde{a}\tilde{\phi}) = \tilde{a}(\tilde{A}\tilde{\phi}) = (\tilde{a}\tilde{A})\tilde{\phi}, \qquad (\tilde{a}\tilde{A})^* = \tilde{a}^*\tilde{A}^*, \qquad < \tilde{a}\tilde{\phi}, \tilde{b}\tilde{\psi} >= \tilde{a}^*\tilde{b} < \tilde{\phi}, \tilde{\psi} > \tag{4.20}$$

and

$$\tilde{s}(\tilde{a}\tilde{A}) = \tilde{a}\tilde{s}(\tilde{A}). \tag{4.21}$$

Also the physical pre Hilbert space $\tilde{\mathcal{H}}$ and the algebra of observables $\tilde{\mathcal{A}}(\mathcal{O})$ are $\tilde{\mathbf{C}}$-modules, and the multiplications by a 'scalar'

$$\tilde{\mathbf{C}} \times \tilde{\mathcal{A}}(\mathcal{O}) \to \tilde{\mathcal{A}}(\mathcal{O}) : (\tilde{a}, [\tilde{A}]) \to \tilde{a}[\tilde{A}] = [\tilde{a}\tilde{A}] = [\tilde{A}]\tilde{a} \tag{4.22}$$

$$\tilde{\mathbf{C}} \times \tilde{\mathcal{H}} \to \tilde{\mathcal{H}} : (\tilde{a}, [\tilde{\phi}]) \to \tilde{a}[\tilde{\phi}] = [\tilde{a}\tilde{\phi}] = [\tilde{\phi}]\tilde{a} \tag{4.23}$$

satisfy (4.20). We are now going to prove that every $[\tilde{\phi}] \in \tilde{\mathcal{H}}$ can be normalized:

Corollary 5: For every $[\tilde{\phi}] \in \tilde{\mathcal{H}}$, $[\tilde{\phi}] \neq 0$, there exist $[\tilde{\psi}] \in \tilde{\mathcal{H}}$ and $\tilde{a} \in \tilde{\mathbf{C}}$ such that

$$[\tilde{\phi}] = \tilde{a}[\tilde{\psi}] \qquad \text{and} \qquad < [\tilde{\psi}], [\tilde{\psi}] >= 1. \tag{4.24}$$

Proof: We set $\tilde{b} :=< [\tilde{\phi}], [\tilde{\phi}] >\in \tilde{\mathbf{C}}$. From Theorem 4 we know $\tilde{b} = \sum_{n=2k}^{\infty} b_n g^n$, $b_n \in \mathbf{R}$, $b_{2k} > 0$.

Case (1), $k = 0$: There exists an invertible $\tilde{a} \in \tilde{\mathbf{C}}$ with $\tilde{a}^*\tilde{a} = \tilde{b}$. Then $[\tilde{\psi}] := \tilde{a}^{-1}[\tilde{\phi}]$ satisfies the assertion (4.24).

Case (2), $k > 0$: We consider a representative $\tilde{\phi} = \sum_n \phi_n g^n$ of $[\tilde{\phi}]$. Due to $< \phi_0, \phi_0 >= b_0 = 0$ and $Q_0\phi_0 = 0$, there exists $\eta_0 \in \mathcal{K}$ with $Q_0\eta_0 = \phi_0$. Then we can define $\tilde{\tau}_1$ by $g\tilde{\tau}_1 := \tilde{\phi} - \tilde{Q}\eta_0$ which fulfills $\tilde{\tau}_1 \in \tilde{\mathcal{K}}_0$ and $[\tilde{\phi}] = g[\tilde{\tau}_1]$. Hence $< [\tilde{\tau}_1], [\tilde{\tau}_1] >= g^{-2}\tilde{b}$. If $k > 1$ we repeat this procedure (starting with $[\tilde{\tau}_1]$ instead of $[\tilde{\phi}]$) until we obtain a $\tilde{\tau}_k \in \tilde{\mathcal{K}}_0$ with $[\tilde{\phi}] = g^k[\tilde{\tau}_k]$ and hence $< [\tilde{\tau}_k], [\tilde{\tau}_k] >= g^{-2k}\tilde{b}$. Similarly to case (1) we conclude that there exists an invertible $\tilde{c} \in \tilde{\mathbf{C}}$ with $\tilde{c}^*\tilde{c} = g^{-2k}\tilde{b}$. Then $[\tilde{\psi}] := \tilde{c}^{-1}[\tilde{\tau}_k]$ satisfies $< [\tilde{\psi}], [\tilde{\psi}] >= 1$ and $[\tilde{\phi}] = g^k\tilde{c}[\tilde{\psi}]$, i.e. (4.24) is satisfied for $\tilde{a} := g^k\tilde{c}$. □

A state ω on the algebra of observables $\tilde{\mathcal{A}}(\mathcal{O})$ is defined by
(i) ω: $\tilde{\mathcal{A}}(\mathcal{O}) \to \tilde{\mathbf{C}}$ is linear, i.e. $\omega(\tilde{a}[\tilde{A}] + [\tilde{B}]) = \tilde{a}\omega([\tilde{A}]) + \omega([\tilde{B}])$,
(ii) $\omega([\tilde{A}]^*) = \omega([\tilde{A}])^* \qquad \forall [\tilde{A}] \in \tilde{\mathcal{A}}(\mathcal{O})$,
(iii) $\omega([\tilde{A}]^*[\tilde{A}]) \geq 0 \qquad \forall [\tilde{A}] \in \tilde{\mathcal{A}}(\mathcal{O})$ and
(iv) $\omega(\tilde{\mathbf{1}}) = \tilde{1}$.
The constructed physical states, i.e. the vector states

$$\omega_{\tilde{\phi}}([\tilde{A}]) =< [\tilde{\phi}], [\tilde{A}][\tilde{\phi}] >, \qquad [\tilde{\phi}] \in \tilde{\mathcal{H}}, \tag{4.25}$$

satisfy obviously (i), (ii) and, if $< [\tilde{\phi}], [\tilde{\phi}] >= 1$, also (iv). The positivity (iii) of the states $\omega_{\tilde{\phi}}$ is ensured by

Corollary 6 (Positivity of the Wightman distributions of gauge invariant fields): Let the algebra $\tilde{\mathcal{A}}$ be generated by the $\tilde{s}$-invariant fields $\tilde{\phi}_1, ..., \tilde{\phi}_l$ and let

$$\tilde{A} := \sum_{n=0}^{k} \int \sum_{j_1,...,j_n} f_{j_1...j_n}(x_1, ..., x_n)\tilde{\phi}_{j_1}(x_1)...\tilde{\phi}_{j_n}(x_n)dx_1...dx_n, \qquad f_{j_1...j_n} \in \mathcal{D}(\mathbf{R}^{4n}), \tag{4.26}$$

and $\tilde{\phi} \in \tilde{\mathcal{K}}_0$. Then

$$< \tilde{\phi}, \tilde{A}^*\tilde{A}\tilde{\phi} >\geq 0. \tag{4.27}$$

Proof: Note $\tilde{A}\tilde{\phi} \in \tilde{\mathcal{K}}_0$ and apply part (i) of Theorem 4. □

Remark: Let us assume $\tilde{Q} = Q_0$. This situation occurs if the adiabatic limit exists ([KO], see also sect. 5.2), e.g. in massive gauge theories. Then $\tilde{Q}\tilde{\phi} = 0$ means $Q_0\phi_k = 0, \forall k$. Therefore, in this case the physical pre Hilbert space $\tilde{\mathcal{H}}$ is the space of formal power series with coefficients in $\mathcal{H}$,

$$\tilde{\mathcal{H}} = \tilde{\mathbf{C}}\mathcal{H} \qquad (\text{if} \quad \tilde{Q} = Q_0). \tag{4.28}$$

But the states ω_ϕ on $\tilde{\mathcal{A}}(\mathcal{O})$ induced by vectors $\phi \in \mathcal{H}$ remain $\tilde{\mathbf{C}}$-valued functionals.

5. Verification of the assumptions in the example of QED

The construction in the previous section relies on some assumptions, which we are now going to verify in QED. The deformation is given by going over from the free theory to the interacting fields discussed in sections 2 and 3. For the free and the interacting theory we will first define the BRST-transformation s ($\tilde{s}$ resp.) and then we will construct a nilpotent and hermitean operator Q ($\tilde{Q}$) which implements s ($\tilde{s}$) in a representation space with indefinite inner product. Then the local observables (defined by (4.4)) are naturally represented on $\mathcal{H} = \frac{\mathrm{Ke}\, Q}{\mathrm{Ra}\, Q}$ ($\tilde{\mathcal{H}} = \frac{\mathrm{Ke}\, \tilde{Q}}{\mathrm{Ra}\, \tilde{Q}}$) by (4.10). It remains to prove the positivity of the inner product induced in $\mathcal{H}$ ($\tilde{\mathcal{H}}$). For the free theory we will do this by giving explicitly (distinguished) representatives of the equivalence classes in $\mathcal{H}$. Then we conclude from Theorem 4 that positivity holds true also for $\tilde{\mathcal{H}}$.

5.1 The free theory

We consider the field algebra $\mathcal{F}$ which is generated by the free fields A^μ, ψ, $\overline{\psi}$, u, $\tilde{u}$, the Wick monomials $j^\mu =: \overline{\psi}\gamma^\mu\psi :$, $\gamma_\mu A^\mu\psi$, $\overline{\psi}\gamma_\mu A^\mu$, $\mathcal{L}^0 = j_\mu A^\mu$ and the derivatives of free fields $\partial_\mu A^\mu$, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. This algebra has a faithful representation on the Fock space $\mathcal{K} = \mathcal{K}_{\mathcal{A}} \otimes \mathcal{K}_\psi \otimes \mathcal{K}_g$ of free fields (appendix A). The $\mathbf{Z}_2$-gradiation is $(-1)^{\delta(F)}$, where F is a monomial in $\mathcal{F}$ and $\delta(F)$ is the ghost number

$$[Q_u, F] = \delta(F)F, \qquad Q_u \stackrel{\mathrm{def}}{=} i \int_{x_0=\mathrm{const.}} d^3x : \tilde{u}(x) \overset{\leftrightarrow}{\partial}_0 u(x) : . \tag{5.1}$$

Note $\delta(u) = -1$, $\delta(\tilde{u}) = 1$. The graded derivation s is the BRST-transformation of free fields

$$s(A^\mu) = i\partial^\mu u, \quad s(\psi) = 0, \quad s(\overline{\psi}) = 0, \quad s(u) = 0, \quad s(\tilde{u}) = -i\partial_\mu A^\mu. \tag{5.2}$$

The transformation of Wick monomials and derivated free fields is given by

$$s(: \phi_1(x)\phi_2(x)\ldots :) =: s(\phi_1)(x)\phi_2(x)\ldots : + (-1)^{\delta(\phi_1)} : \phi_1(x)s(\phi_2)(x)\ldots : + \ldots \tag{5.3}$$

and by translation invariance of s

$$s(\partial_\mu \phi(x)) = \partial^x_\mu s(\phi)(x). \tag{5.4}$$

This transformation is implemented by the free Kugo-Ojima charge [DHKS1]

$$Q \stackrel{\mathrm{def}}{=} \int_{x_0=\mathrm{const.}} d^3x \, (\partial_\nu A^\nu(x)) \overset{\leftrightarrow}{\partial}_0 u(x), \tag{5.5}$$

which fulfills $Q^* = Q$, [5] $[Q_u, Q] = -Q$ and $Q^2 = 0$. This is verified in appendix A. Moreover, it is shown there that the inner product $< ., . >$ is positive semidefinite on $\mathrm{Ke}\, Q$ and that the space of null vectors in $\mathrm{Ke}\, Q$ is precisely $\mathrm{Ra}\, Q$ [DHS1].

[5] We restrict all operators (resp. formal power series of operators) to the dense invariant domain $\mathcal{D}$ and, therefore, there is no difference between symmetric and self-adjoint operators.

The existence of the integral in (5.5) can be proven by the following method due to Requardt [R]. We smear out $J^0 = \partial_\nu A^\nu \partial^{\leftrightarrow}{}_0 u$ with $k(x_0)h(\mathbf{x}) \in \mathcal{D}(\mathbf{R}^4)$, where $\int dx_0\, k(x_0) = 1$ and h is a smeared characteristic function of $\{\mathbf{x} \in \mathbf{R}^3, |\mathbf{x}| \leq R\}$ for some $R > 0$. By scaling $k_\lambda(x_0) := \lambda k(\lambda x_0)$, $h_\lambda(\mathbf{x}) := h(\lambda \mathbf{x})$, $Q_\lambda := \int d^4x\, k_\lambda(x_0) h_\lambda(\mathbf{x}) J^0(x)$ one easily finds $\lim_{\lambda \to 0} \|Q_\lambda \Omega\| = 0$ w.r.t. a suitable Krein space norm (cf Appendix A). In addition, due to current conservation, $\lim_{\lambda \to 0} [Q_\lambda, \phi(y)]_\mp = \int d^3x\, [J^0(x), \phi(y)]_\mp$ (note that the latter integral exists since the region of integration is bounded) for R big enough compared to the support of k. Therefore, the strong limit $\lim_{\lambda \to 0} Q_\lambda$ exists on a dense subspace and agrees with (5.5).

Unfortunately, the representation (4.10) of the observables $\mathcal{A} = \frac{\mathrm{Ke}\, s}{\mathrm{Ra}\, s}$ on the physical pre Hilbert space $\mathcal{H} = \frac{\mathrm{Ke}\, Q}{\mathrm{Ra}\, Q}$ is *not faithful* . The counterexample is $[u(f)]$, $f \in \mathcal{D}(\mathbf{R^4})$ real-valued, which induces a non-trivial element of $\mathcal{A}$ if $\int f d^4x \neq 0$. Namely, due to $u(f)^* u(f) = u(f)^2 = 0$, it is represented by zero on $\mathcal{H}$. (This holds true for each representation in which $< \cdot, \cdot >$ is positive definite.) Since $u(\partial_\mu h) = i\, s(A_\mu(h))$, $h \in \mathcal{D}(\mathbf{R^4})$, $\mathcal{A}$ has the following structure:

$$\mathcal{A} = \mathcal{A}^{(0)} \oplus u_0 \mathcal{A}^{(0)}, \tag{5.6}$$

where u_0 is the rest class of $u(f)$, $\int f d^4x = 1$ and where $\mathcal{A}^{(0)}$ is the subalgebra with ghost number zero.

The representation (4.10) of $\mathcal{A}^{(0)}$ on $\mathcal{H}$ is faithful. To make this plausible we mention that $\mathcal{A}^{(0)}$ is generated by $[F^{\mu\nu}]$, $[\psi]$, $[\overline{\psi}]$ and Wick monomials thereof, whereas the 'canonical' representatives of $\mathcal{H}$ are the states containing transversal photons, electrons and positrons only (A.39).

The interaction Lagrangian of QED is s-invariant up to a divergence of a local field,

$$[Q, \mathcal{L}^0(x)] = i \partial_\mu \mathcal{L}^{1\,\mu}(x), \qquad \mathcal{L}^{1\,\mu} \stackrel{\mathrm{def}}{=} : \overline{\psi} \gamma^\mu \psi : u. \tag{5.7}$$

Thus in the formal adiabatic limit the integral of the Lagrangian becomes invariant. In [DHKS1,DHKS2] the following Ward identities were postulated:

$$[Q, T(\mathcal{L}^0(x_1) ... \mathcal{L}^0(x_n))] = \sum_{l=1}^{n} \partial_\mu^{x_l} T(\mathcal{L}^0(x_1) ... \mathcal{L}^{1\,\mu}(x_l) ... \mathcal{L}^0(x_n)). \tag{5.8}$$

('free (perturbative) operator gauge invariance', compare (3.12)). Provided the adiabatic limit exists this condition implies the s-invariance of the S-matrix; hence the S-matrix induces a unitary operator on the physical Hilbert space $\mathcal{H} = \frac{\mathrm{Ke}\, Q}{\mathrm{Ra}\, Q}$ [DHS1,K].

The nice feature of the condition (5.8) is that its formulation makes sense independent of the adiabatic limit. So, if the normalizations are suitably chosen, the free (perturbative) operator gauge invariance (5.8) (more precisely the corresponding C-number identities which imply (5.8)) could been proven to hold to all orders in QED [DHS2,S] and also in $SU(N)$ Yang-Mills theories [DHS1,D1] and to imply (in the latter case) the usual Slavnov-Taylor identities [D2]. In addition, it determines to a large extent the possible structure of the model. Stora [St1] found that making a general ansatz for the interaction Lagrangian of selfinteracting gauge fields, the Ward identities (5.8) require the coupling parameters to be totally antisymmetric structure constants of a Lie group. Moreover, (5.8) was used for a derivation of all the couplings of the standard model of electroweak interactions (especially the Higgs potential) [DS].

We emphasize that (5.8) is a *pure quantum formulation of gauge invariance*, without reference to classical physics.

5.2 The interacting theory: construction of the interacting Kugo-Ojima charge

We now replace the free fields (including Wick monomials and derivatives) considered in the previous subsection by the corresponding interacting fields, which are formal power series of unbounded operators in the Fock space $\mathcal{K}$ of free fields. Due to $[Q_u, \mathcal{L}(x)] = 0$ (3.1), we can normalize the time ordered products such that

$$\textbf{(N6)} \qquad [Q_u, T(\mathcal{L}(x_1)...\mathcal{L}(x_n))] = 0$$

$$\text{and} \qquad [Q_u, T(\mathcal{L}(x_1)...\mathcal{L}(x_n)F(x))] = \delta(F)T(\mathcal{L}(x_1)...\mathcal{L}(x_n)F(x)).$$

Hence $[Q_u, F_{\text{int}\,\mathcal{L}}] = \delta(F)F_{\text{int}\,\mathcal{L}}$ by (2.8-9). The fundamental normalization condition concerning the ghost number is (B.6) in combination with **(N3)**; they imply **(N6)**. Again we fix the region $\mathcal{O}$ to be the double cone $\mathcal{O} \stackrel{\text{def}}{=} \mathcal{O}_{(-r,\mathbf{0}),(r,\mathbf{0})}$, $(r > 0)$ (3.18) and assume the switching function $g \in \mathcal{D}(\mathbf{R}^4)$ to be constant on $\mathcal{O}$ (3.19). We study the algebra $\tilde{\mathcal{F}}(\mathcal{O})$ (3.17) of interacting fields localized in $\mathcal{O}$. The ghost fields do not couple in QED, hence

$$u_{\text{int}\,\mathcal{L}}(x) = u(x), \qquad \tilde{u}_{\text{int}\,\mathcal{L}}(x) = \tilde{u}(x). \tag{5.9}$$

The abelian BRST-transformation $\tilde{s} = s_0 + gs_1$ [BRS] should be a graded derivation with zero square and compatible with the $*$-operation. In addition it shall induce the following transformations on the basic fields,[6]

$$\tilde{s}(A^\mu_{\text{int}\,\mathcal{L}}(x)) = i\partial^\mu u(x), \qquad \tilde{s}(u(x)) = 0, \qquad \tilde{s}(\tilde{u}(x)) = -i\partial_\mu A^\mu_{\text{int}\,\mathcal{L}}(x),$$

$$\tilde{s}(\psi_{\text{int}\,\mathcal{L}}(x)) = -g(x)\psi_{\text{int}\,\mathcal{L}}(x)u(x), \qquad \tilde{s}(\overline{\psi}_{\text{int}\,\mathcal{L}}(x)) = g(x)\overline{\psi}_{\text{int}\,\mathcal{L}}(x)u(x). \tag{5.10}$$

(The pointwise products are well defined.) Let us assume that we have constructed the interacting Kugo-Ojima charge $\tilde{Q} = Q_{\text{int}}(g)$. Then we shall define $\tilde{s}$ in terms of the corresponding current such that $Q_{\text{int}}(g)$ implements $\tilde{s}$

$$\tilde{s}(F) = Q_{\text{int}}(g)F - (-1)^{\delta(F)}FQ_{\text{int}}(g), \qquad F \in \tilde{\mathcal{F}}(\mathcal{O}). \tag{5.11}$$

If $Q_{\text{int}}(g)$ is hermitian, $\tilde{s}$ is compatible with the $*$-operation, and $\tilde{s}^2 = 0$ is implied by $Q_{\text{int}}(g)^2 = 0$.

To get $Q_{\text{int}}(g)$ we follow Kugo and Ojima [KO] and replace the current in the free charge Q (5.5) by the corresponding interacting field $\partial_\mu A^\mu_{\text{int}\,\mathcal{L}}(x)\partial^{\leftrightarrow x}_\nu u(x)$. By means of the field equation (3.8) and the Ward identity (3.13) we find

$$\partial^\nu_x[\partial_\mu A^\mu_{\text{int}\,\mathcal{L}}(x)\overset{\leftrightarrow}{\partial}{}^x_\nu u(x)] = -(\Box\partial_\mu A^\mu_{\text{int}\,\mathcal{L}}(x))u(x) = (\partial_\mu g)(x)j^\mu_{\text{int}\,\mathcal{L}}(x)u(x). \tag{5.12}$$

Hence the current is conserved in the region where g is constant. We may therefore define $\tilde{s}$ on an algebra $\tilde{\mathcal{F}}(\mathcal{O})$ in the following way: we choose $g(x) = e = \text{const}$ on a neighbourhood $\mathcal{U}$ of $\bar{\mathcal{O}}$ and set

$$\tilde{s}(F) \stackrel{\text{def}}{=} \int_{x_0=0} d^3x\,[\partial_\mu A^\mu_{\text{int}\,\mathcal{L}}(x)\overset{\leftrightarrow}{\partial}{}_0 u(x), F]_\mp, \qquad F \in \tilde{\mathcal{F}}(\mathcal{O}). \tag{5.13}$$

[6] In contrast to the free case $\psi_{\text{int}\,\mathcal{L}}$ and $\overline{\psi}_{\text{int}\,\mathcal{L}}$ are not observables in the sense of sect. 4.1. This different behaviour can be understood physically by the accompanying soft photon cloud and mathematically by Gauss' law.

Because of current conservation, $\tilde{s}$ is implemented by the operators

$$Q_{\rm int}(g,k) = \int d^4x\, k^\mu(x)\,(\partial_\nu A^\nu_{{\rm int}\,\mathcal{L}}(x))\overset{\leftrightarrow}{\partial}{}^x_\mu u(x) \tag{5.14}$$

with $k^\mu \in \mathcal{D}(\mathcal{U})$ where $k^\mu - \delta^\mu_0 h = \partial^\mu f$ for some $f \in \mathcal{D}(\mathcal{U})$ and where $h \in \mathcal{D}(\mathcal{U})$ is a suitably chosen smeared characteristic function of the surface $\{(0,\mathbf{x}), |\mathbf{x}| \leq r\}$.

Now we are well prepared to prove that the definition (5.13) of $\tilde{s}$ agrees with the usual expressions (5.10) on the basic fields, and to compute all further commutators of $Q_{\rm int}(g)$ with the interacting sub Wick monomials of $\mathcal{L}^0$ (3.15).

Theorem 7: We assume that the interacting fields are normalized as described in sections 2 and 3, especially that they fulfil the field equations (3.8-9) and the Ward identities **(N5)**. Furthermore we assume $g = e =$ const on the double cone $\mathcal{O} = \mathcal{O}_{(-r,\mathbf{0}),(r,\mathbf{0})}$. Let k^μ as before. Then we find the commutation rules

$$[Q_{\rm int}(g,k), A^\mu_{{\rm int}\,\mathcal{L}}(y)] = i\partial^\mu u(y), \qquad [Q_{\rm int}(g,k), \partial_\mu A^\mu_{{\rm int}\,\mathcal{L}}(y)] = 0, \tag{5.15a,b}$$

$$[Q_{\rm int}(g,k), \psi_{{\rm int}\,\mathcal{L}}(y)] = -e\psi_{{\rm int}\,\mathcal{L}}(y)u(y), \qquad [Q_{\rm int}(g,k), \overline{\psi}_{{\rm int}\,\mathcal{L}}(y)] = e\overline{\psi}_{{\rm int}\,\mathcal{L}}(y)u(y), \tag{5.16a,b}$$

$$\{Q_{\rm int}(g,k), u(y)\} = 0, \qquad \{Q_{\rm int}(g,k), \tilde{u}(y)\} = -i\partial_\mu A^\mu_{{\rm int}\,\mathcal{L}}(y), \tag{5.17a,b}$$

$$[Q_{\rm int}(g,k), F^{\mu\nu}_{{\rm int}\,\mathcal{L}}(y)] = 0, \qquad [Q_{\rm int}(g,k), j^\mu_{{\rm int}\,\mathcal{L}}(y)] = 0, \tag{5.18a,b}$$

$$[Q_{\rm int}(g,k), (\gamma_\mu A^\mu\psi)_{{\rm int}\,\mathcal{L}}(y)] = -e(\gamma_\mu A^\mu\psi)_{{\rm int}\,\mathcal{L}}(y)u(y) + i\gamma_\mu\psi_{{\rm int}\,\mathcal{L}}(y)\partial^\mu u(y), \tag{5.19}$$

$$[Q_{\rm int}(g,k), (\overline{\psi}\gamma_\mu A^\mu)_{{\rm int}\,\mathcal{L}}(y)] = e(\overline{\psi}\gamma_\mu A^\mu)_{{\rm int}\,\mathcal{L}}(y)u(y) + i\overline{\psi}_{{\rm int}\,\mathcal{L}}(y)\gamma_\mu\partial^\mu u(y), \tag{5.20}$$

$$[Q_{\rm int}(g,k), (:\overline{\psi}\gamma_\mu\psi : A^\mu)_{{\rm int}\,\mathcal{L}}(y)] = (:\overline{\psi}\gamma_\mu\psi :)_{{\rm int}\,\mathcal{L}}(y)i\partial^\mu u(y), \tag{5.21}$$

where always $y \in \mathcal{O}$.

Proof: Since the ghost fields are not influenced by the interaction, we know that the ghost and antighost fields commute with all other interacting fields. Moreover, the pointwise products of these fields with a ghost or antighost field are well defined and behave in commutators as ordinary products of operators in spite of their character as operator valued distributions. (This may be verified by using techniques from microlocal analysis as exposed in [BF].) Thus the above relations follow from the commutation rules with $\partial_\nu A^\nu_{{\rm int}\,\mathcal{L}}(x)$ in Proposition 3 and the ghost antighost anticommutation relations in equation (3.2). With these preparations the commutators (5.15-16) can easily be computed, for example

$$[Q_{\rm int}(g), \psi_{{\rm int}\,\mathcal{L}}(y)] = \int_{x_0=0} d^3x\,[\partial_\mu A^\mu_{{\rm int}\,\mathcal{L}}(x), \psi_{{\rm int}\,\mathcal{L}}(y)]\overset{\leftrightarrow}{\partial}{}^x_0 u(x) =$$

$$= e\psi_{{\rm int}\,\mathcal{L}}(y)\int d^3x\, D(x-y)\overset{\leftrightarrow}{\partial}{}^x_0 u(x) = -e\psi_{{\rm int}\,\mathcal{L}}(y)u(y), \tag{5.22}$$

where we have inserted Proposition 3, (3) and (A.31). By using other commutators of $\partial_\mu A^\mu_{{\rm int}\,\mathcal{L}}$ (Proposition 3) we analogously prove (5.15a,b), (5.16b) and (5.21). Alternatively (5.15b) can be obtained by taking the divergence ∂^y_μ of (5.15a). Part (a) of (5.17) is obvious due to $\{u(x), u(y)\} = 0$; let us compute part (b)

$$\{Q_{\rm int}(g), \tilde{u}(y)\} = -i\int_{x_0=0} d^3x\,\partial_\mu A^\mu_{{\rm int}\,\mathcal{L}}(x)\overset{\leftrightarrow}{\partial}{}^x_0 D(x-y). \tag{5.23}$$

From (5.12) we know $\Box\partial_\mu A^\mu_{\mathrm{int}\,\mathcal{L}}(x) = 0$, $\forall x \in \mathcal{O}$. Therefore, we may apply (A.31) to (5.23), which yields (5.17b). By applying ∂^ν_y to (5.15a) and using $F^{\mu\nu}_{\mathrm{int}\,\mathcal{L}} = \partial^\mu A^\nu_{\mathrm{int}\,\mathcal{L}} - \partial^\nu A^\mu_{\mathrm{int}\,\mathcal{L}}$ we get (5.18a). Analogously by working with the field equations for $A^\mu_{\mathrm{int}\,\mathcal{L}}$ (3.8) and $\psi_{\mathrm{int}\,\mathcal{L}}$ (3.9), we obtain (5.18b) and (5.19-20) from (5.15a) and (5.16a,b). □

In the formal adiabatic limit, $\partial_\nu A^\nu_{\mathrm{int}\,\mathcal{L}}$ converges to $\partial_\nu A^\nu$ (3.14) and therefore one expects that $Q_{\mathrm{int}}(g)$ will converge to the free Kugo-Ojima charge Q. Whereas in QED this reasoning seems to be correct, the corresponding argument does not work in nonabelian gauge theories (as can be seen by an explicit calculation of the first order of $Q_{\mathrm{int}}(g)$) [BDF].

We therefore prefer not to work in the adiabatic limit. The price to pay is that $Q_{\mathrm{int}}(g)$ does not agree with Q, so for the construction of the physical Hilbert space we have to check the conditions of Section 4. We easily find that $Q_{\mathrm{int}}(g,k)$ is hermitian for real valued k and we even get the nilpotency of $Q_{\mathrm{int}}(g,k)$,

$$Q_{\mathrm{int}}(g,k)^2 = \frac{1}{2}\{Q_{\mathrm{int}}(g,k), Q_{\mathrm{int}}(g,k)\} =$$

$$= \frac{1}{2}\int d^4x\, h(x) \int d^4y\, h(y) [\partial_\mu A^\mu_{\mathrm{int}\,\mathcal{L}}(x), \partial_\nu A^\nu_{\mathrm{int}\,\mathcal{L}}(y)] \overset{\leftrightarrow}{\partial}{}^x_0 \overset{\leftrightarrow}{\partial}{}^y_0 u(x)u(y) = 0, \tag{5.24}$$

by means of $k^\mu = \delta^\mu_0 h + \partial^\mu f$ and Proposition 3, (2).[7]

But we need in addition that the zeroth order term $Q_0(k)$ of $Q_{\mathrm{int}}(g,k)$ (5.14) satisfies the positivity assumption (4.8). There seems to be no reason why this should hold for a generic choice of k. One might try to control the limit when k tends to a smeared characteristic function of the $t = 0$ hyperplane (in order that $Q_0(k)$ becomes equal to the free charge Q (5.5)), but without a priori information on the existence of an $\tilde{s}$-invariant state this appears to be a hard problem.

There is a more elegant way to get rid of these problems which relies on the local character of our construction. We may embed our double cone $\mathcal{O}$ isometrically into the cylinder $\mathbf{R} \times C_L$, where C_L is a cube of length L, $\quad L \gg r$, with suitable boundary conditions (see appendix A), and where the first factor denotes the time axis. If we choose the compactification length L big enough, the physical properties of the local algebra $\tilde{\mathcal{F}}(\mathcal{O})$ are not changed.

The quantization of the free fields on this cube is worked out in appendix A. The inductive construction of the perturbation series for the S-matrix or the interacting fields is not changed by the compactification, sections 2 and 3 can be adopted without any modification [BF]. We assume the switching function g to fulfil

$$g(x) = e = \text{constant} \qquad \forall x \in \mathcal{O} \cup \{(x_0, \mathbf{x}) | \, |x_0| < \epsilon\} \qquad (r \gg \epsilon > 0) \tag{5.25}$$

on $\mathbf{R} \times C_L$ and to have compact support in timelike directions. Now we may insert

$$k^\mu(x) := \delta^\mu_0 h(x_0), \qquad h \in \mathcal{D}([-\epsilon, \epsilon]), \qquad \int dx_0\, h(x_0) = 1$$

into the expression (5.14) for Q_{int}, because $(x_0, \mathbf{x}) \to h(x_0)$ is an admissible test function on $\mathbf{R} \times C_L$. We define

$$Q_{\mathrm{int}}(g) : \mathcal{D} \longrightarrow \mathcal{D}, \qquad Q_{\mathrm{int}}(g) \overset{\mathrm{def}}{=} \int dx_0\, h(x_0) \int_{C_L} d^3x\, \partial_\mu A^\mu_{\mathrm{int}\,\mathcal{L}}(x) \overset{\leftrightarrow}{\partial}{}^x_0 u(x). \tag{5.26}$$

[7] We recall that the commutator $[\partial A_{\mathrm{int}\,\mathcal{L}}(x), \partial A_{\mathrm{int}\,\mathcal{L}}(y)]$ vanishes for *all* x and y for which supp $\partial_\mu g$ does not intersect $\overline{\mathcal{O}_{x,y} \cup \mathcal{O}_{y,x}}$.

By means of (5.12) and the fact that g is constant on the region of integration (the time-slice $[-\epsilon,\epsilon]\times C_L$), we conclude that the result of the integration over C_L is independent of x_0 and hence the arbitrariness in the choice of $h(x_0)$ drops out,

$$Q_{\rm int}(g)=\int_{C_L,\,x_0=\text{const.},\,|x_0|<\epsilon} d^3x\;\partial_\mu A^\mu_{{\rm int}\,\mathcal{L}}(x)\overset{\leftrightarrow}{\partial}{}^x_0\,u(x)\,. \tag{5.27}$$

By construction, $Q_{\rm int}(g)$ implements the BRST-transformation (5.13) and fulfills (see (5.24))

$$Q_{\rm int}(g)^2=0,\qquad Q_0=Q,\qquad [Q_u,Q_{\rm int}(g)]=-Q_{\rm int}(g)\quad\text{and}\quad Q_{\rm int}(g)^*=Q_{\rm int}(g), \tag{5.28}$$

where Q_0 is the zeroth order and Q is the free charge (5.5). The last property relies on the *-selfadjointness of u and $A^\mu_{{\rm int}\,\mathcal{L}}$.

We emphasize that our construction describes locally QED also in the *non-compactified* Minkowski space (this is the main concern of the paper) and, therefore, should not depend on the compactification length L. On the level of the algebras this is evident. The local algebras of interacting fields or observables belonging to different values of L are *isomorphic*. We conjecture that also the state space (i.e. the set of expectation functionals induced by vectors in the physical Hilbert space) is independent of L, but this remains to be proven.

6. Outlook

In an abstract setting, Buchholz has developed concepts for the treatment of scattering of infraparticles [Bu1]. It would be interesting to apply his ideas to perturbative QED. Our construction may be helpful for such an investigation which might lead to a more satisfactory understanding of the infrared problem in QED.

The importance of a *local* construction of the observables becomes even more evident in nonabelian gauge theories. There, in the absence of Higgs fields, the adiabatic limit seems not to exist and, hence, only a *local* model makes sense in the framework of perturbation theory. In the application of our construction to nonabelian gauge theories some technical problems appear, but we see no principal obstacle [BDF].

This is also the perspective for the generalization to curved space-times, where the techniques of [BF] can be used.

An open question is the physical meaning of the remaining normalization conditions in a local perturbative construction, after the restrictions from gauge invariance and other symmetries are taken into account. The parameters involved may be considered as structure constants of the algebra of observables, but their usual interpretation as charge and mass involve the adiabatic limit.

Appendix A: Implementation of the free BRST-transformation on the spatially compactified Minkowski space

In this appendix we quantize the free gauge, ghost and spinor fields in a finite spatial volume. Special care is needed in the choice of boundary conditions. In the second part of this appendix we prove that the free Kugo-Ojima charge Q (5.5) is nilpotent, implements the BRST-transformation of free fields and fufills our positivity assumption (4.8).

Let C_L be the open cube of length L. The algebra of a free scalar field φ_L with mass $m \geq 0$ on $\mathbf{R} \times C_L$ with *Dirichlet boundary conditions* [8] is the unital *-algebra generated by elements $\varphi_L(f)$, $f \in \mathcal{D}(\mathbf{R} \times C_L)$ with the relations

$$f \mapsto \varphi_L(f) \text{ is linear} \tag{A.1}$$

$$\varphi_L((\Box + m^2)f) = 0 \tag{A.2}$$

$$\varphi_L(f)^* = \varphi_L(\bar{f}) \tag{A.3}$$

$$[\varphi_L(f), \varphi_L(g)] = -i \int d^4x\, d^4y\, f(x)g(y)\Delta_L(x,y) \tag{A.4}$$

where Δ_L is the fundamental solution of the Klein Gordon equation on $\mathbf{R} \times C_L$ with Dirichlet boundary conditions, which has the explicit form

$$\Delta_L(x^0, \mathbf{x}, y^0, \mathbf{y}) = \sum_{s \in S} (-1)^{n(s)} \Delta(x^0 - y^0, \mathbf{x} - s(\mathbf{y})) \tag{A.5}$$

where S is the group generated by the reflections on the planes which bound C_L and $n(s)$ is the number of reflections occuring in s (which is well defined modulo 2). In particular one sees that Δ_L coincides with Δ on $\mathcal{O} \times \mathcal{O}$ if the closure of the double cone $\mathcal{O}$ is contained in $\mathbf{R} \times C_L$, considered as a region in Minkowski space. *Hence the algebra $\mathcal{F}(\mathcal{O})$ associated to $\mathcal{O}$ is independent of the boundary conditions.*

Since Δ_L depends only on time differences, the algebra is invariant under time translations. The ground state is the quasifree state whose 2-point function is the positive frequency part Δ_L^+ of Δ_L. In the massive case it is given in terms of the Minkowski space 2-point functions by a formula analogous to the formula for the commutator function. In the massless case a corresponding sum might not converge, instead we exploit the fact that the possible frequencies must have squares which are eigenvalues of the Laplace operator on C_L with Dirichlet boundary conditions. In particular zero modes do not appear. Therefore the frequency splitting can be done by convolution with the distribution

$$H(t) = \frac{ih(t)}{(2\pi)^{\frac{1}{2}}(t + i\varepsilon)} \tag{A.6}$$

[8] We first tried *periodic* boundary conditions, but this seems not to work for massless particles because of the existence of zero modes [DF]. For bosonic particles the algebra of the zero mode agrees with the algebra of a free Schrödinger particle in one dimension. There is no ground state on this algebra, and this makes it impossible to define the physical Hilbert space as the cohomology of the free Kugo-Ojima charge Q. Therefore, we choose boundary conditions which exclude the zero mode.

where h is a test function from the Schwartz space $\mathcal{S}(\mathbf{R})$ with $h(0) = 1$ and a Fourier transform with support in the interval $(-\omega, \omega)$ where ω^2 is the smallest eigenvalue of the Laplace operator. $H * \Delta$ differs from Δ^+ by a smooth function and decays fast in spatial directions. Hence Δ_L^+ admits a representation in terms of $H * \Delta$,

$$\Delta_L^+(x^0, \mathbf{x}, y^0, \mathbf{y}) = \sum_{s \in S} (-1)^{n(s)} H * \Delta(x^0 - y^0, \mathbf{x} - s(\mathbf{y})), \tag{A.7}$$

and $\Delta_+ - \Delta_L^+$ is smooth on $\mathcal{O} \times \mathcal{O}$. Due to a result of Verch [V], this implies that the representation of $\mathcal{F}(\mathcal{O})$ induced by the ground state on $\mathbf{R} \times C_L$ is unitarily equivalent to the Minkowski space vacuum representation.

We also need to compare the Wick products in both representations. Since the two point functions differ by a smooth function we may use the Minkowski space definition of Wick products also on $\mathbf{R} \times C_L$. [9] In [BF] a domain of definition of Wick polynomials was found which depends only on the equivalence class of the representation, hence also the Wick products can be identified.

For the electromagnetic field we may use *metallic boundary conditions*, i.e. the pullback of the 2-form F vanishes at the boundary (which means that the tangential components of the electric field and the normal component of the magnetic field vanish). In addition we assume that the auxiliary Nakanishi-Lautrup field $B = \partial^\mu A_{L\,\mu}$ (in Feynman gauge) satisfies Dirichlet boundary conditions.

The corresponding commutator function is

$$D_{\mu\nu,L}(x^0, \mathbf{x}, y^0, \mathbf{y}) = \sum_{s \in S} (-1)^{n(s)} s_\mu^\lambda g_{\lambda\nu} D(x^0 - y^0, \mathbf{x} - s(\mathbf{y})) \tag{A.8}$$

where the matrix (s_μ^λ) describes the action of s on covectors, e.g. for a reflection s on a plane $x_2 = \text{const.}$ we have $s_\mu^\lambda g_{\lambda\nu} = \text{diag}(1, -1, 1, -1)_{\mu\nu}$. The algebra generated by the vector potential $A_{L\,\mu}$ can then be defined as in the scalar case, and again the subalgebra associated to the double cone $\mathcal{O}$ is independent of the boundary conditions. A ground state can also be defined in terms of the positive frequency part of the two point function; as on Minkowski space, it violates the positivity condition.

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[9] To see this we consider

$$: \phi(x_1) \ldots \phi(x_n) :_\omega \Psi = \frac{\delta^n}{i^n \delta f(x_1) \ldots \delta f(x_n)} e^{\frac{1}{2}\omega(f,f)} e^{i\phi(f)} \Psi|_{f=0} =$$

$$= \sum_{l=0}^{[\frac{n}{2}]} \sum \frac{\delta^{2l} e^{\frac{1}{2}[\omega(f,f) - \omega'(f,f)]}}{i^{2l} \delta f(x_{i_1}) \delta f(x_{j_1}) \ldots \delta f(x_{i_l}) \delta f(x_{j_l})}|_{f=0} \cdot \frac{\delta^{(n-2l)} e^{\frac{1}{2}\omega'(f,f)} e^{i\phi(f)} \Psi}{i^{n-2l} \delta f(x_1) \ldots \hat{i}_1 \hat{j}_1 \ldots \hat{i}_l \hat{j}_l \ldots \delta f(x_n)}|_{f=0} =$$

$$=: \phi(x_1) \ldots \phi(x_n) :_{\omega'} \Psi + \sum_{l=1}^{[\frac{n}{2}]} (-1)^l \sum (\omega - \omega')(x_{i_1} - x_{j_1}) \ldots (\omega - \omega')(x_{i_l} - x_{j_l}) \cdot$$

$$\cdot : \phi(x_1) \ldots \hat{i}_1 \hat{j}_1 \ldots \hat{i}_l \hat{j}_l \ldots \phi(x_n) :_{\omega'} \Psi,$$

where ω and ω' denote quasifree states or the corresponding two-point functions, Ψ is a suitable state vector and the hat means the omission of the corresponding factor. Hence if $(\omega - \omega')$ is a smooth function, the limit $(x_i - x_j) \to 0 \quad \forall i \neq j$ exists in $: \phi(x_1) \ldots \phi(x_m) :_\omega$ $\forall m \in \mathbf{N}$ iff it exists in $: \phi(x_1) \ldots \phi(x_m) :_{\omega'} \quad \forall m \in \mathbf{N}$.

The ghost and antighost fields are quantized with Dirichlet boundary conditions, i.e. by the relation

$$(u_L(f) + i\tilde{u}_L(g))^2 = \int d^4x\, d^4y\, f(x)g(y)D_L(x,y) \tag{A.9}$$

which replaces the commutator condition. (D_L is obtained from Δ_L (A.5) by setting $m = 0$.) The ground state is obtained from the two point function

$$\omega_u(\tilde{u}_L(x)u_L(y) = iD_L^+(x,y) \tag{A.10}$$

which again violates positivity w.r.t. the *-operation, which is defined by (cf. (3.5))

$$(u_L(f) + i\tilde{u}_L(g))^* \stackrel{\text{def}}{=} (u_L(\bar{f}) + i\tilde{u}_L(\bar{g})). \tag{A.11}$$

Finally, we have to find suitable boundary conditions for the electron field. For simplicity, we choose *periodic boundary conditions*. Because they are invariant under charge conjugation, the expectation value of the electric current (normal ordered w.r.t. the Minkowski vacuum) vanishes in the ground state (of the cube), hence the interaction Lagrangian $\mathcal{L}^0$ (3.15) keeps the same form as on Minkowski space.

We are now going to represent these algebras in Fock spaces. The one-particle Hilbert space $\mathcal{H}$ for a massless free scalar field is the completion with respect to the scalar product

$$(f,g) = i \int_{C_L, x_0 = \text{const.}} d^3x\, f(x)^* \overset{\leftrightarrow}{\partial}_0^x g(x) \tag{A.12}$$

of the space of all smooth functions $f : \mathbf{R} \times C_L \to \mathbf{C}$ which are positive frequency solutions of the wave equation and fulfil the considered boundary conditions. For Dirichlet boundary conditions the mode decomposition for the functions $f \in \mathcal{H}$ reads

$$f(x) = \sum_{n_1,n_2,n_3=1}^{\infty} f_{\mathbf{n}} v_{\mathbf{n}}(x), \qquad f_{\mathbf{n}} = \text{const.}, \tag{A.13}$$

where

$$v_{\mathbf{n}}(x) \stackrel{\text{def}}{=} \frac{2}{(\omega_{\mathbf{n}} L^3)^{\frac{1}{2}}} \sin k_{n_1} x_1 \, \sin k_{n_2} x_2 \, \sin k_{n_3} x_3 \; e^{-i\omega_{\mathbf{n}} x_0}, \quad \mathbf{k}_{\mathbf{n}} \stackrel{\text{def}}{=} \frac{\pi}{L}\mathbf{n}, \quad \omega_{\mathbf{n}} \stackrel{\text{def}}{=} \|\mathbf{k}_{\mathbf{n}}\| = \frac{\pi}{L}\|\mathbf{n}\| \tag{A.14}$$

(the normalization is such that $(v_{\mathbf{n}}, v_{\mathbf{m}}) = \delta_{\mathbf{n},\mathbf{m}}$).

The representation of gauge fields in Feyman gauge requires an indefinite inner product space. We describe it in terms of a Krein operator

$$J_L = (-1)^{N_0} \otimes 1 \otimes J_g \qquad \text{on the Hilbert space} \qquad \mathcal{K}_L = \mathcal{K}_{\mathcal{A}} \otimes \mathcal{K}_{\psi} \otimes \mathcal{K}_g. \tag{A.15}$$

The latter is the tensor product of the photon-, spinor- and ghost-Fock space. N_0 is the particle number operator for scalar photons and J_g will be defined below. The Krein operator (A.15) fulfils

$$J_L^2 = 1, \qquad J_L^+ = J_L \tag{A.16}$$

($^+$ denotes the adjoint in $\mathcal{K}_L$), and the dense invariant domain $\mathcal{D}_L$ can be chosen such that $J_L\mathcal{D}_L = \mathcal{D}_L$. The indefinite inner product is given by

$$< a, b > \stackrel{\text{def}}{=} (a, J_L b), \qquad a, b \in \mathcal{K}_L, \tag{A.17}$$

where $(.,.)$ denotes the (positive definite) scalar product in $\mathcal{K}_L$, and the *-operation with respect to (A.17) is

$$O^* \stackrel{\text{def}}{=} J_L O^+ J_L, \qquad < Oa, b >=< a, O^* b > . \tag{A.18}$$

Let $a^\mu_{\mathbf{n}}, a^{\mu +}_{\mathbf{n}}, c_{j\mathbf{n}}, c^+_{j\mathbf{n}}$ $(j = 1, 2)$ be the usual annihilation and creation operators of the Fock spaces $\mathcal{K}_{\mathcal{A}}$ and $\mathcal{K}_g$ which fulfil the (anti-)commutation relations

$$[a^\mu_{\mathbf{n}}, a^{\nu +}_{\mathbf{m}}] = \delta_{\mathbf{n},\mathbf{m}} \delta^{\mu\nu} L^3 2\omega_{\mathbf{n}} \tag{A.19}$$

and

$$\{c_{j\mathbf{n}}, c^+_{l\mathbf{m}}\} = \delta_{\mathbf{n},\mathbf{m}} \delta_{jl} L^3 2\omega_{\mathbf{n}}. \tag{A.20}$$

The ghost fields u_L and $\tilde{u}_L$ and the zeroth component of the photon field A^0_L are scalar fields with Dirichlet boundary conditions and some unusual sign conventions

$$\tilde{u}_L(x) = \sum_{n_1,n_2,n_3=1}^{\infty} \frac{1}{(2\omega_{\mathbf{n}} L^3)^{\frac{1}{2}}} (-c_{1\mathbf{n}} v_{\mathbf{n}}(x) + c^+_{2\mathbf{n}} v_{\mathbf{n}}(x)^*), \tag{A.21}$$

$$u_L(x) = \sum_{n_1,n_2,n_3=1}^{\infty} \frac{1}{(2\omega_{\mathbf{n}} L^3)^{\frac{1}{2}}} (c_{2\mathbf{n}} v_{\mathbf{n}}(x) + c^+_{1\mathbf{n}} v_{\mathbf{n}}(x)^*), \tag{A.22}$$

$$A^0_L(x) = \sum_{n_1,n_2,n_3=1}^{\infty} \frac{1}{(2\omega_{\mathbf{n}} L^3)^{\frac{1}{2}}} (a^0_{\mathbf{n}} v_{\mathbf{n}}(x) - a^{0+}_{\mathbf{n}} v_{\mathbf{n}}(x)^*). \tag{A.23}$$

The normalizations are such that they go over into the usual Lorentz covariant conventions of the non-compactified space by replacing $(\frac{2\pi}{L})^3 \sum_{\mathbf{n}\in\mathbf{Z}^3,\mathbf{n}\neq\mathbf{0}}$ by $\int d^3k$. For the spatial components of the photon field A^μ_L we have a mixture of Dirichlet and von Neumann boundary conditions. For example for $\mu = 2$ we define

$$v_{2\mathbf{n}}(x) \stackrel{\text{def}}{=} \frac{\eta_{\mathbf{n}} 2}{(\omega_{\mathbf{n}} L^3)^{\frac{1}{2}}} \sin k_1 x_1 \, \cos k_2 x_2 \, \sin k_3 x_3 \, e^{-i\omega_{\mathbf{n}} x_0}, \quad n_1, n_3 = 1, 2, ..., \quad n_2 = 0, 1, 2, ..., \tag{A.24}$$

where $\eta_{\mathbf{n}} = 1$ for $n_2 \geq 1$ and $\eta_{\mathbf{n}} = 2^{-\frac{1}{2}}$ for $n_2 = 0$ and similar for $\mu = 1, 3$. Then we set

$$A^l_L(x) = \sum_{\mathbf{n}} \frac{1}{(2\omega_{\mathbf{n}} L^3)^{\frac{1}{2}}} (a^l_{\mathbf{n}} v_{l\mathbf{n}}(x) + a^{l+}_{\mathbf{n}} v_{l\mathbf{n}}(x)^*), \qquad l = 1, 2, 3. \tag{A.25}$$

J_g (A.15) is defined implicitly by (A.11), (A.16) and (A.21-22), i.e. we have

$$c^*_{1\mathbf{n}} = c^+_{2\mathbf{n}}, \qquad c^*_{2\mathbf{n}} = c^+_{1\mathbf{n}}.$$

For the photon two-point function we obtain

$$(\Omega_A, A^{\mu\,+}_L(x) A^\nu_L(y) \Omega_A) =< \Omega_A, A^\mu_L(x) J_L A^\nu_L(y) \Omega_A >= \delta^{\mu\nu} \sum_{\mathbf{n}} v_{\mu\mathbf{n}}(x) v_{\mu\mathbf{n}}(y)^*, \quad (v_{0\mathbf{n}} \stackrel{\text{def}}{=} v_{\mathbf{n}}) \tag{A.26}$$

which is obviously positive.

We now transfer the construction of the interacting fields (sections 2 and 3) from Minkowski space to $\mathbf{R} \times C_L$. Due to [BF] there is no principle obstacle and there are only few changes in the formulas. Since spatial translation invariance is lost, the commutator functions and propagators do not only depend on the relative coordinates, they must be replaced by the above given expressions (A.5), (A.7) (A.8) etc.. Some care is required in the proof of Proposition 3. To get (3.22) we use the identity

$$\partial_\mu^x D_L^{\mathrm{av}\,\mu\rho}(x, z_m) = -\partial_{z_m}^\rho D_L^{\mathrm{av}}(x, z_m) \tag{A.27}$$

(which is an immediate consequence of (A.5), (A.8)) and the fact that the boundary terms of the 'partial integration' vanish because D_L^{av} (A.5) fulfills Dirichlet boundary conditions. By Lemma 1 **(A)** we obtain

$$A^\mu_{\mathrm{int}\,\mathcal{L}}(x) = A^\mu(x) + \sum_{n=1}^{\infty} \frac{i^{n+1}}{n!} \int d^4x_1 ... d^4x_n \cdot$$

$$\cdot \sum_{l=1}^{n} g(x_1)...g(x_n) D_L^{\mu\nu\,\mathrm{ret}}(x, x_l) R(\mathcal{L}^0(x_1)...\hat{l}...\mathcal{L}^0(x_n); j^\mu(x_l)). \tag{A.28}$$

Since $D_L^{\mu\nu\,\mathrm{ret}}(x, x_l)$ fulfills the boundary conditions of $A^\mu(x)$ we conclude that $A^\mu_{\mathrm{int}\,\mathcal{L}}$ *obeys the same boundary conditions as the corresponding free field*, and similar for $\psi_{\mathrm{int}\,\mathcal{L}}$, $u_{\mathrm{int}\,\mathcal{L}}$, etc..

Let us turn to the implementation of the free BRST-transformation. In the following and in the main text we omit the lower index 'L'. Due to $\partial^\mu[(\partial_\nu A^\nu)\partial^{\leftrightarrow}{}_\mu u] = 0$ the definition

$$Q \stackrel{\mathrm{def}}{=} \int_{C_L,\, x_0 = \mathrm{const.}} d^3x\, (\partial_\nu A^\nu(x)) \overset{\leftrightarrow}{\partial}{}_0 u(x) \tag{A.29}$$

of the free Kugo-Ojima charge (5.5) is independent of x_0. Because of $A^\mu{}^* = A^\mu$, $u^* = u$ (3.5) we immediately see $Q^* = Q$. By means of

$$\int_{C_L,\, x_0 = \mathrm{const.}} d^3x\, D(y - x) \overset{\leftrightarrow}{\partial}{}_0^x \phi(x) = \phi(y), \qquad \forall \phi \qquad \text{fulfilling} \qquad \square\phi = 0, \tag{A.30}$$

one proves that the charge Q implements the BRST-transformation (5.2) of the free fields, e.g.

$$[Q, A^\mu(y)] = \int_{C_L,\, x_0 = \mathrm{const.}} d^3x\, [\partial_\nu A^\nu(x), A^\mu(y)] \overset{\leftrightarrow}{\partial}{}_0^x u(x) =$$

$$= -i\partial_y^\mu \int_{C_L,\, x_0 = \mathrm{const.}} d^3x\, D(x - y) \overset{\leftrightarrow}{\partial}{}_0^x u(x) = i\partial^\mu u(y). \tag{A.31}$$

The transformation (5.3-4) of Wick monomials and derivated fields is also implemented by Q, because of

$$[Q, :\phi_1(x)\phi_2(x)...:]_\mp =: [Q, \phi_1(x)]_\mp \phi_2(x)... : + (-1)^{\delta(\phi_1)} :\phi_1(x)[Q, \phi_2(x)]_\mp ... : + ... \tag{A.32}$$

and

$$[Q, \partial_\mu \phi(x)]_\mp = \partial_\mu^x [Q, \phi(x)]_\mp.$$

We easily find that Q is nilpotent

$$Q^2 = \frac{1}{2}\{Q,Q\} = \int_{C_L,\, x_0=\text{const.}} d^3x\ \{Q, (\partial_\nu A^\nu) \overset{\leftrightarrow}{\partial}_0 u\} = 0. \tag{A.33}$$

Inserting (A.22-23) and (A.25) into (A.29) we obtain

$$Q = \frac{1}{L^3 2^{\frac{1}{2}}} \sum_{n_1,n_2,n_3=1}^{\infty} [c^+_{1\mathbf{n}} b_{1\mathbf{n}} + b^+_{2\mathbf{n}} c_{2\mathbf{n}}] \tag{A.34}$$

in a straightforward way (the sum in Q converges in the topology of the Krein space on the dense invariant domain $\mathcal{D}$), where

$$b_{1\mathbf{n}} \overset{\text{def}}{=} \frac{1}{2^{\frac{1}{2}}}(a^0_{\mathbf{n}} + i\frac{k^j_{\mathbf{n}} a^j_{\mathbf{n}}}{\omega_{\mathbf{n}}}), \qquad b_{2\mathbf{n}} \overset{\text{def}}{=} \frac{-1}{2^{\frac{1}{2}}}(a^0_{\mathbf{n}} - i\frac{k^j_{\mathbf{n}} a^j_{\mathbf{n}}}{\omega_{\mathbf{n}}}), \tag{A.35}$$

which implies

$$[b_{j\mathbf{n}}, b^+_{l\mathbf{m}}] = \delta_{\mathbf{n},\mathbf{m}} \delta_{jl} L^3 2\omega_{\mathbf{n}} \tag{A.36}$$

(cf. sect. 5 of [DHS1]). By means of $Q^2 = 0$ one finds similarly to [K] that the dense invariant domain $\mathcal{D}$ has the decomposition

$$\mathcal{D} = \text{Ra}\, Q \oplus (\text{Ke}\, Q \cap \text{Ke}\, Q^+) \oplus \text{Ra}\, Q^+ \tag{A.37}$$

and these three subspaces are pairwise orthogonal with respect to the scalar product $(.,.)$ (cf. (A.17-18)). Additionally one easily verifies

$$\text{Ke}\, Q \cap \text{Ke}\, Q^+ = \text{Ke}\, \{Q, Q^+\}. \tag{A.38}$$

Inserting (A.34) we find

$$\{Q, Q^+\} = \frac{1}{L^3} \sum_{n_1,n_2,n_3=1}^{\infty} \omega_{\mathbf{n}} (b^+_{1\mathbf{n}} b_{1\mathbf{n}} + b^+_{2\mathbf{n}} b_{2\mathbf{n}} + c^+_{1\mathbf{n}} c_{1\mathbf{n}} + c^+_{2\mathbf{n}} c_{2\mathbf{n}}), \tag{A.39}$$

which agrees up to factors $2\omega^2_{\mathbf{n}}$ with the particle number operator of the ghosts and the longitudinal and scalar photons; however, the kernels completely agree. Obviously the Krein operator J (A.15) is the identity on $\text{Ke}\{Q, Q^+\}$. Additionally J maps $\text{Ra}Q$ onto $\text{Ra}Q^+$, due to $JQ = Q^+ J$. From the decomposition (A.37) we conclude that our positivity assumption (4.8) is in fact satisfied, i.e. the indefinite product $< .,. >$ is positive semidefinite on $\text{Ke}\, Q$ and the null vectors in $\text{Ke}\, Q$ are precisely $\text{Ra}\, Q$.

The vectors in $\text{Ke}\, \{Q, Q^+\}$ are distinguished representatives of the equivalence classes in the physical space $\mathcal{H} = \frac{\text{Ke}\, Q}{\text{Ra}\, Q}$ (4.9). They provide the usual physical picture, namely that the states in $\mathcal{H}$ are built up from electrons, positrons and transversal photons only.

Appendix B: Proof of the Ward identities

We recall the Ward identites (3.16)

$$\partial^y_\mu T\Big(j^\mu(y) A_1(x_1) \ldots A_n(x_n)\Big) = i \sum_{j=1}^{n} \delta(y - x_j) T\Big(A_1(x_1) \ldots (\theta A_j)(x_j) \ldots A_n(x_n)\Big), \tag{B.1}$$

where

$$(\theta A_j) \stackrel{\text{def}}{=} \frac{d}{d\alpha}|_{\alpha=0} A_{j\alpha} = i(r_j - s_j)A_j \qquad \text{for} \qquad A_j =: \psi^{r_j} \overline{\psi}^{s_j} B_1 ... B_l : \tag{B.2}$$

($B_1, ..., B_l$ are non-spinorial free fields, i.e. photon or ghost operators), and $A_{j\alpha}$ is given by the global $U(1)$-transformation (3.17). Note

$$(\theta A_j) = -i[Q_\psi, A_j] \qquad \text{with} \qquad Q_\psi \stackrel{\text{def}}{=} \int d^3x \ : \overline{\psi}(x)\gamma^0\psi(x) :, \tag{B.3}$$

i.e. Q_ψ is the infinitesimal generator of the transformation (3.17).

There exist several proofs of the Ward identities in QED, e.g. [FHRW,S]. Here we want to show that they can be fulfilled in our framework; in particular we have to check that they are compatible with our normalization conditions. We follow ideas from [St2].

First we point out a consequence of the Ward identities (B.1). For a given $(x_1, ..., x_n)$ let $\mathcal{O} \subset \mathbf{R}^4$ be a double cone which contains the points $x_1, \ldots, x_n$ and let g be a test function which is equal to 1 on a neighbourhood of $\overline{\mathcal{O}}$. We decompose $\partial^\mu g = a^\mu - b^\mu$ such that $\operatorname{supp} a^\mu \cap (\overline{V}_- + \mathcal{O}) = \emptyset$ and $\operatorname{supp} b^\mu \cap (\overline{V}_+ + \mathcal{O}) = \emptyset$. We smear out (B.1) with this g in y. Then, by causal factorization, the left hand side of (B.1) becomes

$$-j^\mu(a_\mu)T\Big(A_1(x_1)...A_n(x_n)\Big) + T\Big(A_1(x_1)...A_n(x_n)\Big)j^\mu(b_\mu) =$$

$$= -[j^\mu(a_\mu), T\Big(A_1(x_1)...A_n(x_n)\Big)] - T\Big(A_1(x_1)...A_n(x_n)\Big)j^\mu(\partial_\mu g). \tag{B.4}$$

The second term vanishes because j^μ is a conserved current. Since $T(A_1(x_1)...A_n(x_n))$ is localized in $\mathcal{O}$, the term $-j^\mu(a_\mu)$ in the commutator can be replaced by Q_ψ. [10] Hence the validity of the following lemma is necessary for the Ward identities:

Lemma 8: In agreement with **(N1-4)** and **(N6)** the normalizations can be chosen such that the vacuum expectation values of the time ordered products vanish, if the sum of the charges of the factors is different from zero

$$< \Omega | T(A_1 ... A_n) | \Omega >= 0 \qquad \text{for} \qquad \sum_j (r_j - s_j) \neq 0. \tag{B.6}$$

[10] This may be seen as follows. Different choices of a_μ differ only in the spacelike complement of $\mathcal{O}$ and therefore do not affect the commutator. We may choose

$$a_\mu(x) = \partial_\mu g(x) \int_{-\infty}^{ex} h(t)dt \tag{B.5}$$

where e is a suitable timelike unit vector and h is a test function of one variable with sufficiently small support and total integral 1, i.e. the integral in (B.5) is a $\mathcal{C}^\infty$-approximation to $\Theta(ex - c)$ ($c \in \mathbf{R}$ is a suitable constant). Then by current conservation and partial integration we obtain

$$-j^\mu(a_\mu) = j^\mu(e_\mu g h(e\cdot)) = \int dt h(t) \int_{ex=t} g(x) j^\mu(x) \varepsilon_{\mu\nu\rho\sigma} dx^\nu dx^\rho dx^\sigma,$$

hence the statement follows from $g \equiv 1$ on $\mathcal{O}$.

Under this condition the following identity becomes true

$$[Q_\psi, T\Big(A_1(x_1)...A_n(x_n)\Big)] = \sum_{j=1}^{n} T\Big(A_1(x_1)...[Q_\psi, A_j(x_j)]...A_n(x_n)\Big) \equiv$$

$$\equiv i\sum_{j=1}^{n} T\Big(A_1(x_1)...(\theta A_j)(x_j)...A_n(x_n)\Big). \qquad (B.7)$$

Proof of Lemma 8: The lemma is certainly fulfilled for $n = 1$, and we proceed inductively with respect to the order n. For each fixed n we consider a second induction in the sum of the degrees of the Wick monomials A_j, $j = 1, ..., n$. We commute the assertion (B.7) with the free fields. After inserting **(N3)** we can use the inductive assumption and find that these commutators vanish. Therefore, the identity (B.7) can only be violated by a C-number. (An analogous computation is given below in step 1 of the proof of the Ward identities.) To determine this C-number we consider the vacuum expectation value of (B.7). Since $Q_\psi = Q_\psi^*$ annihilates the vacuum we find $< \Omega|[Q_\psi, T(A_1...A_n)]|\Omega >= 0$. Moreover note

$$\sum_j < \Omega|T(A_1...(\theta A_j)...A_n)|\Omega >= i\Big(\sum_j (r_j - s_j)\Big) < \Omega|T(A_1...A_n)|\Omega > . \qquad (B.8)$$

Due to the causal factorization and the validity of (B.7) in lower orders, the expression (B.8) must be local. Hence we can require (B.6) as a normalization condition, i.e. we extend zero by zero to the total diagonal. Obviously this prescription is compatible with **(N1-4)** and **(N6)**. This completes the proof of the lemma. □

Proof of the Ward identities: We show that all Ward identities can be satisfied by choosing a suitable normalization of the vacuum expectation values of the time ordered products which contain no free field factor and with vanishing total charge (B.6). We work with the same double inductive procedure as in the previous proof.

Step 1: Again we commute the assertion with the free fields. By means of **(N3)** we obtain

$$\Big[\{\partial_\mu^y T\Big(j^\mu(y)A_1(x_1)...A_n(x_n)\Big) - i\sum_m \delta(y-x_m)T\Big(A_1(x_1)...(\theta A_m)(x_m)...A_n(x_n)\Big)\}, \phi_j(z)\Big] =$$

$$= i\sum_k \Big\{\partial_\mu^y T\Big(j^\mu(y)A_1...\frac{\partial A_k}{\partial\phi_l}...A_n\Big) - i\sum_{m\,(m\neq k)} \delta(y-x_m)T\Big(A_1...\frac{\partial A_k}{\partial\phi_l}...(\theta A_m)...A_n\Big) -$$

$$-i\delta(y-x_k)T\Big(A_1...\frac{\partial(\theta A_k)}{\partial\phi_l}...A_n\Big)\Big\}\Delta_{lj}(x_k - z) + i\partial_\mu^y T\Big(\frac{\partial j^\mu}{\partial\phi_l}(y)A_1...A_n\Big)\Delta_{lj}(y-z). \qquad (B.9)$$

For $\phi_j \neq \psi, \overline{\psi}$ the last term vanishes and we obtain zero, due to

$$\frac{\partial(\theta A_k)}{\partial\phi_l}\Delta_{lj} = \theta\Big(\frac{\partial A_k}{\partial\phi_l}\Big)\Delta_{lj} \qquad (\text{for} \quad \phi_j \neq \psi, \overline{\psi}) \qquad (B.10)$$

and the inductive assumption.

If $\phi_j = \overline{\psi}$ ($\phi_j = \psi$ is analogous) the last term is equal to

$$i\partial_\mu^y\Big\{T\Big(\overline{\psi}(y)A_1...A_n\Big)\gamma^\mu S(y-z)\Big\} = -i\sum_k T\Big(A_1...\frac{\partial A_k}{\partial\psi}...A_n\Big)\delta(x_k - y)S(x_k - z) \qquad (B.11)$$

according to **(N4)**. Because of

$$\frac{\partial(\theta A_k)}{\partial\psi} = i(r_k - s_k)\frac{\partial A_k}{\partial\psi}, \qquad \theta\Big(\frac{\partial A_k}{\partial\psi}\Big) = i(r_k - 1 - s_k)\frac{\partial A_k}{\partial\psi} \tag{B.12}$$

and the inductive assumption, the commutator (B.9) vanishes also in this case.

Again we conclude that a possible violation of a Ward identity (we call it an anomaly) can only appear in the vacuum sector, i.e. in the vacuum expectation values.

$$a(y, x_1, ..., x_n) \stackrel{\text{def}}{=} \partial_\mu^y T\Big(j^\mu(y)A_1(x_1)...A_n(x_n)\Big) -$$

$$-i\sum_{j=1}^{n}\delta(y - x_j)T\Big(A_1(x_1)...(\theta A_j)(x_j)...A_n(x_n)\Big) =$$

$$= \partial_\mu^y < \Omega|T\Big(j^\mu(y)A_1...A_n\Big)|\Omega > -i\sum_{j=1}^{n}\delta(y - x_j) < \Omega|T\Big(A_1...(\theta A_j)...A_n\Big)|\Omega > . \tag{B.13}$$

Moreover the anomalies are local, i.e.

$$a(y, x_1, ..., x_n) = P(\partial)\delta(x_1 - y)...\delta(x_n - y), \tag{B.14}$$

where $P(\partial)$ is a polynomial in $\partial \equiv (\partial_{x_1}, ..., \partial_{x_n})$. The latter is a consequence of the induction with respect to the order n and the causal factorization (2.2) of the time ordered products.

Step 2: Next we prove the Ward identities with a free field factor. We only need to consider their vacuum expectation values. The normalization condition **(N4)** implies the well known identity

$$< \Omega|T(A_1(x_1)...A_n(x_n)\phi_i(x))|\Omega >=$$

$$= i\sum_{k=1}^{n}\sum_{l}\Delta_{il}^F(x - x_k) < \Omega|T(A_1(x_1)...\frac{\partial A_k}{\partial\phi_l}(x_k)...A_n(x_n))|\Omega >, \tag{B.15}$$

where Δ^F is the Feynman propagator. By inserting this formula we obtain

$$\partial_\mu^y < \Omega|T\Big(j^\mu(y)A_1(x_1)...A_m(x_m)\phi_i(x)\Big)|\Omega > -i\sum_{j=1}^{n}\delta(y - x_j) < \Omega|T\Big(A_1(x_1)...(\theta A_j)(x_j)...$$

$$...A_m(x_m)\phi_i(x)\Big)|\Omega > -i\delta(y - x) < \Omega|T\Big(A_1(x_1)...A_m(x_m)(\theta\phi_i)(x)\Big)|\Omega >=$$

$$= i\sum_k \Delta_{il}^F(x - x_k)\partial_\mu^y < \Omega|T\Big(j^\mu(y)A_1...\frac{\partial A_k}{\partial\phi_l}...A_m\Big)|\Omega > +$$

$$+i\partial_\mu^y\Big\{\Delta_{il}^F(x - y) < \Omega|T\Big(\frac{\partial j^\mu}{\partial\phi_l}(y)A_1...A_m\Big)|\Omega >\Big\}+$$

$$+\sum_j \delta(y - x_j)\sum_{k\,(k\neq j)}\Delta_{il}^F(x - x_k) < \Omega|T\Big(A_1...\frac{\partial A_k}{\partial\phi_l}...(\theta A_j)...A_m\Big)|\Omega > +$$

$$+\sum_k \delta(y - x_k)\Delta_{il}^F(x - x_k) < \Omega|T\Big(A_1...\frac{\partial(\theta A_k)}{\partial\phi_l}...A_m\Big)|\Omega > +$$

$$+\delta(y-x)\sum_k \Delta^F_{(\theta i)l}(x-x_k) < \Omega|T\Big(A_1...\frac{\partial A_k}{\partial\phi_l}...A_m\Big)|\Omega>, \qquad (B.16)$$

where $m=n-1$ and $\Delta^F_{(\theta i)l}(x-y) \stackrel{\rm def}{=} i<\Omega|T((\theta\phi_i)(x)\phi_l(y))|\Omega>$.

If $\phi_i \neq \psi, \overline{\psi}$ the second and the last term vanish ($\alpha_i=0$). Due to (B.10) and the Ward identities in lower order we get zero.

If $\phi_i=\overline{\psi}$ ($\phi_i=\psi$ is analogous) the second term is equal to

$$i\sum_k <\Omega|T\Big(A_1...\frac{\partial A_k}{\partial\psi}...A_m\Big)|\Omega> [S^F(x_k-y)\delta(y-x)-\delta(x_k-y)S^F(y-x)] \qquad (B.17)$$

by means of **(N4)**. Because of (B.12) and the Ward identities in lower order all terms cancel in this case, too.

Step 3: By choosing g as in (B.4) we conclude from Lemma 8

$$0=\int d^4y\, g(y)a(y,x_1,...,x_n) = \int d^4y\, a(y,x_1,...,x_n) \qquad (B.18)$$

in $\mathcal{D}'(\mathbf{R}^{4n})$. This restricts the remaining anomalies. We want to remove them by finite renormalizations of $<\Omega|T\Big(j^\mu(y)A_1(x_1)...A_n(x_n)\Big)|\Omega>$. This can only be done if the polynomials $P(\partial)$ (B.14) have the form

$$P(\partial)=(\sum_{i=1}^n \partial_\mu^{x_i})P_1^\mu(\partial), \qquad P_1^\mu(\partial) \quad \text{polynomial in} \quad \partial\equiv(\partial_{x_1},...,\partial_{x_n}). \qquad (B.19)$$

To prove this we consider the Fourier transformation of the anomaly (B.14),

$$\hat{a}(y,p_1,...,p_n)=(2\pi)^{-2n}\int dx_1...dx_n\, a(y,x_1,...,x_n)e^{i(p_1x_1+...+p_nx_n)} =$$

$$=(2\pi)^{-2n}P(-ip_1,....,-ip_n)e^{i(p_1+...+p_n)y}. \qquad (B.20)$$

From (B.18) we know that $P(-ip_1,....,-ip_n)$ vanishes on the submanifold $\sum_{i=1}^n p_i=0$,

$$P(-ip_1,....,-ip_n)\delta(\sum_{i=1}^n p_i)=0. \qquad (B.21)$$

Let $\tilde{P}(q,p_1,...,p_{n-1}) \stackrel{\rm def}{=} P(-ip_1,....,-ip_n)$, where $q\stackrel{\rm def}{=}\sum_{i=1}^n p_i$. We consider the Taylor series of $\tilde{P}$

$$\tilde{P}(q,p_1,...,p_{n-1})=\sum_{k=1}^{\text{degree }\tilde{P}}\sum_{|\alpha|+|\beta|=k}\frac{q^\alpha p^\beta}{\alpha!\beta!}\Big(\frac{\partial^{|\alpha|}}{\partial q^\alpha}\frac{\partial^{|\beta|}}{\partial p^\beta}\tilde{P}\Big)(0), \quad p\equiv(p_1,...,p_{n-1}). \qquad (B.22)$$

The terms $|\alpha|=0$ vanish because $\Big(\frac{\partial^{|\beta|}}{\partial p^\beta}\tilde{P}\Big)(0)$ is obtained by varying $\tilde{P}$ on the submanifold $q=0$. There remain only terms with a factor q^α, $|\alpha|\geq 1$. This proves (B.19).

Step 4: But there is still a problem. The renormalization

$$<\Omega|T\Big(j^\mu(y)A_1(x_1)...A_n(x_n)\Big)|\Omega> \to <\Omega|T\Big(j^\mu A_1...A_n\Big)|\Omega> + P_1^\mu(\partial)\delta(x_1-y)...\delta(x_n-y), \qquad (B.23)$$

(which removes the anomaly) is only admissible if $P_1^\mu(\partial)\delta(x_1-y)...\delta(x_n-y)$ has the same symmetries as required for $<\Omega|T\Big(j^\mu A_1...A_n\Big)|\Omega>$. Especially if there are factors $j^{\mu_l}(x_l)$ among $A_1(x_1)...A_n(x_n)$ the permutation symmetry with respect to $(y,\mu)\leftrightarrow(x_l,\mu_l)$ must be maintained (for all l). There is a prominent counterexample where this is impossible: the axial anomaly, i.e. $<\Omega|T\Big(j_A^\mu(y)j_A^{\mu_1}(x_1)j_A^{\mu_2}(x_2)\Big)|\Omega>$, where $j_A^\mu \stackrel{\rm def}{=} :\overline{\psi}\gamma^\mu\gamma^5\psi:$.

We have not found a general argument (for non-axial QED) that all possible anomalies factorize $P(\partial)=(\sum_i\partial_\mu^{x_i})P_1^\mu(\partial)$ such that $P_1^\mu(\partial)$ has the wanted symmetries. However, taking step 2 into account, and also the fact that the scaling degree at $x_j-y=0$ $(\forall j=1,...,n)$ [BF] of the anomaly cannot be higher than the scaling degree of the terms in the corresponding Ward identity, the number of Ward identities which can be violated is strongly reduced. In addition, due to (B.18) terms of singular order $\omega=0$ (i.e. scaling degree $\delta=4n$) are excluded in the anomalies. The famous Ward identity which connects the vertex function with the electron self-energy has only one factor j and, hence, the renormalization (B.23) maintains the symmetries in that case. There remain the following anomalies

$$\partial_\mu^y<\Omega|T\Big(j^\mu(y)\mathcal{L}(x_{11})...\mathcal{L}(x_{1m})j^{\mu_1}(x_{21})\Big)|\Omega>=\sum_{1\le|a|\le3}C_{1a}^{\mu_1}\partial^a\prod_{h,j}\delta(x_{hj}-y),\qquad(B.24)$$

$$\partial_\mu^y<\Omega|T\Big(j^\mu(y)\mathcal{L}(x_{11})...\mathcal{L}(x_{1m})j^{\mu_1}(x_{21})j^{\mu_2}(x_{22})j^{\mu_3}(x_{23})\Big)|\Omega>=$$
$$=\sum_{|a|=1}C_{2a}^{\mu_1\mu_2\mu_3}\partial^a\prod_{h,j}\delta(x_{hj}-y).\qquad(B.25)$$

The unknown constants $C_{l...}^{...}$, $l=1,2,3,4$ are restricted by Lorentz covariance and the permutation symmetry in $x_{11},...,x_{1m}$. The analogous Ward identity with three factors j is trivially fulfilled, due to Furry's theorem, by imposing C-invariance as a further normalization condition. [11] The anomalies in (B.24) and (B.25) can be further restricted by the symmetry in the factors j of the terms on the l.h.s., e.g. $\partial_{\mu_1}^{x_{21}}\partial_\mu^y<\Omega|T\Big(j^\mu(y)\mathcal{L}(x_{11})...\mathcal{L}(x_{1m})j^{\mu_1}(x_{21})\Big)|\Omega>$ is symmetrical in y,x_{21}. By working this out one finds that the factorization (B.19) of the anomalies can be done in such a way that the symmetries are preserved in the renormalizations (B.23) [DHS2]. □

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[11] In the inductive construction of the time ordered products C-invariance can only get lost in the extension to the total diagonal, because of the causal factorization (2.2). Starting with an extension which fulfills all other normalization condition **(N1-4)**, **(N6)**, (B.6) and symmetrizing it with respect to C-invariance, we obtain an extension which satisfies all requirements.

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