DESY 99-006 ISSN 0418-9833
January 1999

An Algorithm for Calculating the Spin Tune in Circular Accelerators

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arXiv:physics/9902068 24 Feb 1999

Abstract

A new algorithm for calculating the spin tune and the $\boldsymbol{n}$-axis for circular accelerators is presented. The method resembles the one employed in the existing program code SODOM in that one-turn numerical spin maps at equally spaced orbit angle variables are used but it is more efficient than the latter. Furthermore, it is applicable at large openning angles of the $\boldsymbol{n}$-axis, whereas the existing SODOM only converges for small angles.

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1 Introduction

The spin tune, the spin precession frequency divided by the orbit revolution frequency, is an important parameter in the description of spin motion in circular accelerators. When a particle is on the closed orbit, the definition of the spin tune is obvious; it is the spin precession angle over one turn divided by 2π. However, when orbit oscillations are involved, the definition of the spin tune becomes more complicated. One needs the concept of the $\boldsymbol{n}$-axis which was first introduced by Derbenev and Kondratenko[1] for radiative polarization phenomena in electron storage rings.

We assume that we have complete knowledge about the orbit motion, i.e. that we know the action and angle variables, $\boldsymbol{J} = (J_x, J_y, J_z)$ and $\boldsymbol{\phi} = (\phi_x, \phi_y, \phi_z)$, corresponding to the three degrees of freedom of the orbit motion, which can be in general nonlinear.

A particle with initial coordinates $(\boldsymbol{J}, \boldsymbol{\phi})$ at a machine azimuth θ executes orbit oscillations and comes to $(\boldsymbol{J}, \boldsymbol{\phi} + \boldsymbol{\mu})$ after one turn $(\theta \to \theta + 2\pi)$, where $\boldsymbol{\mu} = (\mu_x, \mu_y, \mu_z)$ is the orbit tune, $\boldsymbol{\nu}$, times 2π. The spin motion over one turn can in general be expressed by a 3×3 rotation matrix $R(\boldsymbol{J}, \boldsymbol{\phi}, \theta)$. Obviously, it is a periodic function of θ and $\boldsymbol{\phi}$ with period 2π. On the next turn the rotation is expressed by $R(\boldsymbol{J}, \boldsymbol{\phi} + \boldsymbol{\mu}, \theta + 2\pi) = R(\boldsymbol{J}, \boldsymbol{\phi} + \boldsymbol{\mu}, \theta)$ which differs from $R(\boldsymbol{J}, \boldsymbol{\phi}, \theta)$ unless the orbit tunes are integers.

A particle on the closed orbit sees the same rotation $R_0(\theta)$ for every turn. $R_0(\theta)$ has eigenvalues 1 and $e^{\pm i\mu_{s0}}$ and the spin tune ν_{s0} is $\mu_{s0}/2\pi$. One can show that μ_{s0} is independent of θ. The eigenvector belonging to the eigenvalue 1 is denoted by $\boldsymbol{n}_0$, i.e., $R_0\boldsymbol{n}_0 = \boldsymbol{n}_0$. It depends only on θ. A spin parallel to $\boldsymbol{n}_0(\theta)$ remains unchanged after one turn, and all other spins attached to closed orbit trajectories precess by the angle μ_{s0} around $\boldsymbol{n}_0$ during one turn.

The vector $\boldsymbol{n}$ is a generalization of $\boldsymbol{n}_0$ for particles off the closed orbit. It is a function of $(\boldsymbol{J}, \boldsymbol{\phi}, \theta)$ periodic in $\boldsymbol{\phi}$ and θ and satisfies

$$R(\boldsymbol{J}, \boldsymbol{\phi}, \theta)\boldsymbol{n}(\boldsymbol{J}, \boldsymbol{\phi}, \theta) = \boldsymbol{n}(\boldsymbol{J}, \boldsymbol{\phi} + \boldsymbol{\mu}, \theta). \tag{1}$$

When $\boldsymbol{J} = 0$, $\boldsymbol{n}$ reduces to $\boldsymbol{n}_0$. To define the spin tune for nonzero $\boldsymbol{J}$, we need two more vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ which form an orthonormal basis together with $\boldsymbol{n}$. They are functions of $(\boldsymbol{J}, \boldsymbol{\phi}, \theta)$ and periodic in $\boldsymbol{\phi}$ and θ like $\boldsymbol{n}$. The spin tune is defined as the precession angle in the frame $(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{n})$ divided by 2π.

The concept of the vector $\boldsymbol{n}$ has been playing an important role in the description and calculation of radiative polarization in electron/positron storage rings since[1]. Recently, it has also turned out to be useful for proton rings[2].

To calculate the vector $\boldsymbol{n}$ several algorithms have been invented. S. Mane[3] developed a computer code SMILE using a perturbation expansion with respect to the orbit action variable. The present author suggested a perturbation algorithm using Lie algebra[4] and Eidelmann and Yakimenko[5] coded a program SPINLIE with (low order) orbit nonlinearity. Balandin, Golubeva and Barber[6] also wrote a Lie Algebra code.

The present author considered another method which does not employ a perturbation expansion and wrote a program SODOM[7]. Heinemann and Hoffstaetter[8] use tracking

and 'stroboscopic averaging' in the code SPRINT. The programs SODOM, SPRINT and [6] additionally compute the spin tune.

The new method which we are going to describe is based on SODOM.

We shall briefly summarize the SODOM algorithm in the next section and describe the new method in Sec.3.

2 The SODOM Algorithm

Let us first briefly summarize the algorithm employed in SODOM. (See Sec.3 of [7].) Denote the one-turn SU2 spin transport map starting at a fixed prechosen azimuth θ_0 for particles with initial orbital phase $\boldsymbol{\phi}$ by $M(\boldsymbol{\phi})$ and the spinor representing the $\boldsymbol{n}$-axis at θ_0 by $\psi(\boldsymbol{\phi})$. (Here, we simply write $\psi(\boldsymbol{\phi})$ instead of $\psi_+(\boldsymbol{\phi})$ [7]. We also omit the arguments $\boldsymbol{J}$ and θ_0 since we shall deal with one set of $\boldsymbol{J}$ and consider the one-turn map from the origin θ_0 only.) The fact that $\boldsymbol{n}$ is 'invariant' means

$$M(\boldsymbol{\phi})\psi(\boldsymbol{\phi}) = e^{-iv(\boldsymbol{\phi})/2}\psi(\boldsymbol{\phi}+\boldsymbol{\mu}), \tag{2}$$

where $v(\boldsymbol{\phi})$ is a real periodic function. Once a solution $(\psi(\boldsymbol{\phi}),v(\boldsymbol{\phi}))$ is obtained, we solve the equation

$$v(\boldsymbol{\phi}) + u(\boldsymbol{\phi}+\boldsymbol{\mu}) - u(\boldsymbol{\phi}) = \mu_s \tag{3}$$

and define

$$\Psi(\boldsymbol{\phi}) \equiv e^{iu(\boldsymbol{\phi})/2}\psi(\boldsymbol{\phi}) \tag{4}$$

Then, $\Psi(\boldsymbol{\phi})$ satisfies

$$M(\boldsymbol{\phi})\Psi(\boldsymbol{\phi}) = e^{-i\mu_s/2}\Psi(\boldsymbol{\phi}+\boldsymbol{\mu}), \tag{5}$$

where μ_s is the spin tune times 2π. The $\boldsymbol{u}_{1,2}$ axes are represented by a spinor

$$\Psi_\varphi \equiv \frac{1}{\sqrt{2}}\left[e^{-i\varphi/2}\Psi + e^{i\varphi/2}\hat{\Psi}^*\right] \tag{6}$$

where we define the operation $\hat{}$ as

$$\hat{\Psi} \equiv i\sigma_2\Psi^*, \tag{7}$$

which was denoted by Ψ_- in [7]. Note that $\hat{\hat{\Psi}} = -\Psi$ and $\hat{\Psi}^\dagger\Psi = 0$. The three spinors, Ψ_0, $\Psi_{\pi/2}$, Ψ, represent the three vectors $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{n}$. The phase of Ψ is irrelevant for defining $\boldsymbol{n}$ but it is important for $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

The original SODOM algorithm parametrizes ψ as

$$\psi = \frac{1}{\sqrt{1+|\zeta(\boldsymbol{\phi})|^2}}\begin{pmatrix} 1 \\ \zeta(\boldsymbol{\phi}) \end{pmatrix}. \tag{8}$$

The SU2 matrix $M(\boldsymbol{\phi})$ can be parametrized by two complex functions $f(\boldsymbol{\phi})$ and $g(\boldsymbol{\phi})$ as

$$M(\boldsymbol{\phi}) = \begin{pmatrix} -ig(\boldsymbol{\phi}) & -if^*(\boldsymbol{\phi}) \\ -if(\boldsymbol{\phi}) & ig^*(\boldsymbol{\phi}) \end{pmatrix} \tag{9}$$

Then, one gets an equation for ζ:

$$g^*(\boldsymbol{\phi})\zeta(\boldsymbol{\phi}) + g(\boldsymbol{\phi})\zeta(\boldsymbol{\phi}+\boldsymbol{\mu}) = f(\boldsymbol{\phi}) - f^*(\boldsymbol{\phi})\zeta(\boldsymbol{\phi})\zeta(\boldsymbol{\phi}+\boldsymbol{\mu}). \tag{10}$$

By expanding $f(\boldsymbol{\phi})$, $g(\boldsymbol{\phi})$, and $\zeta(\boldsymbol{\phi})$ into Fourier series like $\sum f_{\boldsymbol{m}} e^{i\boldsymbol{m}\cdot\boldsymbol{\phi}}$, we get a nonlinear equation for $\zeta_{\boldsymbol{m}}$.

A key component of SODOM is the calculation of the Fourier coefficients $f_{\boldsymbol{m}}$ and $g_{\boldsymbol{m}}$ from the tracking data over one turn for several particles having the same $\boldsymbol{J}$ but equally-spaced $\boldsymbol{\phi}$ ($0 \le \phi < 2\pi$).

The parametrization (8) is good only when $\zeta(\boldsymbol{\phi})$ is small. Because of its up-down asymmetric form, many more Fourier terms are needed than required by the physics, when $\zeta(\boldsymbol{\phi})$ is large.[2] Also, the iterative method of solving the nonlinear equation easily fails when ζ is large.

3 The Matrix Eigenvalue Method

The new algorithm is much simpler and involves solving eq.(5) directly rather than eq.(10). By expanding $M(\boldsymbol{\phi})$ (actually the functions $f(\boldsymbol{\phi})$ and $g(\boldsymbol{\phi})$) and $\Psi(\boldsymbol{\phi})$ into Fourier series as

$$M(\boldsymbol{\phi}) = \sum_{\boldsymbol{m}} M_{\boldsymbol{m}} e^{i\boldsymbol{m}\cdot\boldsymbol{\phi}}, \qquad \Psi(\boldsymbol{\phi}) = \sum_{\boldsymbol{m}} \Psi_{\boldsymbol{m}} e^{i\boldsymbol{m}\cdot\boldsymbol{\phi}} \tag{11}$$

eq.(5) can be written as

$$e^{-i\boldsymbol{m}\cdot\boldsymbol{\mu}} \sum_{\boldsymbol{m}'} M_{\boldsymbol{m}-\boldsymbol{m}'} \Psi_{\boldsymbol{m}'} = e^{-i\mu_s/2} \Psi_{\boldsymbol{m}}. \tag{12}$$

This is simply a matrix eigenvalue equation. Thus, the spin tune comes out as an eigenvalue.

However, obviously, eq.(12) has many eigenvalues. Which one gives the spin tune? What do the other eiganvalues and eigenvectors mean? In order to answer these questions let us return to eq.(5) and examine it as an eigenvalue system

$$M(\boldsymbol{\phi})\Psi(\boldsymbol{\phi}) = \lambda \Psi(\boldsymbol{\phi}+\boldsymbol{\mu}) \tag{13}$$

Note that this is not a simple 2×2 algebraic equation because of the $\boldsymbol{\phi}+\boldsymbol{\mu}$.

Before going further we have to think about subtle problems associated with the '2-to-1' correspondence between SU2 and SO3. Note that we use 2-component spinors and SU2

[2] For example, $\Psi = (\cos\phi, \sin\phi)$ is a mild function but leads to $\zeta = \tan\phi$ which is hard to Fourier-expand.

matrices instead of 3-vectors and SO3 matrices to achieve computational speed and to minimize storage but not because the particles have spin $\hbar/2$. The classical spin motion can be completely described by 3-vectors and SO3 matrices.

What does the periodicity of a spinor with respect to $\boldsymbol{\phi}$ mean? The physical object is the 3-vector $\Psi^\dagger \boldsymbol{\sigma} \Psi = \boldsymbol{n}$ rather than the spinor Ψ. In this sense a complex phase factor in Ψ is irrelevant. However, a complex phase factor is still relevant when one constructs the $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ axes from Ψ via Ψ_φ.

On the other hand, a sign change of Ψ does not cause a change of $\boldsymbol{n} = \Psi^\dagger \boldsymbol{\sigma} \Psi$ nor a change of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ defined by $\Psi_\varphi^\dagger \boldsymbol{\sigma} \Psi_\varphi$.

Thus, as the periodicity condition for Ψ with respect to ϕ_j (one of the orbit angle variables), we have to allow both $\Psi(\phi_j + 2\pi) = \Psi(\phi_j)$ and $\Psi(\phi_j + 2\pi) = -\Psi(\phi_j)$. Then with 3 degrees of freedom for orbit motion, we have 8 types of solutions $\Psi(\boldsymbol{\phi})$ differing by their sign change behaviour under the transformation $\phi_j \to \phi_j + 2\pi$. In Fourier expansion language, this means that Ψ can be expanded as

$$\Psi(\boldsymbol{\phi}) = e^{i\boldsymbol{m}^0 \cdot \boldsymbol{\phi}/2} \sum_{\boldsymbol{m}} \Psi_{\boldsymbol{m}} e^{i\boldsymbol{m}\cdot\boldsymbol{\phi}} \tag{14}$$

where $\boldsymbol{m}^0$ is a set of three integers each of which is either 0 or 1.

We now define the scalar product of two arbitrary spinors Ψ_1 and Ψ_2 by

$$(\Psi_1, \Psi_2) \equiv \frac{1}{(4\pi)^3} \int_0^{4\pi} \Psi_1^\dagger(\boldsymbol{\phi}) \Psi_2(\boldsymbol{\phi}) d\boldsymbol{\phi} = \delta_{\boldsymbol{m}_1^0, \boldsymbol{m}_2^0} \sum_{\boldsymbol{m}} \Psi_{1,\boldsymbol{m}}^\dagger \Psi_{2,\boldsymbol{m}} \tag{15}$$

Obviously, solutions of different types in eq.(14) are always orthogonal. In the following we consider the solutions of eq.(13) which are 'periodic' and smooth in $\boldsymbol{\phi}$.

Now, let us list a few lemmas.

[a] $|\lambda| = 1$

[b] $(\Psi_1, \Psi_2) = 0$ *if* $\lambda_1 \neq \lambda_2$

From the unitarity of $M(\boldsymbol{\phi})$ we get

$$\begin{aligned} \Psi_i^\dagger(\boldsymbol{\phi}) \Psi_j(\boldsymbol{\phi}) &= \Psi_i^\dagger(\boldsymbol{\phi}) M(\boldsymbol{\phi})^\dagger M(\boldsymbol{\phi}) \Psi_j(\boldsymbol{\phi}) = [M(\boldsymbol{\phi})\Psi_i(\boldsymbol{\phi})]^\dagger M(\boldsymbol{\phi}) \Psi_j(\boldsymbol{\phi}) \\ &= \lambda_i^* \lambda_j \Psi_i^\dagger(\boldsymbol{\phi} + \boldsymbol{\mu}) \Psi_j(\boldsymbol{\phi} + \boldsymbol{\mu}). \end{aligned}$$

Integrating over $\boldsymbol{\phi}$ and using the definition (15), we get [a] for $i = j$ and [b] for $\lambda_i \neq \lambda_j$. Note that [b] does not imply $\Psi_1(\boldsymbol{\phi})^\dagger \Psi_2(\boldsymbol{\phi}) = 0$ for $\lambda_i \neq \lambda_j$.

[c] $|\Psi(\boldsymbol{\phi})|$ *is independent of* $\boldsymbol{\phi}$ *(and can be normalized to unity).*

The unitarity condition $|\Psi(\boldsymbol{\phi})| = |\Psi(\boldsymbol{\phi} + \boldsymbol{\mu})|$, together with the smoothness of $\Psi(\boldsymbol{\phi})$, and the non-commensurability of $\boldsymbol{\mu}$ are enough to guarantee [c].

[d] *If* $(\lambda, \Psi(\boldsymbol{\phi}))$ *is a solution, so is* $(\lambda^*, \hat{\Psi}(\boldsymbol{\phi}))$

Take the complex conjugate of eq.(13) and use $\sigma_2 M^* \sigma_2 = M$. If Ψ corresponds to $\boldsymbol{n}$, then $\hat{\Psi}$ corresponds to $-\boldsymbol{n}$ and the spin tune changes sign. (Since $\sigma_2 \boldsymbol{\sigma} \sigma_2 = -\boldsymbol{\sigma}^*$, $\hat{\Psi}^\dagger \boldsymbol{\sigma} \hat{\Psi} = -\Psi^\dagger \boldsymbol{\sigma} \Psi$.) Note that not only $(\hat{\Psi}, \Psi) = 0$ but also $\hat{\Psi}^\dagger \Psi = 0$ at every $\boldsymbol{\phi}$.

[e] *If λ is an eigenvalue, then so is $\lambda e^{i\boldsymbol{m}\cdot\boldsymbol{\mu}/2}$, where $\boldsymbol{m}$ is a set of any integers.*

Multiply eq.(13) by $e^{-i\boldsymbol{m}\cdot\boldsymbol{\phi}/2}$ and define $\tilde{\Psi}(\boldsymbol{\phi}) \equiv e^{-i\boldsymbol{m}\cdot\boldsymbol{\phi}/2}\Psi(\boldsymbol{\phi})$. Then

$$M(\boldsymbol{\phi})\tilde{\Psi}(\boldsymbol{\phi}) = \lambda e^{-i\boldsymbol{m}\cdot\boldsymbol{\phi}/2}\Psi(\boldsymbol{\phi}+\boldsymbol{\mu}) = \lambda e^{i\boldsymbol{m}\cdot\boldsymbol{\mu}/2}\tilde{\Psi}(\boldsymbol{\phi}+\boldsymbol{\mu})$$

Thus, $\tilde{\Psi}$ is an eigenvector belonging to the eigenvalue $\lambda e^{i\boldsymbol{m}\cdot\boldsymbol{\mu}/2}$.

This gives an ambiguity in the spin tune: $\mu_s \to \mu_s + \boldsymbol{m}\cdot\boldsymbol{\mu}$. However, all the eigenvalues of the form $\lambda e^{i\boldsymbol{m}\cdot\boldsymbol{\mu}/2}$ give the same vector $\boldsymbol{n} = \tilde{\Psi}^\dagger\boldsymbol{\sigma}\tilde{\Psi} = \Psi^\dagger\boldsymbol{\sigma}\Psi$. The $\boldsymbol{u}_{1,2}$ axes corresponding to $\tilde{\Psi}$ are

$$\tilde{\Psi}_\varphi = \frac{1}{\sqrt{2}}\left[e^{-i\varphi/2}e^{-i\boldsymbol{m}\cdot\boldsymbol{\phi}/2}\Psi + e^{i\varphi/2}e^{i\boldsymbol{m}\cdot\boldsymbol{\phi}/2}i\sigma_2\Psi^*\right] = \Psi_{\varphi+\boldsymbol{m}\cdot\boldsymbol{\phi}}$$

Thus, the new $\boldsymbol{u}_{1,2}$ axes rotate by $\boldsymbol{m}\cdot\boldsymbol{\phi}$ with respect to the original ones.

From the lemmas above, we know that once a solution (λ, Ψ) is found, we can construct infinitely many solutions of the form $(\lambda e^{i\boldsymbol{m}\cdot\boldsymbol{\mu}/2}, e^{-i\boldsymbol{m}\cdot\boldsymbol{\phi}/2}\Psi)$ and $(\lambda^* e^{-i\boldsymbol{m}\cdot\boldsymbol{\mu}/2}, e^{i\boldsymbol{m}\cdot\boldsymbol{\phi}/2}\hat{\Psi})$, and that they all correspond to the same vector $\boldsymbol{n}$ or $-\boldsymbol{n}$.

A natural question is then 'are there any other eigenvalues?'. The answer is 'No':

[f] *If λ is an eigenvalue, all other eigenvalues are either $\lambda e^{i\boldsymbol{m}\cdot\boldsymbol{\mu}/2}$ or $\lambda^* e^{-i\boldsymbol{m}\cdot\boldsymbol{\mu}/2}$*

If (λ_1, Ψ_1) and (λ_2, Ψ_2) are solutions, $a(\boldsymbol{\phi}) \equiv \Psi_2^\dagger(\boldsymbol{\phi})\Psi_1(\boldsymbol{\phi}) = (M\Psi_2)^\dagger M\Psi_1 = \lambda_2^*\lambda_1 a(\boldsymbol{\phi}+\boldsymbol{\mu})$. From the periodicity and smoothness of $a(\boldsymbol{\phi})$ and the non-commensurability of $\boldsymbol{\mu}$ one finds either that [1] $a(\boldsymbol{\phi}) = e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\phi}}$ and $\lambda_2^*\lambda_1 = e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{\mu}}$, $\boldsymbol{\alpha}$ being a constant 3-vector, or that [2] $a = 0$ (Ψ_1 and Ψ_2 are locally orthogonal). In the case [1] $\boldsymbol{\alpha}$ must be of the form $\boldsymbol{m}/2$ from the periodicity requirement, where $\boldsymbol{m}$ is a set of three integers. Therefore, $\lambda_2 = \lambda_1 e^{i\boldsymbol{m}\cdot\boldsymbol{\mu}/2}$. In the case [2], examine $\hat{\Psi}_2$ in place of Ψ_2. Then we get either $\lambda_2 = \lambda_1^* e^{-i\boldsymbol{m}\cdot\boldsymbol{\mu}/2}$ or $\hat{\Psi}_2^\dagger\Psi_1 = 0$. However, if both $\Psi_2^\dagger\Psi_1$ and $\hat{\Psi}_2^\dagger\Psi_1$ vanish, then $\Psi_1 = 0$ because Ψ_2 and $\hat{\Psi}_2$ are orthogonal. Therefore $\hat{\Psi}_2^\dagger\Psi_1 = 0$ cannot be the case. Thus, the cases [1] and [2] correspond to $\lambda_2 = \lambda_1 e^{i\boldsymbol{m}\cdot\boldsymbol{\mu}/2}$ and $\lambda_1^* e^{-i\boldsymbol{m}\cdot\boldsymbol{\mu}/2}$, respectively.

Let us consider the spin tune $\nu_s = \mu_s/2\pi$. It is obtained from the definition $\lambda = e^{-i\mu_s/2} = e^{-i\pi\nu_s}$. From the above arguments we find that if the set $[\nu_s, \boldsymbol{n}, \boldsymbol{u}_1 + i\boldsymbol{u}_2]$ is a solution, then $[\nu_s - \boldsymbol{m}\cdot\boldsymbol{\nu}, \boldsymbol{n}, e^{-i\boldsymbol{m}\cdot\boldsymbol{\phi}}(\boldsymbol{u}_1 + i\boldsymbol{u}_2)]$ and $[-\nu_s - \boldsymbol{m}\cdot\boldsymbol{\nu}, -\boldsymbol{n}, -e^{i\boldsymbol{m}\cdot\boldsymbol{\phi}}(\boldsymbol{u}_1 + i\boldsymbol{u}_2)]$ are also solutions. Thus, the spin tune has ambiguities up to a multiple of the orbit tunes and up to a sign. The latter is related to the choice of sign of $\boldsymbol{n}$.

When obtaining ν_s from λ, one finds an ambiguity only up to an even integer rather than up to an integer. At first sight this is puzzling but it is also due to the '2-to-1' correspondence between SU2 and SO3. Obviously,

[g] *If Ψ is an eigenvector of M with eigenvalue λ, it is also an eigenvector of $-M$ with eigenvalue $-\lambda$.*

Since M and $-M$ represent the same SO3 rotation, we have also to include the solutions to $-M$. However, $-M$ has exactly the same eigenvectors as $+M$ (therefore the same $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{n}$) with spin tunes ν_s shifted by one. This solves the above puzzle. Thus, we can define the spin tune in the interval [0,1) or (-0.5,0.5] and, if the sign of $\boldsymbol{n}$ is irrelevant, we can reduce the interval to [0,0.5].

Thus, we have found that we only need one of the sets of eigenvector and eigenvalue of eq.(5). All others can be constructed from this. Therefore one can Fourier expand Ψ as in eq.(11) rather than as in the general from (14). (Note, however, that one will find tunes of the form $\pm\nu_s + 2\boldsymbol{m}\cdot\boldsymbol{\nu}$ although odd-multiple solutions can be easily reconstructed.)

Let us briefly discuss the degeneracy. Within the eigenvalue group $\lambda e^{i\boldsymbol{m}\cdot\boldsymbol{\mu}/2}$ of the same sign of $\boldsymbol{n}$, a degeneracy is possible when $\boldsymbol{m}\cdot\boldsymbol{\nu}$ is an even integer, i.e., when the orbit motion is in resonance, which we are not interested in. We may assume this is not the case.

On the other hand, a degeneracy between solutions of different signs of $\boldsymbol{n}$ corresponds to a spin-orbit resonance. When the two solutions (λ, Ψ) and $(\lambda^*, \widehat{\Psi})$ degenerate $(\lambda = \lambda^*)$, the spin tune becomes an integer. Taking into account the ambiguity of spin tune, this is equivalent to the relation $\nu_s = \boldsymbol{m}\cdot\boldsymbol{\nu} + \text{integer}$.

4 Choice of the Spin Tune

We have shown in the previous section that there are many eigenvalues (spin tunes) representing the same vector $\boldsymbol{n}$ and different $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ axes. Now, we must finally decide which eigenvalue to choose for the spin tune. Theoretically speaking there is no reason to choose one particular value. As pointed out in [9] spin tune is intrinsically ambiguous up to a multiple of the orbit tunes. The choice of the spin tune, which is equivalent to a choice of $(\boldsymbol{u}_1, \boldsymbol{u}_2)$, is a matter of convention.

In practice, however, a solution is not desirable if $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ is a strong function of $\boldsymbol{\phi}$. When one solves the equation by Fourier expansion, the most natural choice is to take the solution having the largest zero-Fourier harmonic $\left|\Psi_{\boldsymbol{m}=(0,0,0)}\right|$.

If one plots all the eigenvalues as a function of any parameter (beam energy, betatron amplitude, etc), one will find continuous curves. If one plots the spin tune selected as just described as a function of these parameters one may occasionally find a jump of spin tune although the whole spectrum content is continuous.

Let us give an example from a test calculation. The test ring consists of 100 FODO cells, each of which has two thin-lens quadrupole magnets and two bending magnets filling the entire space between quadrupoles. The focusing effect of the bending magnets is ignored. In order to avoid a too high symmetry of the orbit motion, an artificial phase advance of 90 degrees in both horizontal and vertical planes is introduced at one point in the ring. The tunes are $\nu_x = 15.3827$ and $\nu_y = 25.6482$. Only the vertical betatron oscillation is excited. The beam energy is so chosen that $\nu_{s0} = \gamma a = 1520.72$.

Fig.1 shows the eigenvalue (spin tune) spectrum as a function of the betatron action J_y. Only those with small $\boldsymbol{m}$ are plotted. The points linked by a solid line correspond to the spin tune selected by the criterion mentioned above. As one can see, each eigenvalue is a continuous function of J_y (A few curves appear broken because not all the eigenvalues are plotted.) but the selected tune shows a jump at $J_y \approx 0.7 \times 10^{-8}$m·rad. The spin tune before and after the jump, ν_{s1} and ν_{s2} satisfies $\nu_{s1} + \nu_{s2} = -2\nu_y + \text{integer}$. The dashed

line (with the same scale) is the upper limit of polarization, i.e.,

$$P_{lim} = |\langle \boldsymbol{n} \rangle| , \qquad \langle \boldsymbol{n} \rangle = \frac{1}{2\pi} \int_0^{2\pi} \boldsymbol{n}(\phi_y) d\phi_y \tag{16}$$

The minimum of P_{lim} coincides with the point of the spin tune jump.

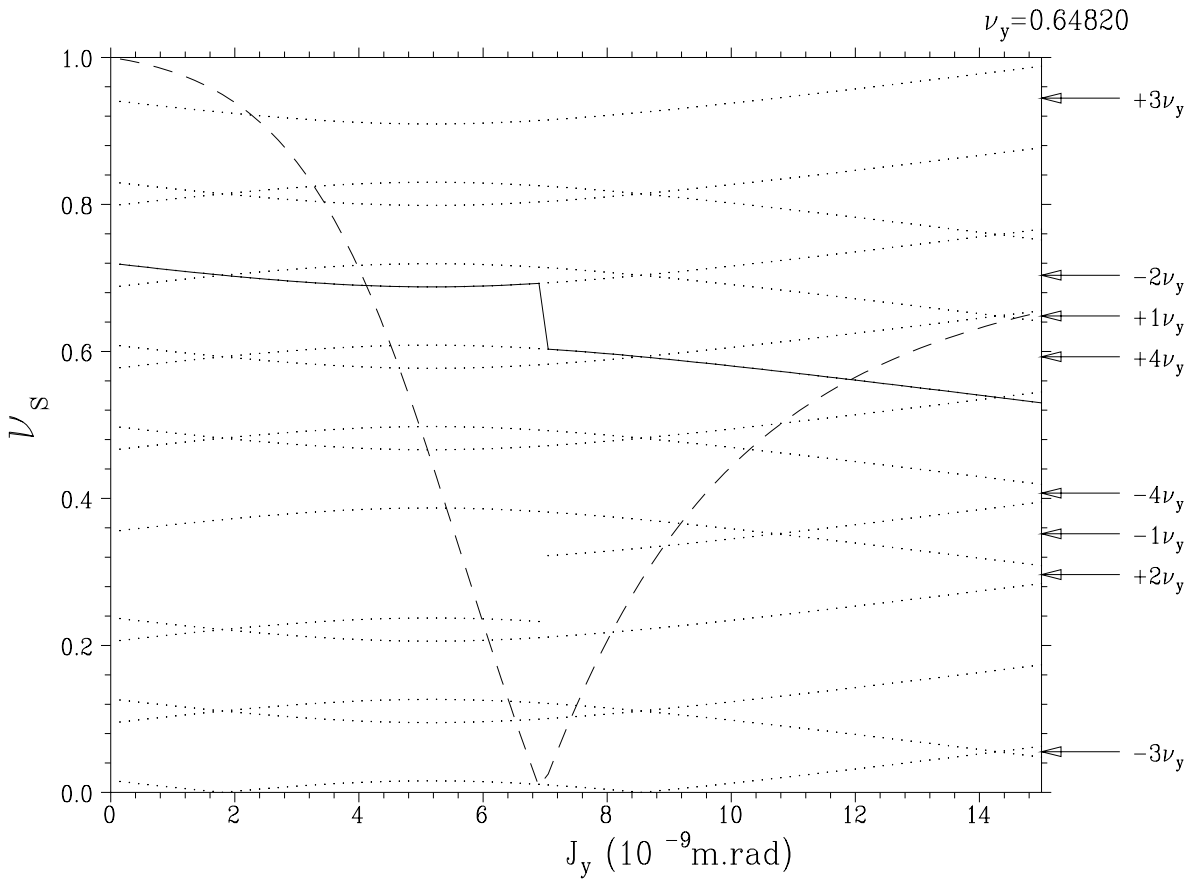

Figure 1: An example of a spin tune spectrum as a function of betatron action. The dashed line is the upper limit of polarization P_{lim}.

We have compared the results of our program with SPRINT for the amplitude dependence of the spin tune in the HERA ring. The agreement of the $\boldsymbol{n}$ axis and the spin tune was excellent. Not only the occurrence of spin tune jumps but also their location agree, which means that taking the largest zero-harmonic and the stroboscopic averaging are almost equivalent.

5 Truncation of Fourier Series

In numerical calculations one has to truncate the Fourier expansion. There are a few problems associated with the truncation.

When N values of ϕ are used (we deal with one degree of freedom for illustration. The extension to 3 degrees of freedom is obvious.), the range of the harmonics should be $-M \leq m \leq M$ ($N = 2M + 1$).[3] For a discrete Fourier transform the range can also be $0 \leq m \leq N - 1$ (as in standard FFT routines), but this choice is not good when other values of ϕ are needed (for example when calculating $\boldsymbol{n}$ for arbitary values after the problem is solved). N must be large enough to ensure that the Fourier components M_m (actually f_m and g_m) are small enough outside the region $[-M, M]$.[4]

The matrix $e^{-im\mu} M_{m-m'}$ in eq.(12) is then a $(2M+1)\times(2M+1)$ matrix (each element is a 2×2 matrix and we are dealing with one degree of freedom.). One finds that the diagonal elements ($m = m'$) are normally large and that the elements with large $|m - m'|$ are small. The elements in the upper-right and lower-left triangles ($|m - m'| > M$) are exactly zero because they require the harmonics outside $[-M, M]$. Owing to this truncation, the matrix does not exactly satisfy the lemmas in the previous section even if N is very large. (For example, in the first row ($m = -M$) even the first harmonic $m - m' = 1$ is lost because m' would be $-M - 1$.) Although the solution with the largest zero-harmonic is not affected much by this truncation, it is not easy to confirm the accuracy of the solutions.

On the other hand, one can fill the upper-right and lower-left triangles by treating the harmonics in a cyclic manner as in a discrete Fourier transformation (i.e. one identifies the $(M+1)$-th harmonic with the $(-M)$-th.). With this prescription the truncated matrix becomes exactly unitary even if N is not large enough. The solution with the largest zero-harmonic does not change much. The appearance of eigenvalues with modulus far from unity then means that the eigenvalue solver is not accurate.

When one adopts the cyclic use of the harmonics, the lemmas [a], [b], [d] and [g] hold exactly apart from round off errors, but [c] and [e] (and accordingly [f]) become inaccurate.

6 Conclusion

We have shown that the spin tune can be obtained as an eigenvalue of a matrix which is created from the one-turn maps calculated by particle tracking. The method is applicable to any system with linear or nonlinear orbit motion as long as the orbit action variables exist. The convergence is much better than with perturbation methods and the previous SODOM algorithm. The computation is very fast because it makes full use of the fact that the spin motion is linear and that we know the orbit tunes.

Acknowledgements The author thanks to Drs. D. Barber, K. Heinemann, G. Hoffstaetter, and M. Vogt for stimulating discussions.

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[3] If N is even, we have to change the upper or lower limit by one.

[4] This is not a sufficient condition for accuracy. Even if M_m is small outside $[-M, M]$, the solution Ψ_m can still be large in some cases.

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