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Non-commutative World-volume Geometries: Branes on SU(2) and Fuzzy Spheres

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Abstract

The geometry of D-branes can be probed by open string scattering. If the background carries a non-vanishing B-field, the world-volume becomes non-commutative. Here we explore the quantization of world-volume geometries in a curved background with non-zero Neveu-Schwarz 3-form field strength $H = dB$. Using exact and generally applicable methods from boundary conformal field theory, we study the example of open strings in the SU(2) Wess-Zumino-Witten model, and establish a relation with fuzzy spheres or certain (non-associative) deformations thereof. These findings could be of direct relevance for D-branes in the presence of Neveu-Schwarz 5-branes; more importantly, they provide insight into a completely new class of world-volume geometries.

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1 Introduction

It was observed by Douglas and Hull [1] that D-branes on $\mathbb{T}^2$ with a constant Neveu-Schwarz (NS) two-form potential B give rise to an effective world-volume theory on a non-commutative torus. Even though this initial observation was re-considered and generalized by many authors [2–4], all the subsequent work is restricted to flat backgrounds. A perturbative analysis along the lines of [4], on the other hand, shows that the quantization of world-volume geometries should be a much more general phenomenon which persists in the case of curved backgrounds.

In this work we shall present the first non-perturbative (in α') investigation of world-volume geometries in a curved string background with non-vanishing NS 3-form field $H = dB$. [1] An exact treatment of D-branes in curved backgrounds is possible within the framework of boundary conformal field theory. Here we illustrate the basic techniques and some general features of the resulting world-volume geometries in a particular example, namely the SU(2) WZW theory, and study D-branes in the WZW model associated with the gluing condition $J^a = \bar{J}^a$. We shall argue that their world-volumes may be regarded as fuzzy two-spheres when the level k is sent to infinity, i.e. when the background becomes flat. For finite level, H is non-zero and we shall find non-associative deformations of these fuzzy spheres, which are closely linked to the theory of quantum groups. While the infinite level result can be predicted from the semi-classical analysis in [5] together with the general phenomenon of world-volume quantization in flat backgrounds [1], our results on the finite level provide a non-trivial extension of the standard rules. Apparently, many features of the world-volume geometry are not captured by the perturbative treatment of D-branes on group manifolds that was suggested recently in [6].

We shall follow a general procedure which allows us to extract world-volume geometry from the world-sheet description of any (generalized) D-brane, even when it is given in purely algebraic terms. The essential input data are the operator product expansions (OPE) of boundary fields (open string vertex operators). Since they depend on the ordering of the operators, it is not surprising that the brane world-volume obtained in this way is a non-commutative space, in general. We shall see that non-associativity may show up as well.

Our approach is inspired by a project initiated by J. Fröhlich and K. Gawędzki in [7] (see also [8] for earlier ideas in the same direction), where it was proposed to construct non-commutative target space geometries from OPEs of closed string vertex operators. This was developed further in [9, 10]. It appears, however,

[1] Recall that the curvature is linked to the field strength H by the string's equation of motion.

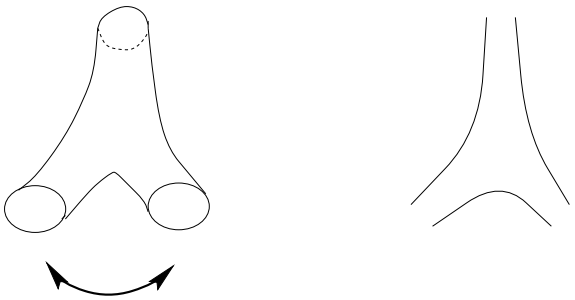

Figure 1: World-sheet diagrams for closed resp. open string interaction. Having assigned vertex operators to the legs, they can be read as structure constants for the multiplication of two operators, projected on the third channel. In the closed string case, the in-coming operators can be interchanged with the help of world-sheet diffeomorphisms, while the ordering of open string vertices is fixed up to cyclic permutations.

that non-commutative geometry emerges in a more natural way and on a more fundamental level in the open string case, cf. the picture below. Our findings add to the growing evidence that brane physics surpasses classical geometry – even though the emergence of a non-commutative world-volume need not necessarily mean that a D-brane behaves non-geometrically in the sense of the criterion formulated in [11]. This criterion rests on a comparison of low-energy effective field theories in the stringy and in the large-volume regime, and we do not attempt to test it in the present paper. But we would like to point out that the structures contained in the non-commutative world-volume also form the main ingredient of the effective action of the brane.

While we have chosen the $\mathrm{SU}(2)_{\mathsf{k}}$ example mainly because of its simplicity and because there exists a semi-classical curved background picture, it is also an important ingredient of the CFT formulation of the Neveu-Schwarz 5-brane, see e.g. [12]. Given that questions like stability of the configuration can be clarified, our findings should be relevant for the geometry of D-branes in the presence of a stack of 5-branes. Similarly, our SU(2) WZW results could be applicable in the study of branes on an $\mathrm{AdS}_3 \times S^3$ string background, see e.g. [13, 14].

2 World-volume geometry – from the flat case to arbitrary backgrounds

Before we show how one can read off fuzzy geometry from branes in the WZW model, let us briefly review the emergence of non-commutative spaces in the more standard case of branes in flat n-dimensional Euclidean space $\mathbb{R}^n$, or on a flat torus $\mathbb{T}^n$. Consider a D-brane which is localized along a p-dimensional hyper-plane V_p

in the target, with tangent space TV_p. The conformal field theory associated with such a Euclidean D-brane is defined on the upper half of the complex plane. It contains an n-component free bosonic field $X = (X^\mu(z,\bar{z}))$, $\mu = 1,\ldots,n$, subject to Neumann boundary conditions in the directions along TV_p and Dirichlet boundary conditions for components perpendicular to the world-volume of the brane. From the free bosons, one may obtain various new fields, in particular the open string vertex operators

$$V_k(x) = :\exp(ikX(x)): \qquad \text{for all} \quad k \in TV_p \quad ,$$

which can be inserted at any point x on the real line. When there is no magnetic field on the brane, the OPE of these U(1)-primaries reads (with $\alpha' = \frac{1}{2}$ and for $x_1 > x_2$)

$$V_{k_1}(x_1)\, V_{k_2}(x_2) \;=\; (x_1 - x_2)^{k_1 k_2/2}\; V_{k_1+k_2}(x_2) + \ldots \quad , \tag{2.1}$$

where the dots indicate less singular non-primary contributions. We can rewrite this relation by introducing the objects

$$f(X(x)) \equiv V[f](x) \;:=\; \frac{1}{(2\pi)^{p/2}} \int_{TV_p} d^p k\; \hat{f}(k)\; V_k(x)$$

for each function $f : V_p \to \mathbb{C}$ with Fourier transform $\hat{f}(k)$. Then the boundary OPE (2.1) translates into a "definition" of pointwise multiplication of functions,

$$V[\,f\,](1)\; V[\,g\,](0) = V[\,f \cdot g\,](0) + \ldots \quad . \tag{2.2}$$

We have specialized to coordinates $x_1 = 1$ and $x_2 = 0$ for convenience, arbitrary insertion points can be recovered via conformal covariance.

The effect of switching on a B-field is described by adding the term

$$S_B \;=\; \frac{1}{2\pi} \int dz d\bar{z}\, B_{\mu\nu} \partial X^\mu(z,\bar{z}) \bar{\partial} X^\nu(z,\bar{z}) \tag{2.3}$$

to the action of the original theory without B-field. One can easily see that this is a pure boundary term with no influence on the bulk properties of the theory. It only changes the boundary conditions. If we assume for definiteness that V_p is spanned by the first p coordinates x^μ, $\mu = 1,\ldots,p$, the new boundary conditions read (with $z = x + iy$)

$$\partial_y X^\mu(z,\bar{z}) \;=\; B^\mu{}_\nu\, \partial_x X^\nu(z,\bar{z}) \quad \text{for} \quad z = \bar{z} \quad \text{and} \quad \mu,\nu = 1,\ldots,p \quad . \tag{2.4}$$

This means that the (exact) free boson propagator becomes ($x_1, x_2 \in \mathbb{R}$)

$$\langle\, X^\mu(x_1) X^\nu(x_2)\,\rangle_B = -\,(\delta^{\mu\nu} + \Theta_S^{\mu\nu})\; \log|x_1 - x_2| - i\,\frac{\pi}{2}\,\Theta_A^{\mu\nu}\; \mathrm{sign}(x_1 - x_2) \tag{2.5}$$

where Θ_S and Θ_A denote the symmetric resp. anti-symmetric part of the matrix $\Theta = (1-B)(1+B)^{-1}$. Explicitly,

$$\Theta_A = \frac{2}{B - B^{-1}} \ . \tag{2.6}$$

In particular, when B is large we obtain $\Theta_A \approx 2B^{-1}$, which means that Θ_A is the Poisson bi-vector corresponding to the symplectic form B. Eq. (2.5) immediately yields the boundary OPE for a non-vanishing B-field,

$$V_{k_1}(1)\, V_{k_2}(0) \ = \ e^{-i\frac{\pi}{2}\, k_1^t \Theta_A k_2}\ V_{k_1+k_2}(0) + \dots \quad .$$

As before, this can be used to define a (deformed) product $\star$ for functions through $V[f](1)V[g](0) = V[f\star g](0) + \dots$, where now

$$(\, f \star g\,)(x) := e^{i\,\frac{\pi}{2}\,\Theta_A^{\mu\nu}\partial_\mu^x \partial_\nu^y}\ f(x)g(y)\ |_{y=x} \quad . \tag{2.7}$$

This is the associative, non-commutative Moyal-Weyl product of functions f, g on the world-volume V_p of the brane. In the context of the derivation we have given, non-commutativity of $\star$ arises because the ordering of boundary fields in general does matter, cf. the sign-term in eq. (2.5). The algebra of functions with product (2.7) is, of course, the non-commutative brane world-volume uncovered by Douglas and Hull using a different approach. It is a deformation of the ordinary algebra of functions, with deformation parameter(s) given by (the matrix) Θ_A.

In [4], the term (2.3) was viewed as a bulk perturbation of the $B = 0$ theory, i.e. techniques of conformal perturbation theory were applied to the operator $\exp(-S_B)$ being inserted into arbitrary correlation functions of the $B = 0$ theory. This perturbative analysis, which can be extended to arbitrary σ-models (at least in the case $dB = 0$), leads to a string theoretic picture of Kontsevich's quantization of Poisson manifolds [15], see also the work of Cattaneo and Felder [16]. It clearly displays that the quantization of world-volume geometries should be expected beyond the case of constant B-fields. This will be confirmed through our exact analysis of the WZW model (see discussion of the limit $\mathsf{k} \to \infty$ below). As we remarked in the introduction, new phenomena are bound to occur when dB does not vanish. In such cases, the classical world-volume of a brane comes equipped with some generalization of an ordinary Poisson-structure, and there exists no general notion of "quantization" for such geometries. Hence, the investigation of branes in a non-vanishing NS 3-form field strength $H = dB$ can teach us new lessons on how to quantize certain non-Poisson geometries. In our example of branes on SU(2) we shall recover some variants of well-known quantum group algebras.

Our formulation of the simple example of flat branes in a constant B-field motivates the following *general procedure*: When we want to associate non-commutative spaces to branes which are given as boundary conditions on the world-sheet, we take the OPE of boundary fields (open string vertex operators corresponding to internal excitations of the brane) as a basic input. Then we choose a suitable subset of boundary fields (e.g. primaries as above) and use them as abstract generators of an algebra of "functions" on the (non-commutative) world-volume of the brane, with multiplication table given by the boundary OPE (projected onto the subset, and evaluated at $x_1 = 1$ and $x_2 = 0$, say).

Further comments on this general prescription will be given later, but now we would like to test it in the case of SU(2) WZW models, where the semi-classical picture provides certain expectations as to how the "quantized world-volume" of branes should look like.

3 D-branes in the SU(2) WZW model

3.1 Semi-classical analysis. The SU(2) WZW model at level k describes strings moving on a three-sphere S^3 of radius $R \sim \sqrt{\mathsf{k}}$, which is equipped with a constant NS 3-form field strength

$$H \;\sim\; \frac{1}{\sqrt{\mathsf{k}}}\,\Omega \;=\; \frac{1}{\sqrt{\mathsf{k}}}\,f_{abc}\,\theta^a \wedge \theta^b \wedge \theta^c \;,$$

where Ω denotes the usual volume form on the unit sphere, and θ^a are components of the 1-form dgg^{-1}. In superstring theory, this geometry appears in the space transverse to a stack of k NS 5-branes. These branes act as sources for k units of NS 3-form flux through a three-sphere surrounding their (5+1)-dimensional world-volume.

The world-sheet swept out by an open string in S^3 is parametrized by a map $g : \mathrm{H} \to \mathrm{SU}(2)$ from the upper half-plane H into the group manifold $\mathrm{SU}(2) \cong S^3$. From this field g one obtains Lie algebra valued chiral currents

$$J(z) \;=\; -\mathsf{k}\,(\partial g)g^{-1} \quad , \qquad \bar{J}(\bar{z}) \;=\; \mathsf{k}\,g^{-1}\bar{\partial}g$$

as usual. We shall be interested in maximally symmetric D-branes on SU(2), which are characterized by the gluing condition $J(z) = \bar{J}(\bar{z})$ along the boundary $z = \bar{z}$. They were analyzed from a semi-classical point of view in [5], and we shall briefly recall the findings of this approach. (For a detailed path integral description of branes in SU(2), see [17].)
We first decompose the tangent space $T_h\mathrm{SU}(2)$ at each point $h \in \mathrm{SU}(2)$ into a part

$T_h^{||}\mathrm{SU}(2)$ tangential to the conjugacy class through h and its orthogonal complement $T_h^{\perp}\mathrm{SU}(2)$ (with respect to the Killing form). In [5], the following two basic observations were made:

1. With gluing conditions of the type $J = \bar{J}$, the endpoints of open strings on SU(2) are confined to conjugacy classes, i.e.

$$(g^{-1}\partial_x g)^{\perp} \;=\; 0 \quad .$$

2. Along the individual branes, i.e. along the conjugacy classes of SU(2), the gluing condition becomes

$$(g^{-1}\partial_y g)^{||} \;=\; \frac{\mathrm{Ad}(g)+1}{\mathrm{Ad}(g)-1}(g^{-1}\partial_x g)^{||} \quad .$$

Except for two degenerate cases, namely the points e and $-e$ on the group manifold, the conjugacy classes are two-spheres in SU(2). Taking into account the usual correspondence between $\sqrt{\mathsf{k}}g^{-1}\partial g$ and the flat space coordinate ∂X, recalling that the metric on the three-sphere scales with k, and comparing with the gluing conditions (2.4), we infer that the D-branes associated with $J = \bar{J}$ carry a non-vanishing 2-form potential (B-field)

$$B \;=\; \frac{1+\mathrm{Ad}(g)}{1-\mathrm{Ad}(g)} \; . \tag{3.1}$$

In the limit $\mathsf{k} \to \infty$ the three-sphere grows and approaches flat 3-space. One can parameterize it by a parameter X taking values in the Lie algebra su(2), such that $g \approx 1 + X$. Then, the formula for the B-field reads

$$B \;\approx\; -2\,(\mathrm{ad}(X))^{-1} \; .$$

This is the Kirillov 2-form on the spheres in the algebra $\mathrm{su}(2) = \mathbb{R}^3$.

Extrapolating formula (2.6) to our curved background, we can construct a bi-vector

$$\Theta_A \;=\; \frac{2}{B - B^{-1}} \;=\; \frac{1}{2}\,(\,\mathrm{Ad}(g^{-1}) - \mathrm{Ad}(g)\,) \; .$$

Introducing an orthonormal basis e^a in su(2), and the left- and right-invariant vector fields e_L^a, e_R^a on the group manifold, one can give an elegant formula for the bi-vector Θ_A,

$$\Theta_A \;=\; \frac{1}{2}\, e_L^a \wedge e_R^a \; .$$

The Schouten bracket of Θ_A (which generally characterizes the deviation from the Jacobi identity) is of the form

$$\phi \;:=\; [\,\Theta_A, \Theta_A\,] \;=\; \frac{1}{6}\, f_{abc}\, (e^a_L - e^a_R)(e^b_L - e^a_R)(e^c_L - e^c_R)\;.$$

Here f_{abc} are the Lie algebra structure constants, the same as those in the expression for the field strength H. This calculation makes sense for an arbitrary simple Lie group. In general, the right hand side does not vanish and gives the obstruction for the Jacobi identity. In the case of $G = \mathrm{SU}(2)$, ϕ vanishes for dimensional reasons: It is a 3-vector tangent to the 2-dimensional conjugacy classes. In the infinite volume limit $\mathsf{k} \to \infty$, the bi-vector Θ_A becomes

$$\Theta_A = \mathrm{ad}(X)\;,$$

which is the Kirillov-Kostant Poisson bi-vector. Consequently, the geometry of the limiting theory $\mathsf{k} = \infty$ is very close to the well-known situation of flat branes in a flat background with constant B-field, and we expect that the world-volume algebras of our branes in the WZW model will be quantizations of two-spheres.
For finite k, however, the background is curved and carries a non-vanishing NS 3-form H. This will result in a non-associative deformation of the $\mathsf{k} = \infty$ theory. Since the three indices of the new object H can relate three-fold products with different positions of brackets, the violation of associativity will turn out to be rather mild. The semi-classical extension of the above analysis shows that, for fixed gluing conditions, only a finite number of SU(2) conjugacy classes satisfy a Dirac-type flux quantization condition [5]. These "integer" conjugacy classes are the two points e and $-e$ along with $\mathsf{k}-1$ of the spherical conjugacy classes (those passing through the points $\mathrm{diag}(\exp(i\pi j/\mathsf{k}), \exp(-i\pi j/\mathsf{k}))$ for $j = 1, \ldots, \mathsf{k}-1$).

3.2 Exact CFT description. The WZW model on the upper half-plane is known in enough detail to support and specify the rather crude arguments of the previous subsection by an exact CFT analysis. In fact, for the situation we are dealing with (gluing conditions $J = \bar{J}$ in a "parent" CFT on the full complex plane with diagonal modular invariant partition function), Cardy [19] was able to list all [20] possible boundary conditions. There exist $\mathsf{k}+1$ of them, differing in the bulk field one-point functions (brane charges) and labeled by an index $\alpha = 0, \frac{1}{2}, \ldots, \frac{\mathsf{k}}{2}$. Without entering a detailed description of these boundary theories [19], we recall that their state spaces have the form

$$\mathcal{H}_\alpha \;=\; \bigoplus\nolimits_J N^J_{\alpha\alpha}\; \mathcal{H}^J \tag{3.2}$$

where $\mathcal{H}^J$, $J = 0, \frac{1}{2}, \ldots, \frac{\mathsf{k}}{2}$, denote irreducible highest weight representations of the affine Lie algebra $\widehat{\mathrm{SU}}(2)_{\mathsf{k}}$, and where N_{IJ}^K are the associated fusion rules. Note that only integer spins J appear on the right hand side of (3.2).

There exists a variant of the state-field correspondence which assigns a boundary field $\psi(x)$ to each element $|\psi\rangle \in \mathcal{H}_\alpha$ (see e.g. [21]). In particular, the SU(2) WZW boundary theory labeled by α contains SU(2)-multiplets associated to primary boundary fields, namely

$$\Psi^J(x) \;=\; (\psi^J_m(x)) \qquad \text{with} \qquad J = 0, 1, \ldots, \min(2\alpha, \mathsf{k} - 2\alpha)$$

and $m = -J, \ldots, J$. All these boundary fields are defined for arguments x on the real line and their correlators have, in general, no unique analytic continuation into the upper half-plane.

In the flat target case, we chose U(1)-primaries as generating elements of the world-volume algebra. Now, it is more appropriate not to break the group symmetry by hand and, therefore, to keep the full SU(2)-multiplets $\Psi^J(x)$. For a fixed order $x > y$ of arguments on the real line, the OPE of two such boundary fields reads

$$\psi_i^I(x)\; \psi_j^J(y) \;\sim\; \sum\nolimits_{K,k} (x-y)^{h_I + h_J - h_K} \; \left[\begin{smallmatrix} I & J & K \\ i & j & k \end{smallmatrix}\right] \; c_{IJK}^{\mathsf{k},\alpha} \; \psi_k^K(y) \quad , \tag{3.3}$$

where h_J is the conformal dimension of Ψ^J and [:::] denote the Clebsch-Gordan coefficients of the group SU(2). The latter simply compensate for the different transformation behavior of the fields on the left and right hand side under the action of the zero-mode subalgebra of $\widehat{\mathrm{SU}}(2)_{\mathsf{k}}$. Hence, the non-trivial information in (3.3) is contained in the new structure constants $\mathcal{C} = (c_{IJK}^{\mathsf{k},\alpha})$.
In a consistent theory, these must obey sewing constraints, which were first analyzed by Lewellen in [22]; see also [20]. Recently, these constraints were reconsidered by Runkel [23] for the A-series of Virasoro minimal models. His findings carry over to SU(2) WZW models on the upper half-plane and show that the only possible solution to the sewing constraints is given by the fusing matrix F of the WZW theory,

$$c_{IJK}^{\mathsf{k},\alpha} \;=\; F_{\alpha K}\left[\begin{smallmatrix} \alpha & \alpha \\ I & J \end{smallmatrix}\right]_{\mathsf{k}} \quad . \tag{3.4}$$

It is one of the fundamental results on the relation between quantum groups and conformal field theory (see e.g. [24]) that the fusing matrix of the WZW model is obtained from the $6J$ symbols of the quantum group algebra $U_q(\mathrm{su}(2))$ according to

$$F_{\alpha K}\left[\begin{smallmatrix} \alpha & \alpha \\ I & J \end{smallmatrix}\right]_{\mathsf{k}} \;=\; \left\{\begin{smallmatrix} I & J & K \\ \alpha & \alpha & \alpha \end{smallmatrix}\right\}_q \qquad \text{where} \qquad q = e^{\frac{2\pi i}{\mathsf{k}+2}} \quad . \tag{3.5}$$

In the limit $q \to 1$, the $6J$ symbols of the quantum group algebra approach those of the classical algebra $U(\mathrm{su}(2))$, thus the structure constants $c_{IJK}^{\mathsf{k},\alpha}$ of the boundary OPE become $6J$ symbols of the group SU(2) when the level k is sent to infinity. Note that in this limit, the conformal dimensions $h_J = J(J+1)/(k+2)$ tend to zero so that the OPEs (3.3) of boundary fields become regular as in a topological theory.

4 D-brane geometry, fuzzy two-spheres, and quantum groups

We are now prepared to follow the procedure sketched at the end of Section 2 and to read off the world-volume geometry of branes in the SU(2)-WZW model. So let us think of the boundary fields $\psi_i^I = V(Y_i^I)$ as being assigned to elements Y_i^I of some vector space, and let us use the operator product expansion (3.3,3.4,3.5) to define a multiplication by the prescription

$$Y_i^I \star Y_j^J \;=\; \sum\nolimits_{K,k} \left[\begin{smallmatrix} I & J & K \\ i & j & k \end{smallmatrix}\right] c_{IJK}^{\mathsf{k},\alpha}\; Y_k^K \quad . \tag{4.1}$$

As in (3.3), the summation on the right hand side runs from $K = 0$ to a maximal spin $K_{\max} = \min(I+J, \mathsf{k}-I-J, 2\alpha, \mathsf{k}-2\alpha)$. First, we shall investigate this product in the limiting case $\mathsf{k} = \infty$, where it produces a familiar algebraic structure. Passing to finite levels leads to the following two changes: There is a k-dependent deformation of structure constants $\mathcal{C}$, cf. (3.5), and the range of the summation in (4.1) becomes a function of the level, $K_{\max} = K_{\max}(\mathsf{k})$. We shall separate these two phenomena by looking at an intermediate case where k is non-rational and where we omit the k-dependent restriction on the K-summation.

Infinite level $\mathsf{k} = \infty$: Recall that, in the case of infinite level, the structure constants $\mathcal{C}$ in eq. (4.1) are given by the $6J$ symbols of the group SU(2). The semi-classical analysis showed that $H \to 0$, so we expect the world-volume algebra to be associative. Indeed this can be confirmed using the Biedenharn-Elliot (or pentagon) relation for the $6J$ symbols, along with the fact that $6J$ symbols of the form (3.5) vanish whenever $K > 2\alpha$. Hence, for infinite level our relations define an infinite set of associative algebras S_α^2, $\alpha = 0, \frac{1}{2}, \ldots$, with finite linear bases consisting of $\dim(S_\alpha^2) = (2\alpha+1)^2$ elements.
Since the dimension of each of these algebras is a perfect square, one may already suspect that they are full matrix algebras, i.e. that $S_\alpha^2 \cong M_N(\mathbb{C})$ with $N = 2\alpha+1$. To describe the isomorphism, we first note that $M_N(\mathbb{C})$ admits an action of the group SU(2) by conjugation with group elements evaluated in the N-dimensional representation of SU(2). Under this action, the SU(2)-module $M_N(\mathbb{C})$ decomposes

into a direct sum of irreducible representations V^J,

$$M_N(\mathbb{C}) \;\cong\; \bigoplus\nolimits_{J=0}^{N-1} V^J \quad . \tag{4.2}$$

Only integer J appear, so this agrees with the decomposition of the state space $\mathcal{H}_\alpha,\ \alpha = (N-1)/2$, in eq. (3.2) for boundary WZW models at sufficiently large (or infinite) level k. Thus, we can identify our elements Y_j^J with a basis of the spaces V^J. The isomorphism (4.2) allows to work out multiplication rules for any two such basis elements from the multiplication of $N\times N$-matrices. The result [25] turns out to coincide with our formula (4.1), which shows that S^2_α and $M_N(\mathbb{C}),\ N = 2\alpha+1$, are indeed isomorphic as associative algebras.
The non-commutative spaces S^2_α are known as *fuzzy spheres* and are obtained when one quantizes functions on a two-sphere with the usual Poisson structure (see e.g. [26] and references therein). The two-spheres may also be identified with co-adjoint orbits of SU(2). According to Kirillov, their quantization gives all representations of the Lie algebra su(2) or of its universal enveloping algebra $U(\mathrm{su}(2))$. Note that the size $N = 2\alpha+1$ of our matrices agrees with the number of components for an su(2)-multiplet of spin α. Hence, through the investigation of maximally symmetric branes on SU(2) at $\mathsf{k} = \infty$, we have recovered Kirillov's theory of co-adjoint orbits.

Finite non-rational level k: Let us stress that this case does not appear among the exact boundary theories above (for non-compact WZW models, it is the generic situation). We include it here merely as an intermediate step before presenting the structure for finite integer level k. To be more precise, we consider the algebras spanned by Y_j^J with relations (4.1) in which the structure constants $\mathcal{C}$ are given by the $6J$ symbols (3.5) of the quantum group algebra $U_q(\mathrm{su}(2))$, but with summation over the same range as in the case $\mathsf{k} = \infty$.
The resulting algebras $S^2_{\alpha,q}$ with $q = \exp(2\pi i/(\mathsf{k}+2))$ not a root of unity cease to be associative. But they are still quasi-associative in the sense that

$$Y_i^I \star (Y_j^J \star Y_k^K)(\tau^I_{in}\otimes\tau^J_{jm}\otimes\tau^K_{kl})(\varphi) \;=\; (Y_n^I \star Y_m^J)\star Y_l^K \tag{4.3}$$

where the τ^L denote representations of $U(\mathrm{su}(2))$ and where $\varphi \in U(\mathrm{su}(2))^{\otimes 3}$ is Drinfeld's "re-associator" [27]. The proof of this statement is sketched in the appendix.

When we perform a standard quasi-classical limit, commutators are replaced by the brackets corresponding to the bi-vector Θ_A. For a general compact simple Lie group Θ_A fails to satisfy the Jacobi identity. This corresponds to the leading non-vanishing term in the $\frac{1}{\mathsf{k}}$-expansion of the re-associator φ,

$$\varphi = 1 + \frac{1}{6\mathsf{k}} f_{abc} e^a \otimes e^b \otimes e^c + \ldots$$

where e^a is, as above, an orthonormal basis in the Lie algebra, and f_{abc} are the corresponding structure constants. When applied to the relation (4.3), the Lie algebra generators e^a act by the adjoint vector fields $(e^a_L - e^a_R)$. In the case of $G = \mathrm{SU}(2)$ this leads to vanishing of the first order correction to the associativity law. This is in accordance with vanishing of $[\Theta_A, \Theta_A]$ in this case. Note that even in the SU(2) case higher order corrections to the associativity law do not vanish.

Let us briefly mention that our quasi-associative algebras $S^2_{\alpha,q}$ are closely connected to associative deformations of the fuzzy sphere which employ the Clebsch-Gordan coefficients of the deformed $U_q(\mathrm{su}(2))$ instead of their classical analogs. Some details on these algebras and their associativity can be found in the appendix. For now, let us only remark that they are factors of the quantum spheres introduced by Podleś in [28]. Their relation to our algebras $S^2_{\alpha,q}$ is based on the fact that one can obtain the Clebsch-Gordon maps of classical Lie algebras from their q-deformed counterparts with the help of Drinfeld's "twist element" $F \in U(\mathrm{su}(2))^{\otimes 2}$. The latter provides the following factorization formula for the re-associator:

$$\varphi \;=\; (\mathrm{id} \otimes \Delta)(F^{-1})\,(e \otimes F^{-1})\,(F \otimes e)\,(\Delta \otimes \mathrm{id})(F)$$

where Δ denotes the co-product of $U(\mathrm{su}(2))$. Combining these two roles of the twist element F, one can show that our algebras $S^2_{\alpha,q}$ are "twist equivalent" to associative factors of a Podleś sphere or, more explicitly, to the same matrix algebras $M_N(\mathbb{C})$, $N = 2\alpha + 1$, as in the case of infinite level. Hence, we simply recover the representations for the usual q-deformation of $U(\mathrm{su}(2))$ at generic values of the deformation parameter.

Finite integer level k: The associated algebras A^{k}_α are spanned by the generators Y^J_m with the label J chosen from the set $J = 0, 1, \ldots, \min(2\alpha, \mathsf{k} - 2\alpha)$. Multiplication of these elements is defined through eq. (4.1) with structure constants $\mathcal{C}$ now given by the $6J$ symbols of $U_q(\mathrm{su}(2))$ at the root of unity $q = \exp(2\pi i/(\mathsf{k}+2))$. In addition, the summation on the right hand side is now restricted to run from $K = 0$ to $\min(I + J, \mathsf{k} - I - J, 2\alpha, \mathsf{k} - 2\alpha)$. Viewed as SU(2)-modules, the linear spaces A^{k}_α decompose as follows:

$$A^{\mathsf{k}}_\alpha \;\cong\; \begin{cases} S^2_\alpha & \text{for} \quad 0 \le \alpha \le \frac{\mathsf{k}}{4} \\ S^2_{\mathsf{k}/2-\alpha} & \text{for} \quad \frac{\mathsf{k}}{4} \le \alpha \le \frac{\mathsf{k}}{2} \end{cases} \quad .$$

Again, the algebras A^{k}_α are only quasi-associative, and they provide examples of the geometries considered in [29]. Using the concept of representations introduced in [30], it is not difficult to show that each of the quasi-associative algebras A^{k}_α possesses precisely one indecomposable representation on a vector space W^α of

dimension

$$\dim W^\alpha \;=\; \begin{cases} 2\alpha+1 & \text{for} \quad 0 \le \alpha \le \frac{\mathsf{k}}{4} \\ \mathsf{k}-2\alpha+1 & \text{for} \quad \frac{\mathsf{k}}{4} \le \alpha \le \frac{\mathsf{k}}{2} \end{cases} \quad .$$

According to our previous discussion, the algebras A^{k}_α and their representations on W^α, $\alpha = 0, \frac{1}{2}, \ldots, \frac{\mathsf{k}}{2}$, generalize Kirillov's theory of co-adjoint orbits to quantum groups at roots of unity. In other words, the algebras A^{k}_α we obtain are "quantizations" of integer conjugacy classes on SU(2). Summing over all possible brane sectors, i.e. over the index α, we construct a deformed universal enveloping algebra.

Of course, quantum group algebras were constructed within the framework of chiral conformal field theory before, see e.g. [27,31–33]. As long as we avoid roots of unity, our new derivation from boundary conformal field theory reproduces well-known algebraic structures. Differences between the two approaches occur only when q is a root of unity. In that case, boundary conformal field theory improves upon the old constructions in two respects. First of all, the theory gives "physical" representations exclusively so that there is no need for additional truncations. Furthermore, the dimensions $\dim W^\alpha$ of the representation spaces are invariant under the simple current symmetry which interchanges α and $\mathsf{k}/2 - \alpha$.

When we increase the level k, the radius of the three-sphere grows and we can fit more and more branes into the background. At the same time, the 3-form field strength decreases and the world-volume algebras become "more associative" – while their non-commutativity survives.
This is to be compared to the non-commutative targets obtained in [7, 9, 10] from closed strings: The $\mathsf{k} \to \infty$ limit of these targets is simply the classical group SU(2). The different behavior of closed and open string geometry may be explained as follows: Both closed and open strings feel the presence of the NS 3-form field H at finite level. Open strings are also sensitive to the concrete choice of a 2-form potential B, while closed strings "see" only its cohomology class. In the flat space limit $\mathsf{k} = \infty$, the cohomology becomes trivial while B itself stays non-zero and is responsible for non-commutativity on the brane.

5 Summary and outlook

We have derived non-commutative world-volume algebras for D-branes in the SU(2) WZW model, using a general scheme that can be applied to arbitrary branes given as conformal boundary conditions, including supersymmetric cases. In the process, we have seen how abstract objects from the CFT description, like Cardy's boundary states and Runkel's OPE coefficients, acquire a geometrical meaning – if in terms of non-commutative (and sometimes non-associative) spaces. The SU(2)

WZW model provides just the simplest example of a string background with a non-vanishing 3-form field strength H, but we think that it illustrates quite nicely much of the behavior one should expect from more complicated backgrounds. In particular, the discussion of SU(2) branes carries over to boundary WZW models with other structure groups G (at least in the compact case) and leads to a quantization of integer conjugacy classes in G. It might be interesting to investigate also branes that are not maximally symmetric, i.e. where the gluing conditions respect only a subalgebra of the maximal chiral symmetry algebra [34].

Boundary CFT yields world-volumes independently of whether limiting classical pictures are available or not, and it actually provides more structure than a mere set of non-commutative algebras. Connes' program [35] shows that, in order to talk about the geometry of a non-commutative space, it is necessary to fix further "spectral data", including a Hilbert space on which the (associative) world-volume algebra and a generalized Dirac or Laplace operator act. How these data can be extracted from a CFT has been discussed, for the bulk case, in [7, 9]. The importance of the Laplace operator, which is related to the conformal Hamiltonian L_0, can also be seen in the context of our definition of non-commutative world-volumes: In order to re-derive the OPE of boundary operators from the algebraic structure of the world-volume, the spectrum of conformal dimensions must be known, cf. the remark after eq. (2.2).
In a CFT on the upper half-plane, additional structure is available, e.g. in the form of boundary condition changing operators which induce transitions between two different boundary conditions α, β. The OPE of the boundary fields $\Psi^I(x)$ with boundary condition changing operators gives rise to bi-modules $B_{\alpha\beta}$ over the world-volume algebras of the two associated branes. In the case of D-branes on a group manifold, these bi-modules allow to construct tensor products for representations of the associated quantum group. OPEs involving two boundary condition changing operators provide even more data, namely a full braided tensor category.

Some comments on our general scheme to extract a world-volume algebras from the boundary CFT description of branes are in order. It involves a choice of "generating elements" among the boundary fields. From a pure CFT perspective, one could restrict to primary operators only, or one could work with all boundary operators and thus with an infinite-dimensional world-volume. In a sense, the latter algebra would include all internal excitations of the "static" space defined using primary fields. The WZW case, where it proved natural to keep the full group multiplets associated with primary boundary fields, suggests that there are distinguished "intermediate" choices. For a large class of CFTs, the appropriate generalization of the lowest-dimension spaces of WZW models is likely to be given by the special subspaces introduced in [36]; see also [37].

Placing the CFT into a string theory context can remove the arbitrariness and provide clear guidelines as to which world-volume generators to select from the boundary fields: String theory contains additional parameters like α', and the relevant generators of the world-volume algebra are those surviving in some limiting regime. E.g. in the flat background case, one can remove all higher excitations by sending α' to zero while keeping the B-field finite; see [38] and also [1]. It may be possible that a number of interesting limits exists; then one expects that the world-volume of a brane can look very different in different regimes, and that full string theory can "interpolate" between those geometries.

The next task would be to calculate the effective action on the – in general non-commutative – world-volume of the brane. The lowest-order terms are, of course, already given by our "multiplication table" (the OPE coefficients). In principle, higher-order contributions can be computed from the same data, but in practice one still needs to integrate over world-sheet moduli.
In the context of the Douglas-Hull model, the effective field theories were found to be non-commutative supersymmetric gauge theories with some amount of non-locality [1–3,39–41]. Seiberg and Witten could show that these models are equivalent to ordinary gauge theories on a flat brane [38]. It remains to be seen whether classical structures are stretched further when more general CFT backgrounds are taken as a starting point. Perhaps it is worthwhile to compare the induced field theories with existing models on fuzzy geometries (see e.g. [42]).

It would also be interesting to investigate further the relation between world-volume non-commutativity as introduced in [1] and non-commuting moduli as discovered by Witten [43]. Both phenomena can be traced back to failures in locality properties of boundary fields – see [44,45] for the case of moduli – so that there exists a direct connection between the brane's intrinsic "fuzziness" and the way it "perceives" its ambient target.

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Note added: After this work was completed, another approach to the geometry of branes in WZW models based on exact CFT methods was presented in [47].

Appendix: (Quasi-)associativity

Here we collect some basic material on Clebsch-Gordan maps, $6J$-symbols and the (quasi-)associativity of various algebras mentioned in the main text. Let us denote by τ^I the irreducible representation of $U_q(\mathrm{su}(2))$ with spin I. By definition, Clebsch-Gordan maps $C_q(IJ|K) : V^I \otimes V^J \to V^K$ intertwine between the actions of $U_q(\mathrm{su}(2))$ on the product module $V^I \otimes V^J$ and the irreducible module V^K. $6J$ symbols enter the theory through the basic relation

$$C_q(MK|L)\,(C_q(IJ|M) \otimes \mathrm{id}^K) \;=\; \sum_P \left\{ \begin{smallmatrix} L & K & M \\ I & J & P \end{smallmatrix} \right\}_q C_q(IP|L)\,(\mathrm{id}^I \otimes C_q(JK|P)) \quad . \tag{A.1}$$

They obey a number of fundamental equations. For our purposes, the Biedenharn-Elliot (pentagon) relation is the most important one. With the spin labels set to the values that we need below, it implies

$$\sum_M \left\{ \begin{smallmatrix} L & K & M \\ I & J & P \end{smallmatrix} \right\}_q \left\{ \begin{smallmatrix} I & J & M \\ \alpha & \alpha & \alpha \end{smallmatrix} \right\}_q \left\{ \begin{smallmatrix} M & K & L \\ \alpha & \alpha & \alpha \end{smallmatrix} \right\}_q \;=\; \left\{ \begin{smallmatrix} J & K & P \\ \alpha & \alpha & \alpha \end{smallmatrix} \right\}_q \left\{ \begin{smallmatrix} I & P & L \\ \alpha & \alpha & \alpha \end{smallmatrix} \right\}_q \tag{A.2}$$

Relations (A.1,A.2) hold for generic q and at the classical point $q = 1$ where we are dealing with representation theory of ordinary Lie algebras.

Let us now study the algebra generated by Y_i^I for $I = 0, 1, \ldots 2\alpha$ and $|i| \leq I$ with the multiplication rules

$$Y_i^I \star Y_j^J \;=\; \sum_{K,k} \left[\begin{smallmatrix} I & J & K \\ i & j & k \end{smallmatrix} \right]_q \left\{ \begin{smallmatrix} I & J & K \\ \alpha & \alpha & \alpha \end{smallmatrix} \right\}_q Y_k^K \quad . \tag{A.3}$$

The Clebsch-Gordan coefficients on the right hand side are obtained from the maps $C(IJ|K)$ once we have selected a basis in each representation space V^L. Associativity of this algebra is rather easy to prove with the help of eqs. (A.1) and (A.2):

$$\begin{aligned}
(Y_i^I \star Y_j^J) \star Y_k^K \;&=\; \sum_{L,l,M,m} \left[\begin{smallmatrix} I & J & M \\ i & j & m \end{smallmatrix} \right]_q \left[\begin{smallmatrix} M & K & L \\ m & k & l \end{smallmatrix} \right]_q \left\{ \begin{smallmatrix} I & J & M \\ \alpha & \alpha & \alpha \end{smallmatrix} \right\}_q \left\{ \begin{smallmatrix} M & K & L \\ \alpha & \alpha & \alpha \end{smallmatrix} \right\}_q Y_l^L \\
&=\; \sum_{L,l,M,P,p} \left[\begin{smallmatrix} J & K & P \\ j & k & p \end{smallmatrix} \right]_q \left[\begin{smallmatrix} I & P & L \\ i & p & l \end{smallmatrix} \right]_q \left\{ \begin{smallmatrix} L & K & M \\ I & J & P \end{smallmatrix} \right\}_q \left\{ \begin{smallmatrix} I & J & M \\ \alpha & \alpha & \alpha \end{smallmatrix} \right\}_q \left\{ \begin{smallmatrix} M & K & L \\ \alpha & \alpha & \alpha \end{smallmatrix} \right\}_q Y_l^L \\
&=\; \sum_{L,l,P,p} \left[\begin{smallmatrix} J & K & P \\ j & k & p \end{smallmatrix} \right]_q \left[\begin{smallmatrix} I & P & L \\ i & p & l \end{smallmatrix} \right]_q \left\{ \begin{smallmatrix} J & K & P \\ \alpha & \alpha & \alpha \end{smallmatrix} \right\}_q \left\{ \begin{smallmatrix} I & P & L \\ \alpha & \alpha & \alpha \end{smallmatrix} \right\}_q Y_l^L \\
&=\; Y_i^I \star (Y_j^J \star Y_k^K)
\end{aligned}$$

For the special case $q = 1$ this computation proves the associativity of the world-volume algebra in the limit $\mathsf{k} = \infty$. When the level k is finite and non-rational, however, the defining relation for our algebra $S^2_{\alpha,q}$ from Sect. 4 employs the *undeformed* Clebsch-Gordan maps along with the deformed $6J$ symbols. Hence, using relation (A.1) for $q = 1$, we generate an undeformed $6J$ symbol in our computation above. The latter cannot be absorbed with the help of the pentagon identity, since we have to deal with a product of one undeformed and two deformed $6J$ symbols.

At this point, Drinfeld's re-associator $\varphi \in U_q(\mathrm{su}(2))^{\otimes}_3$ plays a decisive role because of its fundamental property

$$C(MK|L)\,(C(IJ|M)\otimes \mathrm{id}^K)(\varphi^{-1})^{IJK} \;=\; \sum_P \left\{ \begin{smallmatrix} L & K & M \\ I & J & P \end{smallmatrix} \right\}_q C(IP|L)\,(\mathrm{id}^I \otimes C(JK|P))$$

$$\text{where} \quad (\varphi^{-1})^{IJK} \;=\; (\tau^I \otimes \tau^J \otimes \tau^K)(\varphi^{-1}) : V^I \otimes V^J \otimes V^K \to V^I \otimes V^J \otimes V^K \ .$$

Note that this relation involves Clebsch-Gordan maps of the Lie algebra and q-deformed $6J$-symbols at the same time. φ allows to modify the proof we have given for the associativity of the algebra (A.3) such that we obtain the quasi-associativity property (4.3).

A relation between our quasi-associative algebra $S^2_{\alpha,q}$ and the associative q-deformation of the fuzzy sphere can be established with the help of Drinfeld's twist element F. By definition, it maps the deformed and undeformed Clebsch Gordan maps onto each other,

$$C_q(IJ|K)(\tau^I \otimes \tau^J)(F) \;=\; C(IJ|K) \ .$$

This property becomes crucial in showing that the quasi-associative algebra for non-rational k is "twist-equivalent" to the associative q-deformed fuzzy sphere. Some details on the notion of twist equivalence can be found e.g. in Section 7.3 of [46].

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