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# When does a detector click?

R. Brunetti[1] and K. Fredenhagen[2]

[1] *Dip. di Scienze Fisiche, Univ. Napoli "Federico II", Com. Univ. Monte Sant'Angelo, Via Cintia, I-80126 Napoli, Italy.*
[2] *II Inst. f. Theoretische Physik, Universität Hamburg, 149 Luruper Chaussee, D-22761 Hamburg, Germany.*

We propose a general construction of an observable measuring the time of occurence of an effect in quantum theory. Time delay in potential scattering theory is computed as a straightforward application.

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## I. INTRODUCTION

Time measurements play an important role in experiments, but in quantum theory the corresponding observables are not easy to find. The difficulties are connected with the fact that the dual observable, the energy, has a nontrivial spectrum, on which shifts are in general not well defined.

It is well known and easy to see that no selfadjoint operator exists which decribes the measurement of time [9]. Therefore one has to rely on a more general concept of observables, and a natural option is the concept of positive operator valued measures [7]. This concept has meanwhile often been applied to the problem of time measurements (see e.g. [4]). What is missing, however, so far we know, is a discussion of the *effect* whose occurence time is described. A description of the latter is the main contribution of this letter.

## II. THE CONSTRUCTION OF A TIME OPERATOR

Let $A > 0$ be a bounded operator on a Hilbert space $\mathcal{H}$ which measures the occurence of an effect at time $t = 0$. In Quantum Mechanics one might think of the projection operator $P(x \in M)$ for the position being in the region $M$, see e.g. [1], in Quantum Field Theory a choice would be an Araki-Haag counter [2], i.e. an almost local operator with vanishing vacuum expectation value, an (unbounded) example, in free massive field theory, being the partial number operator $a(f)^* a(f)$ where $a(f)$ denotes the annihilation operator for a particle with a smooth momentum space wave function $f$.

Given a state $\omega$ and a time evolution $\alpha_t$, one may consider the expectation values $\omega(\alpha_t(A))$ as a nonnegative function of $t$ (assumed to be continuous). At times $t$ when this function is big the occurence of the effect will be more probable than at times $t$ when it is small, and one may ask whether one can derive from this a probability distribution. It may happen that $\int \omega(\alpha_t(A))dt = \infty$, so the time spent in the detector is infinitely long, and a probability distribution cannot be defined (in this case an arrival time might possibly be introduced (see e.g. [11])). It may also happen that $\omega(\alpha_t(A)) = 0$ for all $t$, which means that the effect never takes place; also then it does not make sense to discuss a probability distribution. But in the intermediate case when the integral is finite a probability distribution can be found.

The question arises whether such a probability distribution can be written as the expectation value of a positive operator valued measure. This means essentially that the normalization should be done on the level of operators, not on the level of expectation values.

We start from the integrals

$$B(I) = \int_I \alpha_t(A)dt \ ,$$

where $I$ is some bounded interval of the real line $\mathbf{R}$. Hence $B(I)$ is a positive bounded operator, and $B(J) \geq B(I)$ if $J \supset I$. We want to construct the operator

$$B = \int_{\mathbf{R}} \alpha_t(A)dt \ .$$

Up to the normalization of $A$, $B$ may be interpreted as the total time duration of the effect.

For this purpose we consider the operators $(B(I) + 1)^{-1}$. They form a decreasing net of positive bounded operators with greatest lower bound $C \leq 1$. Let $\mathcal{H}_\infty$ be the kernel of $C$ and $\mathcal{H}_0$ the joint kernel of $\alpha_t(A)$. In the corresponding states the time of occurence of the effect is infinite, resp. 0. Let $\mathcal{H}_{\text{finite}}$ be the orthogonal complement of $\mathcal{H}_\infty + \mathcal{H}_0$ in $\mathcal{H}$. On $\mathcal{H}_{\text{finite}}$ we define $B$ as the operator

$$B = C^{-1} - 1 \ .$$

$B$ is a positive selfadjoint, in general unbounded operator, which dominates all operators $B(I)$ (restricted to

operators in $\mathcal{H}_{\text{finite}}$). We set

$$P(I) = B^{-\frac{1}{2}} B(I) B^{-\frac{1}{2}} \ ,$$

and get positive operators bounded by 1 with

$$P(I \cup J) = P(I) + P(J) \ ,$$

for disjoint intervals $I$ and $J$ and with

$$P(\mathbf{R}) = 1 \ .$$

One may readily check that the countable additivity property of $P$ is also fulfilled. So we obtained a positive operator valued measure which transforms in the right way under time translations,

$$\alpha_t(P(I)) = P(I + t) \ .$$

To compare our concept with other definitions of time measurements we consider the first moment of the measure (the time operator associated to $A$),

$$T_A = \int t P(dt) \ .$$

We assume that the time translations are implemented by unitaries $e^{itH}$ with a selfadjoint Hamiltonian $H$ with absolutely continuous spectrum and that the Hilbert space of our model can be represented as the space of sections of a (trivial) bundle of finite dimensional Hilbert spaces over the spectrum of $H$.

Let $A$ be a positive operator with a smooth integral kernel $a(E, E')$ with $a(E, E) > 0, \forall E$. Then the integral kernel of $B$ is

$$\begin{aligned} b(E, E') &= \int e^{it(E-E')} a(E, E') dt \\ &= 2\pi \delta(E - E') a(E, E') \ . \end{aligned}$$

For $P(I)$ we find the integral kernel

$$\frac{1}{2\pi} c(E, E') \int_I e^{it(E-E')} dt \ ,$$

with

$$c(E, E') = a(E, E)^{-\frac{1}{2}} a(E, E') a(E', E')^{-\frac{1}{2}} \ .$$

For the first moment (the time operator) we obtain on smooth sections $\Phi$ with compact support which vanish at the boundaries of the spectrum of $H$

$$T_A \Phi(E) = (-i\partial_E + d_A(E)) \Phi(E) \ ,$$

with

$$d_A(E) = -i \partial_{E'} c(E, E')|_{E'=E} \ .$$

Note that $d_A$ is hermitian because of the positivity of $c$.

We see that the time operator $T_A$ has the expected form as a covariant derivative in our bundle. Different effects give rise to different connections $d_A$.

From this result we can immediately obtain the observable which measures the time in between two effects. If $d_1$ and $d_2$ are the corresponding connections (we may assume that they are bounded, so the domain of the time operators in question coincide), the transition time between both effects is

$$d_1 - d_2 \ .$$

The transition time is therefore (under the above conditions) a selfadjoint bounded operator showing none of the problems associated with the time operator. (This is in conflict with the statement in [8] that also transition times have an intrinsic uncertainty. The reason for this discrepancy, as far we can see, is that in [8] the transition time is analyzed only in eigenstates of the momentum whereas the eigenstates of the transition time as defined above are superpositions of different momentum eigenstates with the same energy.)

## III. APPLICATION TO SCATTERING THEORY

A famous example for an analysis of times in quantum theory is the time delay in scattering theory (see, e.g., [1] [6]). Here a general formula can be derived which is valid for all interactions. In the literature [6] it is done using the so-called sojourn time. The sojourn time is defined as the expectation value of the operator $B$ introduced above where the effect $A$ is the projection operator characterizing the probability that the particle is in a certain region of space which eventually tends to the whole space. One then compares the sojourn times of the interacting case with the noninteracting case. This analysis requires delicate arguments of convergence. We want to show that our concept leads to a much more direct and elementary derivation.

Let us consider a typical situation of scattering in quantum mechanics: The initial state $\phi_{\text{in}}$ evolves according to the interacting Hamiltonian and asymptotically tends to $\phi_{\text{out}} = S\phi_{\text{in}}$ where $S$ is the scattering matrix (assumed to be unitary). The time delay of the event measured by the positive operator $A$ is given by the formula

$$\langle \phi_{\text{out}}, T_A \phi_{\text{out}} \rangle - \langle \phi_{\text{in}}, T_A \phi_{\text{in}} \rangle$$

(provided the domain of $T_A$ is invariant under $S$). Hence,

$$\begin{aligned} t_{\text{delay}} &= \langle \phi_{\text{in}}, (S^{-1} T_A S - T_A) \phi_{\text{in}} \rangle \\ &= \langle \phi_{\text{in}}, S^{-1} [T_A, S] \phi_{\text{in}} \rangle \ . \end{aligned}$$

According to what has been exposed in the previous section we obtain

$$t_{\rm delay}(E) = S(E)^{-1}(-i\partial_E)S(E) + S(E)^{-1}[d_A(E), S(E)] \ ,$$

so the time delay is just the change of the connection under the action of the on-shell scattering matrix (considered as a gauge transformation). If $d_A$ commutes with $S$ (this is always the case when the energy spectrum is nondegenerate) one obtains the well known formula of Eisenbud and Wigner (see [12], but especially [6] and [1]).

As a simple explicit example let us analyze the case of a central repulsive and short range potential.

The radial wave function of a scattering state of the particle with sharp angular momentum has the form

$$\psi(t,r) = \int e^{-itE} u_k(r)\varphi(k)dk \ , \ E = \frac{k^2}{2m} \ ,$$

where the functions $u_k$ are solutions of the radial Schrödinger equation with the asymptotic behavior for $r \to \infty$

$$u_k(r) \sim e^{-ikr} + e^{ikr + i\delta(k)} \ ,$$

with a real function $\delta$ (the phase shift).

We first have to find an observable which indicates the passage of the particle through a spherical shell, of thickness $\rho$ (which eventually will tend to zero) with a sufficiently large radius $R$, in outward direction. Such an observable is

$$A = Q^* P Q \ ,$$

where $P$ is the projection onto states with positions inside the shell and where $Q$ simulates the (nonexistent) projection on states with positive radial momentum. We may choose

$$Q = -i\partial_r (2mH)^{-\frac{1}{2}} - 1 \ ,$$

where $\partial_r$ is the derivative operator with boundary condition $\psi(r=0) = 0$ (then $-i\partial_r$ is maximally symmetric but not selfadjoint). Note that $Q$ is bounded and that it selects for large $r$ the outgoing component of $u_k$.

We now insert $A$ into the formulas derived before and get

$$a(E,E') = \frac{m}{\sqrt{kk'}} \langle Qu_k, PQu_{k'} \rangle \ ,$$

$$c(E,E') = \langle Qu_k, PQu_k \rangle^{-\frac{1}{2}} \langle Qu_k, PQu_{k'} \rangle \langle Qu_{k'}, PQu_{k'} \rangle^{-\frac{1}{2}} \ ,$$

$$d(E) = -\tfrac{im}{2k} \langle Qu_k, PQu_k \rangle^{-1} \big( \langle Q\partial_k u_k, PQu_k \rangle - \langle Qu_k, PQ\partial_k u_k \rangle \big) \ .$$

Provided $r$ is sufficiently large, we can replace $u_k$ by its asymptotic form and find

$$c(E,E') = \frac{\sin \rho(k-k')/2}{\rho(k-k')/2} e^{iR(k-k')} e^{-i(\delta(k)-\delta(k'))} \ .$$

We see that we can safely take the limit $\rho \to 0$.

As a test we evaluate our positive operator valued measure in a state

$$\psi(r) = \int \varphi(k) u_k(r) dk \ ,$$

where $\varphi$ has support in a small neighbourhood. The probability density of passage times is given by the integral

$$p(t) = \int \overline{\varphi(k)} \varphi(k') e^{i((k-k')R + \delta(k) - \delta(k') + t(E-E'))} \ dk \ dk' \ .$$

Since the support of $\varphi$ is small, we may linearize $\delta$ as a function of $E$ and find that $p(t + \partial_E \delta)$ is independent of the interaction. Hence (in this approximation) the distribution of the times when the particle passes through the shell in outward direction is delayed and we get the well known time delay

$$t_{\rm delay} = \partial_E \delta \ .$$

The same conclusion can also be obtained from the general definition of the time delay with the time operator $T_A$ as derived previously. A straightforward computation gives

$$d_A(E) = \frac{mR}{k} - \partial_E \delta \ ,$$

which yields again the logarithmic derivative of the on-shell scattering matrix. Note, however, that the first moment of a positive operator valued measure does in general not contain all informations on the measure (in contrast to projection valued measures).

As a byproduct we see that the connection $d_A$, in the case of the free time evolution, is a function of the energy (independent of the angular momentum). Therefore it commutes with the scattering matrix also for noncentral potentials, and we obtain the Eisenbud and Wigner formula in the general case.

## IV. CONCLUSIONS AND OUTLOOK

We presented a general construction of a time operator measuring the occurence time of an effect and used it for a new derivation of the well known formula for time delay in scattering theory. Our approach may be generalized in several directions. The first is the study of coincidence arrangements of detectors, either in quantum mechanics or in quantum field theory, leading to a distribution of multiple times resembling the general framework of the consistent histories approach. Another generalization is an analysis of times in a periodic or quasiperiodic situation related to bound states of the Hamiltonian. There the positive operator valued measure is concentrated on

a compact space on which time translations act, quite similar to Bellissard's action of translations on homogenous disordered systems [3]. One may also use a similar construction to characterize the localization of an event in *spacetime*. This may be compared with the somewhat different ansatz in [5,10]. We hope to report soon elsewhere on these interesting topics.

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[1] W. Amrein, J.M. Jauch and K. Sinha, Scattering Theory in Quantum Mechanics, Benjamin, Reading Mass. (1977).
[2] H. Araki and R. Haag, Commun. Math. Phys. **4**, 77 (1961).
[3] J. Bellissard, A. van Elst and H. Schulz-Baldes, Jour. Math. Phys. **35**, 5373 (1994).
[4] P. Bush, M. Grabowski and P.J. Lahti, Operational Quantum Physics, Springer-Verlag, Berlin (1995).
[5] R. Giannitrapani, Int. J. Theor. phys. **36**, 1575 (1997).
[6] Ph.A. Martin, Acta Phys. Austriaca, Suppl. XXIII, 157 (1981).
[7] M.A. Naimark, Izv. Acad. Nauk SSSR Ser. Mat. **4**, 277 (1940)
[8] J. Oppenheim, B. Reznik and W. G. Unruh, *Minimum uncertainty for transit-time*, quant-ph/9801034 (1998).
[9] W. Pauli, General Principles of Quantum Mechanics, Springer-Verlag, Berlin (1980).
[10] M. Toller, *On the quantum space-time coordinates of an event*, quant-ph/9702060 (1997).
[11] R. Werner, Ann. Inst. H. Poincaré **47**, 429 (1987).
[12] E. P. Wigner, Phys. Rev. **98**, 145 (1955).