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Testing perturbation theory on the $N_{\rm f} = 0$ static quark potential

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Abstract

The perturbative expansion of static force and potential is reanalyzed concerning its practical applicability. A well behaved perturbative prediction is given by the integration of the renormalization group equation for the coupling $\alpha_{\rm qq}(\mu = 1/r) = (C_{\rm F})^{-1} r^2 F(r)$. Since the Lambda-parameter of the $N_{\rm f} = 0$ theory is known from [1], the perturbative prediction contains no free parameter. It is confronted with recent non-perturbative results. For $\alpha \lesssim 0.3$ where the truncation error of the perturbative expression is naively estimated to be moderate, it is really quite accurate and large "non-perturbative terms" are excluded.

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1. The perturbative expansion in the coupling α_S is the most important theoretical tool for analyzing strong interaction effects in high energy scattering experiments. Since the coupling decreases with increasing momentum transfer, perturbation theory becomes accurate in the high energy regime. In practical applications, in particular in the determination of the running coupling itself, it is tempting to apply the perturbative series already where the coupling is not so small. Popular examples are the determinations of α_S from τ-lepton decays and the perturbative evolution of deep inelastic scattering structure functions starting at a renormalization point below 1 GeV. In order to establish the applicability of perturbation theory, it would be very desirable to study such processes systematically as a function of the energy, but either they involve a fixed energy (τ decays, hadronic Z-decays) or the precision is not sufficient over a larger energy range (deep inelastic scattering, e^+e^- total cross section, Adler function). Our main phenomenological test of perturbation theory therefore is the overall consistency of the determinations of α_S from different processes (see [2] and references therein).

Complementary information may be obtained from suitable observables computed as a function of Euclidean external momenta (or distance) using lattice QCD. Here an important limitation is implied by the necessarily finite lattice spacing a and the corresponding momentum cutoff $\mathrm{O}(1/a)$. For QCD with dynamical quarks (and in large volume), one currently reaches $a^{-1} \approx 2\mathrm{GeV}$. This limitation may be overcome by considering a finite size effect as the physical observable [3] which defines a renormalized coupling[1]. In the Schrödinger functional framework, the method has been completely developed and results for the running coupling are available [4,5,1,6], including in particular the Lambda parameter expressed in terms of the low energy scale $r_0 \approx 0.5\,\mathrm{fm}$ [7] in the theory without quarks [1] (pure Yang-Mills theory). These results refer to the continuum limit, reached by a controlled extrapolation from finite a. In this theory the scale dependence of the Schrödinger functional coupling, $\alpha_{\mathrm{SF}}(\mu)$, is in remarkable qualitative agreement with perturbation theory for $\alpha < 0.3$ and for $\alpha < 0.2$ the 3-loop expression describes $\alpha_{\mathrm{SF}}(\mu)$ within better than 2%. It is an interesting question, whether the coupling in this scheme is a special case or whether this is a more "general property of the theory".

If one restricts oneself to $N_{\mathrm{f}} = 0$, also observables in large volume but still relatively small distances may be computed employing very large lattices (64^4). We have done so for the potential between static quarks in the fundamental representation, reaching distances of $r \approx 0.05\mathrm{fm}$ with small discretization errors and

[1] In finite volume, with no other scale involved but the size of the space-time itself, one may of course keep the lattice spacing small compared to this scale.

in fact extrapolating to the continuum for $r \gtrsim 0.1$fm[8]. In perturbation theory, the potential has been computed to two loops [9,10,11,12,13], but at the same time the usefulness of perturbation theory even at distances as short as 0.1fm has been doubted [11,14]. We shall explain below that this is a question of a suitable renormalization scheme. When the most natural scheme (defined in terms of the force) is adopted, perturbation theory is well behaved at such distances and it is interesting to compare perturbation theory to the non-perturbative force. This is a rather stringent test of perturbation theory since besides the scale dependence of the coupling, its absolute value is predicted by perturbation theory (we remind the reader that the Lambda parameter is known).

A previous exploratory investigation, concentrated on the question, whether there are "large non-perturbative" terms in the potential at short distances [15] as they had been argued to exist [16]. It is not easy to give a definition of "large non-perturbative term". We assume here that roughly the following is meant by this statement.

(i) A certain quantity, here the potential $V(r)$, is considered in a region where its perturbative expansion looks well behaved, i.e. the n-loop contribution is a small correction and significantly smaller than the $(n-1)$-loop contribution (unless the latter is accidentally small itself).

(ii) The difference between the full non-perturbative observable and the truncated perturbative series is much larger than the last term in the series.

With such a definition, necessarily somewhat phenomenological in character, we shall demonstrate below that there are definitely no large non-perturbative terms in the potential. To the contrary, perturbation theory works remarkably well where the criterion (i) is satisfied.

In the following, we first investigate the perturbative expressions and find that a stable perturbative prediction satisfying (i) in an accessible region of r is given by the integration of the renormalization group equation for the coupling $\alpha_{\rm qq}(\mu)$. We then compare perturbation theory to our numerical results, considering also the direct relation between the Schrödinger functional coupling $\alpha_{\rm SF}$ and $\alpha_{\rm qq}(r)$. For completeness we also show the potential itself compared to perturbation theory.

2. In single scale problems, such as the static potential depending only on the separation r, the best perturbative prediction is expected to be the integration of the renormalization group equation. This is equivalent to defining a physical renormalized coupling, often denoted effective charge [17,18]. In particular the

coupling $\alpha_{\rm qq}(\mu)$ may be defined through

$$F(r) = \frac{{\rm d}V}{{\rm d}r} = C_{\rm F}\frac{\alpha_{\rm qq}(\mu)}{r^2}\,,\ \mu = 1/r. \tag{1}$$

(It will become clear below, why we here consider the force rather than the potential.) The running of the coupling defines the associated β-function,

$$\mu\frac{\rm d}{{\rm d}\mu}\bar{g} = \beta(\bar{g})\,,\ \bar{g} = (4\pi\alpha)^{1/2} \tag{2}$$

with a perturbative expansion

$$\beta(\bar{g}) \stackrel{\bar{g}\to 0}{\sim} -\bar{g}^3\{b_0 + b_1\bar{g}^2 + b_2\bar{g}^4 + \ldots\} \tag{3}$$

$$b_0 = \tfrac{1}{(4\pi)^2}\left(11 - \tfrac{2}{3}N_{\rm f}\right),\ b_1 = \tfrac{1}{(4\pi)^4}\left(102 - \tfrac{38}{3}N_{\rm f}\right). \tag{4}$$

The solution of eq. (2),

$$\Lambda_S = \mu(b_0\bar{g}^2)^{-b_1/(2b_0^2)}{\rm e}^{-1/(2b_0\bar{g}^2)}\exp\left\{-\int_0^{\bar{g}} dx\left[\tfrac{1}{\beta(x)} + \tfrac{1}{b_0x^3} - \tfrac{b_1}{b_0^2x}\right]\right\}, \tag{5}$$

relates the coupling $\bar{g} = \bar{g}(\mu)$ to the Lambda-parameter. This general expression turns into a perturbative one by inserting the expansion eq. (3) for the β-function. Truncating in eq. (3) after the term b_{n-1} and solving eq. (5) (numerically) for $\bar{g}$ at given μ (in units of Λ) defines the n-loop RG solution for the coupling. In contrast to the frequently used expansion of $F(r)$ (or $V(r)$) in terms of $\alpha_{\overline{\rm MS}}(\mu)$, one does not need to choose the scale μ. For the Schrödinger functional coupling, this perturbative prediction ($n = 3$) has been shown to be rather accurate for $\alpha < 0.3$ by comparison to non-perturbative results [5,1].

In order to obtain $\alpha_{\rm qq}$ from eq. (5) we need to insert the Lambda-parameter in this scheme. We start from

$$\Lambda^{(0)}_{\overline{\rm MS}}r_0 = 0.602(48) \tag{6}$$

referring to the case $N_{\rm f} = 0$ and extracted at sufficiently high scale μ where the perturbative error is negligible [1]. With the coefficient c_0 (known from [9,10]) in the expansion

$$\alpha_{\rm qq}(\mu) = \alpha_{\overline{\rm MS}}(\mu) + c_0\alpha_{\overline{\rm MS}}(\mu)^2 + c_1\alpha_{\overline{\rm MS}}(\mu)^3 + \ldots \tag{7}$$

we then relate $\Lambda_{\rm qq}$ to $\Lambda_{\overline{\rm MS}}$ via

$$\Lambda_{\rm qq} = \Lambda_{\overline{\rm MS}}\,{\rm e}^{c_0/(8\pi b_0)}\,. \tag{8}$$

For convenience the ratio of Lambda-parameters is listed in Table 1 together with that ratio for other schemes.

The coupling $\alpha_{\rm qq}$ from 2- and 3-loop RG is illustrated in Fig. 1, using the central value $\Lambda_{\overline{\rm MS}} r_0 = 0.602$ (the 8% overall uncertainty of this number corresponds to a common small horizontal shift of all curves in the figure). The perturbative expansion appears quite well behaved up to distances $r \sim 0.25$fm. At $r \sim 0.2$fm one would expect the 3-loop curve to have an accuracy of about 10%. Since the force is completely equivalent to $\alpha_{\rm qq}$ it is given with the same relative accuracy; the potential may be obtained by integration of the force.

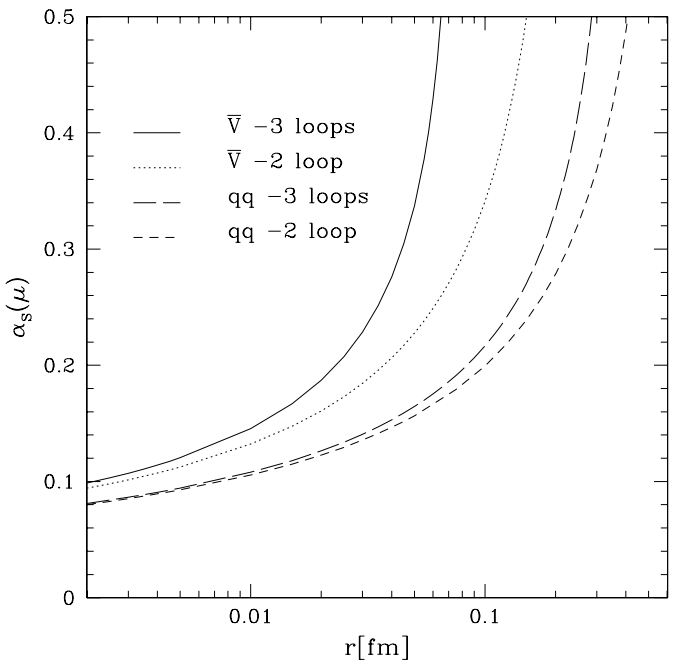

Figure 1: Running couplings obtained by integration of the RG with truncation of the β-functions at 2- and 3-loop and with $\Lambda^{(0)}_{\overline{\rm MS}} = 238$MeV [1].

Alternatively one may define couplings $\alpha_{\overline{V}}$ and $\alpha_{\rm V}$ through the potential

$$V(r) = -C_{\rm F}\frac{\alpha_{\overline{V}}(\mu)}{r}\,, \quad \mu = 1/r \tag{9}$$

and its Fourier transform

$$\tilde{V}(Q) = -4\pi C_{\rm F}\frac{\alpha_{\rm V}(Q)}{Q^2}\,. \tag{10}$$

We note, however, that the 3-loop coefficients b_2 are larger in these cases (see Table 1), in particular in the $\overline{V}$ scheme. As a consequence, the difference between

the 2-loop and the 3-loop running coupling in this scheme is only small at very short distances and this perturbative expansion appears to be applicable only up to $\alpha \sim 0.15$. This is also illustrated in Fig. 1.

	S =	qq	V	$\overline{V}$	SF
$\Lambda_S/\Lambda_{\overline{\rm MS}}$		$\exp(\gamma - 35/66)$	$\exp(31/66)$	$\exp(31/66+\gamma)$	0.48811(1)
$b_2^S \times (4\pi)^3$		1.6524	2.1287	4.3353	0.483(9)

Table 1: Ratio of Λ-parameters and 3-loop coefficient of the β-function for various schemes for $N_{\rm f} = 0$. These results follow from [9,10,19,20,11,12,13,21].

Note that the couplings in the two schemes which are defined in terms of the potential are restricted to perturbation theory. Non-perturbatively it is not clear how to subtract the self energy term in the potential and, in addition, performing the Fourier transformation of numerical data known in a finite range of r is possible only in a model dependent way. Also for this reason, $\alpha_{\rm qq}$ is the natural observable for the comparison between perturbation theory and non-perturbative QCD.

S'	S =	$\overline{\rm MS}$	qq	V	$\overline{V}$
qq		1.0653			
V		1.6095	0.5441		
$\overline{V}$		4.1303	3.0650	2.5208	
SF		$-0.271(10)$	$-1.336(10)$	$-1.880(10)$	$-4.401(10)$

Table 2: Coefficients $f_2^{S'S}(s_0)$ for $s_0 = \Lambda_{S'}/\Lambda_S$ and $N_{\rm f} = 0$.

3. When two different couplings are known non-perturbatively, it is further of interest to study how well perturbation theory predicts their direct relation. This means matching the two couplings at finite μ instead of through the Lambda-parameter, which corresponds to matching for $\mu \to \infty$. The perturbative relation

$$\alpha_{S'}(s\mu) = \alpha_S(\mu) + f_1^{S'S}(s)\alpha_S(\mu)^2 + f_2^{S'S}(s)\alpha_S(\mu)^3 + \ldots \qquad (11)$$

contains a freedom of relative scale, s. Indeed, the choice of s is in general very important for the quality of the perturbative prediction [11,22]. The only viable

criterion for fixing s appears to be to demand that the coefficients $f_i(s)$ are small ("fastest apparent convergence"). The choice $s = s_0 = \Lambda_{S'}/\Lambda_S$ yields $f_1(s_0) = 0$, and in addition the values of $|f_2(s_0)|$ are close to the minimum of $|f_2(s)|$. The coefficients $f_2(s_0)$ connecting selected schemes are listed in Table 2. One observes that the SF-scheme is very close to the $\overline{\rm MS}$-scheme, the qq scheme is not very far, but the other schemes have quite large values of $f_2(s_0)$ in their relation to the $\overline{\rm MS}$-scheme. In particular, the large coefficient between the $\overline{\rm MS}$ scheme and the $\overline{V}$ scheme means that the direct expansion of the coordinate space potential in terms of $\alpha_{\overline{\rm MS}}$ (or $\alpha_{\rm SF}$) is badly behaved, as it has been pointed out in [11,14,13].

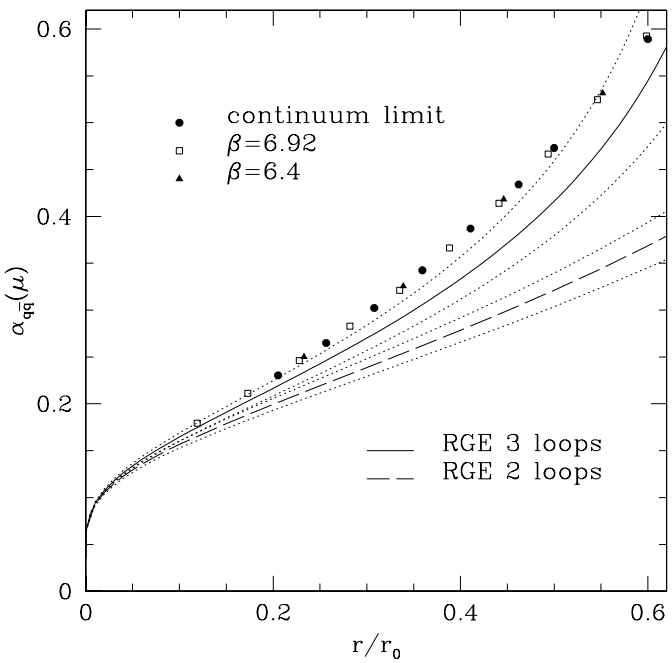

Figure 2: Running coupling in the qq scheme. Results for the continuum limit as well as additional points at finite β, corresponding to finite lattice spacing are shown. In the latter case the discretization errors were estimated to be smaller than the size of the symbols. The perturbative curves use $\Lambda_{\overline{\rm MS}} r_0$ from [1] with the dotted lines corresponding to the 1-σ uncertainties of this combination.

We emphasize the following point. Although the three different schemes qq, V, $\overline{V}$ differ only by kinematics (differentiation, the Fourier transformation) it makes a big difference for the applicability of perturbation theory which one is chosen to represent the potential. The analysis of the perturbative series themselves suggests that potential and force should be reconstructed from $\alpha_{\rm qq}$.

4. In Fig. 2 we compare the non-perturbative results of [8] to perturbation theory. The 3-loop RG expression with $\Lambda_{\overline{\rm MS}}$ at the upper end of the error bar of eq. (6) is in

very close agreement with the non-perturbative coupling. In fact the agreement extends up to values of $\alpha_{\rm qq}$ where perturbation theory is not to be trusted a priory. For $\alpha_{\rm qq}\lesssim 0.3$ our criterion (i) above is satisfied but there is no evidence for non-perturbative terms in this region.

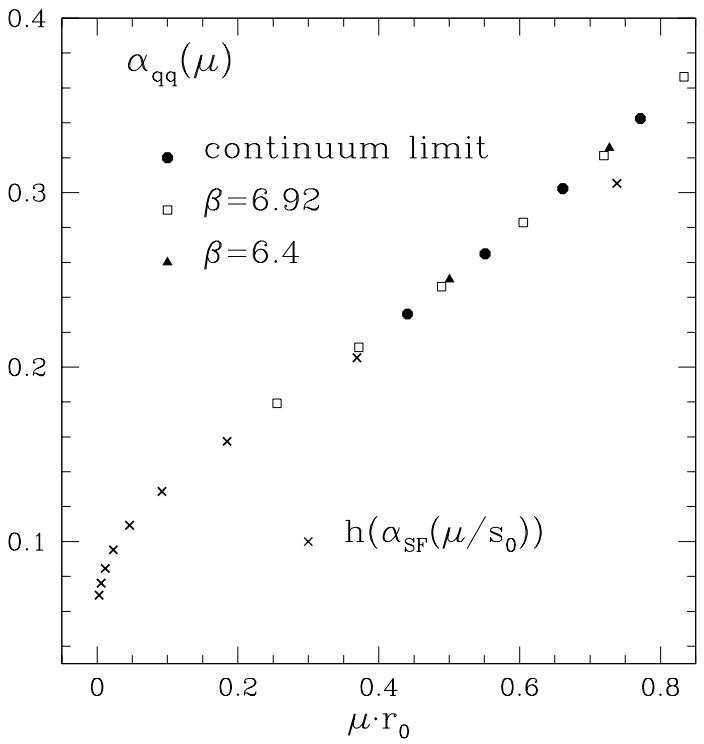

Figure 3: Test of eq. (13). The uncertainty in the combination μr_0 has been translated into an uncertainty for $h(\alpha_{\rm SF}(\mu/s_0))$ and $\alpha_{\rm qq}(\mu)$. The non-perturbative values for $\alpha_{\rm SF}(\mu)$ are constructed from the data of [1]. Errors are smaller than the sizes of the symbols.

The same conclusion is reached when we consider the relation between $\alpha_{\rm SF}(\mu)$ and $\alpha_{\rm qq}(\mu)$ at finite μ: We define the 3-loop expression

$$h(\alpha_{\rm SF}(\mu)) \;=\; \alpha_{\rm SF}(\mu) + 1.336\,[\alpha_{\rm SF}(\mu)]^3 \qquad (12)$$

such that

$$\alpha_{\rm qq}(\mu) \;=\; h(\alpha_{\rm SF}(\mu/s_0)) + {\rm O}([\alpha_{\rm SF}(\mu/s_0)]^4) \qquad (13)$$

as explained above. Then the non-perturbative values of $\alpha_{\rm qq}(\mu)$ and of $h(\alpha_{\rm SF}(\mu/s_0))$ are compared in Fig. 3. If the higher order terms in eq. (13) are negligible, the two different quantities should agree. At $\alpha_{\rm qq}(\mu)\approx 0.3$ a difference is visible but this is only about $3\times\alpha^4$, not far from the expected size of the next order term in the

series.[2] At $\alpha_{\rm qq}(\mu) \approx 0.2$, the difference $\alpha_{\rm qq}(\mu) - h(\alpha_{\rm SF}(\mu/s_0))$ is not significant at all. We conclude that also in eq. (13) a large non-perturbative term at short distances is excluded.

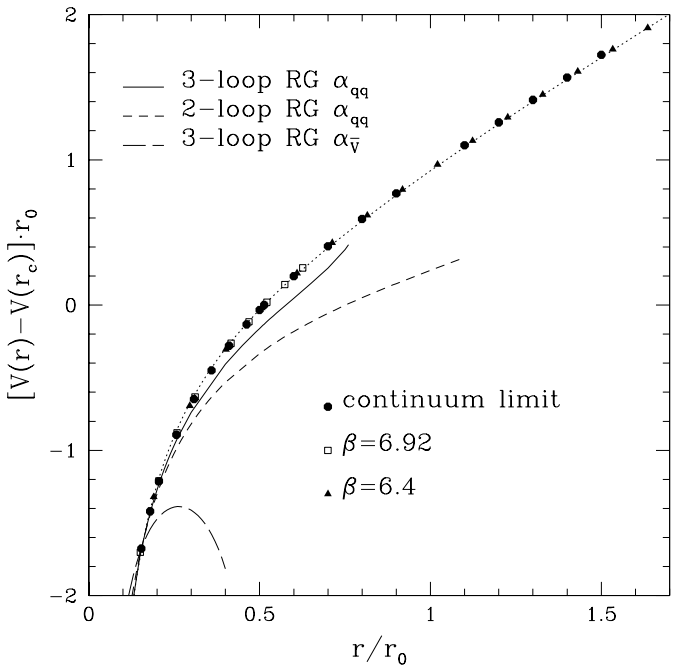

Figure 4: The potential compared to different perturbative expressions. Here, $r_{\rm c} = 0.54 r_0$ [8].

Finally we show in Fig. 4 the static potential itself [8] compared to different perturbative approximations. Full line and short dashes are given by

$$V(r) = V(0.15 r_0) + \int_{0.15 r_0}^{r} {\rm d}y F(y)\,, \; F(r) = C_{\rm F} r^2 \alpha_{\rm qq}(1/r) \tag{14}$$

with the 3-loop and 2-loop RG-solution for $\alpha_{\rm qq}$. Since we know that the 3-loop RG-solution for $\alpha_{\rm qq}$ is accurate, this also hold for $V(r)$ computed through eq. (14). Again, the full line moves very close to the data points ($r < 0.5 r_0$), when Λr_0 at the upper end of the error bar of eq. (6) is inserted. The long dashes represent eq. (9) with the 3-loop RG-solution for $\alpha_{\overline{V}}$. As it was to be expected due to the missing stability of this perturbative expression, it fails in describing the potential.

[2] Note that the next order correction is formally enhanced by a logarithm of α, which originates from a resummation of IR divergent terms. It reads $(A\log(\alpha) + B)\alpha^4$ [23,24,14]. While $A = 9/(4\pi)$ has recently been calculated [25,26], B is not known. The $A\alpha^4\log(\alpha)$ term by itself constitutes a small correction in the figure, which would slightly enlarge the difference between $h(\alpha_{\rm SF}(\mu/s_0))$ and $\alpha_{\rm qq}(\mu)$.

A similarly bad perturbative expression (not shown here) is the direct expansion of the potential in terms of $\alpha_{\overline{\rm MS}}$.

5. In summary, care has to be taken which perturbative prediction (scheme) is adopted to describe the potential. However, perturbation theory does its best in the following sense. As usual in an asymptotic expansion, one should first investigate the apparent "convergence" by comparing subsequent orders and checking that they decrease significantly. If this is not the case, one is obviously outside the domain of applicability of perturbation theory or has chosen a bad truncation (scheme). According to this criterion the β-function in the qq-scheme may be trusted up to $\alpha_{\rm qq} \approx 0.3$. Other truncations of perturbation theory for the potential that we investigated are applicable for much smaller values of the coupling only. Therefore perturbation theory suggests that the qq scheme should be used in order to obtain a reliable perturbative expression.[3]

Our comparison with non-perturbative results, obtained in the continuum limit of lattice simulations ($N_{\rm f} = 0$), does confirm that such a perturbative analysis is a good guideline – at least in the case at hand. Of course one should not expect miracles when one goes up to $\alpha \approx 0.3$. At such values of the coupling we only confirm that the 3-loop perturbative prediction is good to within about 10% and indeed in Fig. 3 one sees explicitly that the truncated perturbative series has *errors* of this order. Similar results have been found for the $N_{\rm f} = 2$ coupling in the SF-scheme (see Fig. 5 in [6]).

Which lessons can we learn for QCD with quarks? Compared to $N_{\rm f} = 0$, the relevant perturbative coefficients, b_2 and $f_2(s_0)$, which are listed in the appendix, are roughly a factor two smaller in magnitude for $N_{\rm f} = 3$. This suggests that with quarks the perturbative prediction for the potential computed through $\alpha_{\rm qq}$ is also applicable up to $\alpha \approx 0.3$ and furthermore in full QCD the issue of the appropriate scheme is somewhat less important. A direct lattice QCD check of these expectations is unfortunately not possible at present and here we had to boldly generalize from the $N_{\rm f} = 0$ case. In addition, these remarks apply to the massless theory (we have not investigated mass effects). Current phenomenological research concentrates on the application of a velocity dependent potential beyond the static limit for phenomenological applications to top-quark physics[27]. On the one hand, in this application the potential is needed for quite short distances, where perturbation theory is intrinsically more precise [28], on the other hand, with the velocity entering as a new scale, this represents a more difficult multi-

[3]Of course, other similarly well behaved truncations of perturbation theory might be found. The important point is that a scheme with a large 3-loop coefficient such as $\overline{V}$ is of no use in the region $\alpha > 0.15$.

S	e_1	e_2	e_3
$\overline{\rm MS}$	$-5033/18$	$325/54$	0
V	$\frac{3}{2}\pi^4 - 24\pi^2 - \frac{2239}{6} - \frac{704}{3}\zeta(3)$	$\frac{377}{54} + \frac{104}{9}\zeta(3)$	0
$\overline{\rm V}$	$-\frac{2239}{6} - \frac{704}{3}\zeta(3) + \frac{3}{2}\pi^4 - \frac{314}{3}\pi^2$	$\frac{377}{54} + \frac{104}{9}\zeta(3) + \frac{44}{9}\pi^2$	$-\frac{8}{81}\pi^2$
$\rm q\bar{q}$	$\frac{3}{2}\pi^4 - \frac{314}{3}\pi^2 + \frac{3569}{6} - \frac{704}{3}\zeta(3)$	$-\frac{2791}{54} + \frac{44}{9}\pi^2 + \frac{104}{9}\zeta(3)$	$\frac{32}{27} - \frac{8}{81}\pi^2$
SF	$-0.275(5) \times (4\pi)^3$	$0.0361(4) \times (4\pi)^3$	$-0.00175(1) \times (4\pi)^3$

Table 3: Coefficients e_i of eq. (15).

scale problem. Indeed the renormalization group has already been applied to deal with this complication [29]. Nevertheless, the lessons learnt in our investigation may be useful in this context as well; the type of renormalization group improvement which we found to increase the reliability of perturbation theory (see Fig. 4) has not been applied in [27] so far.

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Appendix. In order to ease the comparison of different schemes, we here list some perturbative coefficients for general $N_{\rm f}$ whose numerical values for $N_{\rm f} = 0$ were quoted in the tables above. Some of them could be taken directly from the literature [9,10,19,20,11,12,13,30], others such as $b_2^{\rm qq}$ had to be computed by straight forward algebra.

The three-loop coefficient of the β-function can be expressed as

$$b_2^S = b_2^S|_{N_{\rm f}=0} + \frac{1}{(4\pi)^6}\left(e_1 N_{\rm f} + e_2 N_{\rm f}^2 + e_3 N_{\rm f}^3\right) \tag{15}$$

with e_i listed in Table 3.

The one-loop coefficient in eq. (11) assumes the general form

$$f_1^{S'S}(s) = (a_1 + a_2 N_{\rm f})/4\pi - 8\pi b_0 \log(s)), \tag{16}$$

and vanishes for $s = s_0 = \exp\left((a_1 + a_2 N_{\rm f})/(32\pi^2 b_0)\right)$. a_1 and a_2 are listed in Table 4, with $\overline{\rm MS}$ as reference scheme. The other coefficients can be evaluated by $f_1^{SS'}(s) = f_1^{SS''}(s) + f_1^{S''S'}(s)$ and

$$f_2^{S'S}(s) = \frac{(4\pi)^2}{b_0}\left\{b_2^{S'} - b_2^S + b_1\frac{f_1^{S'S}(s)}{4\pi} - b_0\left[\frac{f_1^{S'S}(s)}{4\pi}^2\right]\right\}, \tag{17}$$

S'	a_1	a_2
V	$31/3$	$-10/9$
$\overline{\mathrm{V}}$	$31/3 + 22\gamma$	$-10/9 - 4\gamma/3$
$\mathrm{q\bar{q}}$	$-35/3 + 22\gamma$	$2/9 - 4\gamma/3$
SF	$-1.25562 \times (4\pi)$	$-0.03986 \times (4\pi)$

Table 4: Coefficients a_i of eq. (16) for $S = \overline{\mathrm{MS}}$

which reduces to

$$f_2^{S'S}(s_0) = \tfrac{(4\pi)^2}{b_0}\left[b_2^{S'} - b_2^{S}\right] \tag{18}$$

when we set $s = s_0$.

References

[1] ALPHA, S. Capitani, M. Lüscher, R. Sommer and H. Wittig, Nucl. Phys. B544 (1999) 669, hep-lat/9810063,

[2] S. Bethke, J. Phys. G G26 (2000) R27, hep-ex/0004021,

[3] M. Lüscher, P. Weisz and U. Wolff, Nucl. Phys. B359 (1991) 221.

[4] M. Lüscher, R. Sommer, U. Wolff and P. Weisz, Nucl. Phys. B389 (1993) 247, hep-lat/9207010.

[5] M. Lüscher, R. Sommer, P. Weisz and U. Wolff, Nucl. Phys. B413 (1994) 481, hep-lat/9309005.

[6] ALPHA, A. Bode et al., (2001), hep-lat/0105003,

[7] R. Sommer, Nucl. Phys. B411 (1994) 839, hep-lat/9310022.

[8] S. Necco and R. Sommer, (2001), hep-lat/0108008,

[9] W. Fischler, Nucl. Phys. B129 (1977) 157,

[10] A. Billoire, Phys. Lett. B92 (1980) 343,

[11] M. Peter, Nucl. Phys. B501 (1997) 471, hep-ph/9702245,

[12] Y. Schröder, Phys. Lett. B447 (1999) 321, hep-ph/9812205,

[13] M. Melles, Phys. Rev. D62 (2000) 074019, hep-ph/0001295,

[14] Y. Schröder, DESY-THESIS-1999-021.

[15] G.S. Bali, Phys. Lett. B460 (1999) 170, hep-ph/9905387,

[16] R. Akhoury and V.I. Zakharov, Phys. Lett. B438 (1998) 165, hep-ph/9710487,

[17] G. Grunberg, Phys. Lett. B95 (1980) 70,

[18] G. Grunberg, Phys. Rev. D29 (1984) 2315,

[19] O.V. Tarasov, A.A. Vladimirov and A.Y. Zharkov, Phys. Lett. 93B (1980) 429.

[20] S.A. Larin and J.A.M. Vermaseren, Phys. Lett. B303 (1993) 334, hep-ph/9302208.

[21] ALPHA, A. Bode, U. Wolff and P. Weisz, Nucl. Phys. B540 (1999) 491, hep-lat/9809175,

[22] ALPHA, G. de Divitiis et al., Nucl. Phys. B437 (1995) 447, hep-lat/9411017,

[23] T. Appelquist, M. Dine and I.J. Muzinich, Phys. Rev. D17 (1978) 2074,

[24] T. Appelquist, M. Dine and I.J. Muzinich, Phys. Lett. B69 (1977) 231,

[25] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Phys. Rev. D60 (1999) 091502, hep-ph/9903355,

[26] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Nucl. Phys. B566 (2000) 275, hep-ph/9907240,

[27] A.H. Hoang et al., Eur. Phys. J. direct C3 (2000) 1, hep-ph/0001286,

[28] J.H. Kühn, Acta Phys. Polon. B12 (1981) 347,

[29] A.H. Hoang, A.V. Manohar and I.W. Stewart, Phys. Rev. D64 (2001) 014033, hep-ph/0102257,

[30] ALPHA, A. Bode, P. Weisz and U. Wolff, Nucl. Phys. B576 (2000) 517, Erratum-ibid.B600:453,2001, Erratum-ibid.B608:481,2001, hep-lat/9911018,