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# Universal Higher Order QED Corrections to Polarized Lepton Scattering

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**Abstract**

We calculate the universal radiative QED corrections to polarized lepton scattering for general scattering cross sections in analytic form. The flavor non–singlet and singlet radiation functions are calculated to $O((\alpha \mathrm{L})^5)$. The resummation of the non–singlet and singlet contributions to the QED–anomalous dimensions $\propto (\alpha \ln^2(x))^k$ is performed to all orders. Numerical results are presented for the individual radiation functions. Applications to polarized deeply inelastic lepton–nucleon scattering are given.

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# 1 Introduction

QED corrections to integral and differential cross sections of light charged lepton–lepton scattering or deeply inelastic lepton–nucleon scattering can be quite large in some kinematic regions [1–3]. In particular Bremsstrahlung contributions lead to significant shifts in the sub–system kinematics and cause large logarithmic contributions in differential distributions. The radiative corrections can be grouped into universal pieces, which are process–independent, and the remainder process dependent part. The former corrections are called universal corrections as they are associated with the radiating particle irrespective of the process. Two classes of universal corrections are known : *i)* the leading–logarithmic $O\left[(\alpha L)^k\right]$ corrections, where $L = \ln(s/m^2)$. Here $s$ is a typical energy scale which describes a time– or a space–like virtuality and $m$ denotes the light fermion mass. This type of corrections can be obtained by solving the renormalization group equations for mass–factorization [4] in lowest order completely; *ii)* the QED–exponentiation $\left[\alpha \ln^2(x)\right]^k$–terms in the all–order anomalous dimensions due to infrared evolution equations for initial–state radiation in lowest order [5,6] [2].

Unlike the case in QCD, for QED radiation the radiation sources are no extended distributions, $f_i(x,Q^2)$, which fall rapidly as $x \to 1$. The sources are given by a $\delta(1-x)$–distribution instead. A numerical solution of the evolution equations via a Mellin transform with a complex argument $N$ and a subsequent numerical inverse Mellin transform is thus not possible since the latter requires substantial damping as $\mathsf{Re}(N) \to -\infty$, which is not the case in general here. To determine the solution of the respective evolution equations in QED one rather evaluates the required Mellin transforms of the evolution kernels analytically in perturbative form up to an order in $(\alpha L)^k$ at which sufficient numerical stability is achieved. In the past the analytic resummation of the soft– and virtual leading order contributions was performed [9]. In the flavor non–singlet case analytic results were obtained to 5th order in $\alpha L$ [10,11] [3]. We compactify this description using the small–$x$ resummations of terms $\propto \alpha \ln(x)$, as carried out in [13] to 3rd order, to 5th order. However, these terms do not yield the complete description yet. The flavor singlet contributions have to be calculated to the same order, which is one of the objectives of the present paper.

The second class of universal corrections concerns the resummation of contributions of the type $(\alpha \ln^2(x))^k$ in the cross section. These terms are resummed by infrared evolution equations. The flavor non–singlet contributions were studied earlier in [14,15]. As the comparison of the first two terms of the resummation with QED corrections [16] shows, the resummation makes a correct prediction for the terms which occur in initial state radiation (ISR), whereas it remains an open problem, whether this is the case for final state radiation (FSR) [4]. For this reason we apply the resummation only for initial state radiation. In extension of earlier work in which the non–singlet resummation was dealt with [15] the resummation is carried out also for the singlet contributions.

The paper is organized as follows. In section 2 we revisit the non–singlet solution to $O\left[(\alpha L)^5\right]$ and compactify previous results w.r.t. exponentiations for small and large values of $x$. Section 3 contains the general formalism to derive the singlet solution both for the leading order $O\left[(\alpha L)^k\right]$ terms and those due to the resummation of contributions $\propto O\left[(\alpha \ln^2(x))^k\right]$ consistently. The iterated singlet evolution kernels are presented to $O\left[(\alpha L)^5\right]$ in section 4. In section 5 we derive

[2] Numerical studies in the case of QCD were performed in [7]. For a review see [8].

[3] Earlier results up to 3rd order were given in [12].

[4] Note that these terms do potentially violate the Gribov–Lipatov relation, see e.g. [17,18].

the coefficients for the resummation of the $O\left[(\alpha \ln^2(x))^k\right]$ terms for singlet evolution. In section 6 numerical results are presented and section 7 contains the conclusions. Technical aspects of the calculation being of interest also for other higher order QED and QCD calculations are summarized in the Appendices where convolution tables, which were used in the present calculation, are given in Appendix A. As well we summarize a series of higher order relations between Mellin transforms and harmonic sums, or polynomials of them, which were used to calculate the more complicated convolutions in Appendix B and C.

# 2 The Non–Singlet Solution

The universal radiative corrections discussed in the present paper can be derived using the renormalization group equations for mass factorization. We consider a one–flavor system out of electrons (positrons) and photons. The flavor non–singlet and singlet distributions are defined by the following QED–distributions

$$D^e_{\rm NS}(a,x) = D^{e^-}(a,x) - D^{e^+}(a,x) \tag{1}$$
$$D^e_{\Sigma}(a,x) = D^{e^-}(a,x) + D^{e^+}(a,x) \; . \tag{2}$$

Here $x$ denotes the momentum fraction of the collinear daughter–particle radiated off the source, $0 \leq x \leq 1$, and $a \equiv a(Q^2) = e^2/(4\pi)^2$ is the running coupling constant with $e$ the electric charge. For this system only one non–singlet 'minus' distributions occurs.

The non–singlet contribution to the polarized QED evolution kernel is the same as in the unpolarized case since the splitting functions

$$P^{ff}_{\rm NS}(x) = 2\left(\frac{1+x^2}{1-x}\right)_+ \tag{3}$$

are identical. The +-operation is defined by

$$\int_0^1 dx \left[F(x)\right]_+ \phi(x) = \int_0^1 dx F(x) \left[\phi(x) - \phi(1)\right] \tag{4}$$

for the functions $\phi(x)$, $F(x)$ with $x \; \epsilon \; [0,1]$.

The leading order non–singlet evolution equation reads

$$\frac{\partial}{\partial \ln(Q^2)} D^e_{\rm NS}(a,x) = a(Q^2) P^{ff}_{\rm NS}(x) \otimes D^e_{\rm NS}(a,x) \; . \tag{5}$$

Here $\otimes$ denotes the Mellin–convolution

$$A(x) \otimes B(x) = \int_0^1 dx_1 \int_0^1 dx_2 \; \delta(x - x_1 x_2) A(x_1) B(x_2) \; . \tag{6}$$

One may solve (5) using the Mellin–transform

$$\boldsymbol{M}[F(x)](N) = \int_0^1 dx x^{N-1} F(x) \; . \tag{7}$$

The Mellin–transform of the splitting function $P^{ff}_{\rm NS}(x)$ is the anomalous dimension $\gamma^{ff}_{\rm NS}(N)$.

The QED running coupling constant is given by

$$\frac{da}{d\ln(Q^2)} = -\sum_{k=0}^{\infty} \beta_k a^k \tag{8}$$

with the first coefficients of the $\beta$–function $\beta_0 = -4/3, \beta_1 = -4$. The solution of Eq. (8) in leading order reads

$$a(Q^2) = \frac{a_0}{1 - \frac{4}{3} a_0 L}, \tag{9}$$

with $a_0 \equiv a(m_e^2)$ and $L = \ln(Q^2/m_e^2)$. For $a_0 L \ll 1$ all later results obtained in $a$ may be expanded in the parameter $a_0 L$ to a finite order.

In Mellin–space the solution of the non–singlet evolution equation reads

$$D_{\rm NS}^{ff}(N, Q^2) = \left[1 + \beta_0 a_0 \ln\left(\frac{Q^2}{m_e^2}\right)\right]^{\gamma_{\rm NS}^{ff}(N)/\beta_0} . \tag{10}$$

Here we used the boundary condition $D_{\rm NS}^{ff}(N, m_e^2) = 1$ which corresponds to

$$D_{\rm NS}^{ff}(x, m_e^2) = \delta(1-x) . \tag{11}$$

For Eq. (10) one often defines $\hat{\beta} \equiv (2\alpha_0/\pi)\ln(Q^2/m_e^2)$.

Unlike the case of QCD evolution of extended parton densities [19] or structure functions [18], which fall rapidly as $x \to 1$, the input distributions in QED are of the form (11) and are distribution–valued at $x = 1$. Therefore, the Mellin–inversion of (10) cannot be performed by a numerical contour integral but has to be worked out calculating all the convolutions up to a finite order in $x$–space in to $(\alpha L)^k$ at which sufficient convergence of the series is being obtained. In the present paper the iteration is performed up to $k = 5$. The corresponding convolutions are evaluated in explicit form in terms of Nielsen integrals in Appendix A. The calculation does partly require the use of multiple harmonic sums [20–24], which can be expressed as Mellin transforms of Nielsen integrals. In a previous paper [21] a series of these Mellin transforms has been calculated already. In Appendix B additional representations, which were derived and used in the present calculation, are given.

We compared our results to those given previously in [10, 11]. We agree with [10] but find a deviation from [11] in the $O((\alpha L)^5)$ term. The result of [11] is very lengthy and can be compactified by exponentiating the soft– and virtual contributions to all orders in $(\alpha L)^k$, which was done before in Ref. [10]. Moreover, also the latter representation can be shortened in exponentiating also the contributions $\propto (\hat{\beta}\ln(x))^l$, well–known from QCD [25], which has been carried out for the first three orders in [13], but is not contained in [10, 11].

The non-singlet splitting kernel (no running coupling) can be represented exponentiating the soft and small $x$ contributions to all orders and treating the other contributions to finite order as

$$D_{\rm NS}(x, \hat{\beta}) = \left[\frac{\exp[(\hat{\beta}/2)(3/4 - \gamma_E)]}{\Gamma(1+\hat{\beta}/2)} \frac{\hat{\beta}}{2} (1-x)^{\hat{\beta}/2-1} \frac{{\rm I}_1\left((-\hat{\beta}\ln(x))^{1/2}\right)}{[-\hat{\beta}\ln(x)]^{1/2}} \sum_{n=0}^{\infty} \left(\frac{\hat{\beta}}{2}\right)^n \Psi_n(x)\right]_+ \tag{12}$$

with ${\rm I}_1(z)$ the associated Bessel function, $\gamma_E$ the Euler–Mascheroni constant, and

$$\Psi_0(x) \;=\; 1 + x^2 \tag{13}$$

$$\Psi_1(x) = -\frac{1}{2}\left[(1-x)^2 + x^2\ln(x)\right] \tag{14}$$

$$\Psi_2(x) = \frac{1}{4}(1-x)\left[1-x-x\ln(x)+(1+x)\mathrm{Li}_2(1-x)\right] \tag{15}$$

$$\begin{aligned}\Psi_3(x) = &-\frac{1}{48}\Bigg\{6(1-x^2)[2\mathrm{Li}_3(1-x)+\mathrm{Li}_2(1-x)]+5(x-1)^2 \\ &+(1+7x^2)[\ln(x)\mathrm{Li}_2(1-x)+2\mathrm{S}_{1,2}(1-x)]-\left(\frac{1}{2}+6x-\frac{13}{2}x^2\right)\ln(x)+\frac{1}{12}x^2\ln^3(x)\Bigg\}\end{aligned} \tag{16}$$

$$\begin{aligned}\Psi_4(x) = &\frac{1}{96}\Bigg\{(1-x^2)\Bigg[24\mathrm{Li}_4(1-x)+12\mathrm{Li}_3(1-x)-\frac{5}{2}\mathrm{S}_{1,3}(1-x) \\ &\qquad -12\mathrm{S}_{2,2}(1-x)-\frac{3}{2}\ln(x)\mathrm{S}_{1,2}(1-x)-\frac{1}{4}\ln^2(x)\mathrm{Li}_2(1-x)+7\mathrm{Li}_2(1-x)\Bigg] \\ &+4(1+x^2)\mathrm{Li}_2^2(1-x)+(1-8x+7x^2)\Bigg[\ln(x)\mathrm{Li}_2(1-x)+2\mathrm{S}_{1,2}(1-x)\Bigg] \\ &+2(1+7x^2)\ln(x)\mathrm{Li}_3(1-x)-\left(\frac{3}{4}+5x-\frac{23}{4}x^2\right)\ln(x) \\ &-\frac{1}{12}x(1-x)\ln^3(x)-\frac{1}{48}x^2\ln^4(x)+(1-x)^2\left[\frac{7}{2}+\frac{1}{8}\ln^2(x)\right]\Bigg\}\,.\end{aligned} \tag{17}$$

Except the 0th term all the functions $\Psi_k(x)$ behave at least as $\propto (1-x)$ for $x \to 1$. Therefore in these orders the contribution $\propto 1/(1-x)$ in the function $D_{\rm NS}(x,\hat{\beta})$ is canceled. As mentioned before the leading small $x$ terms $\propto (\hat{\beta}\ln(x))^k$ are resummed into the function $I_1\left[(-\hat{\beta}\ln(x))^{1/2}\right]$. Indeed many more logarithmic contributions are absorbed in this way from the expanded expression. To 3rd order in $\hat{\beta}$ the functions $\Psi_k(x)$ are finite as $x \to 0$. In the higher orders some terms of the order $\hat{\beta}^k \ln^{k-l}(x), l > 0$ remain, but contain a small weight factor only.

# 3 The Solution of Singlet Evolution Equations

In this paragraph we derive the explicit solution of the singlet evolution equation in terms of the running coupling constant $a = a(Q^2)$, with $a = \alpha/(4\pi)$ and account both for the leading order terms as well the leading terms at small $x$ $\propto a^k \ln^{2(k-1)}(x)$. Since the latter terms stem from the higher order evolution kernels, which do not commute with the (resummed) leading order solution in general, the perturbative solution has to be constructed accordingly. A framework of this kind was derived for Mellin $N$-space in Ref. [19] and for the lowest three orders in [26]. Here we will work in $x$–space and have therefore to extend a prescription given to second order in [27] to all orders.

We substitute the scale dependence of the (singlet) evolution equation[5] in terms of $a$, Eq. (8). One obtains

$$\begin{aligned}\frac{\partial \boldsymbol{D}_S(a,x)}{\partial a} &= -\frac{1}{a}\frac{\boldsymbol{P}_0(x)+a\,\boldsymbol{P}_1(x)+a^2\,\boldsymbol{P}_2(x)+\ldots}{\beta_0+a\beta_1+a^2\beta_2\ldots}\otimes \boldsymbol{D}_S(a,x) \\ &= -\frac{1}{\beta_0 a}\left[\boldsymbol{P}_0(x)+a\left(\boldsymbol{P}_1(x)-\frac{\beta_1}{\beta_0}\boldsymbol{P}_0(x)\right)+O(a^2)\right]\otimes \boldsymbol{D}_S(a,x)\end{aligned}$$

[5]The corresponding structures for the non–singlet evolution equations are those of entry $(1,1)$ and are not given separately. We will use this structures also resumming the small $x$ terms $\propto \ln^2(x)$ later on.

$$= -\frac{1}{a}\left[\boldsymbol{R}_0(x) + \sum_{k=1}^{\infty} a^k\, \boldsymbol{R}_k(x)\right] \otimes \boldsymbol{D}_S(a,x)\ . \tag{18}$$

The functions $\boldsymbol{R}_k(x)$ are given by

$$\boldsymbol{R}_0(x) = \frac{1}{\beta_0}\,\boldsymbol{P}_0(x) \tag{19}$$

$$\boldsymbol{R}_k(x) = \frac{1}{\beta_0}\,\boldsymbol{P}_k(x) - \sum_{i=1}^{k}\frac{\beta_i}{\beta_0}\,\boldsymbol{R}_{k-i}(x)\ . \tag{20}$$

The leading order solution is

$$\boldsymbol{D}_{S,0}(a,x) = [\exp(-\boldsymbol{R}_0(x)\ln(a/a_0))_\otimes] \otimes \boldsymbol{D}_S(a_0,x) \equiv \boldsymbol{E}_0(a,a_0,x) \otimes \boldsymbol{D}_S(a_0,x)\ , \tag{21}$$

where $\boldsymbol{E}_0$ is the leading order singlet evolution operator. We use the short–hand notation

$$[f(g(x))_\otimes] = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\otimes_{l=1}^{k} g(x)\ , \tag{22}$$

with $\otimes_{l=1}^{k}$ the $k$-fold convolution. The distribution at the scale $a = a_0$ is

$$\boldsymbol{D}_S(a_0,x) = \mathbf{1}\ \delta(1-x)\ . \tag{23}$$

Higher than leading order contributions are found perturbativly by

$$\boldsymbol{D}_S(a,x) = \boldsymbol{U}(a,x) \otimes \boldsymbol{E}_0(a,a_0,x) \otimes \boldsymbol{U}(a_0,x)^{-1}\ , \tag{24}$$

with

$$\boldsymbol{U}(a,x) = \mathbf{1}\ \delta(1-x) + \sum_{k=1}^{\infty} a^k\, \boldsymbol{U}_k(x)\ . \tag{25}$$

The matrices $\boldsymbol{U}_k(x)$ are obtained recursively from the functions $\boldsymbol{R}_k(x)$,

$$\widehat{\boldsymbol{R}}_1(x) = \boldsymbol{R}_1(x) \tag{26}$$

$$\widehat{\boldsymbol{R}}_k(x) = \boldsymbol{R}_k(x) + \sum_{l=1}^{k-1} \boldsymbol{R}_{k-l} \otimes \boldsymbol{U}_k(x)\ . \tag{27}$$

In general the matrices $\boldsymbol{U}_k$ do not commute. The higher than leading order contributions dealt with in the present paper concern contributions to the splitting functions $\propto a^k \ln^{2k}(x)$. If one would work in $x$–space many convolutions of these functions with those representing the fixed order solutions $\boldsymbol{E}_0$ would have to be calculated. The series in $a^k \ln^{2k}(x)$ may need more terms than in the case of the fixed order solution in $O[(\alpha L)^k]$. The specific structure of the terms $a^k \ln^{2k}(x)$ allows to work in $N$–space since the Mellin transform

$$\boldsymbol{M}[\ln^{2k}(x)](N) = \frac{(2k)!}{N^{2k+1}} \tag{28}$$

is sufficiently damped for $k \geq 1$ in the limit $\mathsf{Re}(N) \to -\infty$. Due to this we may calculate the $\boldsymbol{U}$–matrix in $N$–space and use the solution

$$\begin{aligned}\boldsymbol{D}_S(a,x) = {}& \boldsymbol{E}_0(a,a_0,x) + \widehat{\boldsymbol{U}}(a,x) \otimes \boldsymbol{E}_0(a,a_0,x) + \boldsymbol{E}_0(a,a_0,x) \otimes \widetilde{\boldsymbol{U}}(a_0,x) \\ & + \widehat{\boldsymbol{U}}(a,x) \otimes \boldsymbol{E}_0(a,a_0,x) \otimes \widetilde{\boldsymbol{U}}(a_0,x)\ ,\end{aligned} \tag{29}$$

where

$$\widehat{\boldsymbol{U}}(a,x) = \boldsymbol{U}(a,x) - \mathbf{1}\delta(1-x) \tag{30}$$

$$\widetilde{\boldsymbol{U}}(a,x) = \boldsymbol{U}(a,x)^{-1} - \mathbf{1}\delta(1-x)\ . \tag{31}$$

# 4 The Leading Order Solution to $O[(\alpha L)^5]$

In the present paper we limit the explicit representations to order $n = 5$ in $(a_0 L)$ [6]. It is therefore convenient to expand (21) as

$$\begin{aligned}
\boldsymbol{E}_0(a,x) &= \mathbf{1}\,\delta(1-x) + \boldsymbol{P}_0(x) a_0 L + \left(\frac{1}{2}\,\boldsymbol{P}_0^{(1)}(x) + \frac{2}{3}\,\boldsymbol{P}_0(x)\right)(a_0 L)^2 \\
&+ \left(\frac{1}{6}\,\boldsymbol{P}_0^{(2)}(x) + \frac{2}{3}\,\boldsymbol{P}_0^{(1)}(x) + \frac{16}{27}\,\boldsymbol{P}_0(x)\right)(a_0 L)^3 \\
&+ \left(\frac{1}{24}\,\boldsymbol{P}_0^{(3)}(x) + \frac{1}{3}\,\boldsymbol{P}_0^{(2)}(x) + \frac{22}{27}\,\boldsymbol{P}_0^{(1)}(x) + \frac{16}{27}\,\boldsymbol{P}_0(x)\right)(a_0 L)^4 \\
&+ \left(\frac{1}{120}\,\boldsymbol{P}_0^{(4)}(x) + \frac{1}{9}\,\boldsymbol{P}_0^{(3)}(x) + \frac{14}{27}\,\boldsymbol{P}_0^{(2)}(x) + \frac{80}{81}\,\boldsymbol{P}_0^{(1)}(x) + \frac{256}{405}\,\boldsymbol{P}_0(x)\right)(a_0 L)^5 \ .
\end{aligned} \tag{32}$$

Eq. (32) makes the effect of the running coupling (9) explicit. The matrices $\boldsymbol{P}_0^{(k)}$ are

$$\boldsymbol{P}_0^{(k)}(x) = \otimes_{l=1}^{k}\,\boldsymbol{P}_0(x)\ . \tag{33}$$

The corresponding expressions for the $(1,1)$–components of $\boldsymbol{P}_0^{(k)}(x)$ are given relative to the non–singlet components $\boldsymbol{P}_{\rm NS}^{(k)}(x)$ of section 2. The singlet–matrix components $\boldsymbol{P}_{i,j}^{(k)}(x)$ given below (34–53) were calculated using the convolution formulae of Appendix A. [7] The projections $P_{i,j}^{(k)}$ describe the splitting of an electron into an electron (1,1), of a photon into an electron, positron, respectively (1,2), an electron into a photon (2,1), and a photon into a photon (2,2) in $k$th order in the renormalized coupling constant.

The individual components for the different orders are given by :

0th order :

$$P_{1,1}^{(0)}(x) = P_{\rm NS}^{(0)}(x) \tag{34}$$
$$P_{1,2}^{(0)}(x) = -4(1-2x) \tag{35}$$
$$P_{2,1}^{(0)}(x) = 2(2-x) \tag{36}$$
$$P_{2,2}^{(0)}(x) = -\frac{4}{3}\delta(1-x)\ . \tag{37}$$

Here a non–vanishing contribution to $P_{2,2}^{(0)}(x)$ emerges due to 4–momentum conservation, see also [29].

1st order :

$$P_{1,1}^{(1)}(x) = P_{\rm NS}^{(1)} + 8[5(1-x) + 2(1+x)\ln(x)] \tag{38}$$
$$P_{1,2}^{(1)}(x) = 4\left[2(2x-1)[2\ln(1-x) - \ln(x)] + \frac{13-8x}{3}\right] \tag{39}$$
$$P_{2,1}^{(1)}(x) = 4\left[(2-x)[2\ln(1-x) + \ln(x)] - \frac{8-13x}{6}\right] \tag{40}$$
$$P_{2,2}^{(1)}(x) = 8\left[5(1-x) + 2(1+x)\ln(x) + \frac{2}{9}\delta(1-x)\right] \tag{41}$$

[6] Beyond this level we will use only the exponentiation for the non–singlet terms.

[7] Note that the second order contribution $P_{1,1}^{(1)}(x)$ given in the literature is partly incorrect [28] for polarized scattering.

2nd order :

$$P_{1,1}^{(2)}(x) = P_{\rm NS}^{(2)} + 16\Big[-\frac{49}{3}(1-x) + 20(1-x)\ln(1-x) - \frac{4}{3}(7-8x)\ln(x) \tag{42}$$
$$+8(1+x)\ln(x)\ln(1-x) - 2(1+x)\ln^2(x) + 8(1+x){\rm Li}_2(1-x)\Big]$$

$$P_{1,2}^{(2)}(x) = 8\Big[\frac{1379-1318x}{18} + 8(1-2x)\zeta(2) + \frac{44-16x}{3}\ln(1-x) + \frac{98+158x}{3}\ln(x) \tag{43}$$
$$-8(1-2x)\ln^2(1-x) + 8(1-2x)\ln(x)\ln(1-x) + 3(1-2x)\ln^2(x)\Big]$$

$$P_{2,1}^{(2)}(x) = 8\Big[-\frac{659}{18} + \frac{1379}{36}x - 4(2-x)\zeta(2) - \frac{8-22x}{3}\ln(1-x) - \frac{71}{3}(1+x)\ln(x) \tag{44}$$
$$+4(2-x)\ln^2(1-x) - 4(2-x)\ln(x)\ln(1-x) - \frac{3}{2}(2-x)\ln^2(x)\Big]$$

$$P_{2,2}^{(2)}(x) = 8\Big[-\frac{79}{3}(1-x) + 20(1-x)\ln(1-x) - \frac{20}{3}(2-x)\ln(x) \tag{45}$$
$$+8(1+x)\ln(x)\ln(1-x) - 2(1+x)\ln^2(x) + 8(1+x){\rm Li}_2(1-x) - \frac{8}{27}\delta(1-x)\Big] \ .$$

3rd order :

$$P_{1,1}^{(3)}(x) = P_{\rm NS}^{(3)} + 16\Big[-\Big(\frac{5593}{18} + 120\zeta(2)\Big)(1-x) - \frac{548}{3}(1-x)\ln(1-x)$$
$$-\Big(\frac{1445+3089x}{9} + 48(1+x)\zeta(2)\Big)\ln(x) + 120(1-x)\ln^2(1-x)$$
$$-\frac{80}{3}(4-5x)\ln(x)\ln(1-x) - \frac{85-65x}{3}\ln^2(x) - 24(1+x)\ln^2(x)\ln(1-x) \tag{46}$$
$$+48(1+x)\ln(x)\ln^2(1-x) - \frac{2}{3}(1+x)\ln^3(x) + \frac{40}{3}(1+x){\rm Li}_2(1-x)$$
$$+96(1+x)\ln(1-x){\rm Li}_2(1-x) - 96(1+x){\rm Li}_3(1-x) + 48(1+x){\rm S}_{1,2}(1-x)\Big]$$

$$P_{1,2}^{(3)}(x) = 16\Big[-\frac{42587}{108} + \frac{10768}{27}x - \frac{124-32x}{3}\zeta(2) - 32(1-2x)\zeta(3)$$
$$+\Big(\frac{2693-2506x}{9} + 48(1-2x)\zeta(2)\Big)\ln(1-x)$$
$$-\Big(\frac{3389-1384x}{18} + 24(1-2x)\zeta(2)\Big)\ln(x) + \frac{124-32x}{3}\ln^2(1-x) \tag{47}$$
$$-\frac{380+644x}{3}\ln(x)\ln(1-x) - \frac{217+304x}{6}\ln^2(x) - 16(1-2x)\ln^3(1-x)$$
$$+24(1-2x)\ln(x)\ln^2(1-x) + 10(1-2x)\ln^2(x)\ln(1-x)$$
$$-\frac{7}{3}(1-2x)\ln^3(x) + (168+204x){\rm Li}_2(1-x) + 44(1-2x)\ln(x){\rm Li}_2(1-x)$$
$$+52(1-2x){\rm S}_{1,2}(1-x)\Big]$$

$$P_{2,1}^{(3)}(x) = 16\Big[\frac{5384}{27} + \frac{42587}{216}x + \frac{16-62x}{3}\zeta(2) + 16(2-x)\zeta(3)$$
$$-\Big(\frac{1253}{9} - \frac{2693}{18}x + 24(2-x)\zeta(2)\Big)\ln(1-x) \tag{48}$$

$$
\begin{aligned}
&+\left(\frac{907}{9}-\frac{1997}{36}x+12(2-x)\zeta(2)\right)\ln(x)-\frac{16-62x}{3}\ln^2(1-x)\\
&-\frac{290+314x}{3}\ln(x)\ln(1-x)+\left(\frac{77}{3}+\frac{287}{12}x\right)\ln^2(x)+8(2-x)\ln^3(1-x)\\
&-12(2-x)\ln(x)\ln^2(1-x)-5(2-x)\ln^2(x)\ln(1-x)+\frac{7}{6}(2-x)\ln^3(x)\\
&-22(2-x)\ln(x)\mathrm{Li}_2(1-x)-(102+84x)\mathrm{Li}_2(1-x)-26(2-x)\mathrm{S}_{1,2}(1-x)\Big]\\
P^{(3)}_{2,2}(x) &= 16\left[-\left(\frac{2177}{6}+40\zeta(2)\right)(1-x)-\frac{236}{3}(1-x)\ln(1-x)\right.\\
&-\left(183+\frac{785}{3}x+16(1+x)\zeta(2)\right)\ln(x)+40(1-x)\ln^2(1-x) \qquad (49)\\
&-\frac{128-112x}{3}\ln(x)\ln(1-x)-\frac{103-107x}{3}\ln^2(x)\\
&+16(1+x)\ln(x)\ln^2(1-x)-8(1+x)\ln^2(x)\ln(1-x)-2(1+x)\ln^3(x)\\
&+32(1+x)\ln(1-x)\mathrm{Li}_2(1-x)-\frac{8}{3}(1+x)\mathrm{Li}2(1-x)-32(1+x)\mathrm{Li}_3(1-x)\\
&\left.+16(1+x)\mathrm{S}_{1,2}(1-x)+\frac{16}{81}\delta(1-x)\right]\ .
\end{aligned}
$$

4th order :

$$
\begin{aligned}
P^{(4)}_{1,1}(x) &= P^{(4)}_{\rm NS}+32\left[\left(\frac{73678}{27}+704\zeta(2)+640\zeta(3)\right)(1-x)\right.\\
&-\left(\frac{15212}{9}+960\zeta(2)\right)(1-x)\ln(1-x)\\
&+\left(\frac{36568}{27}-\frac{9068}{27}x+416\zeta(2)-544\zeta(2)x+256(1+x)\zeta(3)\right)\ln(x)\\
&-704(1-x)\ln^2(1-x)-\left(\frac{8000+20672x}{9}+384(1+x)\zeta(2)\right)\ln(x)\ln(1-x)\\
&+\left(\frac{2696+4472x}{9}+96(1+x)\right)\ln^2(x)+320(1-x)\ln^3(1-x)\\
&-(416-544x)\ln(x)\ln^2(1-x)-(152-88x)\ln^2(x)\ln(1-x) \qquad (50)\\
&+\frac{40}{9}(8-7x)\ln^3(x)+128(1+x)\ln(x)\ln^3(1-x)-96(1+x)\ln^2(x)\ln^2(1-x)\\
&+\frac{4}{3}(1+x)\ln^4(x)-\left(\frac{14336}{9}+384\zeta(2)\right)(1+x)\mathrm{Li}_2(1-x)\\
&+128(1+x)\ln(1-x)\mathrm{Li}_2(1-x)-720(1-x)\ln(x)\mathrm{Li}_2(1-x)\\
&+384(1+x)\ln^2(1-x)\mathrm{Li}_2(1-x)-96(1+x)\ln^2(x)\mathrm{Li}_2(1-x)\\
&-128(1+x)\mathrm{Li}_3(1-x)-768(1+x)\ln(1-x)\mathrm{Li}_3(1-x)+768(1+x)\mathrm{Li}_4(1-x)\\
&-(816-944x)\mathrm{S}_{1,2}(1-x)+384(1+x)\ln(1-x)\mathrm{S}_{1,2}(1-x)\\
&\left.-256(1+x)\ln(x)\mathrm{S}_{1,2}(1-x)-384(1+x)\mathrm{S}_{2,2}(1-x)-192(1+x)\mathrm{S}_{1,3}(1-x)\right]\\
P^{(4)}_{1,2}(x) &= 32\left[-\frac{1912189}{648}+\frac{958405}{324}x-\frac{7852-7064x}{9}\zeta(2)+\frac{640-128x}{3}\zeta(3)-48(1-2x)\zeta(4)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left(-\frac{57280}{27}+\frac{58364}{27}x-64(5-x)\zeta(2)-256(1-2x)\zeta(3)\right)\ln(1-x)\\
&+\left(-\frac{37633}{27}-\frac{100213}{27}x-368\zeta(2)-656\zeta(2)x+128(1-2x)\zeta(3)\right)\ln(x)\\
&+\left(\frac{7852-7064x}{9}+192(1-2x)\zeta(2)\right)\ln^2(1-x)\\
&-\left(\frac{9220-4616x}{9}+192(1-2x)\zeta(2)\right)\ln(x)\ln(1-x)\\
&+\left(-\frac{4703}{18}+\frac{4442}{9}x-24(1-2x)\zeta(2)\right)\ln^2(x)+\frac{64}{3}(5-x)\ln^3(1-x)\\
&+(368+656x)\ln(x)\ln^2(1-x)-\frac{616+1000x}{3}\ln^2(x)\ln(1-x)\\
&-\left(20+\frac{89}{3}x\right)\ln^3(x)-32(1-2x)\ln^4(1-x)+64(1-2x)\ln(x)\ln^3(1-x)\\
&+24(1-2x)\ln^2(x)\ln^2(1-x)-\frac{40}{3}(1-2x)\ln^3(x)\ln(1-x)-\frac{5}{12}(1-2x)\ln^4(x)\\
&-(152+272x)\mathrm{Li}_2(1-x)+(1056+1248x)\ln(1-x)\mathrm{Li}_2(1-x)\\
&-\frac{32}{3}(4+x)\ln(x)\mathrm{Li}_2(1-x)+288(1-2x)\ln(x)\ln(1-x)\mathrm{Li}_2(1-x)\\
&-16(1-2x)\ln^2(x)\mathrm{Li}_2(1-x)-(1056+1248x)\mathrm{Li}_3(1-x)\\
&-288(1-2x)\ln(x)\mathrm{Li}_3(1-x)+352(1-2x)\ln(1-x)\mathrm{S}_{1,2}(1-x)\\
&+80(1-2x)\ln(x)\mathrm{S}_{1,2}(1-x)-32(1-2x)\mathrm{Li}_2^2(1-x)+48(9+13x)\mathrm{S}_{1,2}(1-x)\\
&-224(1-2x)\mathrm{S}_{2,2}(1-x)+224(1-2x)\mathrm{S}_{1,3}(1-x)\Bigg]
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
P_{2,1}^{(4)}(x) \;=\; & 32\Bigg[\frac{958405}{648}-\frac{1912189}{1296}x+\frac{3532-3926x}{9}\zeta(2)-\frac{64}{3}(1-5x)\zeta(3)+24(2-x)\zeta(4)\\
&+\left(\frac{41849+94913x}{54}+296\zeta(2)+344\zeta(2)x-64(2-x)\zeta(3)\right)\ln(x)\\
&+\left(\frac{29182-28640x}{27}+32(1-5x)\zeta(2)+128(2-x)\zeta(3)\right)\ln(1-x)\\
&+\left(-\frac{3532-3926x}{9}-96(2-x)\zeta(2)\right)\ln^2(1-x)\\
&+\left(\frac{4756-3242x}{9}+96(2-x)\zeta(2)\right)\ln(x)\ln(1-x)\\
&+\left(\frac{1609}{9}-\frac{6071}{36}x+12(2-x)\zeta(2)\right)\ln^2(x)-\frac{32}{3}(1-5x)\ln^3(1-x)\\
&-(296+344x)\ln(x)\ln^2(1-x)+\frac{436+484x}{3}\ln^2(x)\ln(1-x)+\left(\frac{35}{2}+\frac{14}{3}x\right)\ln^3(x)\\
&+16(2-x)\ln^4(1-x)-32(2-x)\ln(x)\ln^3(1-x)-12(2-x)\ln^2(x)\ln^2(1-x)\\
&+\frac{20}{3}(2-x)\ln^3(x)\ln(1-x)+\frac{5}{24}(2-x)\ln^4(x)+(136+76x)\mathrm{Li}_2(1-x)\\
&-(624+528x)\ln(1-x)\mathrm{Li}_2(1-x)-\frac{16}{3}(1+4x)\ln(x)\mathrm{Li}_2(1-x)\\
&-144(2-x)\ln(x)\ln(1-x)\mathrm{Li}_2(1-x)+8(2-x)\ln^2(x)\mathrm{Li}_2(1-x)\\
&+48(13+11x)\mathrm{Li}_3(1-x)+144(2-x)\ln(x)\mathrm{Li}_3(1-x)+16(2-x)\mathrm{Li}_2^2(1-x)\\
&-312(1+x)\mathrm{S}_{1,2}(1-x)-176(2-x)\ln(1-x)\mathrm{S}_{1,2}(1-x)
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
& -40(2-x)\ln(x)\mathrm{S}_{1,2}(1-x) + 112(2-x)\mathrm{S}_{2,2}(1-x) - 112(2-x)\mathrm{S}_{1,3}(1-x)\Bigg] \\
P_{2,2}^{(4)}(x) \;=\; & 32\Bigg[\left(\frac{256273}{108} + \frac{628}{3}\zeta(2) + 160\zeta(3)\right)(1-x) - (1417 + 240\zeta(2))(1-x)\ln(1-x) \\
& + \left(\frac{32002 - 6257x}{27} + \frac{352 - 368x}{3}\zeta(2) + 64(1+x)\zeta(3)\right)\ln(x) \\
& - \frac{628}{3}(1-x)\ln^2(1-x) - \left(\frac{2150 + 3406x}{3} + 96(1+x)\zeta(2)\right)\ln(x)\ln(1-x) \\
& + \left(\frac{763 + 626x}{3} + 24(1+x)\zeta(2)\right)\ln^2(x) + 80(1-x)\ln^3(1-x) \\
& - \frac{352 - 368x}{3}\ln(x)\ln^2(1-x) - \frac{394 - 386x}{3}\ln^2(x)\ln(1-x) \\
& + \frac{50}{9}(5-4x)\ln^3(x) + 32(1+x)\ln(x)\ln^3(1-x) \\
& - 24(1+x)\ln^2(x)\ln^2(1-x) - \frac{20}{3}(1+x)\ln^3(x)\ln(1-x) + \frac{7}{6}(1+x)\ln^4(x) \\
& - (926 + 96\zeta(2))(1+x)\mathrm{Li}_2(1-x) + \frac{16}{3}(1+x)\ln(1-x)\mathrm{Li}_2(1-x) \\
& - 380(1-x)\ln(x)\mathrm{Li}_2(1-x) + 96(1+x)\ln^2(1-x)\mathrm{Li}_2(1-x) \\
& - 44(1+x)\ln^2(x)\mathrm{Li}_2(1-x) - \frac{16}{3}(1+x)\mathrm{Li}_3(1-x) \\
& - 192(1+x)\ln(1-x)\mathrm{Li}_3(1-x) + 192(1+x)\mathrm{Li}_4(1-x) \\
& - \frac{1252 - 1268x}{3}\mathrm{S}_{1,2}(1-x) + 96(1+x)\ln(1-x)\mathrm{S}_{1,2}(1-x) \\
& - 104(1+x)\ln(x)\mathrm{S}_{1,2}(1-x) - 96(1+x)\mathrm{S}_{2,2}(1-x) - 88(1+x)\mathrm{S}_{1,3}(1-x) \\
& - \frac{32}{243}\delta(1-x)\Bigg] \; .
\end{aligned}
\tag{53}
$$

We expressed the above splitting functions in terms of poly–logarithms and Nielsen integrals [30],

$$
\mathrm{Li}_n(x) \;=\; S_{n-1,1}(x) = \frac{(-1)^{n-1}}{(n-2)!}\int_0^1 \frac{dz}{z}\ln^{n-2}(z)\ln(1-xz) \tag{54}
$$

$$
S_{n,p}(x) \;=\; \frac{(-1)^{n+p-1}}{(n-1)!p!}\int_0^1 \frac{dz}{z}\ln^{n-1}(z)\ln^p(1-zx) \; . \tag{55}
$$

# 5 $\alpha^{n+1}\ln^{2n}(x)$ Corrections to The Evolution Kernel

The resummation of the leading small-$x$ terms to all orders in the coupling constant were considered in QCD in Refs. [7, 14] and [6] for the singlet case. Very recently the results of [5] in the interpretation of Ref. [6a, 12] were verified to be correct in the 3rd order in the coupling constant in the unpolarized non–singlet case [32]. This gives further confidence in the method used. The QED analogue is obtained by setting $C_F = T_F = 1, C_A = 0$. We will furthermore consider only one flavor, the electron. The corresponding contribution to the matrix of singlet splitting functions reads

$$
\mathbf{P}(x,a)_{x\to 0} = \sum_{l=0}^{\infty}\mathbf{P}_{x\to 0}^{(l)}a^{l+1}\ln^{2l}(x) = \frac{1}{8\pi^2}\mathcal{M}^{-1}\left[F_0(N,a)\right](x). \tag{56}
$$

The matrix $\mathbf{F}_0(N,a)$ obeys the equation

$$\mathbf{F}_0(N,a) = 16\pi^2 \frac{a}{N}\mathbf{M}_0 - \frac{8a}{N^2}\mathbf{F}_8(N,a)\mathbf{G}_0 + \frac{1}{8\pi^2}\frac{1}{N}\mathbf{F}_0^2(N,a), \tag{57}$$

where

$$\mathbf{F}_8(N,a) = 16\pi^2 \frac{a}{N}\mathbf{M}_8 + \frac{1}{8\pi^2}\frac{1}{N}\mathbf{F}_8^2(N,a). \tag{58}$$

The generating matrices are

$$\mathbf{M}_0 = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}, \qquad \mathbf{M}_8 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{G}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{59}$$

Since $\mathbf{M}_8^n = \mathbf{M}_8$ in the case of QED, $\mathbf{F}_8(N,a)$ can be given in analytic form

$$\mathbf{F}_8(N,a) = 4\pi^2 \left(1 - \sqrt{1 - \frac{8a}{N^2}}\right) \mathbf{M}_8 \ . \tag{60}$$

| $l$ | $P_{ff}^{(l)}$ | $P_{fg}^{(l)}$ | $P_{gg}^{(l)}$ |
|---|---|---|---|
| 0 | 0.2000000000E+01 | −0.4000000000E+01 | 0.0000000000E+00 |
| 1 | −0.1400000000E+01 | −0.4000000000E+01 | −0.8000000000E+01 |
| 2 | −0.8666666667E+01 | 0.6666666667E+01 | −0.2666666667E+01 |
| 3 | 0.1444444444E+01 | 0.2444444444E+01 | 0.2044444444E+01 |
| 4 | 0.7888888889E+00 | −0.2825396825E+00 | 0.4634920635E+00 |
| 5 | 0.5537918871E−02 | −0.1008112875E+00 | −0.3626102293E−01 |
| 6 | −0.1216503661E−01 | −0.5536849981E−03 | −0.9307893752E−02 |
| 7 | −0.5063924905E−03 | 0.8396483000E−03 | −0.3947845218E−04 |
| 8 | 0.4517396184E−04 | 0.2736857499E−04 | 0.4555056936E−04 |
| 9 | 0.2611470148E−05 | −9.1960462641E−05 | 0.1192943787E−05 |
| 10 | −0.4136771391E−07 | −0.9306140011E−07 | −0.6994805938E−07 |
| 11 | −0.4259181787E−08 | 0.1228221436E−08 | −0.2774664782E−08 |
| 12 | −0.1716660622E−10 | 0.1073980741E−09 | 0.3096908050E−10 |
| 13 | 0.2808782930E−11 | 0.3728781778E−12 | 0.2327857737E−11 |
| 14 | 0.3713452748E−13 | −0.5274589147E−13 | 0.7034915420E−14 |

Table 1: The coefficients of the matrices $\mathbf{P}_{x\to 0}^{(l)}$ in Eq. (56).

Note that Eq. (60) has no pole singularities but a branch cut for complex values of $N$. Due to this only moderate effects are expected. The coefficient matrices $\mathbf{P}^{(l)}_{x\to 0}$ are obtained by iterative solution of Eqs. (57). The first coefficients read

$$\mathbf{P}^{(0)}_{x\to 0} = 2\begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix} \tag{61}$$

$$\mathbf{P}^{(1)}_{x\to 0} = 2\begin{pmatrix} -4 & -2 \\ 2 & -4 \end{pmatrix} \tag{62}$$

$$\mathbf{P}^{(2)}_{x\to 0} = \frac{2}{3}\begin{pmatrix} -13 & 10 \\ -10 & -4 \end{pmatrix} . \tag{63}$$

These matrices agree with the constant contribution to the leading order polarized singlet splitting function and the coefficients of the $\ln^2(x)$ contribution to the next-to-leading order splitting function [31] adjusting the color factors as above. As already noted in the case of QCD the off-diagonal matrix elements are related by

$$\mathbf{P}^{(l)}_{f\gamma} = -\mathbf{P}^{(l)}_{\gamma f}. \tag{64}$$

The first 15 coefficients are listed in Table 1. The corrections $\propto a^{n+1}\ln^{2n}(x)$ are due to the most singular parts of the non–leading order anomalous dimensions and are universal because they emerge as part of a conformal solution. For each power in $a$ we have to construct the corresponding matrices $\boldsymbol{U}_k$, Eq. (25). We use Eqs. (30, 31) and work in $N$–space. The Mellin–inversion is carried out by a numerical contour integral around the singularities in the complex $N$–plane. Finally the solution is given by Eq. (29).

# 6 Numerical Results

The calculation of the universal radiators is carried out to a precision which will not need further improvement in the range of collider energies at present facilities and those in the foreseeable future. These corrections need to be supplemented by the non–universal contributions of the respective processes under consideration. In various cases the universal contributions dominate in some kinematic regions. They do, however, not substitute any complete calculation. In the above we presented the respective expressions in analytic form which are easily implemented into QED radiators used in analysis programs. Now we turn to numerical illustrations.

The universal radiator function $D_{\rm NS}(x,Q^2)$, Eq. (12), summing the leading log corrections up to $O((\alpha L)^5)$ is shown in Figure 1 as a function of the momentum fraction $x$ for different values of $Q = \sqrt{Q^2} = 10, 100, 1000\,\text{GeV}$, the range of relevant scales at high energy colliders. The function rises strongly for growing values of $x$ and reaches 1 for $x \simeq 0.9$. $D_{\rm NS}(x,Q^2)$ grows with $Q$, however, the scale dependence is weaker. In Figure 2 we compare the soft photon resummation beyond the 5th order with $D_{\rm NS}(x,Q^2)$. Although these terms become more significant for large values of $x$ the correction is widely below 3 ppm in the range of $Q$ considered above, which shows that summing the leading orders to $O((\alpha L)^5)$ for the full radiator is already a sufficiently accurate approximation. In Figure 3 we show the relative impact of the radiator in first order compared to the resummed radiator for the same values of $Q$. This ratio varies between 90 to 95 % for small values of $x$ to 120 to 130 % for $x \to 1$. Also the the 4th and 5th order approximation for $D^{k,\rm res}(x,Q^2)$ are compared, including the respective soft exponentiation. The

deviation is smaller than $1.5 \cdot 10^{-5}$ at small values of $x$ and vanishes for large values of $x$, showing the degree of convergence.

In Figure 4 we show the singlet contributions for the leading logarithmic radiators in case of longitudinally polarized electrons or positrons. Here, $D_{11}(x,Q^2)$ denotes the pure singlet part for the fermion–radiator to which the non–singlet contribution has to be added. The spread of the functions w.r.t. the scale variation in $Q$ is larger for the off-diagonal radiators $D_{12}$ and $D_{21}$, as well as for $D_{22}$ if compared to the fermionic radiator $D_{11}$. The off–diagonal radiators $D_{12}(x)$ and $D_{21}(x)$ vary between $-6$ and $+6$ %, and $+7$ and $+2$ %, respectively from low to large values of $x$. The pure singlet part of the diagonal radiators is smaller. It grows from values of $-0.2$ %($-0.1$ %) for small values of $x$ to $+0.05$ % and vanish as $x \to 1$. The relative smallness of these contributions results from the fact that they start with $O((\alpha L)^2)$ only, aside of contributions $\propto \delta(1-x)$ not shown in the Figures.

Figure 5 compares the size of the first order contribution to $D_{11}, D_{12}$ and $D_{22}$ for the chosen values of $Q$ with the total contribution. For $D_{11}$ the first order contribution is 20 % larger than the total contribution at small $x$ while it basically yields the full contribution for large values of $x$. The first order contribution to $D_{12}$ dominates $D_{12}$, except of the small region $x \sim 0.5$, where higher order contributions are significant. $D_{21}$ receives higher order corrections. $D_{21}^1/D_{21}^5 - 1$ varies from $-5$ % to $+15$ % from small to large values of $x$. In Figure 6 we compare the convergence of the approximation showing the impact of the 5th order term w.r.t. the first four orders. This effect is of $O(10^{-5})$ with some variations, where the effect in the case of $D_{22}$ is an order stronger for large values of $x$. This shows that the this level of resummation is sufficient.

In Figure 7 we compare the radiators due to the resummation of the $\alpha \ln^2(x) L$–terms for the same choice of scales $Q$ as before. $D_{11}(x,Q^2)$ denotes the complete fermionic radiator. The non–singlet contributions were dealt with before in Ref. [15]. The pure singlet term contributes only with $O(\alpha^2)$. Comparing Table 1 with Table 1 [15] one sees that the pure singlet contributions change the non–singlet result and is not subleading. We display the contributions starting with $O(\alpha^2)$, i.e. those containing at least one term $\propto \ln^2(x)$. The first order contribution, the small $x$ limit of the leading order splitting functions, has been contained in the radiators of the fixed order calculation already. The respective part of the radiators $|D_{ij}(x,Q^2)|$ is of $O(1$ %) at small values of $x$ and vanishes towards large values of $x$.

We finally apply the corrections to polarized deeply inelastic charged lepton scattering off polarized protons as an example. The hadronic tensor of the polarized part of the $eN$–scattering cross section in case of pure photon exchange[8] reads

$$W_{\mu\nu}^{(A)} = i\varepsilon_{\mu\nu\lambda\sigma}\frac{q^\lambda S^\sigma}{p.q} g_1(x,Q^2) + i\varepsilon_{\mu\nu\lambda\sigma}\frac{q^\lambda (p.q S^\sigma - S.q p^\sigma)}{(p.q)^2} g_2(x,Q^2) \; . \tag{65}$$

$S_\sigma$ denotes the nucleon spin vector, $p$ the nucleon momentum, and $q$ the vector of the 4–momentum transfer, with $Q^2 = -q^2$ and $x = Q^2/(2p.q)$. The polarized part of the scattering cross section for longitudinal nucleon polarization $S_L$, integrated over the azimuthal angle $\phi$, is

$$\frac{d^2\sigma(\lambda, \pm S_L)_{NC}^B}{dxdy} = \pm 2\pi \mathsf{S} \frac{\alpha^2}{Q^4}\left[-2\lambda y\left(2 - y - \frac{2xyM^2}{\mathsf{S}}\right) x g_1(x,Q^2) + 8\lambda \frac{yx^2M^2}{\mathsf{S}} g_2(x,Q^2)\right] \; . \tag{66}$$

Here $M$ is the nucleon mass, $\mathsf{S}$ the cms energy, $\alpha$ the fine structure constant, $y = 2p.q/\mathsf{S}$, $\lambda$ is the degree of lepton lepton polarization, $x$ and $y$ are the Bjorken variables, and $g_{1,2}(x,Q^2)$

[8]The corresponding expressions in the case of additional weak boson exchange are given in [33].

are the two electromagnetic polarized structure functions, which are represented at twist–2 level referring to the parameterization [34]. Very similar results are obtained using other recent parameterizations [35, 36]. We consider the case of initial state radiation and sum over the final state bremsstrahlung. Furthermore, we cut for the Compton peak, [3]. The kinematic variables chosen are the variables for leptonic corrections. The rescaled variables $\hat{x}, \hat{y}$, and $\hat{s}$ are

$$\hat{x} = \frac{xyz}{z+y-1}, \qquad \hat{y} = \frac{z+y-1}{z}, \qquad \hat{s} = z\mathsf{S} \ , \tag{67}$$

and the radiation threshold is $z_0 = (1-y)/(1-xy)$ . The radiative correction to the differential cross section (66) is given by :

$$\frac{d^2\sigma^{isr}_{NC}}{dxdy} = \frac{d^2\sigma^{B}_{NC}}{dxdy} + \int_0^1 dz D(z,Q^2) \left\{ \theta(z-z_0) \mathcal{J}(x,y,z) \left. \frac{d^2\sigma^{B}_{NC}}{dxdy} \right|_{x=\hat{x},y=\hat{y},s=\hat{s}} - \frac{d^2\sigma^{B}_{NC}}{dxdy} \right\} \ . \tag{68}$$

$D(z,Q^2)$ denotes the respective radiator function for fermion–fermion transitions for $z<1$, and the Jacobian $\mathcal{J}$ is given by

$$\mathcal{J}(z,y,z) = \left| \begin{array}{cc} \partial\hat{x}/\partial x & \partial\hat{y}/\partial x \\ \partial\hat{x}/\partial y & \partial\hat{y}/\partial y \end{array} \right| \ . \tag{69}$$

Previously, the leptonic radiative corrections for this process have been calculated in $O(\alpha)$ completely in Ref. [37], where the complete calculation was also compared to the leading log result in $O(\alpha L)$. In the case of longitudinal polarization studied in the present paper, both at HERMES energies, $\sqrt{S} = 7.4\,\mathrm{GeV}$, and for $ep$ collider energies, $\sqrt{S} = 314$ GeV, the complete $O(\alpha)$ corrections deviate at most by a few % for small values of $x$ and very large values of $y$ from the leading log result.

In Figure 8 the initial state QED corrections for the differential scattering cross section (66) are shown for the kinematics of the HERMES experiment at DESY with $\sqrt{S} = 7.4\,\mathrm{GeV}$, as a ratio to the double differential Born cross section using the resummed radiator as a function of $y$ for characteristic values of $x$. The radiative corrections grow with $y$ and towards smaller values of $x$ and may reach values of $O(80-100\ \%)$ at large $y$. The effect of the higher than leading log corrections to those of leading order, normalized to the differential Born cross section is illustrated in the right figure. Depending on the value of $x$, the corrections in the lower $y$ range are of up to $O(\pm 0.5\ \%)$ and grow rapidly in the high $y$ region reaching $O(4\ \%)$.

For a hypothetical future $ep$ collider operating both with longitudinally polarized electrons and protons at a cms energy of $\sqrt{S} = 1$ TeV the double differential radiative corrections due to initial state radiation are shown in Figure 9. For fixed values of $x$ all the corrections grow strongly towards large value of $y$. For large values of $x$ they are negative for smaller values of $y$ and are everywhere positive for $x$ values below $x \sim 0.1$ . Due to the larger kinematic range the scaling violations due to the running fine–structure constant, $\alpha(Q^2)$, are larger than in the case of the HERMES experiment. Even for $x$ values of $O(10^{-2})$ radiative corrections of +160 % may be reached for $y \sim 0.9$. We compare the impact of the higher than leading log corrections to those of leading order, normalized to the differential Born cross section. Depending on the value of $x$, the corrections in the lower $y$ range are of up to $O(\pm 2.5\ \%)$ and grow rapidly in the high $y$ region reaching $O(20\ \%)$.

# 7 Conclusions

Mass factoriztion at the one hand and infrared evolution equations on the other hand, allow to resum two different classes of universal QED corrections $\propto (\alpha \ln(s/m_e^2)^k$ and $\alpha^{k+1} \ln^{2k}(z)$,

respectively, for characteristic invariant masses $s$ and radiation fractions $z$. The corresponding radiator functions are single–particle quantities and describe the universal part of the transitions between the light species, electrons $e$ and photons $\gamma$, via amplitudes $D_{ij},\ (i,j) = e,\gamma$, for collinear kinematics, similar to the parton model in QCD. In this way these radiators resum the universal part of the QED parton cascades. We calculkated the non–singlet and singlet contributions for the case of polarized scattering to $O((\alpha L)^5)$ and resuummed the contributions $\propto \alpha^{k+1}\ln^{2k}(z)$. Both radiators are calculated to an accuracy which is sufficient for the foreseeable applications in present and future high energy experiments. While the calculation of the resummation of the $\alpha^{k+1}\ln^{2k}(z)$–terms can be done using Mellin space techniques, since the functions contributing are damped towards large Mellin parameters $\mathsf{Re}(N) \to -\infty$, this is not the case of for the radiator, resumming the $(\alpha\ln(s/m_e^2)^k$–terms. Due to this the corresponding convolutions had to be calculated analytically in terms of Nielsen functions. The $\alpha^{k+1}\ln^{2k}(z)$–terms are leading small $z$ pieces of the higher order splitting functions. To deal with them in the context of the resummation of the $(\alpha\ln(s/m_e^2)^k$–terms, one has to invoke the mechanism of general higher order solutions of the singlet QED evolution equations for fermionic and photonic densites. Since the respective matrices of the splitting functions of the different orders in the coupling constant do not commute, it is practically excluded to write the solution in terms of analytic functions in $z$–space. However, one may solve the problem numerically and adds these contributions to the analytic radiators found for the $(\alpha\ln(s/m_e^2)^k$–terms. The main objective of the present paper was to provide all necessary radiators and to simplify results worked out earlier. The radiators obtained can easily be adopted for experimental analysis and simulation programs.

# 8 Appendix A: Mellin Convolutions: Basic Functions

In this appendix we list the convolutions of functions up to depth 5 [9], which were calculated recursively in explicit form, since they are of general interest for other higher order calculations in QED and QCD. Some of the integrals require to use Mellin transforms and algebraic relations between the finite harmonic sums [21], being associated to them, to be solved. The respective expressions which were used in this calculation are given in appendix B and C, as long they were not contained in Ref. [21].

$$f(x) \otimes \delta(1-x) = f(x), \quad \forall\, f(x) \,\epsilon\, \mathcal{S}'[0,1] \tag{70}$$

$$1 \otimes 1 = -\ln(x) \tag{71}$$

$$1 \otimes x = 1 - x \tag{72}$$

$$x \otimes x = -x \ln(x) \tag{73}$$

$$\left(\frac{1}{1-x}\right)_+ \otimes \left(\frac{1}{1-x}\right)_+ = \left[2\frac{\ln(1-x)}{1-x} - \frac{\ln(x)}{1-x}\right]_+ \tag{74}$$

$$\left(\frac{1}{1-x}\right)_+ \otimes 1 = \ln(1-x) - \ln(x) \tag{75}$$

$$\left(\frac{1}{1-x}\right)_+ \otimes x = x\left[\ln(1-x) - \ln(x)\right] + 1 - x \tag{76}$$

$$\left(\frac{\ln(1-x)}{1-x}\right)_+ \otimes \left(\frac{1}{1-x}\right)_+ = \left[\frac{3}{2}\frac{\ln^2(1-x)}{1-x} - \frac{\zeta(2)}{1-x} - \ln(x)\frac{\ln(1-x)}{1-x}\right]_+ \tag{77}$$

$$\left(\frac{\ln(1-x)}{1-x}\right)_+ \otimes 1 = \mathrm{Li}_2(x) - \zeta(2) + \frac{1}{2}\ln^2(1-x) \tag{78}$$

$$\left(\frac{\ln(1-x)}{1-x}\right)_+ \otimes x = x\left[\mathrm{Li}_2(x) - \zeta(2) + \frac{1}{2}\ln^2(1-x) - \ln(1-x) + \ln(x)\right] + \ln(1-x) \tag{79}$$

$$\left(\frac{1}{1-x}\right)_+ \otimes \ln(1-x) = \ln^2(1-x) - \ln(x)\ln(1-x) - \zeta(2) \tag{80}$$

$$\left(\frac{1}{1-x}\right)_+ \otimes x\ln(1-x) = x\left[\ln^2(1-x) - \ln(x)\ln(1-x) - \zeta(2)\right] + (1-x)[\ln(1-x) - 1] \tag{81}$$

$$\left(\frac{1}{1-x}\right)_+ \otimes \ln(x) = -\frac{1}{2}\ln^2(x) + \mathrm{Li}_2(1-x) + \ln(x)\ln(1-x) \tag{82}$$

$$\left(\frac{1}{1-x}\right)_+ \otimes x\ln(x) = x\left[-\frac{1}{2}\ln^2(x) + \mathrm{Li}_2(1-x) + \ln(x)\ln(1-x)\right] - 1 + x - x\ln(x) \tag{83}$$

$$\left(\frac{1}{1-x}\right)_+ \otimes \frac{\ln(x)}{1-x} = \ln(x)\frac{\ln(1-x)}{1-x} - \frac{1}{2}\frac{\ln^2(x)}{1-x} \tag{84}$$

$$1 \otimes \ln(1-x) = \mathrm{Li}_2(x) - \zeta(2) \tag{85}$$

$$1 \otimes x\ln(1-x) = (1-x)\left[\ln(1-x) - 1\right] \tag{86}$$

$$x \otimes \ln(1-x) = x\ln(x) + (1-x)\ln(1-x) \tag{87}$$

$$x \otimes x\ln(1-x) = x\left[\mathrm{Li}_2(x) - \zeta(2)\right] \tag{88}$$

[9] A series of other useful convolutions has been given in Ref. [18] recently.

$$1 \otimes \frac{\ln(x)}{1-x} = -\mathrm{Li}_2(1-x) - \frac{1}{2}\ln^2(x) \tag{89}$$

$$x \otimes \frac{\ln(x)}{1-x} = x\left[-\mathrm{Li}_2(1-x) - \frac{1}{2}\ln^2(x)\right] + 1 - x + \ln(x) \tag{90}$$

$$1 \otimes \ln(x) = -\frac{1}{2}\ln^2(x) \tag{91}$$

$$1 \otimes x\ln(x) = -1 + x - x\ln(x) \tag{92}$$

$$x \otimes \ln(x) = 1 - x + \ln(x) \tag{93}$$

$$x \otimes x\ln(x) = -\frac{1}{2}x\ln^2(x) \tag{94}$$

$$1 \otimes \ln^2(x) = -\frac{1}{3}\ln^3(x) \tag{95}$$

$$x \otimes \ln^2(x) = \ln^2(x) + 2(\ln(x) - x + 1) \tag{96}$$

$$1 \otimes x\ln^2(x) = -x\ln^2(x) + 2(x\ln(x) - x + 1) \tag{97}$$

$$x \otimes x\ln^2(x) = -\frac{1}{3}x\ln^3(x) \tag{98}$$

$$1 \otimes \ln^3(x) = -\frac{1}{4}\ln^4(x) \tag{99}$$

$$x \otimes \ln^3(x) = \ln^3(x) + 3\ln^2(x) + 6(\ln(x) - x + 1) \tag{100}$$

$$1 \otimes x\ln^3(x) = -x\ln^3(x) + 3x\ln^2(x) - 6(x\ln(x) - x + 1) \tag{101}$$

$$x \otimes x\ln^3(x) = -\frac{1}{4}x\ln^4(x) \tag{102}$$

$$1 \otimes \ln^4(x) = -\frac{1}{5}\ln^5(x) \tag{103}$$

$$x \otimes \ln^4(x) = \ln^4(x) + 4\ln^3(x) + 12\ln^2(x) + 24(\ln(x) - x + 1) \tag{104}$$

$$\begin{aligned} 1 \otimes x\ln^4(x) = {} & -x\ln^4(x) + 4x\ln^3(x) - 12x\ln^2(x) \\ & +24(x\ln(x) - x + 1) \end{aligned} \tag{105}$$

$$x \otimes x\ln^4(x) = -\frac{1}{5}x\ln^5(x) \tag{106}$$

$$\begin{aligned} \left(\frac{1}{1-x}\right)_+ \otimes \ln(x)\ln(1-x) = {} & 2\mathrm{S}_{1,2}(1-x) - \mathrm{Li}_3(1-x) \\ & + [\ln(x) + \ln(1-x)]\,\mathrm{Li}_2(1-x) \\ & -\frac{1}{2}\ln^2(x)\ln(1-x) + \ln(x)\ln^2(1-x) \\ & -\zeta(2)\ln(x) \end{aligned} \tag{107}$$

$$\begin{aligned} \left(\frac{1}{1-x}\right)_+ \otimes x\ln(x)\ln(1-x) = {} & x\Big\{2\mathrm{S}_{1,2}(1-x) - \mathrm{Li}_3(1-x) \\ & + [\ln(x) + \ln(1-x)]\,\mathrm{Li}_2(1-x) \\ & -\frac{1}{2}\ln^2(x)\ln(1-x) + \ln(x)\ln^2(1-x) - \zeta(2)\ln(x)\Big\} \\ & +2(1-x) - x\ln(x)\ln(1-x) + x\ln(x) \\ & -(1-x)\ln(1-x) - \mathrm{Li}_2(1-x) \end{aligned} \tag{108}$$

$$\left(\frac{1}{1-x}\right)_+ \otimes \left(\frac{\ln^2(1-x)}{1-x}\right)_+ = \left[\frac{4}{3}\frac{\ln^3(1-x)}{1-x} - \ln(x)\frac{\ln^2(1-x)}{1-x}\right.$$

$$-2\zeta(2)\frac{\ln(1-x)}{1-x}+2\zeta(3)\frac{1}{1-x}\Bigg]_+ \tag{109}$$

$$\left(\frac{1}{1-x}\right)_+\otimes \mathrm{Li}_2(1-x) = -\mathrm{S}_{1,2}(1-x)-\mathrm{Li}_3(1-x)$$
$$+\mathrm{Li}_2(1-x)[\ln(1-x)-\ln(x)] \tag{110}$$

$$\left(\frac{1}{1-x}\right)_+\otimes x\mathrm{Li}_2(1-x) = -x\left[\mathrm{S}_{1,2}(1-x)+\mathrm{Li}_3(1-x)\right.$$
$$\left.-\ln\left(\frac{1-x}{x}\right)\mathrm{Li}_2(1-x)\right]$$
$$-(1-x)[1-\mathrm{Li}_2(1-x)]-x\ln(x) \tag{111}$$

$$\left(\frac{1}{1-x}\right)_+\otimes \ln^2(1-x) = \ln^3(1-x)-\ln(x)\ln^2(1-x)$$
$$-2\zeta(2)\ln(1-x)+2\zeta(3) \tag{112}$$

$$\left(\frac{1}{1-x}\right)_+\otimes x\ln^2(1-x) = x\Big[\ln^3(1-x)-\ln(x)\ln^2(1-x)$$
$$-2\zeta(2)\ln(1-x)+2\zeta(3)\Big] \tag{113}$$
$$+(1-x)\left[\ln^2(1-x)-2\ln(1-x)+2\right]$$

$$\left(\frac{1}{1-x}\right)_+\otimes \ln^2(x) = -2\left[\mathrm{Li}_3(x)-\zeta(3)-\ln(x)\zeta(2)+\frac{1}{6}\ln^3(x)\right] \tag{114}$$

$$\left(\frac{1}{1-x}\right)_+\otimes x\ln^2(x) = -2x\mathrm{Li}_3(x)+2\left[\zeta(3)-1\right]x+2\left[\zeta(2)+1\right]x\ln(x)$$
$$-x\ln^2(x)-\frac{1}{3}x\ln^3(x)+2 \tag{115}$$

$$\left(\frac{1}{1-x}\right)_+\otimes \frac{\ln(x)\ln(1-x)}{1-x} = 2\frac{\mathrm{S}_{1,2}(1-x)}{1-x}+\frac{\ln(x)\mathrm{Li}_2(1-x)}{1-x}-\zeta(2)\frac{\ln(x)}{1-x}$$
$$-\frac{1}{2}\frac{\ln^2(x)\ln(1-x)}{1-x}+\frac{\ln(x)\ln^2(1-x)}{1-x} \tag{116}$$

$$\left(\frac{1}{1-x}\right)_+\otimes \frac{\ln^2(x)}{1-x} = 2\frac{\mathrm{S}_{1,2}(1-x)}{1-x}-\frac{1}{3}\frac{\ln^3(x)}{1-x}$$
$$-2\frac{\mathrm{Li}_3(x)-\zeta(3)}{1-x}+2\zeta(2)\frac{\ln(x)}{1-x} \tag{117}$$

$$1\otimes \ln(x)\ln(1-x) = \zeta(3)+\mathrm{Li}_2(x)\ln(x)-\mathrm{Li}_3(x) \tag{118}$$

$$1\otimes x\ln(x)\ln(1-x) = -\mathrm{Li}_2(1-x)+(1-x)[2-\ln(1-x)]$$
$$+x\ln(x)[1-\ln(1-x)] \tag{119}$$

$$1\otimes\left(\frac{\ln^2(1-x)}{1-x}\right)_+ = -2\left[\mathrm{S}_{1,2}(x)-\zeta(3)\right]+\frac{1}{3}\ln^3(1-x) \tag{120}$$

$$1\otimes \mathrm{Li}_2(1-x) = -2\mathrm{S}_{1,2}(1-x)-\ln(x)\mathrm{Li}_2(1-x) \tag{121}$$

$$1\otimes x\mathrm{Li}_2(1-x) = (1-x)\mathrm{Li}_2(1-x)-1+x-x\ln(x) \tag{122}$$

$$1\otimes \ln^2(1-x) = 2[\zeta(3)-\mathrm{S}_{1,2}(x)] \tag{123}$$

$$1\otimes x\ln^2(1-x) = (1-x)\left[\ln^2(1-x)+2-2\ln(1-x)\right] \tag{124}$$

$$1 \otimes \frac{\ln^2(x)}{1-x} = 2\mathrm{S}_{1,2}(1-x) - \frac{1}{3}\ln^3(x) \tag{125}$$

$$\begin{aligned} 1 \otimes \frac{\ln(x)\ln(1-x)}{1-x} &= \mathrm{S}_{1,2}(1-x) - \mathrm{S}_{1,2}(x) \\ &\quad + \frac{1}{2}\ln(x)\ln(1-x)\left[\ln(1-x) - \ln(x)\right] + \zeta(3) \end{aligned} \tag{126}$$

$$\begin{aligned} x \otimes \ln(x)\ln(1-x) &= x\mathrm{Li}_2(1-x) + \frac{x}{2}\ln^2(x) + (1-x)\ln(1-x) \\ &\quad + \ln(x)[x + \ln(1-x)] \end{aligned} \tag{127}$$

$$x \otimes x\ln(x)\ln(1-x) = x\left[\zeta(3) + \ln(x)\mathrm{Li}_2(x) - \mathrm{Li}_3(x)\right] \tag{128}$$

$$\begin{aligned} x \otimes \frac{\ln^2(1-x)}{1-x} &= 2x[\zeta(3) - \mathrm{S}_{1,2}(x)] + (1-x)\ln^2(1-x) \\ &\quad + 2x[\zeta(2) - \mathrm{Li}_2(x)] + \frac{x}{3}\ln^3(1-x) \end{aligned} \tag{129}$$

$$x \otimes \mathrm{Li}_2(1-x) = (1-x)\mathrm{Li}_2(1-x) - \frac{1}{2}x\ln^2(x) \tag{130}$$

$$x \otimes x\mathrm{Li}_2(1-x) = -x\left[2\mathrm{S}_{1,2}(1-x) + \ln(x)\mathrm{Li}_2(1-x)\right] \tag{131}$$

$$x \otimes \ln^2(1-x) = (1-x)\ln^2(1-x) - 2x\left[\mathrm{Li}_2(x) - \zeta(2)\right] \tag{132}$$

$$x \otimes x\ln^2(1-x) = 2x[\zeta(3) - \mathrm{S}_{1,2}(x)] \tag{133}$$

$$\begin{aligned} x \otimes \frac{\ln(x)\ln(1-x)}{1-x} &= x\left\{\mathrm{S}_{1,2}(1-x) - \mathrm{S}_{1,2}(x)\right. \\ &\quad \left. + \frac{1}{2}\ln(x)\ln(1-x)\left[\ln(1-x) - \ln(x)\right] + \zeta(3)\right\} \\ &\quad + \ln(x)\ln(1-x) + \frac{x}{2}\ln^2(x) + (1-x)\ln(1-x) \\ &\quad + x\ln(x) + x\mathrm{Li}_2(1-x) \end{aligned} \tag{134}$$

$$\begin{aligned} x \otimes \frac{\ln^2(x)}{1-x} &= x\left[2\mathrm{S}_{1,2}(1-x) - \frac{1}{3}\ln^3(x)\right] + \ln^2(x) \\ &\quad + 2[\ln(x) + 1 - x] \end{aligned} \tag{135}$$

$$\begin{aligned} \left(\frac{1}{1-x}\right)_+ \otimes \left(\frac{\ln^3(1-x)}{1-x}\right)_+ &= \left[\frac{5}{4}\frac{\ln^4(1-x)}{1-x} - \ln(x)\frac{\ln^3(1-x)}{1-x} - 3\zeta(2)\frac{\ln^2(1-x)}{1-x}\right. \\ &\quad \left. + 6\zeta(3)\frac{\ln(1-x)}{1-x} - 6\zeta(4)\frac{1}{1-x}\right]_+ \end{aligned} \tag{136}$$

$$\begin{aligned} \left(\frac{1}{1-x}\right)_+ \otimes \frac{\ln(x)\ln^2(1-x)}{1-x} &= \frac{1}{1-x}\left[-2\mathrm{S}_{2,2}(x) - 2\mathrm{S}_{2,2}(1-x) + \frac{\zeta(4)}{2}\right. \\ &\quad - \mathrm{Li}_2^2(1-x) + 6\ln(1-x)\mathrm{S}_{1,2}(1-x) + 4\ln(x)\mathrm{S}_{1,2}(x) \\ &\quad - 2\ln^2(x)\ln^2(1-x) + \ln(x)\ln^3(1-x) \\ &\quad \left. - 2\zeta(2)\ln(x)\ln(1-x)\right] \end{aligned} \tag{137}$$

$$\begin{aligned} \left(\frac{1}{1-x}\right)_+ \otimes \frac{\ln^2(x)\ln(1-x)}{1-x} &= \frac{1}{1-x}\left[4\ln(1-x)\mathrm{S}_{1,2}(1-x) - \mathrm{Li}_2(1-x)^2\right. \\ &\quad + 2\ln(x)[\mathrm{S}_{1,2}(1-x) + \mathrm{S}_{1,2}(x) - \zeta(3)] \\ &\quad \left. - \ln^2(x)\mathrm{Li}_2(x) - \frac{4}{3}\ln^3(x)\ln(1-x)\right] \end{aligned} \tag{138}$$

$$\left(\frac{1}{1-x}\right)_+ \otimes \frac{\ln^3(x)}{1-x} = \frac{6}{1-x}\Bigg[-\mathrm{S}_{1,3}(1-x)+\zeta(4)-\mathrm{Li}_4(x) + \zeta(3)\ln(x)+\frac{1}{2}\zeta(2)\ln^2(x)-\frac{1}{24}\ln^4(x)\Bigg] \quad (139)$$

$$\left(\frac{1}{1-x}\right)_+ \otimes \frac{\mathrm{S}_{1,2}(1-x)}{1-x} = \frac{1}{1-x}\Bigg[2\mathrm{S}_{2,2}(1-x)-2\mathrm{S}_{1,3}(1-x) -\frac{1}{2}\mathrm{Li}_2^2(1-x)+\ln\left(\frac{1-x}{x}\right)\mathrm{S}_{1,2}(1-x)\Bigg] \quad (140)$$

$$\left(\frac{1}{1-x}\right)_+ \otimes \frac{\mathrm{Li}_3(1-x)}{1-x} = \frac{1}{1-x}\left[\ln\left(\frac{1-x}{x}\right)\mathrm{Li}_3(1-x)-\mathrm{S}_{2,2}(1-x)\right] \quad (141)$$

$$\left(\frac{1}{1-x}\right)_+ \otimes \frac{\ln(x)\mathrm{Li}_2(1-x)}{1-x} = \frac{1}{1-x}\Bigg[2\mathrm{S}_{2,2}(x)-2\mathrm{S}_{2,2}(1-x)+4\mathrm{S}_{1,3}(1-x) +\frac{1}{2}\mathrm{Li}_2^2(1-x)+\ln(x)[\mathrm{S}_{1,2}(1-x)-\mathrm{S}_{1,2}(x)-\zeta(3)] -\frac{1}{2}\ln^2(x)\mathrm{Li}_2(1-x)-2\ln(1-x)\mathrm{S}_{1,2}(1-x) -\frac{1}{2}\zeta(4)\Bigg] \quad (142)$$

$$\left(\frac{1}{1-x}\right)_+ \otimes \frac{\mathrm{Li}_3(x)-\zeta(3)}{1-x} = \frac{1}{1-x}\Bigg[-2\mathrm{S}_{2,2}(x)-2\mathrm{S}_{1,3}(1-x) \quad (143)$$
$$-\ln(x)[\mathrm{S}_{1,2}(1-x)-2\zeta(3)]$$
$$-\ln(1-x)[\mathrm{S}_{1,2}(1-x)-\zeta(2)\ln(x)]+\frac{1}{2}\zeta(4)$$
$$+\frac{1}{2}\mathrm{Li}_2(1-x)^2+\frac{1}{6}\ln^3(x)\ln(1-x)\Bigg]$$

$$\left(\frac{1}{1-x}\right)_+ \otimes \mathrm{S}_{1,2}(x) = \mathrm{S}_{2,2}(x)-3\mathrm{S}_{1,3}(x)-\ln(x)\mathrm{S}_{1,2}(x) +\zeta(2)\mathrm{Li}_2(x)+\zeta(3)\ln(1-x)+\frac{1}{4}\zeta(4) \quad (144)$$

$$\left(\frac{1}{1-x}\right)_+ \otimes x\mathrm{S}_{1,2}(x) = x\Bigg\{\mathrm{S}_{2,2}(x)-3\mathrm{S}_{1,3}(x)-\ln(x)\mathrm{S}_{1,2}(x) \zeta(2)\mathrm{Li}_2(x)+\zeta(3)\ln(1-x)+\frac{1}{4}\zeta(4)\Bigg\} +\zeta(3)-x\mathrm{S}_{1,2}(x) -(1-x)\left[1-\ln(1-x)+\frac{1}{2}\ln^2(1-x)\right] \quad (145)$$

$$\left(\frac{1}{1-x}\right)_+ \otimes \mathrm{Li}_3(1-x) = -\mathrm{S}_{2,2}(1-x)-\mathrm{Li}_4(1-x)+\ln\left(\frac{1-x}{x}\right)\mathrm{Li}_3(1-x) \quad (146)$$

$$\left(\frac{1}{1-x}\right)_+ \otimes x\mathrm{Li}_3(1-x) = x\Bigg\{-\mathrm{S}_{2,2}(1-x)-\mathrm{Li}_4(1-x) +\ln\left(\frac{1-x}{x}\right)\mathrm{Li}_3(1-x)\Bigg\} +x\ln(x)+(1-x)[1-\mathrm{Li}_2(1-x)+\mathrm{Li}_3(1-x)] \quad (147)$$

$$\left(\frac{1}{1-x}\right)_+ \otimes \ln(1-x)\mathrm{Li}_2(1-x) = 2\mathrm{S}_{2,2}(x) - \mathrm{S}_{1,3}(x) + \frac{\zeta(4)}{2} \quad (148)$$
$$-3\ln(1-x)\mathrm{S}_{1,2}(1-x) - 2\ln(x)\mathrm{S}_{1,2}(x) + \frac{1}{2}\mathrm{Li}_2(1-x)^2$$
$$+\left[\frac{1}{2}\ln^2(1-x) - \ln(x)\ln(1-x) - \zeta(2)\right]\mathrm{Li}_2(1-x)$$
$$-\frac{1}{6}\ln(x)\ln^3(1-x) + \frac{1}{2}\ln^2(x)\ln^2(1-x)$$

$$\left(\frac{1}{1-x}\right)_+ \otimes x\ln(1-x)\mathrm{Li}_2(1-x) = x\Bigg\{2\mathrm{S}_{2,2}(x) - \mathrm{S}_{1,3}(x) + \frac{\zeta(4)}{2} \quad (149)$$
$$-3\ln(1-x)\mathrm{S}_{1,2}(1-x) - 2\ln(x)\mathrm{S}_{1,2}(x) + \frac{1}{2}\mathrm{Li}_2(1-x)^2$$
$$+\left[\frac{1}{2}\ln^2(1-x) - \ln(x)\ln(1-x) - \zeta(2)\right]\mathrm{Li}_2(1-x)$$
$$-\frac{1}{6}\ln(x)\ln^3(1-x) + \frac{1}{2}\ln^2(x)\ln^2(1-x)\Bigg\}$$
$$+(1-x)\left[\ln(1-x) - 1\right]\mathrm{Li}_2(1-x) - \mathrm{Li}_2(1-x)$$
$$+3(1-x) + 2x\ln(x) - \left[1 - x + x\ln(x)\right]\ln(1-x)$$

$$\left(\frac{1}{1-x}\right)_+ \otimes \ln(x)\mathrm{Li}_2(1-x) = 2\mathrm{S}_{2,2}(x) - 2\mathrm{S}_{2,2}(1-x) + 4\mathrm{S}_{1,3}(1-x) + \mathrm{Li}_2^2(1-x)$$
$$+\ln(x)[\mathrm{S}_{1,2}(1-x) - \mathrm{S}_{1,2}(x) - \zeta(3)]$$
$$-\frac{1}{2}\ln^2(x)\mathrm{Li}_2(1-x)$$
$$-2\ln(1-x)\mathrm{S}_{1,2}(1-x) - \frac{1}{2}\zeta(4) \quad (150)$$

$$\left(\frac{1}{1-x}\right)_+ \otimes x\ln(x)\mathrm{Li}_2(1-x) = x\Bigg\{2\mathrm{S}_{2,2}(x) - 2\mathrm{S}_{2,2}(1-x) + 4\mathrm{S}_{1,3}(1-x)$$
$$+\ln(x)[\mathrm{S}_{1,2}(1-x) - \mathrm{S}_{1,2}(x) - \zeta(3)]$$
$$+\mathrm{Li}_2^2(1-x) - \frac{1}{2}\ln^2(x)\mathrm{Li}_2(1-x)$$
$$-2\ln(1-x)\mathrm{S}_{1,2}(1-x) - \frac{1}{2}\zeta(4)\Bigg\} \quad (151)$$
$$+[1 - x + x\ln(x)](3 - \mathrm{Li}_2(1-x)) - x\ln^2(x)$$
$$-2\mathrm{S}_{1,2}(1-x)$$

$$\left(\frac{1}{1-x}\right)_+ \otimes \mathrm{S}_{1,2}(1-x) = -2\mathrm{S}_{1,3}(1-x) + \mathrm{S}_{2,2}(1-x) - \frac{1}{2}\mathrm{Li}_2^2(1-x)$$
$$+\ln\left(\frac{1-x}{x}\right)\mathrm{S}_{1,2}(1-x) \quad (152)$$

$$\left(\frac{1}{1-x}\right)_+ \otimes x\mathrm{S}_{1,2}(1-x) = x\Bigg[-2\mathrm{S}_{1,3}(1-x) + \mathrm{S}_{2,2}(1-x) - \frac{1}{2}\mathrm{Li}_2^2(1-x)$$
$$+\ln\left(\frac{1-x}{x}\right)\mathrm{S}_{1,2}(1-x)\Bigg]$$
$$+(1-x)[\mathrm{S}_{1,2}(1-x) - 1] - x\ln(x)\left[1 - \frac{1}{2}\ln(x)\right] \quad (153)$$

$$\left(\frac{1}{1-x}\right)_+ \otimes \mathrm{Li}_3(x) = -2\mathrm{S}_{2,2}(x) - 2\mathrm{S}_{1,3}(1-x) - \ln(x)\mathrm{S}_{1,2}(1-x)$$
$$-\zeta(2)\mathrm{Li}_2(x) + \frac{1}{6}\ln^3(x)\ln(1-x) + 3\zeta(4) \qquad (154)$$
$$+\zeta(3)[\ln(x) + \ln(1-x)]$$

$$\left(\frac{1}{1-x}\right)_+ \otimes x\mathrm{Li}_3(x) = x\Bigg[-2\mathrm{S}_{2,2}(x) - 2\mathrm{S}_{1,3}(1-x) - \ln(x)\mathrm{S}_{1,2}(1-x)$$
$$-\zeta(2)\mathrm{Li}_2(x) + \frac{1}{6}\ln^3(x)\ln(1-x) + 3\zeta(4) \qquad (155)$$
$$+\zeta(3)[\ln(x) + \ln(1-x)]\Bigg]$$
$$+\zeta(3) - \zeta(2) + x[\mathrm{Li}_2(x) - \mathrm{Li}_3(x)]$$
$$+(1-x)[1 - \ln(1-x)]$$

$$\left(\frac{1}{1-x}\right)_+ \otimes \ln(x)\mathrm{Li}_2(x) = -2\mathrm{S}_{2,2}(x) - 4\mathrm{S}_{1,3}(1-x)$$
$$+\ln(x)[3\zeta(3) - \mathrm{S}_{1,2}(x) - 3\mathrm{S}_{1,2}(1-x)] \qquad (156)$$
$$+\frac{1}{2}\ln^2(x)\mathrm{Li}_2(x) + 3\zeta(4) - \zeta(2)\mathrm{Li}_2(x)$$
$$+\frac{5}{6}\ln^3(x)\ln(1-x)$$

$$\left(\frac{1}{1-x}\right)_+ \otimes x\ln(x)\mathrm{Li}_2(x) = x\Bigg\{-2\mathrm{S}_{2,2}(x) - 4\mathrm{S}_{1,3}(1-x)$$
$$+\ln(x)[3\zeta(3) - \mathrm{S}_{1,2}(x) - 3\mathrm{S}_{1,2}(1-x)]$$
$$+\frac{1}{2}\ln^2(x)\mathrm{Li}_2(x) + 3\zeta(4) - \zeta(2)\mathrm{Li}_2(x) \qquad (157)$$
$$+\frac{5}{6}\ln^3(x)\ln(1-x)\Bigg\} + [1 + x - x\ln(x)]\mathrm{Li}_2(x)$$
$$+(1-x)(2 - \ln(x))[1 - \ln(1-x)]$$
$$-2\zeta(2) + 1 - x + \ln(x)$$

$$\left(\frac{1}{1-x}\right)_+ \otimes \ln^3(1-x) = \ln^4(1-x) - \ln(x)\ln^3(1-x) - 3\zeta(2)\ln^2(1-x)$$
$$+6\zeta(3)\ln(1-x) - 6\zeta(4) \qquad (158)$$

$$\left(\frac{1}{1-x}\right)_+ \otimes x\ln^3(1-x) = x\Bigg[\ln^4(1-x) - \ln(x)\ln^3(1-x) - 3\zeta(2)\ln^2(1-x)$$
$$+6\zeta(3)\ln(1-x) - 6\zeta(4)\Bigg]$$
$$-6(1-x)\Bigg[1 - \ln(1-x) + \frac{1}{2}\ln^2(1-x)$$
$$-\frac{1}{6}\ln^3(1-x)\Bigg] \qquad (159)$$

$$\left(\frac{1}{1-x}\right)_+ \otimes \ln(x)\ln^2(1-x) = -2\mathrm{S}_{2,2}(x) - 2\mathrm{S}_{2,2}(1-x) - 2\mathrm{S}_{1,3}(x) + \frac{5}{2}\zeta(2)$$
$$-\mathrm{Li}_2^2(1-x) + 6\ln(1-x)\mathrm{S}_{1,2}(1-x) \qquad (160)$$

$$
\begin{aligned}
& -4\ln(x)[\mathrm{Li}_3(1-x)-\zeta(3)] \\
& +2\ln(x)\ln(1-x)\left[2\mathrm{Li}_2(1-x)-\zeta(2)\right] \\
& +\frac{2}{3}\ln(x)\ln^3(1-x) \\
\left(\frac{1}{1-x}\right)_+ \otimes x\ln(x)\ln^2(1-x) &= x\Bigg\{-2\mathrm{S}_{2,2}(x)-2\mathrm{S}_{2,2}(1-x)-2\mathrm{S}_{1,3}(x)+\frac{5}{2}\zeta(2) \\
& -\mathrm{Li}_2^2(1-x)+6\ln(1-x)\mathrm{S}_{1,2}(1-x) \qquad (161) \\
& -4\ln(x)[\mathrm{Li}_3(1-x)-\zeta(3)] \\
& +2\ln(x)\ln(1-x)\left[2\mathrm{Li}_2(1-x)-\zeta(2)\right] \\
& +\frac{2}{3}\ln(x)\ln^3(1-x)\Bigg\}-2x\ln(x) \\
& +(1-x)\ln(x)\ln(1-x)[\ln(1-x)-2] \\
& -(1-x)[\ln(1-x)^2-4\ln(1-x)+6] \\
& +2[\zeta(2)-\mathrm{Li}_2(x)]+2[\zeta(3)-\mathrm{S}_{1,2}(x)] \\
\left(\frac{1}{1-x}\right)_+ \otimes \ln^2(x)\ln(1-x) &= 2\mathrm{S}_{2,2}(1-x)+2[\ln(x)+\ln(1-x)]\mathrm{S}_{1,2}(1-x) \qquad (162) \\
& +2\ln(x)[\mathrm{S}_{1,2}(x)-\zeta(3)]-\mathrm{Li}_2(1-x)^2 \\
& +\ln^2(x)[\mathrm{Li}_2(1-x)-\zeta(2)]-\frac{1}{3}\ln^3(x)\ln(1-x) \\
\left(\frac{1}{1-x}\right)_+ \otimes x\ln^2(x)\ln(1-x) &= x\Bigg\{2\mathrm{S}_{2,2}(1-x)+2[\ln(x)+\ln(1-x)]\mathrm{S}_{1,2}(1-x) \\
& +2\ln(x)[\mathrm{S}_{1,2}(x)-\zeta(3)]-\mathrm{Li}_2(1-x)^2 \qquad (163) \\
& +\ln^2(x)[\mathrm{Li}_2(1-x)-\zeta(2)]-\frac{1}{3}\ln^3(x)\ln(1-x)\Bigg\} \\
& +2\mathrm{S}_{1,2}(1-x)+2\mathrm{Li}_2(1-x) \\
& -x[\ln^2(x)-2\ln(x)]\ln(1-x)+2(1-x)\ln(1-x) \\
& -4x\ln(x)+x\ln^2(x)+6(x-1) \\
\left(\frac{1}{1-x}\right)_+ \otimes \ln^3(x) &= 6\left[\zeta(4)-\mathrm{Li}_4(x)+\zeta(3)\ln(x)+\frac{1}{2}\zeta(2)\ln^2(x)\right] \\
& -\frac{1}{4}\ln^4(x) \qquad (164) \\
\left(\frac{1}{1-x}\right)_+ \otimes x\ln^3(x) &= 6x\Bigg[\zeta(4)-\mathrm{Li}_4(x)+\zeta(3)\ln(x)+\frac{1}{2}\zeta(2)\ln^2(x) \\
& +1-\ln(x)+\frac{1}{2}\ln^2(x)-\frac{1}{6}\ln^3(x)\Bigg] \\
& -6-\frac{x}{4}\ln^4(x) \qquad (165) \\
1\otimes\left(\frac{\ln^3(1-x)}{1-x}\right)_+ &= 6\left[\mathrm{S}_{1,3}(x)-\zeta(4)\right]+\frac{1}{4}\ln^4(1-x) \qquad (166) \\
1\otimes\frac{\ln(x)\ln^2(1-x)}{1-x} &= 2\Bigg[\mathrm{S}_{2,2}(x)+\mathrm{S}_{1,3}(x)-\ln(x)\mathrm{S}_{1,2}(x)-\frac{5}{4}\zeta(4)
\end{aligned}
$$

$$+\frac{1}{6}\ln(x)\ln^3(1-x)\Bigg] \tag{167}$$

$$1 \otimes \frac{\ln^2(x)\ln(1-x)}{1-x} = -2\Bigg[\mathrm{S}_{2,2}(1-x) + \mathrm{S}_{1,3}(1-x) \tag{168}$$
$$-\ln(1-x)\mathrm{S}_{1,2}(1-x) + \frac{1}{6}\ln^3(x)\ln(1-x)\Bigg]$$

$$1 \otimes \frac{\ln^3(x)}{1-x} = -\frac{1}{4}\ln^4(x) - 6\mathrm{S}_{1,3}(1-x) \tag{169}$$

$$1 \otimes \frac{\mathrm{S}_{1,2}(1-x)}{1-x} = \mathrm{S}_{2,2}(1-x) - \ln(x)\mathrm{S}_{1,2}(1-x) - 3\mathrm{S}_{1,3}(1-x) \tag{170}$$

$$1 \otimes \frac{\mathrm{Li}_3(1-x)}{1-x} = -\frac{1}{2}\mathrm{Li}_2^2(1-x) - \ln(x)\mathrm{Li}_3(1-x) + \mathrm{Li}_4(1-x) \tag{171}$$

$$1 \otimes \frac{\ln(x)\mathrm{Li}_2(1-x)}{1-x} = -\frac{1}{2}\mathrm{Li}_2^2(1-x) - \frac{1}{2}\ln^2(x)\mathrm{Li}_2(1-x) + 3\mathrm{S}_{1,3}(1-x) \tag{172}$$

$$1 \otimes \frac{\mathrm{Li}_3(x) - \zeta(3)}{1-x} = \zeta(4) - \mathrm{Li}_4(x) + \zeta(3)\ln(x) + \ln(1-x)[\mathrm{Li}_3(x) - \zeta(3)]$$
$$-\frac{1}{2}[\zeta(2)^2 - \mathrm{Li}_2^2(x)] \tag{173}$$

$$1 \otimes \mathrm{S}_{1,2}(x) = \frac{1}{4}\zeta(4) - \mathrm{S}_{2,2}(x) \tag{174}$$

$$1 \otimes x\mathrm{S}_{1,2}(x) = \zeta(3) - x\mathrm{S}_{1,2}(x)$$
$$-(1-x)\left[1 - \ln(1-x) + \frac{1}{2}\ln^2(1-x)\right] \tag{175}$$

$$1 \otimes \mathrm{Li}_3(1-x) = -\ln(x)\mathrm{Li}_3(1-x) - \frac{1}{2}\mathrm{Li}_2^2(1-x) \tag{176}$$

$$1 \otimes x\mathrm{Li}_3(1-x) = x\ln(x) + (1-x)[1 - \mathrm{Li}_2(1-x) + \mathrm{Li}_3(1-x)] \tag{177}$$

$$1 \otimes \ln(1-x)\mathrm{Li}_2(1-x) = 2\mathrm{S}_{2,2}(1-x) - \frac{1}{2}\mathrm{Li}_2^2(1-x) - 2\ln(1-x)\mathrm{S}_{1,2}(1-x)$$
$$-\ln(x)\ln(1-x)\mathrm{Li}_2(1-x) \tag{178}$$

$$1 \otimes x\ln(1-x)\mathrm{Li}_2(1-x) = -\mathrm{Li}_2(1-x) + (1-x)[\ln(1-x) - 1]\mathrm{Li}_2(1-x)$$
$$+(1-x)[3 - \ln(1-x)] + x\ln(x)[2 - \ln(1-x)] \tag{179}$$

$$1 \otimes \ln(x)\mathrm{Li}_2(1-x) = -\frac{1}{2}\ln^2(x)\mathrm{Li}_2(1-x) + 3\mathrm{S}_{1,3}(1-x) \tag{180}$$

$$1 \otimes x\ln(x)\mathrm{Li}_2(1-x) = 3(1-x) + x\ln(x)[3 - \ln(x)]$$
$$-x[\ln(x) - 1]\mathrm{Li}_2(1-x) - 2\mathrm{S}_{1,2}(1-x) - \mathrm{Li}_2(1-x) \tag{181}$$

$$1 \otimes \mathrm{S}_{1,2}(1-x) = -\ln(x)\mathrm{S}_{1,2}(1-x) - 3\mathrm{S}_{1,3}(1-x) \tag{182}$$

$$1 \otimes x\mathrm{S}_{1,2}(1-x) = (1-x)[\mathrm{S}_{1,2}(1-x) - 1) - x\left[\ln(x) - \frac{1}{2}\ln^2(x)\right] \tag{183}$$

$$1 \otimes \mathrm{Li}_3(x) = \zeta(4) - \mathrm{Li}_4(x) \tag{184}$$

$$1 \otimes x\mathrm{Li}_3(x) = (1-x)[1 - \ln(1-x)] + x[\mathrm{Li}_2(x) - \mathrm{Li}_3(x)]$$
$$+\zeta(3) - \zeta(2) \tag{185}$$

$$1 \otimes \ln(x)\mathrm{Li}_2(x) = \mathrm{Li}_4(x) - \ln(x)\mathrm{Li}_3(x) - \zeta(4) \tag{186}$$

$$1 \otimes x\ln(x)\mathrm{Li}_2(x) = [3 - 2\ln(1-x)](1-x) - \mathrm{Li}_2(1-x) - \zeta(2)$$

$$
\begin{aligned}
& & & +x\left\{\mathrm{Li}_2(x)\left[1-\ln(x)\right]+\ln(x)\left[1-\ln(1-x)\right]\right\} && (187)\\
1\otimes\ln^3(1-x) &=& & -6[\zeta(4)-\mathrm{S}_{1,3}(x)] && (188)\\
1\otimes x\ln^3(1-x) &=& & -(1-x)[6-6\ln(1-x)+3\ln^2(1-x) \\
& & & -\ln^3(1-x)] && (189)\\
1\otimes\ln(x)\ln^2(1-x) &=& & -2\left[\frac{1}{4}\zeta(4)-\mathrm{S}_{2,2}(x)+\ln(x)\mathrm{S}_{1,2}(x)\right] && (190)\\
1\otimes x\ln(x)\ln^2(1-x) &=& & 2[\zeta(3)-\mathrm{S}_{1,2}(x)]+2[\zeta(2)-\mathrm{Li}_2(x)]-2\ln(x) \\
& & & -(1-x)[6-4\ln(1-x)+\ln^2(1-x)] \\
& & & +(1-x)\ln(x)[2-2\ln(1-x)+\ln^2(1-x)] && (191)\\
1\otimes\ln^2(x)\ln(1-x) &=& & -2[\mathrm{S}_{1,3}(1-x)+\frac{1}{6}\ln^3(x)\ln(1-x)] && (192)\\
1\otimes x\ln^2(x)\ln(1-x) &=& & 2\left[\mathrm{S}_{1,2}(1-x)+\mathrm{Li}_2(1-x)\right] \\
& & & +x\ln(x)\left\{-2+\left[2-\ln(x)\right]\left[\ln(1-x)-1\right]\right\} && (193)\\
& & & +2(1-x)\left[\ln(1-x)-3\right] \\
x\otimes\left(\frac{\ln^3(1-x)}{1-x}\right)_+ &=& & 6x\left[\mathrm{S}_{1,3}(x)+\mathrm{S}_{1,2}(x)-\zeta(4)-\zeta(3)\right] \\
& & & +\frac{x}{4}\ln^4(1-x)+(1-x)\ln^3(1-x) && (194)\\
x\otimes\frac{\ln(x)\ln^2(1-x)}{1-x} &=& & 2x\Big[\mathrm{S}_{2,2}(x)+\mathrm{S}_{1,3}(x)-\ln(x)\mathrm{S}_{1,2}(x)-\frac{5}{4}\zeta(4) \\
& & & -\mathrm{S}_{1,2}(1-x)+\mathrm{S}_{1,2}(x)+\frac{1}{6}\ln(x)\ln^3(1-x) && (195)\\
& & & -\frac{1}{2}\ln(x)\ln(1-x)\left[\ln(1-x)-\ln(x)\right]-\zeta(3)\Big] \\
& & & +(1-x)\ln^2(1-x)-2x\left[\mathrm{Li}_2(x)-\zeta(2)\right] \\
& & & +\ln(x)\ln^2(1-x) \\
x\otimes\frac{\ln^2(x)\ln(1-x)}{1-x} &=& & -2x\Big[\mathrm{S}_{2,2}(1-x)+\mathrm{S}_{1,3}(1-x)-\ln(1-x)\mathrm{S}_{1,2}(1-x) \\
& & & +\mathrm{S}_{1,2}(1-x)-\mathrm{Li}_2(1-x)+\frac{1}{6}\ln^3(x)\ln(1-x) && (196)\\
& & & -\frac{1}{6}\ln^3(x)-\frac{1}{2}\ln^2(x)+\ln(1-x)-\ln(x)\Big] \\
& & & +2[1+\ln(x)]\ln(1-x)+\ln^2(x)\ln(1-x) \\
x\otimes\frac{\ln^3(x)}{1-x} &=& & -x\left[\frac{1}{4}\ln^4(x)+6\mathrm{S}_{1,3}(1-x)\right] \\
& & & +6\left[1-x+\ln(x)+\frac{1}{2}\ln^2(x)+\frac{1}{6}\ln^3(x)\right] && (197)\\
x\otimes\frac{\mathrm{S}_{1,2}(1-x)}{1-x} &=& & x\left[\mathrm{S}_{2,2}(1-x)-\ln(x)\mathrm{S}_{1,2}(1-x)-3\mathrm{S}_{1,3}(1-x)\right] \\
& & & +(1-x)\,\mathrm{S}_{1,2}(1-x)+\frac{x}{6}\ln^3(x) && (198)
\end{aligned}
$$

$$x \otimes \frac{\mathrm{Li}_3(1-x)}{1-x} = -x\left[\frac{1}{2}\mathrm{Li}_2^2(1-x) + \ln(x)\mathrm{Li}_3(1-x) - \mathrm{Li}_4(1-x)\right] + (1-x)\mathrm{Li}_3(1-x) + x\ln(x)\mathrm{Li}_2(1-x) + 2x\mathrm{S}_{1,2}(1-x) \tag{199}$$

$$x \otimes \frac{\ln(x)\mathrm{Li}_2(1-x)}{1-x} = x\left[-\frac{1}{2}\mathrm{Li}_2^2(1-x) - \frac{1}{2}\ln^2(x)\mathrm{Li}_2(1-x) + 3\mathrm{S}_{1,3}(1-x)\right] + [1+\ln(x)]\mathrm{Li}_2(1-x) - \frac{x}{3}\ln^3(x) - \frac{x}{2}\ln^2(x) + 2x\mathrm{S}_{1,2}(1-x) - x\mathrm{Li}_2(1-x) \tag{200}$$

$$x \otimes \frac{\mathrm{Li}_3(x) - \zeta(3)}{1-x} = \mathrm{Li}_3(x) + \mathrm{Li}_2(x) - \zeta(3) - x\zeta(2) - x\ln(x) - (1-x)\ln(1-x) + x\Big\{\zeta(4) - \mathrm{Li}_4(x) + \zeta(3)\ln(x) + \ln(1-x)[\mathrm{Li}_3(x) - \zeta(3)] - \frac{1}{2}[\zeta(2)^2 - \mathrm{Li}_2^2(x)]\Big\} \tag{201}$$

$$x \otimes \mathrm{S}_{1,2}(x) = \mathrm{S}_{1,2}(x) - x\zeta(3) + \frac{1}{2}(1-x)\ln^2(1-x) + x[\zeta(2) - \mathrm{Li}_2(x)] \tag{202}$$

$$x \otimes x\mathrm{S}_{1,2}(x) = x\left[\frac{1}{4}\zeta(4) - \mathrm{S}_{2,2}(x)\right] \tag{203}$$

$$x \otimes \mathrm{Li}_3(1-x) = 2x\mathrm{S}_{1,2}(1-x) + (1-x)\mathrm{Li}_3(1-x) + x\ln(x)\mathrm{Li}_2(1-x) \tag{204}$$

$$x \otimes x\mathrm{Li}_3(1-x) = -x\left[\ln(x)\mathrm{Li}_3(1-x) + \frac{1}{2}\mathrm{Li}_2^2(1-x)\right] \tag{205}$$

$$x \otimes \ln(1-x)\mathrm{Li}_2(1-x) = 3x\mathrm{S}_{1,2}(1-x) - \frac{x}{2}\ln^2(x)\ln(1-x) + [x\ln(x) + (1-x)\ln(1-x)]\,\mathrm{Li}_2(1-x) \tag{206}$$

$$x \otimes x\ln(1-x)\mathrm{Li}_2(1-x) = x\Big[2\mathrm{S}_{2,2}(1-x) - \frac{1}{2}\mathrm{Li}_2^2(1-x) - 2\ln(1-x)\mathrm{S}_{1,2}(1-x) - \ln(x)\ln(1-x)\mathrm{Li}_2(1-x)\Big] \tag{207}$$

$$x \otimes \ln(x)\mathrm{Li}_2(1-x) = 2x\mathrm{S}_{1,2}(1-x) + [1-x+\ln(x)]\mathrm{Li}_2(1-x) - \frac{x}{3}\ln^3(x) - \frac{x}{2}\ln^2(x) \tag{208}$$

$$x \otimes x\ln(x)\mathrm{Li}_2(1-x) = -x\left[\frac{1}{2}\ln^2(x)\mathrm{Li}_2(1-x) - 3\mathrm{S}_{1,3}(1-x)\right] \tag{209}$$

$$x \otimes \mathrm{S}_{1,2}(1-x) = (1-x)\mathrm{S}_{1,2}(1-x) + \frac{x}{6}\ln^3(x) \tag{210}$$

$$x \otimes x\mathrm{S}_{1,2}(1-x) = x\left[-\ln(x)\mathrm{S}_{1,2}(1-x) - 3\mathrm{S}_{1,3}(1-x)\right] \tag{211}$$

$$x \otimes \mathrm{Li}_3(x) = \mathrm{Li}_3(x) + \mathrm{Li}_2(x) - x[\zeta(3) + \zeta(2)] - x\ln(x)$$

$$-(1-x)\ln(1-x) \tag{212}$$

$$x \otimes x\mathrm{Li}_3(x) = x\left[\zeta(4) - \mathrm{Li}_4(x)\right] \tag{213}$$

$$\begin{aligned} x \otimes \ln(x)\mathrm{Li}_2(x) &= -x\mathrm{Li}_2(1-x) + [1+\ln(x)]\mathrm{Li}_2(x) - x\zeta(2) \\ &\quad - \ln(1-x)[\ln(x)+2] \\ &\quad -2x\left[\ln(x) - \ln(1-x) + \frac{1}{4}\ln^2(x)\right] \end{aligned} \tag{214}$$

$$x \otimes x\ln(x)\mathrm{Li}_2(x) = x\left[\mathrm{Li}_4(x) - \ln(x)\mathrm{Li}_3(x) - \zeta(4)\right] \tag{215}$$

$$x \otimes \ln^3(1-x) = 6x[\mathrm{S}_{1,2}(x) - \zeta(3)] + (1-x)\ln^3(1-x) \tag{216}$$

$$x \otimes x\ln^3(1-x) = -6x\left[\zeta(4) - \mathrm{S}_{1,3}(x)\right] \tag{217}$$

$$\begin{aligned} x \otimes \ln(x)\ln^2(1-x) &= [1-x+\ln(x)]\ln^2(1-x) + 2x[\zeta(2) - \mathrm{Li}_2(x)] \\ &\quad -2x\Bigg[\zeta(3) - \mathrm{Li}_3(x) + \ln(x)\mathrm{Li}_2(x) + \mathrm{Li}_3(1-x) \\ &\quad - \ln(1-x)\mathrm{Li}_2(1-x)\Bigg] \end{aligned} \tag{218}$$

$$x \otimes x\ln(x)\ln^2(1-x) = -2x\left[\frac{1}{4}\zeta(4) - \mathrm{S}_{2,2}(x) + \ln(x)\mathrm{S}_{1,2}(x)\right] \tag{219}$$

$$\begin{aligned} x \otimes \ln^2(x)\ln(1-x) &= 2x\left[\mathrm{Li}_2(1-x) - \mathrm{S}_{1,2}(1-x)\right] \\ &\quad + \left[2[1-x+\ln(x)] + \ln^2(x)\right]\ln(1-x) \\ &\quad + x\ln(x)\left[2 + \ln(x) + \frac{1}{3}\ln^2(x)\right] \end{aligned} \tag{220}$$

$$x \otimes x\ln^2(x)\ln(1-x) = -2x[\mathrm{S}_{1,3}(1-x) + \frac{1}{6}\ln^3(x)\ln(1-x)] \tag{221}$$

For some applications one wishes to give explicit account for soft gluon or soft photon resummation using $x$–space formulae. Here one has to perform convolutions of the type

$$\left[\frac{\ln^k(1-x)}{1-x}\right]_+ \otimes \left[\frac{\ln^l(1-x)}{1-x}\right]_+ = D_k(x) \otimes D_l(x), \quad k,l \geq 0\ . \tag{222}$$

The Mellin convolution of two $(...)_+$–distributions is again a $(...)_+$–distribution. The convolutions $D_k \otimes D_0, k\ \epsilon\ [0,3]$ are given in Eqs. (69,72,104) and (131) above and can easily be extended to general values of $k$ and $l$ using relations given in Ref. [21]. The Mellin transform of $D_k(x)$ reads

$$\boldsymbol{M}[D_k(x)](N) = (-1)^k(k-1)!\underbrace{S_{1,\ldots,1}}_{k}(N-1)\ . \tag{223}$$

The harmonic sum in (223) obeys a determinant–representation, Eq. (158) [21], and is a polynomial of only single harmonic sums. One obtains the Mellin transform of $[\ln(x)\ln^{(k)}(1-x)/(1-x)]_+$ by the relation

$$\boldsymbol{M}\left\{\left[\ln(x)\frac{\ln^k(1-x)}{1-x}\right]_+\right\}(N) = \frac{\partial}{\partial N}(-1)^k(k-1)!\underbrace{S_{1,\ldots,1}}_{k}(N-1)\ . \tag{224}$$

Since the single harmonic sums can be represented by Euler's $\psi$–function and its derivatives the differentiation for $N$ is performed easily. Furthermore, one may re-express the result again in a polynomial of single harmonic sums.

# 9 Appendix B: Mellin Transforms

Some of the convolutions calculated in the previous section were determined using harmonic sums and Mellin transformations of specific functions, such as Nielsen–integrals. The Mellin transform of a convolution is given by

$$\boldsymbol{M}\left\{[A\otimes B](x)\right\}(N) = \boldsymbol{M}\left\{[A](x)\right\}(N)\cdot\boldsymbol{M}\left\{[B](x)\right\}(N)\ . \tag{225}$$

Below we list Mellin transforms which were used and were not contained in Ref. [21].

$$\boldsymbol{M}[\ln^p(1-x)](N) = \frac{1}{N}(-1)^p p! S_{\underbrace{1,\ldots,1}_{p}}(N) \tag{226}$$

$$\begin{aligned}\boldsymbol{M}[\ln^4(1-x)](N) &= \frac{1}{N}\Big[S_1^4(N)+6S_1^2(N)S_2(N)+3S_2^2(N)+8S1(N)S_3(N)\\ &\quad +6S_4(N)\Big]\end{aligned} \tag{227}$$

$$\boldsymbol{M}[\mathrm{S}_{1,p}(x)](N) = \frac{1}{N}\left\{\zeta(p)-\frac{1}{N}S_{\underbrace{1,\ldots,1}_{p}}(N)\right\} \tag{228}$$

$$\boldsymbol{M}[\mathrm{S}_{1,p}(1-x)](N) = -\frac{1}{N}\left[S_p(N)-\zeta(p)\right] \tag{229}$$

$$\boldsymbol{M}[\mathrm{S}_{1,3}(x)](N) = \frac{1}{N}\left\{\zeta(4)-\frac{1}{6N}\left[S_1^3(N)+3S_1(N)S_2(N)+2S_3(N)\right]\right\} \tag{230}$$

$$\boldsymbol{M}[\mathrm{S}_{1,3}(1-x)](N) = -\frac{1}{N}\left[S_4(N)-\zeta(4)\right] \tag{231}$$

$$\begin{aligned}\boldsymbol{M}[\mathrm{S}_{1,4}(x)](N) &= \frac{1}{N}\Bigg\{\zeta(5)-\frac{1}{24N}\Big[S_1^4(N)+6S_1^2(N)S_2(N)+8S_1(N)S_3(N)\\ &\quad +3S_2^2(N)+6S_4(N)\Big]\Bigg\}\end{aligned} \tag{232}$$

$$\boldsymbol{M}[\mathrm{S}_{1,4}(1-x)](N) = -\frac{1}{N}\left[S_5(N)-\zeta(5)\right] \tag{233}$$

$$\boldsymbol{M}[\mathrm{S}_{2,2}(x)](N) = \frac{\zeta(4)}{4N}-\frac{\zeta(3)}{N^2}+\frac{S_1^2(N)+S_2(N)}{2N^3} \tag{234}$$

$$\boldsymbol{M}[\mathrm{S}_{2,2}(1-x)](N) = \frac{1}{N}\left[\frac{1}{4}\zeta(4)+S_{1,3}(N)-\zeta(3)S_1(N)\right] \tag{235}$$

$$\begin{aligned}\boldsymbol{M}\left[\frac{\mathrm{S}_{2,2}(1-x)}{1-x}\right](N) &= 2\zeta(5)-\zeta(2)\zeta(3)-\frac{1}{4}\zeta(4)S_1(N-1)-S_{1,1,3}(N-1)\\ &\quad +\zeta(3)S_{1,1}(N-1)\end{aligned} \tag{236}$$

$$\begin{aligned}\boldsymbol{M}\left[\mathrm{Li}_2^2(x)\right](N) &= \frac{1}{N}\left\{\frac{5}{2}\zeta(4)-4\frac{\zeta(3)}{N}-2\zeta(2)\frac{S_1(N)}{N}+2\frac{S_{2,1}(N)}{N}\right\}\\ &\quad +\frac{2}{N^3}\Big[S_1^2(N)+S_2(N)\Big]\end{aligned} \tag{237}$$

$$\begin{aligned}\boldsymbol{M}\left[\ln(1-x)\mathrm{Li}_3(x)\right](N) &= \frac{1}{N}\Bigg[-\frac{1}{2}\zeta(2)^2+\frac{2\zeta(3)}{N}+\zeta(2)\frac{S_1(N)}{N}-\frac{S_1^2(N)+S_2(N)}{N^2}\\ &\quad -\frac{S_{2,1}(N)}{N}-\zeta(3)S_1(N)+\zeta(2)S_2(N)-S_{3,1}(N)\Bigg]\end{aligned} \tag{238}$$

$$\boldsymbol{M}\left[\ln(x)\mathrm{Li}_3(x)\right](N) = -\frac{1}{N^2}\left[\zeta(3) - \frac{3\zeta(2)}{N} + \frac{S_2(N)}{N} + \frac{3S_1(N)}{N^2}\right] \tag{239}$$

$$\boldsymbol{M}\left[\ln(x)\mathrm{S}_{1,2}(1-x)\right](N) = \frac{1}{N}\left\{3S_4(N) - 3\zeta(4) + \frac{1}{N}\left[S_3(N) - \zeta(3)\right]\right\} \tag{240}$$

$$\begin{aligned}\boldsymbol{M}\left[\ln(x)\mathrm{S}_{1,2}(x)\right](N) = \frac{1}{N^2}\Bigg\{& S_1(N)S_2(N) + S_3(N) - \zeta(2)S_1(N) - 2\zeta(3) \\ &+ \frac{1}{N}\left[S_1^2(N) + S_2(N)\right]\Bigg\}\end{aligned} \tag{241}$$

$$\boldsymbol{M}\left[\mathrm{Li}_4(x)\right](N) = \frac{\zeta(4)}{N} - \frac{\zeta(3)}{N^2} + \frac{\zeta(2)}{N^3} - \frac{S_1(N)}{N^4} \tag{242}$$

$$\begin{aligned}\boldsymbol{M}\left[\ln^2(x)\mathrm{Li}_2(1-x)\right](N) = \frac{1}{N}\Bigg\{& 6[\zeta(4) - S_4(N)] + \frac{4}{N}[\zeta(3) - S_3(N)] \\ &+ \frac{2}{N^2}[\zeta(2) - S_2(N)]\Bigg\}\end{aligned} \tag{243}$$

$$\boldsymbol{M}\left[\frac{\mathrm{S}_{2,2}(x) - \zeta(4)/4}{1-x}\right](N) = -S_{3,1,1}(N-1) + \zeta(3)S_2(N-1) - \frac{\zeta(5)}{2} \tag{244}$$

$$\boldsymbol{M}\left[\frac{\mathrm{S}_{1,3}(x) - \zeta(4)}{1-x}\right](N) = S_{2,1,1,1}(N-1) - 4\zeta(5) \tag{245}$$

$$\boldsymbol{M}\left[\frac{\mathrm{S}_{1,3}(1-x)}{1-x}\right](N) = S_{1,4}(N-1) - \zeta(4)S_1(N-1) + 2\zeta(5) - \zeta(2)\zeta(3) \tag{246}$$

# 10 Appendix C: Representation of Sums

To obtain the inverse Mellin transforms of more complicated convolutions back in $x$–space products of harmonic sums and powers of $1/N$ have to be expressed as Mellin transforms. A series of representations has been given in [21] before. Here we list additional representations which were used in the present calculation.

$$\frac{S_1(N)}{N} = -\boldsymbol{M}[\ln(1-x)](N) \tag{247}$$

$$\frac{S_1(N)}{N^2} = -\boldsymbol{M}[\mathrm{Li}_2(x) - \zeta(2)](N) \tag{248}$$

$$\frac{S_1(N)}{N^3} = \boldsymbol{M}[\mathrm{Li}_3(x) - \zeta(2)\ln(x) - \zeta(3)](N) \tag{249}$$

$$\frac{S_1(N)}{N^4} = -\boldsymbol{M}\left[\mathrm{Li}_4(x) - \frac{\zeta(2)}{2}\ln^2(x) - \zeta(3)\ln(x) - \zeta(4)\right](N) \tag{250}$$

$$\frac{S_2(N)}{N} = -\boldsymbol{M}[\mathrm{Li}_2(1-x) - \zeta(2)](N) \tag{251}$$

$$\frac{S_2(N)}{N^2} = \boldsymbol{M}[2\mathrm{S}_{1,2}(1-x) + \ln(x)[\mathrm{Li}_2(1-x) - \zeta(2)]](N) \tag{252}$$

$$\frac{S_2(N)}{N^3} = -\boldsymbol{M}\left[3\mathrm{S}_{1,3}(1-x) + 2\ln(x)\mathrm{S}_{1,2}(1-x) + \frac{1}{2}\ln^2(x)[\mathrm{Li}_2(1-x) - \zeta(2)]\right](N) \tag{253}$$

$$\frac{S_3(N)}{N} = -\boldsymbol{M}[\mathrm{S}_{1,2}(1-x)-\zeta(3)](N) \tag{254}$$

$$\frac{S_3(N)}{N^2} = \boldsymbol{M}[3\mathrm{S}_{1,3}(1-x)+\ln(x)\mathrm{S}_{1,2}(1-x)-\zeta(3)\ln(x)](N) \tag{255}$$

$$\frac{S_4(N)}{N} = -\boldsymbol{M}[\mathrm{S}_{1,3}(1-x)-\zeta(4)](N) \tag{256}$$

$$\frac{S_1^2(N)}{N} = \boldsymbol{M}[\ln^2(1-x)+\mathrm{Li}_2(1-x)-\zeta(2)](N) \tag{257}$$

$$\frac{S_1^2(N)}{N^2} = -\boldsymbol{M}[2\mathrm{S}_{1,2}(x)+2\mathrm{S}_{1,2}(1-x)+\ln(x)[\mathrm{Li}_2(1-x)-\zeta(2)]-2\zeta(3)](N) \tag{258}$$

$$\begin{aligned}\frac{S_1^2(N)}{N^3} &= \boldsymbol{M}\left[2\mathrm{S}_{2,2}(x)-\frac{\zeta(4)}{2}-2\zeta(3)\ln(x)\right](N)\\ &+\boldsymbol{M}\left[3\mathrm{S}_{1,3}(1-x)+2\ln(x)\mathrm{S}_{1,2}(1-x)+\frac{1}{2}\ln^2(x)[\mathrm{Li}_2(1-x)-\zeta(2)]\right](N)\end{aligned} \tag{259}$$

$$\begin{aligned}\frac{S_2^2(N)}{N} &= \boldsymbol{M}\left[4\mathrm{S}_{22}(x)+\mathrm{S}_{13}(1-x)-4\ln(x)\mathrm{S}_{1,2}(x)-2\ln(x)\ln(1-x)\mathrm{Li}_2(x)\right]\\ &+\boldsymbol{M}\left[4\ln(1-x)\mathrm{Li}_3(x)+\mathrm{Li}_2^2(x)-4\zeta(3)\ln(1-x)-\zeta(4)\right]\end{aligned} \tag{260}$$

$$\frac{S_2(N)S_1(N)}{N} = \boldsymbol{M}\left[\mathrm{S}_{1,2}(1-x)-\mathrm{Li}_3(1-x)+\ln(1-x)[\mathrm{Li}_2(1-x)-\zeta(2)]\right](N) \tag{261}$$

$$\begin{aligned}\frac{S_2(N)S_1(N)}{N^2} &= \boldsymbol{M}\left[\ln(x)[\mathrm{S}_{1,2}(x)-\mathrm{S}_{1,2}(1-x)]-3\mathrm{S}_{1,3}(1-x)\right](N)\\ &-\boldsymbol{M}\left[2\mathrm{S}_{2,2}(x)+\zeta(2)\mathrm{Li}_2(x)-\zeta(3)\ln(x)-3\zeta(4)\right](N)\end{aligned} \tag{262}$$

$$\begin{aligned}\frac{S_3(N)S_1(N)}{N} &= \boldsymbol{M}\left[\mathrm{S}_{2,2}(1-x)+\mathrm{S}_{1,3}(1-x)-\frac{1}{2}\mathrm{Li}_2^2(1-x)\right](N)\\ &+\boldsymbol{M}\left[[\ln(1-x)[\mathrm{S}_{1,2}(1-x)-\zeta(3)]]\right](N)\end{aligned} \tag{263}$$

$$\begin{aligned}\frac{S_2(N)S_1^2(N)}{N} &= \boldsymbol{M}\left[-4\mathrm{S}_{22}(x)-2\mathrm{S}_{22}(1-x)+2\mathrm{S}_{13}(x)-\mathrm{S}_{13}(1-x)+4\ln(x)\mathrm{S}_{1,2}(x)\right]\\ &+\boldsymbol{M}\left[-2\ln(1-x)\mathrm{S}_{1,2}(1-x)-4\ln(1-x)\mathrm{Li}_3(x)+\mathrm{Li}_2^2(1-x)-\mathrm{Li}_2^2(x)\right]\\ &+\boldsymbol{M}\left[2\ln(x)\ln(1-x)\mathrm{Li}_2(x)+\frac{1}{3}\ln(x)\ln^3(1-x)+\zeta(2)\ln^2(1-x)\right]\\ &+\boldsymbol{M}\left[4\zeta(3)\ln(1-x)-\zeta(4)\right]\end{aligned} \tag{264}$$

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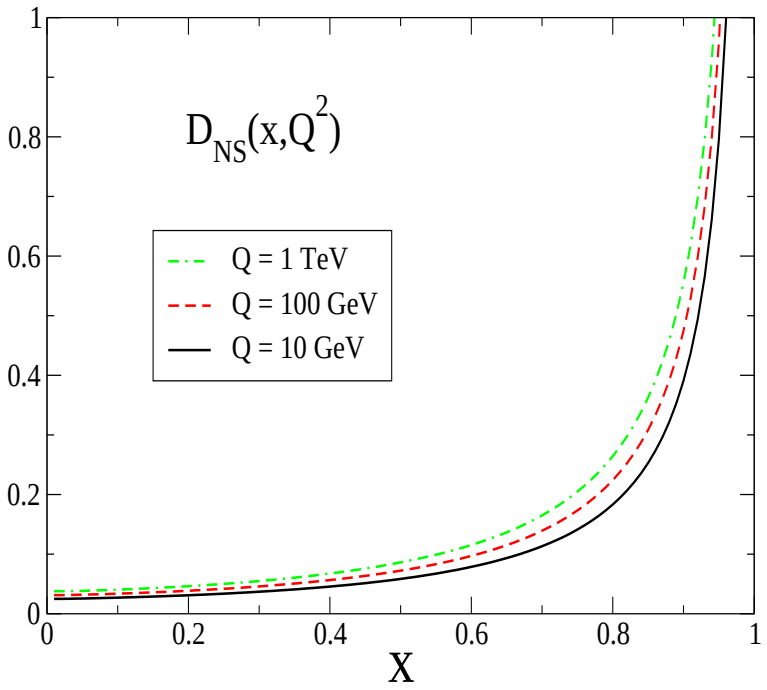


Figure 1: The non–singlet radiation function $D_{\rm NS}(x,Q^2)$ to $O((\alpha L)^5)$ as a function of $x$ for $Q = 10\,\text{GeV}, 100\,\text{GeV}$ and $1\,\text{TeV}$.

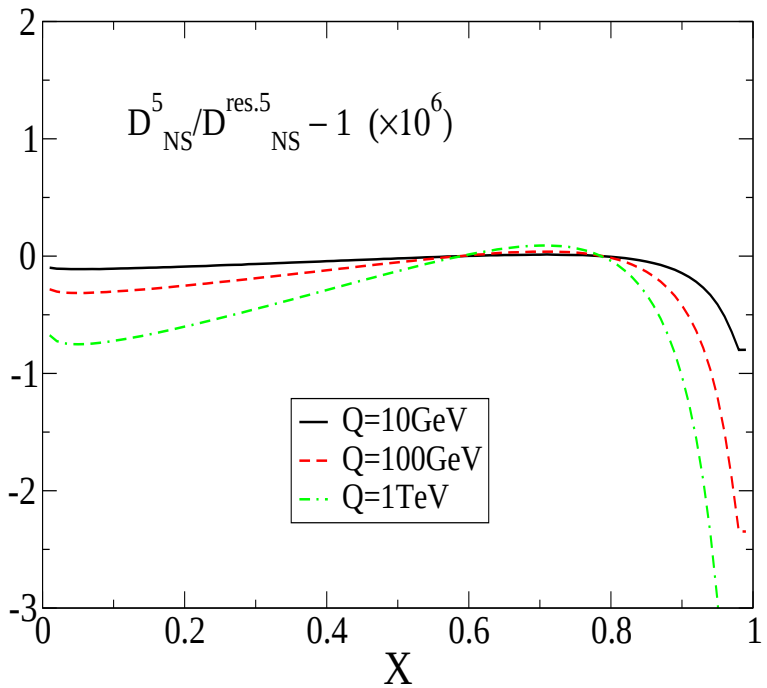


Figure 2: The effect of the resummed contributions beyond $O((\alpha L)^5)$ compared to the contributions up to $O((\alpha L)^5)$ for $D_{\rm NS}(x,Q^2)$ as a function of $x$ and $Q$.

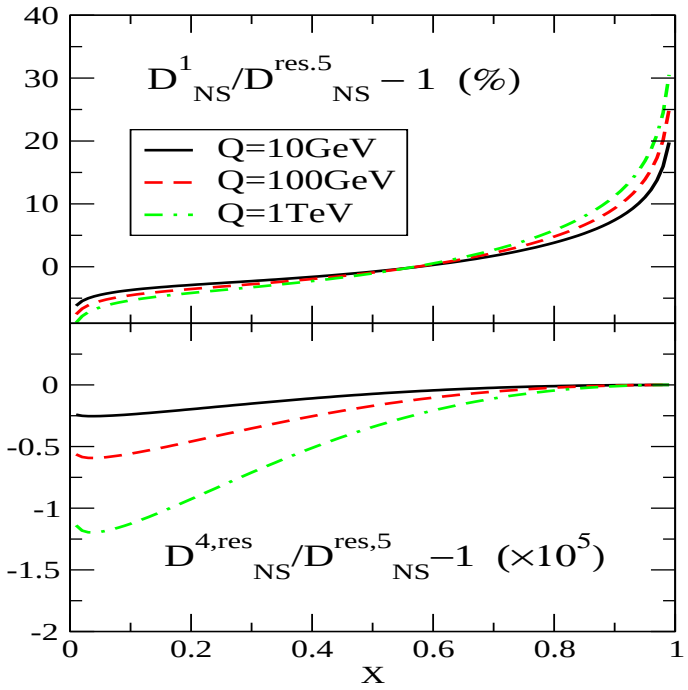


Figure 3: Relative contribution of the first order non-singlet radiator compared to all terms up to $O((\alpha L)^5)$ as a function of $x$ and $Q$ and relative correction of all contributions up to $O((\alpha L)^4)$ in comparison to the terms up to $O((\alpha L)^5)$.

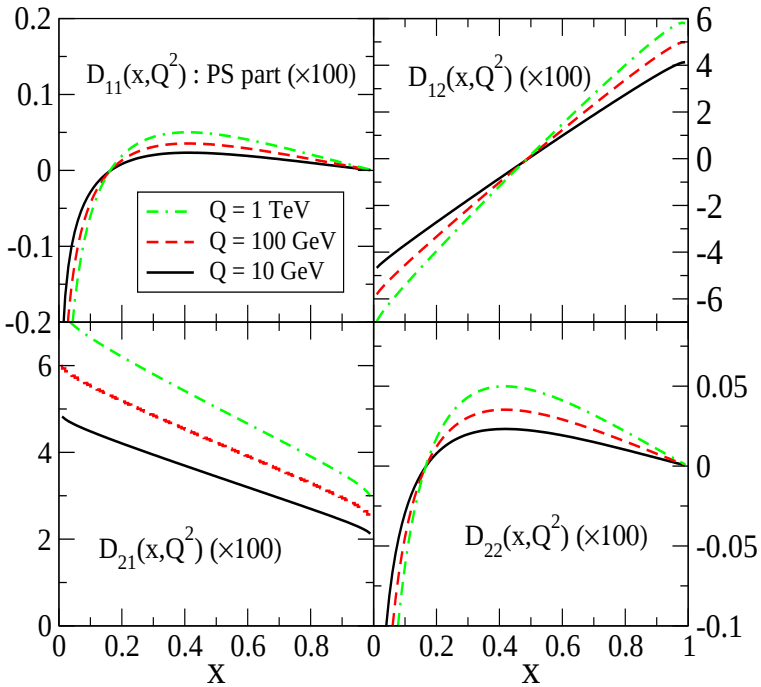


Figure 4: The polarized singlet contributions $D_{ij}$ as a function of $x$ and $Q$ in %. Here $D_{11}$ denotes the pure singlet term, which has to be added to the non–singlet contribution.

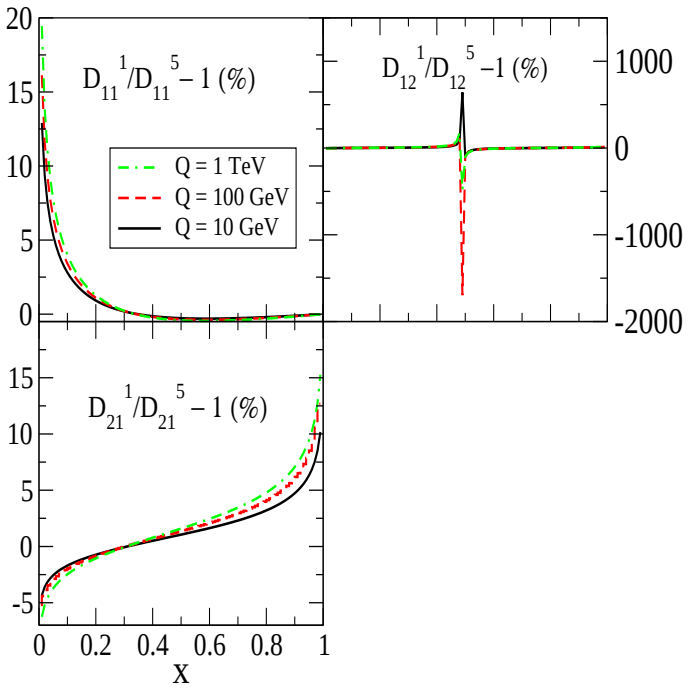


Figure 5: Relative contribution of the first order singlet radiators $D_{ij}^1$ in the terms to $O((\alpha L)^5)$.

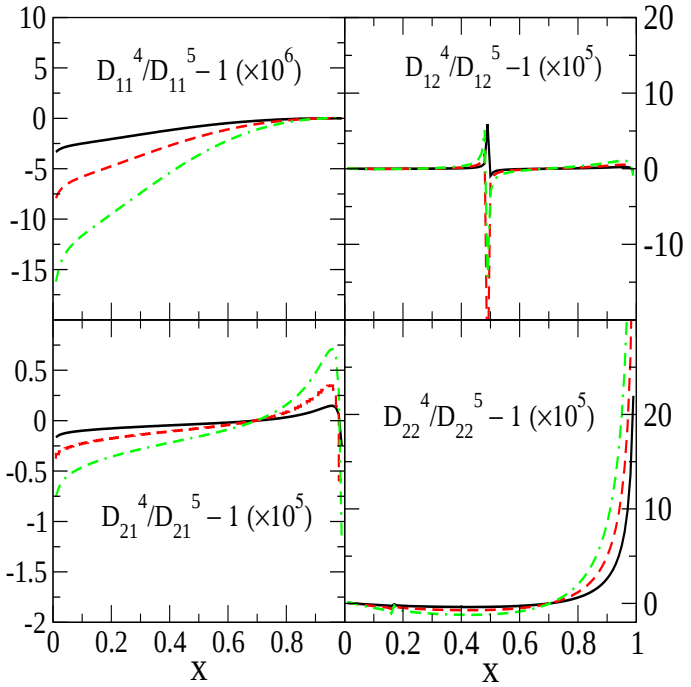


Figure 6: Relative contribution of all terms of the singlet radiators $D_{ij}$ to the 4th order in $\alpha L$ if compared to all terms to 5th order as a function of $x$ and $Q$.

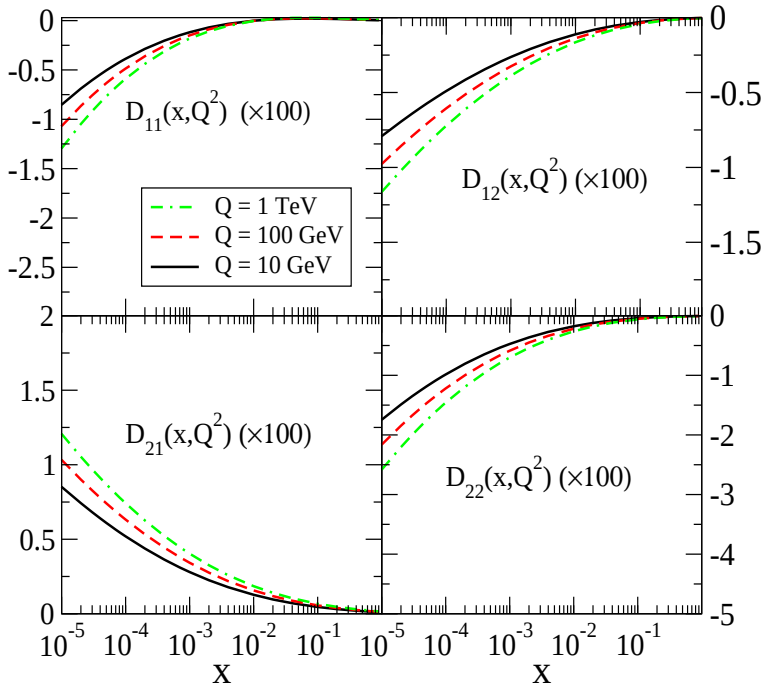


Figure 7: The polarized singlet contributions $D_{ij}(x,Q^2)$ corresponding to the resummation of the $O(\alpha \ln^2(x))$ terms as a function of $x$ and $Q$ in % starting with $O(\alpha^2)$. Here $D_{11}$ contains also the non–singlet contribution.

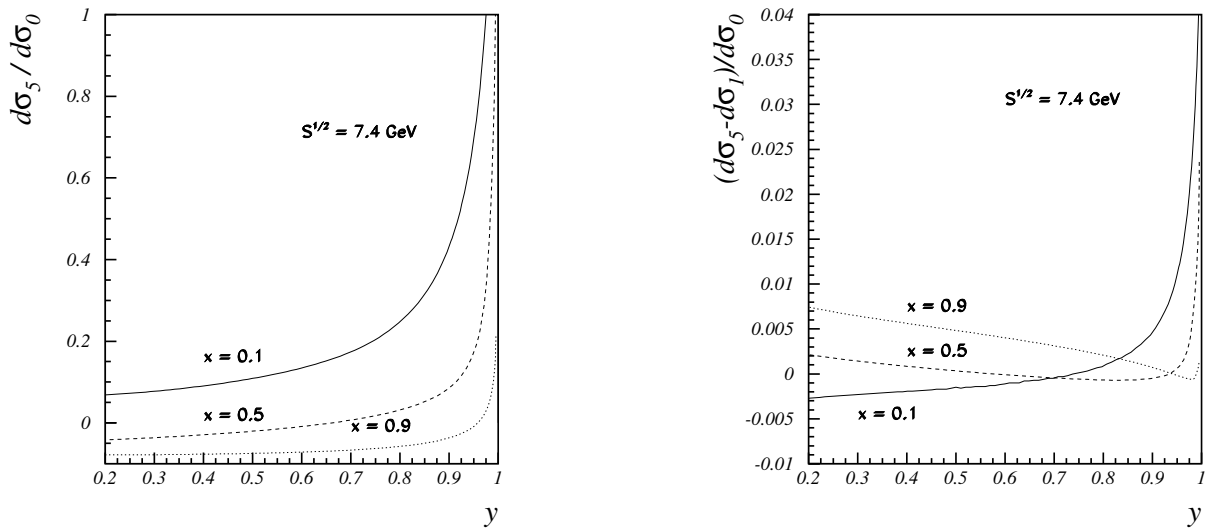


Figure 8: The QED radiative initial state correction to the differential cross section of polarized deep–inelastic lepton–proton scattering including the $O((\alpha L)^5)$ corrections as a function of $x$ and $y$ in a fixed target experiment ($E_e = 27.5\,\mathrm{GeV}$). Left : the correction factor. Right : the contributions due to non–leading orders.

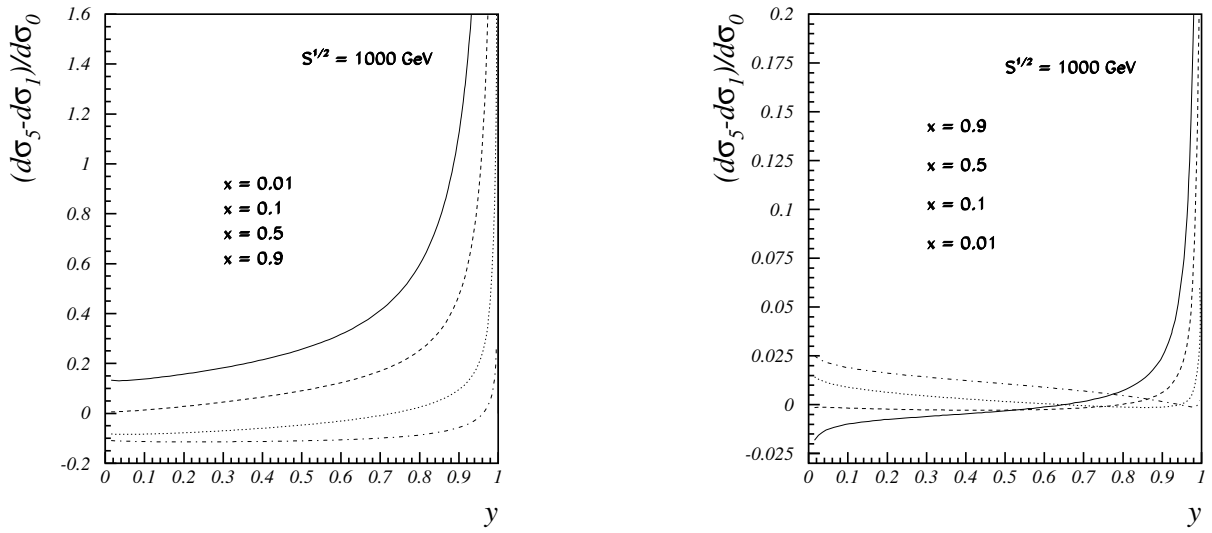


Figure 9: The QED radiative initial state correction to the differential cross section of polarized deep–inelastic lepton–proton scattering including the $O((\alpha L)^5)$ corrections as a function of $x$ and $y$ for a collider experiment at $S = 1\,\mathrm{TeV}^2$. Left : the correction factor. Right : the contributions due to non–leading orders.

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