The QCD equation of state at nonzero densities: lattice result

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Abstract

In this letter we give the equation of state of QCD at finite temperatures and densities. The recently proposed overlap improving multi-parameter reweighting technique is used to determine observables at nonvanishing chemical potentials. Our results are obtained by studying n_f=2+1 dynamical staggered quarks with semi-realistic masses on $N_t = 4$ lattices.

According to the standard picture of QCD, at high temperatures and/or high densities there is a change from a state dominated by hadrons to a state dominated by partons. This transition happened in the early Universe (at essentially vanishing density) and probably happens in heavy ion collisions (at moderate but non-vanishing density) and in neutron stars (at large density, for which a rich phase structure is conjectured [1, 2, 3, 4]). There are well established nonperturbative lattice techniques to study this transition at vanishing density, at which the equation of state were determined as a function of the temperature. Due to the sign problem (oscillating signs lead to cancellation in results, a phenomena which appears in many fields of physics) nothing could have been said about the experimentally/observationally relevant case at non-vanishing densities. By using our recently proposed lattice technique [5] we calculate the equation of state at non-vanishing temperatures and densities. In this letter we present the technique and show the results. A detailed version of the present work will be published elsewhere [6].

QCD at nonzero density can be studied by introducing the chemical potential (μ). Such a theory is easily formulated on space-time lattice [7, 8, 9]. However; standard, importance sampling based Monte-Carlo techniques can not be used at $\mu \neq 0$. Up to now, no technique was suggested capable of giving the equation of state (EOS) at non-vanishing μ, which would be essential to describe the quark gluon plasma (QGP) formation at heavy ion collider experiments. Results are only available for μ=0 [10, 11, 12] at non-vanishing temperatures (T) (a recent review is Ref. [13]).

The overlap improving multi-parameter reweighting [5] opened the possibility to study lattice QCD at nonzero T and μ. First one produces an ensemble of QCD configurations at μ=0 and at T$\neq$0. Then the Ferrenberg-Swendsen type reweighting factors [14] of these configurations are determined at $\mu \neq 0$ and at a lowered T. The idea can be easily expressed in terms of the partition function

$$Z(\mu,\beta) = \int \mathcal{D}U \exp[-S_g(\beta,U)] \det M(\mu,m,U) = \int \mathcal{D}U \exp[-S_g(\beta_0,U)] \det M(\mu=0,m,U) \left\{ \exp[-S_g(\beta,U)+S_g(\beta_0,U)] \frac{\det M(\mu,m,U)}{\det M(\mu=0,m,U)} \right\}, \quad (1)$$

where S_g is the action of the gluonic field (U), $\beta = 6/g^2$ fixes the coupling of the strong interactions (g). Note that for a given lattice T is an increasing function of β. The quark mass parameter is m and $\det M$ comes from the integration over the quark fields. At nonzero μ one gets a complex $\det M$ which has no probability interpretation, thus it spoils any importance sampling. Therefore, the first expression of eq. (1), $\mu \neq$0, is rewritten in a way that the second line of eq. (1) is used as an integration measure (at μ=0, for which importance sampling works) and the remaining part in the curly bracket is measured on each independent configuration and interpreted as a weight factor $\{w(\beta,\mu,m,U)\}$. In order to maximise the accuracy of the reweighting is performed along the best weight lines on the μ–β plane (or equivalently on the μ–T plane). These best weight lines are defined by minimising the spread of $\log w$.

Similar reweighting can be done in the mass parameter, too. Using the above technique, transition (or hadronic/QGP) configurations are reweighted to transition (or hadronic/QGP) configurations as illustrated by Fig. 1; thus, a much better overlap can be obtained than by reweighting pure hadronic configurations to transition ones as done by single μ-reweighting [15]. As we emphasised, the technique works for temperatures at, below and above the transition temperature (T_c). By using the reweighting technique, the phase diagram [5] and the location of the critical endpoint [16] was given (for other approaches see e.g. [17, 18]). Using a Taylor expansion around μ=0, T$\neq$0 for small μ is a variant of our multi-parameter reweighting

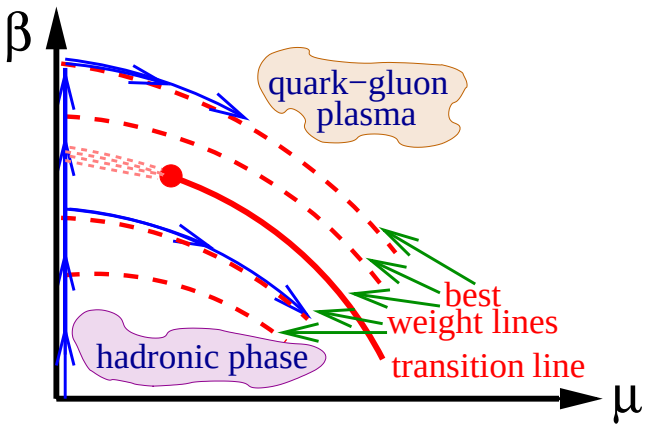

Figure 1: The best weight lines on the μ–β plane, along which reweighting is performed. In the middle we indicate the transition line. Its first dotted part is the crossover region. The blob represents the critical endpoint, after which the transition is of first order. Below the transition line the system is in the hadronic phase, above the transition line we find the QGP. The integration paths used to calculate the pressure are shown by the arrows along the $\mu = 0$ axis and the best weight lines.

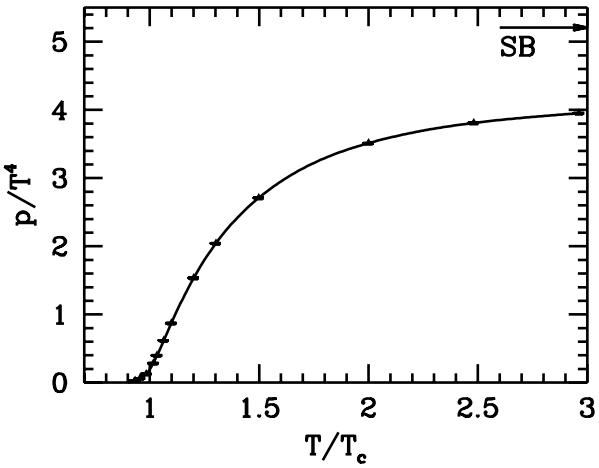

Figure 2: The pressure normalised by T^4 as a function of T/T_c at $\mu = 0$ (to help the continuum interpretation the raw lattice result is multiplied with c_p). The continuum SB limit is also shown.

method, which can be used to determine thermal properties [19]. A completely different method, analytic continuation from imaginary μ, confirmed the result of the reweighting technique on the phase diagram [20].

In the present analysis we use $N_t \cdot N_s^3$ finite T lattices with N_t=4 and N_s=8,10,12 for reweighting and we extrapolate to the thermodynamical limit using the available volumes (V). At T=0 lattices of $24 \cdot 14^3$ are taken for vacuum subtraction and to connect lattice parameters to physical quantities. 16 different β values are used, which correspond to $T/T_c = 0.8, \ldots, 3$ and at T_c to a lattice spacing (a) of ≈0.28 fm. We use 2+1 flavours of dynamical staggered quarks. Only μ values of the light quarks are studied. While varying β (thus the temperature) we keep the physical quark masses constant at $m_{ud} \approx 65$ MeV and $m_s \approx 135$ MeV (the pion to rho mass ratio is m_π/m_ρ ≈0.66). From now on we usually omit the different quark mass indices.

The determination of the equation of state at $\mu \neq 0$ needs several observables $\mathcal{O}$ at non-vanishing μ values. This can be calculated by using the weights of eq. (1)

$$\overline{\mathcal{O}}(\beta, \mu, m) = \frac{\sum\{w(\beta, \mu, m, U)\}\mathcal{O}(\beta, \mu, m, U)}{\sum\{w(\beta, \mu, m, U)\}}. \tag{2}$$

We use the following notation for subtracting the vacuum term: $\langle \mathcal{O}(\beta, \mu, m) \rangle = \overline{\mathcal{O}}(\beta, \mu, m)_{T \neq 0} - \overline{\mathcal{O}}(\beta, \mu = 0, m)_{T=0}$.

The pressure (p) can be obtained from the partition function as $p = T \cdot \partial \log Z / \partial V$ which can be written as $p = (T/V) \cdot \log Z$ for large homogeneous systems. On the lattice we can only determine the derivatives of $\log Z$ with respect to the parameters of the action (β, m, μ), so p can be written as an integral[21]:

$$\frac{p}{T^4} = \frac{1}{T^3 V} \int d(\beta, m, \mu) \left(\left\langle \frac{\partial(\log Z)}{\partial \beta} \right\rangle, \left\langle \frac{\partial(\log Z)}{\partial m} \right\rangle, \left\langle \frac{\partial(\log Z)}{\partial \mu} \right\rangle \right). \tag{3}$$

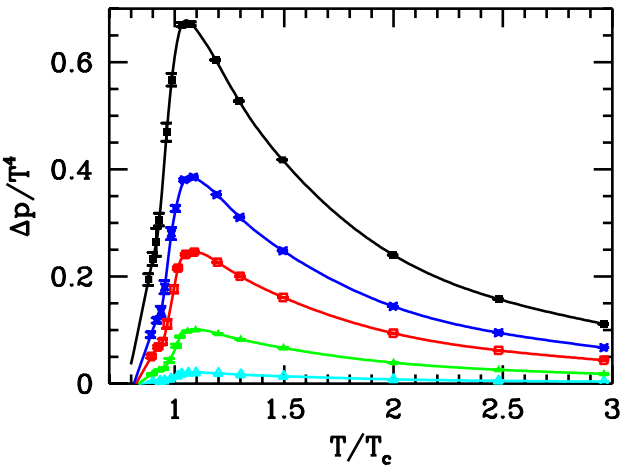

Figure 3: $\Delta p = p(\mu \neq 0, T) - p(\mu = 0, T)$ normalised by T^4 as a function of T/T_c for μ_B=100,210,330,410 MeV and μ_B=530 MeV (from top to bottom; to help the continuum interpretation the raw lattice result is multiplied with c_μ).

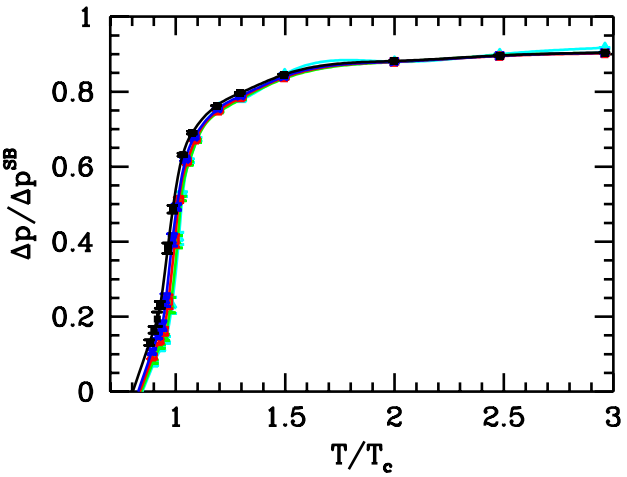

Figure 4: Δp of the interacting QCD plasma normalised by Δp of the free gas (SB) as a function of T/T_c for μ_B=100,210,330,410 MeV and μ_B=530 MeV. The result is essentially μ-independent.

The integral is by definition independent of the integration path. The chosen integration paths are shown on Fig 1.

The energy density can be written as $\epsilon = (T^2/V) \cdot \partial(\log Z)/\partial T + (\mu T/V) \cdot \partial(\log Z)/\partial \mu$. By changing the lattice spacing T and V are simultaneously varied. The special combination $\epsilon - 3p$ contains only derivatives with respect to a and μ:

$$\frac{\epsilon - 3p}{T^4} = -\frac{a}{T^3 V} \frac{\partial \log(Z)}{\partial a}\bigg|_\mu + \frac{\mu}{T^3 V} \frac{\partial(\log Z)}{\partial \mu}\bigg|_a . \tag{4}$$

The quark number density is $n = (T/V) \cdot \partial \log(Z)/\partial \mu$ which can be measured either directly or obtained from the pressure (note, that the baryon density is n_B=$n/3$ and the baryonic chemical potential is μ_B=3μ).

We present lattice results on $p(\mu = 0, T)$, $\Delta p(\mu, T) = p(\mu \neq 0, T) - p(\mu = 0, T)$, $\epsilon(\mu, T)$-$3p(\mu, T)$ and $n_B(\mu, T)$. Our statistical errorbars are also shown. They are rather small, in many cases they are even smaller than the thickness of the lines. On the figures we multiply the lattice results with the dominant correction factors between N_t=4 and the continuum in the $T \to \infty$ case: $c_p = p(\mu = 0, T \to \infty, \text{continuum})/\text{p}(\mu = 0, \text{T} \to \infty, \text{N}_\text{t} = 4)$=0.518 and $c_\mu = \Delta p(\mu, T \to \infty, \text{continuum})/\Delta\text{p}(\mu, \text{T} \to \infty, \text{N}_\text{t} = 4)$=0.446. The well-known continuum expressions in the $T \to \infty$ Stefan-Boltzmann (SB) case are $p(\mu = 0, T \to \infty, \text{continuum}) = (16 + 21\text{n}_\text{f}/2)\pi^2\text{T}^4/90$ and $\Delta p(\mu, T \to \infty, \text{continuum}) = \text{n}_\text{f}\mu^2\text{T}^2/2 + \mathcal{O}(\mu^4)$. This way the results presented on the figures might be interpreted as continuum estimates and could be directly used in phenomenological applications. Direct N_t=4 lattice results can be obtained by dividing the presented values with these c_p , c_μ correction factors.

Fig. 2 shows the pressure at μ=0. On Fig. 3 we present $\Delta p/T^4$ for five different μ values. Fig. 4 gives $\Delta p(\mu, T/T_c)$ normalised by $\Delta p^{SB} = \Delta p(\mu, T \to \infty)$. Notice the interesting scaling behaviour. $\Delta p/\Delta p^{SB}$ depends only on T and it is practically independent of μ in the analysed region. Fig. 5 shows ϵ-$3p$ normalised by T^4, which tends to zero for large T. Fig. 6 gives the baryonic density as a function of T/T_c for different μ-s. As it can be seen the densities exceed the nuclear

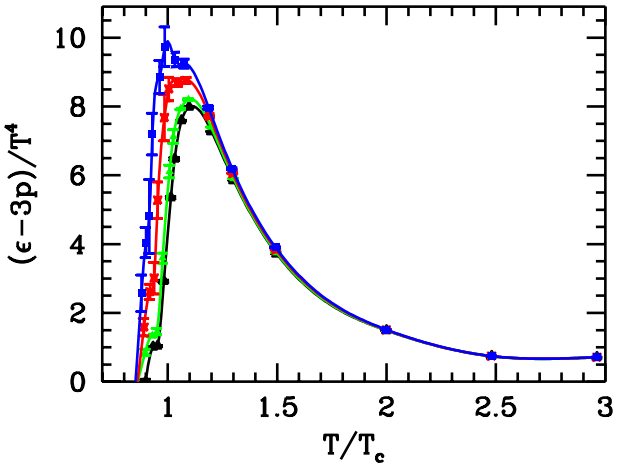

Figure 5: $(\epsilon - 3p)/T^4$ at μ_B=0,210,410 MeV and μ_B=530 MeV as a function of T/T_c (from bottom to top; to help the continuum interpretation the raw lattice result is multiplied with c_p)

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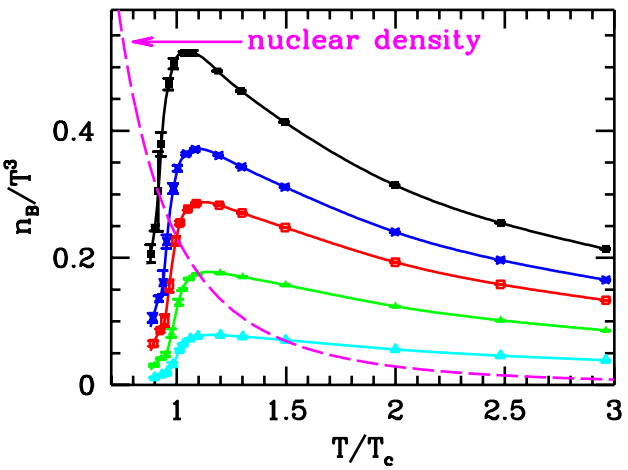

Figure 6: The quark number density normalized T^3 as a function of T/T_c for μ_B =100,210,330,410 MeV and μ_B=530 MeV. (to help the continuum interpretation the raw lattice result is multiplied with c_μ). As a reference value the line starting in the left upper corner indicates the nuclear density.

density by up to an order of magnitude.

As an important finding we mention that in the present analysis the applicability of our reweighting method, the maximal μ value scales with the volume as $\mu_{\rm max} \cdot a \sim (N_t \cdot N_s^3)^{-0.25}$. If this behaviour persists, one could –in principle– approach the true continuum limit (note, that for $N_t \to \infty$ $a \sim 1/N_t \sim (N_t \cdot N_s^3)^{-0.25}$, thus $\mu_{\rm max}$ is constant).

In this paper we studied the thermodynamical properties of QCD at nonzero μ. We used staggered QCD with 2+1 quarks on N_t=4 lattices. We determined for the first time the equation of state at nonzero temperature and chemical potentials. Future analyses should be performed at smaller lattice spacings and quark masses. A detailed version of the present work will be published elsewhere [6].

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References

[1] M. G. Alford, K. Rajagopal and F. Wilczek, Phys. Lett. B **422** (1998) 247 [arXiv:hep-ph/9711395].

[2] M. G. Alford, K. Rajagopal and F. Wilczek, Nucl. Phys. B **537** (1999) 443 [arXiv:hep-ph/9804403].

[3] R. Rapp, T. Schafer, E. V. Shuryak and M. Velkovsky, Phys. Rev. Lett. **81** (1998) 53 [arXiv:hep-ph/9711396].

[4] K. Rajagopal and F. Wilczek, arXiv:hep-ph/0011333.

[5] Z. Fodor and S. D. Katz, Phys. Lett. B **534** (2002) 87 [arXiv:hep-lat/0104001].

[6] F. Csikor et al., in preparation.

[7] P. Hasenfratz and F. Karsch, Phys. Lett. B **125** (1983) 308.

[8] J. B. Kogut, H. Matsuoka, M. Stone, H. W. Wyld, S. H. Shenker, J. Shigemitsu and D. K. Sinclair, Nucl. Phys. B **225** (1983) 93.

[9] I. Montvay and G. Münster, Quantum fields on a lattice. Cambridge, UK, University Press (1994).

[10] S. Gottlieb *et al.*, Phys. Rev. D **55** (1997) 6852 [arXiv:hep-lat/9612020].

[11] F. Karsch, E. Laermann and A. Peikert, Phys. Lett. B **478** (2000) 447 [arXiv:hep-lat/0002003].

[12] A. Ali Khan *et al.* [CP-PACS collaboration], Phys. Rev. D **64** (2001) 074510 [arXiv:hep-lat/0103028].

[13] S. Ejiri, Nucl. Phys. Proc. Suppl. **94** (2001) 19 [arXiv:hep-lat/0011006].

[14] A. M. Ferrenberg and R. H. Swendsen, Phys. Rev. Lett. **61** (1988) 2635.

[15] I. M. Barbour, S. E. Morrison, E. G. Klepfish, J. B. Kogut and M. P. Lombardo, Nucl. Phys. Proc. Suppl. **60A** (1998) 220 [arXiv:hep-lat/9705042].

[16] Z. Fodor and S. D. Katz, JHEP **0203** (2002) 014 [arXiv:hep-lat/0106002].

[17] J. Berges and K. Rajagopal, Nucl. Phys. B **538** (1999) 215 [arXiv:hep-ph/9804233].

[18] M. A. Halasz, A. D. Jackson, R. E. Shrock, M. A. Stephanov and J. J. Verbaarschot, Phys. Rev. D **58** (1998) 096007 [arXiv:hep-ph/9804290].

[19] C. R. Allton *et al.*, arXiv:hep-lat/0204010.

[20] P. de Forcrand and O. Philipsen, arXiv:hep-lat/0205016.

[21] J. Engels, J. Fingberg, F. Karsch, D. Miller and M. Weber, Phys. Lett. B **252** (1990) 625.