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Quantization of the Optical Phase Space $\mathcal{S}^2 = \{\varphi \bmod 2\pi, I > 0\}$ in Terms of the Group $SO^{\uparrow}(1,2)$

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Abstract

The problem of quantizing properly the canonical pair "angle and action variables ", φ and I, is almost as old as quantum mechanics itself and since decades an intensively debated but still unresolved issue in quantum optics. The present paper proposes a new approach to the problem, namely quantization in terms of the group $SO(1,2)$: The crucial point is that the phase space $\mathcal{S}^2 = \{\varphi \bmod 2\pi,\, I > 0\}$ has the global structure $S^1 \times \mathbb{R}^+$ (a simple cone) and cannot be quantized in the conventional manner. As the group $SO(1,2)$ acts transitively, effectively and Hamilton-like on that space its irreducible unitary representations of the positive discrete series provide the appropriate quantum theoretical framework. The phase space $\mathcal{S}^2$ has the conic structure of an orbifold $\mathbb{R}^2/Z_2$. That structure is closely related to a Z_2 gauge symmetry which corresponds to the center of a 2-fold covering of $SO(1,2)$, the symplectic group $Sp(2,\mathbb{R})$. The basic variables on the phase space are the functions $h_0 = I$, $h_1 = I\cos\varphi$ and $h_2 = -I\sin\varphi$ the Poisson brackets of which obey the Lie algebra $\mathfrak{so}(1,2)$. In the quantum theory they are represented by the self-adjoint Lie algebra generators K_0, K_1 and K_2 of a unitary representation, where K_0 has the spectrum $\{k+n,\, n = 0,1,\ldots; k > 0\}$. A crucial prediction is that the classical Pythagorean relation $h_1^2 + h_2^2 = h_0^2$ can be violated in the quantum theory. For each representation one can define three different types of coherent states the complex phases of which may be "measured" by means of the operators K_1 and K_2 alone without introducing any new phase operators! The $SO(1,2)$ structure of optical squeezing and interference properties as well as that of the harmonic oscillator are analyzed in detail. The additional coherent states can be used for the introduction of (Husimi type) "Q" distributions and (Sudarshan-Glauber type) "P" representations of the density operator. The three operators K_0, K_1 and K_2 are fundamental in the sense that one can construct composite position and momentum operators out of them! The new framework poses quite a number of fascinating experimental and theoretical challenges.

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Chapter 1

Introduction and overview

The problem how to quantize modulus and phase of a wave as some kind of canonically conjugate action and angle variables and relate them to genuine self-adjoint operators in Hilbert spaces is a very old one and appears - according to the still ongoing controversial discussions in the field of quantum optics - not yet settled.

(See, e.g. the reviews [1–10] and the textbooks [11–18])

A solution of that theoretical problem becomes more and more urgent, however, because the fascinating experiments in quantum optics become increasingly more refined and allow to differentiate between different theoretical schemes.

The present paper addresses the problem from a new point of view, namely quantization of classical phase spaces in terms of groups and their irreducible unitary representations [20, 21]. This approach to the quantization of classical systems provides a genuine extension of the conventional quantization procedure which is applicable to phase spaces of the type $\mathbb{R}^{2n}$ only.

The more general approach used in this paper allows for a quantization of phase spaces which have a *global* topological structure which is different from that of $\mathbb{R}^{2n}$ and which makes predictions which can be tested experimentally.

1.1 The problem

Let me illustrate the essential origin of the difficulties with the conventional quantization procedure applied to the phase-modulus pair by the harmonic oscillator with the Hamiltonian (the mass m and the frequency ω are scaled to 1)

$$H(q,p) = \frac{1}{2}\,p^2 + \frac{1}{2}\,q^2 \;\;, (q,p) \in \mathbb{R}^2 \,,\;\; \{q,p\}_{q,p} = 1 \,, \tag{1.1}$$

where

$$\{f_1, f_2\}_{q,p} \equiv \partial_q f_1\, \partial_p f_2 - \partial_p f_1\, \partial_q f_2 \tag{1.2}$$

denotes the Poisson bracket for the phase space functions $f_1(q,p)$ and $f_2(q,p)$. It is intimately connected to the symplectic form (see Appendix A.1)

$$\omega = dq \wedge dp \,. \tag{1.3}$$

The transformation to action and angle variables $I > 0$ and $\varphi \in (-\pi\, \pi]$,

$$q(\varphi, I) \;\; = \;\; \sqrt{2\,I}\, \cos\varphi \,, \tag{1.4}$$

$$p(\varphi, I) \;\; = \;\; -\sqrt{2\,I}\, \sin\varphi \,, \tag{1.5}$$

is locally canonical because

$$\frac{\partial(q,p)}{\partial(\varphi,I)} = 1\,, \quad \text{or equivalently} \quad : \; dq \wedge dp = d\varphi \wedge dI\,, \tag{1.6}$$

for its functional determinant.

The transformation 1.4 - 1.5 yields

$$H = I\,, \quad \{\varphi, I\}_{\varphi,I} = 1\,, \tag{1.7}$$

where

$$(\varphi, I) \in \mathcal{S}^2_{\varphi,I} = \{(\varphi \in \mathbb{R} \bmod 2\pi, I \; > \; 0)\}\,, \tag{1.8}$$

and where now the Poisson bracket applies to functions $h_j(\varphi, I)\,,\; j = 1, 2$:

$$\{h_1, h_2\}_{\varphi,I} \equiv \partial_\varphi h_1\, \partial_I h_2 - \partial_I h_1\, \partial_\varphi h_2\,. \tag{1.9}$$

This new phase space $\mathcal{S}^2_{\varphi,I}$ no longer has the global topology $\mathbb{R}^2$ we started from, 1.1, but is homeomorphic to

$$S^1 \times \mathbb{R}^+\,, \quad \mathbb{R}^+ = \{r \in \mathbb{R}, r > 0\}\,, \tag{1.10}$$

where S^1 denotes the unit circle. This is the topology of a simple cone with the vertex deleted or that of $\mathbb{R}^2 - (0,0)$, i.e. the plane without the origin!

The reason for deleting the origin or to demand $I > 0$ is the following:

At first sight that deletion does not to appear necessary because the functional determinant 1.6 is regular at $I = 0$. However, the transformation formulae 1.4 and 1.5 contain the factor $\sqrt{I}$ which is *not analytic* at $I = 0$!

Another way of looking at the problem is to introduce polar coordinates φ and $\rho = \sqrt{I}$. Now we get the functional determinant

$$\frac{\partial(q,p)}{\partial(\varphi,\rho)} = 2\,\rho\,, \tag{1.11}$$

which shows that the transformation is not regular for $\rho = 0$!

Thus, the symplectic space 1.8 confronts us with a non-trivial topology which prevents a "naive" quantization approach even though we *locally* still have the equality 1.6 of the symplectic forms!

We know from the Aharonov-Bohm effect [22] that a "punctured plane" can yield very interesting physical effects and that one should take the hole seriously, being it as small as it may! The crucial point is that one cannot contract loops around it to a point.

We shall see below that the picture of a simple cone with its vertex point deleted will be more adequate than the picture of a punctured plane.

Before pointing out explicitly the well-known difficulties with quantizing the canonical pair (φ, I) naively, let me write down the complex "amplitudes"

$$a \;=\; \frac{1}{\sqrt{2}}(q + i\,p) = I^{1/2} e^{-i\,\varphi}\,, \tag{1.12}$$

$$\bar{a} \;=\; \frac{1}{\sqrt{2}}(q - i\,p) = I^{1/2} e^{i\,\varphi}\,, \tag{1.13}$$

$$I \;=\; \bar{a}\,a\,, \tag{1.14}$$

which become annihilation and creation operators for the quantized system.

(Here and in the following $\bar{a}$ denotes the complex conjugate of a!)

The conventional recipe for quantizing a 2-dimensional classical phase space - like that of the harmonic oscillator - consists in replacing the pair (q,p) of canonical variables by self-adjoint operators (Q,P) in an appropriate Hilbert space, where the classical Poisson bracket 1.1 is replaced by the quantum mechanical commutator[1]:

$$(q,p) \to (Q,P)\,, \quad \{q,p\} = 1 \to [Q,P] = i\,. \tag{1.15}$$

This commutator implies that *self-adjoint* Q and P both have the full real line $\mathbb{R}$ as their spectrum, reflecting the fact that we started from a classical phase space with the global structure $\mathbb{R}^2$.

We shall see more details of this below when we discuss the Weyl-Heisenberg group generated by the operators 1.15.

Instead of the classical Hamiltonian 1.1 we now get the Hamilton operator

$$\hat{H} = \frac{1}{2}P^2 + \frac{1}{2}Q^2 \tag{1.16}$$

$$= \hat{N} + \frac{1}{2}\,, \quad \hat{N} = \hat{a}^+\hat{a}\,, \tag{1.17}$$

$$\hat{a} = \frac{1}{\sqrt{2}}(Q + i\,P)\,, \tag{1.18}$$

$$\hat{a}^+ = \frac{1}{\sqrt{2}}(Q - i\,P)\,, \tag{1.19}$$

$$[\hat{a}, \hat{a}^+] = 1\,. \tag{1.20}$$

$\hat{H}$ has the well-known normalizable eigenstates $|n\rangle\,, n = 0, 1, \ldots$, and the spectrum $E_n = n + 1/2$.

We now come to the crucial point:

In view of the canonical character of the transformations 1.4 and 1.5 and the Poisson bracket 1.7 one might be tempted to quantize φ and I by the replacements [23, 24]

$$\varphi \to \hat{\varphi}\,, \quad I \to \hat{N}\,, \quad \{\varphi, I\} = 1 \to [\hat{\varphi}, \hat{N}] = i\,. \tag{1.21}$$

That commutator, however, implies an immediate contradiction when one writes down its number state matrix elements:

$$\langle n_2|[\hat{\varphi}, \hat{N}]|n_1\rangle = (n_1 - n_2)\langle n_2|\hat{\varphi}|n_1\rangle = i\,\delta_{n_2 n_1}\,. \tag{1.22}$$

For $n_2 = n_1$ we get $0 = i$!

1.2 Some history

That the commutator 1.21 does not make sense was noticed very early - even before Dirac proposed it - by London [25] who essentially used the argument just presented in the framework of the then just invented matrix mechanics.

London had seen two earlier papers of Dirac [27] in which he also dealt with the problem of quantizing angle and action variables of classical mechanics and in which Dirac suggested operator versions of the complex amplitudes 1.12 and 1.13, without yet postulating the commutator 1.21.

[1] I assume $\hbar = 1$ throughout the whole paper

London took up the issue of quantizing the canonical pair w and J of angle and action variables and started by observing that the quantized quantity $J\,w - w\,J$ cannot be a diagonal matrix in the framework of matrix mechanics if J is diagonal. This is exactly the argument from the end of the last section above.

Then London goes on, introduces an operator

$$E(i\,n\,w) = \sum_{\nu=0}^{\infty} \frac{(i\,n\,w)^{\nu}}{\nu!} = E^{n}(iw)\,,\ n = 1, 2, \ldots\,, \tag{1.23}$$

and discusses properties of that formal power series, e.g.

$$E^{-1}(iw) J\, E(iw) - J = \mathbf{1}\,. \tag{1.24}$$

For the harmonic oscillator he writes down the operator transformation

$$P = \frac{1}{\sqrt{2}}(\sqrt{J}\,E + E^{-1}\sqrt{J})\,,\ Q = \frac{1}{i\sqrt{2}}(\sqrt{J}\,E - E^{-1}\sqrt{J})\,,\ E = E(iw)\,, \tag{1.25}$$

which he calls canonical because it yields the commutator 1.15.

(Like London I have used the same letters w and J for the classical and the quantized quantities.)

In his second paper on the subject [26] London used wave functions, Hilbert spaces and unitary transformations between Hilbert spaces: He transformed, e.g. the wave functions $\exp[i\,n\,(w = \varphi)]$ of the harmonic oscillator (described in the Hilbert space 1.28) into the usual Hermite functions (see Ch. 4 of the present paper for details). He realized that there was no such operator like $w = \hat{\varphi}$ (I change notations now) but that operators like

$$E_{-} = \widehat{e^{-i\,\varphi}}\,,\ E_{+} = \widehat{e^{i\,\varphi}}\ \ (= E(i\,w)) \tag{1.26}$$

can make sense according to the formal power series 1.23 and that they have the properties

$$E_{-}|n\rangle = |n-1\rangle\,,\ E_{+}|n\rangle = |n+1\rangle\,,\ E_{-}\hat{N}E_{+} = \hat{N} + 1\ . \tag{1.27}$$

London also made the correct mathematical observation that $\hat{\varphi}$ cannot be a self-adjoint (multiplication) operator in a Hilbert space with a scalar product

$$(\psi_2, \psi_1) = \frac{1}{2\,\pi}\int_0^{2\,\pi} d\varphi\,\bar{\psi}_2(\varphi)\psi_1(\varphi)\ , \tag{1.28}$$

because φ is not periodic. (As to a modern discussion of that problem see Ref. [28]).

The mathematical inconsistency 1.22 of the commutator 1.21 was rediscovered – without the knowledge of London's previous work – in the sixties [29,45] as well as the possible usefulness of the operators E_{-} and E_{+} from 1.26 and their properties 1.27.

If one inverts Dirac's formal operator polar decompositions

$$\hat{a} = E_{-}\,\sqrt{\hat{N}}\,,\quad \hat{a}^{+} = \sqrt{\hat{N}}\,E_{+}\,, \tag{1.29}$$

naively, one gets the representations

$$E_{-} = \hat{a}\,\hat{N}^{-1/2}\,,\quad E_{+} = \hat{N}^{-1/2}\,\hat{a}^{+}\,. \tag{1.30}$$

As to the introduction of these operators the almost only new element which was added in the sixties of the last century, compared to that what was known to London (and Dirac), was

the observation that the operator E_- from Eq. 1.30 is not defined when applied to the ground state $|n=0\rangle$. This deficiency was cured in the following way [29, 30]:

If $f(\hat{N})$ is an "appropriate" function of the number operator $\hat{N}$ then – as a consequence of the Eqs. 1.17 and 1.20 – the relations

$$\hat{a}\, f(\hat{N}) = f(\hat{N}+1)\,\hat{a}\ ,\ \ f(\hat{N})\,\hat{a}^+ = \hat{a}^+\, f(\hat{N}+1) \tag{1.31}$$

hold.

"Appropriate" here means that both sides of the Eqs. 1.31 are well-defined operators in the Hilbert space they act in. Though the operator E_- from Eq. 1.30 is not appropriate in this sense, the relations 1.31 were used in order to make E_- - and E_+ - well-defined:

$$E_- = (\hat{N}+1)^{-1/2}\hat{a}\ ,\ \ E_+ = \hat{a}^+(\hat{N}+1)^{-1/2}\ , \tag{1.32}$$

where $E_-|n=0\rangle = 0$ is now obvious.

(Already Dirac used the relations 1.31 in special cases [23]!)

The operators 1.32 still have the properties 1.27 from which it follows that

$$E_-E_+ = 1\,,\ \ E_+E_- = 1 - \hat{P}_0\,, \tag{1.33}$$

where $\hat{P}_0$ is the projection operator onto the ground state $|n=0\rangle$.

The second of the relations 1.33 shows that the operator E_- is not unitary, but merely isometric.

In mathematics operators with the properties 1.27 are called "shift operators" and they have been studied extensively [31].

Susskind and Glogower [29] defined operators for *cosine* and *sine* as

$$\hat{C} \equiv \widehat{\cos\varphi} \ = \ \frac{1}{2}(E_+ + E_-)\ , \tag{1.34}$$

$$\hat{S} \equiv \widehat{\sin\varphi} \ = \ \frac{1}{2i}(E_+ - E_-) \tag{1.35}$$

and discussed several of their properties.

The development of those years was nicely summarized in a thorough review by Carruthers and Nieto [1]. For a personal account by Nieto of those times and their problems see Ref. [32].

1.3 Central elements of the new approach

If one tries to analyze and trace back all the apparent and real problems and difficulties which have been discussed over the decades since the early days of quantum mechanics in connection with the quantization of moduli and phases, one always ends up with the non-trivial global structure of the symplectic (phase) space 1.8.

Quantizing this symplectic space requires a new approach and group theoretical quantization [20, 21] provides such an approach!

I shall first outline how this approach works in the case of the conventional quantization scheme and then discuss the appropriate generalizations for the phase space 1.8.

1.3.1 Group theoretical background of the conventional quantization procedure

The classical phase space

Let me briefly recall essential group-theoretical properties of the 2-dimensional phase space

$$\mathcal{S}^2_{q,p} = \{(q,p) \in \mathbb{R}^2\}\,, \tag{1.36}$$

associated with the system 1.1 or similar ones which have that phase space:

Because of the Poisson bracket 1.2 the phase space 1.36 does not only form a vector space, but also has a 3-dimensional (nilpotent) Lie algebra $\tilde{\mathfrak{t}}^{2+1}_{WH}$ associated with it.

The index "WH" stands for "Weyl" and "Heisenberg", because it has become customary to speak of the corresponding group as the "Weyl-Heisenberg group" (or "Weyl group" or "Heisenberg" group). To name that group after Weyl is certainly justified, but it might be debatable to single out the name of Heisenberg in view of the equally important contributions of Born, Jordan and Dirac to the commutator structure of the quantized canonical variables!

The letter t in $\tilde{\mathfrak{t}}^{2+1}_{WH}$ stands for "translations" (see below) and the "tilde" for "central extension" (see below, too).

The Lie algebra is 3-dimensional because the Poisson (Lie) bracket of q and p is a fixed real number - compared to the *variables* q and p - ,

$$\{q,p\}_{q,p} = 1\,, \quad \{q,1\}_{q,p} = 0\,, \quad \{p,1\}_{q,p} = 0\,, \tag{1.37}$$

i.e. the Lie algebra is generated by q, p and the real number 1.

The commutator of 2 general Lie algebra elements

$$l_j = a_j\, q + b_j\, p + r_j\ ,\ j = 1,2, \tag{1.38}$$

is given by

$$\{l_1, l_2\}_{q,p} = a_1\, b_2 - a_2\, b_1\ , \tag{1.39}$$

i.e. a real number.

As

$$\{l, \{l_1, l_2\}_{q,p}\}_{q,p} = 0 \tag{1.40}$$

for an arbitrary $l \in \tilde{\mathfrak{t}}^{2+1}_{WH}$, the Lie algebra is nilpotent, see, e.g. Ref. [33].

The center of the Lie algebra is generated by the number 1.

If we consider q, p and 1 as basis of the Lie algebra, we may characterize a general element 1.38 by a triple of real numbers

$$(a,b,r)\,. \tag{1.41}$$

According to Eq. 1.39 we then have the following general Lie algebra commutator structure

$$[(a_1,b_1,r_1),(a_2,b_2,r_2)] = (0,0,a_1\, b_2 - a_2\, b_1)\ . \tag{1.42}$$

for two of the column vectors 1.41.

If we merely look at the first 2 components of the elements (a,b,r) on both sides of the Eq. 1.42 and ignore for a moment the 3rd component, then we have a Lie algebra of the 2-dimensional abelian group of translations which acts on functions on $\mathbb{S}^2_{q,p} \cong \mathbb{R}^2$ as follows:

For any smooth function $f(q,p)$ we have

$$\{a\, q + b\, p + r, f(q,p)\}_{q,p} = a\, \partial_p f(q,p) - b\, \partial_q f(q,p)\ . \tag{1.43}$$

Thus, the Lie algebra generator q generates (infinitesimal) translations in momentum space and p generates (infinitesimal) translations in coordinate space!

Actually, however, that Lie algebra of the 2-dimensional translation group is *not* a subalgebra of $\tilde{\mathfrak{t}}^{2+1}_{WH}$, because the Poisson commutator of the two translation generators q and p does not vanish, but gives the 3rd generator, see 1.37.

The Lie algebra $\tilde{\mathfrak{t}}^{2+1}_{WH}$ generates a group, the so-called "Weyl-Heisenberg" group $\tilde{T}^{2+1}_{WH}$. Its group law is obtained by exponentiating the Lie algebra elements l_1 and l_2 and multiplying the result:

$$e^{l_1} \circ e^{l_2} = e^{l_1 + l_2 + \{l_1, l_2\}_{q,p}/2} \ . \tag{1.44}$$

Here the Baker-Campbell-Hausdorff formula [34]

$$e^A \cdot e^B = e^{A+B+[A,B]/2} \tag{1.45}$$

for the product of the exponentials of two operators A and B has been used, in the special case that their commutator $[A, B]$ commutes with both A and B.

The relation 1.44 for the exponentials shows that one can characterize a group element by the tripel (a, b, r) which describes the exponentiated Lie algebra element.

From Eq. 1.44 one reads off the following group law:

$$(a_1, b_1, r_1) \circ (a_2, b_2, r_2) = (a_1 + a_2, b_1 + b_2, r_1 + r_2 + (a_1\, b_2 - a_2\, b_1)/2) \ . \tag{1.46}$$

The group $\tilde{T}^{2+1}_{WH}$ consists of 2-dimensional translations enlarged by a 1-dimensional "central extension" $\mathbb{R}$ of the additive group of real numbers! (As to central extensions of groups in physics see, e.g. the Refs. [35])

The action of the 2-dimensional translations

$$q \to q - b \, , \quad p \to p + a \, , \quad a, b \in \mathbb{R} , \tag{1.47}$$

generated by the group $\tilde{T}^{2+1}_{WH}$ on the phase space 1.36 has the following properties:

1. It is *symplectic*: $d(q - b) \wedge d(p + a) = dq \wedge dp$.

2. It is is *transitive*: Any two points (q_1, p_1) and (q_2, p_2) can be transformed into each other by an element of $\tilde{T}^{2+1}_{WH}(a, b) : (b = q_1 - q_2, a = p_2 - p_1)$. Here the point $(q = 0, p = 0)$ is in no way special!

3. It is *effective*: If $(a, b) \cdot (q, p) = (q - b, p + a) = (q, p)\ \forall (q, p)$, then $(a, b) = (0, 0)$.

4. The abstract Lie algebra $\tilde{\mathfrak{t}}^{2+1}_{HW}$ defined by the commutators

$$[A_1, A_2] = A_3 \, , \quad [A_1, A_3] = 0 \, , \quad [A_2, A_3] = 0 \, , \tag{1.48}$$

of the "canonical" group $\tilde{T}^{2+1}_{WH}$ is isomorphic to the Poisson algebra of 3 *globally* defined functions $f_{A_1}(q, p)$, f_{A_2} and f_{A_3} on 1.36, namely

$$\{f_{A_1}, f_{A_2}\}_{q,p} = f_{A_3} \, , \quad \{f_{A_1}, f_{A_3}\}_{q,p} = 0 \, , \quad \{f_{A_2}, f_{A_3}\}_{q,p} = 0 \, . \tag{1.49}$$

The required 3 functions are obviously (see Eqs. 1.37)

$$f_{A_1}(q, p) = q \, , \quad f_{A_2}(q, p) = p \, , \quad f_{A_3}(q, p) = 1 \, . \tag{1.50}$$

These properties are essential for a generalization of the following quantization procedure to other phase spaces like 1.8!

Group theoretical quantization of the phase space $\mathcal{S}^2_{q,p} \cong \mathbb{R}^2$

Quantization of the phase space 1.36 now consists in determining the irreducible unitary representations of the Weyl-Heisenberg group 1.46 !

The important point for this approach is that in such a representation the unitary operators $U(a)$ and $V(b)$ which implement the translations 1.47 are generated by *self-adjoint* operators Q and P:

$$U(a) = e^{-i\,a\,Q} \ , \ V(b) = e^{-i\,b\,P} \ . \tag{1.51}$$

Because of 1.15 and 1.45 we have

$$V(b)\,U(a) \ = \ e^{i\,a\,b/2}\,e^{-i\,(a\,Q+b\,P)}\,, \tag{1.52}$$

$$U(a)\,V(b) \ = \ e^{-i\,a\,b/2}\,e^{-i\,(a\,Q+b\,P)}\,, \tag{1.53}$$

and therefore Weyl's integrated *group* commutator relation [36]

$$U(a_1)\,U(a_2) \ = \ U(a_2)\,U(a_1)\ , \tag{1.54}$$

$$V(b_1)\,V(b_2) \ = \ V(b_2)\,V(b_1)\ , \tag{1.55}$$

$$U^{-1}(a)\,V(b)\,U(a)\,V^{-1}(b) \ = \ e^{i\,a\,b}\,, \tag{1.56}$$

instead of the *Lie algebra* relation 1.15 as introduced by Born, Heisenberg and Jordan and by Dirac as well.

Weyl's approach has the mathematical advantage that one is dealing with *bounded* operators in Hilbert space and the famous Stone - von Neumann theorem [37] asserts that all irreducible unitary representations of the operators $U(a)$ and $V(b)$ with the properties 1.54-1.56 are unitarily equivalent to the Schrödinger representation in the L^2-space of square-integrable functions on the real line $\mathbb{R}$.

Notice, however, that the fundamental "observables" of the system are the *Lie algebra* basis elements q, p and 1 on the classical level and the corresponding operators Q, P and $\mathbf{1}$ on the quantum level, the Lie algebra being that of the group $\tilde{T}^{2+1}_{WH}$!

The group law 1.46 is implemented again by applying the relation 1.45 to the operator product

$$e^{-i\,(a_1\,Q+b_1\,P+r_1)}\cdot e^{-i\,(a_2\,Q+b_2\,P+r_2)}\ . \tag{1.57}$$

At first sight the group-theoretical approach to the usual quantization of the phase space 1.36 may appear complicated and even far-fetched. The reason is that the quantizing "canonical" Weyl-Heisenberg group 1.46 is unusual in the sense that it has the structure of a *central extension* ([35]) of an abelian translation group which acts on that phase space 1.36. The resulting group is nilpotent and as such somewhat "singular".

Essential is, however, that the basic underlying "canonical" transformation group on 1.36 is the 2-dimensional translation group of that space, with the properties listed above.

1.3.2 The group $SO^{\uparrow}(1,2)$ as the canonical group of the phase space $\mathcal{S}^2_{\varphi,I}$

Contrary to the seemingly somewhat complicated Weyl-Heisenberg group 1.46 of the phase space 1.36, the corresponding "canonical" group of the phase space 1.8 is much simpler, namely the group

$$SO^{\uparrow}(1,2)\,, \tag{1.58}$$

the "orthochronous proper" Lorentz group in $2+1$ (space-time) dimensions (see Eq. B.4 for the precise definition). The group is also "simple" in the mathematical sense [38].

The role of the symplectic group $Sp(2,\mathbb{R})$ as the canonical group of the phase space $\mathbb{R}^2-(0,0)$ was first discussed in the context of a $U(1)$-gauge model by Loll [39] and more recently – without the knowledge of Loll's paper – in connection with the quantization of Schwarzschild black holes [40, 41].

After finishing the paper [40] I realized that the quantization formalism employed there also sheds new light on the old – still mainly unsolved – problem of how to describe phase and modulus in terms of self-adjoint operators in a suitable Hilbert space associated with a corresponding physical system [42–44].

The general idea of a group theoretical quantization of a given phase space is outlined in Appendix A.1. Appendices A.2 and A.3 describe the application of that general approach to the symplectic space 1.8 in terms of the group 1.58 in detail. Mathematical properties of that group, its double covering groups $SU(1,1) \cong SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$ and the universal covering group of all of them are discussed in Appendix B.

In the following I briefly sketch and summarize the essential results in order to illustrate the power and the richness of the theory and to indicate the points for possible experimental tests of the framework.

The action of the transformation group 1.58 on the phase space 1.8 should have all the general properties listed after Eq. 1.47. We shall see that it indeed does have them. Most of the proofs can be found in Appendix A.

In order to describe the action of $SO^{\uparrow}(1,2)$ the following parametrization of the space 1.8 is convenient:

A point $s \in \mathcal{S}^2_{\varphi,I}$ can be represented by the matrix

$$\underline{s} = \begin{pmatrix} I & I\,e^{-i\varphi} \\ I\,e^{i\varphi} & I \end{pmatrix} \equiv \begin{pmatrix} h_0 & h_1 + i\,h_2 \\ h_1 - i\,h_2 & h_0 \end{pmatrix} . \tag{1.59}$$

As

$$\det(\underline{s}) = h_0^2 - h_1^2 - h_2^2 = I^2 - (I\,\cos\varphi)^2 - (-I\,\sin\varphi)^2 = 0 \tag{1.60}$$

and $h_0 > 0$ the three (dependent) coordinates

$$h_0 = I\,, \quad h_1 = I\,\cos\varphi\,, \quad h_2 = -I\,\sin\varphi\,, \tag{1.61}$$

parametrize a 2-dimensional "forward light cone" with the vertex deleted. It is well known that the group $SO^{\uparrow}(1,2)$ acts transitively on that cone.

As in the case of the Lorentz or rotation group the transformation of $\underline{s}$ with respect to $SO^{\uparrow}(1,2)$ is best implemented in terms of the 2-fold covering group

$$SU(1,1) = \left\{ \tilde{g} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} , \quad |\alpha|^2 - |\beta|^2 = 1 \right\} , \tag{1.62}$$

namely

$$\underline{s} \to \hat{\underline{s}} = \tilde{g} \cdot \underline{s} \cdot \tilde{g}^{+} , \tag{1.63}$$

where $\tilde{g}^{+}$ denotes the hermitian conjugate of $\tilde{g}$.

The center

$$Z_2 \; : \quad \{E_2, -E_2\} , \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{1.64}$$

of $SU(1,1)$ leaves all points $\underline{s}$ invariant, i.e. the group $SU(1,1)$ acts only almost effectively on the phase space 1.8, but $SO^{\uparrow}(1,2) \cong SU(1,1)/Z_2$ itself acts effectively, i.e. only the identity element of $SO^{\uparrow}(1,2)$ leaves all points 1.59 fixed.

More important, the transformations 1.63 leave the symplectic form

$$\omega_{\varphi,I} = d\varphi \wedge dI \tag{1.65}$$

invariant (Appendix A.2):

$$d\hat{\varphi} \wedge d\hat{I} = d\varphi \wedge dI \ . \tag{1.66}$$

The Lie algebra $\mathfrak{so}(1,2)$ may be spanned by three generators A_j , $j = 0, 1, 2$, with the commutators

$$[A_0, A_1] = -A_2 \ , \quad [A_0, A_2] = A_1 \ , \quad [A_1, A_2] = A_0 \ . \tag{1.67}$$

Here A_0 generates rotations and A_1 and A_2 generate "Lorentz boosts"! (See Eqs. A.81-A.89 of Appendix A.2.)

Each of those 1-dimensional subgroups generates a global vector field $\tilde{A}_j$ on the phase space 1.8. These have the form (Appendix A.2)

$$\tilde{A}_0 = \partial_\varphi \ , \tag{1.68}$$

$$\tilde{A}_1 = \cos\varphi \, \partial_\varphi + I \, \sin\varphi \, \partial_I \ , \tag{1.69}$$

$$\tilde{A}_2 = \sin\varphi \, \partial_\varphi - I \, \cos\varphi \, \partial_I \ . \tag{1.70}$$

The vector fields generate the same Lie algebra 1.67 as the A_j themselves.

A crucial point now is that those vector fields are global *Hamiltonian* ones, i.e. they have the form

$$-X_h = \partial_\varphi h(\varphi, I) \, \partial_I - \partial_I h(\varphi, I) \, \partial_\varphi \ , \tag{1.71}$$

where $h(\varphi, I)$ is a smooth function on the phase (symplectic) space 1.8.

The most essential result of all this (details are in Appendix A.2) is that the generating functions for the Hamiltonian vector fields 1.68-1.70 are just the three coordinate functions h_j introduced in Eq. 1.59:

$$h_0(\varphi, I) = I \ , \quad h_1(\varphi, I) = I \, \cos\varphi \ , \quad h_2(\varphi, I) = -I \, \sin\varphi \ . \tag{1.72}$$

These functions obey the Lie algebra $\mathfrak{so}(1,2)$ in terms of the Poisson brackets 1.9, too:

$$\{h_0, h_1\}_{\varphi,I} = -h_2 \ , \quad \{h_0, h_2\}_{\varphi,I} = h_1 \ , \quad \{h_1, h_2\}_{\varphi,I} = h_0 \ . \tag{1.73}$$

As in the case of the Weyl-Heisenberg group above where the basic observables on the phase space $\mathcal{S}^2_{q,p}$ are given by the generators q and p (and 1) of the Lie algebra $\mathfrak{t}^{2+1}_{WH}$, now the three functions $h_j(\varphi, I)$ from 1.72 are to be chosen as the *basic* classical observables on the phase space 1.8 ! They obviously suffice to expand any "decent" function $f(\varphi \bmod 2\pi, I)$ on $\mathcal{S}^2_{\varphi,I}$!

It is important to understand the following point: The resulting Hamiltonian functions 1.72 are solely determined by the vector fields 1.68 - 1.70 the form of which is a consequence of the action of the associated 1-parameter subgroups on the phase space 1.8 (see Eqs. A.57- A.61 and A.65-A.67 of Appendix A), with the invariance property 1.66. Their form 1.72 is *not* a consequence of the very convenient but not cogent parametrization 1.59 of the conic space 1.8.

We have to conclude that the canonical group $SO^{\uparrow}(1,2)$ and its action on the symplectic space 1.8 determine the basic "observables" h_j on that space all by itself!

This systematic result justifies an early suggestion by Louisell [45] that one should use $\cos\varphi$ and $\sin\varphi$ instead of φ itself when trying to quantize the latter.

At this point one might ask:

Why then not use the functions

$$\tilde{h}_1 = \cos\varphi \ , \ \tilde{h}_2 = \sin\varphi \ , \ \tilde{h}_3 = I \ , \tag{1.74}$$

as basic observables?

These functions, however, generate the Lie algebra of the *Euclidean group* $E(2)$ *in the plane*:

$$\{\tilde{h}_3, \tilde{h}_1\}_{\varphi,I} = \tilde{h}_2, \quad \{\tilde{h}_3, \tilde{h}_2\}_{\varphi,I} = -\tilde{h}_1, \quad \{\tilde{h}_1, \tilde{h}_2\}_{\varphi,I} = 0\,, \tag{1.75}$$

where $\cos\varphi$ and $\sin\varphi$ commute now!

The latter property might be welcome, but the group $E(2)$ is not suitable at all for our purpose:

First, the group consists of rotations $O(2)$ and two translations on the plane which do *not* avoid the origin!

Second, the spectrum of the self-adjoint operator corresponding to the modulus I in any irreducible unitary representation of $E(2)$ corresponds to the integers $\mathbb{Z}$ [46, 47], not to the positive numbers $\mathbb{N}$, and therefore the quantized I would have the wrong spectrum!

The deeper reason is that the group $E(2)$ is the quantizing group of $S^1 \times \mathbb{R}$ [46], not of $S^1 \times \mathbb{R}^+$!

The situation is quite different for the group $SO^{\uparrow}(1,2)$, where the positive discrete series of irreducible unitary representations provides a positive definite operator for the quantized action variable I!

1.4 The relationship between the action of the group $SO^{\uparrow}(1,2)$ on $\mathcal{S}^2_{\varphi,I}$ and the action of its covering group $Sp(2,\mathbb{R})$ on $\mathcal{S}^2_{q,p}$; the Z_2 gauge symmetry

We have just seen how the group $SO^{\uparrow}(1,2)$ acts on the phase space 1.8. On the other hand, its double covering group, the symplectic group $Sp(2,\mathbb{R})$, see B.13, acts on the space 1.36 as follows:

$$\begin{pmatrix} q \\ p \end{pmatrix} \to g_1 \cdot \begin{pmatrix} q \\ p \end{pmatrix}\,, \; g_1 \in Sp(2,\mathbb{R})\,. \tag{1.76}$$

The transformations 1.76 leave the symplectic form 1.3 invariant, transform the point $(q = 0, p = 0)$ into itself and act transitively on the compliment

$$\mathcal{S}^2_{q,p;0} = \mathcal{S}^2_{q,p} - \{(0,0)\}\,. \tag{1.77}$$

In order to see the difference between the simultaneous actions of the symplectic group $Sp(2,\mathbb{R})$ on the spaces 1.77 and 1.8 (via $SO^{\uparrow}(1,2) = Sp(2,\mathbb{R})/Z_2$) let us look at the following two examples:

The groups B.20 and B.21 act on the space $\mathcal{S}^2_{q,p}$ as

$$R_1: \quad q \to \cos(\theta/2)\, q + \sin(\theta/2)\, p\,, \tag{1.78}$$

$$p \to -\sin(\theta/2)\, q + \cos(\theta/2)\, p\,, \; \theta \in (-2\pi, 2\pi]\,, \tag{1.79}$$

$$A_1: \quad q \to e^{t/2}\, q\,, \tag{1.80}$$

$$p \to e^{-t/2}\, p\,, \quad t \in \mathbb{R}\,. \tag{1.81}$$

The transformations leave the symplectic form 1.3 invariant.

The groups induce simultaneous transformations on $\mathcal{S}^2_{\varphi,I}$: This space is parametrized by the coordinates 1.61 which transform as a 3-vector under $SO^{\uparrow}(1,2)$ (see the formulae A.81-A.86):

$$
\begin{aligned}
R_1: \quad h_0 &\to h_0\,, && (1.82)\\
h_1 &\to \cos\theta\, h_1 - \sin\theta\, h_2\,, && (1.83)\\
h_2 &\to \sin\theta\, h_1 + \cos\theta\, h_2\,, && (1.84)\\
A_1: \quad h_0 &\to \cosh t\, h_0 + \sinh t\, h_2\,, && (1.85)\\
h_1 &\to h_1\,, && (1.86)\\
h_2 &\to \sinh t\, h_0 + \cosh t\, h_2\,. && (1.87)
\end{aligned}
$$

Here the transformations leave $h_0^2 - h_1^2 - h_2^2$ invariant!

The crucial point now is the following:

If $\theta = 2\pi$ for the R_1 transformations, then the pair (q,p) changes sign, but for the triple (h_0, h_1, h_2) we have the identity transformation! This is due to the fact that $Sp(2,\mathbb{R})$ is a double covering of $SO^{\uparrow}(1,2)$ and that the kernel of the homomorphism

$$Sp(2,\mathbb{R}) \to SO^{\uparrow}(1,2) \tag{1.88}$$

is the center 1.64: As to its transformation properties with regards to the group $SO^{\uparrow}(1,2)$ the pair (q,p) transforms as a "spinor", namely as a vector with respect to the double covering $Sp(2,\mathbb{R} \cong SU(1,1)$, whereas the h_j transform as a vector with respect to $SO^{\uparrow}(1,2)$. The relationships here parallel completely those for the well-known rotation group $SO(3) = SU(2)/Z_2$ and its spinor group $SU(2)$!

We come here to an essential point of the whole paper:

The center 1.64 of the group $Sp(2,\mathbb{R})$ acts on the space $\{(q,p) \in \mathbb{R}^2\}$ as the identity, too, if we identify the points (q,p) and $(-q,-p)$!

That is so say, if we pass from the space 1.36 to the quotient space

$$\check{\mathcal{S}}^2_{q,p} = \{(-q,-p) \equiv (q,p) \in \mathbb{R}^2\} \cong \mathbb{R}^2/Z_2\,. \tag{1.89}$$

The resulting space 1.89 is a simple cone with its tip (vertex) at the origin!

This can be seen as follows: Consider the (q,p)-plane: Rotate the lower half of that plane around the q-axis till it lies on the upper half plane such that the negative part of the p-axis coincides with the positive one. Afterwards rotate the left half of the upper half plane around the positive p-axis till the negative part of the q-axis lies on the positive one. Then glue these two q-half-axis together. The result is the cone just mentioned.

We now recognize the essential point of the difference between the phase spaces 1.8 and 1.36: The phase space 1.8 is globally equivalent (homeomorphic and even diffeomorphic) to the quotient space 1.89 if the point $(0,0)$, the "tip" or vertex of the cone, is deleted!

We started from the local equality 1.6 and see now the global difference between the spaces 1.36 and 1.8. Notice, however, that the local symplectic form 1.3 is invariant under the action of the center 1.64, too.

Quotient spaces of the type 1.89 where points of a given space are identified by means of a discontinuous transformation group are called "orbifolds" [48].

The equivalence of the spaces 1.8 and 1.89 has far-reaching consequences, because the Z_2 group 1.64 acts as kind of gauge group on the phase space 1.36:

With regard to the phase space 1.8 only those functions of (q,p) are "observables" which are invariant under the Z_2 transformations 1.64, i.e. only even powers of q and (or) p. Thus, the *original* q or p themselves are no observables in this sense!

There is a surprise, however: We may define "composite" canonical coordinates

$$\tilde{q}(\varphi, I) = \sqrt{2}\,\frac{h_1(\varphi, I)}{\sqrt{h_0(\varphi, I)}} = \sqrt{2\,I}\,\cos\varphi\,, \tag{1.90}$$

$$\tilde{p}(\varphi, I) = \sqrt{2}\,\frac{h_2(\varphi, I)}{\sqrt{h_0(\varphi, I)}} = -\sqrt{2\,I}\,\sin\varphi\,, \tag{1.91}$$

on the symplectic space 1.8!

They obey the usual relation

$$\{\tilde{q}, \tilde{p}\}_{\varphi,I} = 1\,, \tag{1.92}$$

but are Z_2 *gauge invariant* functions now:

According to their definition and according to the Eqs. 1.82- 1.84 they transform as

$$\tilde{q} \;\to\; \cos\theta\,\tilde{q} - \sin\theta\,\tilde{p}\,, \tag{1.93}$$

$$\tilde{p} \;\to\; \sin\theta\,\tilde{q} + \cos\theta\,\tilde{p}\,. \tag{1.94}$$

One realizes the important difference to the transformation formulae 1.78 and 1.79:

The crucial point is that the coordinates 1.90 and 1.91 are functions on $\mathcal{S}^2_{\varphi,I} \cong \mathcal{S}^2_{q,p;0}/Z_2$, whereas the original q and p are functions on $\mathcal{S}^2_{q,p}$!

Notice that $\tilde{q}$ and $\tilde{p}$ have the property $(\tilde{q}, \tilde{p}) \neq (0, 0)$.

The quantized version of the composite coordinates 1.90 and 1.91 will be discussed in Ch. 2.

More about the consequences of the gauge group 1.64 can be found in Secs. 6.3, 8.1.3 and Appendix A.3.

Gauge symmetries of the Z_2 type appearing here have been discussed by Prokhorov and Shabanov [49, 50].

1.5 Overview

In the following I briefly sketch the main topics and results of the following chapters. The bulk of the references will be given within the chapters and the appendices. Due to the length of the paper I could (did) not always avoid using the same symbol (letter) for different things, but their meaning will be clear from the context!

1.5.1 Chapter 2

The quantized versions of the basic classical observables 1.72 are the corresponding self-adjoint generators

$$K_0 = \hat{I} = \hat{h}_0\,, \quad K_1 = \widehat{I\,\cos\varphi} = \hat{h}_1\,, \quad K_2 = \widehat{-I\,\sin\varphi} = \hat{h}_2\,, \tag{1.95}$$

of the unitary 1-dimensional subgroups in the positive discrete series of the unitary irreducible representations of $SO^{\uparrow}(1,2)$ or one of its covering groups. The essential advantage of this procedure is: Given a unitary representation of a *group*, then the *Lie algebra generators* of its 1-dimensional subgroups are *self-adjoint*!

The K_j obey the relations

$$[K_0, K_1] = iK_2, \quad [K_0, K_2] = -iK_1, \quad [K_1, K_2] = -iK_0 \quad . \tag{1.96}$$

A salient feature of the positive series representations is that K_0, the generator of the $O(2)$ subgroup, has the spectrum

$$\mathrm{spec}(K_0) = \{k+n, k>0, n=0,1,\ldots\}\ , \tag{1.97}$$

where the number $k > 0$ characterizes the representation (like the number j in the case of $SU(2)$!) Its possible values depend on the group: For the group $SO^{\uparrow}(1,2)$ k can take the positive integer values $k = 1, 2, \ldots$ and for the, e.g. 2-fold covering $SU(1,1) \cong SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$ the values $k = 1/2, 1, 3/2, 2, \ldots$.

The eigenstates $|k,n\rangle$, $n = 0, 1, \ldots$, of K_0 are normalized elements of the associated Hilbert space and can be used as a (infinite) basis. That basis can be generated with the help of raising and lowering operators

$$K_+ = K_1 + i\,K_2\ , \quad K_- = K_1 - i\,K_2\ , \tag{1.98}$$

with

$$K_-|k,n=0\rangle = 0\,, \quad K_0|k,n=0\rangle = k\,|k,n=0\rangle\,. \tag{1.99}$$

For an irreducible representation the Casimir operator

$$L = K_1^2 + K_2^2 - K_0^2 \tag{1.100}$$

has the eigenvalues

$$l = k(1-k)\,. \tag{1.101}$$

This has an immediate surprising quantum physical implication: Classically we have the trigonometric Pythagorean relation

$$(h_1)^2 + (h_2)^2 = h_0^2\,, \tag{1.102}$$

but - because of Eq. 1.100 - quantum theoretically we get for an irreducible representation

$$K_1^2 + K_2^2 = K_0^2 + L = K_0^2 + k\,(1-k)\,. \tag{1.103}$$

As the eigenvalues 1.101 of L vanish only for $k = 1$, we see that *the quantum effects can violate Pythagoras' theorem*!

In the case of the harmonic oscillator we have $k = 1/2, l = 1/4$. So we have "quantum trigonometrical deviations" for that system!

Testing the quantum relation 1.103 is one of the major experimental challenges of the whole approach discussed here!

Another important point is the following:

It has been realized [51] that the generators K_j may be constructed from the operators 1.18 and 1.19 in a non-linear (Holstein-Primakoff type [52]) manner:

$$K_+ = a^+\sqrt{N+2k}\,,\ K_- = \sqrt{N+2k}\,a\,,\ K_0 = N+k\,. \tag{1.104}$$

Here I have - as is usual - dropped the "hat" on the operators a, a^+ and N.

However, it is far more interesting to turn the argument around:

Given the self-adjoint operators K_j of a positive discrete series irreducible unitary representation of the group $SU(1,1) \cong SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$, then one can define annihilation and creation operators

$$a = (K_0+k)^{-1/2}K_-\,, \quad a^+ = K_+(K_0+k)^{-1/2}\,, \quad N = K_0 - k\,, \tag{1.105}$$

which have the usual properties

$$[a, a^+] = 1 \ , \quad N = a^+ a \ . \tag{1.106}$$

The *composite* position and momentum operators

$$\tilde{Q} = \frac{1}{\sqrt{2}}[(K_0 + k)^{-1/2} K_- + K_+ (K_0 + k)^{-1/2}] , \tag{1.107}$$

$$\tilde{P} = \frac{1}{i\sqrt{2}}[(K_0 + k)^{-1/2} K_- - K_+ (K_0 + k)^{-1/2}] , \tag{1.108}$$

are the quantized counterparts of the classical composite coordinates 1.90 and 1.91!

Thus, in a sense the operators K_0, K_1 and K_0 are at least as fundamental as the operators Q and P and it appears possible - at least in principle - to base the structures of quantum mechanics on the group $SO^{\uparrow}(1,2)$, its covering groups and corresponding higher dimensional generalizations!

1.5.2 Chapter 3

Ch. 3 discusses the problem how the operators K_1 and K_2 may be used in order to "measure" the phase φ appearing in physically interesting state vectors.

The state vectors of that analysis are three different types of coherent states (Barut-Girardello [53], Perelomov [54,264] and the conventional Schrödinger-Glauber [55,56] coherent states) which are associated with the Lie algebra of $SO^{\uparrow}(1,2)$ and all of which are characterized by a complex number with a non-vanishing phase.

The three types can be defined by the relations

$$K_- |k, z\rangle = z \, |k, z\rangle \ , \quad z = |z| \, e^{i\phi} \in \mathbb{C} \ , , \tag{1.109}$$

$$(K_0 + k)^{-1} K_- |k, \lambda\rangle = \lambda \, |k, \lambda\rangle \ , \quad \lambda = |\lambda| \, e^{i\,\theta} \in \mathbb{D} \ , \tag{1.110}$$

$$\mathbb{D} = \{\lambda \in \mathbb{C} \ , \ |\lambda| < 1\} \ , \tag{1.111}$$

$$(K_0 + k)^{-1/2} K_- |k, \alpha\rangle = \alpha \, |k, \alpha\rangle \ , \quad \alpha = |\alpha| \, e^{i\,\beta} \in \mathbb{C} \ . \tag{1.112}$$

(As to the definition 1.112 see the first of the Eqs. 1.105.)

Ch. 3 contains a large number of matrix elements of the operators K_0, K_1 and K_2, as well as functions of them, with respect to the coherent states 1.109-1.112.

The results obtained show very clearly that the operators K_1 and K_2 are well suited for "measuring" the phase content of those states and that their matrix elements provide a large number of predictions for experimental tests.

1.5.3 Chapter 4

That chapter deals with $SU(1,1)$-related properties of the harmonic oscillator.

For $k = 1/2$ we can identify the Hamilton operator H with the operator K_0 (the frequency ω being normalized to 1).

The group $SU(1,1)$ has the following explicit irreducible unitary representation for $k = 1/2$:

$$
\begin{aligned}
(f_2, f_1) &= \frac{1}{2\,\pi} \int_0^{2\,\pi} d\varphi \; \bar{f}_2(\varphi)\, f_1(\varphi) \;, && (1.113)\\
|k = 1/2, n\rangle &= e^{i\,n\,\varphi} \;, n = 0, 1, \ldots \;, && (1.114)\\
K_0 &= \frac{1}{i}\,\partial_\varphi + 1/2 \;, && (1.115)\\
K_- &= e^{-i\,\varphi}\,\frac{1}{i}\,\partial_\varphi \;, && (1.116)\\
K_+ &= e^{i\,\varphi}\,(\frac{1}{i}\,\partial_\varphi + 1) \;, && (1.117)\\
H &= K_0 \;. && (1.118)
\end{aligned}
$$

It is possible to describe the whole quantum theory of the harmonic oscillator in the Hilbert space with the scalar product 1.113 !

According to the general relations 1.109-1.112 we get *three* different types of coherent states for the harmonic oscillator! In the Hilbert space above they have the forms

$$
\begin{aligned}
f_z(\varphi) &= \frac{e^{z\,e^{i\varphi}}}{\sqrt{I_0(2|z|)}} \;, && (1.119)\\
& \qquad I_0 \;:\; \text{modified Bessel function of 1st kind} \,,\\
f_\lambda(\varphi) &= (1 - |\lambda|^2)^{1/2} (1 - \lambda\, e^{i\varphi})^{-1} \;, && (1.120)\\
f_\alpha(\varphi) &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha\, e^{i\varphi})^n}{\sqrt{n!}} \;, && (1.121)
\end{aligned}
$$

and they have a number of interesting properties!

The use of this Hilbert space also allows for a critical evaluation of the widespread - rather controversial - notion of "phase states" [57].

Despite the superficial impression, the phase φ above is *not* the *quantum mechanical observable canonically conjugate to the Hamilton operator* H ! It is merely the mathematical variable used in the Hilbert space with the scalar product 1.113 and the basis 1.114 !

The two genuine *quantum mechanical* observables K_1 and K_2 here have the form

$$
\begin{aligned}
K_1 &= \frac{1}{2}(K_+ + K_-) = \cos\varphi\,\frac{1}{i}\partial_\varphi + \frac{1}{2}\,e^{i\,\varphi} \;, && (1.122)\\
K_2 &= \frac{1}{2i}(K_+ - K_-) = \sin\varphi\,\frac{1}{i}\partial_\varphi + \frac{1}{2i}\,e^{i\,\varphi} \;. && (1.123)
\end{aligned}
$$

They are self-adjoint with respect to the scalar product 1.113 and have a continuous spectrum.

The rest of that chapter discusses the quantum mechanics of the harmonic oscillator in terms of the Hardy space $H^2(\mathbb{R}, d\xi)$ on the real line and the relationship of that Hilbert space to the usual $L^2(\mathbb{R}, d\xi)$.

1.5.4 Chapter 5

That chapter discusses possible attempts to define operators $\widehat{\cos\varphi}$ and $\widehat{\sin\varphi}$ in the sense of London, Susskind and Glogower by combining the operators K_j in a non-linear way.

At first sight suitable generalizations of the operators 1.34 and 1.35 appear to be possible. But this is only so as long as one uses them in lowest order. Already their squares are of doubtful use! The reason is the following:

If one expresses the operators 1.34 and 1.35 and their generalizations for $k \neq 1/2$ in terms of the K_1, K_2 and K_0, then the *cosine*-operator does not only depend on K_1 - as one would expect - but also on K_2. Similarly, the *sine*-operator not only depends on K_2 but also on K_1. This is unsatisfactory and even contradictory in the present framework, where K_1 stands for cosine- and K_2 for sine-properties!

Actually it is not necessary to define such cosine- and sine-operators at all! The results of Ch. 3 clearly show that K_1 and K_2 themselves serve all the desired purposes!

1.5.5 Chapter 6

Ch. 6 discusses possible applications of the general group theoretical framework to quantum optical problems. The group $SU(1,1) \cong SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$, its Lie algebra and associated coherent states entered quantum optics in the middle of the eighties of the last century [58–63]. This came in the course of generating squeezed states [64] from the well-known coherent states [65, 66, 68–71, 264] with their "minimum uncertainty" properties: the product of the r.m.s. fluctuations of the operators Q and P with repect to the Schrödinger-Glauber coherent states satisfies the *equality* in the Heisenberg uncertainty relation. In the case of squeezed states one of the r.m.s. fluctuations is made smaller ("squeezed") at the expense of the other factor in the product.

As a preparation for dealing with squeezing properties Sec. 6.1 discusses the adjoint representation of the group $SO^{\uparrow}(1,2)$, i.e. the representation in the vector space of its Lie algebra.

Sec. 6.2 deals with Schwarz's inequality for scalar products as the basis for the different inequalities discussed in connection with squeezing.

Squeezed states can be generated by "squeezing operators" which are bilinear in photon creation and annihilation operators and it was realized that certain combinations obey the Lie algebra of $SO^{\uparrow}(1,2)$:

Sec. 6.3 discusses the case for one mode where

$$K_+^{(1)} = \frac{1}{2}(a^+)^2\,,\ K_-^{(1)} = \frac{1}{2}a^2\,,\ K_0^{(1)} = \frac{1}{2}(a^+a + 1/2)\,. \tag{1.124}$$

As $K_-^{(1)}$ here annihilates $|n=0\rangle$ as well as $|n=1\rangle$ the representation decomposes into one with states having an even number of quanta and one with states having an odd number of quanta. For even numbers one has $k = 1/4$ and for odd numbers $k = 3/4$.

Here only the even states are invariant under the gauge transformation 1.64 which leads to experimentally interesting selection rules.

Another realization of the Lie algebra is provided by a *pair* $a_j^+\,, a_j\,,\ j = 1,2$, of creation and annihilation operators:

$$K_0^{(2)} = \frac{1}{2}(a_1^+a_1 + a_2^+a_2 + 1)\ ,\quad K_+^{(2)} = a_1^+a_2^+\ ,\quad K_-^{(2)} = a_1a_2\ . \tag{1.125}$$

This is discussed in Sec. 6.4 where the relation of such a system to the problem of simultaneous measurements of non-commuting operators [72] is indicated, too.

1.5.6 Chapter 7

It has become popular in quantum optics to express quantum mechanical expectation values

$$\langle A\rangle_\rho = \mathrm{tr}(\rho\, A)\,,\ \rho\ :\ \text{density operator}\,, \tag{1.126}$$

of self-adjoint operators A in terms of of a density function $w(q,p)$ for a phase space like 1.36 and a phase space function $\tilde{A}(q,p)$ corresponding to A such that

$$\langle A\rangle_\rho = \int_{\mathcal{S}^2_{q,p}} dqdp\, w(q,p)\, \tilde{A}(q,p)\,. \tag{1.127}$$

Several such approaches are in use:

The oldest one is due to Weyl and especially Wigner [183]. It makes essential use of Fourier transformation between coordinate and momentum space and is very closely related to properties of the Weyl-Heisenberg group.

A second approach is due to Husimi [192] and centers around the function

$$Q(\alpha,\bar{\alpha}) = \langle\alpha|\rho|\alpha\rangle\,, \tag{1.128}$$

where $|\alpha\rangle$ is a conventional Schrödinger-Glauber coherent state.

A third scheme, due to Sudarshan [194] and Gauber [196], postulates the following "diagonal P-representation" for the density operator:

$$\rho = \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha\, P(\alpha,\bar{\alpha})|\alpha\rangle\langle\alpha|\,. \tag{1.129}$$

Here the "density" $P(\alpha,\bar{\alpha})$ can become negative and highly singular!

As we have now two more types of coherent states, namely $|k,z\rangle$ and $|k,\lambda\rangle$, we can define the corresponding distributions on the phase space 1.8 :

$$S_k(z,\bar{z}) = \langle k,z|\rho|k,z\rangle\,, \tag{1.130}$$
$$T_k(\lambda,\bar{\lambda}) = \langle k,\lambda|\rho|k,\lambda\rangle\,, \tag{1.131}$$

and the density operator representations

$$\rho = \int_{\mathbb{C}} d\mu_k(z)\, F_k(z,\bar{z})\, |k,z\rangle\langle k,z|\,, \tag{1.132}$$
$$\rho = \int_{\mathbb{D}} d\mu_k(\lambda)\, G_k(\lambda,\bar{\lambda})\, |k,\lambda\rangle\langle k,\lambda|\,. \tag{1.133}$$

The distributions 1.130 and 1.131 as well as the representations 1.132 and 1.133 can be discussed and analysed in a very similar manner as has been done extensively for the distribution 1.128 and the representation 1.129.

As we now have altogether six "densities", there exist a large number of relations between them which might be very useful for future applications.

I have not attempted to introduce distributions corresponding to Wigner's one, because it is so closely related to the Weyl-Heisenberg group and its action as a translation group.

1.5.7 Chapter 8

Ch. 8 deals with the symplectic properties of interference patterns, especially the $SO^\uparrow(1,2)$ structure of the main observable quantities:

Consider the sum

$$A = A_1 + A_2 = |A_1|\, e^{-i\,\varphi_1} + |A_2|\, e^{-i\,\varphi_2} \tag{1.134}$$

of two complex amplitudes A_j.

The quantities $|A_j|$ and φ_j may be functions of other parameters, e.g. space or/and time variables etc., depending on the concrete experimental situation.

The absolute square of A has the form

$$\begin{aligned} w_3(I_1, I_2, \varphi) &= |A|^2 = I_1 + I_2 + 2\sqrt{I_1\, I_2}\,\cos\varphi\ , \\ & \quad I_j = |A_j|^2\ ,\ j = 1, 2\ , \quad \varphi = \varphi_1 - \varphi_2\ . \end{aligned} \tag{1.135}$$

Phase shifting one of the two amplitudes A_j by appropriate devices yields new intensities:

$$w_4(I_1, I_2, \varphi) = w_3(\varphi + \pi) = I_1 + I_2 - 2\sqrt{I_1\, I_2}\cos\varphi\ , \tag{1.136}$$

$$w_5(I_1, I_2, \varphi) = w_3(\varphi + \pi/2) = I_1 + I_2 - 2\sqrt{I_1\, I_2}\sin\varphi\ , \tag{1.137}$$

$$w_6(I_1, I_2, \varphi) = w_3(\varphi - \pi/2) = I_1 + I_2 + 2\sqrt{I_1\, I_2}\sin\varphi\ . \tag{1.138}$$

The typical quantities for a *classical* description of the interference pattern are then

$$4\, h_1(\varphi, I_{1,2}) = w_3 - w_4 = 4\, I_{1,2}\,\cos\varphi\ ,\ I_{1,2} = \sqrt{I_1\, I_2}\ , \tag{1.139}$$

$$4\, h_2(\varphi, I_{1,2}) = w_5 - w_6 = -4\, I_{1,2}\,\sin\varphi\ , \tag{1.140}$$

$$4\, h_0(\varphi, I_{1,2}) = 4\, I_{1,2} = \sqrt{(w_4 - w_3)^2 + (w_6 - w_5)^2} > 0\ . \tag{1.141}$$

We see that the essential part of the interference pattern is characterized by the three classical observables $h_j(\varphi, I_{1,2})$ with their associated $\mathfrak{so}(1,2)$ Lie algebra structure we know already.

The system has a number of interesting gauge properties, symplectic reductions and intriguing relations to the symplectic group $Sp(4, \mathbb{R})$ which all play a role in the quantum theory of the system.

The quantum version of the classical relation 1.141 poses critical questions as to the validity of the "operational phase" analysis of the interesting experiments by Mandel et al. [245], because we know from Ch. 2 that such a trigonomical Pythagorean relation can be violated on the quantum level.

Sec. 8.2.2 discusses how one may test experimentally - at least in principle - the different predictions for the observable quantities of interference patterns mentioned above, e.g. by multiport homodyning [243]. *That section is mainly an appeal to the experts*!

One last general remark in this context:

It is essential to realize that a group theoretical quantization does *not* assume that the generators of the basic Lie algebra themselves can be expressed by some conventional canonical operators (Q_j, P_j), $j = 1, \ldots$, or in terms of associated annihilation and creation operators a_j and a_j^+.

A well-known similar example is the angular momentum: In the case of an orbital angular momentum the components $\hat{l}_j$, $j = 1, 2, 3$, can be expressed in terms of three pairs (Q_j, P_j), but this is not essential at all for the quantum theory. That can be derived from the single property that the 3 operators $\hat{l}_j$ generate the Lie algebra of the group $SO(3)$ or that of its covering group $SU(2)$!

The representations of the latter allow for half-integer spins not expected from semi-classical arguments! *Classically* the Poisson brackets of the 3 components $l_1 = q_2 p_3 - q_3 p_2\ , l_2 = \ldots$ fulfill the $SO(3)$ Lie algebra, too. However, this applies to the *orbital* angular momentum only.

Correspondingly, though some $SO^{\uparrow}(1,2)$ Lie algebra representations may be constructed in terms of creation and annihilation operators, this is not possible for other interesting ones and also definitely not necessary!

1.6 Apologies

This article has been written by an outsider and non-specialist as to the impressively active and successful quantum optics community! I have tried rather hard to understand some central topics and problems of that growing and fascinating field. But I am sure I missed important contributions and essential papers ! So I would like to apologize to all those authors the work of which I failed to appreciate or simply overlooked. I shall certainly value being informed about any oversight or worse.

In addition I apologize to the mathematical physicists for my pedestrian way of dealing with the mathematics of the problems. But I wanted to focus on the physical aspects of those.

Chapter 2

The $SO^{\uparrow}(1,2)$ Lie algebra generators K_0, K_1, and K_2 as quantum "observables" and the associated number states

2.1 General structures

According to the Introduction and Appendix A.2 the basic classical "canonical observables" on the phase space

$$\mathcal{S}^2_{\varphi,I} = \{(\varphi, I);\, \varphi \in \mathbb{R} \bmod 2\pi,\ \ I > 0\,\} \tag{2.1}$$

are

$$h_0(\varphi, I) = I\,, \quad h_1(\varphi, I) = I\,\cos\varphi\,, \quad h_2(\varphi, I) = -I\,\sin\varphi\,. \tag{2.2}$$

They obviously are not independent, but obey the quadratic Pythagorean relation

$$(I\cos\varphi)^2 + (I\sin\varphi)^2 = I^2\,. \tag{2.3}$$

They also obey the Lie algebra $\mathfrak{so}(1,2)$ of the group $SO^{\uparrow}(1,2)$, namely

$$\{h_0, h_1\}_{\varphi,I} = -h_2\,, \quad \{h_0, h_2\}_{\varphi,I} = h_1\,, \quad \{h_1, h_2\}_{\varphi,I} = h_0\,, \tag{2.4}$$

where

$$\{h_0, h_1\}_{\varphi,I} \equiv \partial_\varphi h_0\,\partial_I h_1 - \partial_I h_0\,\partial_\varphi h_1\,, \ldots\,. \tag{2.5}$$

The relations 2.2-2.4 constitute the main structure properties for any kind of analysis, uses or applications as to the *classical* phase space 2.1.

The important fact for a quantization of that phase space now is that the group $SO^{\uparrow}(1,2)$ acts transitively, effectively and globally Hamilton-like on it (see the Introduction and Appendix A.2). This allows for a consistent group theoretical quantization in terms of the positive discrete series of irreducible unitary representations of the group $SO^{\uparrow}(1,2)$ or one of its covering groups (Appendix B.3).

The quantization is implemented by replacing the classical observables 2.2 by the corresponding self-adjoint operators K_0, K_1 and K_2 which represent the Lie algebra $\mathfrak{so}(1,2)$ in the Hilbert space of the unitary representation under consideration:

$$h_0 \to K_0 \quad h_1 \to K_1\,, \quad h_2 \to K_2\,. \tag{2.6}$$

The self-adjoint K_j obey the commutation relations

$$[K_0, K_1] = i\,K_2\,, \quad [K_0, K_2] = -i\,K_1\,, \quad [K_1, K_2] = -i\,K_0\,, \tag{2.7}$$

$$[K_+, K_-] = -2K_0\,, \; [K_0, K_\pm] = \pm K_\pm\,, \tag{2.8}$$

where

$$K_+ = K_1 + iK_2\,, \quad K_- = K_1 - iK_2\,. \tag{2.9}$$

In order to calculate expectation values and other matrix elements we have to know the actions of the operators $K_j, j = 0, 1, 2$, on the Hilbert spaces associated with the positive discrete series representations of $SO^\uparrow(1,2)$ (or its covering groups).

In the following I frequently use properties which are discussed in more detail in Appendix B and in the literature quoted there.

As the eigenfunctions of K_0 – the generator of the compact subgroup $O(2)$ of $SO^\uparrow(1,2)$ – form a complete basis of the associated Hilbert spaces, it is convenient to use them as a starting point. The operators 2.9 act as ladder operators. The positive discrete series is characterized by the property that there exists a state $|k,0\rangle$ for which

$$K_-|k,0\rangle = 0\,, \quad K_0|k,0\rangle = k\,|k,0\rangle\,. \tag{2.10}$$

The number $k > 0$ characterizes the representation:

For a general normalized eigenstate $|k,n\rangle$ of K_0 we have

$$K_0|k,n\rangle = (k+n)|k,n\rangle\,, \; n = 0, 1, \ldots, \; k > 0\;, \tag{2.11}$$

$$K_+|k,n\rangle = \omega_n\,[(2k+n)(n+1)]^{1/2}|k,n+1\rangle\;, \quad |\omega_n| = 1\;, \tag{2.12}$$

$$K_-|k,n\rangle = \frac{1}{\omega_{n-1}}[(2k+n-1)n]^{1/2}|k,n-1\rangle\;. \tag{2.13}$$

In irreducible unitary representations the operator K_- is the adjoint operator of K_+ : $(f_1, K_+ f_2) = (K_- f_1, f_2)$. The phases ω_n serve to guarantee this property. Their choice depends on the concrete realization of the representations. In the examples discussed in Ref. [40] and in Appendix B.3 they have the values 1 or i. In the following I assume ω_n to be independent of n : $\omega_n = \omega$. But then it can be absorbed into the definition of $K_\pm$ and we can ignore it in the following discussions.

The Casimir operator

$$L = K_1^2 + K_2^2 - K_0^2 = K_+K_- + K_0(1 - K_0) = \tag{2.14}$$

$$= K_-K_+ - K_0(1 + K_0) \tag{2.15}$$

is a multiple of the unit operator in an irreducible unitary representation and has the eigenvalues

$$l = k(1-k)\,. \tag{2.16}$$

The allowed values of k depend on the group:

For $SO^\uparrow(1,2)$ itself one can have $k = 1, 2, \ldots$ and for the double coverings $SU(1,1) \cong SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$ the values $k = 1/2, 1, 3/2, \ldots$. For the universal covering group k may be any real number > 0 (for details see Refs. [40, 259]). The appropriate choice will depend on the physics to be described.

In any case, for any unitary representation the number k has to be non-vanishing and positive!

Writing the Casimir operator for an irreducible representation in the form

$$K_1^2 + K_2^2 = K_0^2 + l\ ,\ l = k - k^2 = \frac{1}{4} - (k - \frac{1}{2})^2 \tag{2.17}$$

and comparing it with the corresponding classical Pythagorean (trigonomical) relation 2.3 one sees immediately how that important geometrical property *may be violated quantum theoretically*[1] !

Only for $k = 1$ there is no such violation!

But already for the important harmonic oscillator where $k = 1/2$ (see Ch. 4 below) we have $l = 1/4$ and the r.h. side of Eq. 2.17 is enlarged compared to the classical case 2.3. This is the largest value l may take.

We shall later (Ch. 6) encounter representations with $k = 1/4$ and $k = 3/4$. In both cases $l = 3/16$.

On the other hand, if $k > 1$ then the r.h. side of Eq. 2.17 is reduced.

The relation 2.17 implies that we have the following constraint on the mean-square fluctuations $(\Delta K_j)_k^2$ and the averages $\langle K_j\rangle_k \equiv \langle k|K_j|k\rangle$ of the operators $K_j\,, j = 0, 1, 2$, with respect to any state $|k\rangle$ of an irreducible unitary representation with index k:

$$(\Delta K_1)_k^2 + (\Delta K_2)_k^2 - (\Delta K_0)_k^2 + \langle K_1\rangle_k^2 + \langle K_2\rangle_k^2 - \langle K_0\rangle_k^2 = k - k^2 \tag{2.18}$$

The modification 2.17 of the classical trigonomical relation 2.3 is one of the most striking predictions of the present approach to group quantizing the phase space 2.1 It is, of course, extremely important to test such a prediction experimentally. I shall come back to this point on several occasions.

2.2 Matrix elements of the number states

Eq. 2.12 implies

$$|k, n\rangle = \frac{1}{\sqrt{(2k)_n\, n!}}(K_+)^n |k, 0\rangle\ . \tag{2.19}$$

Here the definition

$$2k\,(2k+1)\cdots(2k+n-1) \equiv (2k)_n = \frac{\Gamma(2k+n)}{\Gamma(2k)} \tag{2.20}$$

has been used, where $(a)_n$ denotes Pochhammer's symbol [73, 74].

The completeness relation for the states $|k, n\rangle$ may formally be written as

$$\sum_{n=0}^{\infty} |k, n\rangle\langle k, n| = \mathbf{1}. \tag{2.21}$$

The expectation values of the self-adjoint operators

$$K_1 = (K_+ + K_-)/2\,, \quad K_2 = (K_+ - K_-)/2i\,, \tag{2.22}$$

(which correspond to the classical observables $I\cos\varphi$ and $-I\sin\varphi$) with respect to the eigenstates $|k, n\rangle$ and the associated mean-square fluctuations can be calculated with the help of the relations 2.11-2.13:

$$\langle k, n|K_j|k, n\rangle = 0\ ,\quad j = 1, 2\,;\ n = 0, 1, \ldots\ . \tag{2.23}$$

[1] The additional term $l = k - k^2$ in Eq. 2.17 is a genuine quantum effect associated with the non-classical properties of the ground state - see Eqs. 2.10 - as shown explicitly by the example of the harmonic oscillator. It is not to be compared with the constant $a \neq 0$ in the equation $x_1^2 + x_2^2 = x_0^2 + a$ of a classical hyperboloid!

Thus, for the number states the expectation values of the "quadrature" operators K_1 and K_2 vanish. This is what one expects.

The corresponding mean-square fluctuations are

$$\begin{aligned}(\Delta K_j)^2_{k,n} &= \langle k,n|K_j^2|k,n\rangle = \frac{1}{2}(n^2+2nk+k) = \qquad (2.24)\\ &= \frac{1}{2}[(n+k)^2+l]\,,\ j=1,2\,,\\ (\Delta K_j)^2_{k,n=0} &= \langle k,n=0|K_j^2|k,n=0\rangle = \frac{k}{2}\,. \qquad (2.25)\end{aligned}$$

Because of $[K_1,K_2]=-iK_0$, the general uncertainty relation

$$\Delta A\,\Delta B \geq \frac{1}{2}|\langle\,|[A,B]|\,\rangle| \qquad (2.26)$$

for self-adjoint operators A and B here takes the special form

$$(\Delta K_1)_{k,n}\,(\Delta K_2)_{k,n} = \frac{1}{2}(n^2+2kn+k) \geq \frac{1}{2}|\langle k,n|K_0|k,n\rangle| = \frac{1}{2}(n+k)\ . \qquad (2.27)$$

The equality sign holds for the ground state $|k,n=0\rangle$ only.

The Eqs. 2.24 imply further that

$$\begin{aligned}\langle k,n|K_1^2|k,n\rangle + \langle k,n|K_2^2|k,n\rangle &= (\Delta K_1)^2_{k,n} + (\Delta K_2)^2_{k,n} = \qquad (2.28)\\ = (n+k)^2+l &= \langle k,n|K_0^2|k,n\rangle + l\ , \qquad (2.29)\end{aligned}$$

which is, of course, just special case of the Casimir operator relation 2.17. But it shows in addition that for very large n the term l on the r.h.s. of Eq. 2.29 is negligible and the *correspondence principle* holds.

We also have

$$\langle k,n|K_1\,K_2 + K_2\,K_1|k,n\rangle = 0\,. \qquad (2.30)$$

2.3 Position and momentum operators as functions of the Lie algebra generators K_0, K_1 and K_2

If $f(K_0)$ is an "appropriate" function of the operator K_0, then the commutation relations 2.7 imply relations which are completely analogous to those in Eq. 1.31:

$$K_-\,f(K_0) = f(K_0+\mathbf{1})\,K_-\,,\ f(K_0)\,K_+ = K_+\,f(K_0+\mathbf{1})\,. \qquad (2.31)$$

"Appropriate" means that the application of the operators $f(K_0)$ and $f(K_0+\mathbf{1})$ to all number states 2.19 is well-defined, including the application to the ground state $|k,n=0\rangle$!

The relations 2.31 imply the following important observation (see also Sec. 1.4):

According to Eq. 2.11 the operator K_0 is a positive definite one, in an irreducible unitary representation of the positive discrete series! Therefore the operator $(K_0+k)^{-1/2}$ does always exist in the Hilbert space for such a representation and we can define

$$a = (K_0+k)^{-1/2}K_-\ ,\ a^+ = K_+(K_0+k)^{-1/2}\ ,\ N = K_0 - k\ , \qquad (2.32)$$

which, according to the relations 2.12 and 2.13 have the properties

$$a|k,n\rangle = \sqrt{n}\,|k,n-1\rangle\ ,\quad a^+|k,n\rangle = \sqrt{n+1}\,|k,n+1\rangle\ . \qquad (2.33)$$

It follows that

$$[a, a^+] = 1 \ . \tag{2.34}$$

The relation 2.34 can also be shown in the following way: From Eq. 2.14 we have

$$K_- K_+ = L + K_0(K_0 + 1) \,, \quad L = k(1-k) \,, \tag{2.35}$$

so that

$$a \cdot a^+ = \frac{k(1-k) + K_0(K_0+1)}{K_0 + k} = K_0 - k + 1 = N + 1 \,. \tag{2.36}$$

On the other hand, because of

$$K_+ K_- = L + K_0(K_0 - 1) \tag{2.37}$$

and the relations 2.31, we get

$$a^+ \cdot a = K_+(K_0 + k)^{-1} K_- = (K_0 + k - 1)^{-1} K_+ K_- = K_0 - k = N \,. \tag{2.38}$$

Thus, it always possible to associate creation and annilation operators 2.32 with a given irreducible unitary representation of the positive discrete series of $SO^{\uparrow}(1,2)$ or $SU(1,1) \cong SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$ or any other of the covering groups!

As a consequence we can introduce the *composite self-adjoint position and momentum operators* (with $\omega = 1$)

$$\begin{aligned}
\tilde{Q} &= \frac{1}{\sqrt{2}}(a^+ + a) = \frac{1}{\sqrt{2}}[K_+(K_0+k)^{-1/2} + (K_0+k)^{-1/2} K_-] && (2.39)\\
&= \frac{1}{\sqrt{2}}[(K_0+k)^{-1/2} K_1 + K_1 (K_0+k)^{-1/2}] - && (2.40)\\
&\quad -\frac{i}{\sqrt{2}}[(K_0+k)^{-1/2} K_2 - K_2 (K_0+k)^{-1/2}] \,, \\
\tilde{P} &= \frac{i}{\sqrt{2}}(a^+ - a) = \frac{i}{\sqrt{2}}[K_+(K_0+k)^{-1/2} - (K_0+k)^{-1/2} K_-] && (2.41)\\
&= -\frac{i}{\sqrt{2}}[(K_0+k)^{-1/2} K_1 - K_1 (K_0+k)^{-1/2}] - && (2.42)\\
&\quad -\frac{1}{\sqrt{2}}[(K_0+k)^{-1/2} K_2 + K_2 (K_0+k)^{-1/2}] \,.
\end{aligned}$$

This is possible[2] for any allowed k.

The operators $\tilde{Q}$ and $\tilde{P}$ are the quantum versions of the classical coordinates 1.90 and 1.91. They are "observables" in the sense of Secs. 1.4 and 6.3 because they transform according to the group $SO^{\uparrow}(1,2)$ (adjoint representation from Sec. 6.1), and not according to $Sp(2,\mathbb{R})$ (Sec. 6.3)!

As has been stressed in Sec. 1.4.1, *the possibility to construct position and momentum operators from them makes the operators K_1, K_2 and K_0 as least as fundamental as the Q and P themselves!*

The following point is important, too: The matrix elements of the operators 2.32, 2.39 and 2.41 in general will depend on the index k, i.e. one the choice of the unitary representation. This will become evident and explicitly shown in Secs. 3.1 and 3.2 in connection with coherent states.

[2]As to to the relationship of the operators 2.39-2.42 to the Stone–von Neumann theorem [37] see Subsec. 4.4.4.

2.4 A contraction of the Lie algebra $\mathfrak{so}(1,2)$ to the Weyl-Heisenberg algebra

There is another relationship between the Lie algebra 2.7 and the Lie algebra

$$[a,a]=0\,,\quad [a^+,a^+]=0\,,\quad [a,a^+]=1\,, \tag{2.43}$$

of the Heisenberg-Weyl group 1.46 [53]:

Define

$$\check{K}_\pm=(2k)^{-1/2}K_\pm\ ,\ \check{K}_3=k^{-1}K_0 \tag{2.44}$$

Then

$$[\check{K}_+,\check{K}_-]=-\check{K}_3\ ,\ [\check{K}_3,\check{K}_\pm]=\pm k^{-1}\check{K}_\pm\ . \tag{2.45}$$

If we now take the limit $k\to\infty$, then - formally - the limit of the Lie algebra 2.45 is the Weyl-Heisenberg Lie algebra 2.43! This may be justified by looking at the matrix elements of the new operators 2.44: From Eqs. 2.11- 2.13 we get

$$\langle k,n_2|\check{K}_3|k,n_1\rangle = (1+\frac{n_1}{k})\delta_{n_2,n_1}, \tag{2.46}$$

$$\langle k,n_2|\check{K}_+|k,n_1\rangle = [(1+\frac{n_1}{2k})(n_1+1)]^{1/2}\,\delta_{n_2,n_1+1}, \tag{2.47}$$

$$\langle k,n_2|\check{K}_-|k,n_1\rangle = [(1+\frac{n_1-1}{2k})n_1]^{1/2}\,\delta_{n_2,n_1-1}\ . \tag{2.48}$$

Taking the limit $k\to\infty$ shows that

$$\lim_{k\to\infty}\check{K}_3\to 1\ ,\ \lim_{k\to\infty}\check{K}_+\to a^+\ ,\ \lim_{k\to\infty}\check{K}_-\to a\ . \tag{2.49}$$

However, we have seen in the previous section that it is not necessary to go to the limit $k\to\infty$ in order to arrive at the Lie algebra 2.43 if a representation of the Lie algebra 2.7 is given. We may just use the relations 2.32 or 2.39 and 2.41!

Chapter 3

Three types of coherent state matrix elements of the "observables" K_0, K_1 and K_2

Next I want to discuss matrix elements of the "observables" K_j, $j = 0, 1, 2$, with respect to coherent states. The purpose is to see how the properties of their classical counterparts

$$h_0 = I\,, \quad h_1 = I\,\cos\varphi\,, \quad h_2 = -I\,\sin\varphi\,,$$

are incorporated into those matrix elements, especially how the angle φ is represented.

The three different types of coherent states discussed below are all characterized by a complex number and it is interesting to see how the phase of these numbers is "measured" by the operators K_1 and K_2. Many of the mathematical results discussed in the following are well-known. They are repeated here in order to keep the discussion of the intended aim rather self-contained. Most of the related References will be given in the course of the chapter.

The three types of coherent states – those of Barut-Girardello [53], Perelomov [54,110,264] and Schrödinger-Glauber [55, 56] – can all be characterized as eigenstates of the annihilation operators $K_-, (K_0 + k)^{-1}K_-$ and $(K_0 + k)^{-1/2}K_-$, as defined in the Eqs. 1.109-1.112.

3.1 Barut-Girardello coherent states

3.1.1 General properties

From the relation 2.19 we get for the Barut-Girardello states 1.109

$$\langle k, n|k, z\rangle = \left[\frac{1}{(2k)_n\, n!}\right]^{1/2} z^n\, \langle k, n = 0|k, z\rangle\ , \tag{3.1}$$

so that

$$\begin{aligned} \langle k, z|k, z\rangle &= \sum_{n=0}^{\infty} = \langle k, z|k, n\rangle\langle k, n|k, z\rangle \qquad (3.2)\\ &= |\langle k, n = 0|k, z\rangle|^2\, g_k(|z|^2)\ , \\ g_k(|z|^2) &= \sum_{n=0}^{\infty} \frac{|z|^{2n}}{(2k)_n\, n!} \qquad (3.3)\\ &= \Gamma(2k)|z|^{1-2k}\, I_{2k-1}(2|z|)\ . \end{aligned}$$

Here

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(\nu+n+1)} \left(\frac{x}{2}\right)^{2n} \tag{3.4}$$

is the usual modified Bessel function of the first kind [75] which has the asymptotic expansion

$$I_\nu(x) \asymp \frac{e^x}{\sqrt{2\pi\,x}} \left[1 - \frac{4\nu^2-1}{8\,x} + 2\frac{(4\nu^2-1)(4\nu^2-9)}{16^2\,x^2} + O(x^{-3})\right] \tag{3.5}$$
$$\text{for } x \to +\infty \ .$$

For the function $g_k(|z|^2)$ this implies the asymptotic behaviour

$$g_k(|z|^2) \asymp \frac{\Gamma(2k)}{2\sqrt{\pi}} |z|^{\frac{1}{2}-2k}\ e^{2|z|} \left[1 - \frac{4(2k-1)^2-1}{16\,|z|} + \right. \tag{3.6}$$
$$+ \left. \frac{[4(2k-1)^2-1][4(2k-1)^2-9]}{2\,16^2\,|z|^2} + O(|z|^{-3})\right]$$
$$\text{for } |z| \to +\infty \ .$$

If $\langle k,z|k,z\rangle = 1$ we have

$$|\langle k, n=0|k,z\rangle|^2 = \frac{1}{g_k(|z|^2)} \ . \tag{3.7}$$

Choosing the phase of $\langle k, n=0|k,z\rangle$ to be zero we finally get the expansion

$$|k,z\rangle = \frac{1}{\sqrt{g_k(|z|^2)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{(2k)_n\,n!}}\,|k,n\rangle \quad . \tag{3.8}$$

Notice that $|k, z=0\rangle = |k, n=0\rangle$.
In order to avoid explicit factors like $|z|^{2k-1}$ etc. it is convenient to introduce the function $g_k(w)$ which in addition is an entire holomorphic function of the complex variable w. This last property does not hold for the function I_ν if ν is not an integer.
In the general language of hypergeometric series [76] we have the relation

$$g_k(w) = {}_0F_1(2k; w) \ . \tag{3.9}$$

${}_0F_1(2k;w)$ is an entire function of order $1/2$ and type 2, i.e. it behaves for large $|w|$ as $\leq O(\exp(2\,|w|^{1/2}))$ (see Eq. 3.6). As to the notion of "order" and "type" of an entire function see Ref. [77].

The function ${}_0F_1(2k;w)$ obeys the differential equation

$$w\frac{d^2({}_0F_1)}{dw^2} + 2k\,\frac{d({}_0F_1)}{dw} - {}_0F_1 = 0 \ . \tag{3.10}$$

Two different coherent states are not orthogonal:

$$\langle k, z_2|k,z_1\rangle = \sum_{n=0}^{\infty} \langle k,z_2|k,n\rangle\langle k,n|k,z_1\rangle = \tag{3.11}$$
$$= \frac{g_k(\bar{z}_2\,z_1)}{\sqrt{g_k(|z_2|^2)\,g_k(|z_1|^2)}} \ .$$

They are complete, however, in the sense that, with $z = |z|\, e^{i\,\phi}$, we have the relation

$$\frac{2}{\pi\,\Gamma(2k)} \int_0^\infty d|z|\, |z|^{2k}\, K_{2k-1}(2|z|)\, g_k(|z|^2) \times$$
$$\times \int_0^{2\pi} d\phi\, \langle k, n_2|k, |z|\, e^{i\phi}\rangle \langle k, |z|\, e^{i\phi}|k, n_1\rangle =$$
$$= \langle k, n_2|k, n_1\rangle = \delta_{n_2 n_1}\ , \quad (3.12)$$

because [78]

$$\int_0^\infty d|z|\, |z|^{2(n+k)} K_{2k-1}(2|z|) = \frac{1}{4} n!\, \Gamma(2k+n)\ . \quad (3.13)$$

Here $K_\nu(x)$ is the modified Bessel function of the third kind [75]. It has the asymptotic expansion

$$K_\nu(x) \asymp$$
$$\asymp \sqrt{\frac{\pi}{2\,x}}\, e^{-x}\, \left[1 + \frac{4\nu^2 - 1}{8\,x} + 2\frac{(4\nu^2-1)(4\nu^2-9)}{16^2\, x^2} + O(x^{-3})\right]$$
$$\text{for } x \to +\infty \quad (3.14)$$

and a singularity for $x \to 0$ like

$$K_\nu(x) \ \to\ \frac{\Gamma(\nu)}{2} \left(\frac{x}{2}\right)^{-\nu} (1 + O(x^2))\ \text{ for } \nu > 0\ , \quad (3.15)$$
$$K_0(x) \ \to\ -[\gamma + \ln\frac{x}{2}] + O(x^2\, \ln x)\ , \quad (3.16)$$
$$\gamma \ =\ 0.57\ldots\ :\ \text{Euler's constant}\,.$$

That singularity is, however, removed by the additional factor $|z|^{2k}$ in the measure of the integral 3.12.

Formally we may express the completeness relation 3.12 ("resolution of the identity") as

$$\int_{\mathbb{C}} d\mu_k(z)\, |k, z\rangle\langle k, z| \ =\ \mathbf{1}\,, \quad (3.17)$$
$$d\mu_k(z) \ =\ m_k(z)\, d^2 z\ , \quad (3.18)$$
$$m_k(z) \ =\ \frac{2}{\pi\,\Gamma(2k)}\, |z|^{2k-1}\, K_{2k-1}(2|z|)\, g_k(|z|^2)\ ,$$
$$d^2 z = |z|\, d|z|\, d\phi\ .$$

The completeness relation implies the following convolution property for the scalar product 3.11:

$$\int_{\mathbb{C}} d\mu_k(z) \langle k, z_2|k, z\rangle\langle k, z|k, z_1\rangle = \langle k, z_2|k, z_1\rangle\ . \quad (3.19)$$

3.1.2 Associated Hilbert space of holomorphic functions

It is worthwhile to have a brief look at the Hilbert space of holomorphic functions associated with the coherent states $|k, z\rangle$: The coefficients

$$\tilde{f}_{k,n}(z) = \frac{z^n}{\sqrt{(2k)_n\, n!}}\ ,\ n = 0, 1, \ldots \quad (3.20)$$

in the expansion 3.8 form an orthonormal basis of the Hilbert space of holomorphic square-integrable functions $\tilde{f}(z)$ with the scalar product

$$(\tilde{f}_2, \tilde{f}_1)_{k,\tilde{\mu}} = \int_{\mathbb{C}} d\tilde{\mu}_k(z)\, \bar{\tilde{f}}_2(z)\, \tilde{f}_1(z)\ , \tag{3.21}$$

$$d\tilde{\mu}_k(z) = \tilde{m}_k(|z|)\, d^2z\ , \tag{3.22}$$

$$\tilde{m}_k(|z|) = \frac{2}{\pi\,\Gamma(2k)}\,|z|^{2k-1}\,K_{2k-1}(2|z|)\ . \tag{3.23}$$

The weight function $\tilde{m}_k(|z|)$ of the measure $d\tilde{\mu}_k(z)$ has the following limiting properties:

$$\tilde{m}_k(|z|=0) = \frac{1}{(2k-1)\,\pi} \text{ for } 2k-1>0\ , \tag{3.24}$$

$$\tilde{m}_{1/2}(|z|) \to -\frac{2}{\pi}[\gamma + \ln|z|] \text{ for } |z|\to 0. \tag{3.25}$$

Thus, the weight $\tilde{m}_{1/2}(|z|)$ has a logarithmic singularity for $|z|\to 0$. Like above, this singularity is, however, suppressed by the additional factor $|z|$ in $d^2z = |z|\,d|z|\,d\phi$.
For $|z|\to\infty$ we get from 3.14

$$\tilde{m}_k(|z|) \asymp \frac{|z|^{2k-3/2}}{\sqrt{\pi}\,\Gamma(2k)}\,e^{-2|z|}\,\left[1+\frac{4(2k-1)^2-1}{16\,|z|}+O(|z|^{-2})\right]\ . \tag{3.26}$$

The same arguments which lead to the relation 3.12 show that

$$(\tilde{f}_{k,n_2}, \tilde{f}_{k,n_1})_{k,\tilde{\mu}} = \delta_{n_2\,n_1}\ . \tag{3.27}$$

The so-called "reproducing kernel" [79–81] for this Hilbert space is given by

$$\tilde{\Delta}_k(\bar{z}_2,\, z_1) = \sum_{n=0}^{\infty} \bar{\tilde{f}}_{k,n}(z_2)\, \tilde{f}_{k,n}(z_1) = g_k(\bar{z}_2 z_1)\,. \tag{3.28}$$

Its most essential property

$$\int_{\mathbb{C}} \tilde{\mu}_k(z_2)\, \tilde{\Delta}_k(\bar{z}_2, z_1)\, z_2^n = z_1^n\ ,\ n\geq 0\,, \tag{3.29}$$

follows immediately from the orthogonality relations 3.27. Thus, if

$$\tilde{f}(z) = \sum_{n=0}^{\infty} c_n\, z^n \tag{3.30}$$

is a convergent series, then the relations 3.29 imply

$$\int_{\mathbb{C}} \tilde{\mu}(z_2)\, \tilde{\Delta}_k(\bar{z}_2, z_1)\, \tilde{f}(z_2) = \tilde{f}(z_1)\ . \tag{3.31}$$

Therefore, the kernel $\tilde{\Delta}(\bar{z}_2, z_1)$ here plays the same role as the usual δ-function in other circumstances. It has the property

$$\bar{\tilde{\Delta}}_k(\bar{z}_2, z_1) = \tilde{\Delta}_k(\bar{z}_1, z_2)\ , \tag{3.32}$$

so that

$$\int_{\mathbb{C}} \tilde{\mu}_k(z_2)\, \tilde{\Delta}_k(\bar{z}_1, z_2)\, (\bar{z}_2)^n = (\bar{z}_1)^n\ . \tag{3.33}$$

Notice, however, that we do not have the reproducing property 3.31 if a function f is not holomorphic, i.e. $\partial_{\bar{z}} f \neq 0$. In this case the generalization of the relation 3.29 is

$$\int_{\mathbb{C}} \tilde{\mu}_k(z_2)\, \tilde{\Delta}(\bar{z}_2, z_1) \bar{z}_2^{\bar{n}}\, z_2^n = 0 \text{ for } \bar{n} > n \ , \tag{3.34}$$

$$= \frac{n!\,(2k)_n}{(n-\bar{n})!\,(2k)_{n-\bar{n}}}\, z_1^{n-\bar{n}} \text{ for } n \geq \bar{n} \ . \tag{3.35}$$

(here $\bar{n} = 0, 1, \ldots$ denotes an independent natural number, not the complex conjugate of n.)

The corresponding generalization of 3.33 is obtained by complex conjugation of the relations 3.34 and 3.35

The convolution 3.19 implies

$$\int_{\mathbb{C}} d\tilde{\mu}_k(z)\, \tilde{\Delta}_k(\bar{z}_2, z)\, \tilde{\Delta}_k(\bar{z}, z_1) = \tilde{\Delta}_k(\bar{z}_2, z_1) \ , \tag{3.36}$$

or

$$\int_{\mathbb{C}} d\tilde{\mu}_k(z)\, g_k(\bar{z}_2\, z)\, g_k(\bar{z}\, z_1) = g_k(\bar{z}_2\, z_1) \ . \tag{3.37}$$

One can also implement a - positive discrete series - irreducible unitary representations with index k of $SU(1,1)$ etc. in the present Hilbert space. The corresponding Lie algebra generators are

$$K_+ = z \ , \tag{3.38}$$

$$K_- = 2k\frac{d}{dz} + z\frac{d^2}{dz^2} \ , \tag{3.39}$$

$$K_0 = z\frac{d}{dz} + k \ . \tag{3.40}$$

They obey the commutation relations 2.7 and the relations 2.11-2.13 when applied to the basis functions 3.20.

That K_- is the adjoint operator of K_+ may be seen as follows:

The scalar product of two functions

$$\tilde{f}_j(z) = \sum_{n=0}^{\infty} c_n^{(j)} \tilde{f}_{k,n}(z) \ , \ j = 1, 2, \tag{3.41}$$

is given by the series

$$(\tilde{f}_2, \tilde{f}_1)_{\tilde{\mu}} = \sum_{n=0}^{\infty} \bar{c}_n^{(2)} c_n^{(1)}. \tag{3.42}$$

Applying K_+ to $\tilde{f}_1$, observing the action 2.12 of K_+ on $\tilde{f}_{k,n}$ and the orthonormality 3.27 gives

$$(\tilde{f}_2, K_+\tilde{f}_1)_{\tilde{\mu}} = \sum_{n=0}^{\infty} \sqrt{(2k+n)(n+1)}\, \bar{c}_{n+1}^{(2)} c_n^{(1)} \ . \tag{3.43}$$

Applying K_- to $\tilde{f}_2$ and calculating $(K_-\tilde{f}_2, \tilde{f}_1)$ gives the same result:

$$(K_-\tilde{f}_2, \tilde{f}_1)_{\tilde{\mu}} = (\tilde{f}_2, K_+\tilde{f}_1)_{\tilde{\mu}} \ . \tag{3.44}$$

The operator K_- has the eigenfunctions $\tilde{f}_{k,\zeta}(z)$:

$$K_-\tilde{f}_{k,\zeta}(z) = \zeta\, \tilde{f}_{k,\zeta}(z) \ , \ \zeta \in \mathbb{C}. \tag{3.45}$$

The normalizable solution of this differential equation is (see also Eq. 3.10)

$$C \sum_{n=0}^{\infty} \frac{\zeta^n}{\sqrt{(2k)_n \, n!}} \tilde{f}_{k,n}(z) = C \, g_k(\zeta z) \; . \tag{3.46}$$

Normalization finally gives

$$\tilde{f}_{k,\zeta}(z) = \frac{1}{g_k(|\zeta|^2)} \, g_k(\zeta z) \; . \tag{3.47}$$

Let

$$\tilde{f}(z) = \sum_{n=0}^{\infty} c_n \, \tilde{f}_{k,n}(z) \tag{3.48}$$

be a normalized function,

$$(\tilde{f}, \tilde{f})_{\tilde{\mu}} = \sum_{n=0}^{\infty} |c_n|^2 = 1 \; . \tag{3.49}$$

Then we have the estimate

$$|f(z)|^2 \leq \sum_{n=0}^{\infty} |c_n \tilde{f}_{k,n}(z)|^2 \leq (\sum_{n=0}^{\infty} |c_n|^2)(\sum_{n=0}^{\infty} |\tilde{f}_{k,n}(z)|^2) = g_k(|z|^2) \; . \tag{3.50}$$

Thus

$$|\tilde{f}(z)| \leq \sqrt{g_k(|z|^2)}. \tag{3.51}$$

This describes the allowed upper bound and the possible growth properties for the functions $\tilde{f}(z)$.

The regular solutions of the eigenvalue equations

$$K_1 \tilde{f}_{k,h_1}(z) \;\; = \;\; h_1 \, \tilde{f}_{k,h_1}(z) \, , \tag{3.52}$$

$$K_2 \tilde{f}_{k,h_2}(z) \;\; = \;\; h_2 \, \tilde{f}_{k,h_2}(z) \, , \tag{3.53}$$

where the operators $K_1 = (K_+ + K_-)/2$ and $K_2 = (K_+ - K_-)/2i$ follow from the Eqs. 3.38 and 3.39, may easily be determined with the help of Ref. [82]:

$$\tilde{f}_{k,h_1}(z) \;\; = \;\; \tilde{C}_{k,h_1} \, e^{-iz} \, \Phi(k - ih_1, 2k; 2iz) \, , \;\; h_1 \in \mathbb{R} \, , \tag{3.54}$$

$$\tilde{f}_{k,h_2}(z) \;\; = \;\; \tilde{C}_{k,h_2} \, e^{-z} \, \Phi(k - ih_2, 2k; 2z) \, , \;\; h_2 \in \mathbb{R} \, . \tag{3.55}$$

Here C_{k,h_1} and C_{k,h_2} are (normalization) constants and $\Phi(a, c; z)$ is the at the origin regular basic solution ($\Phi(a, c; z = 0) = 1$) of the differential equation for confluent hypergeometric functions [83].

The convolution relation 3.19 shows that the scalar product 3.11 is a reproducing kernel, too:

$$\Delta_k(\bar{z}_2, z_1) = \langle z_2 | z_1 \rangle = \sum_{n=0}^{\infty} \bar{f}_{k,n}(z_2) \, f_{k,n}(z_1) = \frac{g_k(\bar{z}_2 \, z_1)}{\sqrt{g_k(|z_2|^2) \, g_k(|z_1|^2)}} \; , \tag{3.56}$$

where the functions

$$f_{k,n}(z) \frac{1}{\sqrt{g_k(|z|^2)}} \, \frac{z^n}{\sqrt{(2k)_n \, n!}} \; , \;\; n = 0, 1, \ldots \tag{3.57}$$

form an orthonormal basis with respect to a scalar product with the integration measure 3.18:

$$(f_{k,n_2}, f_{k,n_1})_{k,\mu} = \int_{\mathbb{C}} d\mu_k(z) \bar{f}_{k,n_2}(z) f_{k,n_1}(z) = \delta_{n_2 n_1} \; . \tag{3.58}$$

Due to the coefficient $[g_k(|z|^2)]^{-1/2}$ the functions 3.57 are no longer holomorphic. Instead of 3.29 we now have the relations

$$\int_{\mathbb{C}} d\mu_k(z_2)\Delta_k(\bar{z}_2, z_1)z_2^n/\sqrt{g_k(|z_2|^2)} = z_1^n/\sqrt{g_k(|z_1|^2)}\ . \tag{3.59}$$

Notice also that

$$|k,n\rangle = \int_{\mathbb{C}} d\mu_k(z)\bar{f}_{k,n}(z)\,|k,z\rangle\,. \tag{3.60}$$

For a normalizable function

$$f(z) = \sum_{n=0}^{\infty} c_n f_{k,n}(z) \tag{3.61}$$

the same arguments as above lead to the bound

$$|f(z)| \leq 1\ . \tag{3.62}$$

This appears to impose a severe restriction on possible allowed functions $f(z)$. But this restriction is spurious, because the two Hilbert spaces with the different measures 3.18 and 3.22 are unitarily equivalent. Obviously the Hilbert space with the measure 3.22 is mathematically the "cleaner" one, because its basic functions are genuinely holomorphic!

There is a completely analogous situation for the conventional ("Schrödinger-Glauber") normalized coherent states [55, 56]

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\ ,\ \alpha\in\mathbb{C}\ , \tag{3.63}$$

$$\langle\alpha_2|\alpha_1\rangle = e^{-(|\alpha_2|^2+|\alpha_1|^2)/2}\,e^{\bar{\alpha}_2\,\alpha_1}\ . \tag{3.64}$$

The integration measure here is

$$d^2\alpha/\pi = d\,\Re(\alpha)\,d\,\Im(\alpha)/\pi\ , \tag{3.65}$$

with respect to which the functions

$$h_n(\alpha) = e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}} \tag{3.66}$$

form an orthonormal basis.

The completeness relation may be written as

$$\frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha|\alpha\rangle\langle\alpha| = \mathbf{1}\,. \tag{3.67}$$

The reproducing kernel is

$$\Delta(\alpha_2,\alpha_1) = \langle\alpha_2|\alpha_1\rangle = \sum_{n=0}^{\infty}\bar{h}_n(\alpha_2)h_n(\alpha_1) = e^{-(|\alpha_2|^2+|\alpha_1|^2)/2}\,e^{\bar{\alpha}_2\alpha_1}\ . \tag{3.68}$$

Square-integrable functions

$$f(\alpha) = \sum_{n=0}^{\infty} c_n h_n(\alpha) \tag{3.69}$$

are bounded as

$$|f(\alpha)| \leq 1 \ . \tag{3.70}$$

On the other hand we have the associated Bargmann-Segal Hilbert space [71, 80, 84] with the integration measure

$$d\tilde{\mu}(\alpha) = \frac{1}{\pi} d^2\alpha \, e^{-|\alpha|^2} \tag{3.71}$$

and the orthonormal basis of holomorphic functions

$$\tilde{h}_n(\alpha) = \frac{\alpha^n}{\sqrt{n!}} \ , \ n = 0, 1, \dots \ . \tag{3.72}$$

Here the reproducing kernel is given by

$$\tilde{\Delta}(z_2, z_1) = e^{\bar{\alpha}_2 \, \alpha_1} \ . \tag{3.73}$$

It has the property

$$\int_{\mathbb{C}} d\tilde{\mu}(\alpha_2) \, \tilde{\Delta}(\alpha_2, \alpha_1) \, \alpha_2^n = \alpha_1^n \ . \tag{3.74}$$

The growth of square-integrable holomorphic functions

$$\tilde{h}(\alpha) = \sum_{n=0}^{\infty} c_n \tilde{h}_n(\alpha) \tag{3.75}$$

is restricted by

$$|\tilde{h}(\alpha)| \leq e^{|\alpha|^2/2} \ , \tag{3.76}$$

i.e. is of order 2 and type 1/2.

The above discussions about the different Hilbert space and their associated reproducing kernels will be of special interest in Ch. 7 when we discuss (quasi)-probability distributions related to the different types of coherent states associated with the Lie algebra $\mathfrak{so}(1,2)$.

3.1.3 Expectation values of quantum observables

The following expectation values are associated with the states $|k, z\rangle$:

$$\langle K_0 \rangle_{k,z} \equiv \langle k, z | K_0 | k, z \rangle = k + |z| \, \rho_k(|z|) \, , \tag{3.77}$$

$$\rho_k(|z|) = \frac{I_{2k}(2|z|)}{I_{2k-1}(2|z|)} < 1 \ , \tag{3.78}$$

$$\langle K_0^2 \rangle_{k,z} = k^2 + |z|^2 + |z| \, \rho_k(|z|) \ , \tag{3.79}$$

so that

$$(\Delta K_0)^2_{k,z} = |z|^2[1 - \rho_k^2(|z|)] + (1 - 2k)|z| \, \rho_k(|z|) \ . \tag{3.80}$$

The inequality $\rho_k < 1$ in Eq. 3.78 will be justified in Appendix D.1.

For the number operator

$$N = K_0 - k \tag{3.81}$$

this implies

$$\langle N \rangle_{k,z} \equiv \bar{n}_{k,z} = |z| \rho_k(|z|) \ , \tag{3.82}$$

$$\langle N^2 \rangle_{k,z} = |z|^2 + (1 - 2k)|z| \, \rho_k \ , \tag{3.83}$$

$$(\Delta N)^2_{k,z} = |z|^2(1 - \rho_k^2) + (1 - 2k)|z| \, \rho_k \ . \tag{3.84}$$

The quantity

$$R = \frac{(\Delta n)^2 - \bar{n}}{\bar{n}^2} \ , \tag{3.85}$$

is used in quantum optical discussions [85] as a (rough) measure for deviations from Poisson statistics for which $R = 0$. It is closely related to Mandel's Q-parameter [86]

$$Q = \frac{(\Delta n)^2 - \bar{n}}{\bar{n}} = \bar{n}\, R \ . \tag{3.86}$$

R and Q here have the values

$$R_{k,z} = \frac{1}{\rho_k^2} - \frac{2k}{|z|\,\rho_k} - 1 \ , \tag{3.87}$$

$$Q_{k,z} = |z|\,(\frac{1}{\rho_k} - \rho_k) - 2k \ . \tag{3.88}$$

It follows from the expansion 3.8 that the probability

$$w_{k,z\leftrightarrow n} \equiv w(|k,n\rangle \leftrightarrow |k,z\rangle) = |\langle k,n|k,z\rangle|^2 \tag{3.89}$$

is given by

$$w_{k,z\leftrightarrow n} = \frac{|z|^{2n}}{(2k)_n\, n!\, g_k(|z|^2)} \ . \tag{3.90}$$

The following limits are of interest:
For $|z| \to 0$ one has

$$\rho_k(|z|) \to \frac{|z|}{2\,k}\left(1 - \frac{|z|^2}{2\,k\,(2k+1)}\right) \quad \text{for } |z| \to 0\,, \tag{3.91}$$

and for very large $|z|$, the *correspondence limit*, we get from Eq. 3.5 that

$$\rho_k(|z|) \asymp 1 - \frac{4k-1}{4|z|} + \frac{16\,(k^2-k)+3}{32|z|^2} + O(|z|^{-3}) \ , \tag{3.92}$$

$$\rho_k^2(|z|) \asymp 1 - \frac{4k-1}{2|z|} + \frac{8\,k^2 - 6\,k + 1}{4\,|z|^2} + O(|z|^{-3}) \tag{3.93}$$

$$\text{for } |z| \to \infty \ .$$

This implies

$$\langle K_0\rangle_{k,z} \asymp |z| + \frac{1}{4} + O(|z|^{-1}) \ , \tag{3.94}$$

$$(\Delta K_0)^2_{k,z} \asymp \frac{1}{2}|z| + O(|z|^{-1}) \ , \tag{3.95}$$

$$\bar{n}_{k,z} \asymp |z| + 1/4 - k + O(|z|^{-1}) \ , \tag{3.96}$$

$$(\Delta N)^2_{k,z} \asymp \frac{1}{2}|z| + O(|z|^{-1}) \asymp \frac{1}{2}\bar{n}_{k,z} \ , \tag{3.97}$$

$$R_{k,z} \asymp -\frac{1}{2\,\bar{n}_{k,z}} + O(|z|^{-2}) \ , \tag{3.98}$$

$$Q_{k,z} \asymp -\frac{1}{2} \ , \tag{3.99}$$

$$w_{k,z\leftrightarrow n} \asymp 2\sqrt{\pi}\frac{|z|^{2(k+n)-1/2}}{n!\,\Gamma(2k+n)}\, e^{-2|z|}\,(1 + O(|z|^{-1})) \ . \tag{3.100}$$

The last relations show that that distribution does not become Poisson-like in the classical limit but stays sub-Poisson-like, even though $R_{k,z}$ tends to 0 in this limit.

For $|z| \to 0$ the limiting behaviour of the expectation values 3.77 etc. can be derived with the help of the relation 3.91.

For K_1 and K_2 we have the following expectation values:

$$\langle K_1\rangle_{k,z} = \frac{1}{2}(\bar{z}+z) = |z|\,\cos\phi\;, \tag{3.101}$$

$$\langle K_2\rangle_{k,z} = \frac{1}{2i}(\bar{z}-z) = -|z|\,\sin\phi\;, \tag{3.102}$$

$$\langle K_1^2\rangle_{k,z} = |z|^2\cos^2\phi + \frac{1}{2}\langle K_0\rangle_{k,z}\;, \tag{3.103}$$

$$\langle K_2^2\rangle_{k,z} = |z|^2\sin^2\phi + \frac{1}{2}\langle K_0\rangle_{k,z}\;, \tag{3.104}$$

$$\langle K_1^2+K_2^2\rangle_{k,z} = \langle K_0^2\rangle_{k,z} + l = |z|^2 + \langle K_0\rangle_{k,z}\;, \tag{3.105}$$

$$(\Delta K_1)^2_{k,z} = (\Delta K_2)^2_{k,z} = \frac{1}{2}\langle K_0\rangle_{k,z}\;, \tag{3.106}$$

$$\langle K_1\,K_2 + K_2\,K_1\rangle_{k,z} = \frac{1}{2i}(\bar{z}^2 - z^2) = -|z|^2\sin 2\phi\;, \tag{3.107}$$

$$\langle S(K_1,K_2)_{k,z}\rangle_{k,z} = 0\;, \tag{3.108}$$

$$S(K_1,K_2)_{k,z} = \frac{1}{2}(K_1\,K_2 + K_2\,K_1) - \langle K_1\rangle_{k,z}\langle K_2\rangle_{k,z}\;, \tag{3.109}$$

$$\langle (K_0+k)^{-1}\rangle_{k,z} = |z|^{-1}\,\rho_k(|z|)\;, \tag{3.110}$$

$$\langle a = (K_0+k)^{-1/2}K_-\rangle_{k,z} = z\,\langle (K_0+k)^{-1/2}\rangle_{k,z}\;, \tag{3.111}$$

$$\langle (K_0+k)^{-1/2}\rangle_{k,z} = \frac{1}{g_k(|z|^2)}\sum_{n=0}^{\infty}\frac{|z|^{2n}}{\sqrt{2k+n}\,(2k)_n\,n!} = \tag{3.112}$$

$$= \frac{1}{\sqrt{\pi}\,g_k(|z|^2)}\times \\ \times\int_0^\infty dt\,t^{-1/2}\,e^{-2k\,t}\,g_k(|z|^2e^{-t})\,, \tag{3.113}$$

$$\langle (K_0+k)^{-1/2}\rangle_{k,z} \to |z|^{-1/2}\left[1 - \frac{2k-1/4}{4|z|} + O(|z|^{-2})\right] \\ \text{for } |z|\to\infty\;. \tag{3.114}$$

In deriving the relations 3.103 and 3.104 the equality $K_-K_+ = K_+K_- + 2K_0$ (Eq. 2.14) has been used.

The Eqs 3.101 and 3.102 show clearly that the expectation values of K_1 and K_2 have the same form as their classical counterparts

$$h_1 = I\,\cos\varphi\,, \quad h_2 = -I\,\sin\varphi\,, \tag{3.115}$$

the difference being that $|z|$ and $\langle K_0\rangle_{k,z}$, see Eq. 3.77, coincide only for very large $|z|$.

But the operators K_1 and K_2 do measure the phase of z for *all* values of $|z|$ in a straightforward way:

$$\tan\phi = -\frac{\langle K_2\rangle_{k,z}}{\langle K_1\rangle_{k,z}}\;. \tag{3.116}$$

Notice also that the expectation values 3.101 and 3.102 of K_1 and K_2 do not depend on the index k at all, contrary to that of K_0.

We see that we may identify the complex number z as

$$z = h_1 - i\,h_2 \ , \tag{3.117}$$

where h_1 and h_2 are the classical versions 3.115 of the quantum observables K_1 and K_2.

The situation is analogous to Schrödinger-Glauber coherent states $|\alpha\rangle$ where the complex number α is interpreted as $(q + i\,p)/\sqrt{2}$. A difference is, however, that in our case we have a third observable K_0 - or h_0, respectively - the role of which is less trivial than that of the identity operator in the usual quantum mechanical descriptions.

Eq. 3.105 shows how the deviations from the Pythagorean relation 2.3 are determined by the expectation value of K_0!

The relations 3.106 imply that the general inequality

$$(\Delta K_1)^2(\Delta K_2)^2 \geq \frac{1}{4}|\langle K_0\rangle|^2 \ . \tag{3.118}$$

becomes an equality here, i.e. the states $|k,z\rangle$ are " minimal uncertainty" states, but they are not squeezed (see Ch. 6 for details, also as to the quantity $S(K_1,K_2)_{k,z}$).

In turning the series 3.112 into an integral the relation

$$(2k+n)^{-1/2} = \frac{1}{\sqrt{\pi}}\int_0^\infty dt\, t^{-1/2}\, e^{-(2k+n)\,t} \tag{3.119}$$

has been used. The asymptotic behaviour of the integral 3.113 will be justified in Appendix D.2.

As the Barut-Girardello coherent states $|k,z\rangle$ are not squeezed the interest in them grew only slowly - compared to the Perelomov coherent states discussed below - within the quantum optical community [87–105]. It was realized that an appropriate dynamics could evolve Barut-Girardello states into squeezed states [88, 105] and that a superposition of 2 of them can be squeezed, too [94, 97].

3.2 Perelomov coherent states

3.2.1 General Properties

The Perelomov coherent states as defined by Eq. 1.110 have always been introduced in a different way [54]. I shall discuss the relationship between the two approaches below. Let me first turn to the implications of the definition 1.110:

Multiplying Eq. 1.110 from the left with $\langle k,n|$ and using the relations 2.11 and 2.12 yields the recursion formula

$$\langle k,n+1|k,\lambda\rangle = \left(\frac{2k+n}{n+1}\right)^{1/2} \lambda\,\langle k,n|k,\lambda\rangle \ , \tag{3.120}$$

which implies

$$\langle k,n|k,\lambda\rangle = \left(\frac{(2k)_n}{n!}\right)^{1/2} \lambda^n\,\langle k,n=0|k,\lambda\rangle. \tag{3.121}$$

Using the relation [106, 107]

$$(2k)_n = \frac{\Gamma(2k+n)}{\Gamma(2k)} = (-1)^n\,n!\binom{-2k}{n} \tag{3.122}$$

we obtain a summable binomial power series:

$$\begin{aligned}\langle k,\lambda|k,\lambda\rangle &= \sum_{n=0}^{\infty}\langle k,\lambda|k,n\rangle\langle k,n|k,\lambda\rangle \\ &= |\langle k,n=0|k,\lambda\rangle|^2\sum_{n=0}^{\infty}\binom{-2k}{n}(-|\lambda|^2)^n \\ &= |\langle k,n=0|k,\lambda\rangle|^2\,(1-|\lambda|^2)^{-2k}\ .\end{aligned} \tag{3.123}$$

The series 3.123 converges only for $|\lambda|^2<1$.

From $\langle k,\lambda|k,\lambda\rangle=1$ we get

$$|\langle k,n=0|k,\lambda\rangle| = (1-|\lambda|^2)^k\ . \tag{3.124}$$

Choosing the phase of $\langle k,n=0|k,\lambda\rangle$ to be zero finally gives

$$|k,\lambda\rangle = (1-|\lambda|^2)^k\sum_{n=0}^{\infty}\left(\frac{(2k)_n}{n!}\right)^{1/2}\lambda^n\,|k,n\rangle\,. \tag{3.125}$$

Since their introduction [54] the existing definitions of the states 3.125 are as follows:

From the Lie algebra 2.7 it follows that [108–110, 264]

$$U(w)\equiv e^{(w/2)\,K_+-(\bar w/2)\,K_-} = e^{\lambda\,K_+}\,e^{\ln(1-|\lambda|^2)\,K_0}\,e^{-\bar\lambda\,K_-}\ , \tag{3.126}$$

where

$$w=|w|\,e^{i\,\theta}\in\mathbb{C}\ ,\ \ \lambda=\tanh(|w|/2)\,e^{i\,\theta}\,. \tag{3.127}$$

Notice that the complex numbers w and λ have the same phase!

For later convenience I collect some elementary relations:

$$|w| = \ln\left(\frac{1+|\lambda|}{1-|\lambda|}\right)\ , \tag{3.128}$$

$$1-|\lambda|^2 = \frac{1}{\cosh^2(|w|/2)}\ , \tag{3.129}$$

$$1+|\lambda|^2 = \frac{\cosh|w|}{\cosh^2(|w|/2)}\ , \tag{3.130}$$

$$\frac{|\lambda|}{1-|\lambda|^2} = \frac{1}{2}\sinh|w|\,. \tag{3.131}$$

Applying the operator relation 3.126 to the ground state $|k,n=0\rangle$ and using the Eqs. 2.13 and 2.19 then gives

$$e^{(w/2)\,K_+-(\bar w/2)\,K_-}\,|k,n=0\rangle = |k,\lambda\rangle\,. \tag{3.132}$$

That this state is an eigenstate of $(K_0+k)^{-1}K_-$ has been noticed before [111, 112] (as to the general algebraic structure see also the Refs. [113, 114]). When taking the first of the relations in Eq. 2.32 into account, the special case $k=1/2$ has been discussed previously, too, as an eigenstate of the operator E_- from Eq. 1.32, without emphazising its property as a Perelomov coherent state [115–118].

In this paper I use the property 1.110 as the defining one!

The definition 3.132 was motivated by the property of the conventional coherent states 3.63 to be definable as

$$|\alpha\rangle = D(\alpha)|0\rangle\ ,\ \ D(\alpha)=e^{\alpha\,a^+-\bar\alpha\,a} = e^{-|\alpha|^2/2}\,e^{\alpha\,a^+}\,e^{-\bar\alpha\,a}\,. \tag{3.133}$$

The states $|k,\lambda\rangle$ have properties similar to those of the states $|k,z\rangle$ from above: They are not orthogonal,

$$\langle k,\lambda_2|k,\lambda_1\rangle = (1-|\lambda_1|^2)^k\,(1-|\lambda_2|^2)^k\,(1-\bar{\lambda}_2\,\lambda_1)^{-2\,k}\,, \tag{3.134}$$

they are, however, complete in the following sense: From

$$\int d\mu_k(\lambda)\,\langle k,n|k,\lambda\rangle\langle k,\lambda|k,n\rangle = \frac{(2k)_n}{n!}\int d\mu_k(\lambda)(1-|\lambda|^2)^{2\,k}\,|\lambda|^{2\,n}\,, \tag{3.135}$$

where

$$d\mu_k(\lambda) = \frac{2k-1}{\pi}(1-|\lambda|^2)^{-2}|\lambda|\,d|\lambda|\,d\theta\,, \tag{3.136}$$

and [120]

$$\int_0^1 d\,x(1-x)^{2k-2}\,x^n = \mathrm{B}(n+1,2k-1) = \frac{n!\,\Gamma(2k-1)}{\Gamma(2k+n)}\,, \tag{3.137}$$

we have

$$\int_{\mathbb{D}} d\,\mu_k(\lambda)\,\langle k,n_2|k,\lambda\rangle\langle k,\lambda|k,n_1\rangle = \delta_{n_2 n_1}\,, \tag{3.138}$$

where in addition

$$\int_0^{2\pi} d\theta e^{i\,(n_1-n_2)} = 0 \quad \text{for } n_2 \neq n_1 \tag{3.139}$$

has been used.

It follows from Eq. 3.137 that the integral 3.135 exists for $k = 1/2$, too, because $(2k-1)\,\Gamma(2k-1) = \Gamma(2k)\,!$

In terms of the variable w from Eq. 3.127 the measure 3.136 has the form

$$d\mu_k = \frac{2k-1}{4\pi}\sinh|w|\,d|w|d\theta\,. \tag{3.140}$$

Formally we may write the completeness relation 3.138 as

$$\int_{\mathbb{D}} d\,\mu_k(\lambda)\,|\lambda\rangle\langle\lambda| = \mathbf{1}. \tag{3.141}$$

It is obviously advantageous to introduce the measure

$$d\tilde{\mu}_k(\lambda) = (1-|\lambda|^2)^{2k}\,d\mu_k(\lambda)\,, \tag{3.142}$$

because (see, e.g. [40, 257])

$$(\lambda^{n_2},\lambda^{n_1})_{\mathbb{D},k} = \int_{\mathbb{D}} d\tilde{\mu}_k(\lambda)\,\bar{\lambda}^{n_2}\,\lambda^{n_1} = \frac{n_1!}{(2k)_{n_1}}\,\delta_{n_2\,n_1}\,. \tag{3.143}$$

The integral 3.143 provides the well-known scalar product for a Hilbert space of holomorphic functions on the unit disc $\mathbb{D}$, a so-called Hardy space [31].

We have the orthonormal basis

$$e_{k,n}(\lambda) = \sqrt{\frac{(2k)_n}{n!}}\,\lambda^n \tag{3.144}$$

The role of the usual δ-function here plays the "reproducing kernel" [71, 79–81], too (see subsection 3.1.2):

$$\Delta_k(\bar{\lambda}_2,\lambda_1) = \sum_{n=0}^{\infty}\bar{e}_{k,n}(\lambda_2)\,e_{k,n}(\lambda_1) = (1-\bar{\lambda}_2\,\lambda_1)^{-2\,k}\,, \tag{3.145}$$

with

$$\int_{\mathbb{D}} d\tilde{\mu}_k(\lambda_2)\,\Delta_k(\bar{\lambda}_2,\lambda_1)\,\lambda_2^n = \lambda_1^n\ . \tag{3.146}$$

In summing the series 3.145 the relation 3.122 has been used.

For a given unitary representation the scalar product between the normalized coherent states $|k,z\rangle$, Eq. 3.8, and $|k,\lambda\rangle$, Eq. 3.125, is given by

$$\langle k,\lambda|k,z\rangle = \frac{(1-|\lambda|^2)^k}{\sqrt{g_k(|z|^2)}}\,e^{\bar{\lambda}\,z}\ . \tag{3.147}$$

With $z = |z|\exp(i\,\phi)$ and $\lambda = |\lambda|\exp(i\,\theta)$ we have

$$\langle k,\lambda|k,z\rangle = \frac{(1-|\lambda|^2)^k}{\sqrt{g_k(|z|^2)}}\,e^{|\lambda||z|e^{i\,(\phi-\theta)}}\ . \tag{3.148}$$

This shows: The scalar product 3.148 decreases for (the classical limit, see below) $|\lambda| \to 1$ and for (the classical limit) $|z| \to \infty$ (because $|\lambda| < 1$ and because of the asymptotic behaviour 3.6). The decrease is maximal for $\phi - \theta = \pm\pi$ and minimal for $\phi = \theta$.

From the completeness relations 3.17 and 3.141 we get the integral transforms

$$|k,z\rangle = \int_{\mathbb{D}} d\mu_k(\lambda)\langle k,\lambda|k,z\rangle\,|k,\lambda\rangle\,,\quad |k,\lambda\rangle = \int_{\mathbb{C}} d\mu_k(z)\langle k,z|k,\lambda\rangle\,|k,z\rangle\,. \tag{3.149}$$

With the kernel 3.147 these become

$$\begin{aligned}
|k,z\rangle &= \frac{1}{\sqrt{g_k(|z|^2)}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{(2k)_n\,n!}}\,|k,n\rangle = && (3.150)\\
&= \int_{\mathbb{D}} d\mu_k(\lambda)\frac{(1-|\lambda|^2)^k\,e^{\bar{\lambda}\,z}}{\sqrt{g_k(|z|^2)}}\,|k,\lambda\rangle = && (3.151)\\
&= \frac{1}{\sqrt{g_k(|z|^2)}}\int_{\mathbb{D}} d\tilde{\mu}_k(\lambda)\,e^{\bar{\lambda}\,z}\sum_{n=0}^{\infty}\left(\frac{(2k)_n}{n!}\right)^{1/2}\lambda^n\,|k,n\rangle\,; && (3.152)\\
|k,\lambda\rangle &= (1-|\lambda|^2)^k\sum_{n=0}^{\infty}\left(\frac{(2k)_n}{n!}\right)^{1/2}\lambda^n\,|k,n\rangle = && (3.153)\\
&= \int_{\mathbb{C}} d\mu_k(z)\frac{(1-|\lambda|^2)^k\,e^{\bar{z}\,\lambda}}{\sqrt{g_k(|z|^2)}}\,|k,z\rangle = && (3.154)\\
&= (1-|\lambda|^2)^k\int_{\mathbb{C}} d\tilde{\mu}_k(z)\,e^{\bar{z}\,\lambda}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{(2k)_n\,n!}}|k,n\rangle\,. && (3.155)
\end{aligned}$$

One immediately can read off the following unitary mappings of the bases 3.20 and 3.144:

$$\begin{aligned}
\tilde{f}_{k,n}(z) &= \int_{\mathbb{D}} d\tilde{\mu}_k(\lambda)\,e^{\bar{\lambda}\,z}\,e_{k,n}(\lambda)\ ,\ n = 0,1,\ldots\,, && (3.156)\\
e_{k,n}(\lambda) &= \int_{\mathbb{C}} d\tilde{\mu}_k(z)\,e^{\bar{z}\,\lambda}\,\tilde{f}_{k,n}(z)\ . && (3.157)
\end{aligned}$$

Notice that the kernels $\exp(\bar{\lambda}\,z)$ and $\exp(\bar{z}\,\lambda)$ do not depend on the index k.

For a discussion of related integral transforms see Ref. [121]

Multiplying the relations 3.150 and 3.153 from the left with the states $\langle k,z_2|$ and $\langle k,\lambda_2|$, respectively, yields the following useful integral transforms

$$g_k(\bar{z}_2\, z_1) \;=\; \int_{\mathbb{D}} d\tilde{\mu}_k(\lambda)\, e^{\bar{z}_2\,\lambda+\bar{\lambda}\,z_1}\ , \tag{3.158}$$

$$(1-\bar{\lambda}_2\,\lambda_1)^{-2k} \;=\; \int_{\mathbb{C}} d\tilde{\mu}_k(z)\, e^{\bar{\lambda}_2\,z+\bar{z}\,\lambda_1}\ . \tag{3.159}$$

Differentiating these relations with respect to $\bar{z}_2$ etc. generates new relations, e.g.

$$\frac{dg_k(\bar{z}_2\, z_1)}{d\bar{z}_2} = \frac{z_1}{2k}\, g_{k+1/2}(\bar{z}_2\, z_1) = \int_{\mathbb{D}} d\tilde{\mu}_k(\lambda)\,\lambda\, e^{\bar{z}_2\,\lambda+\bar{\lambda}\,z_1}\ , \tag{3.160}$$

where the relation $(2k)_{n+1} = 2k\,(2k+1)_n$ has been used [122].

The Hilbert spaces with the scalar product 3.143 "carry" irreducible unitary representations of $SU(1,1)$ etc.

The generators of the Lie algebra are [40, 257]

$$K_+ = 2k\,\lambda + \lambda^2\frac{d}{d\lambda}\ ,\ K_- = \frac{d}{d\lambda}\ ,\ K_0 = \lambda\frac{d}{d\lambda} + k\ . \tag{3.161}$$

The basis functions 3.144 are the eigenfunctions of K_0. The - unnormalized - eigenfunctions of K_- are

$$f_{k,z}(\lambda) = C_{k,z}\, e^{z\,\lambda}\ ,\ K_- f_{k,z} = z\, f_{k,z}\ . \tag{3.162}$$

As

$$\begin{aligned}(f_{k,z}, f_{k,z}) \;&=\; |C_{k,z}|^2 \int_{\mathbb{D}} d\tilde{\mu}_k(\lambda)\,|e^{z\,\lambda}|^2 = \qquad (3.163)\\ &=\; |C_{k,z}|^2\, g_k(|z|^2) \int_{\mathbb{D}} d\mu_k(\lambda)\langle z|\lambda\rangle\langle\lambda|z\rangle = |C_{k,z}|^2\, g_k(|z|^2)\ ,\end{aligned}$$

we have the normalized eigenfunctions

$$f_{k,z}(\lambda) = \frac{1}{\sqrt{g_k(|z|^2)}}\, e^{z\,\lambda}\ . \tag{3.164}$$

(Notice that the integration over θ in Eq. 3.163 yields the same result for the exponents $z\,\bar{\lambda} + \bar{z}\,\lambda = 2\,|z||\lambda|\,\cos(\theta-\phi)$ and $z\,\lambda + \bar{z}\,\bar{\lambda} = 2\,|z||\lambda|\,\cos(\theta+\phi)$).

The unnormalized eigenfunctions of K_1 and K_2 are

$$f_{k,h_1}(\lambda) \;=\; C_{k,h_1}\left(\frac{1+i\lambda}{1-i\lambda}\right)^{-i\,h_1}(1+\lambda^2)^{-k}\ , \tag{3.165}$$

$$K_1 f_{k,h_1}(\lambda) \;=\; h_1\, f_{k,h_1}(\lambda)\ ,\ h_1\in\mathbb{R}\ ; \tag{3.166}$$

$$f_{k,h_2}(\lambda) \;=\; C_{k,h_2}\left(\frac{1+\lambda}{1-\lambda}\right)^{-i\,h_2}(1-\lambda^2)^{-k}\ , \tag{3.167}$$

$$K_2 f_{k,h_2}(\lambda) \;=\; h_2\, f_{k,h_2}(\lambda)\ ,\ h_2\in\mathbb{R}\ , \tag{3.168}$$

where C_{k,h_1} and C_{k,h_2} are (normalization) constants.

3.2.2 Expectation values of quantum observables

Next we come to the expectation values and the fluctuations of the operators K_j with respect to the states 3.125. We have [60, 88, 92] (using the relations 3.128-3.131):

$$\begin{aligned}
\langle K_0\rangle_{k,\lambda} &\equiv \langle k,\lambda|K_0|k,\lambda\rangle = k\,\frac{1+|\lambda|^2}{1-|\lambda|^2} = k\,\cosh|w|\ , && (3.169)\\
\langle K_0^2\rangle_{k,\lambda} &= k^2\,\frac{(1+|\lambda|^2)^2}{(1-|\lambda|^2)^2} + 2\,k\,\frac{|\lambda|^2}{(1-|\lambda|^2)^2} = && (3.170)\\
&= k^2\,\cosh^2|w| + \frac{k}{2}\,\sinh^2|w|\ ,\\
(\Delta K_0)^2_{k,\lambda} &= 2\,k\,\frac{|\lambda|^2}{(1-|\lambda|^2)^2} = \frac{k}{2}\,\sinh^2|w|\ ,\\
&= \frac{1}{2}(\frac{\langle K_0\rangle^2_{k,\lambda}}{k} - k)\,.
\end{aligned}$$

Here $|\lambda| = \tanh|w/2|$ (see Eq. 3.127).

For the number operator 3.81, the parameter R defined in Eq. 3.85 and Mandel's Q-parameter 3.86 we get

$$\begin{aligned}
\langle N\rangle_{k,\lambda} &\equiv \bar{n}_{k,\lambda} = k\,(\cosh|w| - 1)\ , && (3.171)\\
(\Delta N)^2_{k,\lambda} &= \frac{k}{2}\,\sinh^2|w| = \bar{n}_{k,\lambda}\,(1+\bar{n}_{k,\lambda}/2k)\ , && (3.172)\\
R_{k,\lambda} &= \frac{1}{2k}\ , && (3.173)\\
Q_{k,\lambda} &= \frac{\bar{n}_{k,\lambda}}{2k}\ . && (3.174)
\end{aligned}$$

The probability

$$w_{k,\lambda\leftrightarrow n} \equiv w(|k,n\rangle \leftrightarrow |k,\lambda\rangle) = |\langle k,n|k,\lambda\rangle|^2 \qquad (3.175)$$

is given by

$$w_{k,\lambda\leftrightarrow n} = (1-|\lambda|^2)^{2k}\,\frac{(2k)_n\,|\lambda|^{2n}}{n!}\ . \qquad (3.176)$$

For $k = 1/2$ this distribution represents the Bose - statistics [118, 119] of free quanta with energies $E_\nu\,, \nu = 0, 1, \ldots$, in a heat bath of (inverse) temperature $\beta = 1/k_BT$ and chemical potential μ:

$$w_{1/2,\lambda\leftrightarrow n} = (1-|\lambda|^2)\,|\lambda|^{2n}\ ,\ |\lambda|^2 = e^{-\beta\,(E_\nu-\mu)} \qquad (3.177)$$

is the probability to find n quanta in a state with energy E_ν.

For the expectation values and fluctuations of K_1 and K_2 we get:

$$\begin{aligned}
\langle K_1\rangle_{k,\lambda} &= 2\,k\,\frac{|\lambda|}{1-|\lambda|^2}\,\cos\theta = k\,\sinh|w|\,\cos\theta\ , && (3.178)\\
&= \sqrt{2\,k}\,(\Delta\,K_0)_{k,\lambda}\,\cos\theta = \frac{2\,|\lambda|}{1+|\lambda|^2}\langle K_0\rangle_{k,\lambda}\cos\theta\ , \\
\langle K_2\rangle_{k,\lambda} &= -2\,k\,\frac{|\lambda|}{1-|\lambda|^2}\,\sin\theta = -k\,\sinh|w|\,\sin\theta\ , && (3.179)\\
&= -\sqrt{2\,k}\,(\Delta\,K_0)_{k,\lambda}\,\sin\theta\ , \\
\langle K_1^2\rangle_{k,\lambda} &= 4\,k^2\,\cos^2\theta\,\frac{|\lambda|^2}{(1-|\lambda|^2)^2} + \frac{k}{2}\frac{|1+\lambda^2|^2}{(1-|\lambda|^2)^2}\ , && (3.180)\\
& |1+\lambda^2|^2 = 1+2\,|\lambda|^2\,\cos 2\theta + |\lambda|^4\ , && (3.181)\\
(\Delta K_1)^2_{k,\lambda} &= \frac{k}{2}\,\frac{|1+\lambda^2|^2}{(1-|\lambda|^2)^2}\ , && (3.182)\\
\langle K_2^2\rangle_{k,\lambda} &= 4\,k^2\,\sin^2\theta\,\frac{|\lambda|^2}{(1-|\lambda|^2)^2} + \frac{k}{2}\frac{|1-\lambda^2|^2}{(1-|\lambda|^2)^2}\ , && (3.183)\\
& |1-\lambda^2|^2 = 1-2\,|\lambda|^2\,\cos 2\theta + |\lambda|^4\ , && (3.184)\\
(\Delta K_2)^2_{k,\lambda} &= \frac{k}{2}\,\frac{|1-\lambda^2|^2}{(1-|\lambda|^2)^2}\ , && (3.185)\\
(\Delta K_1)^2_{k,\lambda}\,(\Delta K_2)^2_{k,\lambda} &= \frac{k^2}{4}\,\frac{|1+\lambda^2|^2\,|1-\lambda^2|^2}{(1-|\lambda|^2)^4} \geq \frac{1}{4}\,|\langle K_0\rangle_{k,\lambda}|^2 = && (3.186)\\
&= \frac{k^2}{4}\,\frac{(1+|\lambda|^2)^2}{(1-|\lambda|^2)^2}\ , \\
\langle K_1\rangle^2_{k,\lambda} + \langle K_2\rangle^2_{k,\lambda} &= \langle K_0\rangle^2_{k,\lambda} - k^2\,, && (3.187)\\
(\Delta K_1)^2_{k,\lambda} + (\Delta K_2)^2_{k,\lambda} &= (\Delta K_0)^2_{k,\lambda} + k\ . && (3.188)
\end{aligned}$$

Here again - like in the cases 3.101 and 3.102 - the expectation values 3.178 and 3.179 have the simple structure

$$r\,\cos\theta\,,\quad -r\,\sin\theta\,,\quad r = k\,\sinh|w|\,. \tag{3.189}$$

The modulus r approaches the expectation value $\langle K_0\rangle_{k,\lambda}$, 3.169, for large $|w|$. The Eqs. 3.178 and 3.179 show again how the operators K_1 and K_2 "measure" the phase of the complex numbers λ or w. We have

$$\tan\theta = -\frac{\langle K_2\rangle_{k,\lambda}}{\langle K_1\rangle_{k,\lambda}}\ . \tag{3.190}$$

Sometimes the following relations are useful:

$$\begin{aligned}
\langle K_+^2\rangle_{k,\lambda} &= 2k\,(2k+1)\,\frac{\bar\lambda^2}{(1-|\lambda|^2)^2}\ , && (3.191)\\
\langle K_+K_-\rangle_{k,\lambda} &= 2k\,|\lambda|^2\,\frac{2\,k+|\lambda|^2}{(1-|\lambda|^2)^2}\ , && (3.192)\\
\langle S_{k,\lambda}(K_1,K_2)\rangle_{k,\lambda} &= \frac{k}{2i}\,\frac{\bar\lambda^2-\lambda^2}{(1-|\lambda|^2)^2} = -k\frac{|\lambda|^2\sin 2\theta}{(1-|\lambda|^2)^2}\ , && (3.193)\\
S_{k,\lambda}(K_1,K_2) &= \frac{1}{2}(K_1K_2+K_2K_1) - \langle K_1\rangle_{k,\lambda}\,\langle K_2\rangle_{k,\lambda}\ . && (3.194)
\end{aligned}$$

We note the equality

$$(\Delta K_1)^2_{k,\lambda}\,(\Delta K_2)^2_{k,\lambda} = \frac{1}{4}\,|\langle K_0\rangle_{k,\lambda}|^2 + |\langle S_{k,\lambda}(K_1,K_2)\rangle_{k,\lambda}|^2\ , \tag{3.195}$$

and shall discuss the background of this relation in Ch. 6 in more detail.

As to the fluctuations the following θ-values are of special interest:

$$(\Delta K_1)^2_{k,\lambda}(|\lambda| > 0, \theta = 0, \pi) = \frac{k}{2}\frac{(1+|\lambda|^2)^2}{(1-|\lambda|^2)^2} > \tag{3.196}$$

$$> (\Delta K_2)^2_{k,\lambda}(\theta = 0, \pi) = \frac{k}{2} < \frac{1}{2}\langle K_0\rangle_{k,\lambda}\,, \tag{3.197}$$

$$(\Delta K_2)^2_{k,\lambda}(|\lambda| > 0, \theta = -\pi/2\,, \pi/2) = \frac{k}{2}\frac{(1+|\lambda|^2)^2}{(1-|\lambda|^2)^2} > \tag{3.198}$$

$$> (\Delta K_1)^2_{k,\lambda}(\theta = -\pi/2\,, \pi/2) = \frac{k}{2} < \frac{1}{2}\langle K_0\rangle_{k,\lambda}\,. \tag{3.199}$$

For all 4 cases we get the equality sign in the "uncertainty" inequality 3.118. The same follows from Eq. 3.195 because the correlations 3.193 vanish for those θ-values.

In addition we have squeezing - for $|\lambda| > 0$ - in these cases because of the last inequalities in the relations 3.197 and 3.199! Squeezing will be discussed in more detail in Ch. 6.

Because of the squeezing properties the Perelomov coherent states attracted the attention of the quantum optics community earlier than the Barut-Girardello ones. Here is a selection of the early papers: [58–62,88]

For the expectation value of the annihilation operator $a = (K_0 + k)^{-1/2}K_-$ we get

$$\langle a = (K_0 + k)^{-1/2}K_-\rangle_{k,\lambda} = \lambda\,\langle (K_0 + k)^{1/2}\rangle_{k,\lambda}\,, \tag{3.200}$$

$$\langle (K_0 + k)^{1/2}\rangle_{k,\lambda} = (1-|\lambda|^2)^{2k}\sum_{n=0}^{\infty}\frac{(2k)_n\,(2k+n)^{1/2}}{n!}|\lambda|^{2\,n}\,.$$

Observing that $(2k+n)^{1/2} = (2k+n)(2k+n)^{-1/2}$ and again using the relation 3.119 gives for the sum in Eq. 3.200

$$\sum_{n=0}^{\infty}\frac{(2k)_n\,(2k+n)^{1/2}}{n!}|\lambda|^{2\,n} = |\lambda|^2\,\frac{du_{2k}(|\lambda|^2)}{d\,|\lambda|^2} + 2k\,u_{2k}(|\lambda|^2)\,, \tag{3.201}$$

$$u_{2k}(|\lambda|^2) = \frac{1}{\sqrt{\pi}}\int_0^{\infty} dt\,t^{-1/2}(e^t - |\lambda|^2)^{-2k}\,. \tag{3.202}$$

Notice that

$$\frac{d^m\,u_{2k}}{d(|\lambda|^2)^m} = (2k)_m\,u_{2k+m}\,,\ m = 1, 2, \ldots\,. \tag{3.203}$$

As $2k$ in general is a positive integer we may generate the necessary u_{2k} from

$$u_1(|\lambda|^2) = \Phi(|\lambda|^2, 1/2, 1) = (|\lambda|^2)^{-1}\,F(|\lambda|^2, 1/2) = (|\lambda|^2)^{-1}\,\mathrm{Li}_{\frac{1}{2}}(|\lambda|^2)\,, \tag{3.204}$$

where

$$\Phi(z, s, a) = \sum_{n=0}^{\infty}\frac{z^n}{(a+n)^s} \tag{3.205}$$

is Lerch's function [123] and

$$\mathrm{Li}_s(z) = F(z, s) = z\,\Phi(z, s, a = 1) = \sum_{n=1}^{\infty}\frac{z^n}{n^s} \tag{3.206}$$

the so-called "polylogarithm" of index s [124].

Expectation values of the type

$$\langle (K_0+b)^{-1}\rangle_{k,\lambda} = s_k^{(b)}(|\lambda|^2) = (1-|\lambda|^2)^{2k}\, v_{2k,b}(|\lambda|^2)\,, \tag{3.207}$$

$$v_{2k,b}(|\lambda|^2) = \sum_{n=0}^{\infty} \frac{(2k)_n}{(k+b+n)\,n!}\,|\lambda|^{2\,n}\;,\; b\geq 0\,, \tag{3.208}$$

may be calculated in a similar fashion:

With

$$\frac{1}{k+b+n} = \int_0^1 du\, u^{k+b+n-1} \tag{3.209}$$

we get

$$v_{2k,b}(|\lambda|^2) = \frac{1}{|\lambda|^{2(k+b)}} \int_0^{|\lambda|^2} dx\, x^{k+b-1}(1-x)^{-2k}\;. \tag{3.210}$$

The sum $v_{2k,b}(|\lambda|^2)$ may be expressed by a hypergeometric function [125]: ${}_2F_1(2k, k+b; k+b+1; |\lambda|^2)/(k+b)$, but in practice it will be more convenient to (partially) integrate:

$$\hat{v}_{2k,b}(|\lambda|^2) \equiv \int_0^{|\lambda|^2} dx\, x^{k+b-1}(1-x)^{-2k} = \tag{3.211}$$

$$= \frac{|\lambda|^{2(k+b-1)}}{2k-1}(1-|\lambda|^2)^{-2k} - \frac{k+b-1}{2k-1}\,\hat{v}_{2k-1,b-1}(|\lambda|^2)\,, \tag{3.212}$$

$$k>1/2\,,\; k+b-1>0\;. \tag{3.213}$$

(The index $2k$ of $v_{2k,b}$ refers to the integrand $(1-x)^{-2k}$ only!)
For, e.g. $k=1/2$ and $b=0, 1/2$ we can integrate directly:

$$\hat{v}_{1,0}(|\lambda|^2) = \ln\left(\frac{1+|\lambda|}{1-|\lambda|}\right) = |w|\,, \tag{3.214}$$

$$\hat{v}_{1,\frac{1}{2}}(|\lambda|^2) = -\ln(1-|\lambda|^2) = 2\,\ln\cosh(|w|/2)\,, \tag{3.215}$$

(see the relations 3.128 and 3.129) so that

$$\langle (K_0)^{-1}\rangle_{k=1/2,\lambda} = \frac{1-|\lambda|^2}{|\lambda|}\,\ln\left(\frac{1+|\lambda|}{1-|\lambda|}\right) = \frac{2\,|w|}{\sinh|w|}\;, \tag{3.216}$$

$$\langle (K_0+1/2)^{-1}\rangle_{k=1/2,\lambda} = -\frac{1-|\lambda|^2}{|\lambda|^2}\,\ln(1-|\lambda|^2) \tag{3.217}$$

$$= \frac{2\,\ln\cosh(|w|/2)}{\sinh^2(|w|/2)}\,.$$

The *classical (correspondence) limits* of the above formulae are obtained for $|w|\to\infty$ or $|\lambda|\to 1$. As to the behaviour of the function 3.205 in this limit see the Refs. [123, 126]

3.3 Schrödinger-Glauber coherent states

As the properties of these states are very well-known I confine myself to those properties which are of special interest in the context of the groups $SU(1,1)$ etc. and their irreducible unitary representations.

3.3.1 Some general properties

We know already from the discussion of the Eqs. 2.32 in Ch. 2 that we may define annihilation and creation operators

$$a = (K_0 + k)^{-1/2} K_- \,, \quad a^+ = K_+ (K_0 + k)^{-1/2} \,, \tag{3.218}$$

for any given self-adjoint operators K_0, K_1 and K_2 from an irreducible unitary representation of the positive discrete series of $SU(1,1)$. It follows from the definition 1.112 and the relations 2.13 and 2.11 that

$$|k, \alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \, |k, n\rangle \;, \; \alpha = |\alpha| \, e^{i\,\beta} \;, \tag{3.219}$$

is an eigenstate of a from Eq. 3.218. It has all the general properties of the usual Schrödinger-Glauber coherent states! Because of the additional dependence on the index k, we now get a whole family of such states! One has to realize, however, that for $k \neq 1/2$ the states $|k, n\rangle$ are not the usual Hermite functions, eigenfunctions of the harmonic oscillator. I shall discuss the case $k = 1/2$ in the next chapter.

We have the usual completeness relation

$$\frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha \, |k, \alpha\rangle\langle k, \alpha| = \mathbf{1} \,. \tag{3.220}$$

Of special interest are the following scalar products between the coherent states $|k, z\rangle$, $|k, \lambda\rangle$ and $|k, \alpha\rangle$:

From their number state representations 3.8, 3.125, and 3.219 we obtain

$$\langle k, \alpha | k, z\rangle = \frac{e^{-|\alpha|^2/2}}{\sqrt{g_k(|z|^2)}} \, C_k(\bar{\alpha}; z) \,, \tag{3.221}$$

$$C_k(\bar{\alpha}; z) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{(2k)_n}} \, \frac{(\bar{\alpha}\, z)^n}{n!} = \bar{C}_k(\bar{z}; \alpha) \;, \tag{3.222}$$

$$\langle k, \alpha | k, \lambda\rangle = e^{-|\alpha|^2/2} (1 - |\lambda|^2)^k \, D_k(\bar{\alpha}; \lambda) \,, \tag{3.223}$$

$$D_k(\bar{\alpha}; \lambda) = \sum_{n=0}^{\infty} \sqrt{(2k)_n} \, \frac{(\bar{\alpha}\, \lambda)^n}{n!} = \bar{D}_k(\bar{\lambda}; \alpha) \,. \tag{3.224}$$

From the completeness relations 3.17, 3.141 and 3.220 one gets a number of mappings in

terms of integral transforms

$$
\begin{aligned}
|k,z\rangle &= \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha\,\langle k,\alpha|k,z\rangle\,|k,\alpha\rangle = && (3.225)\\
&= \frac{1}{\sqrt{g_k(|z|^2)}}\int_{\mathbb{C}} d\tilde{\mu}(\alpha)\,C_k(\bar{\alpha};z)\,e^{|\alpha|^2/2}|k,\alpha\rangle\ ;\\
|k,\alpha\rangle &= \int_{\mathbb{C}} d\mu_k(z)\,\langle k,z|k,\alpha\rangle\,|k,z\rangle = && (3.226)\\
&= e^{-|\alpha|^2/2}\int_{\mathbb{C}} d\tilde{\mu}_k(z)\,C_k(\bar{z};\alpha)\sqrt{g_k(|z|^2)}\,|k,z\rangle\ ;\\
|k,\lambda\rangle &= \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha\,\langle k,\alpha|k,\lambda\rangle\,|k,\alpha\rangle = && (3.227)\\
&= (1-|\lambda|^2)^k\int_{\mathbb{C}} d\tilde{\mu}(\alpha)\,D_k(\bar{\alpha};\lambda)\,e^{|\alpha|^2/2}|k,\alpha\rangle\ ;\\
|k,\alpha\rangle &= \int_{\mathbb{D}} d\mu_k(\lambda)\,\langle k,\lambda|k,\alpha\rangle\,|k,\lambda\rangle = && (3.228)\\
&= e^{-|\alpha|^2/2}\int_{\mathbb{D}} d\tilde{\mu}_k(\lambda)\,D_k(\bar{\lambda};\alpha)(1-|\lambda|^2)^{-k}\,|k,\lambda\rangle\ ;
\end{aligned}
$$

These mappings imply the following unitary transformations of the basis functions

$$
\begin{aligned}
\tilde{f}_{k,n}(z) &= \int_{\mathbb{C}} d\tilde{\mu}(\alpha)\,C_k(\bar{\alpha};z)\,\tilde{h}_n(\alpha)\ , && (3.229)\\
\tilde{h}_n(\alpha) &= \int_{\mathbb{C}} d\tilde{\mu}_k(z)\,C_k(\bar{z};\alpha)\tilde{f}_{k,n}(z)\ ; && (3.230)\\
e_{k,n}(\lambda) &= \int_{\mathbb{C}} d\tilde{\mu}(\alpha)\,D_k(\bar{\alpha};\lambda)\,\tilde{h}_n(\alpha)\ , && (3.231)\\
\tilde{h}_n(\alpha) &= \int_{\mathbb{D}} d\tilde{\mu}_k(\lambda)\,D_k(\bar{\lambda};\alpha)\,e_{k,n}(\lambda)\ . && (3.232)
\end{aligned}
$$

Like in subsection 3.2.1 we get additional interesting integral transforms by multiplying the relations 3.225-3.228 from the left with an appropriate $\langle k,\cdot|$. I list only a few examples: Generalizations are straightforward.

$$
\begin{aligned}
e^{\bar{\alpha}_2\,\alpha_1} &= \int_{\mathbb{C}} d\tilde{\mu}_k(z)\,C_k(\bar{\alpha}_2;z)\,C_k(\bar{z};\alpha_1) = && (3.233)\\
&= \int_{\mathbb{D}} d\tilde{\mu}_k(\lambda)\,D_k(\bar{\alpha}_2;\lambda)\,D_k(\bar{\lambda};\alpha_1)\ , && (3.234)\\
e^{\bar{\lambda}\,z} &= \int_{\mathbb{C}} d\tilde{\mu}_k(\alpha)\,D_k(\bar{\lambda};\alpha)\,C_k(\bar{\alpha};z)\ , && (3.235)\\
C_k(\bar{\alpha};z) &= \int_{\mathbb{D}} d\tilde{\mu}_k(\lambda)\,D_k(\bar{\alpha};\lambda)\,e^{\bar{\lambda}\,z}\ , && (3.236)\\
D_k(\bar{\alpha};\lambda) &= \int_{\mathbb{C}} d\tilde{\mu}_k(z)\,C_k(\bar{\alpha};z)\,e^{\bar{z}\,\lambda}\ . && (3.237)
\end{aligned}
$$

3.3.2 Expectation values of the observables K_1, K_2 and K_0

Again using the relations 2.11-2.13 we get

$$\langle K_1\rangle_{k,\alpha} = |\alpha|\,\cos\beta\,\langle k,\alpha|\sqrt{N+2k}|k,\alpha\rangle\,, \tag{3.238}$$

$$\langle K_2\rangle_{k,\alpha} = -|\alpha|\,\sin\beta\,\langle k,\alpha|\sqrt{N+2k}|k,\alpha\rangle\,, \tag{3.239}$$

$$\langle K_0\rangle_{k,\alpha} = \langle N\rangle_{k,\alpha}+k = |\alpha|^2+k = \bar{n}_{k,\alpha}+k\,, \tag{3.240}$$

where

$$\langle k,\alpha|\sqrt{N+2k}|k,\alpha\rangle = h_1^{(k)}(|\alpha|) = e^{-|\alpha|^2}\sum_{n=0}^{\infty}\sqrt{2k+n}\,\frac{|\alpha|^{2n}}{n!}\,. \tag{3.241}$$

Notice that the variable $|\alpha|^2$ here, too, equals the observable quantity $\bar{n}_{k,\alpha}$, the average number of quanta associated with the state $|k,\alpha\rangle$!

Contrary to the mean numbers $\bar{n}_{k,z}$ and $\bar{n}_{k,\lambda}$ which depend on the index k (Eqs. 3.82 and 3.171) *the average $\bar{n}_{k,\alpha}$ does not depend on k* and we can omit the label k:

$$|\alpha|^2 = \bar{n}_\alpha\,. \tag{3.242}$$

In addition, we see again that the operators K_1 and K_2 measure the phase of the complex number α associated with the state $|k,\alpha\rangle$ in a very similar way as in the previous two cases $|k,z\rangle$ and $|k,\lambda\rangle$. Like there we have

$$\tan\beta = -\frac{\langle K_2\rangle_{k,\alpha}}{\langle K_1\rangle_{k,\alpha}}\,. \tag{3.243}$$

We further get

$$\langle K_1^2\rangle_{k,\alpha} = |\alpha|^2\,\cos^2\beta\,h_2^{(k)}(|\alpha|)+h^{(k)}(|\alpha|)\,, \tag{3.244}$$

$$h_2^{(k)}(|\alpha|) = e^{-|\alpha|^2}\sum_{n=0}^{\infty}\sqrt{(2k+n)(2k+n+1)}\,\frac{|\alpha|^{2n}}{n!}\,, \tag{3.245}$$

$$h^{(k)}(|\alpha|) = \frac{|\alpha|^4}{2}-\frac{1}{2}(h_2^{(k)}-2k-1)\,|\alpha|^2+\frac{k}{2}\,, \tag{3.246}$$

$$(\Delta K_1)^2_{k,\alpha} = |\alpha|^2\,\cos^2\beta\,[h_2^{(k)}-(h_1^{(k)})^2]+h^{(k)}\,; \tag{3.247}$$

$$\langle K_2^2\rangle_{k,\alpha} = |\alpha|^2\,\sin^2\beta\,h_2^{(k)}(|\alpha|)+h^{(k)}(|\alpha|)\,, \tag{3.248}$$

$$(\Delta K_2)^2_{k,\alpha} = |\alpha|^2\,\sin^2\beta\,[h_2^{(k)}-(h_1^{(k)})^2]+h^{(k)}\,; \tag{3.249}$$

$$(\Delta K_1)^2_{k,\alpha}\,(\Delta K_1)^2_{k,\alpha} = \frac{|\alpha|^4\,\sin^2 2\beta}{4}\,[h_2^{(k)}-(h_1^{(k)})^2]^2+ \tag{3.250}$$

$$+|\alpha|^2\,[h_2^{(k)}-(h_1^{(k)})^2]\,h^{(k)}+(h^{(k)})^2\,,$$

$$(\Delta K_1)^2_{k,\alpha}+(\Delta K_2)^2_{k,\alpha} = |\alpha|^2\,[h_2^{(k)}-(h_1^{(k)})^2]+2h^{(k)}\,; \tag{3.251}$$

$$S(K_1,K_2)_{k,\alpha} = -\frac{|\alpha|^2}{2}\,\sin 2\beta\,[h_2^{(k)}-(h_1^{(k)})^2]\,; \tag{3.252}$$

$$\langle K_0^2\rangle_{k,\alpha} = (|\alpha|^2+k)^2+|\alpha|^2\,, \tag{3.253}$$

$$(\Delta K_0)^2_{k,\alpha} = |\alpha|^2 = \langle N\rangle_{k,\alpha} = \bar{n}_\alpha\,. \tag{3.254}$$

Using the methods described in Appendix D.3 we obtain the following asymptotic expansions for the functions $h_1^{(k)}(|\alpha|)$ and $h_2^{(k)}(|\alpha|)$ and certain combinations of them in the case of large

$|\alpha|$:

$$h_1^{(k)}(|\alpha|) \asymp |\alpha|\,[1+(k-\frac{1}{8})\,|\alpha|^{-2}+c_{1;-4}\,|\alpha|^{-4}+ \tag{3.255}$$
$$+O(|\alpha|^{-6})]\,,\ c_{1;-4}=-\frac{1}{2}\,k^2-\frac{3}{8}\,k+\frac{7}{128}\ ,$$
$$(h_1^{(k)})^2 \asymp |\alpha|^2\,[1+(2k-\frac{1}{4})|\alpha|^{-2}-(k-\frac{1}{8})|\alpha|^{-4}+ \tag{3.256}$$
$$+O(|\alpha|^{-6}]\ ,$$
$$h_2^{(k)}(|\alpha|) \asymp |\alpha|^2\,[1+(2k+1/2)\,|\alpha|^{-2}-\frac{1}{8}\,|\alpha|^{-4}+ \tag{3.257}$$
$$+O(|\alpha|^{-6})]\,,$$
$$h_2^{(k)}(|\alpha|)-(h_1^{(k)})^2(|\alpha|) \asymp \frac{3}{4}+(k-\frac{1}{4})|\alpha|^{-2}+O(|\alpha|^{-4})\ , \tag{3.258}$$
$$h^{(k)}(|\alpha|) \asymp \frac{|\alpha|^2}{4}\,[1+(2k+\frac{1}{4})|\alpha|^{-2}+O(|\alpha|^{-4})]\,. \tag{3.259}$$

This gives the following asymptotic approximations for large $|\alpha|$

$$\langle K_1\rangle_{k,\alpha} \asymp |\alpha|^2\,\cos\beta\,[1+(k-\frac{1}{8})\,|\alpha|^{-2}+c_{1;-4}\,|\alpha|^{-4}+O(|\alpha|^{-6})]\,, \tag{3.260}$$
$$(\Delta K_1)^2_{k,\alpha} \asymp \frac{|\alpha|^2\,\cos^2\beta}{4}\,[3+(4k-1)|\alpha|^{-2}+O(|\alpha|^{-4})]+ \tag{3.261}$$
$$+\frac{|\alpha|^2}{4}\,[1+(2k+\frac{1}{4})|\alpha|^{-2}+O(|\alpha|^{-4})];$$
$$\langle K_2\rangle_{k,\alpha} \asymp -|\alpha|^2\,\sin\beta\,[1+(k-\frac{1}{8})\,|\alpha|^{-2}+c_{1;-4}\,|\alpha|^{-4}+O(|\alpha|^{-6})]\,, \tag{3.262}$$
$$(\Delta K_2)^2_{k,\alpha} \asymp \frac{|\alpha|^2\,\sin^2\beta}{4}\,[3+(4k-1)|\alpha|^{-2}+O(|\alpha|^{-4})]+ \tag{3.263}$$
$$+\frac{|\alpha|^2}{4}\,[1+(2k+\frac{1}{4})|\alpha|^{-2}+O(|\alpha|^{-4})]\ .$$

Whereas the operators

$$Q=\frac{1}{\sqrt{2}}(a^+ + a)\ ,\ P=\frac{i}{\sqrt{2}}(a^+ - a)\ ,\ [Q,P]=i\ ,$$

have very simple properties with respect to the states $|k,\alpha\rangle$, namely

$$(\Delta Q)^2_{k,\alpha}=(\Delta P)^2_{k,\alpha}=\frac{1}{2}\ ,\ (\Delta Q)^2_{k,\alpha}(\Delta P)^2_{k,\alpha}=\frac{1}{4}|\langle k,\alpha|[Q,P]|k,\alpha\rangle|^2\ , \tag{3.264}$$

the operators K_1 and K_2 obviously have not.

Because $[K_1,K_2]=-i\,K_0$ we have for arbitrary states $|\psi\rangle$

$$(\Delta K_1)^2_\psi\,(\Delta K_2)^2_\psi \geq \frac{1}{4}\,|\langle\psi|K_0|\psi\rangle|^2\ . \tag{3.265}$$

One speaks of "squeezing" (see Ch. 6) if

$$(\Delta K_1)^2_\psi<\frac{1}{2}|\langle\psi|K_0|\psi\rangle| \text{ or } (\Delta K_2)^2_\psi<\frac{1}{2}|\langle\psi|K_0|\psi\rangle|\ . \tag{3.266}$$

Because

$$\langle k,\alpha|K_0|k,\alpha\rangle=|\alpha|^2+k=|\alpha|^2(1+k|\alpha|^{-2})\ , \tag{3.267}$$

we see from Eqs. 3.261 and 3.263 that for large $|\alpha|$ the squared uncertainties 3.247 or 3.249 are squeezed in leading order if either

$$\cos^2\beta < \frac{1}{3} \text{ or } \sin^2\beta < \frac{1}{3} \ . \tag{3.268}$$

Chapter 4

Harmonic oscillator

There are many interrelations between the operators a and a^+ (or Q and P) of the harmonic oscillator the Lie algebra generators K_j, $j = 0, 1, 2$, of the group $SU(1,1)$ etc. Several of those relations we encountered already in previous chapters.

Of special interest are the non-linear ones [51]

$$K_+ = a^+ \sqrt{N + 2k} \ , \ K_- = \sqrt{N + 2k}\, a \ , \ K_0 = N + k \ , \tag{4.1}$$

which allows the construction of self-adjoint K_j, once the $a\,, a^+$ are given.

But even more important is the inversion (see Secs. 1.5.1 and 2.3): Given self-adjoint K_j, an irreducible unitary representation of the positive discrete series with index k and the associated Hilbert space, we can define

$$a = (K_0 + k)^{-1/2} K_- \ , \ a^+ = K_+ (K_0 + k)^{-1/2} \ , \ N = K_0 - k = a^+ a \ . \tag{4.2}$$

Another interesting relationship between the $a^+\,, a$ and the K_j is the following: The bilinear expressions

$$K_+ = \frac{1}{2}(a^+)^2 \ , \ K_- = \frac{1}{2} a^2 \ , \ K_0 = \frac{1}{2}(a^+ a + 1/2) \tag{4.3}$$

obey the Lie algebra 2.7. As the operator K_- here annihilates the states $|n = 0\rangle$ *and* $|n = 1\rangle$, we get two different representations of the Lie algebra, one with $k = 1/4$ and one with $k = 3/4$. Details of these will be discussed Sec. 6.3, also in Sec. 6.4 realizations of that Lie algebra by a pair a_1, a_2 of annihilation operators and the related creation operators.

As the harmonic oscillator has the Hamiltonian $H = \omega\,(N + 1/2)$ the third of the relations 4.1 strongly suggests to identify

$$H_{osc} = K_0 \text{ for } k = \frac{1}{2} \tag{4.4}$$

(frequency ω scaled to $= 1$) and to implement the quantum mechanics of the harmonic oscillator in a Hilbert space which contains a unitary irreducible representation of $SU(1,1)$ or $SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$ with index $k = 1/2$.

All the results and predictions of the last chapter, derived for general indices k, apply, of course, for $k = 1/2$, too, and will therefore not be listed here again. They contain, to be sure, a wealth of new informations about the quantum mechanical harmonic oscillator! One just has to take $k = 1/2$ in all the formulae of Ch. 3! In some cases one has to take the limit $k \to 1/2$, starting from $k > 1/2$

The present chapter focuses on properties which are special for unitary representations with $k = 1/2$.

4.1 Quantum mechanics of the harmonic oscillator in the Hardy space of the unit circle

In subsection 3.2.1 we encountered the scalar product

$$(f,g)_{\mathbb{D},\,k} = \frac{2k-1}{\pi}\int_{\mathbb{D}} \bar{f}(\lambda)g(\lambda)\,(1-|\lambda|^2)^{2k-2}|\lambda|\,d|\lambda|\,d\theta\ , \tag{4.5}$$

for the important (Bargmann) Hilbert space $\mathcal{H}_{\mathbb{D},\,k}$ of holomorphic functions on the unit disc $\mathbb{D}$ (Eq. 1.111). It can be used for any real $k > 1/2$ and also - as we have seen - in the limiting case $k \to 1/2$.
Any holomorphic function in $\mathbb{D}$ can be expanded in powers of λ and we saw that the functions

$$e_{k,n}(\lambda) = \sqrt{\frac{(2k)_n}{n!}}\,\lambda^n \quad , \quad n = 0,1,2,\ldots \quad , \tag{4.6}$$

form an orthonormal basis of $\mathcal{H}_{\mathbb{D},\,k}$.

If we have the functions

$$f(\lambda) = \sum_{n=0}^{\infty} a_n\,\lambda^n\ , \quad g(\lambda) = \sum_{n=0}^{\infty} b_n\,\lambda^n\ , \tag{4.7}$$

then, according to Eq. 3.143, their scalar product $(f,g)_{\mathbb{D},\,k}$ is given by the series

$$(f,g)_{\mathbb{D},\,k} = \sum_{n=0}^{\infty} \frac{n!}{(2k)_n}\,\bar{a}_n\,b_n \quad . \tag{4.8}$$

This series can be used to extend the definition of the scalar product for Hilbert spaces $\mathcal{H}_{\mathbb{D},\,k}$ to all real $k > 0$!

For the special case $k = 1/2$ the coefficient in front of $\bar{a}_n\,b_n$ in Eq. 4.8 has the value 1. This allows for an interesting reinterpretation of the Hilbert space $\mathcal{H}_{\mathbb{D},\,\frac{1}{2}}$:

Consider the Hilbert space $L^2(S^1, d\varphi)$ on the unit circle with the scalar product

$$(f_2, f_1) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\ \bar{f}_2(\varphi)\,f_1(\varphi)\ , \tag{4.9}$$

an orthonormal basis of which is given by the functions $\exp(i\,n\,\varphi)\ ,\ n \in \mathbb{Z}$.

That subspace of functions $f(\varphi) \in L^2(S^1, d\varphi)$ which have only non-negative Fourier coefficients, i.e. $a_n = 0$ for $n < 0$, is being called the "Hardy space $H^2(S^1, d\varphi)$ of the unit disc" $\mathbb{D}$ or "unit cirle" ($S^1 = \partial\mathbb{D}$) [127], and the corresponding scalar product will be denoted by $(f_1, f_2)_+$. The Hilbert space $H^2(S^1, d\varphi)$ has the orthonormal basis

$$e_n(\varphi) = e^{i\,n\,\varphi}\ ,\ n = 0,1,2,\ldots\ . \tag{4.10}$$

Hardy spaces have a number of interesting properties and are closely related to Hilbert spaces of holomorphic functions [127, 257] because the unit circle is the boundary $\partial\mathbb{D}$ of $\mathbb{D}$! Notice that the eigenfunctions 4.10 may be considered as limits of those in Eq. 4.6:

$$e_n(\varphi) = \lim_{|\lambda|\to 1} e_{k=1/2,n}(|\lambda|\,e^{i\,\varphi})\ ,\ \lambda \in \mathbb{D}\ . \tag{4.11}$$

(Mathematically the limit should be taken in terms of the appropriate norm [127]!)

If we have two Fourier series $\in H^2(S^1\,, d\varphi)$,

$$f_1(\varphi) = \sum_{n=0}^{\infty} a_n\, e^{i\,n\,\varphi}\ , \quad f_2(\varphi) = \sum_{n=0}^{\infty} b_n\, e^{i\,n\,\varphi}\quad , \tag{4.12}$$

they have the scalar product

$$(f_1, f_2)_+ = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, \bar{f}_1(\varphi) f_2(\varphi) = \sum_{n=0}^{\infty} \bar{a}_n\, b_n\quad . \tag{4.13}$$

Comparing with 4.8 for $k = 1/2$ we see that we may realize the Hilbert space $\mathcal{H}_{\mathbb{D},\,1/2}$ by using the Hardy space $H^2(S^1,\, d\varphi)$!

The $SU(1,1)$ Lie algebra generators 3.161 for $k = 1/2$ are now [40]

$$K_+ \;=\; e^{i\,\varphi}\,(\frac{1}{i}\partial_\varphi + 1)\ , \tag{4.14}$$

$$K_- \;=\; e^{-i\,\varphi}\,\frac{1}{i}\partial_\varphi\ , \tag{4.15}$$

$$K_0 \;=\; \frac{1}{i}\partial_\varphi + \frac{1}{2}\ , \tag{4.16}$$

for which the relations 2.11-2.13 take the form

$$K_0\, e_n(\varphi) \;=\; (n + \frac{1}{2})\, e_n(\varphi)\ , \tag{4.17}$$

$$K_+\, e_n(\varphi) \;=\; (n+1)\, e_{n+1}(\varphi)\ , \tag{4.18}$$

$$K_-\, e_n(\varphi) \;=\; n\, e_{n-1}(\varphi)\ . \tag{4.19}$$

The operators

$$a = (K_0 + 1/2)^{-1/2} K_-\,, \quad a^+ = K_+ (K_0 + 1/2)^{-1/2} \tag{4.20}$$

have the usual properties

$$a\, e_n(\varphi) = \sqrt{n}\, e_{n-1}(\varphi)\,, \quad a^+\, e_n(\varphi) = \sqrt{n+1}\, e_{n+1}(\varphi)\,, \tag{4.21}$$

and have the usual matrix elements [128]. The same applies, of course, to those of the operators Q and P:

$$Q = \frac{1}{\sqrt{2}}\,(a^+ + a)\,, \quad P = \frac{i}{\sqrt{2}}\,(a^+ - a)\,. \tag{4.22}$$

Before I turn to the harmonic oscillator itself let me add a few mathematical remarks:

The reproducing kernel (see Eqs. 3.145 and 3.146) here has the form

$$\Delta(\varphi_2, \varphi_1) = \sum_{n=0}^{\infty} \overline{e_n(\varphi_2)}\, e_n(\varphi_1) = (1 - e^{i\,(\varphi_1 - \varphi_2)})^{-1}\ , \tag{4.23}$$

$$\frac{1}{2\,\pi}\int_0^{2\,\pi} d\varphi_2\, \Delta(\varphi_2, \varphi_1)\ e_n(\varphi_2) = e_n(\varphi_1)\ .$$

The kernel has a singularity (pole) for $\varphi_2 = \varphi_1$. In calculations one has to replace $\exp(i\,(\varphi_1 - \varphi_2))$ by $(1-\epsilon)\,\exp(i\,(\varphi_1 - \varphi_2))$ and then take the limit $\epsilon \to 0$ at the end.

The operators $(K_0 + 1/2)^{-s}, s = 1/2, 1$ etc., as needed, e.g. in Eqs. 1.107 and 1.110, are represented by integral kernels

$$\begin{aligned}(K_0 + \frac{1}{2})^{-s}(\varphi_2, \varphi_1) &= \sum_{n=0}^{\infty}(n+1)^{-s}(e^{i\,(\varphi_1-\varphi_2)})^n = &(4.24)\\ &= \Phi(e^{i\,(\varphi_1-\varphi_2)}, s, 1) = &(4.25)\\ &= e^{-i\,(\varphi_1-\varphi_2)}\sum_{n=1}^{\infty} n^{-s}(e^{i\,(\varphi_1-\varphi_2)})^n &(4.26)\\ &= e^{-i\,(\varphi_1-\varphi_2)}\mathrm{Li}_s(e^{i\,(\varphi_1-\varphi_2)})\ , &(4.27)\end{aligned}$$

where $\Phi(z, s, a)$ is Lerch's function and $\mathrm{Li}_s(z)$ the polylogarithm (see Eqs. 3.205 and 3.206 of subsection 3.2.2).

For $s > 0$, $a = 1$ and $\varphi_2 \neq \varphi_1$ one has the integral representation [123]

$$(K_0 + \frac{1}{2})^{-s}(\varphi_2, \varphi_1) = \frac{1}{\Gamma(s)}\int_0^{\infty} dt\,\frac{t^{s-1}}{e^t - e^{i\,(\varphi_1-\varphi_2)}}\ . \tag{4.28}$$

According to the Eqs. 4.4 and 4.16 we now get for the Hamiltonian of the harmonic oscillator

$$H_{osc} = (\frac{1}{i}\,\partial_\varphi + \frac{1}{2})\ . \tag{4.29}$$

As here $|k = 1/2, n\rangle = \exp(i\,n\,\varphi)$, we can sum the coherent state series 3.8, 3.125 and 3.219 as functions of φ:

For the coherent state 3.8 we get

$$f_z(\varphi) = \frac{1}{\sqrt{I_0(2\,|z|)}}\,e^{z\,e^{i\,\varphi}} = \frac{1}{\sqrt{I_0(2\,|z|)}}\,e^{|z|\,e^{i\,(\varphi+\phi)}}\ . \tag{4.30}$$

Thus, $f_z(\varphi)$ is essentially the generating function for the basis 4.10.

Applying the time evolution operator

$$U(t) = e^{-iHt} = e^{-iK_0 t} \tag{4.31}$$

to the function $f_z(\varphi)$ (in their number basis 3.8) yields

$$U(t)\cdot f_z(\varphi) = e^{-it/2}\,f_{z(t)}(\varphi)\ ,\ z(t) = z\,e^{-it}\ . \tag{4.32}$$

Thus, the time evolution essentially translates the phase ϕ of z by $-t$ (recall that the frequency ω has been scaled to the value 1).

Similarly we get for the coherent states 3.125:

$$f_\lambda(\varphi) = (1 - |\lambda|^2)^{1/2}\,(1 - \lambda\,e^{i\,\varphi})^{-1} = (1 - |\lambda|^2)^{1/2}\,(1 - |\lambda|\,e^{i\,(\varphi+\theta)})^{-1} \tag{4.33}$$

and

$$U(t)\cdot f_\lambda(\varphi) = e^{-it/2}\,f_{\lambda(t)}(\varphi)\ ,\ \lambda(t) = \lambda\,e^{-it}\ . \tag{4.34}$$

For the coherent states 3.219 the resulting series cannot be summed in an elementary way:

$$f_\alpha(\varphi) = e^{-|\alpha|^2/2}\,\sum_{n=0}^{\infty}\frac{(\alpha\,e^{i\,\varphi})^n}{\sqrt{n!}} = e^{-|\alpha|^2/2}\,\sum_{n=0}^{\infty}\frac{(|\alpha|\,e^{i\,(\varphi+\beta)})^n}{\sqrt{n!}}\ , \tag{4.35}$$

$$U(t)\cdot f_\alpha(\varphi)=e^{-it/2}\,f_{\alpha(t)}(\varphi)\ ,\ \alpha(t)=\alpha\,e^{-it}\,. \tag{4.36}$$

(Recall that the series 3.219 can be summed to a Gaussian function if one takes for $|n\rangle$ the Hermite functions 4.67.)

The above results for the time evolution of the 3 different coherent states show that the phases of $z\,,\lambda$ and α are the dynamical variables, *not* the mathematical auxiliary phase φ!

An asymptotic expansion of the function 4.35 has been given in Ref. [129] as

$$f_\alpha(\varphi)\asymp(2\,\pi)^{1/4}\,(2\,|\alpha|)^{1/2}\,e^{-[|\alpha|^2\,(\varphi+\beta)^2\,-\,i\,(|\alpha|^2\,-\,1/2)\,(\varphi+\beta)]}\ . \tag{4.37}$$

The above discussions show explicitly that we can associate *three* different coherent states with the harmonic oscillator, all of which stay coherent with time! They have different "squeezing" and many different other properties, already discussed in Ch. 3 in a more general context.

I would like to stress again that - contrary to all possible appearances - the phase φ of the Hilbert space $H^2(S^1,d\varphi)$ with the basis 4.10 and the scalar product 4.13 is *not* the quantum mechanical canonically conjugate observable with respect to the operator K_0 of Eq. 4.16. The reason is that φ as a multiplication operator is not self-adjoint with respect to the scalar product 4.13 [28].

The quantity φ is merely a mathematical auxiliary variable which parametrizes the Hilbert space. The information about the physical phases of the states 4.30, 4.33 and 4.35 has to be extracted by means of the operators K_1 and K_2 as discussed in detail in the previous Ch. 3.

In addition, the multiplication operator $\exp(i\,\varphi)$ is not unitary on $H^2(S^1,d\varphi)$, because it acts on the eigenstate basis as an isometric shift operator [31]:

$$e^{i\,\varphi}\,e_n(\varphi)=e_{n+1}(\varphi)\ , \tag{4.38}$$

where the inverse transformation

$$e^{-i\,\varphi}\,e_n(\varphi)=e_{n-1}(\varphi) \tag{4.39}$$

is, however, not defined for $e_{n=0}$! This implies again that φ cannot be a self-adjoint operator, because otherwise $\exp(i\,\varphi)$ would be unitary!

In the present approach to the quantum theory of the harmonic oscillator the basic observables are the self-adjoint operators K_0, K_1 and K_2. Even the position and momentum operators Q and P are functions of them (see Eq. 4.22)!

4.2 Some critical remarks on "phase states"

At this point a few remarks about the so-called "phase states" [57] may be appropriate:

In the search for a possible phase operator it was surmised [29] that the following ("phase") state might be a candidate for an eigenstate of the yet to be found phase operator:

$$|\varphi\rangle=\sum_{n=0}^{\infty}e^{i\,n\,\varphi}\,|n\rangle\ , \tag{4.40}$$

where $|n\rangle$ are the usual eigenstates (Hermite functions) of the harmonic oscillator and the $e_n(\varphi)=\exp(i\,n\,\varphi)$ constitute a basis for the Hilbert space with the scalar product 4.13. The situation is the same as in the case of the three types 3.8, 3.125 and 3.219 of coherent states which are introduced as series in the number states the coefficients of which form a basis in an associated Hilbert space!

In the literature, however, the functions $\exp(i\,n\,\varphi)$ are treated as mere coefficients multiplying the basis vectors $|n\rangle$. The norm of the state 4.40 is then obviously infinite.

For the formal scalar product with respect to the basis $|n\rangle$ of two such states one gets

$$\langle\varphi_2|\varphi_1\rangle = \sum_{n=0}^{\infty} e^{i\,n\,(\varphi_1-\varphi_2)}\,, \tag{4.41}$$

which is just the reproducing kernel 4.23, and not a delta-function! The latter property usually is then interpreted as an indication that there is no such self-adjoint phase operator which has the state $|\varphi\rangle$ as an eigenstate, because states $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are not orthogonal for $\varphi_2 \neq \varphi_1$! The argument, however, is due to misunderstandings:

First, let me rewrite the expression 4.40 in a more symmetrical way:

$$|\varphi\rangle \to F(\varphi, x) = \sum_{n=0}^{\infty} e_n(\varphi)\,u_n(x)\,, \tag{4.42}$$

where $|n\rangle = u_n(x)$ are Hermite's functions [128] with the scalar product

$$(g_2, g_1) = \int_{-\infty}^{\infty} dx\,\bar{g}_2(x)\,g_1(x)\,. \tag{4.43}$$

Eq. 4.42 is the - formal - sum over the products $e_n(\varphi)\,u_n(x)$ of the eigenfunctions of the oscillator Hamilton operator H_{osc}, represented in two different Hilbert spaces, namely $H^2(S^1, \varphi)$ and $L^2(\mathbb{R}, dx)$! Thus, the products in the sum are just the "diagonal" elements of the the basis $\{e_{n_1}(\varphi)\,u_{n_2}(x)\,,\ n_1\,, n_2 = 0, 1, 2, \ldots\}$ for the tensor product of the two Hilbert spaces!

Instead of taking the formal "scalar product" of two "states" $F_j(\varphi_j, x_j)\,,\ j = 1, 2$ with respect to the basis $u_n(x)$ – i.e. $x_2 = x_1 = x$ and integration over x, (Eq. 4.66), – with the $e_n(\varphi_j)$ as coefficients, like what is being done in Eq. 4.41, we may as well do it the other way round and obtain

$$\frac{1}{2\,\pi}\int_0^{2\,\pi} d\,\varphi\,\bar{F}_2(\varphi, x_2)\,F_1(\varphi, x_1) = \sum_{n=0}^{\infty} \bar{u}_n(x_2)\,u_n(x_1) = \delta(x_2 - x_1)\,. \tag{4.44}$$

Here the δ-function is the reproducing kernel for the basis $\{u_n(x)\}$!

One of the misunderstandings is the following:

A reproducing kernel represents the properties of the *completeness relation* for the functions of the basis in a concrete Hilbert space. The completeness does not have to be expressed by a δ-function as the examples of the coherent states $|k, z\rangle\,, |k, \lambda\rangle$ and $|k, \alpha\rangle$ in the last chapter clearly show. They also show that the completeness relation is independent of the orthogonality of the associated eigenfunctions.

Another well-known example for such a situation is the Hilbert space of positive frequency solutions of the free Klein-Gordon equation [130]. They have the Fourier representation

$$\begin{aligned} \psi^{(+)}(x) &= \frac{1}{i\,(2\,\pi)^{3/2}}\int_{p^0>0} d^4p\,e^{-ip\cdot x}\,\delta(p^2 - m^2)\,a(p)\,, \\ & \qquad x = (x^0, \vec{x})\,,\ p\cdot x = p^0x^0 - \vec{p}\cdot\vec{x}\,, \end{aligned} \tag{4.45}$$

and their scalar product is

$$\begin{aligned} (\psi_2^{(+)}, \psi_1^{(+)}) &= i\int_{x^0=t} d^3x\,\bar{\psi}_2^{(+)}\partial_0\psi_1^{(+)} - (\partial_0\bar{\psi}_2^{(+)})\,\psi_1^{(+)} & (4.46) \\ &= \int_{p^0>0} d^4p\,\delta(p^2 - m^2)\bar{a}_2(p)\,a_1(p)\,. & (4.47) \end{aligned}$$

For an orthonormal basis $\{f_n\}$, $(f_{n_2}, f_{n_1}) = \delta_{n_2 n_1}$, one has the completeness relation

$$\sum_{n=0}^{\infty} \bar{f}_n(x_2)\, f_n(x_1) = i\, \Delta_+(x_2 - x_1) \ , \tag{4.48}$$

where Δ_+ is the distribution (generalized function)

$$\Delta_+(x_2 - x_1) = \frac{1}{i\,(2\,\pi)^3} \int_{p^0>0} d^4p\, \delta(p^2 - m^2)\, e^{-i\,p\cdot x} \ , \tag{4.49}$$

which is not a δ-function, either.

In order to avoid the (pseudo-) problems mentioned above it has been proposed [9,115,131] to truncate the sum 4.40 at some finite $n = s$ and start with a finite dimensional phase state space with a discretized phase variable, where everything is under mathematical control. Barnett and Pegg suggested to do all required calculations in the finite dimensional space first and let the dimension $s + 1$ go to ∞ at the very end. This proposal has led to a large number of follow-up papers [57]. But it is unsatisfactory for several reasons: In finite dimensional vector spaces all Hermitian operators are also self-adjoint, i.e. have a complete set of eigenfunctions and therefore a spectral representation. This is no longer true in infinite dimensional Hilbert spaces (see, e.g. Ref. [28]) and therefore one has to expect problems for the limiting theory, which may be reached - if at all - by weak convergence only. As to a discussion of the mathematical problems involved see Ref. [132].

Actually it is not necessary at all to employ the additional oscillator basis $|n\rangle$ in Eq. 4.40. We have seen that it suffices to work with the functions 4.10 and the associated Hilbert space $H^2(S^1, d\varphi)$ alone! *All* the usual quantum theory of the harmonic oscillator can be described by means of that space.

Furthermore, we have seen in the previous section, that the phase φ is a mere auxiliary mathematical variable and *not* a canonical quantity!

As to the relationship between the Hilbert space $H^2(S^1, d\varphi)$ with its basis $e_n(\varphi)$ and the usual Hilbert space $L^2(\mathbb{R}, dx)$ with its basis of Hermite functions $u_n(x)$ see Sec. 4.5.

4.3 Eigenfunctions of K_1 and K_2

In the present framework the quantum theoretical properties of the phase are incorporated into the operators K_1 and K_2 which, according to the Eqs. 4.14 and 4.15, here have the form

$$K_1 = \frac{1}{2}(K_+ + K_-) = \cos\varphi\, \frac{1}{i}\partial_\varphi + \frac{1}{2}\, e^{i\,\varphi} \ , \tag{4.50}$$

$$K_2 = \frac{1}{2i}(K_+ - K_-) = \sin\varphi\, \frac{1}{i}\partial_\varphi + \frac{1}{2i}\, e^{i\,\varphi} \ . \tag{4.51}$$

The determination of their eigenfunctions $f_{h_1}(\varphi)$ and $f_{h_2}(\varphi)$ is straightforward. Let me start with K_2. The case K_1 can be dealt with by a shift $\varphi \to \varphi + \pi/2$.

Integrating the differential equation

$$K_2 f_{h_2}(\varphi) = h_2 f_{h_2}(\varphi) \tag{4.52}$$

yields [133]

$$f_{h_2}(\varphi) = C\,(\sin\varphi)^{-1/2} (\tan(\varphi/2))^{i\,h_2}\, e^{-i\,\varphi/2} \ , \ h_2 \in \mathbb{R} \ , \ \varphi \in (0, \pi) \ , \tag{4.53}$$

where C is a normalization constant. In the interval $\varphi \in (\pi, 2\pi)$ the functions $\sin\varphi$ and $\tan(\varphi/2)$ are negative. Here we get

$$f_{h_2}(\varphi) = C\,|\sin\varphi|^{-1/2}|\tan(\varphi/2)|^{i\,h_2}\,e^{-i\,\varphi/2}\ ,\ h_2 \in \mathbb{R}\ ,\ \varphi \in (\pi, 2\pi)\ , \tag{4.54}$$

Thus we have

$$f_{h_2}(\varphi) = C\,|\sin\varphi|^{-1/2}|\tan(\varphi/2)|^{i\,h_2}\,e^{-i\,\varphi/2}\ ,\ h_2 \in \mathbb{R}\ ,\ \varphi \in (0, 2\pi)\ . \tag{4.55}$$

The normalization constant C is determined from

$$\frac{1}{2\,\pi}\int_0^\pi d\varphi\,\bar{f}_{h_2'}(\varphi)\,f_{h_2}(\varphi) = \frac{|C|^2}{2\,\pi}\int_0^\pi \frac{d\,\varphi}{\sin\varphi}(\tan(\varphi/2))^{i\,(h_2 - h_2')}\ . \tag{4.56}$$

The substitution

$$u = \ln\tan(\varphi/2)\ ,\ du = \frac{d\varphi}{\sin\varphi}\ ,\ u(\varphi \to 0^+, \pi^-) \to -\infty\,, +\infty, \tag{4.57}$$

then gives

$$\frac{1}{2\,\pi}\int_0^\pi d\varphi\,\bar{f}_{h_2'}(\varphi)\,f_{h_2}(\varphi) = |C|^2\,\delta(h_2' - h_2)\ . \tag{4.58}$$

Taking the integral 4.58 from π to $2\,\pi$ gives the same contribution, so that $|C|^2 = 1/2$. Thus, we finally have

$$f_{h_2}(\varphi) = |2\sin\varphi|^{-1/2}|\tan(\varphi/2)|^{i\,h_2}\,e^{-i\,\varphi/2}\ ,\ h_2 \in \mathbb{R}\ ,\ \varphi \in (0, 2\pi)\ . \tag{4.59}$$

The substitution $\varphi \to \varphi + \pi/2$ transforms the operator K_2 of Eq. 4.51 into the operator K_1 of Eq. 4.50. Its normalized eigenfunctions therefore are

$$f_{h_1}(\varphi) = |2\cos\varphi|^{-1/2}|\tan(\varphi/2 + \pi/4)|^{i\,h_1}\,e^{-i\,\varphi/2}\ ,\ h_1 \in \mathbb{R}\ ,\ \varphi \in (0, 2\pi)\ . \tag{4.60}$$

The hypothetical ansatz

$$f_{h_1}(\varphi) = \sum_{n=0}^{\infty} c_n^{(1)}\,e_n(\varphi) \tag{4.61}$$

for the eigenfunctions of K_1 from 4.50 leads to the recursion formula

$$c_{n+1}^{(1)} = \frac{2\,h_1}{n+1}\,c_n^{(1)} - \frac{n}{n+1}\,c_{n-1}^{(1)}\ , n = 0, 1, 2, \ldots\ . \tag{4.62}$$

The first few terms are the following

$$\begin{aligned} c_1^{(1)}/c_0^{(1)} &= 2\,h_1\ , \\ c_2^{(1)}/c_0^{(1)} &= 2\,h_1^2 - \frac{1}{2}\ , \\ c_3^{(1)}/c_0^{(1)} &= \frac{2^3}{3!}\,h_1^3 - \frac{5}{3}\,h_1\ , \\ c_4^{(1)}/c_0^{(1)} &= \frac{2^4}{4!}\,h_1^4 - \frac{7}{3}\,h_1^2 + \frac{3}{8}\ . \end{aligned} \tag{4.63}$$

For $f_{h_2}(\varphi)$ we get accordingly:

$$c_{n+1}^{(2)} = \frac{2i\,h_1}{n+1}\,c_n^{(2)} + \frac{n}{n+1}\,c_{n-1}^{(2)}\ , n = 0, 1, 2, \ldots\ . \tag{4.64}$$

$$
\begin{aligned}
c_1^{(2)}/c_0^{(2)} &= 2i\,h_2\ , \\
c_2^{(2)}/c_0^{(2)} &= -2\,h_2^2+\frac{1}{2}\ , \\
c_3^{(2)}/c_0^{(2)} &= -\frac{2^3 i}{3!}\,h_2^3-\frac{5i}{3}\,h_2\ , \\
c_4^{(2)}/c_0^{(2)} &= \frac{2^4}{4!}\,h_2^4-\frac{7}{3}\,h_2^2+\frac{3}{8}\ .
\end{aligned}
\tag{4.65}
$$

These formulae allow nothing to say about the possible convergence of the series.

4.4 The harmonic oscillator in the Hardy space of the complex upper half-plane

There is the obvious question to be asked:

The usual quantum mechanical description of the harmonic oscillator is in terms of the Hilbert space $L^2(\mathbb{R},d\xi)$ with the scalar product

$$
(g_2,g_1)=\int_{-\infty}^{\infty}d\xi\,\bar{g}_2(\xi)\,g_1(\xi)\ ,\ \xi=\beta x\ ,\ \beta=\sqrt{m\,\omega/\hbar}\,,
\tag{4.66}
$$

and the Hermite functions - the oscillator eigenfunctions of the stationary Schrödinger equation - as an orthonormal basis (see, e.g. Ref. [128]):

$$
u_n(\xi)=\frac{e^{-\xi^2/2}}{2^{n/2}\,\sqrt{n!}\,\pi^{1/4}}\,H_n(\xi)\,,
\tag{4.67}
$$

where $H_n(\xi)$ is the n-th Hermite polynomial.

If the same quantum mechanics is to be described by the Hilbert space $H^2(S^1,d\varphi)$ with the scalar product 4.13 and the eigenfunctions 4.10, what is the relationship between the two spaces?

The answer is somewhat subtle and not quite straightforward[1]: I shall first discuss the relationship between the Hardy space $H^2(S^1,d\varphi)$ of the circle and the unitarily equivalent Hardy space $H^2(\mathbb{R},d\xi)$ on the real line.

The space $H^2(\mathbb{R},d\xi)$ consists of that closed subspace of functions $g(\xi)\in L^2(\mathbb{R},d\xi)$ wich are limits (in an appropriate topology)

$$
g(\xi)=\lim_{\eta\to 0}g(\xi+i\,\eta)
\tag{4.68}
$$

of functions $g(z=\xi+i\,\eta)$ which are holomorphic in the upper complex half-plane. Afterwards I shall explain how the elements of $L^2(\mathbb{R},d\xi)$ can be projected on the subspace $H^2(\mathbb{R},d\xi)$ and and how the two spaces are unitarily related. The space $H^2(\mathbb{R},d\xi)$ is of physical interest by itself.

4.4.1 The relationship between the Hardy space of the circle and that of the real line

The transformation [127, 259]:

$$
\xi=\frac{1}{i}\,\frac{e^{i\,\varphi}-1}{e^{i\,\varphi}+1}=\tan\frac{\varphi}{2}\ ,\quad e^{i\,\varphi}=\frac{1+i\,\xi}{1-i\,\xi}\ ,\ \varphi=2\arctan\xi\,,
\tag{4.69}
$$

[1] This was, unfortunately, obscured in the first (even printed) version of the present paper, though it is indicated in the frequently quoted Ref. [40].

maps the unit circle $S^1 = \partial\mathbb{D}$ onto the real line $\mathbb{R}$ and vice versa.

Given a function $f(\varphi) \in H^2(S^1, d\varphi)$, we can define a function $g^{(+)}(\xi) \in H^2(\mathbb{R}, d\xi) \subset L^2(\mathbb{R}, d\xi)$ and vice versa:

$$g^{(+)}(\xi) = \frac{1}{\sqrt{\pi}\,(1 - i\,\xi)}\, f\left(e^{i\,\varphi} = \frac{1 + i\,\xi}{1 - i\,\xi}\,,\ \varphi = 2\arctan\xi\right)\,, \tag{4.70}$$

$$f(\varphi) = \frac{2\sqrt{\pi}}{1 + e^{i\,\varphi}}\, g^{(+)}(\,\xi = \tan(\varphi/2)\,)\,, \tag{4.71}$$

$$(1 + e^{i\,\varphi})(1 - i\,\xi) = 2\,. \tag{4.72}$$

The mapping is unitary because

$$(g_2^{(+)}, g_1^{(+)}) = \int_{-\infty}^{\infty} d\xi\, \bar{g}_2^{(+)}(\xi)\, g_1^{(+)}(\xi) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, \bar{f}_2(\varphi) f_1(\varphi)\,,\ \frac{d\varphi}{2} = \frac{d\xi}{1 + \xi^2}\,. \tag{4.73}$$

The unitary transformation 4.70 and 4.71 maps the basis 4.10 onto an orthonormal basis

$$v_n(\xi) = \frac{1}{\sqrt{\pi}(1 - i\,\xi)}\left(\frac{1 + i\,\xi}{1 - i\,\xi}\right)^n\,,\ n = 0, 1, \ldots, \tag{4.74}$$

of $H^2(\mathbb{R}, d\xi)$.

Except for an irrelevant phase, the basis 4.74 coincides with the basis B.105 for $k = 1/2$, $\Im(w) = v = 0$, and the scalar product B.116.

If we replace the real variable ξ in Eq. 4.74 by the complex $z = \xi + i\,\eta$ one, then the functions $v_n(z)$ are holomorphic in the upper half-plane and have a pole of order $n + 1$ in the lower on.

A formal transformation of the type 4.70 was already discussed by London [26], using methods of Jordan and Pauli!

The functions 4.74 are eigenfunctions, with eigenvalues $n + 1/2$, of the (Hamilton) operator

$$\tilde{K}_0 = \tilde{H}_{osc} = \frac{1}{2i}[\xi + (\xi^2 + 1)\frac{d}{d\xi}\,]\,, \tag{4.75}$$

$$\tilde{H}_{osc}\, v_n(\xi) = (n + \frac{1}{2})\, v_n(\xi)\,. \tag{4.76}$$

This follows from Eq. B.111 of Appendix B. The associated Eqs. B.112 give - after an appropriate redefinition of the phases -

$$\tilde{K}_+ = \frac{1 + i\xi}{2}\,[1 - i(1 + i\xi)\frac{d}{d\xi}]\,, \tag{4.77}$$

$$\tilde{K}_+ v_n(\xi) = (n + 1)\, v_{n+1}(\xi)\,, \tag{4.78}$$

$$\tilde{K}_- = -\frac{1 - i\xi}{2}\,[1 + i(1 - i\xi)\frac{d}{d\xi}]\,, \tag{4.79}$$

$$\tilde{K}_- v_n(\xi) = n\, v_{n-1}\,, \tag{4.80}$$

which imply

$$\tilde{K}_1 = \frac{i}{2}\,[\xi + (\xi^2 - 1)\,\frac{d}{d\xi}]\,. \tag{4.81}$$

$$\tilde{K}_2 = \frac{1}{i}\,(\frac{1}{2} + \xi\,\frac{d}{d\xi})\,, \tag{4.82}$$

The eigenfunctions of $\tilde{K}_1$ and $\tilde{K}_2$ are discussed below.

The eigenfunctions 4.67 have a remarkable property: Their density

$$\rho_{v;n}(\xi) = |v_n(\xi)|^2 = \frac{1}{\pi(1+\xi^2)} \tag{4.83}$$

is independent of n : $\rho_{v;n} = \rho_v$!

This is a remarkable difference compared to the densities $|u_n(\xi)|^2$ of the eigenfunctions 4.74 which depend strongly on n ! The property 4.83 is, of course, a consequence of the fact that the eigenfunctions 4.10 have the constant density 1 !

In probability theory and statistics the density 4.83 is called the "density of the Cauchy distribution" [134] (in physics: "Lorentz" [135] or "Breit-Wigner" distribution).

One property of the Cauchy distribution is that its moments

$$\langle \xi^m \rangle = \int_{-\infty}^{\infty} d\xi \, \xi^m \rho_v(\xi) \tag{4.84}$$

do not exist for $m \geq 1$. (For odd m the integral is formally $= 0$ because the integrand is an odd function. However, the characteristic function

$$\phi(t) = \int_{-\infty}^{\infty} d\xi \, \rho_v(\xi) \, e^{i\,t\,\xi} = e^{-|t|} \tag{4.85}$$

of the Cauchy distribution is not differentiable at $t = 0$!)

The divergence of the moment integrals means especially that the functions 4.67 do not belong to the domain of definition of the multiplication operator ξ because $\xi \, v_n(\xi)$ is not square integrable ! On the other hand, they do belong to the domain of the operator $(1/i)d/d\xi$, because

$$\frac{1}{i}\frac{dv_n}{d\xi} = \frac{(1+i\xi+2n)(1+i\xi)^{n-1}}{\sqrt{\pi}\,(1-i\xi)^{n+2}} \, . \tag{4.86}$$

If we introduce the variable x of Eq. 4.66 - which has the dimension of a length, whereas ξ is dimensionless - and the measure dx, then the density 4.83 takes the form

$$\rho_v(x;\lambda) = \frac{\beta}{\pi(1+\beta^2 x^2)} = \frac{\lambda}{\pi(\lambda^2+x^2)} \, , \; \lambda = 1/\beta \, . \tag{4.87}$$

It has its maximum for $x = 0$, with the value

$$\rho_v(x=0;\lambda) = \frac{\beta}{\pi} = \frac{1}{\pi\,\lambda} \, . \tag{4.88}$$

If we denote by $x_{(1/2)}$ the arguments where $\rho_v(x;\lambda)$ takes half its maximum value 4.88, then

$$x_{\pm(1/2)} = \pm\frac{1}{\beta} = \pm\lambda \, . \tag{4.89}$$

The points $x_{(\pm i)}$ of inflexion of the function 4.87 are

$$x_{(\pm i)} = \pm\frac{\lambda}{\sqrt{3}} \, . \tag{4.90}$$

In the (classical!) limits $\omega \to \infty$ or $\hbar \to 0$ we have $\beta \to \infty$, $\lambda \to 0$, where the density 4.87 approaches the delta-function (see, e.g. Ref. [201], vol. I):

$$\lim_{\lambda \to 0} \rho_v(x;\lambda) = \delta(x) \, . \tag{4.91}$$

Shifting the center of the distribution density from $x = 0$ to $x = a$ yields the density (I drop the index "v" in the following)

$$\rho(x;\lambda,a) = \frac{1}{\pi}\frac{\lambda}{\lambda^2 + (x-a)^2}\,, \tag{4.92}$$

which has the characteristic function

$$\phi(t) = e^{i\,a\,t - \lambda\,|t|}\,. \tag{4.93}$$

The characteristic function 4.93 has an important property which relates the Cauchy density 4.92 to an associated Markovian Cauchy process [136] with respect to the parameter λ:

If $\rho_i = \rho(x_i;\lambda_i,a_i)\,,\ i = 1,2$, are the Cauchy densities of two independent random variables x_i with corresponding characteristic functions $\phi_i(t)$, then the product of those functions is given by

$$\phi_3(t) = \phi_2(t)\,\phi_1(t) = e^{i(a_2+a_1) - (\lambda_2+\lambda_1)|t|}\,, \tag{4.94}$$

i.e. ϕ_3 is again the characteristic function of a Cauchy distribution which is a convolution of the two original ones:

$$\rho(x_3;\lambda_3 = \lambda_1+\lambda_2\,,\ a_3 = a_1+a_2) = \int_{-\infty}^{\infty} dx\,\rho(x_3-x;\lambda_2,a_2)\,\rho(x;\lambda_1,a_1)\,. \tag{4.95}$$

This important "Chapman-Kolmogorov" property here is a consequence of the fact that the inverse Fourier transform of the product 4.94 yields the convolution 4.95. Important is the resulting semi-group property in λ. The a_i may be put to zero.

As the moments of the Cauchy distribution do not exist one need some other means in order to characterize salient features of the distribution. One possibility is to use so-called *"quantils of order p"* [137]:

The Cauchy distribution function $F(x)$ of the density ρ is given by

$$F(x) = \int_{-\infty}^{x} du\,\rho(u;\lambda,a) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x-a}{\lambda}\right)\,. \tag{4.96}$$

The pth quantil is defined as the value $x = x_p$ for which

$$F(x_p) = p\,,\ 0 < p < 1\,. \tag{4.97}$$

$x_{1/2}$ is called the "*median*". Here we have

$$x_{1/2} = a\,. \tag{4.98}$$

For the lower and upper "*quartils*" $x_{1/4}$ and $x_{3/4}$ we get

$$x_{1/4} = a-\lambda\,,\ x_{3/4} = a+\lambda\,,\ \lambda = \frac{1}{2}(x_{3/4}-x_{1/2})\,,\ a = \frac{1}{2}(x_{1/4}+x_{3/4}) = x_{1/2}\,. \tag{4.99}$$

Thus, $(x_{3/4}-x_{1/4})/2$ is a measure for the width of the distribution.

Whereas the conventional eigenfunctions 4.67 are eigenfunctions of the parity operation $\xi \to -\xi$, the eigenfunctions 4.74 are not, but they obey the relation

$$\bar{v}_n(-\xi) = v_n(\xi)\,. \tag{4.100}$$

This corresponds to the property

$$\overline{\tilde{\tilde{H}}}_{osc}(-\xi) = \tilde{H}_{osc}(\xi)\,. \tag{4.101}$$

The self-adjoint position and momentum operators 2.39 and 2.41 here take the form

$$\tilde{Q} = \frac{1}{\sqrt{2}}[\tilde{K}_+(\tilde{K}_0+1/2)^{-1/2} + (\tilde{K}_0+1/2)^{-1/2}\tilde{K}_-]\,, \tag{4.102}$$

$$\tilde{P} = \frac{i}{\sqrt{2}}[\tilde{K}_+(\tilde{K}_0+1/2)^{-1/2} - (\tilde{K}_0+1/2)^{-1/2}\tilde{K}_-]\,. \tag{4.103}$$

These operators have the same matrix elements with respect to the basis 4.74 as the operators $Q = x$ and $P = (1/i)d/dx$ have with respect to the basis 4.67!

4.4.2 The eigenfuctions of $\tilde{K}_1$ and $\tilde{K}_2$

I briefly discuss the eigenfunctions $\tilde{f}_{h_1}(\xi)$ and $\tilde{f}_{h_2}(\xi)$ of the operators 4.81 and 4.82. The procedure is similar as in Sec. 4.3: For $\xi^2 > 1$ we have

$$\tilde{f}_{h_1}(\xi) = C_+ \left(\frac{\xi+1}{\xi-1}\right)^{i\,h_1} (\xi^2-1)^{-1/2} \text{ for } \xi^2 > 1\,, h_1 \in \mathbb{R},\, C_+ = \text{const.}\,, \tag{4.104}$$

and for $\xi^2 < 1$ we obtain analogously

$$\tilde{f}_{h_1}(\xi) = C_- \left(\frac{1+\xi}{1-\xi}\right)^{i\,h_1} (1-\xi^2)^{-1/2} \text{ for } \xi^2 < 1\,, h_1 \in \mathbb{R}\,. \tag{4.105}$$

Setting $\xi = 1+\epsilon\,,\ \epsilon > 0$ for the relation 4.104 and $\xi = 1-\epsilon$ for 4.105 near $\xi = 1$ and requiring continuity of the eigenfunctions yields $C_- = C_+ = C$.

In order to determine the constants C it is useful to introduce a new variable:

$$e^u = \frac{\xi+1}{\xi-1}\,,\ \xi^2 > 1\,,\ du = -2(\xi^2-1)^{-1}d\xi\,, \tag{4.106}$$

where $u \to -\infty$ for $\xi \to -1^-$, $u \to 0^-$ for $\xi \to -\infty$; $u \to \infty$ for $\xi \to 1^+$, $u \to 0^+$ for $\xi \to \infty$, so that

$$\int_{-\infty}^{-1} d\xi\, \bar{\tilde{f}}_{h_1'}(\xi)\, \tilde{f}_{h_1}(\xi) = \frac{|C|^2}{2} \int_{-\infty}^{0} du\, e^{i(h_1-h_1')u} \tag{4.107}$$

and

$$\int_{1}^{\infty} d\xi\, \bar{\tilde{f}}_{h_1'}(\xi) \tilde{f}_{h_1}(\xi) = \frac{|C|^2}{2} \int_{0}^{\infty} du\, e^{i(h_1-h_1')u}\,. \tag{4.108}$$

Putting

$$e^u = \frac{1+\xi}{1-\xi}\,,\ du = 2(1-\xi^2)d\xi \tag{4.109}$$

for $\xi^2 < 1$ gives correspondingly

$$\int_{-1}^{1} d\xi\, \bar{\tilde{f}}_{h_1'}(\xi) \tilde{f}_{h_1}(\xi) = \frac{|C|^2}{2} \int_{-\infty}^{\infty} du\, e^{i(h_1-h_1')u}\,. \tag{4.110}$$

Adding the three contributions yields

$$\int_{-\infty}^{\infty} d\xi\, \bar{\tilde{f}}_{h_1'}(\xi) \tilde{f}_{h_1}(\xi) = 2\pi\, |C|^2\, \delta(h_1'-h_1)\,, \tag{4.111}$$

so that

$$C = \frac{1}{\sqrt{2\pi}} \tag{4.112}$$

gives the appropriate normalization:

$$\tilde{f}_{h_1}(\xi) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \left(\dfrac{\xi+1}{\xi-1}\right)^{ih_1} (\xi^2-1)^{-1/2} \text{ for } \xi^2 > 1\,, \\ \dfrac{1}{\sqrt{2\pi}} \left(\dfrac{1+\xi}{1-\xi}\right)^{ih_1} (1-\xi^2)^{-1/2} \text{ for } \xi^2 < 1\,. \end{cases} \tag{4.113}$$

In a similar manner we get

$$\tilde{f}_{h_2}(\xi) = \begin{cases} \dfrac{1}{2\sqrt{\pi}}\, \xi^{ih_2-1/2} \text{ for } \xi > 0\,, \\ \dfrac{1}{2\sqrt{\pi}}\, (-\xi)^{ih_2-1/2} \text{ for } \xi < 0\,, \end{cases} \tag{4.114}$$

with the normalization

$$(\tilde{f}_{h_2'}, \tilde{f}_{h_2}) = \delta(h_2' - h_2)\,. \tag{4.115}$$

4.4.3 The Fourier transform of $g(\xi) \in H^2(\mathbb{R},\, d\xi)$

The Fourier transforms[2]

$$\hat{g}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\xi\, g(\xi)\, e^{-ip\,\xi}\,,\ g(\xi) \in H^2(\mathbb{R},\, d\xi) \tag{4.116}$$

have the important property (Paley-Wiener [138]) that

$$\hat{g}(p) = 0 \text{ for } p < 0\,, \tag{4.117}$$

and we have the inversion

$$g(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} dp\, \hat{g}(p)\, e^{i\,\xi p}\,. \tag{4.118}$$

Furthermore

$$\int_0^{\infty} dp\, |\hat{g}(p)|^2 = \int_{-\infty}^{\infty} d\xi\, |g(\xi)|^2\,. \tag{4.119}$$

The Fourier transforms of the basis functions 4.74 are [139]

$$\hat{v}_n(p) = \sqrt{2}(-1)^n\, e^{-p} L_n(2p)\,,\ n = 0, 1, \ldots\,, \tag{4.120}$$

where the L_n are Laguerre's polynomials [140]. The $\hat{v}_n(p)$ are eigenfunctions of the operator

$$\hat{K}_0 = \frac{1}{2}\left(p - \frac{d}{dp} - p\frac{d^2}{dp^2}\right)\,, \tag{4.121}$$

$$\hat{K}_0\, \hat{v}_n(p) = (n + 1/2)\, \hat{v}_n(p)\,. \tag{4.122}$$

The other generators have the form

$$\hat{K}_1 = \frac{1}{2}\left(p + \frac{d}{dp} + p\frac{d^2}{dp^2}\right)\,, \tag{4.123}$$

$$\hat{K}_2 = i\left(\frac{1}{2} + p\,\frac{d}{dp}\right)\,. \tag{4.124}$$

[2]It is convenient here to have a different sign convention in the exponent compared to the one used in quantum mechanics.

Using the properties of L_n [140] one verifies that

$$\hat{K}_+ \hat{v}_n(p) = (n+1)\,\hat{v}_{n+1}(p)\,, \tag{4.125}$$
$$\hat{K}_- \hat{v}_n(p) = n\,\hat{v}_{n-1}(p)\,. \tag{4.126}$$

The eigenvalue equation

$$\hat{K}_1 \hat{f}_{h_1}(p) = h_1\,\hat{f}_{h_1}(p) \tag{4.127}$$

has the regular solution [82]

$$\hat{f}_{h_1}(p) = C\,e^{-ip}\,{}_1F_1(1/2 - ih_1, 1; 2ip)\,,\ h_1 \in \mathbb{R}, \tag{4.128}$$

where ${}_1F_1(a,c;z)$ is the usual regular confluent hypergeometric ("Kummer's") function [141]. The normalization constant C can be obtained by comparing the relation [142]

$$\lim_{\epsilon\to 0} \frac{1}{\sqrt{2\pi}} \int_0^\infty dp\, e^{-ip}\,{}_1F_1(1/2 - ih_1, 1; 2ip)\, e^{ip\,(\xi + i\epsilon)} = \frac{i}{\sqrt{2\pi}} \left(\frac{\xi+1}{\xi-1}\right)^{ih_1} (\xi^2-1)^{-1/2} \ \text{ for } \xi > 1 \tag{4.129}$$

whith Eq. 4.113. The result is

$$\hat{f}_{h_1}(p) = -i\,e^{-ip}\,{}_1F_1(1/2 - ih_1, 1; 2ip)\,,\ h_1 \in \mathbb{R}\,. \tag{4.130}$$

For the eigenfunctions of $\hat{K}_2$ we get (compare Eq. 4.114)

$$\hat{f}_{h_2}(p) = \frac{1}{\sqrt{2\pi}}\, p^{-ih_2 - 1/2}\,,\ h_2 \in \mathbb{R}\,. \tag{4.131}$$

4.4.4 Relationships between $L^2(\mathbb{R}, d\xi)$ and $H^2(\mathbb{R}, d\xi)$

I briefly mention two essential relationships between the spaces $L^2(\mathbb{R}, d\xi)$ and $H^2(\mathbb{R}, d\xi)$ which are of interest here: $H^2(\mathbb{R}, d\xi)$ as a closed subspace of $L^2(\mathbb{R}, d\xi)$ [127] and the unitary correspondence between the two.

$H^2(\mathbb{R}, d\xi)$ as a subspace of $L^2(\mathbb{R}, d\xi)$

Let me start with $H^2(\mathbb{R}, d\xi)$ as a subspace: it consists of those elements $g^{(+)}(\xi) \in L^2(\mathbb{R}, d\xi)$ which are limits $\lim_{\eta\to 0} g(z = \xi + i\,\eta)$ of functions $g(z)$ which are holomorphic in the upper complex half-plane ($\eta > 0$).

If $g(\xi) \in L^2(\mathbb{R}, d\xi)$ its projection $g^{(+)}(\xi)$ onto the subspace $H^2(\mathbb{R}, d\xi)$ is obtained by (double) Fourier transformation [143], namely

$$\hat{g}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\xi\, g(\xi)\, e^{-ip\,\xi}\,, \tag{4.132}$$
$$g^{(+)}(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} dp\, \hat{g}(p)\, e^{i\,\xi\,p}\,. \tag{4.133}$$

We know already that $\hat{g}(p) = 0$ for $p \le 0$ if $g(\xi) \in H^2(\mathbb{R}, d\xi)$ (Paley–Wiener theorem). So Eq. 4.133 is the appropriate projection. The complementary subspace

$$\bar{H}^2(\mathbb{R}, d\xi) = L^2(\mathbb{R}, d\xi) \setminus H^2(\mathbb{R}, d\xi) \tag{4.134}$$

consists of the functions

$$g^{(-)}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} dp\, \hat{g}(p)\, e^{i\,\xi\,p} \tag{4.135}$$

They are limits of functions $g(z)$ which are holomorphic in the *lower* half-plane.

A basis of $\bar{H}^2(\mathbb{R}, d\xi)$ is given by $\{\bar{v}_n(\xi)\}$.

Obviously we have

$$g(\xi) = g^{(+)}(\xi) + g^{(-)}(\xi)\,. \tag{4.136}$$

For the basis functions 4.67 we have the Fourier transforms [144]

$$\hat{u}_n(p) = \frac{(-i)^n}{(2^n\, n!\, \sqrt{\pi})^{1/2}}\, e^{-p^2/2}\, H_n(p)\,, \tag{4.137}$$

so that

$$u_n^{(+)}(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dp\, \hat{u}_n(p)\, e^{i\,\xi\, p}\,. \tag{4.138}$$

The integrals 4.138 may be evaluated with the help of the generating function [140]

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, H_n(p) = e^{-t^2 + 2\,t\,p}\,, \tag{4.139}$$

which yields

$$F(t;\xi) = \sum_{n=0}^{\infty} \frac{t^n}{\sqrt{n!}} u_n^{(+)}(\xi) = \frac{e^{t^2/2}}{2^{1/2}\, \pi^{3/4}} \int_0^\infty dp\, e^{-p^2/2 - i(\sqrt{2}t - \xi)\, p}\,. \tag{4.140}$$

Evaluation of integral in the last Eq. follows from [145]

$$\int_0^\infty dp\, e^{-p^2/2 - z\,p} = \sqrt{\frac{\pi}{2}}\, e^{z^2/2}\, [1 - \mathrm{erf}(z/\sqrt{2})] \equiv \sqrt{\frac{\pi}{2}}\, e^{z^2/2}\, \mathrm{erfc}(z/\sqrt{2})\,, \tag{4.141}$$

where $\mathrm{erf}(w)$ is the "error function" [146]

$$\mathrm{erf}(w) = \frac{2}{\sqrt{\pi}} \int_0^w du\, e^{-u^2} \tag{4.142}$$

and

$$\mathrm{erfc}(w) = 1 - \mathrm{erf}(w) = \frac{2}{\sqrt{\pi}} \int_w^\infty du\, e^{-u^2} \tag{4.143}$$

the "complementary error function".

For imaginary arguments $w = i\,v$, $v \in \mathbb{R}$, $\mathrm{erf}(i\,v)$ is purely imaginary and generally $\mathrm{erf}(-w) = -\mathrm{erf}(w)$.

For 4.140 we obtain

$$F(t;\xi) = \frac{1}{2\pi^{1/4}}\, e^{-(t^2/2 - \sqrt{2}\,t\,\xi + \xi^2/2)}\, \mathrm{erfc}[i(t - \xi/\sqrt{2})]\,. \tag{4.144}$$

The functions

$$u_n^{(+)}(\xi) = \frac{1}{\sqrt{n!}}\, \left.\frac{d^n F(t;\xi)}{dt^n}\right|_{t\,=\,0} \tag{4.145}$$

may be calculated with the help of the relation [146]

$$\frac{d^{n+1}\mathrm{erf}(b\,t + a)}{dt^{n+1}} = -\frac{2}{\sqrt{\pi}}\, (-b)^{n+1} e^{-(b\,t+a)^2}\, H_n(b\,t + a)\,,\ n = 0, 1, \ldots\,, \tag{4.146}$$

where here $b = i$ and $a = -i\,\xi/\sqrt{2}$. Thus

$$\left.\frac{d^{n+1}\mathrm{erf}[i(t - \xi/\sqrt{2})]}{d\,t^{n+1}}\right|_{t\,=\,0} = \frac{2}{\sqrt{\pi}}\, i^{n+1} e^{\xi^2/2} H_n(i\xi/\sqrt{2})\,. \tag{4.147}$$

As the $H_n(i\xi/\sqrt{2})$ are real for even n and imaginary for odd n the r.h. side of Eq. 4.147 is always purely imaginary.

Examples:

$$u_0^{(+)}(\xi) = \frac{1}{2\,\pi^{1/4}}\,e^{-\xi^2/2}\,[1+\mathrm{erf}(i\,\xi/\sqrt{2})]\,, \tag{4.148}$$

$$u_1^{(+)}(\xi) = \frac{1}{2\,(2\sqrt{\pi})^{1/2}}\,e^{-\xi^2/2}\,H_1(\xi)\,[1+\mathrm{erf}(i\,\xi/\sqrt{2})] - \pi^{-3/4}\,i\,. \tag{4.149}$$

The *real* parts of these functions are just one half of the original $u_0(\xi)$ and $u_1(\xi)$. The other half comes from $u_0^{(-)}(\xi)$ and $u_1^{(-)}(\xi)$ (see Eq. 4.136).

Other interesting quantities are the "transition" amplitudes

$$c_{m,n} = (v_m, u_n^{(+)}) = (\hat{v}_m, \hat{u}_n) \tag{4.150}$$

$$= (-1)^m\,(-i)^n\,\frac{\sqrt{2}}{\pi^{1/4}\,(2^n\,n!)^{1/2}}\int_0^\infty dp\,e^{-p^2/2-p}\,L_m(2p)\,H_n(p)\,. \tag{4.151}$$

They can be calculated with the help of the generating function 4.139 and [140]

$$\sum_{m=0}^{\infty} s^m\,L_m(u) = \frac{e^{u\,s/(s-1)}}{1-s}\,, |s|<1\,, \tag{4.152}$$

which yield

$$c_{m,n} = \frac{1}{m!\,\sqrt{n!}}\,\left.\frac{d^{m+n}G(s,t)}{ds^m\,dt^n}\right|_{s\,=\,0,\;t\,=\,0}\,, \tag{4.153}$$

$$G(s,t) = \sum_{m=0,n=0}^{\infty} c_{m,n}\,s^m\,\frac{t^n}{\sqrt{n!}} \tag{4.154}$$

$$= \frac{\pi^{1/4}}{s+1}\exp\left[\frac{1}{2}\left(\frac{1-s}{1+s}\right)^2 + \sqrt{2}\,\frac{1-s}{1+s}\,t\,i - \frac{1}{2}\,t^2\right]\mathrm{erfc}\left(\frac{1}{\sqrt{2}}\,\frac{1-s}{1+s} + t\,i\right), \tag{4.155}$$

where again the relation 4.141 has been used.

Another possibility in order to evaluate the integrals 4.150 is the use of the relation

$$\int_0^\infty dp\,p^n\,e^{p^2/2-p} = \frac{(-1)^n}{n!}\sqrt{\frac{\pi}{2}}\,\frac{d^n}{dz^n}\,\left[e^{z^2/2}\,\mathrm{erfc}(\frac{z}{\sqrt{2}})\right]\Big|_{z=1}\,, \tag{4.156}$$

which follows from 4.141.

Examples:

$$c_{0,0} = \pi^{1/4}\,\sqrt{e}\,\mathrm{erfc}(\frac{1}{\sqrt{2}}) = 0.6965\,, \tag{4.157}$$

$$|c_{0,0}|^2 = 0.4851\,; \tag{4.158}$$

$$c_{1,0} = \pi^{1/4}\,\left[2\,\sqrt{\frac{2}{\pi}} - 3\,\sqrt{e}\,\mathrm{erfc}(\frac{1}{\sqrt{2}})\right] = 0.0351\,, \tag{4.159}$$

$$|c_{1,0}|^2 = 0.0012\,; \tag{4.160}$$

$$c_{0,1} = \pi^{1/4}\,\left[\sqrt{2\,e}\,\mathrm{erfc}(\frac{1}{\sqrt{2}}) - \frac{2}{\sqrt{\pi}}\right]\,i = -0.5173\,i\,, \tag{4.161}$$

$$|c_{0,1}|^2 = 0.2676\,; \tag{4.162}$$

$$c_{0,2} = \pi^{-1/4}\,\left[1 - \sqrt{\frac{e\,\pi}{2}}\,\mathrm{erfc}(\frac{1}{\sqrt{2}})\right] = 0.2586\,, \tag{4.163}$$

$$|c_{0,2}|^2 = 0.0669\,. \tag{4.164}$$

Here the numerical value [147]

$$\mathrm{erfc}(\frac{1}{\sqrt{2}}) = 0.31731\ldots \tag{4.165}$$

has been used.

The relationship between the Laguerre polynomials $L_m(p)$ and the Hermite polynomials $H_n(p)$ for $p \geq 0$ may briefly sketched as follows:

The Laguerre polynomials form an orthonormal basis of the Hilbert space $L^2_{(+)}(p \geq 0, dp)$ with weight function $\exp(-p)$ [140]:

$$\int_0^\infty dp\, e^{-p}\, L_m(p)\, L_n(p) = \delta_{m\,n}\,. \tag{4.166}$$

The situation with respect to the Hermite functions 4.67 is somewhat more complicated:

From $H_n(-p) = (-1)^n\, H_n(p)$ it follows that

$$\int_{-\infty}^\infty dp\, e^{-p^2} H_m(p)\, H_n(p) = [1 + (-1)^{m+n}] \int_0^\infty dp\, e^{-p^2} H_m(p)\, H_n(p) = 2^m\, m!\, \sqrt{\pi}\, \delta_{m\,n}\,, \tag{4.167}$$

which shows that the functions $H_m(p)\, \exp\,(-p^2/2)$ form an orthogonal set on $L^2_{(+)}(p \geq 0, dp)$ for either m and n both even or m and n both odd. Each of the sets $\{H_{2n}(p)\,,\, n = 0, 1, \ldots\}$ and $\{H_{2n+1}(p)\,,\, n = 0, 1, \ldots\}$ provide a basis in the following sense: They are related to the special generalized Laguerre polynomials [140]

$$H_{2n}(\sqrt{u}) = (-1)^n\, 2^{2n}\, n!\, L_n^{-1/2}(u)\,, \tag{4.168}$$

$$H_{2n+1}(\sqrt{u}) = (-1)^n\, 2^{2n+1}\, n!\, \sqrt{u}\, L_n^{1/2}(u)\,, \tag{4.169}$$

where

$$L_n^\alpha(u) = \binom{n+\alpha}{n}\, {}_1F_1(-n,\, \alpha+1; u)\,,\, \alpha > -1\,, \tag{4.170}$$

and

$$\int_0^\infty du\, e^{-u}\, u^\alpha\, L_m^\alpha(u) L_n^\alpha(u) = \Gamma(\alpha+1)\, \binom{n+\alpha}{n}\, \delta_{m\,n}\,. \tag{4.171}$$

For fixed $\alpha > -1$ the sets $\{L_n^\alpha(u)\,,\, n = 0, 1, \ldots\}$ together with the weight function $u^\alpha\, \exp\,(-u)$ form an orthonormal basis for the Hilbert space $L^2_{(+)}(u \geq 0, du)$. With the help of the relations 4.168 and 4.169 one finds [148]

$$\int_0^\infty dp\, e^{-p^2}\, H_{2m}(p)\, H_{2n+1}(p) = 2^{2(m+n)}\, (m - n + 1/2)_n\, (n - m + 3/2)_m\,, \tag{4.172}$$

which means that the functions $H_{2m}(p)\, \exp\,(-p^2/2)$ and $H_{2n+1}(p)\, \exp\,(-p^2/2)$ are no longer orthogonal on $L^2_{(+)}(p \geq 0, dp)$!

The unitary equivalence of $L^2(\mathbb{R}, d\xi)$ and $H^2(\mathbb{R}, d\xi)$

We have seen that the operators 4.102 and 4.103 have the same matrix elements with respect to the basis 4.74 as the usual operators ξ and $-i\, d/d\xi$ do have with respect to the basis 4.67. Thus, in view of the Stone–von Neumann theorem (see Subsec. 1.3.1) the question of their unitary equivalence arises[3]. The answer is surprisingly simple:

[3] I thank K. Fredenhagen for clarifying remarks concerning this problem.

The genuine subspace $H^2(\mathbb{R}, d\xi) \subset L^2(\mathbb{R}, d\xi)$ can be mapped unitarily onto $L^2(\mathbb{R}, d\xi)$ by the prescription

$$v_n(\xi) \Longleftrightarrow u_n(\xi)\,. \tag{4.173}$$

Then any two functions

$$g_j^{(+)}(\xi) = \sum_{n=0}^{\infty} c_n^{(j)}\, v_n(\xi) \in H^2(\mathbb{R}, d\xi)\,,\ j = 1,\ 2, \tag{4.174}$$

are mapped unitarily onto

$$g_j(\xi) = \sum_{n=0}^{\infty} c_n^{(j)}\, u_n(\xi) \in L^2(\mathbb{R}, d\xi)\,,\ j = 1,\ 2, \tag{4.175}$$

and vice versa. The mapping is unitary because

$$(g_2^{(+)}, g_1^{(+)}) = \sum_{n=0}^{\infty} \bar{c}_n^{(2)}\, c_n^{(1)} = (g_2, g_1)\,. \tag{4.176}$$

This implies the unitary correspondences

$$\tilde{Q} \Longleftrightarrow \xi\,,\quad \tilde{P} \Longleftrightarrow \frac{1}{i}\frac{d}{d\xi}\,. \tag{4.177}$$

This is an explicit example of the Stone–von Neumann theorem [37].

The above arguments may actually be generalized to representations with $k \neq 1/2$: In that case the correspondence 4.173 is replaced by

$$|k, n\rangle \Longleftrightarrow u_n(\xi) \tag{4.178}$$

and the operators $\tilde{Q}$ and $\tilde{P}$ of Eq. 4.177 by the operators 2.39 - 2.42.

The unitary equivalence of the position and momentum operators does, however, not imply that the associated generators K_j for different k are unitarily equivalent! They are not, of course, because they belong to different inequivalent irreducible unitary representations of the group $SO^{\uparrow}(1,2)$ or one of its covering groups!

4.5 A few generalizations for $k \neq 1/2$

Several of the above results related to the Hardy space H^2 of the unit circle may be generalized to unitary representations with $k \neq 1/2$ [40,150,275]:

The idea again is to implement the scalar product 4.5 first in terms of a series expansion and then realize that series expansion by means of H^2.

One starts by defining the self-adjoint operator A_k on H^2 which is diagonal in the basis 4.10 of H^2 and which acts on it as

$$A_k\, e_n(\varphi) = \frac{n!}{(2k)_n}\, e_n(\varphi)\,. \tag{4.179}$$

Then one can define a H^2 related Hilbert space $H^2_{A_k}$ with the scalar product

$$(f_1, f_2)_k \equiv (f_1, A_k\, f_2) = \sum_{n=0}^{\infty} \frac{n!}{(2k)_n}\, \bar{a}_n\, b_n \tag{4.180}$$

for the functions 4.12. The series 4.180 representing the scalar product of $H^2_{A_k}$ is obviously the same as 4.8 which represents the scalar product for $\mathcal{H}_{\mathbb{D},k}$. This exhibits the very close relationship between the two Hilbert spaces. More explicitly this means:

An orthonormal basis for $H^2_{A_k}$ is given by

$$\chi_{k,n}(\varphi) = \sqrt{\frac{(2k)_n}{n!}}\, e_n(\varphi),\ n = 0, 1, \ldots, \tag{4.181}$$

$$(\chi_{k,n_1}, \chi_{k,n_2})_k = \delta_{n_1 n_2}\ . \tag{4.182}$$

The operators K_0, K_+, K_- now have the form [149]

$$K_0 = \frac{1}{i}\,\partial_\varphi + k\ , \tag{4.183}$$

$$K_+ = e^{i\,\varphi}(\frac{1}{i}\partial_\varphi + 2\,k)\ , \tag{4.184}$$

$$K_- = e^{-i\,\varphi}\frac{1}{i}\,\partial_\varphi\ . \tag{4.185}$$

Their action on the basis functions 4.181 is given by

$$K_0\chi_{k,n} = (k+n)\chi_{k,n}\ , \tag{4.186}$$

$$K_+\chi_{k,n} = [(2k+n)(n+1)]^{1/2}\chi_{k,n+1}\ , \tag{4.187}$$

$$K_-\chi_{k,n} = [(2k+n-1)n)]^{1/2}\chi_{k,n-1}\ . \tag{4.188}$$

It is important to realize that the operators K_0, K_+, K_- belong to a representation which is unitary only with respect to the scalar product 4.180, not with respect to the scalar product 4.13! This may be seen explicitly as follows: Applying the operators K_+ and K_- to the series

$$f_1(\varphi) = \sum_{m=0}^{\infty} a_m\,\chi_{k,m}(\varphi)\ ,\quad f_2(\varphi) = \sum_{n=0}^{\infty} b_n\,\chi_{k,n}(\varphi)\ , \tag{4.189}$$

using the relations 4.187 and 4.188 and the orthonormality 4.182 yields

$$(f_2, K_+ f_1)_k = \sum_{n=0}^{\infty}[(2k+n)(n+1)]^{1/2}\,\bar{b}_{n+1}\,a_n = (K_- f_2, f_1)_k\ , \tag{4.190}$$

which says that K_- is the adjoint operator of K_+ with respect to the scalar product 4.180. But one sees immediately that this is not so with respect to the scalar product 4.13!

The reproducing kernel here is

$$\Delta_k(\varphi_2, \varphi_1) = \sum_{n=0}^{\infty}\bar{\chi}_{k,n}(\varphi_2)\,\chi_{k,n}(\varphi_1) = (1 - e^{i\,(\varphi_1 - \varphi_2)})^{-2k}\ . \tag{4.191}$$

Because

$$\lim_{n\to\infty}\left(\frac{(2k)_n}{n!}\right)^{1/n} = 1\,, \tag{4.192}$$

the radius of convergence of the series 4.191 is 1 and the kernel becomes singular for $\varphi_2 = \varphi_1$. In the calculations one has to replace the basic functions $e_n(\varphi)$ here too by $(1-\epsilon)\,e_n(\varphi)$ and take the limit $\epsilon \to 0$ at the end.

For the coherent states 3.8, 3.125 and 3.219 one gets here from the basis 4.181 the functions

$$f_{k,z}(\varphi) = \frac{1}{\sqrt{g_k(|z|^2)}}\, e^{z\, e^{i\,\varphi}} \ , \tag{4.193}$$

$$f_{k,\lambda}(\varphi) = (1-|\lambda|^2)^k\, (1-\lambda\, e^{i\,\varphi})^{-2k} \ , \tag{4.194}$$

$$f_{k,\alpha}(\varphi) = \sum_{n=0}^{\infty} \sqrt{(2k)_n}\frac{(\alpha\, e^{i\,\varphi})^n}{n!} \ . \tag{4.195}$$

From the expressions 4.184 and 4.185 we obtain the operators

$$K_1 = \frac{1}{2}(K_+ - K_-) = \cos\varphi \frac{1}{i}\partial_\varphi + k\, e^{i\,\varphi} \ , \tag{4.196}$$

$$K_2 = \frac{1}{2i}(K_+ - K_-) = \sin\varphi \frac{1}{i}\partial_\varphi + \frac{k}{i}\, e^{i\,\varphi} \ . \tag{4.197}$$

They have the eigenfunctions

$$f_{k,h_1}(\varphi) = C_1\, |\cos\varphi|^{-k} |\tan(\varphi/2+\pi/4)|^{i\,h_1}\, e^{-i\,k\,\varphi} \ , \tag{4.198}$$

$$h_1 \in \mathbb{R} \ , \ \varphi \in (0, 2\pi) \ , \tag{}$$

$$f_{k,h_2}(\varphi) = C_2\, |\sin\varphi|^{-k} |\tan(\varphi/2)|^{i\,h_2}\, e^{-i\,k\,\varphi} \ , \tag{4.199}$$

$$h_2 \in \mathbb{R} \ , \ \varphi \in (0, 2\pi) \ . \tag{4.200}$$

Chapter 5

Operators for $\cos\varphi$ and $\sin\varphi$?

We have seen in Ch. 3 that the operators K_1 and K_2 by themselves are well suited in order to determine the phase content of a state. One may nevertheless ask whether there are "reasonable" operators for $\cos\varphi$ and $\sin\varphi$ in the present framework and how the London-Susskind-Glogower operators 1.34 and 1.35 fit in.

In view of the relations 2.32 the following operators in an irreducible unitary representation with Bargmann index k are a generalization of the ones in Eq. 1.32:

$$E_{k,-} = (K_0 + k)^{-1} K_- \ , \tag{5.1}$$

$$E_{k,+} = K_+ (K_0 + k)^{-1} = (E_{k,-})^+ \ . \tag{5.2}$$

For $k = 1/2$ we get back the the operators 1.32.

Let us look at some properties of the operators 5.1 and 5.2:

Using the relations 2.35 and 2.37 with $L = k(1-k)$ we get

$$E_{k,-} \cdot E_{k,+} = 1 - \frac{2k-1}{K_0 + k} \ , \tag{5.3}$$

$$E_{k,+} \cdot E_{k,-} = 1 - \frac{2k-1}{K_0 + k - 1} \ , \tag{5.4}$$

$$\langle k,0 | E_{k,+} \cdot E_{k,-} | k,0 \rangle = 0 \ \forall \ k \ . \tag{5.5}$$

We see that the operators $E_{k,-}$ and $E_{k,+}$ are *not* isometric for $k \neq 1/2$! The case $k = 1/2$ is a very special one and not generic!

The relation 5.5 follows from 5.4 even for $k = 1/2$ if one takes the limit $k \to 1/2$ after forming the expectation value of 5.4 with respect to $|k,0\rangle$.

The operators 5.1 and 5.2 obey the commutation relation

$$[E_{k,-}, E_{k,+}] = \frac{2k-1}{(K_0 + k)(K_0 + k - 1)} \ , \tag{5.6}$$

$$\langle k,0 | [E_{k,-}, E_{k,+}] | k,0 \rangle = \frac{1}{2k} \ . \tag{5.7}$$

In the case $k = 1/2$ the expectation values of the commutator 5.6 vanish for all states, except for the ground state: 5.7. This is well-known and follows immediately from the relations 1.33, but the situation is obviously different for $k \neq 1/2$!

The obvious generalizations of the $\cos\varphi$ - and $\sin\varphi$ - operators 1.34 and 1.35 are

$$\tilde{C}_k = \frac{1}{2}(E_{k,+} + E_{k,-}) \ , \tag{5.8}$$

$$\tilde{S}_k = \frac{1}{2i}(E_{k,-} - E_{k,+}) \ . \tag{5.9}$$

These have the properties

$$[K_0, \tilde{C}_k] = -i\,\tilde{S}_k \ , \ [K_0, \tilde{S}_k] = i\,\tilde{C}_k \ , \tag{5.10}$$

$$[\tilde{C}_k \ , \ \tilde{S}_k] = \frac{(2k-1)\,i}{2}\,\frac{1}{(K_0+k)(K_0+k-1)} \ , \tag{5.11}$$

$$\langle k,0|[\hat{C}_k \ , \ \hat{S}_k]|k,0\rangle = \frac{i}{4k} \ , \tag{5.12}$$

$$\tilde{C}_k^2 + \tilde{S}_k^2 = 1 - \frac{2k-1}{2}\left(\frac{1}{K_0+k-1} + \frac{1}{K_0+k}\right) \ , \tag{5.13}$$

$$\langle k,0|\tilde{C}_k^2 + \tilde{S}_k^2|k,0\rangle = \frac{1}{4k} \ . \tag{5.14}$$

In view of the following properties with respect to the number states and the coherent states 3.8 and 3.125 these operators may appear to be appealing:

$$\langle k,n|\tilde{C}_k|k,n\rangle = 0 \ , \ \langle k,n|\tilde{S}_k|k,n\rangle = 0 \ , \tag{5.15}$$

$$\langle \tilde{C}_k\rangle_{k,z} = \cos\phi\,\rho_k(|z|) \ , \ \langle \tilde{S}_k\rangle_{k,z} = \sin\phi\,\rho_k(|z|) \ , \tag{5.16}$$

$$\langle \tilde{C}_k\rangle_{k,\lambda} = \cos\theta\,|\lambda| \ , \ \langle \tilde{S}_k\rangle_{k,\lambda} = \sin\theta\,|\lambda| \ , \tag{5.17}$$

where the relations

$$\langle k,z|E_{k,-}|k,z\rangle = \frac{z}{|z|}\,\rho_k(|z|) \ , \tag{5.18}$$

$$\langle k,\lambda|E_{k,-}|k,\lambda\rangle = \lambda \tag{5.19}$$

have been used (see Eqs. 3.110 and 1.110). Notice that the coherent states $|k,\lambda\rangle$ are eigenstates of $E_{k,-}$!

However, problems appear for higher powers of the operators $\tilde{C}_k$ and $\tilde{S}_k$, like $\tilde{C}_k^2$ etc.:

If we express $\tilde{C}_k$ and $\tilde{S}_k$ in terms of the observables K_1 and K_2, instead of K_+ and K_-, we get

$$\begin{aligned}\tilde{C}_k &= \frac{1}{2}[(K_0+k)^{-1}K_1 + K_1(K_0+k)^{-1}] \\ &\quad -\frac{i}{2}[(K_0+k)^{-1}K_2 - K_2(K_0+k)^{-1}] \ ,\end{aligned} \tag{5.20}$$

$$\begin{aligned}\tilde{S}_k &= -\frac{1}{2}[(K_0+k)^{-1}K_2 + K_2(K_0+k)^{-1}] \\ &\quad +\frac{1}{2\,i}[(K_0+k)^{-1}K_1 - K_1(K_0+k)^{-1}] \ .\end{aligned} \tag{5.21}$$

Thus, *the* cos*-operator* $\tilde{C}_k$ *contains contributions from the* sin*-observable* K_2 *and the* sin*-operator* $\tilde{S}_k$ *contains contributions from the* cos*-observable* K_1 .

These contributions do not matter for the expectation values 5.15-5.17 which are linear in $\tilde{C}_k$ and $\tilde{S}_k$, but they will in general matter for higher powers of these operators. That can be seen by comparing the expectation values $\langle k,n|\tilde{C}_k^2|k,n\rangle$ or $\langle \tilde{C}_k^2\rangle_{k,z}$ etc. with corresponding expectation values of suitable variants of the operators 5.20 and 5.21:

The actions of the operators $\tilde{C}_k$ and $\tilde{S}_k$ on the number states $|k,n\rangle$ are

$$\tilde{C}_k|k,n\rangle = \frac{1}{2}(\tilde{f}_n^{(k)}|k,n-1\rangle + \tilde{f}_{n+1}^{(k)}|k,n+1\rangle) \ , \tag{5.22}$$

$$\tilde{S}_k|k,n\rangle = \frac{1}{2i}(\tilde{f}_n^{(k)}|k,n-1\rangle - \tilde{f}_{n+1}^{(k)}|k,n+1\rangle) \ , \tag{5.23}$$

$$\tilde{f}_n^{(k)} = \frac{[n\,(2k+n-1)]^{1/2}}{2k+n-1} \ , \ \tilde{f}_{n=0}^{(k)} = 0 \ . \tag{5.24}$$

This gives the following expectation values for the squared operators

$$\langle k,n|\tilde{C}_k^2|k,n\rangle = \langle k,n|\tilde{S}_k^2|k,n\rangle = \frac{1}{4}((\tilde{f}_n^{(k)})^2 + (\tilde{f}_{n+1}^{(k)})^2 \ , \tag{5.25}$$

$$\langle \tilde{C}_k^2\rangle_{k,n=0} = \langle \tilde{S}_k^2\rangle_{k,n=0} = \frac{1}{8\,k} \ . \tag{5.26}$$

In order to compare these fluctuations with those of cos- and sin-operators which are "pure" ones, the expressions 5.20 and 5.21 suggest to define

$$\hat{C}_k = \frac{1}{2}[(K_0+k)^{-1}K_1 + K_1(K_0+k)^{-1}] \tag{5.27}$$

$$= \frac{1}{4}[E_{k,-} + E_{k+1,-} + E_{k,+} + E_{k+1,+}] \ , \tag{5.28}$$

$$\hat{S}_k = -\frac{1}{2}[(K_0+k)^{-1}K_2 + K_2(K_0+k)^{-1}] \tag{5.29}$$

$$= \frac{1}{4i}[E_{k,-} + E_{k+1,-} - E_{k,+} - E_{k+1,+}] \ . \tag{5.30}$$

Because

$$\langle k,z|E_{k+1,-}|k,z\rangle = z\,\langle k,z|(K_0+k+1)^{-1}|k,z\rangle \tag{5.31}$$

$$= \frac{z}{|z|}\,\rho_k(|z|) - \frac{z}{|z|^2} + \frac{2k\,z}{|z|^3}\rho_k(|z|) \ ,$$

and (see the relation 3.207)

$$\langle k,\lambda|E_{k+1,-}|k,\lambda\rangle = \lambda\, s_k^{(b=k+1)}(|\lambda|^2)\,, \tag{5.32}$$

we have now

$$\langle k,n|\hat{C}_k|k,n\rangle = 0 \ , \ \langle k,n|\hat{S}_k|k,n\rangle = 0 \ , \tag{5.33}$$

$$\langle \hat{C}_k\rangle_{k,z} = \cos\phi\,\left(\rho_k(|z|) - \frac{1}{2|z|} + \frac{k}{|z|^2}\rho_k(|z|)\right) \ , \tag{5.34}$$

$$\langle \hat{S}_k\rangle_{k,z} = \sin\phi\,\left(\rho_k(|z|) - \frac{1}{2|z|} + \frac{k}{|z|^2}\rho_k(|z|)\right) \ , \tag{5.35}$$

$$\langle \hat{C}_k\rangle_{k,\lambda} = \cos\theta\,|\lambda|\,[1 + s_k^{(b=k+1)}(|\lambda|^2)]/2\,, \tag{5.36}$$

$$\langle \hat{S}_k\rangle_{k,\lambda} = \sin\theta\,|\lambda|\,[1 + s_k^{(b=k+1)}(|\lambda|^2)]/2\,. \tag{5.37}$$

The expectation values of $\hat{C}_k$ and $\hat{S}_k$ with respect to the conventional coherent states $|k,\alpha\rangle$ may be dealt with in the same way as with the other ones above.

From the relations

$$\hat{C}_k|k,n\rangle = \frac{1}{4}(\hat{f}_n^{(k)}|k,n-1\rangle + \hat{f}_{n+1}^{(k)}|k,n+1\rangle) \ , \tag{5.38}$$

$$\hat{S}_k|k,n\rangle = \frac{1}{4i}(\hat{f}_n^{(k)}|k,n-1\rangle - \hat{f}_{n+1}^{(k)}|k,n+1\rangle) \ , \tag{5.39}$$

$$\hat{f}_n^{(k)} = [n\,(2k+n-1)]^{1/2}\left(\frac{1}{2k+n-1} + \frac{1}{2k+n}\right) \ , \tag{5.40}$$

$$\hat{f}_{n=0}^{(k)} = 0 \ .$$

we get the fluctuations

$$\begin{aligned}\langle k,n|\hat{C}_k^2|k,n\rangle &= \langle k,n|\hat{S}_k^2|k,n\rangle = \qquad (5.41)\\ &= \frac{1}{16}((\hat{f}_n^{(k)})^2+(\hat{f}_{n+1}^{(k)})^2)\ ,\\ \langle\hat{C}_k^2\rangle_{k,n=0} &= \langle\hat{S}_k^2\rangle_{k,n=0}=\frac{(4\,k+1)^2}{32\,k\,(2k+1)^2}\ . \qquad (5.42)\end{aligned}$$

In order to see now the difference in the consequences of the different definitions of the operators $\tilde{C}_k\,,\tilde{S}_k$ and $\hat{C}_k\,,\hat{S}_k$, respectively, let us look at the special but important case of the ground state expectation values of their squares for $k=1/2$:

$$\langle\tilde{C}_{k=1/2}^2\rangle_{k=1/2,n=0} = \langle\tilde{S}_{k=1/2}^2\rangle_{k=1/2,n=0}=\frac{1}{4}=0.25\ , \qquad (5.43)$$

$$\langle\hat{C}_{k=1/2}^2\rangle_{k=1/2} = \langle\hat{S}_{k=1/2}^2\rangle_{k=1/2,n=0}=\frac{9}{64}\approx 0.14\ . \qquad (5.44)$$

In view of the operators $\hat{C}_k$ and $\hat{S}_k$ one may ask, why divide by the operator K_0+k and not by K_0 itself? This leads to the operators [43]

$$\check{C}_k=\hat{C}_{k-1}\ \text{ and }\ \check{S}_k=\hat{S}_{k-1} \qquad (5.45)$$

and their actions

$$\begin{aligned}\check{C}_k|k,n\rangle &= \frac{1}{4}(\check{f}_n^{(k)}|k,n-1\rangle+\check{f}_{n+1}^{(k)}|k,n+1\rangle)\ , \qquad (5.46)\\ \check{S}_k|k,n\rangle &= \frac{1}{4i}(\check{f}_n^{(k)}|k,n-1\rangle-\check{f}_{n+1}^{(k)}|k,n+1\rangle)\ , \qquad (5.47)\\ \check{f}_n^{(k)} &= [n\,(2k+n-1)]^{1/2}\left(\frac{1}{k+n-1}+\frac{1}{k+n}\right)\ , \qquad (5.48)\\ &\check{f}_{n=0}^{(k)}=0\ . \qquad (5.49)\end{aligned}$$

It follows that

$$\begin{aligned}\langle k,n|\check{C}_k^2|k,n\rangle &= \langle k,n|\check{S}_k^2|k,n\rangle = \qquad (5.50)\\ &= \frac{1}{16}((\check{f}_n^{(k)})^2+(\check{f}_{n+1}^{(k)})^2)\ ,\\ \langle\check{C}_k^2\rangle_{k,n=0} &= \langle\check{S}_k^2\rangle_{k,n=0}=\frac{(2\,k+1)^2}{8\,k\,(k+1)^2}\ , \qquad (5.51)\end{aligned}$$

For $k=1/2$ we get from 5.51

$$\langle\check{C}_{k=1/2}^2\rangle_{k=1/2,n=0}=\langle\check{S}_{k=1/2}^2\rangle_{k=1/2,n=0}=\frac{4}{9}\approx 0.44\ . \qquad (5.52)$$

In the special case $k=1/2$ the operators 5.45 were already discussed some time ago [151,152].

Comparing the different results 5.43, 5.44 and 5.52 one realizes the problems as to finding an appropriate definition of suitable operators $\widehat{\cos}$ and $\widehat{\sin}$!

However, such a search is not necessary in the present $SO^{\uparrow}(1,2)$ framework, because here the primary quantum observables incorporating the properties of $\cos\varphi$ and $\sin\varphi$ are the operators K_1 and K_2, *not* $\tilde{C}_k$ and $\tilde{S}_k$ or $\hat{C}_k$ and $\hat{S}_k$ or $\check{C}_k$ and $\check{S}_k$ as discussed above.

In any case: In view of the analysis made above in connection with the expressions 5.20 and 5.21 we see that the London Susskind-Glowgower operators $\tilde{C}_k$ and $\tilde{S}_k$ are *not* appropriate for measuring angle properties of a state!

Chapter 6

The group $SO^{\uparrow}(1,2)$ as a framework for applications in quantum optics

Structures of the group $SU(1,1) \cong SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$ and $SO^{\uparrow}(1,2) = SU(1,1)/Z_2$ - especially its Lie algebra $\mathfrak{so}(1,2)$ - appear to be around "all over the place" in quantum optics and seem to "loom" behind many corners! It is the purpose of the present and the next chapters to put those structures into the perspective of the present approach. Before I give a selection of references from the quantum optics literature let me start with some general remarks:

6.1 Adjoint representation

Much can be learnt from the adjoint representation associated with any unitary representation of the group $SU(1,1)$, i.e. the representation of the group $SO^{\uparrow}(1,2)$ in the 3-dimensional vector space of the Lie algebra spanned by the basis K_j , $j = 0, 1, 2$, or K_0, K_+, K_-:

If

$$\begin{aligned} U(w) &= e^{(w/2)\,K_+ - (\bar{w}/2)\,K_-} = e^{i\,w_2 K_1 + i\,w_1 K_2} \ , \\ & w = w_1 + i\,w_2 = |w|\,e^{i\,\theta} \ , \end{aligned} \tag{6.1}$$

then it follows from the general formula (see, e.g. Ref. [154])

$$e^{A}\,B\,e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] \ \cdots \tag{6.2}$$

and the commutation relation 2.7 that

$$\begin{aligned} U(-w)\,K_+\,U(w) &= \frac{1}{2}(\cosh|w|+1)\,K_+ + \\ &\quad + \frac{1}{2}e^{-2\,i\,\theta}(\cosh|w|-1)\,K_- + e^{-i\,\theta}\,\sinh|w|\,K_0 \end{aligned} \tag{6.3}$$

$$\begin{aligned} U(-w)\,K_-\,U(w) &= \frac{1}{2}(\cosh|w|+1)\,K_- + \\ &\quad + \frac{1}{2}e^{2\,i\,\theta}(\cosh|w|-1)\,K_+ + e^{i\,\theta}\,\sinh|w|\,K_0 \end{aligned} \tag{6.4}$$

$$\begin{aligned} U(-w)\,K_0\,U(w) &= \cosh|w|\,K_0 + \\ &\quad + \frac{1}{2}\,\sinh|w|(e^{i\,\theta}\,K_+ + e^{-i\,\theta}\,K_-) \ , \end{aligned} \tag{6.5}$$

$$U(-\tau)\,K_+\,U(\tau) = e^{-i\,\tau}\,K_+ \ , \tag{6.6}$$

$$U(-\tau)\,K_-\,U(\tau) = e^{i\,\tau}\,K_- \ , \tag{6.7}$$

$$U(\tau) = e^{i\,\tau\,K_0} \ . \tag{6.8}$$

For the operators K_1 and K_2 this means

$$\begin{aligned} U(-w)\,K_1\,U(w) &= [1+\cos^2\theta(\cosh|w|-1)]\,K_1 - \\ &\quad -\sin\theta\,\cos\theta\,(\cosh|w|-1)\,K_2+\cos\theta\,\sinh|w|\,K_0\ , \end{aligned} \tag{6.9}$$

$$\begin{aligned} U(-w)\,K_2\,U(w) &= [1+\sin^2\theta(\cosh|w|-1)]\,K_2 - \\ &\quad -\sin\theta\,\cos\theta\,(\cosh|w|-1)\,K_1-\sin\theta\,\sinh|w|\,K_0\ , \end{aligned} \tag{6.10}$$

$$U(-w)\,K_0\,U(w) = \sinh|w|\,(\cos\theta\,K_1-\sin\theta\,K_2)+\cosh|w|\,K_0\ , \tag{6.11}$$

$$U(-\tau)\,K_1\,U(\tau) = \cos\tau\,K_1+\sin\tau\,K_2\ , \tag{6.12}$$

$$U(-\tau)\,K_2\,U(\tau) = -\sin\tau\,K_1+\cos\tau\,K_2\ . \tag{6.13}$$

The transformations leave the quadratic Killing form $L=K_1^2+K_2^2-K_0^2$ invariant. This property reflects the fact that the above transformations of the 3-dimensional Lie algebra vector space are (1+2)-dimensional Lorentz transformations with K_0 playing the role of a "time variable". That can be made more explicit in the following way:

Let us define the two "spatial" vectors

$$\vec{K}=(K_1,K_2)\ ,\ \vec{n}=(-\cos\theta,\sin\theta)\ . \tag{6.14}$$

Then the transformation formulae 6.9-6.11 can be written as

$$U(-w)\vec{K}\,U(w) = \vec{K}+(\cosh|w|-1)(\vec{n}\cdot\vec{K})\,\vec{n}-\sinh|w|\,\vec{n}\,K_0\,, \tag{6.15}$$

$$U(-w)K_0\,U(w) = \cosh|w|\,K_0-\sinh|w|\,(\vec{n}\cdot\vec{K})\ . \tag{6.16}$$

These equations describe Lorentz transformations ("boosts") in the direction $\vec{n}$!

Notice that all the above relations are independent of the index k. The k-dependence will appear in the matrix elements with respect to a given Hilbert space. The reason for using the transformations $U(-w)\,K_1\,U(w)$ etc. instead of the usual $U(w)\,K_1\,U(-w)$ etc. is the following: In applications it is frequently useful to generate a (coherent) state $|\psi\rangle$ by applying one of the above unitary operators U to a given reference state $|\psi_0\rangle$, namely $|\psi\rangle=U|\psi_0\rangle$. The best-known examples are the coherent states $|\alpha\rangle$ (Eq. 3.133) and $|k,\lambda\rangle$ (Eq. 3.132) where the reference state $|\psi_0\rangle$ is the ground state $|k,n=0\rangle$.

For the Perelomov state 3.132 the generating unitary operator is the operator $U(w)$ from above. The expectation value of, e.g. K_1, is then given by

$$\langle k,\lambda|K_1|k,\lambda\rangle=\langle k,0|U(-w)\,K_1\,U(w)|k,0\rangle\,. \tag{6.17}$$

According to Eq. 6.9 this expectation value may by calculated by taking the ground state expectation value of the r.h. side of 6.9 which immediately gives $k\,\cos\theta\,\sinh|w|$, in agreement with Eq. 3.178. The other cases can be dealt with accordingly.

The unitary operator $U(w)$ is not a very natural one from a group theoretical point of view, because the vector subspace spanned by the operators K_1 and K_2 does not form a Lie subalgebra, i.e. the unitary transformations $U(w)$ do not form a subgroup.

From a group theoretical aspect one would decompose a general unitary transformation in an irreducible representation either according to Cartan's polar decomposition or according to Iwasawa's decomposition (see Appendix B):

The polar decomposition of a general unitary $SU(1,1)$ transformation is given by

$$U(\tau_2,w_2,\tau_1)=e^{i\,\tau_2\,K_0}\cdot e^{i\,w_2\,K_1}\cdot e^{i\,\tau_1\,K_0}\ , \tag{6.18}$$

and the Iwasawa decomposition by

$$U(\tau,w_2,\nu)=e^{i\,\tau\,K_0}\cdot e^{i\,w_2\,K_1}\cdot e^{i\,\nu\,N}\ ,\ \nu\in\mathbb{R}\ , \tag{6.19}$$

where

$$N = K_2 + K_0 \ , \tag{6.20}$$

$$[N, K_1] = i\,N \ , \ [N, K_0] = i\,K_1 \ , \ [K_1, K_0] = i\,(K_0 - N)\,. \tag{6.21}$$

N is the generator of a nilpotent subgroup. The Iwasawa decomposition appears rarely in quantum optical papers [153].
From the commutation relations 6.21 we get for the adjoint representation with respect to the basis K_1, K_0 and N:

$$e^{-i\,\nu\,N} K_1 e^{i\,\nu\,N} = K_1 + \nu\,N \ , \tag{6.22}$$

$$e^{-i\,\nu\,N} K_0 e^{i\,\nu\,N} = K_0 + \nu\,K_1 + \frac{\nu^2}{2}\,N \ , \tag{6.23}$$

$$e^{-i\,w_2\,K_1} N e^{i\,w_2\,K_1} = e^{-w_2} N \ , \tag{6.24}$$

$$e^{-i\,w_2\,K_1} K_0 e^{i\,w_2\,K_1} = (\cosh w_2 + \sinh w_2)\,K_0 - \sinh w_2\,N \ . \tag{6.25}$$

The rest follows immediately from Eqs. 6.12 and 6.13.

6.2 Schwarz's inequality and $SO^{\uparrow}(1,2)$ squeezing

6.2.1 Uncertainty relations

One of the main purposes of the operators $U(w)$ in quantum optics is to generate "squeezed" states [10, 60, 155–158]. I recall the main features of their definition: Let A and B two self-adjoint operators with the commutator $[A, B] = i\,C$, where C is again self-adjoint. Then the lower limit for the product of the "uncertainties" $(\Delta\,A)_\psi$ and $(\Delta\,B)_\psi$ in the state $|\psi\rangle$, where

$$(\Delta\,A)^2_\psi = \langle\psi|(A - \langle\psi|A|\psi\rangle)^2|\psi\rangle\,, \tag{6.26}$$

is derived from Schwarz's inequality

$$\langle\psi_2|\psi_2\rangle\,\langle\psi_1|\psi_1\rangle \geq |\langle\psi_2|\psi_1\rangle|^2 \tag{6.27}$$

for scalar products as follows [159–164] [70]: Taking for $|\psi_2\rangle$ and $|\psi_1\rangle$ the states

$$\tilde{A}|\psi\rangle \equiv (A - \langle A\rangle_\psi)|\psi\rangle \ \text{ and } \ \tilde{B}|\psi\rangle \equiv (B - \langle B\rangle_\psi)|\psi\rangle\,, \tag{6.28}$$

we get

$$(\Delta A)^2_\psi \cdot (\Delta B)^2_\psi \geq |\langle\psi|\tilde{A}\cdot\tilde{B} = \frac{1}{2}[\tilde{A}, \tilde{B}] + \frac{1}{2}(\tilde{A}\tilde{B} + \tilde{B}\tilde{A})|\psi\rangle|^2. \tag{6.29}$$

As the expectation value of the commutator $[A, B]$ is purely imaginary and that of

$$S_\psi(A, B) = \frac{1}{2}(AB + BA) - \langle A\rangle_\psi\langle B\rangle_\psi \tag{6.30}$$

purely real we can write Eq. 6.29 as

$$(\Delta A)^2_\psi \cdot (\Delta B)^2_\psi \geq \frac{1}{4}|\langle\psi|[A, B]|\psi\rangle|^2 + |\langle\psi|S_\psi(A, B)|\psi\rangle|^2 \geq \frac{1}{4}|\langle\psi|[A, B]|\psi\rangle|^2 \ . \tag{6.31}$$

If $\langle\psi|S_\psi(A, B)|\psi\rangle$ vanishes then the inequality 6.31 reduces to the usual Heisenberg uncertainty relation. If $\langle\psi|S_\psi(A, B)|\psi\rangle \neq 0$ then the first of the inequalities 6.31 is stronger than the second one.

One now says that " the state $|\psi\rangle$ is squeezed with respect to the operator A" if

$$(\Delta A)^2_\psi < [|\langle\psi|[A,B]|\psi\rangle|^2/4 + |\langle\psi|S_\psi(A,B)|\psi\rangle|^2]^{1/2} \ , \tag{6.32}$$

("Robertson-Schrödinger squeezing"), or

$$(\Delta A)^2_\psi < \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle| \tag{6.33}$$

("Heisenberg squeezing").

The latter condition is more restrictive.

In order to preserve the inequalities 6.31 the second uncertainty $(\Delta B)_\psi$ has to be enlarged or "stretched" accordingly.

Depending on the state $|\psi\rangle$ the r.h. side of the inequalities might be quite large. Then it appears appropriate [92, 165] to sharpen the criteria 6.32 and 6.33 to

$$(\Delta A)^2_\psi < \min_{\{|\psi\rangle\}}[|\langle\psi|[A,B]|\psi\rangle|^2/4 + |\langle\psi|S_\psi(A,B)|\psi\rangle|^2]^{1/2} \equiv \Delta^2_{RS} \ , \tag{6.34}$$

("absolute Robertson-Schrödinger squeezing"), or

$$(\Delta A)^2_\psi < \min_{\{|\psi\rangle\}} \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle| \equiv \Delta^2_H \ , \tag{6.35}$$

("absolute Heisenberg squeezing"). Here $\{|\psi\rangle\}$ means a given set of states, e.g. a basis of the Hilbert space.

Example:

If $A = K_1$, $B = K_2$ and $|\psi\rangle = |k,n\rangle$ then $\langle k,n|S_{k,n}(K_1,K_2)|k,n\rangle = 0$ and the minimum $k/2$ of the r.h. side in 6.35 is obtained for the ground state $|k,n=0\rangle$.

6.2.2 Group theoretical generation of squeezed states

In order to see the squeezing properties of the operators $U(w)$ let us specialize to pure special Lorentz transformations in the $(K_1,K_0)-$ and $(K_2,K_0)-$ subspaces, respectively:

$$\begin{aligned}
U(-w_1)\,K_1\,U(w_1) &= \cosh w_1\,K_1 + \sinh w_1\,K_0 \ , && (6.36)\\
U(-w_1)\,K_0\,U(w_1) &= \sinh w_1\,K_1 + \cosh w_1\,K_0 \ , && (6.37)\\
U(w_1) &= e^{i\,w_1\,K_2} \ ; && (6.38)\\
U(-i\,w_2)\,K_2\,U(i\,w_2) &= \cosh w_2\,K_2 - \sinh w_2\,K_0 \ , && (6.39)\\
U(-i\,w_2)\,K_0\,U(i\,w_2) &= -\sinh w_2\,K_2 + \cosh w_2\,K_0 \ , && (6.40)\\
U(i\,w_2) &= e^{i\,w_2\,K_1} \ . && (6.41)
\end{aligned}$$

If we now define the operators

$$K_{1\pm} = K_1 \pm K_0 \ , \ K_{2\pm} = K_2 \pm K_0 \ , \tag{6.42}$$

where

$$[K_{1+},K_{1-}] = 2\,i\,K_2 \ , \ [K_{2+},K_{2-}] = -2\,i\,K_1 \ , \tag{6.43}$$

then we get

$$\begin{aligned}
U(-w_1)\,K_{1\pm}\,U(w_1) &\equiv \hat{K}_{1\pm} = e^{\pm w_1}K_{1\pm} \ , && (6.44)\\
U(-w_1)\,K_2\,U(w_1) &\equiv \hat{K}_2 = K_2 \ , && (6.45)\\
[\hat{K}_{1+},\hat{K}_{1-}] &= 2\,i\,K_2 \ ; && (6.46)\\
U(-i\,w_2)\,K_{2\pm}\,U(i\,w_2) &\equiv \hat{K}_{2\pm} = e^{\mp w_2}K_{2\pm} \ , && (6.47)\\
U(-i\,w_2)\,K_1\,U(i\,w_2) &\equiv \hat{K}_1 = K_1 \ , && (6.48)\\
[\hat{K}_{2+},\hat{K}_{2-}] &= -2\,i\,K_1 \ . && (6.49)
\end{aligned}$$

We have the following correspondences of the operators $K_{1\pm}$ and $K_{2\pm}$ and the basic classical quantities $I, I\cos\varphi$ and $I\sin\varphi$:

$$K_{1\pm} \leftrightarrow I\,(\cos\varphi \pm 1)\,, \qquad K_{2\pm} \leftrightarrow -I\,(\sin\varphi \pm 1)\,. \tag{6.50}$$

The operator K_{2+} is identical with the generator N of Eq. 6.20, K_{2-} is the generator $\bar{N}$ of another nilpotent group (see Eqs. B.61 and B.62).

The operators $U(w_1)$ and $U(i\,w_2)$ generate special Perelomov coherent states 3.132 from a ground state $|k,0\rangle$.

(If $w_2 = 0$, we have $w_1 > 0$, for $\theta = 0$, and $w_1 < 0$ for $\theta = \pi$, and if $w_1 = 0$, then $w_2 > 0$, for $\theta = \pi/2$, and $w_2 < 0$ for $\theta = -\pi/2$, i.e. the variable

$$\lambda = \lambda_1 + i\,\lambda_2 \tag{6.51}$$

in $|k,\lambda\rangle$ is purely real in the first case and purely imaginary in the second.)

When calculating expectation values with respect to these states we can do so by just calculating the ground state expectation values of the r.h. sides of the above Eqs. 6.44-6.49:

$$\begin{aligned}
\langle K_{1\pm}\rangle_{k,\lambda=\lambda_1} &= \langle \hat{K}_{1\pm}\rangle_{k,\lambda=0} = e^{\pm w_1}\langle K_{\pm 1}\rangle_{k,\lambda=0} = \pm k\, e^{\pm w_1}\ , && (6.52)\\
\langle K^2_{1\pm}\rangle_{k,\lambda=\lambda_1} &= \langle \hat{K}^2_{1\pm}\rangle_{k,\lambda=0} = e^{\pm 2\,w_1}\langle K^2_{\pm 1}\rangle_{k,\lambda=0} && (6.53)\\
&= e^{\pm 2\,w_1}(k/2 + k^2)\ , \\
(\Delta K_{1\pm})^2_{k,\lambda=\lambda_1} &= \frac{k}{2}\, e^{\pm 2\,w_1}\ ; && (6.54)\\
\langle K_{2\pm}\rangle_{k,\lambda=i\,\lambda_2} &= \langle \hat{K}_{2\pm}\rangle_{k,\lambda=0} = e^{\mp w_2}\langle K_{\pm 2}\rangle_{k,\lambda=0} = \pm k\, e^{\mp w_2}\ , && (6.55)\\
\langle K^2_{2\pm}\rangle_{k,\lambda=i\,\lambda_2} &= \langle \hat{K}^2_{2\pm}\rangle_{k,\lambda=0} = e^{\mp 2\,w_2}\langle K^2_{\pm 2}\rangle_{k,\lambda=0} && (6.56)\\
&= e^{\mp 2\,w_2}(k/2 + k^2)\ , \\
(\Delta K_{2\pm})^2_{k,\lambda=i\,\lambda_2} &= \frac{k}{2}\, e^{\mp 2\,w_2}\ . && (6.57)
\end{aligned}$$

For the r.h. side of the inequality 6.31 we get in case of the pair $K_{1\pm}$:

$$\begin{aligned}
|\langle [K_{1+}, K_{1-}]\rangle_{k,\lambda=\lambda_1}| &= |\langle [K_{1+}, K_{1-}]\rangle_{k,\lambda=0}| && (6.58)\\
&= 2|\langle K_2\rangle_{k,\lambda=0}| = 0\ ; \\
|\langle S_{k,\lambda=\lambda_1}(K_{1+}, K_{1-})\rangle_{k,\lambda=\lambda_1}| &= |\langle S_{k,\lambda=0}(K_{1+}, K_{1-})\rangle_{k,\lambda=0}| = \frac{k}{2}\ . && (6.59)
\end{aligned}$$

Combined with Eqs. 6.54 we obtain the equality

$$(\Delta K_{1+})_{k,\lambda=\lambda_1}(\Delta K_{1-})_{k,\lambda=\lambda_1} = \frac{k}{2} = |\langle S_{k,\lambda=0}(K_{1+}, K_{1-})\rangle_{k,\lambda=0}|\ . \tag{6.60}$$

That such an equality has to hold we know already from the general result 3.195. Here we learn in addition from 6.54 that one can make one of the uncertainties as small as possible by a corresponding choice of w_1, at the expense of the other uncertainty.

The corresponding discussion for the operators $K_{2\pm}$ gives completely analogous results.

We have here - Eqs. 6.58 and 6.59 - the somewhat unusual case that the expectation value of the commutator in the inequality 6.31 vanishes whereas the expectation value of $S_\psi(A,B)$ is non-vanishing! A deeper reasons for the equality 6.60 will be discussed below.

The examples just discussed may be generalized immediately: Take any normalizable state vector $|k,\sigma\rangle$ of the Hilbert space of a representation with Bargmann index k and apply one of the operators $U(w_1)$, $U(i\,w_2)$ to it. Then the states

$$|k,\sigma;w_1\rangle = U(w_1)|k,\sigma\rangle\ ,\ |k,\sigma;w_2\rangle = U(i\,w_2)|k,\sigma\rangle\ , \tag{6.61}$$

are squeezed as to the operators $K_{1\pm}$ and $K_{2\pm}$, respectively: We have, e.g. from Eqs. 6.44-6.46

$$\begin{aligned}
\langle K_{1\pm}\rangle_{k,\sigma;w_1} &= \langle \hat{K}_{1\pm}\rangle_{k,\sigma} = e^{\pm w_1}\langle K_{\pm 1}\rangle_{k,\sigma} \ , && (6.62)\\
\langle K^2_{1\pm}\rangle_{k,\sigma;w_1} &= \langle \hat{K}^2_{1\pm}\rangle_{k,\sigma} = e^{\pm 2\, w_1}\langle K^2_{\pm 1}\rangle_{k,\sigma} \ , && (6.63)\\
(\Delta K_{1\pm})^2_{k,\sigma;w_1} &= e^{\pm 2\, w_1}(\Delta K_{1\pm})^2_{k,\sigma} \ . && (6.64)
\end{aligned}$$

Because the transformations 6.61 are unitary and because of the relations 6.44-6.46 we get

$$\begin{aligned}
|\langle [K_{1+}, K_{1-}]\rangle_{k,\sigma;w_1}| &= |\langle [K_{1+}, K_{1-}]\rangle_{k,\sigma}| = 2\,|\langle K_2\rangle_{k,\sigma}| \ , && (6.65)\\
|\langle S_{k,\sigma;w_1}(K_{1+}, K_{1-})\rangle_{k,\sigma;w_1}| &= |\langle S_{k,\sigma}(K_{1+}, K_{1-})\rangle_{k,\sigma}| \ . && (6.66)
\end{aligned}$$

The inequality 6.31 holds, of course, for the original state $|k,\sigma\rangle$ and the operators $A = K_{1+}$ and $B = K_{1-}$. The Eqs. 6.64, 6.65 and 6.66 then show that the same inequality holds for the transformed state $|k,\sigma;w_1\rangle$:

$$(\Delta K_{1+})^2_{k,\sigma;w_1}\cdot(\Delta K_{1-})^2_{k,\sigma;w_1} \geq |\langle K_2\rangle_{k,\sigma}|^2 + |\langle S_{k,\sigma}(K_{1+}, K_{1-})\rangle_{k,\sigma}|^2 \,, \qquad (6.67)$$

where the uncertainties on the l.h. side are now squeezed and stretched according to Eqs. 6.64.

The present squeezing procedure can, of course, applied especially to the states $|k,n\rangle$, $|k,z\rangle$, $|k,\lambda\rangle$ and $|k,\alpha\rangle$ discussed in Ch. 3. How this may be done even experimentally will be indicated below in Sec. 6.5.

6.2.3 Schwarz's equality!

We have just seen that we get interesting additional information if the inequality 6.31 becomes an equality. This can be exploited further:

A *necessary and sufficient* condition for Schwarz's inequality 6.27 to become an equality is the linear dependence of the states 6.28 [166].

Let $|\psi_0\rangle$ be a state for which the equality holds. Then

$$\tilde{B}|\psi_0\rangle = \gamma\,\tilde{A}|\psi_0\rangle \ , \ \gamma \in \mathbb{C}, \qquad (6.68)$$

i.e. we have the "eigenvalue" equation

$$(B - \gamma\, A)|\psi_0\rangle = (\langle B\rangle_{\psi_0} - \gamma\,\langle A\rangle_{\psi_0})|\psi_0\rangle \ , \qquad (6.69)$$

where the complex parameter γ may be calculated as [163]

$$\begin{aligned}
\gamma &= \frac{2\langle S_{\psi_0}(A,B)\rangle_{\psi_0} + \langle [A,B]\rangle_{\psi_0}}{2\,(\Delta A)^2_{\psi_0}} \ , && (6.70)\\
\frac{1}{\gamma} &= \frac{2\langle S_{\psi_0}(A,B)\rangle_{\psi_0} - \langle [A,B]\rangle_{\psi_0}}{2\,(\Delta B)^2_{\psi_0}} \ , && (6.71)\\
|\gamma| &= \frac{(\Delta B)_{\psi_0}}{(\Delta A)_{\psi_0}} \ , \ \arg\gamma = \arctan\left(-i\frac{\langle [A,B]\rangle_{\psi_0}}{2\langle S_{\psi_0}(A,B)\rangle_{\psi_0}}\right) \ . && (6.72)
\end{aligned}$$

Thus, γ will in general be complex.

As a first example take the coherent states $|k,z\rangle$ for which the "Schwarz equality" holds (Eqs. 3.106 and 3.108) Here the relation 6.68 takes the form

$$(K_1 - \langle K_1\rangle_{k,z})|k,z\rangle = i\,(K_2 - \langle K_2\rangle_{k,z})|k,z\rangle \ , \ \langle K_j\rangle_{k,z} = z_j \,, j = 1,2 \,;\, z = z_1 + i\,z_2 \,, \qquad (6.73)$$

which is just another version of the defining equation $K_-|k,z\rangle = z|k,z\rangle$.

As a second example consider the operators K_1 and K_2 acting on the coherent state $|k,\lambda\rangle$. According to Eq. 3.195 the uncertainty inequality is an equality. This implies that

$$(K_1 - \gamma\, K_2)|k,\lambda\rangle = (\langle K_1\rangle_{k,\lambda} - \gamma\,\langle K_2\rangle_{k,\lambda})|k,\lambda\rangle\ , \tag{6.74}$$

where γ may be calculated, according to Eqs. 6.72, from the relations 3.185, 3.182, 3.169 (because $[K_1, K_2] = -i\ K_0$), and 3.193:

$$|\gamma| = \frac{|1-\lambda^2|}{|1+\lambda^2|}\,,\ \ \arg\gamma = \arctan\left(\frac{1-|\lambda|^4}{2\,|\lambda|^2\,\sin(2\theta)}\right)\ . \tag{6.75}$$

6.3 The Lie algebra $\mathfrak{so}(1,2)$ in terms of products of 1-mode operators a and a^+

The Lie algebra $\mathfrak{so}(1,2)$ of the group $SU(1,1)$ entered quantum optics - unrecognized as such - in connection with squeezing [155–158]. The group theoretical background was realized later [58–61].

A realization of the Lie algebra $\mathfrak{so}(1,2)$ in terms of annihilation and creation operators a and a^+ is given by [167–170]

$$K_+ = \frac{1}{2}(a^+)^2\ ,\ K_- = \frac{1}{2}\,a^2\ ,\ K_0 = \frac{1}{2}(a^+a + 1/2)\ . \tag{6.76}$$

An alternative form is (we put the frequency $\omega = 1$ in the following)

$$K_0 = \frac{1}{4}(Q^2+P^2)\,,\ \ K_1 = \frac{1}{4}(Q^2-P^2)\,,\ \ K_2 = -\frac{1}{4}(QP+PQ)\,. \tag{6.77}$$

As K_- annihilates the states $|n_{osc}=0\rangle$ *and* $|n_{osc}=1\rangle$,

$$K_-|n_{osc}=0\rangle = 0\,,\ \ K_-|n_{osc}=1\rangle = 0\,, \tag{6.78}$$

we get 2 different irreducible representations of the Lie algebra $\mathfrak{so}(1,2)$, one which is given by states with even numbers of oscillator quanta and one with odd numbers. Because

$$K_0|n_{osc}\rangle = \frac{1}{2}\,(n_{osc}+1/2)|n_{osc}\rangle\,,\ n_{osc} = 0,1,2,\ldots, \tag{6.79}$$

we see that K_0 has the eigenvalues

$$(2\,n_{osc}+1/2)/2 = n + \frac{1}{4}\ \text{ and }\ (2\,n_{osc}+1+1/2)/2 = n+\frac{3}{4}\,,\ n=0,1,\ldots, \tag{6.80}$$

in the cases of even and odd numbers of quanta, respectively.

That is to say, we get one irreducible unitary representation with $k = 1/4$ and one with $k = 3/4$.

As to the related group these are true representations of a 2-fold covering of $SU(1,1) \cong SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$ and a 4-fold covering of $SO^{\uparrow}(1,2)$. Those 2-fold covering groups of the symplectic groups $Sp(2n,\mathbb{R})$ in $2n$ dimensions are called "metaplectic" [279, 293] (see Appendices B and C for more details).

The two representations with $k = 1/4$ and $k = 3/4$ may be realized not only in the 2 subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ of the Fock space of the harmonic oscillator just mentioned, but also in

the Hilbert space with the series inner product 4.8 of Ch. 4 or in the associated Hilbert space with the scalar product 4.180. In the case of $k = 3/4$ - but not for $k = 1/4$ - one can also use the Bargmann Hilbert space with the scalar product 4.5.

If one uses for the harmonic oscillator the conventional Hilbert space with the scalar product 4.66 (with the variable ξ replaced by q and Hermite's functions (see 4.67)

$$f_{n_{osc}}(q) = C_{n_{osc}}\, e^{-q^2/2}\, H_{n_{osc}}(q)\,,\ C_{n_{osc}} = \text{ const.}\,, \tag{6.81}$$

as basis, then the subspace $\mathcal{H}_+$ for the unitary representation with $k = 1/4$ is spanned by the Hermite functions with even Hermite polynomials $H_{n_{osc}}$ (they are invariant under the reflection $q \to -q$) and the subspace $\mathcal{H}_-$ for the representation with $k = 3/4$ is spanned by the Hermite functions with odd Hermite polynomials.

In the "even" subspace $\mathcal{H}_+$ the Hamiltonian

$$H_{osc} = 2\,K_0 \tag{6.82}$$

has the eigenvalues

$$(2n_{osc} + 1/2)\,, \tag{6.83}$$

and in the "odd" subspace $\mathcal{H}_-$ its eigenvalues are

$$(2n_{osc} + 1 + 1/2)\,. \tag{6.84}$$

The two subsystems may be given the following model interpretation:

As the odd Hermite functions do all vanish at $q = 0$ one may interprete them as the energy eigenfunctions of a system with the potential

$$V(q) = \frac{1}{2}q^2 \text{ for } q \geq 0\,,\ V(q) = \infty \text{ for } q < 0\,. \tag{6.85}$$

A more mathematical description is that we obtain a system confined to the half plane $q \geq 0$ by identifying $-q$ with q. The resulting (classical) system has a phase space

$$\mathbb{R}^2/Z_2[\cdot q] = \{(q,p) \in \mathbb{R}^2, (-q,p) \equiv (q,p)\}\,. \tag{6.86}$$

(The notion $Z_2[\cdot q]$ is to indicate that the group $Z_2 = \{e, -e\}$ acts on the variable q.) Such a space is called an "orbifold" [171] (see also Appendix A.3) Note that here we do *not* identify $-p$ with p!

The model interpretation for the even Hermite functions and their Hilbert space $\mathcal{H}_+$ I take from Appendix A.3: If we identify the points $(-q,-p)$ and (q,p), then the resulting orbifold

$$\mathbb{R}^2/Z_2[\cdot(q,p)] = \{(q,p) \in \mathbb{R}^2,\ (-q,-p) \equiv (q,p)\}\,. \tag{6.87}$$

is a cone with its tip at $(q,p) = (0,0)$.

We now may interpret the group Z_2 as a kind of "gauge" group the action of which leaves "observables" like the expressions 6.77 and the even Hermite functions invariant.

The coordinates q and p are *not* observables in this sense, only the equivalence classes

$$\{(q,p) \equiv (-q,-p)\}\,. \tag{6.88}$$

The crucial point may also be seen as follows: With help of the relation 6.2 one can show that the operators 6.76 generate the following rotations:

$$
\begin{aligned}
e^{-i\tau K_0} a\, e^{i\tau K_0} &= e^{i\tau/2}\, a\,, && (6.89)\\
e^{-i\tau K_0} a^+\, e^{i\tau K_0} &= e^{-i\tau/2}\, a^+\,, && (6.90)\\
e^{-i\tau K_0} Q\, e^{i\tau K_0} &= \cos\frac{\tau}{2}\, Q - \sin\frac{\tau}{2}\, P\,, && (6.91)\\
e^{-i\tau K_0} P\, e^{i\tau K_0} &= \sin\frac{\tau}{2}\, Q + \cos\frac{\tau}{2}\, P\,, && (6.92)\\
e^{-i\tau K_0} K_1\, e^{i\tau K_0} &= \cos\tau\, K_1 + \sin\tau\, K_2\,, && (6.93)\\
e^{-i\tau K_0} K_2\, e^{i\tau K_0} &= -\sin\tau\, K_1 + \cos\tau\, K_2. && (6.94)
\end{aligned}
$$

These formulae show that Q and P transform according to the maximal compact subgroup B.20 of the symplectic group $Sp(2,\mathbb{R})$, whereas K_1 and K_2 transform according to the maximal compact subgroup $O(2)$ of $SO^{\uparrow}(1,2) \cong Sp(2,\mathbb{R})/Z_2$, where Z_2 denotes the center of $Sp(2,\mathbb{R})$!

The difference is expressed by the property that for $\tau = 2\pi$ the operators Q and P change sign whereas K_1 and K_2 remain invariant! See also the closely related discussion in Sec. 1.3 ! Gauge transformations like

$$Z_2[\cdot(q,p)]: \quad (q,p) \to (-q,-p) \tag{6.95}$$

have been discussed in a number of papers by Prokhorov and Shabanov [49] and the subject was reviewed recently by Shabanov [50]. There is one crucial difference, however, between one of their quantum mechanical conclusions and the above result 6.83 concerning the energy spectrum:

They also get a doubling of the energy levels of the harmonic oscillator, but *including* the ground state energy, whereas the ground state energy in Eq. 6.83 is the *same as that of the harmonic oscillator*. However, they derive the energy spectrum of the orbifold 6.87 by means of a semi-classical Bohr-Sommerfeld procedure which is known to be quantitatively unreliable as far as ground state energies are concerned.

What does all this mean for quantum optics? The crucial point is that the orbifold 6.87, with the tip deleted, is diffeomorphic to the phase space 1.8 we started from (see also Appendix A.3). So the above unitary representation with $k = 1/4$ may be a possible candidate for a quantization of that phase space!

Squeezing properties of the even and odd oscillator states discussed here are analyzed in Ref. [172].

6.3.1 Squeezing of Q or P

The special forms 6.76 or 6.77 of K_0, K_1 and K_2 have all the properties discussed in general in the previous section, especially those as to squeezing from Sec. 6.2. For completeness I add here their action on the canonical operators

$$Q = \frac{1}{\sqrt{2}}(a^+ + a)\,,\ P = \frac{i}{\sqrt{2}}(a^+ - a)\,, \tag{6.96}$$

namely

$$
\begin{aligned}
e^{-i w_2 K_1} Q\, e^{i w_2 K_1} &= \cosh(w_2/2)\, Q + \sinh(w_2/2)\, P\ , && (6.97)\\
e^{-i w_2 K_1} P\, e^{i w_2 K_1} &= \cosh(w_2/2)\, P + \sinh(w_2/2)\, Q\ , && (6.98)\\
e^{-i w_1 K_2} Q\, e^{i w_1 K_2} &= e^{w_1/2}\, Q\,, && (6.99)\\
e^{-i w_1 K_2} P\, e^{i w_1 K_2} &= e^{-w_1/2}\, P\,. && (6.100)
\end{aligned}
$$

All these transformations and the rotations 6.91-6.92 leave the commutation relations

$$Q\,P - P\,Q = i \tag{6.101}$$

invariant. This is a consequence of the property that the group $SU(1,1)$ is isomorphic to the symplectic group $Sp(2,\mathbb{R})$ in $1+1$ dimensions (Appendix B).

In quantum optics the operators Q and P are frequently denoted by $\hat{X}_1$ and $\hat{X}_2$ or $\hat{X}$ and $\hat{Y}$ and called "quadratures", reflecting the two orthogonal directions in the associated phase space $\mathbb{R}^2$:

$$Q = \hat{X}_1 = \hat{X}\ , \tag{6.102}$$

$$P = \hat{X}_2 = \hat{Y}\ . \tag{6.103}$$

The relations 6.99 and 6.100 show explicitly the squeezing ("Lorentz boost") generated by K_2.

The operators $K_{1\pm}$ and $K_{2\pm}$ from the section 6.2.2 here have the form

$$K_{1+} = K_1 + K_0 = \frac{1}{4}\,(a^+ + a)^2 = \frac{1}{2}\,Q^2\ , \tag{6.104}$$

$$K_{1-} = K_1 - K_0 = \frac{1}{4}\,(a^+ - a)^2 = -\frac{1}{2}\,P^2\ , \tag{6.105}$$

$$K_{2+} = K_2 + K_0 = \frac{1}{4i}\,(a^+ + ia)^2 = \tag{6.106}$$

$$= \frac{1}{4}(\,Q^2 + P^2 - Q\,P - P\,Q)\,, \tag{6.107}$$

$$K_{2-} = K_2 - K_0 = \frac{1}{4i}\,(a^+ - ia)^2 = \tag{6.108}$$

$$= -\frac{1}{4}(\,Q^2 + P^2 + Q\,P + P\,Q)\ , \tag{6.109}$$

They have, of course, all the algebraic properties listed in that section.

6.3.2 The $\mathfrak{so}(1,2)$ structure of squared hermitian amplitudes

The $\mathfrak{so}(1,2)$ Lie algebra realization 6.76 also occurs in so-called "amplitude-squared squeezing" [173–175]. The starting observation is of general interest:

Let

$$E = \lambda\,(\,a\,e^{-i\,\omega\,t} + a^+\,e^{i\,\omega\,t})\,,\quad \lambda \in \mathbb{R}\,, \tag{6.110}$$

be the hermitian quantized amplitude of a 1-mode field. Then its square has the form

$$E^2 = \lambda^2\,\{[(a^+)^2 + (a)^2]\,\cos(2\,\omega\,t) + \tag{6.111}$$

$$+i\,[(a^+)^2 - (a)^2]\,\sin(2\,\omega\,t) + +2\,a^+a + 1\}$$

$$= 4\,\lambda^2\,[K_1\,\cos(2\,\omega\,t) - K_2\,\sin(2\,\omega\,t) + K_0]\ . \tag{6.112}$$

Thus, the Lie algebra 6.76 generates the squared quantized amplitude E^2! We shall come back to the more general case involving interfering amplitudes in Ch. 8.

6.4 The Lie algebra $\mathfrak{so}(1,2)$ in terms of 2-mode operators

The realization 6.76 of the Lie algebra $\mathfrak{so}(1,2)$ by a single pair of annihilation and creation operators leads only to two irreducible unitary representations with the Bargmann indices $k = 1/4\,, 3/4$.

The outcome is much richer if one takes two pairs [60, 61, 176–178]: The operators

$$K_+ = a_1^+ a_2^+ \ , \ K_- = a_1 a_2 \ , \ K_0 = \frac{1}{2}(a_1^+ a_1 + a_2^+ a_2 + 1) \tag{6.112}$$

obey the commutation relations 2.7.

The tensor product $\mathcal{H}_1^{osc} \otimes \mathcal{H}_2^{osc}$ of the two harmonic oscillator Hilbert spaces contains all the irreducible unitary representations of the group
$SU(1,1) \cong SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$ (for which $k = 1/2, 1, 3/2, \ldots$) in the following way:

Let $|n_j\rangle_j$, $n_j = 0, 1, \ldots$, $j = 1, 2$, be the eigenstates of the number operators N_j, generated by a_j^+ from the oscillator ground states.

Then each of those two subspaces of $\mathcal{H}_1^{osc} \otimes \mathcal{H}_2^{osc} = \{|n_1\rangle_1 \otimes |n_2\rangle_2\}$ with fixed $|n_1 - n_2| \neq 0$ contains an irreducible representation with

$$k = 1/2 + |n_1 - n_2|/2 = 1, 3/2, 2, \ldots \tag{6.113}$$

(the operator $N_1 - N_2$ commutes with all 3 operators in Eqs. 6.112) and for which the number n in the eigenvalue $n + k$ of K_0 is given by

$$n = \min\{n_1, n_2\} \ . \tag{6.114}$$

For the "diagonal" case $n_2 = n_1$ one gets the unitary representation with $k = 1/2$.

Actually the operators 6.112 are only 3 of 10 independent (hermitian) bilinear operators one can build from the two pairs a_1, a_1^+ and a_2, a_2^+. Those 10 operators generate the Lie algebra $\mathfrak{sp}(4)$ of the real symplectic group $Sp(4,\mathbb{R})$ which plays a major role in our analysis, too (see Ch. 8 and Appendix C).

Using the realizations one finds [179] that the two pairs $a_1\,, a_1^+$ and $a_2\,, a_2^+$ are $SU(1,1)$- or $Sp(2,\mathbb{R})$- transformed as follows:

$$e^{-i\,w_2\,K_1} a_1\, e^{i\,w_2\,K_1} = \cosh(w_2/2)\, a_1 + i\, \sinh(w_2/2)\, a_2^+ \ , \tag{6.115}$$

$$e^{-i\,w_2\,K_1} a_1^+\, e^{i\,w_2\,K_1} = \cosh(w_2/2)\, a_1^+ - i\, \sinh(w_2/2)\, a_2 \ , \tag{6.116}$$

$$e^{-i\,w_1\,K_2} a_1\, e^{i\,w_1\,K_2} = \cosh(w_1/2)\, a_1 + \sinh(w_2/2)\, a_2^+ \ , \tag{6.117}$$

$$e^{-i\,w_1\,K_2} a_1^+\, e^{i\,w_1\,K_2} = \cosh(w_1/2)\, a_1^+ + \sinh(w_2/2)\, a_2 \ , \tag{6.118}$$

$$e^{-i\,\tau\,K_0} a_1\, e^{i\,\tau\,K_0} = e^{i\,\tau/2}\, a_1 \ , \qquad e^{-i\,\tau\,K_0} a_1^+\, e^{i\,\tau\,K_0} = e^{-i\,\tau/2}\, a_1^+ \ , \tag{6.119}$$

The other half of the resulting relations is obtained by exchanging the indices 1 and 2.

As to squeezing the following implications are of special interest:

$$e^{-i\,w_1\,K_2}(a_1 + a_2^+)\, e^{i\,w_1\,K_2} = e^{w_1/2}\,(a_1 + a_2^+) \ , \tag{6.120}$$

$$e^{-i\,w_1\,K_2}(a_1 - a_2^+)\, e^{i\,w_1\,K_2} = e^{-w_1/2}\,(a_1 - a_2^+) \ . \tag{6.121}$$

The rest is obtained by hermitian conjugation.

Introducing

$$Q_j = \frac{1}{\sqrt{2}}(a_j^+ + a_j)\,, \quad P_j = \frac{i}{\sqrt{2}}(a_j^+ - a_j)\,,\ j = 1, 2, \tag{6.122}$$

and

$$Q_\pm = Q_1 \pm Q_2 \ , \tag{6.123}$$

$$P_\pm = P_1 \pm P_2 \ , \tag{6.124}$$

$$[Q_+, P_+] = 2\,i \ , \tag{6.125}$$

$$[Q_-, P_-] = 2\,i \ , \tag{6.126}$$

$$[Q_+, P_-] = 0 \ , \tag{6.127}$$

$$[Q_-, P_+] = 0 \ , \tag{6.128}$$

implies the squeezing relations

$$e^{-i\,w_1\,K_2} Q_+\, e^{i\,w_1\,K_2} = e^{w_1/2}\, Q_+ \ , \tag{6.129}$$
$$e^{-i\,w_1\,K_2} P_+\, e^{i\,w_1\,K_2} = e^{-w_1/2}\, P_+ \ , \tag{6.130}$$
$$e^{-i\,w_1\,K_2} Q_-\, e^{i\,w_1\,K_2} = e^{-w_1/2}\, Q_- \ , \tag{6.131}$$
$$e^{-i\,w_1\,K_2} P_-\, e^{i\,w_1\,K_2} = e^{w_1/2}\, P_- \ . \tag{6.132}$$

The combinations $Q_\pm\,, P_\pm$ of the two original pairs $(a_1\,, a_1^+)\,, (a_2, a_2^+)$ occur in the discussions [72] of the problem how to measure the values of a non-commuting pair of canonically conjugate observables like, e.g. Q_1 and P_1. The second pair $(Q_2\,, P_2)$ is related to properties of the measuring device. The discussions exploit the fact that the operators Q_- and P_+ (or Q_+ and P_-) commute and may be measured simultaneously with arbitrary precision. The squeezing relations 6.129-6.132 underline this interpretation.

For the product coherent state

$$|\alpha_1, \alpha_2\rangle = |\alpha_1\rangle \otimes |\alpha_2\rangle \tag{6.133}$$

we get the squared uncertainties

$$(\Delta Q_-)^2_{\alpha_1,\alpha_2} = 1 \ , \ (\Delta P_+)^2_{\alpha_1,\alpha_2} = 1 \ , \tag{6.134}$$

According to the relations 6.131 and 6.130 we may "squeeze" these uncertainties and make them *simultaneously* arbitrary small by an appropriate choice of the factor $\exp(-w_1)$!

The value 1 of the squared fluctuations 6.134 instead of $1/2$, which one has for the basic 1-mode case, is a consequence of the definitions 6.123 and 6.124 of $Q_\pm$ and $P_\pm$. A factor $1/\sqrt{2}$ would have yielded $1/2$ instead of 1 for the squared uncertainties 6.134. Such a "renormalization" may appear possible for the coordinates $Q_\pm$ but as $P_+ = P_1 + P_2$ is the the total momentum of the two modes one should not change its normalization arbitrarily.

6.5 Related applications

There are a number of other applications of the groups $SU(1,1) \cong SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$ which I merely mention here without going into details:

1. The Lie algebra realization 1.104 in order to construct multi-boson squeezed states [180].

2. The group $SU(1,1)$ plays also a prominant role in discussions of Mach-Zehnder type interferometers [181].

6.6 A few remarks on $SU(1,1)/SO^{\uparrow}(1,2)$ dynamics

Up to now I have discussed only "kinematical" aspects of the structure group $S(1,1)$ etc. in quantum optics. The relations discussed in the previous sections become physically much more interesting if the group parameters $w_j\,,\ j = 1,2\,,\ \tau$ etc. become dynamical, i.e. functions of time determined by an appropriate Hamiltonian which then generates squeezed or other states. This can be implemented as follows:

The Lie algebra $\mathfrak{so}(1,2)$ realizations 6.76 and 6.112 appear in physically important effective interaction Hamiltonians for nonlinear quantum optical processes [182].

Those interaction Hamiltonians in general take the form

$$H_I = \chi K_+ + \bar{\chi} K_- \ , \tag{6.135}$$

where the complex quantity χ contains the time-dependence of H_I and the amplitude(s) and non-linear susceptibilities of the classical source(s) ("pump") to which those quanta (photons) couple which are described by the creation and annihilation operators occuring in the Lie algebra operators 6.76 and 6.112. The free Hamiltonian H_0 in $H = H_0 + H_I$ is proportional to K_0.

The case 6.76 is called "degenerate" because both quanta appearing in K_+ etc. have the same frequency ω . They may either be annihilated, resulting in *one* quantum (better: classical wave) with frequency 2ω ("harmonic generation"), or created by a classical wave of frequency 2ω ("parametric down-conversion").

The effective interaction Hamiltonian has the same "degenerate" structure 6.135 if one has two incoming or outgoing classical waves ("four-wave mixing") both with frequency ω which interact with the two quanta. The relevant interaction properties of the two classical waves and the nonlinear medium are again incorporated into the quantity χ.

Qualitatively one encounters the same physical situations in the non-degenerate case 6.112 with the essential difference that one has now 2 possible modes which can have different frequencies ω_1 and ω_2, so that e.g. in non-degenerate parametric down-conversions a wave of frequency 2ω generates 2 quanta with frequencies ω_1 and ω_2 such that $\omega_1 + \omega_2 = 2\omega$.

What is especially important in all these processes is the property of the self-adjoint operator 6.135 to generated squeezed states as we have seen in detail in Sec. 6.2. The crucial point is that the interaction Hamiltonian 6.135 is an element of the Lie algebra $\mathfrak{so}(1,2)$. In addition, the free Hamiltonian H_0 in general will be built from K_0. Thus, the Lie algebra $\mathfrak{so}(1,2)$ plays an essential role in the *dynamics* of quantum optical processes, too!

Chapter 7

(Pseudo-) Probability distributions on the phase space $\mathcal{S}^2_{\varphi,I}$

7.1 Preliminaries

A lot of efforts has been put into attempts to find ways for expressing quantum mechanical expectation values

$$\langle A \rangle_\rho = \mathrm{tr}(\rho\, A) \tag{7.1}$$

of a quantum observable (self-adjoint operator) A with respect to a state - characterized by the density operator ρ - in terms of a classical density function $w(q,p)$ on, e.g. the phase space

$$\mathcal{S}^2_{q,p} = \{(q,p) \in \mathbb{R}^2\}\,, \tag{7.2}$$

and a phase space function $\tilde{A}(q,p)$ (corresponding to the operator A) such that

$$\langle A \rangle_\rho = \int_{\mathcal{S}_{(q,p)}} dq\, dp\, w(q,p)\, \tilde{A}(q,p)\ . \tag{7.3}$$

The oldest proposals are those of Weyl [36] and especially Wigner [183]. The corresponding Wigner function

$$w(q,p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{-ipx}\, \langle q + x/2|\rho|q - x/2\rangle \tag{7.4}$$

is widely used in the modern quantum optical literature (see the Refs. [11–19], especially Ref. [18]; additional reviews are [184–190]).

Because of its intensive use of Fourier transformations that approach is closely related to the phase space 7.2, its global structure and the associated harmonic analysis in terms of the translation groups in coordinate and momentum space. As the density 7.4 may become negative, it is actually not a genuine probability density. Its negative values in certain regions of the phase space 7.2 are usually attributed to typical quantum effects.

Inspired by - among others - a paper by Uhlenbeck [191] Husimi in 1940 published a long and very interesting article on various properties of the density matrix [192] in which he suggested to use the "diagonal" matrix elements

$$Q(\alpha, \bar{\alpha}) = \langle \alpha|\rho|\alpha\rangle\ . \tag{7.5}$$

of the "coherent" states $|\alpha = (q + ip)/\sqrt{2}\rangle$ for a reconstruction of the density operator. Husimi used Gaussian wave packets and recognized the non-orthogonality 3.64 of different such states.

The distribution function 7.5 is always non-negative. It was later rediscovered [193].

Of special interest has been another highly singular and non-positive "distribution function" associated with the density operator, the so-called "Sudarshan-Glauber diagonal P-representation" [194, 196] in terms of the coherent states $|\alpha\rangle$:

$$\rho = \frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha \, P(\alpha, \bar{\alpha}) \, |\alpha\rangle\langle\alpha| \ . \tag{7.6}$$

If it exists then the expectation values of appropriate operators B are given by the convenient expression

$$\mathrm{tr}(\rho \, B) = \int_{\mathbb{C}} d^2\alpha \, P(\alpha, \bar{\alpha}) \, \langle\alpha|B|\alpha\rangle \ . \tag{7.7}$$

In general $P(\alpha, \bar{\alpha})$ can be negative in certain regions of the phase space and highly singular [194,196,197,199,200], namely a generalized function (linear functional) belonging to the space Z', which contains, e.g. infinite series where the n-th term is proportional to the n-th derivative of a delta-function [201,202]!

Expressing the Wigner function $w(q,p)$ in terms of the complex variables $\alpha, \bar{\alpha}$,

$$w(\alpha, \bar{\alpha}) = \frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha_1 \ \langle \alpha - \alpha_1 | \rho | \alpha + \alpha_1 \rangle \, e^{\alpha \, \bar{\alpha}_1 - \bar{\alpha} \, \alpha_1} \ , \tag{7.8}$$

one gets the following relations between the densities $w(\alpha, \bar{\alpha}), Q(\alpha, \bar{\alpha})$ and $P(\alpha, \bar{\alpha})$:

$$w(\alpha, \bar{\alpha}) \ = \ \frac{2}{\pi} \int_{\mathbb{C}} d^2\alpha_1 P(\alpha_1, \bar{\alpha}_1) \, e^{-2|\alpha_1 - \alpha|^2} \ , \tag{7.9}$$

$$Q(\alpha, \bar{\alpha}) \ = \ \frac{1}{\pi} \int_{\mathbb{C}} d^2\alpha_1 P(\alpha_1, \bar{\alpha}_1) \, e^{-|\alpha_1 - \alpha|^2} \ . \tag{7.10}$$

Before I generalize essential properties of the $|\alpha\rangle$-related densities $Q(\alpha, \bar{\alpha})$ and $P(\alpha, \bar{\alpha})$ to those associated with the states $|k, z\rangle$ and $|k, \lambda\rangle$ - a corresponding generalization of the Wigner function is not obvious, at least not to me - let me make some general remarks:

We have seen - already in the Introduction - that the three appropriate basic "canonical" functions on the phase space 1.8 are

$$h_0 = I \, , \ h_1 = I \cos\varphi \, , \ h_2 = -I \sin\varphi \, .$$

Thus, one would start by considering classical probability densities and "observables" like

$$w(h_0, h_1, h_2) \ , \ G(h_0, h_1, h_2) \ . \tag{7.11}$$

However, as the three variables h_j are not independent,

$$h_0^2 - h_1^2 - h_2^2 = 0 \ ,$$

the corresponding integration measure is

$$\delta(h_0^2 - h_1^2 - h_2^2) dh_0 dh_1 dh_2 \ . \tag{7.12}$$

As

$$\delta(h_0^2 - h_1^2 - h_2^2) = \frac{1}{2\sqrt{h_1^2 + h_2^2}} \left[\delta(h_0 = \sqrt{h_1^2 + h_2^2}) + \delta(h_0 = -\sqrt{h_1^2 + h_2^2})\right] \tag{7.13}$$

and because $h_0 > 0$ only the first delta-function contributes. In this way the measure 7.12 reduces to (see Eq. 1.6)

$$\frac{1}{2\sqrt{h_1^2+h_2^2}} dh_1 dh_2 = \frac{1}{2}\, d\varphi dI = \frac{1}{2} dqdp \ . \tag{7.14}$$

One just has to replace h_0 in w and G of Eq. 7.11 by $\sqrt{h_1^2+h_2^2}$.

Obviously, the measure 7.14 is equivalent to the canonical measure $dqdp = d\varphi dI$ on phase space.

The situation is more involved for the quantized theories: For an irreducible unitary representation with index k we have - as mentioned several times before - the operator relation

$$K_1^2 + K_2^2 = K_0^2 + l \ , \ l = k(1-k) \ . \tag{7.15}$$

Thus, except for the case $k = 1$, the quantum fluctuations (k characterizes the non-classical ground state) modify the classical Pythagorean relation $h_1^2 + h_2^2 = h_0^2$.

If one treats the relation 7.15 as a constraint - together with $\mathrm{tr}(\rho) = 1$ - in order to determine the density operator ρ by maximizing the associated entropy [203]

$$S = -\mathrm{tr}(\rho \ln \rho) \ , \tag{7.16}$$

subject to the two constraints, then one gets

$$\rho = \frac{e^{-\gamma\,(K_1^2+K_2^2-K_3^2)}}{Z(\gamma)} \ , \ Z(\gamma) = -\ln(\mathrm{tr}[e^{-\gamma\,(K_1^2+K_2^2-K_3^2)}]) \ . \tag{7.17}$$

In possible applications the relation

$$K_1^2 + K_2^2 - K_0^2 = K_+K_- - K_0^2 + K_0 \tag{7.18}$$

will be useful.

I will not pursue this ansatz further here - if l is fixed, a "microcanonical" approach will be more appropriate - and turn to the indicated generalizations of the expressions 7.5 and 7.6. By doing so I shall just scratch the surface of the underlying structures and the substance of the many problems. The main purpose is to point out possible directions for future research. What is needed is a thorough investigation along the lines of Ref. [204] for the conventional coherent states. Several points of the present chapter appear to be related to those of Ref. [205].

7.2 Pseudo-probability distributions associated with Barut-Girardello coherent states

Many concepts developed in connection with pseudo-probability distributions (densities) related to the conventional coherent states $|\alpha\rangle$ may be carried over to the states $|k,z\rangle$. There are, however, at least two important differences:

First, contrary to the states $|\alpha\rangle = D(\alpha)|0\rangle$ which can be generated form the ground state by a unitary (translation) operator - Eq. 3.133 -, this appears not to be the case for the states $|k,z\rangle$.

Second, whereas the measure $d^2\alpha$ is invariant under rotations and translations, the measure $d\mu_k(z)$ - Eq. 3.18 - is invariant only under rotations. The same applies to the measure $\tilde{\mu}_k(z)$ from Eq. 3.23.

In the following I use the notations and definitions from Sec. 3.1.

Let

$$A = \sum_{m,n=0}^{\infty} |k,m\rangle A_{k;m\,n} \langle k,n| \tag{7.19}$$

be a self-adjoint operator in the number state representation 2.11. Then

$$\begin{aligned} \langle k,z_2|A|k,z_1\rangle &= [g_k(|z_2|^2)\, g_k(|z_1|^2)]^{-1/2}\, A(k;\bar{z}_2,z_1)\ , & (7.20)\\ A(k;\bar{z}_2,z_1) &= \sum_{m,n=0}^{\infty} \bar{\tilde{f}}_{k,m}(z_2)\, A_{k;m\,n}\, \tilde{f}_{k,n}(z_1) = & (7.21)\\ &= \sum_{m,n=0}^{\infty} \frac{A_{k;m\,n}}{\sqrt{m!(2k)_m\, n!(2k)_n}} \bar{z}_2^m\, z_1^n\ . \end{aligned}$$

Self-adjointness of $A\,, A^+ = A$, implies

$$\bar{A}(k;\bar{z}_2,z_1) = A(k;\bar{z}_1,z_2)\ . \tag{7.22}$$

The expression 7.21 shows that $A(k;\bar{z}_2,z_1)$ is a double series, holomorphic in z_1 and anti-holomorphic in $\bar{z}_2$. Because of the additional factors $\sqrt{(2k)_m\,(2k)_n}$ in the denominator which for large $m\,,n$ behave like $\sqrt{m!\,n!}$, the convergence of the series 7.21 is better than the corresponding one for the states $|\alpha\rangle$ [194, 196].

Using the orthonormality 3.27 of the functions $\tilde{f}_{k,m}$ we get the inversion

$$A_{k;m\,n} = \int_{\mathbb{C}}\int_{\mathbb{C}} d\tilde{\mu}_k(z_2)\, d\tilde{\mu}_k(z_1)\, \bar{\tilde{f}}_{k,m}(z_2)\, A(k;\bar{z}_2,z_1)\, \bar{\tilde{f}}_{k,n}(z_1)\ . \tag{7.23}$$

From the completeness relation 3.17 we have

$$\begin{aligned} A &= \int_{\mathbb{C}}\int_{\mathbb{C}} d\mu_k(z_2)\, d\mu_k(z_1)\, |k,z_2\rangle\langle k,z_2|A|k,z_1\rangle\langle k,z_1| = & (7.24)\\ &= \int_{\mathbb{C}}\int_{\mathbb{C}} d\tilde{\mu}_k(z_2)\, d\tilde{\mu}_k(z_1) A(k;\bar{z}_2,z_1)\, |k,z_2\rangle\langle k,z_1|\ . & (7.25) \end{aligned}$$

For the product $A_2 \cdot A_1$ of two operators we obtain

$$(A_2\cdot A_1)(k;\bar{z}_2,z_1) = \int_{\mathbb{C}} d\tilde{\mu}_k(z)\, A_2(k;\bar{z}_2,z)\, A_1(k;\bar{z},z_1)\ . \tag{7.26}$$

Let us apply these relations to the density operator

$$\rho = \sum_{m,n=0}^{\infty} |k,m\rangle \rho_{k;m\,n} \langle k,n| \tag{7.27}$$

with its special properties.

$$\begin{aligned} \mathrm{tr}(\rho) = \sum_{n=0}^{\infty} \rho_{k;n\,n} &= \int_{\mathbb{C}}\int_{\mathbb{C}} d\tilde{\mu}_k(z_2)\, d\tilde{\mu}_k(z_1)\, \tilde{\Delta}_k(z_2,\bar{z}_1)\, \rho(k;\bar{z}_2,z_1) = & (7.28)\\ &= \int_{\mathbb{C}} d\tilde{\mu}_k(z_1) \rho(k;\bar{z}_1,z_1) = 1\ . & (7.29) \end{aligned}$$

Eqs. 7.23 with $A = \rho$ and 3.28 were used here and in the last step use was made of the fact that $\tilde{\Delta}_k(z_2, \bar{z}_1)$ is a reproducing kernel (Eq. 3.33) and $\rho(\bar{z}_2, z_1)$ an anti-holomorphic function in $\bar{z}_2$.

We may get the same result more directly from Eq. 7.20:

$$\mathrm{tr}(\rho) = \int_{\mathbb{C}} d\mu_k(z) \langle k, z|\rho|k, z\rangle = \int_{\mathbb{C}} d\tilde{\mu}_k(z)\, \rho(k; \bar{z}, z) \ . \tag{7.30}$$

From the positivity of the (Husimi-type) function

$$S_k(z, \bar{z}) \equiv \langle k, z|\rho|k, z\rangle \geq 0 \tag{7.31}$$

and 7.20 it follows further that

$$\rho(k; \bar{z}, z) \geq 0 \ . \tag{7.32}$$

The expectation value of a self-adjoint operator A is given by

$$\langle A\rangle_{\rho;k,z} = \mathrm{tr}(\rho\, A) = \int_{\mathbb{C}} \int_{\mathbb{C}} d\tilde{\mu}_k(z_2)\, d\tilde{\mu}_k(z_1) \rho(k; \bar{z}_2, z_1)\, A(k; \bar{z}_1, z_2) \ . \tag{7.33}$$

Examples:

$$\rho = |k, n\rangle\langle k, n| \ . \tag{7.34}$$

For this density operator we get

$$\langle k, z_2|\rho|k, z_1\rangle = \frac{\rho(k; \bar{z}_2, z_1)}{\sqrt{g_k(|z_2|^2)\, g_k(|z_1|^2)}} \tag{7.35}$$

$$\rho(k; \bar{z}_2, z_1) = \bar{\tilde{f}}_{k,n}(z_2)\, \tilde{f}_{k,n}(z_1) = \frac{(\bar{z}_2\, z_1)^n}{(2k)_n\, n!} \ . \tag{7.36}$$

For

$$\rho = (1 - a) \sum_{n=0}^{\infty} a^n\, |k, n\rangle\langle k, n| \ , \ 0 < a < 1 \ , \tag{7.37}$$

we have

$$\langle k, z_2|\rho|k, z_1\rangle = \frac{1 - a}{\sqrt{g_k(|z_2|^2)\, g_k(|z_1|^2)}}\, g_k(a\, \bar{z}_2\, z_1) \ . \tag{7.38}$$

This implies that

$$S_k(z, \bar{z}) \equiv \langle k, z|\rho|k, z\rangle = (1 - a)\, \frac{g_k(a\, |z|^2)}{g_k(|z|^2)} \ . \tag{7.39}$$

We further note the relation

$$\int_0^{2\pi} d\phi\ |k, |z|e^{i\,\phi}\rangle\langle k, |z|e^{i\,\phi}| = \frac{2\pi}{g_k(|z|^2)} \sum_{n=0}^{\infty} \frac{|z|^{2\,n}}{(2k)_n\, n!}\, |k, n\rangle\langle k, n| \ . \tag{7.40}$$

Next let us suppose that a certain class of density operators allows for a so-called "diagonal representation" [194, 196, 199]

$$\rho = \int_{\mathbb{C}} d\mu_k(z)\, F_k(z, \bar{z})|k, z\rangle\langle k, z| \ . \tag{7.41}$$

Such a representation will in general require F_k to be a generalized function of type Z', see below.

Because ρ is self-adjoint, F_k has to be real. Furthermore, in general F_k will depend on both variables z and $\bar{z}$.

(Note the frequent change of the measures $d\mu_k(z)$ and $d\tilde{\mu}_k(z)$ in the following!)

If the representation 7.41 is available, it has a number of interesting properties, e.g.

$$\begin{aligned} \mathrm{tr}(\rho\, A) &= \int_{\mathbb{C}} d\mu_k(z_1) \int_{\mathbb{C}} d\mu_k(z)\, F_k(z,\bar{z})\, \langle k, z_1|k, z\rangle\langle k, z|A|k, z_1\rangle = & (7.42)\\ &= \int_{\mathbb{C}} d\tilde{\mu}_k(z)\, F_k(z,\bar{z})\, A(k,\bar{z},z)\ , & (7.43) \end{aligned}$$

where the completeness relation 3.17 has been used.

If we put $A = \mathbf{1}$, then we get the normalization

$$\mathrm{tr}(\rho) = \int_{\mathbb{C}} d\mu_k(z)\, F_k(z,\bar{z}) = 1\ . \tag{7.44}$$

The diagonal representation 7.41 of the density operator is especially convenient as to operators of the type

$$B_N = \sum_{n_+,n_-=0}^{N_+,N_-} b_{n_+,n_-}\, K_+^{n_+}\, K_-^{n_-}\ , \tag{7.45}$$

because

$$\mathrm{tr}(\rho B_N) = \int_{\mathbb{C}} d\mu_k(z)\, F_k(z,\bar{z}) \sum_{n_+,n_-=0}^{N_+,N_-} b_{n_+,n_-}\, \bar{z}^{n_+}\, z^{n_-}\ . \tag{7.46}$$

This is in complete analogy to the usual normal-ordered polynomials of the operators a^+ and a applied to the states $|\alpha\rangle$.

However, normal-ordering is more complicated in the framework of our Lie algebra, because we have a third operator K_0! Here "normal-ordering" may be defined [91, 264] to mean that all the K_- are to be put to the right in a product, all the K_+ to the left and the K_0 in the middle. In rearranging a given product the following relations have to be observed:

$$\begin{aligned} K_-\, K_+ &= K_+\, K_- + 2\, K_0 = & (7.47)\\ &= l + K_0\,(K_0+1)\ , & (7.48)\\ K_0\, K_+ &= K_+\,(K_0+1)\ , & (7.49)\\ K_-\, K_0 &= (K_0+1)\, K_-\ . & (7.50) \end{aligned}$$

They follow from the Eqs. 2.7 and 2.14 The relation 7.48 holds only within an irreducible representation with a value $l = k(1-k)$ of the Casimir operator.

An example for normal ordering is the relation 3.126. Its anti-normally ordered version is [91, 264]

$$U(w) = e^{-\bar{\lambda}\, K_-}\ e^{-\ln(1-|\lambda|^2) K_0}\ e^{\lambda\, K_+}\ . \tag{7.51}$$

Anti-normal ordering here obviously means to put all the K_+ to the right, the K_- to the left and the K_0 again in the middle.

Normal-ordering of more general operators like

$$e^{a_+ K_+ + a_- K_- + a_3 K_0} \tag{7.52}$$

have also been discussed in the literature [91, 111, 214, 215], the most extensive discussion of normal and anti-normal ordering in the present context being contained in Ref. [91].

The expectation value of anti-normally ordered operators like, e.g.

$$B_A = \sum_{n_+,n_-=0}^{N_+,N_-} b_{n_+,n_-}\, K_-^{n_-}\, K_+^{n_+} \ , \tag{7.53}$$

can be expressed in terms of the Husimi type function $S_k(z,\bar{z})$ from Eq. 7.31:

$$\begin{aligned} \mathrm{tr}(\rho B_A) &= \int_{\mathbb{C}} d\mu_k(z)\, \langle k,z|\rho\, B_A|k,z\rangle = \\ &= \int_{\mathbb{C}} d\mu_k(z)\, \langle k,z|\rho|k,z\rangle \sum_{n_+,n_-=0}^{N_+,N_-} b_{n_+,n_-}\, \bar{z}^{n_+}\, z^{n_-} \ . \end{aligned} \tag{7.54}$$

Here the cyclic property of the trace has been used.

If a density operator has the anti-normally ordered form

$$\rho^{(A)} = \sum_{n_+,n_-=0}^{N_+,N_-} \rho^{(A)}_{n_+,n_-}\, K_-^{n_-}\, K_+^{n_+} \ , \tag{7.55}$$

then inserting the completeness relation 3.17 between the two types of operators gives

$$\rho^{(A)} = \int_{\mathbb{C}} d\mu_k(z) \sum_{n_+,n_-=0}^{N_+,N_-} \rho^{(A)}_{n_+,n_-}\, \bar{z}^{n_+}\, z^{n_-} |k,z\rangle\langle k,z| \ , \tag{7.56}$$

i.e. for such density operators the diagonal representation 7.41 exists, at least formally.

Again, formally the density operator

$$|k,z_0\rangle\langle k,z_0| \tag{7.57}$$

may be expressed in terms of the relation

$$m_k(|z|)\, F_k(z,\bar{z}) = \delta(z-z_0) \ , \tag{7.58}$$

where $\delta(z-z_0)$ is the 2-dimensional delta-function with respect to the measure d^2z .

For $k = 1/2$ one has to take care of the logarithmic singularity 3.16. One has [216]

$$K_0(2|z|) == -(\gamma + \ln|z|)\, I_0(2|z|) + 2\sum_{n=1}^{\infty} \frac{1}{n}\, I_{2n}(2|z|) \ , \tag{7.59}$$

so that

$$\hat{K}_0(|z|) \equiv -(\, K_0(2|z|)/(\gamma + \ln|z|) \to 1 \text{ for } |z| \to 0 \tag{7.60}$$

Similarly one can make use of more singular generalized functions in order to find a $F_k(z,\bar{z})$ which yields a diagonal representation for a given density operator. Following Sudarshan [194] one can be tempted to postulate the following "duality" between powers of $r = |z|$ and derivatives of the radial δ-functions $\delta(r)$:

$$(-1)^{n_1} \int_0^{\infty} dr\, r^{n_2}\, \delta^{(n_1)}(r) = n_2!\, \delta_{n_1 n_2} \ . \tag{7.61}$$

Here $\delta^{(n)}(r)$ is the n-th derivative with respect to r. The relation is quite formal because r^{n_2} is not a test function of any of the spaces $D\,, S$ or Z [201, 202] . Actually, r^{λ} is a generalized function $\in S'$ or D' itself [201, 202]!

As such a generalized function the expression $r^{n_2}\,\delta^{(n_1)}(r)$ has the property [217]

$$r^{n_2}\,\delta^{(n_1)}(r) = \begin{cases} 0 & : \quad n_1 < n_2 \ , \\ (-1)^{n_2}\, n_2!\, \delta(r) & : \quad n_2 = n_1 \ , \\ (-1)^{n_2}\, \frac{n_1!}{(n_1-n_2)!}\, \delta^{(n_1-n_2)}(r) & : \quad n_1 > n_2 \ . \end{cases} \tag{7.62}$$

This is compatible with the postulate 7.61 if we apply the generalized function 7.62 to test functions which have the constant value 1 in a compact interval $0 \le r \le a > 0$ and vanish for $r \ge b > a$. Such functions exist [218]. They should have vanishing derivatives of arbitrary order at $r = 0$!

As to different approaches to a mathematical "solidification" of the heuristic ansatz 7.61 see the remarks below. At the moment I just use it:

From the number state representation 3.8 we get the formal expansion

$$|k,z\rangle\langle k,z| = \frac{1}{g_k(|z|^2)} \sum_{\bar{n},n=0}^{\infty} \frac{|z|^{\bar{n}+n}\, e^{i\,(n-\bar{n})\phi}}{\sqrt{(2k)_{\bar{n}}\,(2k)_n\,\bar{n}!\,n!}}\, |k,\bar{n}\rangle\langle k,n| \ . \tag{7.63}$$

It may be reproduced by the ansatz

$$|z|\, m_k(|z|)\, F_k(z,\bar{z}) = \frac{1}{2\pi} \sum_{\bar{m},m=0}^{\infty} \frac{\langle k,\bar{m}|\rho|k,m\rangle}{(\bar{m}+m)!} \sqrt{(2k)_{\bar{m}}\,(2k)_m\,\bar{m}!\,m!}$$
$$(-1)^{\bar{m}+m}\, \delta^{(\bar{m}+m)}(|z|)\, e^{i\,(\bar{m}-m)\phi} \ , \tag{7.64}$$

which formally leads to the desired result:

$$\rho = \int_{\mathbb{C}} d\mu_k(z) F_k(z,\bar{z})\, |k,z\rangle\langle k,z| = \sum_{\bar{m},m=0} \langle k,\bar{m}|\rho|k,n\rangle\, |k,\bar{m}\rangle\langle k,m| \ . \tag{7.65}$$

Here the integration over ϕ impose the condition $m + \bar{n} = \bar{m} + n$ and the integration over $|z|$, according to Eq. 7.61, $\bar{m} + m = \bar{n} + n$. Combined this gives $m = n, \bar{m} = \bar{n}$.

(Here $\bar{m}$ is an independent natural number like m, not the complex conjugate of m !)

If the series 7.64 does not terminate, the resulting generalized function will belong to the type Z' [201, 202] the associated test functions of which are Fourier transforms of the smooth functions with compact support, i.e. they are elements of the space D.

If ρ is diagonal in the number state basis,

$$\langle k,\bar{m}|\rho|k,m\rangle = \rho_{k,m}\, \delta_{\bar{m}\,m} \ , \tag{7.66}$$

then

$$|z|\, m_k(|z|)\, F_k(z,\bar{z}) = \frac{1}{2\pi} \sum_{m=0}^{\infty} \frac{\rho_{k,m}}{(2m)!}\, (2k)_m\, m!\, \delta^{(2m)}(|z|) \ . \tag{7.67}$$

Especially for

$$\rho = |k,n\rangle\langle k,n| \tag{7.68}$$

we have

$$|z|\, m_k(|z|)\, F_k(z,\bar{z}) = \frac{1}{2\pi} \frac{(2k)_m\, m!}{(2m)!}\, \delta^{(2m)}(|z|) \ . \tag{7.69}$$

Remarks:

The first paper of Sudarshan [194] with its formal use of the relation 7.61 led to a number of articles which discussed appropriate mathematical approaches:

As generalized functions may be considered as limits of continuous functions (I leave out the technical details!), Klauder et al. [197, 199] and Rocca [206] clarified this possibility for a solid mathematical framework. Miller and Mishkin [200] discussed the meaning of such infinite series like 7.64 as generalized functions of the space Z'.

An interesting regularization in terms of Laguerre polynomials with appropriate limits was introduced by Peřina and Mišta [207]. The possibility of using polynomials as testfunctions was discussed by Lukš [208], following another paper by Peřina [209]. That approach later was also briefly pointed out by Sudarshan [195]. The subject has been analyzed recently again by Wünsche [210] and Richter [211].

Another tool for finding the quantities F_k and S_k is Fourier transformation [198, 213]. The 2-dimensional Fourier transform for complex variables w and z can be formulated as follows

$$f(w) = \frac{1}{\pi}\int_{\mathbb{C}} d^2z\, e^{w\,\bar{z}-\bar{w}\,z}\,\tilde{f}(z)\ , \tag{7.70}$$

$$\tilde{f}(z) = \frac{1}{\pi}\int_{\mathbb{C}} d^2w\, e^{-w\,\bar{z}+\bar{w}\,z}\,f(w)\ ; \tag{7.71}$$

$$\delta(w) = \delta(\Re(w))\,\delta(\Im(w)) = \frac{1}{\pi^2}\int_{\mathbb{C}} d^2z\, e^{w\,\bar{z}-\bar{w}\,z}\ , \tag{7.72}$$

$$w\,\bar{z}-\bar{w}\,z = 2i[\Re(z)\,\Im(w)-\Re(w)\,\Im(z)]\ . \tag{7.73}$$

One may define a "normally ordered" characteristic function $\chi_N(w,\bar{w})$ by

$$\chi_N(w,\bar{w}) = \mathrm{tr}(\rho\, e^{w\,K_+}\,e^{-\bar{w}\,K_-})\ . \tag{7.74}$$

Implementing the trace in terms of number states $|k,n\rangle$, inserting for ρ the diagonal representation 7.41 and using the completeness relation 2.21 gives

$$\chi_N(w,\bar{w}) = \int_{\mathbb{C}} d\mu_k(z) F_k(z,\bar{z})\, e^{w\,\bar{z}-\bar{w}\,z}\ . \tag{7.75}$$

Fourier transforming yields

$$m_k(|z|)\,F_k(z,\bar{z}) = \frac{1}{\pi^2}\int_{\mathbb{C}} d^2w\,\chi_N(w,\bar{w})\,e^{-w\,\bar{z}+\bar{w}\,z}\ . \tag{7.76}$$

Without the use of the diagonal representation the characteristic function 7.74 can also be evaluated by employing the completeness relation 3.17:

$$\chi_N(w,\bar{w}) = \int_{\mathbb{C}}\int_{\mathbb{C}} d\tilde{\mu}_k(z_2)\,d\tilde{\mu}_k(z_1)\,\rho(k;\bar{z}_2,z_1)\,g_k(\bar{z}_1 z_2)\,e^{w\,\bar{z}_1-\bar{w}\,z_2}\ . \tag{7.77}$$

The anti-normally ordered characteristic function is defined correspondingly:

$$\chi_A(w,\bar{w}) = \mathrm{tr}(\rho\, e^{-\bar{w}\,K_-}\,e^{w\,K_+})\ . \tag{7.78}$$

Again inserting the completeness relation 3.17 between the last two operators inside the trace gives

$$\chi_A(w,\bar{w}) = \int_{\mathbb{C}} d\mu_k(z)\,S_k(z,\bar{z})\,e^{w\,\bar{z}-\bar{w}\,z}\ . \tag{7.79}$$

By Fourier transforming the last equation one can express S_k in terms of χ_A.

From the diagonal representation 7.41 we get the following relation between the quantities F_k and S_k:

$$S_k(z,\bar{z}) \equiv \langle k,z|\rho|k,z\rangle = \int_{\mathbb{C}} d\mu_k(z_1)\, F_k(z_1,\bar{z}_1)\, |\langle k,z|k,z_1\rangle|^2 \tag{7.80}$$

$$= \frac{1}{g_k(|z|^2)} \int_{\mathbb{C}} d\tilde{\mu}_k(z_1)\, F_k(z_1,\bar{z}_1)\, |g_k(\bar{z}_1 z)|^2 \ . \tag{7.81}$$

The kernel $|g_k(\bar{z}_2\, z_1)|^2 = g_k(\bar{z}_2 z_1)\, g_k(z_2 \bar{z}_1)$ may be rewritten as [219]

$$|g_k(\bar{z}_2 z_1)|^2 = \sum_{n=0}^{\infty} \frac{(\bar{z}_2 z_1)^n}{(2k)_n\, n!}\, {}_2F_1(-n,-n-(2k-1);2k;\frac{\bar{z}_2 z_1}{z_2\bar{z}_1}) \ . \tag{7.82}$$

The hypergeometric functions in the sum are essentially Gegenbauer polynomials $C_n^{2k-1/2}$ [220]:

$${}_2F_1(-n,-n-(2k-1);2k;b) = \frac{n!}{(4k-1)_n}(1-b)^n\, C_n^{2k-1/2}(\frac{1+b}{1-b}) \tag{7.83}$$

Whether this is of any use remains to be seen!

From the representation 7.41 we further get (see Eq. 3.221)

$$Q_k(\alpha,\bar{\alpha}) = \langle k,\alpha|\rho|k,\alpha\rangle = \int_{\mathbb{C}} d\mu_k(z)\, F_k(z,\bar{z})\, |\langle k,\alpha|k,z\rangle|^2 \tag{7.84}$$

$$= e^{-|\alpha|^2} \int_{\mathbb{C}} d\tilde{\mu}_k(z)\, F_k(z,\bar{z})\, |C_k(\bar{\alpha};z)|^2 \ .$$

Analogously one derives

$$S_k(z.\bar{z}) = \langle k,z|\rho|k,z\rangle = \int_{\mathbb{C}} d^2\, P_k(\alpha,\bar{\alpha})\, |\langle k,z|k,\alpha\rangle|^2 \tag{7.85}$$

$$= \frac{1}{g_k(|z|^2)} \int_{\mathbb{C}} d^2\alpha\, P_k(\alpha,\bar{\alpha})\, e^{-|\alpha|^2}\, |C_k(\bar{\alpha};z)|^2 \ .$$

Additional relations may be obtained from the equality

$$\rho = \int_{\mathbb{C}} d\mu_k(z)\, F_k(z,\bar{z})\, |k,z\rangle\langle k,z| = \int_{\mathbb{C}} d\mu_k(\alpha)\, P_k(\alpha,\bar{\alpha})\, |k,\alpha\rangle\langle k,\alpha| \ . \tag{7.86}$$

In the case of the conventional coherent states $|\alpha\rangle$ the "symmetric" characteristic function

$$\chi_S(\alpha,\bar{\alpha}) = \mathrm{tr}(\rho\, e^{\alpha\, a^+ - \bar{\alpha}\, a}) \tag{7.87}$$

is the Fourier transform of the Wigner function $w(\alpha,\bar{\alpha})$.

Here the situation is more complicated: If we define

$$\chi_S(w,\bar{w}) = \mathrm{tr}(\rho\, e^{w\, K_+ - \bar{w}\, K_-}) \ , \tag{7.88}$$

the trace may be evaluated with the help of the relations 3.126 and 3.17 as

$$\chi_S(w,\bar{w}) = \chi_S(\lambda,\bar{\lambda}) =$$

$$= (1-|\lambda|^2)^k \int_{\mathbb{C}}\int_{\mathbb{C}} d\tilde{\mu}_k(z_2)\tilde{\mu}_k(z_1)\, e^{\lambda\,\bar{z}_1 - \bar{\lambda}\, z_2} \times \tag{7.89}$$

$$\times\, g_k[(1-|\lambda|^2)\,\bar{z}_1 z_2]\, \langle k,z_2|\rho|k,z_1\rangle \ ,$$

$$\lambda = (w/|w|)\, \tanh|w| \ , \quad 1-|\lambda|^2 = 1/\cosh^2|w| \ , \tag{7.90}$$

It does not seem to be obvious which quantity would correspond to the Wigner function in the present framework. Progress may come from recent proposals for generalizations of that concept to more general Lie groups [221].

In any case one has to deal with the following mathematical problem: As $\lambda \in \mathbb{D}$, the integral transform 7.89 represents a mapping from $\mathbb{C}^2$ onto the unit disc $\mathbb{D}$. The associated Fourier transformation is more sophisticated than the usual one [222] and, unfortunately, beyond the scope of the present paper!

Using the representation 7.41 the chacteristic function 7.89 may also be expressed as

$$\chi_S(\lambda,\bar{\lambda}) = (1-|\lambda|^2)^k \int_{\mathbb{C}}\int_{\mathbb{C}} d\tilde{\mu}_k(z_2)\tilde{\mu}_k(z_1)\, e^{\lambda\,\bar{z}_1-\bar{\lambda}\,z_2}\cdot \\ \cdot F_k(z_1,\bar{z}_1)\, g_k(\bar{z}_2 z_1)\, g_k[(1-|\lambda|^2)\,\bar{z}_1 z_2]\ . \quad (7.91)$$

7.3 Pseudo-probability distributions associated with Perelomov coherent states

It is clear from Sec. 3.2 that the Perelomov coherent states have some qualitatively different properties compared to the ones just discussed. One of the main differences is that the states $|k,\lambda\rangle$ may be generated by the unitary operator 3.126. Another is that the complex numbers λ take only values in the unit disc $\mathbb{D}$. Third, the states $|k,\lambda\rangle$ are eigenstates of the operators $E_{k,-} = (K_0+k)^{-1}K_-$. Together with $E_{k,+} = (E_{k,-})^+$ these operators now play the role the operators K_- and K_+ had in the last section and the operators a and a^+ play in connection with the states $|\alpha\rangle$.

Partially one can go through the same routine as in the last section (in the following the notions of Sec. 3.2 are being used):

From the representation 7.19 and the relations 3.125 and 3.144 we get now

$$\langle k,\lambda_2|A|k,\lambda_1\rangle = (1-|\lambda_2|)^k\,(1-|\lambda_1|^2)^k\, A(k;\bar{\lambda}_2,\lambda_1)\ , \quad (7.92)$$

$$A(k;\bar{\lambda}_2,\lambda_1) = \sum_{m,n=0}^{\infty} \bar{e}_{k,m}(\lambda_2)\, A_{k;m\,n}\, e_{k,n}(\lambda_1) = \\ = \sum_{m,n=0}^{\infty} A_{k;m\,n}\sqrt{\frac{(2k)_m\,(2k)_n}{m!\,n!}}\,\bar{\lambda}_2^m\,\lambda_1^n\ . \quad (7.93)$$

Compared to the series 7.21 one has to realize that $|\lambda_j| < 1$, $j = 1,2$, and that the factor $\sqrt{(2k)_m\,(2k)_n}$ is now in the numerator instead in the nominator! Otherwise one may proceed formally as in the last section: replace the variables z_2, z_1 by λ_2, λ_1, the complex plane $\mathbb{C}$ by the unit disc $\mathbb{D}$, the eigenfunctions $\tilde{f}_{k,n}(z)$ by $e_{k,n}(\lambda)$ and the integration measures accordingly.

One can also define a Husimi type density

$$T_k(\lambda,\bar{\lambda}) \equiv \langle k,\lambda|\rho|k,\lambda\rangle \geq 0\ , \quad (7.94)$$

and a diagonal representation

$$\rho = \int_{\mathbb{D}} d\mu_k(\lambda)\, G_k(\lambda,\bar{\lambda})\, |k,\lambda\rangle\langle k,\lambda|\ . \quad (7.95)$$

Their relationship is

$$\begin{aligned} T_k(\lambda,\bar\lambda) &= \int_{\mathbb{D}} d\mu_k(\lambda_1)\, G_k(\lambda_1,\bar\lambda_1)\, |\langle k,\lambda|k,\lambda_1\rangle|^2 = \\ &= (1-|\lambda|^2)^{2k} \int_{\mathbb{D}} d\mu_k(\lambda_1)(1-|\lambda_1|^2)^{2k}\, G_k(\lambda_1,\bar\lambda_1)\times \\ &\qquad \times [(1-\bar\lambda\,\lambda_1)(1-\lambda\,\bar\lambda_1)]^{-2k}\,. \end{aligned} \tag{7.96}$$

With

$$\lambda = |\lambda| e^{i\theta}\,,\ \lambda_1 = |\lambda_1| e^{i\theta_1}\,,\ \cos(\theta-\theta_1) = t\,,\ |\lambda|\,|\lambda_1| = x\ , \tag{7.97}$$

we have

$$[(1-\bar\lambda\,\lambda_1)(1-\lambda\,\bar\lambda_1)]^{-2k} = (1-2\,t\,x+x^2)^{-2k} = \sum_{n=0}^{\infty} C_n^{2k}(t)\, x^n\ , \tag{7.98}$$

where the $C_n^{2k}(t)$ are again Gegenbauer polynomials [223].

As we have three different coherent states now in a given unitary representation with index k, a large variety of relations may be established (see the integral transforms in subsections 3.2.1 and 3.3.1). I briefly mention only a few examples:

$$S_k(z,\bar z) = \frac{1}{g_k(|z|^2)} \int_{\mathbb{D}} d\tilde\mu_k(\lambda)\, G_k(\lambda,\bar\lambda)\, e^{\bar\lambda\, z+\lambda\,\bar z}\ , \tag{7.99}$$

and the "inverse" relation

$$T_k(\lambda,\bar\lambda) = (1-|\lambda|^2)^{2k} \int_{\mathbb{C}} d\tilde\mu_k(z)\, F_k(z,\bar z)\, e^{\bar\lambda\, z+\lambda\,\bar z}\ . \tag{7.100}$$

As to possible Fourier transforms one may use the trick [224]

$$\langle k,-z|\rho|k,z\rangle = \frac{1}{g_k(|z|^2)} \int_{\mathbb{D}} d\tilde\mu_k(\lambda)\, G_k(\lambda,\bar\lambda)\, e^{\bar\lambda\, z-\lambda\,\bar z}\ , \tag{7.101}$$

where the exponential in the integrand is now that of a complex Fourier transform (see Eq. 7.70). Formally we get from 7.72

$$\begin{aligned} \frac{2k-1}{\pi}\,(1-|\lambda|^2)^{2k-2}\, G_k(\lambda,\bar\lambda) &= \frac{1}{\pi^2} \int_{\mathbb{C}} d^2z\, g_k(|z|^2)\, \langle k,-z|\rho|k,z\rangle\, e^{\lambda\,\bar z-\bar\lambda\, z} \;= \\ &= \frac{1}{\pi^2} \int_{\mathbb{C}} d^2z\, \rho(k;-\bar z,z)\, e^{\lambda\,\bar z-\bar\lambda\, z}\ , \end{aligned} \tag{7.102}$$

where in the last step the relation 7.20 has been used.
Thus, we obtain the quantity $G_k(\lambda,\bar\lambda)$ by Fourier transforming $\rho(k;-\bar z,z)$, a quantity which in general may be calculated readily. For instance, for the density operator 7.34 we have from 7.36 that

$$\rho(k;-\bar z,z) = (-1)^n\, \frac{|z|^{2n}}{(2k)_n\, n!}\ . \tag{7.103}$$

Inserting this into the integral 7.102 yields

$$\frac{2k-1}{\pi}\,(1-|\lambda|^2)^{2k-2}\, G_k(\lambda,\bar\lambda) = \frac{1}{2^{2n}\,(2k)_n\, n!}(\Delta_\lambda)^n\delta(\lambda)\ , \tag{7.104}$$

$$\begin{aligned} \Delta_\lambda &= \partial^2/\partial\sigma_1^2 + \partial^2/\partial\sigma_2^2\,, \\ &\quad \lambda = \sigma_1 + i\,\sigma_2\ . \end{aligned} \tag{7.105}$$

As

$$(\Delta_\lambda)^n |\lambda|^{2n+2m} = 2^{2n} \left[(m+1)_n\right]^2 |\lambda|^{2m} \ , m = 0, 1, \dots \ , \tag{7.106}$$

we see: For the number state projection operator 7.34 the pseudo-density G_k has a very similar singularity structure as the pseudo-density F_k (see Eq. 7.69) though the "derivations" have been different. The latter one shows: in order to obtain the projection operator 7.34 from the relations 7.104 and 7.106 by means of "integrating by parts" under the integral 7.95 one has to treat the powers of $|\lambda|^2$ like test functions (with compact support) when shifting the differential operator Δ_λ from the δ-functions to those powers. This is in line with the duality postulate 7.61.

An important point is, of course , that $|\lambda| < 1$ which makes the Fourier analysis more involved, as was already pointed out at the end of the last section. Take, e.g. instead of Eq. 7.100 the relation

$$\langle k, -\lambda | \rho | k, \lambda \rangle = (1 - |\lambda|^2)^{2k} \int_{\mathbb{C}} d\tilde{\mu}_k(z) \, F_k(z, \bar{z}) \, e^{-\bar{\lambda} z + \lambda \bar{z}} \ . \tag{7.107}$$

This Fourier integral cannot be inverted in the conventional manner but requires the more sophisticated means already mentioned before [222].

As the states $|k, \lambda\rangle$ are eigenstates of the operators $E_{k,-}$ one may deal with them (and $E_{k,+}$) similarly as with the operators K_- and K_+ in the last section. As to possible normal orderings one has to take into account the commutators 5.6 etc. I shall skip the details here.

If the operators $E_{k,-}$ and $E_{k,+}$ would yield convincing cos- and sin- operators 5.8 and 5.9 as discussed critically in Ch. 5, the applications of them would be quite interesting. The discussion in Ch. 5 shows, however, that such an interpretation has its severe problems.

Chapter 8

The $SO^{\uparrow}(1,2)$-structure of interferences

8.1 Classical theory

The analysis and mathematical descriptions of interference pattern play a very essential role in classical optics [225]. Consider the following generic example which will show all the essential problems of an appropriate quantization to be discussed in the next section:

The intensity

$$I = |A_1 + A_2|^2 \tag{8.1}$$

of two "interfering" complex amplitudes $A_j\,, j = 1, 2$, may be expressed in different ways, depending on the coordinates one uses, cartesian or polar ones:

$$A_j = \frac{1}{\sqrt{2}}(q_j + i\,p_j) = |A_j|\,e^{-i\,\varphi_j}\;,\; j = 1, 2\;. \tag{8.2}$$

The two reprensations are classically equivalent, except for the points $(q_j, p_j) = (0, 0)$ where the functional determinants

$$\frac{\partial(q_j,\,p_j)}{\partial(\varphi_j, |A_j|)} = 2\,|A_j|\;,\; j = 1, 2, \tag{8.3}$$

for the mutual transformations vanishes or its inverse becomes singular!

For the intensity 8.1 we get

$$\begin{aligned} I = w_3 &= (\bar{A}_1 + \bar{A}_2)(A_1 + A_2) = && (8.4)\\ &= I_1 + I_2 + \bar{A}_1\,A_2 + A_1\,\bar{A}_2\;, \\ & \quad I_j = |A_j|^2\;,\; j = 1, 2, && (8.5)\\ \bar{A}_1\,A_2 + A_1\,\bar{A}_2 &= 2\,g_1(q_1, p_1; q_2, p_2) \equiv q_1\,q_2 + p_1\,p_2 && (8.6)\\ &= 2\,h_1(|A_1|, |A_2|, \varphi = \varphi_1 - \varphi_2) \equiv && (8.7)\\ &\equiv 2\,|A_1|\,|A_2|\,\cos\varphi \end{aligned}$$

Employing beam splitters and $\lambda/4$ phase shifters (compensators) allows to shift the phase of one amplitude by $\pm\pi$ or $\pm\pi/2$, respectively, relative to that of the other. The result of these

modifications are new intensities:

$$
\begin{aligned}
w_4 &= (\bar{A}_1 - \bar{A}_2)(A_1 - A_2) = I_1 + I_2 - \qquad (8.8)\\
&\quad -(\bar{A}_1\, A_2 + A_1\, \bar{A}_2)\ ,\\
w_5 &= (\bar{A}_1 - i\,\bar{A}_2)(A_1 + i\,A_2) = I_1 + I_2 + \qquad (8.9)\\
&\quad +i(\bar{A}_1\, A_2 - A_1\, \bar{A}_2)\ ,\\
i(\bar{A}_1\, A_2 - A_1\, \bar{A}_2) &= -2\, g_2(q_1, p_1; q_2, p_2) \equiv -q_1\, p_2 + q_2\, p_1 \qquad (8.10)\\
&= 2\, h_2(|A_1|, |A_2|, \varphi = \varphi_1 - \varphi_2) \equiv \qquad (8.11)\\
&\equiv -2\, |A_1|\, |A_2|\, \sin\varphi\ ,\\
w_6 &= (\bar{A}_1 + i\,\bar{A}_2)(A_1 - i\,A_2) = I_1 + I_2 - \qquad (8.12)\\
&\quad -i(\bar{A}_1\, A_2 - A_1\, \bar{A}_2)\ ,
\end{aligned}
$$

From the differences of the intensities

$$
\begin{aligned}
w_3 - w_4 &= 2\,(\bar{A}_1\, A_2 + A_1\, \bar{A}_2)\ , \qquad (8.13)\\
w_5 - w_6 &= 2\,i\,(\bar{A}_1\, A_2 - A_1\, \bar{A}_2) \qquad (8.14)
\end{aligned}
$$

one gets the important functions g_j, $j = 1, 2$, or (and) h_j, $j = 1, 2$, which determine the interference pattern.

I shall discuss the experimental methods (multi-port homodyning etc.) for determining the densities w_j and their differences later (see Sec. 8.2.2 and the literature quoted there).

Before passing to any quantum theory let me first discuss several subtleties of the classical canonical structures because they are important for the quantum theory:

The functions g_j and h_j are functions on different phase (symplectic) spaces. The g_j are functions on the 4-dimensional space

$$
\mathcal{S}^4_{q,p;0,0} = \{(q_1, p_1; q_2, p_2) \in \mathbb{R}^2 \times \mathbb{R}^2\ , (q_1, p_1) \neq (0, 0) \neq (q_2, p_2)\,\} \qquad (8.15)
$$

with the *local* symplectic form

$$
\omega_{q,p} = dq_1 \wedge dp_1 + dq_2 \wedge dp_2\ . \qquad (8.16)
$$

(Though it does not appear to be necessary at this stage, the origin of $\mathcal{S}^4_{q,p}$ has been deleted in order to have a one-to-one relationship between cartesian and polar coordinates. The importance of deleting the origin has been iscussed before in previous chapters and will become evident in this one, too!)

The h_j , however, are functions on the 2-dimensional space

$$
\mathcal{S}^2_{\varphi, I_{1,2}} = \{(\varphi, I_{1,2} = |A_1|\, |A_2|) \in S^1 \times \mathbb{R}^+\}\ , \qquad (8.17)
$$

with the symplectic form

$$
\omega_{\varphi, I_{1,2}} = d\varphi \wedge dI_{1,2}\ , \qquad (8.18)
$$

that is, here we have the same symplectic manifold we started from in the very beginning!

The deeper relationship between $\mathcal{S}^4_{q,p;0,0}$ and $\mathcal{S}^2_{\varphi, I_{1,2}}$ has its origin in the following "gauge" symmetries:

All the densities w_j, $j = 3, 4, 5, 6$, are invariant against a simultaneous phase transformation of the complex amplitudes A_j:

$$
A_j \to e^{i\,\alpha}\, A_j\ ,\ \ j = 1, 2\ . \qquad (8.19)
$$

For $\alpha = \pm\pi$ this gives the reflection

$$R: \quad (q_1, p_1, q_2, p_2) \to (-q_1, -p_1, -q_2, -p_2)\,, \; R^2 = \mathbf{1}\,. \tag{8.20}$$

The reflection symmetry 8.20 is a generalization of the $Z_2[\cdot(q,p)]$-symmetry encountered in Ch. 6 (Eq. 6.87). Though it is contained in the group of phase transformations 8.19, it will be important in the following to consider it for itself!

I therefore shall deal with the continuous symmetry 8.19 and certain generalizations of it and the discrete symmetry 8.20 separately.

8.1.1 Continuous gauge transformations and associated symplectic reductions

For the coordinates q_j and p_j the transformations 8.19 imply the following rotations

$$\begin{aligned} q_j &\to \tilde{q}_j(\alpha) = \cos\alpha\, q_j - \sin\alpha\, p_j \;, \\ p_j &\to \tilde{p}_j(\alpha) = \sin\alpha\, q_j + \cos\alpha\, p_j \;. \end{aligned} \tag{8.21}$$

All observable (physical) quantities occuring in the relations 8.4-8.12 should be invariant under the gauge transformations 8.19 or 8.21 respectively !

The moduli $|A_j|$ and the phase difference $\varphi = \varphi_1 - \varphi_2$ are, of course, invariant!. Thus, the phase space $\mathcal{S}^2_{\varphi,I_{1,2}}$ represents the $U(1)$ gauge invariant part of $\mathcal{S}^4_{q,p;0,0}$, with the $U(1)$-gauge dependent part factored out:

The transformations 8.21 generate circles around the (deleted!) origin in each factor $\mathbb{R}^2 - (0,0)$ of $\mathcal{S}^4_{q,p;0,0}$. Any two points on a given circle are physically equivalent. Two such circles - one in each factor $\mathbb{R}^2 - (0,0)$ - represent just one point in the (gauge invariant) phase space $\mathcal{S}^2_{\varphi,I_{1,2}}$!

The transformations 8.21 induce the following vector field on $\mathcal{S}^4_{q,p;0,0}$:

$$X_{2g_0} = p_1\,\partial_{q_1} - q_1\,\partial_{p_1} + p_2\,\partial_{q_2} - q_2\,\partial_{p_2}\;. \tag{8.22}$$

This follows immediately from the relation

$$\partial f(\tilde{q}_1(\alpha), \tilde{p}_1(\alpha), \tilde{q}_2(\alpha), \tilde{p}_2(\alpha)\,)/\partial\alpha(\alpha = 0) = -X_{2g_0} f\;, \tag{8.23}$$

where f is any smooth function on $\mathcal{S}^4_{q,p;0,0}$. (For this and the following see the notions briefly introduced in Appendix A.1 and the literature mentioned there!)

The vector field field X_{2g_0} is associated with the hamiltonian function

$$2\,g_0(q_1, p_1, q_2, p_2) = \frac{1}{2}\,(q_1^2 + p_1^2 + q_2^2 + p_2^2) = I_1 + I_2\;. \tag{8.24}$$

This may be seen as follows: If we denote by $i_X\rho$ the interior product of a vector field X with a differential form ρ (see, e.g. [226, 227]) then we get for the 2-form 8.16

$$i_{X_{2g_0}}\omega_{q,p} = p_1\,dp_1 + q_1\,dq_1 + p_2\,dp_2 + q_2\,dq_2 = 2\,dg_0\;. \tag{8.25}$$

(Note that $i_{\partial_{q_1}}(dq_1 \wedge dp_1) = dq_1(\partial_{q_1}) \wedge dp_1 = dp_1\,, i_{\partial_{p_1}}(dq_1 \wedge dp_1) = -dp_1(\partial_{p_1}) \wedge dq_1 = -dq_1$ etc.)

A more familiar - but also more "hand-waving" - argument might be the following:

Consider variations δL of the Lagrangean

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 - q_1^2 - q_2^2)\,, \tag{8.26}$$

generated by variations δq_j. Then (by partial differentiation)

$$\delta L = \sum_{j=1}^{2} (\partial L/\partial q_j - \frac{d}{dt}\partial L/\partial \dot{q}_j)\delta q_j + \frac{d}{dt}\sum_{j=1}^{2}(\partial L/\partial \dot{q}_j)\delta q_j) \ . \tag{8.27}$$

If the curves $q_j(t)$ are solutions of the Euler-Lagrange eqs. of motion, the first term on the r.h. side of the relation 8.27 vanishes and we have

$$\delta L = \frac{d}{dt}\sum_{j=1}^{2}(\partial L/\partial \dot{q}_j)\,\delta q_j \ . \tag{8.28}$$

If the variations δq_j are infinitesimal transformations of a 1-parameter group which leaves the Lagrangean invariant – $\delta L = 0$ –, then we have a conservation law (Noether's first theorem). More generally, it suffices that L is invariant up to a total time derivative,

$$\delta L = \frac{d}{dt} C(q, \dot{q}) \, , \tag{8.29}$$

because we now have the conservation law

$$\frac{d}{dt}[\sum_{j=1}^{2} p_j \, \delta q_j - C(q,p)] = 0 \, , \ p_j = \partial L/\partial \dot{q}_j = \dot{q}_j \, . \tag{8.30}$$

The last situation occcurs in the context of the transformations 8.21, where

$$\delta q_j = -p_j \, \alpha \, , \ \delta p_j = q_j \, \alpha \, , \ |\alpha| \ll 1 \ . \tag{8.31}$$

This gives

$$C = (q_1^2 + q_2^2)\, \alpha \ . \tag{8.32}$$

Therefore we have the conservation law

$$\frac{d}{dt}(q_1^2 + q_2^2 + p_1^2 + p_2^2) = 0 \quad \text{or} \ \frac{d}{dt}\, g_0 = 0 \ . \tag{8.33}$$

Thus we get the energy conservation law from the invariance - up to a total time derivative - under the rotations 8.21 !

The relation 8.33 implies the *constraint*

$$\phi_0(q_1, p_1, q_2, p_2) \equiv g_0 - E/2 = 0 \, , \ E > 0 \ . \tag{8.34}$$

Before discussing this constraint let us first analyse the group structures associated with the system:

For smooth functions on $\mathcal{S}^4_{q,p}$ we have the Poisson brackets

$$\{f_1, f_2\}_{q,p} = \partial_{q_1} f_1 \, \partial_{p_1} f_2 - \partial_{p_1} f_1 \, \partial_{q_1} f_2 + \partial_{q_2} f_1 \, \partial_{p_2} f_2 - \partial_{p_2} f_1 \, \partial_{q_2} f_2 \ . \tag{8.35}$$

For the functions g_1 and g_2 from 8.6 and 8.10 we get

$$\{g_1, g_2\}_{q,p} = \frac{1}{4}(q_1^2 + p_1^2 - q_2^2 - p_2^2) \equiv g_3 = \frac{1}{2}(I_1 - I_2) \, . \tag{8.36}$$

The three functions $g_j, j = 1, 2, 3$, generate the Lie algebra of the group $SO(3)$ or of its covering group $SU(2)$:

$$\{g_j, g_k\}_{q,p} = \epsilon_{jkl} \, g_l \ . \tag{8.37}$$

Note that the function g_2, Eq. 8.10, has the form of the usual angular momentum in the plane.

The function g_0 from Eq. 8.24 Poisson commutes with all $g_j, j = 1, 2, 3$,

$$\{g_j, g_0\}_{q,p} = 0\ ,\ j = 1, 2, 3\ . \tag{8.38}$$

Whereas the functions $g_j, j = 1, 2, 3$, generate the group $SU(2)$ or $SO(3)$, the function g_0 generates a $U(1)$ or $O(2)$ which is an invariance (gauge) group of the interference observables g_j, $j = 1, 2, 3$, and g_0 itself.

These 4 functions are not independent:

$$g_1^2 + g_2^2 = I_1 I_2 = g_0^2 - g_3^2\ . \tag{8.39}$$

On the other hand, let $k_a(\varphi, I_{1,2})$, $a = 1, 2$, be functions on the phase space 8.17 with the Poisson bracket

$$\{k_1, k_2\}_{\varphi, I_{1,2}} = \partial_\varphi k_1\, \partial_{I_{1,2}} k_2 - \partial_{I_{1,2}} k_1\, \partial_\varphi k_2\ . \tag{8.40}$$

With

$$I_{1,2} = \sqrt{I_1\, I_2} = |A_1|\, |A_2| \tag{8.41}$$

we have for the functions $h_a, a = 1, 2$, from 8.7 and 8.11 and $I_{1,2}$:

$$\{h_1, h_2\}_{\varphi, I_{1,2}} = I_{1,2}\ ,\ \{h_1, I_{1,2}\}_{\varphi, I_{1,2}} = h_2\ ,\ \{h_2, I_{1,2}\}_{\varphi, I_{1,2}} = -h_1\ , \tag{8.42}$$

which, as we know, constitutes the Lie algebra of the group $SO^\uparrow(1,2)$!

An immediate question is, of course, how these qualitatively different group structures (compact $SO(3)$ and non-compact $SO^\uparrow(1,2)$) are related!

One key lies in the constraint 8.34. Before we exploit it, let me observe that

$$\begin{aligned} \{I_1 I_2, g_1\}_{q,p} &= \{(g_0^2 - g_3^2), g_1\} = -2\, g_3 \{g_3, g_1\} \qquad (8.43)\\ &= 2\, g_3\, g_2\ , \\ \{I_1 I_2, g_2\}_{q,p} &= -2\, g_3\, g_1\ , \qquad (8.44)\\ \{I_1 I_2, g_3\}_{q,p} &= 0\ . \end{aligned}$$

Expressing the real variables q_j and p_j in terms of the amplitudes A_j and their complex conjugates we get for the symplectic form 8.16

$$\omega_{q,p} = i\,(dA_1 \wedge d\bar{A}_1 + dA_2 \wedge d\bar{A}_2) = d\varphi_1 \wedge dI_1 + d\varphi_2 \wedge dI_2\ . \tag{8.45}$$

The constraint 8.34 implies

$$dI_2 = -dI_1\ ,\ d(I_1 I_2) = (I_2 - I_1) dI_1 = -2\, g_3\, dI_1\ , \tag{8.46}$$

so that

$$d\varphi_1 \wedge dI_1 + d\varphi_2 \wedge dI_2 = d\varphi \wedge dI_1 = -\frac{1}{2\, g_3}\, d\varphi \wedge d(I_1 I_2)\ , \varphi = \varphi_1 - \varphi_2\ . \tag{8.47}$$

As $I_1 I_2 = I_{1,2}^2$ we get

$$\omega_{q,p} = -\frac{I_{1,2}}{g_3}\, d\varphi \wedge dI_{1,2}\ . \tag{8.48}$$

The symplectic form 8.48 implies for the corresponding ("dual") Poisson brackets (see Appendix A.1)

$$\{\cdot, \cdot\}_{\varphi, I_{1,2}} = -\frac{I_{1,2}}{g_3}\, \{\cdot, \cdot\}_{q,p}\, . \tag{8.49}$$

We now see that the relations 8.36, 8.43 and 8.44 are equivalent to the relations 8.42! (Recall that $h_2 \leftrightarrow -g_2$.) We also observe that the symplectic reduction of the symplectic space 8.15 to the symplectic space 8.17 is accompanied by a transition from the structure group $SO(3)$ to the structure group $SO^{\uparrow}(1,2)$!

That symplectic reduction needs some more comments: According to Dirac's classification [228] the constraint ϕ_0 of Eq. 8.34 is "first class", because it commutes with the $g_j, j = 1,2,3$, and (the Hamiltonian) g_0. It generates a $U(1)$ - gauge transformation which induces a reduction of the original 4-dimensional phase space to a 2-dimensional one with the symplectic form 8.47. This comes about as follows: The constraint 8.34 reduces the original 4-dimensional symplectic space $\mathcal{S}^4_{q,p}$ - without deletion of the origin - to a 3-dimensional (non-symplectic) one, corresponding to a 3-dimensional sphere S^3. Factoring out the $U(1)$ - gauge transformations yields a 2-dimensional subspace [229]. Mathematically we are dealing with the so-called "Hopf fibration" [230–233][1].

For convenience let us put $E = 1$. Then

$$S^3 = \{(A_1, A_2) \in \mathbb{C}^2 ; |A_1|^2 + |A_2|^2 = 1\} \ . \tag{8.50}$$

Furthermore

$$S^2 = \{(w,t) \in \mathbb{C} \times \mathbb{R}; |w|^2 + t^2 = 1, w = u + iv \} \ . \tag{8.51}$$

The manifold ("chart") $S^2 - \{(0,0,1)\}$ may be mapped stereographically onto the complex plane:

$$p : S^2 - \{(0,0,1)\} \to \mathbb{C} ; \ p(w,t) = z = x + iy = \frac{w}{1-t} \ , \tag{8.52}$$

with the inversion

$$w = z(1-t) \, , \ (1-t) = \frac{2}{1 + x^2 + y^2} \ . \tag{8.53}$$

The mapping (projection) $h : S^3 \to S^2$ is implemented by

$$h(A_1, A_2) = (w = 2 \, \bar{A}_1 A_2, t = |A_2|^2 - |A_1|^2) \ . \tag{8.54}$$

Notice that $h(e^{i\alpha} A_1, e^{i\alpha} A_2) = h(A_1, A_2)$. The orbits of these $U(1) \backsimeq S^1$ transformations which are projected to points on S^2, the so-called "fibers over $(w,t) \in S^2$", are given by

$$\begin{aligned} h^{-1}(w,t) = \{(A_1, A_2) \in \mathbb{C}^2 ; \qquad & \qquad (8.55) \\ |A_1|^2 = \frac{1-t}{2}, \ |A_2|^2 = \frac{|w|^2}{2(1-t)} = \frac{1+t}{2}, \ & w = 2 A_2 \bar{A}_1 \} \ . \end{aligned}$$

Combining the projection h with a consecutive stereographic one, p, yields the mapping $p \circ h : S^3 \to \mathbb{C}$:

$$(p \circ h)(A_1, A_2) = \frac{2 \bar{A}_1 A_2}{1 - (|A_2|^2 - |A_1|^2)} = \frac{\bar{A}_1 A_2}{|A_1|^2} = z = |z| e^{i\vartheta} \in \mathbb{C} \ . \tag{8.56}$$

The relations 8.56 and 8.50 imply

$$|A_1| \ = \ (1 + |z|^2)^{-1/2} \ , \tag{8.57}$$

$$|A_2| \ = \ |z| \, (1 + |z|^2)^{-1/2} \ , \tag{8.58}$$

$$z + \bar{z} = 2 \, |z| \cos\vartheta \ = \ \frac{A_1 \bar{A}_2 + \bar{A}_1 A_2}{|A_1|^2} = 2 \frac{|A_2|}{|A_1|} \cos\varphi \, , \ \varphi = \varphi_1 - \varphi_2 \, , \tag{8.59}$$

$$z - \bar{z} = 2i \, |z| \sin\vartheta \ = \ 2i \, \frac{|A_2|}{|A_1|} \sin\varphi \ . \tag{8.60}$$

[1] I came across the textbook [171] when this article was practically completed. Ch. *I* of that monograph contains a lot of mathematical material which is very closely related to that of the present section! See also the papers [234].

The last Eqs. show that $\vartheta = \varphi$.

Up to now we have ignored the topological fine structures of the manifolds involved: By implementing the stereographic projection 8.36 we have taken out the north pole of the sphere S^2, reducing the sphere topologically to a (complex) plane. By introducing polar coordinates we in addition have to take out the points $|A_1| = 0$ and $|A_2| = 0$. This means that $|z| \neq 0$, too.

The relations 8.59 and 8.60 imply

$$i\, dz \wedge d\bar{z} = \frac{1}{I_1^2}\, d\varphi \wedge dI_1 \ , \tag{8.61}$$

showing the connection between the Hopf fibration and the symplectic structure 8.47.

The above reduction 8.47 of a 4-dimensional symplectic space to a 2-dimenional one by using a single constraint is very special, because in general a first class constraint reduces a symplectic space by just one dimension leaving the additional reduction by an additional dimension to gauge fixing [228], i.e. selecting unique representatives from the equivalence classes formed by the orbits of the gauge transformations.

A mathematically possible gauge fixing in our case is $\varphi_2 = 0$, i.e. we take A_2 to be real ($p_2 = 0$) and $q_2 > 0$.

If we express this constraint on $\mathcal{S}^4_{q,p}$ by

$$\chi_2(q,p) \equiv \arctan\left(\frac{p_2}{q_2}\right) = 0 \ , \tag{8.62}$$

then we have to restrict the arctan to the interval $(-\pi/2, +\pi/2)$ in order to have a unique solution of the Eq. 8.62. Otherwise one would also get the solutions $\varphi_2 = \pm\pi$.

Because

$$\{g_0, \chi_2\}_{q,p} = \{I_2, \chi_2\}/2 = 1/2\,, \tag{8.63}$$

we now have two second class constraints [228].

As to the physics, however, the gauge constraint 8.62 is too restrictive, because it requires A_2 not only to be real, but to be positive, too. This could at most be an initial condition, but cannot hold for a finite time interval which is longer than the time period of the oscillator. The "weaker" constraint $p_2 = 0$ which would allow for negative A_j is unsuitable, too: it would entail the secondary constraint $\{g_0, p_2\}_{q,p} = q_2/2 = 0$, so that $A_2 = 0$. In addition it would violate the postulate $(q_2, p_2) \neq (0,0)$ and the line $p_2 = 0$ would cut all gauge orbits - the circles around $(q_2, p_2) = (0,0)$ - twice.

A less restrictive gauge fixing is

$$\chi_0 = \phi_1 + \phi_2 - \beta = 0\,,\ \beta = \text{ const.} \quad . \tag{8.64}$$

If one tries to express this again by the original coordinates q_j and p_j, namely

$$\chi_0 = \arctan\frac{p_1}{q_1} + \arctan\frac{p_2}{q_2} - \beta = \arctan\left(\frac{p_1 q_2 + q_1 p_2}{q_1 q_2 + p_1 p_2}\right) - \beta = 0 \ , \tag{8.65}$$

with

$$\{\phi_0, \chi_0\}_{q,p} = 1 \ , \tag{8.66}$$

then one encounters (Gribov) ambiguities, e.g. $(q_1, p_1) \to (-q_1, -p_1)$ and $(q_2, p_2) \to (-q_2, -p_2)$ give the same χ_0 in 8.65. Such ambiguities are allowed, however, if we take the discrete gauge symmetry 8.20 into account, which will be discussed below.

In the case of second class constraints the symplectic structure on the reduced space is given by the Dirac brackets [228]:

Suppose that we start from a 4-dimensional manifold $\mathcal{S}^4_{q,p}$ with the the (local) symplectic form 8.16, the associated Poisson brackets 8.35 and two second class constraints $\phi(q,p)$ and $\chi(q,p)$ which obey

$$\{\phi,\chi\}_{q,p} = \delta = \text{ const. } \neq 0 \ , \tag{8.67}$$

then the Dirac brackets are given by

$$\{f_1,f_2\}^* = \{f_1,f_2\}_{q,p} + \frac{1}{\delta}\{f_1,\phi\}_{q,p}\{\chi,f_2\}_{q,p} - \frac{1}{\delta}\{f_1,\chi\}_{q,p}\{\phi,f_2\}_{q,p} \ . \tag{8.68}$$

For $\phi = g_0$ and the special functions $f_1, f_2 = g_1, g_2$ or g_3 the Dirac brackets are equal to the ordinary Poisson brackets because g_0 Poisson commutes with all the $g_j, j = 1,2,3$.

The Dirac brackets play a non-trivial role, however, in the following modification of the above canonical framework: The "gauge" $\varphi_2 = 0$ implies the weaker assumption $A_2 = \bar{A}_2$.

A weak form of this restriction is to replace the amplitude A_2 in 8.1 and 8.2 by its complex conjugate $\bar{A}_2$:

$$\tilde{I} = |A_1 + \bar{A}_2|^2 \ , \tag{8.69}$$

with the corresponding properties

$$\begin{aligned}
\tilde{I} = \tilde{w}_3 &= (A_1+\bar{A}_2)(\bar{A}_1+A_2) = && (8.70)\\
&= I_1 + I_2 + \bar{A}_1\,\bar{A}_2 + A_1\,A_2 \ , &&\\
\bar{A}_1\,\bar{A}_2 + A_1\,A_2 &= 2\,g_4(q_1,p_1;q_2,p_2) \equiv q_1\,q_2 - p_1\,p_2 && (8.71)\\
&= 2\,\tilde{h}_1(|A_1|,|A_2|,\tilde{\varphi} = \varphi_2+\varphi_1) \equiv && (8.72)\\
&\equiv 2\,|A_1|\,|A_2|\,\cos\tilde{\varphi} &&\\
\tilde{w}_4 &= (A_1-\bar{A}_2)(\bar{A}_1-A_2) = I_1+I_2- && (8.73)\\
&\quad -(\bar{A}_1\,\bar{A}_2 + A_1\,A_2) \ , &&\\
\tilde{w}_5 &= (A_1 - i\,\bar{A}_2)(\bar{A}_1 + i\,A_2) = I_1+I_2+ && (8.74)\\
&\quad +i(A_1\,A_2 - \bar{A}_1\,\bar{A}_2) \ , &&\\
i(A_1\,A_2 - \bar{A}_1\,\bar{A}_2) &= -2\,g_5(q_1,p_1;q_2,p_2) \equiv -q_1\,p_2 - q_2\,p_1 && (8.75)\\
&= 2\,\tilde{h}_2(|A_1|,|A_2|,\tilde{\varphi} = \varphi_2+\varphi_1) \equiv && (8.76)\\
&\equiv -2\,|A_1|\,|A_2|\,\sin\tilde{\varphi}\,, &&\\
\tilde{w}_6 &= (A_1 + i\,\bar{A}_2)(\bar{A}_1 - i\,A_2) = I_1+I_2- && (8.77)\\
&\quad -i(A_1\,A_2 - \bar{A}_1\,\bar{A}_2) \ , &&
\end{aligned}$$

The functions g_4, g_5, g_0 and g_3 have the Poisson brackets

$$\{g_4,g_5\}_{q,p} = -g_0 \ , \ \{g_0,g_4\}_{q,p} = -g_5 \ , \ \{g_0,g_5\}_{q,p} = g_4 \ , \tag{8.78}$$

i.e. the g_4, g_5 and g_0 generate the Lie algebra of the group $SO^{\uparrow}(1,2)$! (Replace g_4 or g_5 by its negative and compare with 8.42.) These 3 functions Poisson commute with the function g_3:

$$\{g_j,g_3\}_{q,p} = 0 \ , \ j = 0,4,5 \,. \tag{8.79}$$

The function $2g_3$ generates a group $U(1)$ which is the invariance (gauge) group of the observables $\tilde{w}_j, j = 3,4,5,6$:

$$A_1 \to e^{i\alpha}\,A_1 \ , \ \bar{A}_2 \to e^{i\alpha}\,\bar{A}_2 \ . \tag{8.80}$$

The consequences can be discussed in a completely analogous way as in the case of the transformations 8.19:

Instead of the transformations 8.21 we now have

$$\begin{aligned} q_1 \to \breve{q}_1(\alpha) &= \cos\alpha\, q_1 - \sin\alpha\, p_1 \ , \\ p_1 \to \breve{p}_1(\alpha) &= \cos\alpha\, p_1 + \sin\alpha\, q_1 \ , \\ q_2 \to \breve{q}_2(\alpha) &= \cos\alpha\, q_2 + \sin\alpha\, p_2 \ , \\ p_2 \to \breve{p}_2(\alpha) &= \cos\alpha\, p_2 - \sin\alpha\, q_2 \ , \end{aligned} \tag{8.81}$$

The difference to the rotations 8.21 lies in the property that the simultaneous rotations in the (q_1, p_1)- and (q_2, p_2) - planes have opposite directions now.

Instead of the vector field 8.22 we here have

$$X_{2g_3} = p_1\partial_{q_1} - q_1\partial_{p_1} - p_2\partial_{q_2} + q_2\partial_{p_2} \ , \tag{8.82}$$

with the property

$$i_{X_{2g_3}}\omega_{q,p} = 2dg_3 \ , \ 2g_3 = I_1 - I_2 \ . \tag{8.83}$$

The conservation law 8.33 now is replaced by

$$\frac{d}{dt}\, g_3 = 0 \tag{8.84}$$

and we have the constraint

$$\phi_3 \equiv g_3 - \tilde{\epsilon} = 0 \ , \ \tilde{\epsilon} = \ \text{const.} \quad . \tag{8.85}$$

This constraint implies

$$dI_1 = dI_2 \ , \ dI_{1,2}^2 = d(I_1 I_2) = 2\, g_0\, dI_1 \ , \tag{8.86}$$

leading to a new reduction of the symplectic form 8.16:

$$\omega_{q,p} = d\tilde{\varphi} \wedge dI_1 = \frac{I_{1,2}}{g_0} d\tilde{\varphi} \wedge dI_{1,2} \ , \ \tilde{\varphi} = \varphi_1 + \varphi_2 \ , \tag{8.87}$$

which is to be compared with the reduced form 8.48.
For the functions $\tilde{h}_j(\tilde{\varphi} = \varphi_2 + \varphi_1, I_{1,2})$ we have the Poisson brackets

$$\{\tilde{h}_1, \tilde{h}_2\}_{\tilde{\varphi}, I_{1,2}} = I_{1,2} \ , \ \{\tilde{h}_1, I_{1,2}\}_{\tilde{\varphi}, I_{1,2}} = \tilde{h}_2 \ , \ \{\tilde{h}_2, I_{1,2}\}_{\tilde{\varphi}, I_{1,2}} = -\tilde{h}_1 \ , \tag{8.88}$$

which form the Lie algebra $\mathfrak{so}(1,2)$, too.

They are functions on the phase space

$$\mathcal{S}^2_{\tilde{\varphi}, I_{1,2}} = \{(\tilde{\varphi}, I_{1,2} = |A_1|\,|A_2|) \in S^1 \times \mathbb{R}^+ \} \tag{8.89}$$

Again the first class constraint 8.85 becomes a second class one if one recurs to gauge fixing, e.g. $\varphi_2 = 0$, or the gauge 8.62 which has the property

$$\{\phi_3, \chi_2\} = -1/2 \ . \tag{8.90}$$

As we have a pair of second class constraints now, we have to replace ordinary Poisson brackets by Dirac brackets 8.68.

Because of the relations 8.79 all the additional terms in 8.68 vanishes and thus the Poisson brackets 8.78 can be replaced by Dirac brackets $\{\cdot,\cdot\}^*$ without changes!

This is different for the relations 8.37 and 8.38. Here we have

$$\{g_1, g_2\}^* = -g_0 \ , \ \{g_0, g_1\}^* = g_2 \ , \ \{g_0, g_2\}^* = -g_1 \tag{8.91}$$

and

$$\{g_3, g_j\}^* = 0\,,\ j = 0, 1, 2\,. \tag{8.92}$$

What is surprising is that the "reduced" relations 8.91 form the Lie algebra $\mathfrak{so}(1,2)$, whereas the unreduced ones 8.37 constitute the Lie algebra $\mathfrak{so}(3)$!

Similar to the relation 8.39 we have for g_4 and g_5:

$$g_4^2 + g_5^2 = I_1 I_2 = g_0^2 - g_3^2\ . \tag{8.93}$$

This relation may be used in order to obtain from 8.91

$$\{I_{1,2}, g_1\}^* = \frac{g_0}{I_{1,2}}\, g_2\,,\ \{I_{1,2}, g_2\}^* = -\frac{g_0}{I_{1,2}}\, g_1\,. \tag{8.94}$$

Rewriting the first of the relations 8.91 as

$$\{g_1, g_2\}^* = -\frac{g_0}{I_{1,2}}\, I_{1,2} \tag{8.95}$$

and comparing with the expression 8.87 we see that we have a completely analogous situation as discussed in connection with the relations 8.43, 8.44, 8.48 and 8.49 if we replace $\varphi = \varphi_2 - \varphi_1$ by $\tilde{\varphi} = \varphi_2 + \varphi_1$, the factor $-I_{1,2}/g_3$ by $I_{1,2}/g_0$ and the Poisson brackets by Dirac brackets.

Thus, the relations 8.94 and 8.95 are equivalent to those of 8.88!

8.1.2 Emergence of the symplectic group $Sp(4, \mathbb{R})$

We have seen that the functions g_1, g_2 and g_3 generate the Lie algebra $\mathfrak{so}(3)$ - Eqs. 8.37 - and the functions g_4, g_5 and g_0 the Lie algebra $\mathfrak{so}(1,2)$ - see Eqs. 8.78. These two groups are subgroups of the symplectic group $Sp(4, \mathbb{R})$, the real symplectic group in 4 dimensions which is an invariance group of the symplectic form 8.16. The group is 10-dimensional.

The remaining 4 independent generators of its Lie algebra may be obtained by the following 4 Poisson brackets:

$$\{g_1, g_4\} = g_6 \equiv -\frac{1}{2}(q_1 p_1 + q_2 p_2) \tag{8.96}$$

$$\{g_2, g_4\} = -g_8 \equiv -\frac{1}{4}(q_1^2 - p_1^2 - q_2^2 + p_2^2)\,, \tag{8.97}$$

$$\{g_1, g_5\} = g_9 \equiv \frac{1}{4}(q_1^2 - p_1^2 + q_2^2 - p_2^2)\,, \tag{8.98}$$

$$\{g_2, g_5\} = -g_7 \equiv -\frac{1}{2}(q_1 p_1 - q_2 p_2)\,. \tag{8.99}$$

Properties of the group $Sp(4, \mathbb{R})$ have been discussed extensively in the mathematical and physical literature (see Appendix C for more details).

As the 10 functions g_j, $j = 0, \ldots, 9$, form a complete set of observables for a group theoretical analysis of the phase space 8.15 in the framework of the present approach, let me list the remaining six of them here, too:

$$g_0 = \frac{1}{4}(q_1^2 + p_1^2 + q_2^2 + p_2^2)\,, \quad g_1 = \frac{1}{2}(q_1 q_2 + p_1 p_2)\,, \tag{8.100}$$

$$g_2 = \frac{1}{2}(q_1 p_2 - q_2 p_1)\,, \qquad g_3 = \frac{1}{4}(q_1^2 + p_1^2 - q_2^2 - p_2^2)\,, \tag{8.101}$$

$$g_4 = \frac{1}{2}(q_1 q_2 - p_1 p_2)\,, \qquad g_5 = \frac{1}{2}(q_1 p_2 + q_2 p_1)\,, \tag{8.102}$$

It is also instructive to express the ten functions in terms of the variables I_1, I_2, φ_1 and φ_2:

$$
\begin{aligned}
g_0 &= \frac{1}{2}(I_1 + I_2)\,, && (8.103)\\
g_1 &= \sqrt{I_1 I_2}\cos(\varphi_1 - \varphi_2)\,, && (8.104)\\
g_2 &= \sqrt{I_1 I_2}\sin(\varphi_1 - \varphi_2)\,, && (8.105)\\
g_3 &= \frac{1}{2}(I_1 - I_2)\,, && (8.106)\\
g_4 &= \sqrt{I_1 I_2}\cos(\varphi_1 + \varphi_2)\,, && (8.107)\\
g_5 &= \sqrt{I_1 I_2}\sin(\varphi_1 + \varphi_2)\,, && (8.108)\\
g_6 &= \frac{1}{2}(I_1 \sin 2\varphi_1 + I_2 \sin 2\varphi_2)\,, && (8.109)\\
g_7 &= \frac{1}{2}(-I_1 \sin 2\varphi_1 + I_2 \sin 2\varphi_2)\,, && (8.110)\\
g_8 &= \frac{1}{2}(I_1 \cos 2\varphi_1 - I_2 \cos 2\varphi_2)\,, && (8.111)\\
g_9 &= \frac{1}{2}(I_1 \cos 2\varphi_1 + I_2 \cos 2\varphi_2)\,, && (8.112)
\end{aligned}
$$

The 4 functions $g_0, g_j, j = 1, 2, 3$, are the generators of the (maximal) compact subgroup $U(2) \simeq U(1) \times SU(2)$ of the symplectic group $Sp(4, \mathbb{R})$. (See Sec. 8.2.3 and Appendix C!)

8.1.3 The Z_2 "gauge" symmetry

The "observables" g_j, $j = 0, \ldots, 9$, are all invariant under the Z_2 type "gauge" symmetry 8.20. In case of the explicit form 8.103-8.112 this means invariance under

$$I_j \to I_j\,, \quad \varphi_j \to \varphi_j \pm \pi\,, \; j = 1, 2\,. \tag{8.113}$$

We have here the same but more general situation we encountered in case of the group $Sp(2, \mathbb{R})$, see the discussions in Secs. 1.4, 2.3, 6.3 and in Appendix A.3:

Passing from the phase space

$$\mathcal{S}^4_{q,p} = \{(q_1, p_1, q_2, p_2) \in \mathbb{R}^4\} \tag{8.114}$$

to the phase space

$$\mathcal{S}^4_{\varphi,I} = \{\varphi_j \in \mathbb{R} \bmod 2\pi, \quad I_j > 0,\, j = 1, 2\} \tag{8.115}$$

means to factor out a Z_2 gauge symmetry, i.e. passing to the orbifold

$$\mathcal{S}^4_{q,p}/Z_2\,. \tag{8.116}$$

Actually one can use the Z_2 invariance for a definition of "observables" on the space 8.114. This implies that the original canonical variables q_j and p_j are not *"observables" in such a framework*!

This is especially so for the corresponding quantum theory (see below). The deeper reason is that the deletion of the origin of the phase space 8.15 no longer allows for arbitrary translations within that space. But the functions q_j and p_j, or the corresponding operators Q_j and P_j would generate such translations. For that reason they have to be discarded! But like in the 2-dimensional case one can define Z_2 invariant "composite" coordinates $\tilde{q}_j, \tilde{p}_j$ and operators $\tilde{Q}_j, \tilde{P}_j$!

As to the group theoretical side this means the following: The reflection 8.20 constitutes the non-trivial center element $-E_4$ (E_4: unit matrix in $\mathbb{R}^4$) of the center

$$Z_2 = \{E_4, -E_4\} \subset Sp(4,\mathbb{R})\,. \tag{8.117}$$

As (see Appendix C.3)

$$Sp(4,\mathbb{R})/Z_2 \cong SO^{\uparrow}(2,3)\,, \tag{8.118}$$

we see that the actual structure group is not $Sp(4,\mathbb{R})$, but the pseudo-orthogonal group $SO^{\uparrow}(2,3)$, like in the 2-dimensional case, where the effective structure group is $SO^{\uparrow}(1,2) = Sp(2,\mathbb{R})/Z_2$ not $Sp(2,\mathbb{R})$! This will be important for the selection of the appropriate irreducible unitary representations. (Compare also the well-known example of $SU(2)$ and $SU(2)/Z_2 \cong SO(3)$!)

8.1.4 Two important remarks

Let me conclude this section with two important remarks:

1. During the whole discussions above I have ignored any time- and space-dependence of the quantities involved. In classical interference phenomena, however, the intensity 8.1 and the amplitudes 8.2 in general will vary from point to point (e.g. on a screen) according to the interferences involved. The same may hold for the phase α in the transformations 8.19 or 8.80 which can depend on the time t, too. The same applies to the Z_2 symmetry 8.20.

   Thus, we are actually dealing with field theories and genuine gauge transformations (of the 2nd kind!). Taking this into account will not change the core of the above results in an essential way.

2. The same mathematical structures which emerge from the interference quantities 8.4-8.12 also occur in the description of polarization properties of electromagnetic waves in terms of Stokes parameters [235]: If

$$\begin{aligned} E_x &= a_1\cos(\tau+\delta_1) = \Re(a_1 e^{-i(\tau+\delta_1)})\,, && (8.119)\\ E_y &= a_2\cos(\tau+\delta_2) = \Re(a_2 e^{-i(\tau+\delta_2)})\,, && (8.120)\\ & a_1, a_2 > 0\,,\ \tau = \omega t - \vec{k}\cdot\vec{x}\,, && (8.121) \end{aligned}$$

   then the observable properties of the wave may be characterized by the parameters

$$\begin{aligned} s_1 &= 2a_1a_2\cos\delta\,,\ \delta = \delta_2 - \delta_1\,, && (8.122)\\ s_2 &= 2a_1a_2\sin\delta\,, && (8.123)\\ s_3 &= a_1^2 - a_2^2 = I_1 - I_2\,, && (8.124)\\ s_0 &= a_1^2 + a_2^2 = I_1 + I_2\,, && (8.125) \end{aligned}$$

   which obviously correspond exactly to the quantities g_1 (or h_1), g_2 (or $-h_2$), $2g_3$ and $2g_0$ from above.

   All the formal results derived previously may be applied to the physical observables 8.122-8.125 as well.

8.2 Quantum theory

Quantizing the classical system of the last section requires the group theoretical approach: This follows from the structure of the phase spaces 8.15 and 8.17 both of which have their origin deleted. Let us start with the space 8.17 which has been at the center of the present paper!

8.2.1 $SO^{\uparrow}(1,2)$ quantization of interference patterns

According to the Eqs. 8.7 and 8.11 the functions

$$h_1(\varphi, I_{1,2}) = I_{1,2}\cos\varphi\,,\ h_2(\varphi, I_{1,2}) = -I_{1,2}\sin\varphi\,,\ I_{1,2} = \sqrt{I_1 I_2}\ , \tag{8.126}$$

characterize the classical "observable" interference pattern. (The additional sum $I_1 + I_2$ is a background quantity which does not contribute to the dominant structure of the interference patterns.)

Eqs. 8.42 show that the quantities 8.126 obey the Lie algebra of the group $SU(1,1)$ and the Eqs. 8.88 mean that the same relations hold for the functions $\tilde{h}_j(\tilde{\varphi}, I_{1,2})$ of the interference pattern 8.72 and 8.76!

In order to quantize the (reduced) phase spaces 8.17 and 8.89 we again have to use the unitary irreducible representation of the positive discrete series of the group $SU(1,1)$ etc.

As the "observable" $I_{1,2}$ corresponds to the operator K_0, its eigenvalues in the quantum theory are $n+k, n = 0, 1, \ldots$ where k characterizes the representation (see Ch. 2 and Appendix B for details).

It is, of course, tempting - but certainly not necessary - to use the tensor product $\mathcal{H}_1^{osc} \otimes \mathcal{H}_2^{osc}$ of two harmonic oscillator Hilbert spaces as the carrier space of the unitary representations to be employed here (see the discussion in Sec. 6.4): The SU(1,1) generators then have the form (see 6.112)

$$K_+ = a_1^+ a_2^+\ ,\ K_- = a_1 a_2\ ,\ K_0 = \frac{1}{2}(a_1^+ a_1 + a_2^+ a_2 + 1) = \frac{1}{2}(H_1 + H_2)\,. \tag{8.127}$$

The product $\mathcal{H}_1^{osc} \otimes \mathcal{H}_2^{osc}$ contains all the irreducible unitary representations of the group $SU(1,1)$ (for which $k = 1/2, 1, 3/2, \ldots$) as follows:

Let $|n_i\rangle_j,\ n_j = 0, 1, \ldots,\ j = 1, 2$, be the eigenstates of the number operators N_j, generated by a_j^+ from the oscillator ground states $|n_j = 0\rangle_j, j = 1, 2$.

Each of those two subspaces of $\mathcal{H}_1^{osc} \otimes \mathcal{H}_2^{osc} =$ spanned by $\{|n_1\rangle_1 \otimes |n_2\rangle_2\}$ with fixed $|n_1 - n_2| \neq 0$ contains an irreducible representation with

$$k = 1/2 + |n_1 - n_2|/2 = 1, 3/2, 2, \ldots\ . \tag{8.128}$$

The Casimir operator $L = K_+K_- + K_0(1 - K_0)$ now has the form

$$L = \frac{1}{4} - (H_1 - H_2)^2 = \frac{1}{4} - (N_1 - N_2)^2\ . \tag{8.129}$$

For a given index k according to 8.128 the number $n = 0, 1, \ldots$ in the eigenvalue $n + k$ of the above K_0 is given by

$$n = \min\{n_1, n_2\}\ . \tag{8.130}$$

For the "diagonal" case $n_2 = n_1$ we have the unitary representation with $k = 1/2$.

For $n_1 > n_2$ and $n_1 - n_2 = 2k - 1$ fixed - i.e. for a subspace of $\mathcal{H}_1^{osc} \otimes \mathcal{H}_2^{osc}$ with the basis $\{|n_1 = n_2 + 2k - 1\rangle_1 \otimes |n_2\rangle_2\}$ which carries an irreducible unitary representation with index k - we have $n = n_2$!

A word of caution may be appropriate here: According to 8.41 the quantity $I_{1,2}$ classically is given by $\sqrt{I_1 I_2}$, $I_j = (q_j^2 + p_j^2)/2$, $j = 1, 2$. When quantizing naively one expects that $I_1 I_2 \to H_1 H_2$ so that $I_{1,2} \to \sqrt{H_1 H_2}$. However, this argument is on the level of the phase space 8.15, not on the level of the reduced phase space 8.17! Only for the subspace with $n_1 = n_2$ i.e. $k = 1/2$ according to Eq. 8.128 the naive interpretation just mentioned appears to be possible.

In the other cases the use of the tensor product $\mathcal{H}_1^{osc} \otimes \mathcal{H}_2^{osc}$ is just a convenient way to implement an appropriate irreducible unitary representations of the group $SU(1,1)$ for an analysis of the quantum version of the interference pattern 8.7 and 8.11. It serves the purpose to exploit the group theoretical structures 8.42 or 8.88 of the reduced phase space of "observable" quantities which characterize the interference pattern. Other Hilbert spaces for the irreducible unitary representations in question might be more appropriate.

8.2.2 Experimental aspects

The question is, of course, how to determine the expectation values of the operators $K_j = \hat{h}_j, j = 1, 2$, $K_0 = \hat{I}_{1,2}$, and their statistical distributions experimentally.

Let $|\centerdot\rangle$ be an appropriate state which belongs to the domains of definition of the operators $K_j, j = 1, 2$, and K_0, e.g. a 2-mode state of the type discussed above or a 2-mode generated Barut-Girardello or Perelomov or Schrödinger-Glauber coherent state associated with a unitary representation indexed by k, i.e. eigenstates of the operators K_- or $(K_0 + k)^{-1} K_-$ or $(K_0 + k)^{-1/2} K_-$ as discussed in Ch. 3.

The classical relations 8.13 and 8.14 suggests the following quantum relations

$$\bar{n}_3 - \bar{n}_4 \equiv \langle\centerdot|N_3|\centerdot\rangle - \langle\centerdot|N_4|\centerdot\rangle = 4\,\langle\centerdot|K_1|\centerdot\rangle\ , \tag{8.131}$$

$$\bar{n}_5 - \bar{n}_6 \equiv \langle\centerdot|N_5|\centerdot\rangle - \langle\centerdot|N_6|\centerdot\rangle = 4\,\langle\centerdot|K_2|\centerdot\rangle\ , \tag{8.132}$$

where N_3, N_4, N_5 and N_6 are the number operators corresponding to the classical intensities. Here I have assumed that the quantum versions of the intensities w_3 and w_4 as well as those of w_5 and w_6 have the same vacuum contributions so that these cancel in the corresponding differences 8.131 and 8.132.

Similarly we have for the expectation values of K_1^2 and K_2^2:

$$\langle\centerdot|K_1^2|\centerdot\rangle = \frac{1}{16}\langle\centerdot|(N_3 - N_4)^2|\centerdot\rangle\ ,\ \langle\centerdot|K_2^2|\centerdot\rangle = \frac{1}{16}\langle\centerdot|(N_5 - N_6)^2|\centerdot\rangle\ . \tag{8.133}$$

The corresponding determination of the expectation values of $K_0 = \hat{I}_{1,2}$ and K_0^2 is not so obvious. Let us begin with K_0^2:

Here one has to take into account that the classical Pythagorean relation $I_{1,2}^2 = h_1^2 + h_2^2$ in general is no longer valid on the quantum level. For an irreducible representation with index k we have instead

$$K_0^2 = K_1^2 + K_2^2 + k(k-1)\ . \tag{8.134}$$

So, if we know k we may use that relation in order determine $\langle\centerdot|K_0^2|\centerdot\rangle$ with the help of the relations 8.133. However, in general one will not have a definite value of k in a given experimental situation.

Additional information about $\langle\centerdot|K_0^2|\centerdot\rangle$ may come by exploiting the classical relations

$$2\,I_{1,2}^2 = (I_1 + I_2)^2 - I_1^2 - I_2^2\ ,\ I_1 + I_2 = w_3 + w_4 = w_5 + w_6\ . \tag{8.135}$$

On the quantum level we have to take vacuum contributions to $\hat{I}_j, j = 1, 2$, and $\hat{w}_l, l = 3, 4, 5, 6$, into account which here do not cancel. A well-known experimental approach employs multi-port homodyning to be discussed below.

A new problem is posed as to the determination of

$$\langle .|K_0|.\rangle \,. \tag{8.136}$$

Here the relation

$$K_0 = i\,[K_1, K_2] = [K_-, K_+]/2 \tag{8.137}$$

may be useful: Together with Eq. 8.134 it relates properties of K_1 and K_2 to those of K_0. This can be seen for the special states $|.\rangle = |k, z\rangle$ or $= |k, \lambda\rangle$ from section 3.1 and 3.2:

The Eqs. 3.106 and 3.105 or 3.187, 3.188 and 3.195 show how the expectation values $\langle k, z|K_0|k, z\rangle$ etc. may be obtained from $\langle K_j^2\rangle$ and the mean-square fluctuations $(\Delta K_j)^2_{k,z}\,, j = 1, 2$ etc.

How those coherent states can be generated was indicated in section 6.5 above.

Additional information about the expectation value 8.136 may come from the relation 2.18.

In general, however, one probably needs some new ideas for measuring the expectation values 8.136 of K_0 directly!

The quantities $h_j(\varphi, I_{1,2}), j = 1, 2$, and $I_1 + I_2 = 2g_0$ occuring in the relations 8.7, 8.11 and 8.24 (or the corresponding quantities 8.122, 8.123 and 8.125) may be determined experimentally by "balanced" multi-port homodyning [236–239]; reviews: [240–242]; textbooks [243], especially Ref. [244]!

("Balanced" means that the employed beam splitters are of the 50 : 50 type). Using the relations 8.131 and 8.132 such a device should be suitable for measuring the quantized versions of $h_j \to K_j, j = 1, 2$, simultaneously by making clever use of certain vacuum contributions [72]. The squared fluctuations of these operators are then to be deduced from Eqs. 8.133.

Mandel and coworkers [245] have used an eight-port homodyning experimental setup in order to promote and analyze an "operational" approach to the concept of a "quantized phase".

It is a central assumption in their analysis of the experimental data that the Pythagorean trigonometric relation 8.39 is valid in the quantum regime, too! We have seen that this assumption is not justified in general, especially for small numbers of the quanta involved.

As to examples of other experiments using similar homodyning techniques see, e.g. [246,247].

In the "balanced" homodyning scheme one does not determine the difference $I_1 - I_2 = 2g_3$ or the Stokes parameter s_3 (Eq. 8.124). Several proposals have been made [248] to pass to an "unbalanced" setup in order to determine the quantized counterpart of the quantity $I_1 - I_2 = 2g_3 = s_3$ simultaneously with those of the 3 others measured in the balanced scheme (h_1, h_2 and $I_1 + I_2$ or s_1, s_2 and s_0).

If one knows the 2 quantities $I_1 + I_2$ and $I_1 - I_2$ - or their quantum counterparts - then one may calculate I_1 and I_2 and one may be able to measure $4I_{1,2}^2 = 4I_1I_2 = (I_1 + I_2)^2 - (I_1 - I_2)^2$ or the corresponding quantum expectation value.

But one still needs an idea in order to determine the quantum version of $I_{1,2} = \sqrt{I_1 I_2}$!

8.2.3 Relations to unitary representations of $Sp(4, \mathbb{R})$

In the quantum theory of the interference patterns we are entering again the realm of the symplectic group $Sp(4, \mathbb{R})$ already mentioned in section 8.1.2 .

As explained in appendix C, an irreducible unitary representation of the positive discrete series of $Sp(4, \mathbb{R})$ may be characterized by a pair $(\epsilon_0, j_0)\,, \epsilon_0 > j_0$, of numbers, where ϵ_0 is the lowest eigenvalue of the operator K_0 (now embedded in the 10-dimensional Lie algebra of $Sp(4, \mathbb{R})$) which generates the commuting $U(1)$ subgroup of the maximal compact subgroup $U(2) \simeq U(1) \times SU(2)$. The number ϵ_0 corresponds to the number k which characterizes the irreducible unitary representations of $SU(1, 1)$ etc. The other compact subgroup $SU(2) \subset U(2)$

has the usual finite dimensional unitary representations characterized by $j = 0, 1/2, 1, 3/2, \ldots$. For a given eigenvalue ϵ_0 the associated eigenstates carry an irreducible unitary representation of $SU(2)$ characterized by the number j_0 which may take the values $j_0 = 0, 1/2, 1, \ldots$ but has to be smaller than ϵ_0! (see Appendix C!)

In the case of the tensor product $\mathcal{H}_1^{osc} \otimes \mathcal{H}_2^{osc}$ we can construct two $Sp(4, \mathbb{R})$ irreducible unitary representations of this type:

Employing creation and annihilation operators we have the correspondence [278]

$$g_1 \rightarrow J_1 = \frac{1}{2}(a_1 a_2^+ + a_2 a_1^+) \ , \tag{8.138}$$

$$g_2 \rightarrow J_2 = \frac{i}{2}(a_1 a_2^+ - a_2 a_1^+) \ , \tag{8.139}$$

$$g_3 \rightarrow J_3 = \frac{1}{2}(a_1^+ a_1 - a_2^+ a_2) \ , \tag{8.140}$$

$$g_0 \rightarrow K_0 = \frac{1}{2}(a_1^+ a_1 + a_2^+ a_2 + 1) = H/2 \ . \tag{8.141}$$

The space $\mathcal{H}_1^{osc} \otimes \mathcal{H}_2^{osc}$ may be decomposed into 2 subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$: one subspace is spanned by the basis

$$|n_1, n_2\rangle_+ \equiv \{|n_1\rangle_1 \otimes |n_2\rangle_2; n_1 + n_2 \text{ even } \} \in \mathcal{H}_+ \ , \tag{8.142}$$

the other by

$$|n_1, n_2\rangle_- \equiv \{|n_1\rangle_1 \otimes |n_2\rangle_2; n_1 + n_2 \text{ odd } \} \in \mathcal{H}_- \ . \tag{8.143}$$

In the first case we have for the ground state ($n_1 = n_2 = 0$):

$$K_0|0,0\rangle_+ = \frac{1}{2}|0,0\rangle_+ \ , \tag{8.144}$$

$$J_3|0,0\rangle_+ = 0 \ , \tag{8.145}$$

i.e. we have a representation with $\epsilon_0 = 1/2$, $j_0 = 0$. It may be shown (see Appendix C) that it is irreducible and unitary.

In the second case the ground state is degenerate because we have the two possibilities $n_1 = 1, n_2 = 0$ and $n_1 = 0, n_1 = 1$. These ground states have the properties

$$K_0|1,0\rangle_- = |1,0\rangle_- \ , \tag{8.146}$$

$$K_0|0,1\rangle_- = |0,1\rangle_- \ , \tag{8.147}$$

$$J_3|1,0\rangle_- = \frac{1}{2}|1,0\rangle_- \ , \tag{8.148}$$

$$J_3|0,1\rangle_- = -\frac{1}{2}|0,1\rangle_- \ , \tag{8.149}$$

i.e. here we have $\epsilon_0 = 1$, $j_0 = 1/2$.

As to the higher levels in both representations we have the following situation:

If $n_1 + n_2 = 2n, n = 0, 1, 2, \ldots$ we have the eigenvalues $\epsilon_n = n + 1/2$ for K_0 with a $(2n+1)$-fold degeneracy. The associated $(2n+1)$-dimensional subspace carries an irreducible representation of the group $SU(2)$ with $j = n$. If $n_1 + n_2 = 2n + 1, n = 0, 1, 2, \ldots$ we get the eigenvalues $\epsilon_n = n + 1$ with a $(2n + 2)$-fold degeneracy. The associated subspace carries an irreducible $SU(2)$-representation with $j = n + 1/2$.

In the present context it is of special interest in which way the irreducible unitary representations of the subgroup $SU(1,1) \cong SL(2, \mathbb{R}) = Sp(2, \mathbb{R})$, as given by the relations 8.127-8.130,

are contained in the two irreducible unitary representations $(1/2,0)$ and $(1,1/2)$ of $Sp(2,\mathbb{R})$ just described.

One sees immediately that the $SU(1,1)$ representations with $k = 1/2, 3/2, \ldots$ are contained in $(1/2,0)$ and the representations with $k = 1,2,\ldots$ in $(1,1/2)$. (Recall that we have the correspondences $g_0 \to K_0$, $g_4 \to K_1$, $g_5 \to -K_2$!)

It is important to realize that the use of irreducible unitary representations (of the positive discrete series) of the group $Sp(4,\mathbb{R})$ is mandatory if one wants to quantize the phase space 8.15 with its origin deleted! And it might be necessary to employ other irreducible unitary representations of $Sp(4,\mathbb{R})$ than just the 2 discussed above (see Appendix C)!

Accordingly one has to use the unitary representations of the group $SO^{\uparrow}(2,3)$ for quantizing the orbifold $\mathcal{S}^4_{q,p}/Z_2$!

There is another topic left which is to be discussed: In the previous section when discussing the unquantized classical properties of interference pattern we encountered the typical situation of gauge invariances: The interference observables 8.7 and 8.11 are invariant under phase transformations generated by g_0 and the observables 8.72 and 8.76 are invariant under phase transformations generated by g_3. This lead to the constraints $\phi_0 = 0$ of Eq. 8.34 or $\tilde{\phi}_3 = 0$ of Eq. 8.85, respectively.

According to the ideas of Dirac [228] one has two possibilities in order to quantize such systems:

1. One quantizes the "reduced" system, i.e. the subspace of gauge invariant quantities by associating a canonically conjugate gauge fixing function χ with the originally first class constraint function ϕ (see the discussion around Eq. 8.67) and by eliminating the unphysical gauge degrees of freedom by the conditions $\phi = 0$ and $\chi = 0$. We have seen how we arrived in this way at the reduced phase spaces 8.17 and 8.89 and how these may be quantized in terms of irreducible unitary representations of the group $SU(1,1)$ or $SO^{\uparrow}(1,2)$.

2. The second approach of quantizing such a system with gauge constraints consists in quantizing the original phase space (here 8.15) first (now in terms of irreducible unitary representations of the group $Sp(4,\mathbb{R})$) and constructing the physical Hilbert space by requiring the quantized version $\hat{\phi} \to 0$ of the classical first class constraint $\phi = 0$ to be implemented by the condition that the constraint operator $\hat{\phi}$ annihilates the physical states.

   Let us see how this works in our case:

   The quantized version of the classical constraint function 8.34 is

   $$\phi_0 \to \hat{\phi}_0 = K_0 - \epsilon\,,\ \epsilon = E/2\,. \tag{8.150}$$

   When we apply this $\hat{\phi}_0$ to a state $|n_1,n_2\rangle_+$ of the representation $(1/2,0)$ we see that

   $$\hat{\phi}_0|n_1,n_2\rangle_+ = 0 \text{ iff } \epsilon = \epsilon_n = n + 1/2\,,\ n = (n_1+n_2)/2\,. \tag{8.151}$$

   We know already that the corresponding subspace is (2n+1)-dimensional and carries an irreducible unitary representation of $SU(2)$ with $j = n$.

   Analogous results hold for the representation $(1,1/2)$. The constraint operator $\hat{\phi}_0$ merely implements the conservation of energy and decomposes the Hilbert spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ into energy eigenstate subspaces with unitary representations of $SU(2)$.

   In the case of the constraint 8.85 we get the operator

   $$\hat{\phi}_3 = J_3 - \tilde{\epsilon}\,. \tag{8.152}$$

The physical state condition for $|n_1, n_2\rangle_+$ here is

$$\hat{\phi}_3 |n_1, n_2\rangle_+ = [\frac{1}{2}(n_1 - n_2) - \tilde{\epsilon}] \, |n_1, n_2\rangle_+ = 0 \tag{8.153}$$

which means that the number $\tilde{\epsilon}$ has to be quantized:

$$\tilde{\epsilon} = \frac{1}{2}(n_1 - n_2) \ . \tag{8.154}$$

In view of the relation 8.128 this means that

$$|\tilde{\epsilon}| = k - 1/2 \ , \tag{8.155}$$

i.e. $\hat{\phi}$ essentially projects onto irreducible unitary representations of $SU(1,1)$, here (in the case of the $Sp(4,\mathbb{R})$ representation $(1/2, 0)$) onto representations with $k = 1/2, 3/2, \ldots$. For the $Sp(4,\mathbb{R})$ representation $(1, 1/2)$ the operator $\hat{\phi}_3$ projects onto $SU(1,1)$ or $SO^{\uparrow}(1,2)$ representations with $k = 1, 2, \ldots$.

For related discussions of first class constraints of the type $\hat{\phi}_0$ and $\hat{\phi}_3$ see the Refs. [249].

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Appendix A

Basic properties of group theoretical quantizations

A.1 Generalities

In the following I shall sketch the main ideas of the group theoretical approach to quantizing classical phase spaces (symplectic manifolds). For far more thorough and more detailed discussions I refer to the two main expositions of the subject: Refs. [20, 21]. I shall also borrow heavily from Ref. [40].

I have already indicated in the Introduction how the conventional quantization procedure may be interpreted as a group theoretical quantization in terms of translations in coordinate and momentum space. That interpretation appears complicated by the fact that one hat to extend the abelian group of translations on $\mathbb{R}^{2n}$ by the abelian additive group of the real numbers $\mathbb{R}$ (see the group law 1.46 for $n = 1$). Those complications do not occur for simple groups (i.e. non-abelian Lie groups G the Lie algebra $\mathfrak{g}$ of which does not contain any ideal except $\{0\}$ and $\mathfrak{g}$ itself). The Lie groups we are interested in, namely $SU(1,1) \cong Sp(2,\mathbb{R})$, $SO^{\uparrow}(1,2)$ and $Sp(4,\mathbb{R})$ are all simple.

Group theoretical quantization is a genuine generalization of the conventional quantization procedure to phase spaces (symplectic spaces) $\mathcal{S}^{2n}$ which *globally* are not diffeomorphic to $\mathbb{R}^{2n}$!

On a $2n$–dimensional symplectic space[1] (manifold)

$$\mathcal{S}^{2n} = \{s = (q^1, \ldots, q^n, p_1, \ldots, p_1)\} \tag{A.1}$$

with a non-degenerate (local) symplectic form

$$\omega = dq^1 \wedge dp_1 + \cdots + dq^n \wedge dp_n \ , \ dp_1 \wedge dq^1 = -dq^1 \wedge dp_1 \ , \ \ldots \ , \tag{A.2}$$

one has several Lie algebra structures:

1. Smooth functions $f_j(s), j = 1, 2, \ldots$, on $\mathcal{S}^{2n}$ form a Lie algebra by means of their Poisson brackets:
$$\{f_1, f_2\}_{q,p} = \sum_{j=1}^{n} \partial_{q^j} f_1 \, \partial_{p_j} f_2 - \partial_{p_j} f_1 \, \partial_{q^j} f_2 \, . \tag{A.3}$$

2. The Lie algebra of smooth (tangent) vector fields $X(s)$ on $\mathcal{S}^{2n}$:

[1] In this section I use the "covariant" notation: upper indices for the coordinates, q^j, and lower ones, p_j, for the momenta.

Let $C_{\tau_1\to\tau_2} = \{s(\tau)\,, \tau \in [\tau_1, \tau_2]\,\} \subset \mathcal{S}^{2n}$ be a smooth curve and $f(s)$ a smooth function. Then

$$\frac{d}{d\tau} f[s(\tau)] = \sum_{j=1}^{n} (\partial_{q^j} f(s)) \dot{q}^j(\tau) + (\partial_{p_j} f(s)) \dot{p}_j(\tau)\,,\ \dot{q}^j \equiv \frac{dq^j}{d\tau}\,, \tag{A.4}$$

where $(\dot{q}^1, \ldots, \dot{p}_n)$ determines the tangent vector of $C_{\tau_1\to\tau_2}$ at $s(\tau)$.

The relation A.4 may be interpreted as follows: The $2n$ partial derivatives ∂_{q^j}, ∂_{p_j} form a basis of the tangent space at s. A general tangent vector $X(s)$ at s has the form

$$X(s) = \sum_{j=1}^{n} A^j(s)\partial_{q^j} + B_j(s)\partial_{p_j}\,. \tag{A.5}$$

If $A^j(s), B_j(s), j = 1, \ldots, n$, are smooth functions, then the relation A.5 defines a smooth vector field on $\mathcal{S}^{2n}$.

Comparing with the relation A.4 shows that any such vector field defines a family of curves as solutions of the ordinary differential equations

$$\dot{q}^j = A^j[s(\tau)]\,,\ \dot{p}_j = B_j[s(\tau)]\,,\ j = 1, \ldots, n\,. \tag{A.6}$$

Let $\phi_\tau(s_0)$ be a solution with the initial condition $\phi_{\tau=0}(s_0) = s_0$. Such a solution may be interpreted as a 1-parameter transformation group on $\mathcal{S}^{2n}$ with the property

$$s_0 \to s(\tau) = \phi_\tau(s_0)\,,\ \phi_{\tau_2}[\phi_{\tau_1}(s_0)] = \phi_{\tau_1+\tau_2}(s_0)\,. \tag{A.7}$$

If

$$X_\alpha = \sum_{j=1}^{n} A^j(s;\alpha)\partial_{q^j} + B_j(s;\alpha)\partial_{p_j}\,,\ \alpha = 1, 2\,, \tag{A.8}$$

are two vector fields then their commutator is again a vector field:

$$\begin{aligned}
(X_2\, X_1 - X_1\, X_2)f &= X_3 f\,, \qquad\qquad (A.9)\\
A^j(s;3) &= \sum_{k=1}^{n} [A^k(2)\partial_{q^k} B_j(1) + B_k(2)\partial_{p_k} B_j(1) - \\
&\quad - A^k(1)\partial_{q^k} B_j(2) - B_k(1)\partial_{p_k} B_j(2)]\,,\\
B_j(s;3) &= \sum_{k=1}^{n} [A^k(2)\partial_{q^k} A^j(2) + B_k(2)\partial_{p_k} A^j(1) - \\
&\quad - A^k(1)\partial_{q^k} A^j(1) - B_k(1)\partial_{p_k} A^j(2)]\,.
\end{aligned}$$

In this way we get a Lie algebra of vector fields on $\mathcal{S}^{2n}$.

Recall that the differential 1-forms

$$\rho = \sum_{j=1}^{n} a_j(s) dq^j + b^j(s) dp_j \tag{A.10}$$

are dual to the vector fields A.5, i.e. we have $dq^j(\partial_{q^k}) = \delta^j_k$ etc.

I also briefly recall three important operations on differential forms: exterior differentiation d, interior multiplication i_X by a vector field X and Lie derivation L_X (for a more sytematic introduction into these concepts see, e.g. Refs. [226, 227, 281]):

Exterior differentiation of a function $f(s)$ means

$$df(s) = \sum_{j=1}^{n} \partial_{q^j} f(s) dq^j + \partial_{p_j} f(s) dp_j \ , \tag{A.11}$$

supplemented by the important property $d^2 = 0$. This implies for the differential 1-form A.10

$$d\rho = \sum_{j=1}^{n} da_j(s) \wedge dq^j + db^j(s) \wedge dp_j \ . \tag{A.12}$$

In this way we get the symplectic form A.2 from $\theta = \sum_{j=1}^{n} p_j dq^j$ as $\omega = -d\theta$. We also have $d\omega = 0$.

Exterior differentiation converts a function (a 0-form) into a 1-form, a 1-form into a 2-form etc.

Interior multiplication by a vector field X on the other hand converts a 2-form

$$\begin{aligned} \eta(\cdot,\cdot) \ &= \ \sum_{j,k=1}^{n} \frac{1}{2} a_{jk} dq^j \wedge dq^k + \frac{1}{2} b^{jk} dp_j \wedge dp_k + c_j^k dq^j \wedge dp_k \, , \\ & \quad a_{kj} = -a_{jk} \, , \, b_{kj} = -b_{jk} \, , \end{aligned} \tag{A.13}$$

into the 1-form

$$i_X \eta = \eta(X, \cdot) = \sum_{j,k=1}^{n} (A^j \, a_{jk} - B_j \, c_k^j) \, dq^k + (B_j \, b^{jk} + A^j \, c_j^k) \, dp_k \, . \tag{A.14}$$

For $\eta = \omega$ (Eq. A.2) this simplifies to

$$i_X \omega = \sum_{j=1}^{n} -B_j \, dq^j + A^j \, dp_j \ . \tag{A.15}$$

For functions $f(s)$ one has $i_X(f) = 0$.

The notion of *Lie derivative* L_X *with respect to a vector field* X is important in connection with invariance properties of differential forms: The concept is closely related to the 1-parameter transformation group A.7. It suffices to define L_X for functions $f(s)$ and their differentials $df(s)$ because all differential forms of higher degree may built from them. The definitions are

$$L_X f(s_0) \ = \ Xf(s_0) = \lim_{\tau \to 0} \frac{1}{\tau} [f(\phi_\tau(s_0)) - f(s_0)] \ , \tag{A.16}$$

$$L_X df(s_0) \ = \ d(Xf)(s_0) = d[\lim_{\tau \to 0} \frac{1}{\tau} [f(\phi_\tau(s_0)) - f(s_0)] \, . \tag{A.17}$$

Of considerable practical importance is the identity

$$L_X = d \, i_X + i_X \, d \, . \tag{A.18}$$

If a p-form ρ^p, $p = 0, 1, 2, \ldots$, is invariant under a transformation A.7 this can be expressed as

$$L_X \rho^p = 0 \, . \tag{A.19}$$

3. Of special interest are the so-called "Hamiltonian" vector fields:

   For a Hamiltonian system with Hamilton function H we have the Eqs. of motion

$$\dot{q}^j = \frac{\partial H}{\partial p_j}\,,\ \dot{p}_j = -\frac{\partial H}{\partial q^j}\,,\ j = 1, \ldots, n\,,\ \tau = t\ \text{(time)}. \tag{A.20}$$

   Comparing with Eqs. A.6 we see that the Hamilton function $H(s)$ generates the vector field

$$X_H = \sum_{j=1}^{n} (\partial_{p_j} H)\, \partial_{q^j} - (\partial_{q^j} H)\, \partial_{p_j}\ . \tag{A.21}$$

   We have (see Eq. A.15)

$$i_{X_H}\omega = \sum_{j=1}^{n} -(-\partial_{q^j} H)\, dq^j + (\partial_{p_j} H)\, dp_j = dH\ , \tag{A.22}$$

   i.e. we get from the identity A.18 that (recall $d\omega = 0$)

$$L_{X_H}\omega = d(i_{X_H}\omega) = d(dH) = 0\,. \tag{A.23}$$

   The last equation expresses the important property that the 1-parameter transformations $s_0 = s(t=0) \to s(t) = \phi_t^{(H)}(s_0)$ generated by the Hamilton function H and its associated vector field X_H are "canonical", i.e. they leave the symplectic form ω invariant:

$$\omega_{s(t)} = \omega_{s_0}\,,\ s(t) = \phi_t^{(H)}(s_0)\,. \tag{A.24}$$

   Important is the generalization:

   Any smooth function $f(s)$ generates a Hamilton-type vector field

$$X_f = \sum_{j=1}^{n} (\partial_{p_j} f)\, \partial_{q^j} - (\partial_{q^j} f)\, \partial_{p_j}\,, \tag{A.25}$$

   with the properties

$$i_{X_f}\omega = df\,,\ L_{X_f}\omega = 0\,. \tag{A.26}$$

   If X_{f_1} and X_{f_2} are two such vector fields then one may define the Poisson brackets A.3 of f_1 and f_2 as

$$\{f_1, f_2\} = \omega(X_{f_1}, X_{f_2}) = -X_{f_1}(f_2) = X_{f_2}(f_1)\,. \tag{A.27}$$

   Essential is the following relationship between the Lie algebra structure induced by the Poisson brackets A.3 and the Lie algebra structure A.9 of Hamiltonian vector fields:

$$[X_{f_1}, X_{f_2}] = -X_{\{f_1, f_2\}}\,. \tag{A.28}$$

   These remarks on Hamiltonian vector fields show that we have a homomorphism

$$f \mapsto -X_f \tag{A.29}$$

   of the Lie algebra of smooth functions f on $\mathcal{S}^{2n}$ onto the Lie algebra of smooth Hamiltonian vector fields X_f on $\mathcal{S}^{2n}$.

   It is also important to notice that this mapping has a non-trivial kernel, namely the constant functions:

$$f_0 = \ \text{const.}\ \mapsto -X_{f_0} = 0\ . \tag{A.30}$$

After all these preliminaries we now come closer to the gist of the group theoretical quantization approach:

Let us assume that there is a r-dimensional Lie transformation group G $= \{g\}$ acting on

$$\mathcal{S}^{2n} = \{s\} \ : \ s \to g \cdot s, \tag{A.31}$$

and which has the following properties

1. The transformations $s \to g \cdot s$ leave the symplectic form ω invariant:

$$\omega_{g \cdot s} = \omega_s \ . \tag{A.32}$$

2. Let $g(t) \subset G$ be 1-parameter subgroup generated by an element $A \in \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ of G:

$$g(t) = e^{-A\,t} \ , \ A \in \mathfrak{g} \ . \tag{A.33}$$

   The action of such a subgroup in turn generates a vector field $\tilde{A}(s)$ on $\mathcal{S}^{2n}$:

$$[\tilde{A}f](s) = \lim_{t \to 0} \frac{1}{t}[f(e^{-A\,t} \cdot s) - f(s)] \ . \tag{A.34}$$

   We shall discuss some properties of these G-induced vector fields below.

3. For the implementation of the intended quantization procedure one wants to have an isomorphismen between the r-dimensional Lie algebra $\mathfrak{g}$ and the corresponding Lie algebra $\tilde{\mathfrak{g}}$ of the induced vector fields $\tilde{A}(s)$. This is the case if the action of the group is effective, i.e. if $g \cdot s = s \ \forall s$, then $g = e$ (unit group element).

   The latter condition may be relaxed to almost effective actions, i.e. if $g \cdot s = s \ \forall s$, then g is an element of a discrete center subgroup. This generalization is possible because the existence of such a discrete center subgroup does not affect the structure of the Lie algebra which is the same for a group, all of its covering groups and all groups which may be obtained by factoring out a discrete center group.

4. G should act transitively on $\mathcal{S}^{2n}$, i.e. if s_1 and s_2 are any two points of $\mathcal{S}^{2n}$ then there exist a group element $g_{1\to 2} \in G$ such that $s_2 = g_{1\to 2} \cdot s_1$.

   Transitivity of the group action means that the group G can map any given "state" to any other "state" of the symplectic space, i.e. it takes the global structure of the "phase space" into account!

   The transitivity requirement of the action in general will imply that the dimension r of the group G is larger than the dimension $2n$ of the space $\mathcal{S}^{2n}$. This is certainly so if the latter may be described as a homogenous space G/H, where $H \subset G$ is an appropriate subgroup of G.

5. As the transformations A.33 leave the symplectic form ω invariant its Lie-derivatives $L_{\tilde{A}}$ have the property

$$L_{\tilde{A}}\omega = 0 \, , \tag{A.35}$$

   which – together with $d\omega = 0$ and according to the relation A.18 – implies

$$d\,(i_{\tilde{A}}\omega) = 0 \ . \tag{A.36}$$

   The last relation means that $i_{\tilde{A}}\omega$ is a closed 1-form on $\mathcal{S}^{2n}$. The corresponding vector fields $\tilde{A}$ are called "locally Hamiltonian".

According to Poincaré's famous lemma one has locally

$$i_{\tilde{A}}\omega(s) = dh_A(s)\,, \tag{A.37}$$

where $h_A(s)$ is some function.

Under certain conditions (the first cohomology group $H^1(\mathcal{S}^{2n})$ has to vanishes) $i_{\tilde{A}}\omega$ is even exact and we have a globally defined Hamiltonian vector field, i.e. we have

$$\tilde{A}(s) = -X_{h_A}(s)\ \forall s \in \mathcal{S}^{2n}\ . \tag{A.38}$$

If the Lie algebra element A can be written as the commutator of two other ones, $A = [A_1, A_2]$, then, because of $i_{[\tilde{A}_1,\tilde{A}_2]}\omega = d(i_{\tilde{A}_1} i_{\tilde{A}_2}\omega)$, $\tilde{A}$ is globally Hamiltonian. This is so for semisimple transformation groups G, a case we are mainly interested in in this paper.

We now come to the central part of the group theoretical quantization program:

Up to now we have established

1. an isomorphism between the r-dimensional Lie algebra $\mathfrak{g}$ and a corresponding r-dimensional Lie subalgebra $\tilde{\mathfrak{g}}$ of Hamiltonian vector fields on $\mathcal{S}^{2n}$ and

2. a homomorphism A.29 of functions $f(s)$ on $\mathcal{S}^{2n}$ into the Lie algebra of Hamiltonian vector fields, with the constant functions as kernel.

What we are aiming at is the following: In general one will select some special functions $h_j(s)$, $j = 1, 2, \ldots$, as basic "observables" associated with the given symplectic space $\mathcal{S}^{2n}$, in such a way that these function form a Lie algebra with respect to the Poisson brackets A.3. In the conventional case these special functions are the canonical variables q^j, p_j, $j = 1, \ldots, n$ and the number $1 \in \mathbb{R}$. In the well-known quantization procedure described in Sec. 1.3.1 these special functions become self-adjoint operators representing the generating Lie algebra of the Weyl-Heisenberg group.

We are interested in the following generalizations:

1. We want to introduce an appropriate set of basic functions $h_{A_\rho}, \rho = 1, \ldots, r$, forming a Poisson Lie algebra which is isomorphic to the Lie algebra $\mathfrak{g}$ with basis $\{A_\rho, \rho = 1, \ldots, r\}$:

$$A_\rho \mapsto \tilde{A}_\rho \;=\; -X_{h_{A_\rho}} \leftrightarrow h_{A_\rho}\,, \tag{A.39}$$

$$\{h_{A_\rho}, h_{A_\sigma}\} \;=\; h_{[A_\rho, A_\sigma]}\,,\ \rho, \sigma = 1, \ldots, r\,. \tag{A.40}$$

The relations A.39 and A.40 are by no means trivial and cannot always be satisfied. The deeper reason is that the functions h_A and $h_A + const.$ generate the same Hamiltonian vector field X_{h_A}. This implies that the relation A.40 may acquire an additional constant $c(A_\rho, A_\sigma)$ on the r.h. side which cannot be made to vanish! Such complications occur for the group theoretical quantization approach to the conventional quantization procedure (for details see Refs. [20, 21]) I shall not discuss these important features here, because they do not occur for the (simple) groups we are dealing with in the present paper: $SU(1,1) \cong Sp(2, \mathbb{R})$ and $Sp(4, \mathbb{R})$.

In modern symplectic differential geometry the existence of the isomorphism A.39 and A.40 is closely related to properties of the so-called "momentum map" [250–253]; for a recent historical review see [254]).

If the relations A.39 and A.40 do hold, then one calls the r-dimensional group G the "canonical group" of the symplectic space $\mathcal{S}^{2n}$.

2. Having established the above isomorphism between the Lie algebra $\mathfrak{g}$ and a corresponding Poisson Lie algebra of a system $\{h_A\}$ of preferred observables on $\mathcal{S}^{2n}$, one then can quantize the classical system by using the irreducible unitary representations of the transformation group G where the self-adjoint generators $K_\rho(A_\rho)$ of the unitary 1-parameter subgroups
$$U[g_\rho(t) = \exp(-A_\rho t)] = \exp[-iK_\rho(A_\rho)t]\,,\ \rho = 1, \ldots, r, \tag{A.41}$$
represent the corresponding original classical observables h_{A_ρ}.

3. As there may be different groups with symplectic, transitive and effective action on $\mathcal{S}^{2n}$, one has to make a choice which one to use.

   Here physical considerations come into play: One wants a group such that the corresponding observables $h_{A_\rho}(s)$ constitute basic functions on $\mathcal{S}^{2n}$, so that all physically interesting observables can be expressed by them. For additional discussions of these problems see Refs. [20, 39, 255]

A.2 The canonical group $SO^\uparrow(1,2)$ of the symplectic space $\mathcal{S}^2_{\varphi,I} = \{\varphi \in \mathbb{R} \bmod 2\pi\,, I > 0\}$

I now want to apply the general remarks of the last section to the concrete phase space
$$\mathcal{S}^2_{\varphi,I} = \{\varphi \in \mathbb{R} \bmod 2\pi\,, I > 0\}\,, \tag{A.42}$$
with the Poisson brackets
$$\{k_1, k_2\} = \partial_\varphi k_1 \partial_I k_2 - \partial_I k_1 \partial_\varphi k_2\,. \tag{A.43}$$
As the definition of the space A.42 means that the origin $\{0\}$ of the underlying plane $\mathbb{R}^2$ is deleted, one cannot use the 2-dimensional translations as the canonical quantizing group because it cannot avoid the origin as the result of special elements of the group! One therefore has to find another appropriate group which has all the desired properties listed in the last section.

The appropriate canonical group for the phase space A.42 is the proper orthochronous Lorentz group $SO^\uparrow(1,2)$ which leaves the quadratic form
$$(x^0)^2 - (x^1)^2 - (x^2)^2\ ,\ x^0 > 0\,, \tag{A.44}$$
invariant, has determinant $+1$ and also leaves the time direction unchanged [39, 40].

The first reason for the choice of the group $SO^\uparrow(1,2)$ is that the cone
$$(x^0)^2 - (x^1)^2 - (x^2)^2 = 0\ ,\ x^0 > 0\,, \tag{A.45}$$
is homeomorphic (and diffeomorphic) to the space A.42. In order to see this put
$$x^0 = I > 0\,,\ x^1 = I\cos\varphi\,,\ x^2 = -I\sin\varphi\,, \tag{A.46}$$
which provides a smooth parametrization of that space: any given triple (x^0, x^1, x^2) of Eqs. A.46 uniquely determines $I > 0$ and $\varphi \in (-\pi, \pi]$!

In the following it is advantageous to employ the twofold covering group $SU(1,1)$ of $SO^\uparrow(1,2)$ (see Appendix B) the elements g_0 of which are given by
$$g_0 = \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}\,,\quad \det g_0 = |\alpha|^2 - |\beta|^2 = 1\,. \tag{A.47}$$

If we define the matrix

$$X = \begin{pmatrix} x^0 & x^1 - i\,x^2 \\ x^1 + i\,x^2 & x^0 \end{pmatrix} , \quad \det X = (x^0)^2 - (x^1)^2 - (x^2)^2 , \tag{A.48}$$

the transformations $x^\mu \to \hat{x}^\mu,\ \mu = 0, 1, 2$, under $SO^\uparrow(1,2)$ are implemented by

$$X \to \hat{X} = g_0 \cdot X \cdot g_0^+ , \quad \det \hat{X} = \det X , \tag{A.49}$$

where g_0^+ denotes the hermitian conjugate of the matrix g_0.

Applying a general g_0 to the matrix

$$X = \begin{pmatrix} I & I\,e^{-i\varphi} \\ I\,e^{i\varphi} & I \end{pmatrix} \tag{A.50}$$

yields the mapping:,

$$(I, \varphi) \ \to \ (\hat{I}, \hat{\varphi}) : \tag{A.51}$$

$$\hat{I} \ = \ |\alpha + e^{i\varphi}\,\beta|^2\ I , \tag{A.52}$$

$$e^{i\hat{\varphi}} \ = \ \frac{\bar{\alpha}\,e^{i\varphi} + \bar{\beta}}{\alpha + e^{i\varphi}\,\beta} . \tag{A.53}$$

As

$$\frac{\partial\hat{\varphi}}{\partial\varphi} = |\alpha + e^{i\varphi}\beta|^{-2} , \tag{A.54}$$

we have the equality

$$d\hat{\varphi} \wedge d\hat{I} = d\varphi \wedge dI , \tag{A.55}$$

that is, the transformations A.52 and A.53 are symplectic.

One sees immediately that g_0 and $-g_0$ lead to the same transformations of I and φ. Thus, the group $SU(1,1)$ acts on on the space A.42 only almost effectively with the kernel Z_2 representing the center of the twofold covering group $SU(1,1)$ of $SO^\uparrow(1,2)$. It is well-known that the latter group acts effectively and transitively on the forward light cone and thus on A.42 (see also the remarks below after Eq. A.64).

For later we need the actions of the 1-parametric subgroups $\mathrm{R}_0, \mathrm{A}_0$ and N_0 which form the Iwasawa decomposition $SU(1,1) = \mathrm{R}_0 \cdot \mathrm{A}_0 \cdot \mathrm{N}_0$ (see the Eqs. B.23-B.25), with the general element

$$\begin{aligned} r_0 \cdot a_0 \cdot n_0 \ = \ & \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \cdot \begin{pmatrix} \cosh(t/2) & i\,\sinh(t/2) \\ -i\,\sinh(t/2) & \cosh(t/2) \end{pmatrix} \\ & \cdot \begin{pmatrix} 1 + i\xi/2 & \xi/2 \\ \xi/2 & 1 - i\,\xi/2 \end{pmatrix} , \end{aligned} \tag{A.56}$$

where $\theta \in (-2\pi, +2\pi];\ t, \xi \in \mathbb{R}$. According to A.49 the actions of the 1-parameter subgroups $\mathrm{R}_0, \mathrm{A}_0, \mathrm{N}_0$, respectively, are:

$$\mathrm{R}_0 : \qquad \hat{I} = I , \tag{A.57}$$

$$e^{i\hat{\varphi}} = e^{i(\varphi - \theta)} . \tag{A.58}$$

$$\mathrm{A}_0 : \qquad \hat{I} = \rho(t,\varphi)\,I , \quad \rho(t,\varphi) = \cosh t - \sinh t\,\sin\varphi , \tag{A.59}$$

$$\cos\hat{\varphi} = \cos\varphi/\rho(t,\varphi) , \tag{A.60}$$

$$\sin\hat{\varphi} = (\cosh t \sin\varphi - \sinh t)/\rho(t,\varphi) . \tag{A.61}$$

$$\mathrm{N}_0 : \qquad \hat{I} = \rho(\xi,\varphi)\,I , \ \rho(\xi,\varphi) = 1 + \xi\cos\varphi + \xi^2(1 + \sin\varphi)/2 , \tag{A.62}$$

$$\cos\hat{\varphi} = [\cos\varphi + \xi(1 + \sin\varphi)]/\rho(\xi,\varphi) , \tag{A.63}$$

$$\sin\hat{\varphi} = [\sin\varphi - \xi\cos\varphi - \xi^2(1 + \sin\varphi)/2]/\rho(\xi,\varphi) . \tag{A.64}$$

Transitivity of the $SU(1,1)$ group action on A.42 can be seen as follows:

Any point $s_1 = (\varphi_1, I_1)$ may be transformed into any other point $s_2 = (\varphi_2, I_2)$: first transform (φ_1, I_1) into $(0, I_1)$ by $r_0(\theta = \varphi_1)$, then map this point into $(\varphi_0 = -\arctan(\sinh t_0\,), I_2)$ by $a_0(t_0; \cosh t_0 = I_2/I_1)$ and finally transform (φ_0, I_2) by $r_0(\theta = \varphi_0 - \varphi_2)$ into $s_2 = (\varphi_2, I_2)$.

These transitivity properties reflect the fact that any element g_0 of $SU(1,1)$ may be written as $r_0(\theta_2) \cdot a_0(t) \cdot r_0(\theta_1)$ (see B.30).

The transformation formulae A.63 and A.64 show that the group N_0 leaves the half-line $\varphi = -\pi/2$, $I > 0$, invariant, that is, N_0 is the stability group of these points. This means that the symplectic space A.42 is diffeomorphic to the coset space $SU(1,1)/(Z_2 \times \mathrm{N}_0) \simeq SO^{\uparrow}(1,2)/\mathrm{N}_0$. Notice that N_0, and A_0 as well, does not contain the second center element $-e$ of $SU(1,1)$. The center Z_2 is a subgroup of R_0.

We also give the action of the group B_0,(see B.27), on the points $s = (\varphi, I)$:

$$B_0: \qquad \hat{I} = \rho(s,\varphi)\, I\,, \;\; \rho(s,\varphi) = \cosh t + \sinh t\, \cos\varphi\,, \tag{A.65}$$

$$\cos\hat{\varphi} = (\cosh s \cos\varphi + \sinh s)/\rho(s,\varphi)\,, \tag{A.66}$$

$$\sin\hat{\varphi} = \sin\varphi/\rho(s,\varphi)\,. \tag{A.67}$$

We next determine the Hamiltonian vector fields induced on A.42 by the above $SU(1,1)$ transformations and – most important – the corresponding classical observables h_R, h_A and h_B.

For infinitesimal values of the parameters θ, t and s the transformations A.57–A.61 and A.65–A.67 take the form

$$\mathrm{R}: \quad \delta\varphi = -\theta,\ |\theta| \ll 1, \quad \delta I = 0\ , \tag{A.68}$$

$$\mathrm{A}: \quad \delta\varphi = -(\cos\varphi)\, t, \quad \delta I = -I\,(\sin\varphi)\, t, \quad |t| \ll 1\ , \tag{A.69}$$

$$\mathrm{B}: \quad \delta\varphi = -(\sin\varphi)\, s, \quad \delta I = I\,(\cos\varphi)\, s, \quad |s| \ll 1\ \ . \tag{A.70}$$

According to A.34 they induce on A.42 the vector fields

$$\tilde{A}_R = \partial_\varphi\,, \tag{A.71}$$

$$\tilde{A}_A = \cos\varphi\,\partial_\varphi + I\,\sin\varphi\,\partial_I\,, \tag{A.72}$$

$$\tilde{A}_B = \sin\varphi\,\partial_\varphi - I\,\cos\varphi\,\partial_I\,. \tag{A.73}$$

It is easy to check that their Lie algebra is isomorphic to the Lie algebra of $SO^{\uparrow}(1,2)$ (see Sec. B.2 of Appendix B), and all its covering groups, of course.

According to the general relations A.25, A.29 and A.38 we get the following relations (recall that $X_f = \partial_I f\,\partial_\varphi - \partial_\varphi f\,\partial_I$)

$$\tilde{A}_R = -X_{f_R}\,,\ f_R(\varphi, I) = -I\,, \tag{A.74}$$

$$\tilde{A}_A = -X_{f_A}\,,\ f_A(\varphi, I) = -I\,\cos\varphi\,, \tag{A.75}$$

$$\tilde{A}_B = -X_{f_B}\,,\ f_B(\varphi, I) = -I\,\sin\varphi\,. \tag{A.76}$$

The functions f_R, f_A and f_B obey the Lie algebra $\mathfrak{so}(1,2)$ with respect to the Poisson brackets A.43:

$$\{f_R, f_A\} = -f_B\,,\ \{f_R, f_B\} = f_A\,,\ \{f_A, f_B\} = f_R\,. \tag{A.77}$$

In order to avoid two of the minus signs we finally define as our three basic classical observables the functions

$$h_0(\varphi, I) \equiv -f_R = I\,,\ h_1(\varphi, I) \equiv -f_A = I\,\cos\varphi\,,\ h_2(\varphi, I) \equiv f_B = -I\,\sin\varphi\,. \tag{A.78}$$

Their Poisson brackets

$$\{h_0, h_1\}_{\varphi,I} = -h_2 \,,\ \{h_0, h_2\}_{\varphi,I} = h_1 \,,\ \{h_1, h_2\}_{\varphi,I} = h_0 \,, \tag{A.79}$$

again form the Lie algebra $\mathfrak{so}(1,2)$.

The Eqs. A.78 constitute one of our principal results:

The canonical group $SO^{\uparrow}(1,2)$ of the symplectic space A.42 *determines* the basic "observables" A.78 of that classical space.

The functions A.78 are indeed suitable in order to fulfill the desired purposes:

Any smooth function $f(\varphi, I)$ periodic in φ with period 2π can, under quite general conditions, be expanded in a Fourier series and as $\sin(n\varphi)$ and $\cos(n\varphi)$ can be expressed as polynomials of n-th order in $\sin\varphi = -h_2/I$ and $\cos\varphi = h_1/I$, the observables A.78 are indeed sufficient.

Actually the functions A.78 are just the cone coordinates A.50 we started from! We merely have to identify

$$I = h_0 \,, \quad I\, e^{-i\varphi} = h_1 + i\, h_2 \,. \tag{A.80}$$

The transformations A.49 imply that (h_0, h_1, h_2) transforms as a 3-vector with repect to the group $SO^{\uparrow}(1,2)$. The explicit transformation formulae for the three subgroups B.23, B.24 and B.27 are

$$\begin{aligned}
R_0: \quad h_0 &\to \hat{h}_0 = h_0 \,, && \text{(A.81)}\\
h_1 &\to \hat{h}_1 = \cos\theta\, h_1 - \sin\theta\, h_2 \,, && \text{(A.82)}\\
h_2 &\to \hat{h}_2 = \sin\theta\, h_1 + \cos\theta\, h_2 \,, && \text{(A.83)}
\end{aligned}$$

$$\begin{aligned}
A_0: \quad h_0 &\to \hat{h}_0 = \cosh t\, h_0 + \sinh t\, h_2 \,, && \text{(A.84)}\\
h_1 &\to \hat{h}_1 = h_1 \,, && \text{(A.85)}\\
h_2 &\to \hat{h}_2 = \sinh t\, h_0 + \cosh t\, h_2 \,, && \text{(A.86)}
\end{aligned}$$

$$\begin{aligned}
B_0: \quad h_0 &\to \hat{h}_0 = \cosh s\, h_0 + \sinh s\, h_1 \,, && \text{(A.87)}\\
h_1 &\to \hat{h}_1 = \sinh s\, h_0 + \cosh s\, h_1 \,, && \text{(A.88)}\\
h_2 &\to \hat{h}_2 = h_2 \,. && \text{(A.89)}
\end{aligned}$$

So we have rotations in the $h_1 - h_2$ plane and two Lorentz "boosts", one in the $h_0 - h_2$ plane and the other in the $h_0 - h_1$ plane! All transformations leave the form $h_0^2 - h_1^2 - h_2^2$ invariant.

In addition we know from the general result A.55 that these transformations are symplectic transformations of the phase space A.42.

After all the efforts the quantization of the phase space A.42 is now straightforward:

The irreducible unitary representations of the group $SO^{\uparrow}(1,2)$ and its covering groups are well-known (see Appendix B). Their 1-parameter unitary subgroups are generated by self-adjoint generators K_j, $j = 0, 1, 2$, corresponding to the three observables A.78:

$$h_0 \to K_0 \ ,\ h_1 \to K_1 \ ,\ h_2 \to K_2 \ . \tag{A.90}$$

Because $h_0 \equiv I > 0$ the quantized theory has to use the positive discrete series of the irreducible unitary representations (Appendix B).

A.3 The symplectic space $\mathcal{S}^2_{\varphi,I}$ and the orbifold $\mathbb{R}^2/Z_2$

We now come to a very interesting relationship between the symplectic space A.42 and the original phase space

$$\mathcal{S}^2_{q,p} = \{x = \begin{pmatrix} q \\ p \end{pmatrix} \in \mathbb{R}^2\}\,, \tag{A.91}$$

on which the symplectic group $Sp(2,\mathbb{R})$, B.13, acts as

$$x \to \hat{x} = g_1 \cdot x\,,\ g_1 \in Sp(2,\mathbb{R})\,, \tag{A.92}$$

with the property

$$d\hat{q} \wedge d\hat{p} = dq \wedge dp\,. \tag{A.93}$$

The group action A.92 has some intriguing other properties:

The whole group transforms the point $x = 0$ into itself and acts transitively on the complement

$$\mathcal{S}^2_{q,p;0} \equiv \mathcal{S}^2_{q,p} - \{x = 0\}. \tag{A.94}$$

It also acts effectively on the latter because the second element $-e$ of its center $Z_2 = \{e, -e\}$, where

$$e = E_2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,, \tag{A.95}$$

acts non-trivially on $\mathcal{S}^2_{q,p;0}$:

$$(-e) \cdot x = -x \neq x\,. \tag{A.96}$$

This is in obvious contrast to the action A.49 of the group $SU(1,1) \cong Sp(2,\mathbb{R})$ on the space A.42 for the points s of which one has

$$(-e) \cdot s = s\,,\ e = E_2 \in SU(1,1)\,, \tag{A.97}$$

as the Eqs. A.52 and A.53 show.

How can this be reconciled especially in view of the fact that *locally*

$$d\varphi \wedge dI = dq \wedge dp\,? \tag{A.98}$$

Recall also that the space A.42 is diffeomorphic to a cone with the tip (vertex) deleted, but that the space A.91 is *globally* different!

The neat reconciliation of this apparent difficulty is the following:

The mapping A.92 has the same property A.97 of the mapping A.52,A.53 if we identify the points $-x$ and x of the space A.91, i.e. if we pass from the space A.91 to the quotient space

$$\mathcal{S}^2_{q,p}/Z_2 \equiv \check{\mathcal{S}}^2_{q,p} = \{\check{s} = \pm x\,,\ ,x \in \mathbb{R}^2\} \cong \mathbb{R}^2/Z_2\,. \tag{A.99}$$

Such a space is called an "orbifold" [256].

An orbifold may be generated from a manifold $\mathtt{M}$ by identifying points which are connected by a finite discontinuous group D_n of n elements so that the orbifold is given by the quotient space $\mathtt{M}/D_n$.

An orbifold generally has additional singularities as compared to the manifold from which it is constructed, as we shall see now:

In our case the orbifold A.99 is a cone: Take the lower half of the (q,p)-plane and rotate it around the q-axis till it coincides with the upper half of the plane such that the negative p-axis

lies on the positive one. Then rotate the left half of the upper half plane around the positive p-axis till the negative q-axis coincides with the positive one. Finally glue the two q-half-axis together. The resulting space is a cone with its "tip" (vertex) at $x = 0$. (See, e.g. Fig. 1 in Ref. [50].)

We thus arrive at the cone structure for the symplectic space A.42 by a differerent route and the quantization of that space appears to be equivalent to the quantization of the orbifold A.99, with the vertex deleted!

Next let us see which vector fields are induced on A.91 by the groups B.20, B.21 and B.26 and which are the associated Hamiltonian functions.

The same procedure as in the previous section yields

$$\tilde{A}_{R_1} = \frac{1}{2}(q\,\partial_p - p\,\partial_q)\,, \tag{A.100}$$

$$\tilde{A}_{A_1} = -\frac{1}{2}(q\,\partial_q + p\,\partial_p)\,, \tag{A.101}$$

$$\tilde{A}_{B_1} = -\frac{1}{2}(p\,\partial_q + q\,\partial_p)\,, \tag{A.102}$$

and the corresponding Hamiltonian functions (A.25 and A.26) are

$$\check{g}_0(q,p) = \frac{1}{4}(q^2 + p^2)\,, \tag{A.103}$$

$$\check{g}_1(q,p) = -\frac{1}{2}\,q\,p\,, \tag{A.104}$$

$$\check{g}_2(q,p) = \frac{1}{4}(q^2 - p^2)\,. \tag{A.105}$$

Their Poisson brackets again obey the Lie algebra $\mathfrak{so}(1,2)$:

$$\{\check{g}_0, \check{g}_1\}_{q,p} = -\check{g}_2\,,\ \{\check{g}_0, \check{g}_2\}_{q,p} = \check{g}_1\,,\ \{\check{g}_1, \check{g}_2\}_{q,p} = \check{g}_0\,. \tag{A.106}$$

Notice that

$$\check{g}_0^2 - \check{g}_1^2 - \check{g}_2^2 = 0\,. \tag{A.107}$$

Inserting

$$q = \sqrt{2I}\cos\varphi\,,\ p = -\sqrt{2I}\sin\varphi\,, \tag{A.108}$$

into the expressions A.103-A.105 we get another set of functions $\check{h}_j(\varphi, I)\,, j = 0, 1, 2$, which obey the Lie algebra $\mathfrak{so}(1,2)$ with respect to the Poisson brackets $\{\cdot,\cdot\}_{\varphi,I}$:

$$\check{h}_0(\varphi, I) = \frac{1}{2}I\,, \tag{A.109}$$

$$\check{h}_1(\varphi, I) = \frac{1}{2}I\sin(2\varphi)\,, \tag{A.110}$$

$$\check{h}_2(\varphi, I) = \frac{1}{2}I\cos(2\varphi)\,. \tag{A.111}$$

We observe that the vector fields A.100-A.102 are non-trivial only for $(q,p) \neq (0,0)$ and that the origin (0,0) here has to be excluded, too!

As the vector fields A.100-A.102 and the Hamiltonian functions A.103-A.105 are invariant against the substitution $(q,p) \to -(q,p)$, they are defined on A.91 and on the orbifold A.99 as well.

The identification of the points (q,p) and $(-q,-p)$ implies the identification of φ and $\varphi \pm \pi$ for the angle φ in A.108. It leaves the functions A.109-A.111 invariant!

The last point may also be disscused in terms of the complex amplitude $a = \sqrt{I}\exp(-i\varphi)$ from B.51: Identification of a and $-a$ means identification of a and $\exp(\pm i\pi)\,a$ and passing to the functions A.109-A.111 is equivalent to passing to I and a^2 or $\bar{a}^2$.

The functions $\check{g}_j(q,p)\,, j = 0,1,2$, and $\check{h}_j(\varphi, I)\,, j = 0,1,2$, provide another parametrization of the cone (without its tip) representing the symplectic space A.42 (with φ now $\in (-\pi/2, \pi/2]$). This parametrization is equivalent to that by the functions A.78 from above!

The functions $h_j(\varphi, I)$, $\check{g}_j(q,p)$ and $\check{h}_j(\varphi, I)$, respectively, transform as 3-vectors with respect to the group $SO^{\uparrow}(1,2) = Sp(2,\mathbb{R})/Z_2 = SU(1,1)/Z_2$. This follows immediately from the fact that they obey the Lie algebra $\mathfrak{so}(1,2)$, i.e. they transform according to the adjoint representation. These transformations may be induced, e.g. by the action A.92 of the group B.13.

Take the subgroup B.20 as an example: For the coordinates x, A.91, we have

$$x \to \hat{x} = \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} = \begin{pmatrix} \cos(\theta/2)\, q + \sin(\theta/2)\, p \\ -\sin(\theta/2)\, q + \cos(\theta/2)\, p \end{pmatrix} . \tag{A.112}$$

These transformations induce the following mappings (rotations!) of the functions A.103-A.105:

$$\check{g}_0(q,p) = \frac{1}{2} I(q,p) \to \frac{1}{2} I(q,p)\,, \tag{A.113}$$

$$\check{g}_1(q,p) = -\frac{1}{2} q\, p \to \cos\theta\, \check{g}_1(q,p) + \sin\theta\, \check{g}_2(q,p)\,, \tag{A.114}$$

$$\check{g}_2(q,p) = \frac{1}{4}(q^2 - p^2) \to -\sin\theta\, \check{g}_1(q,p) + \cos\theta\, \check{g}_2(q,p)\,. \tag{A.115}$$

The transformations A.112 and A.114-A.115 illustrate the central message of this section very clearly: the canonical pair x transforms as a "spinor" (see also Sec. 6.3), whereas the pair $\check{g}_1, \check{g}_2$ transforms as a vector as to the group $SO^{\uparrow}(1,2)$!

If one chooses $\theta = 2\pi$, then we have the identity transformation for A.114 and A.115, but x is replaced by $-x$ in A.112!

Appendix B

The group $SO^{\uparrow}(1,2)$, some of its covering groups and their irreducible unitary representations of the positive discrete series

In the present appendix I summarize properties of the group $SO^{\uparrow}(1,2)$, some of its covering groups, their Lie algebra and their irreducible unitary representations, especially those of the positive discrete series. These properties are important for the group theoretical quantization procedures of the phase space A.42.

The following material is essentially taken from Appendix A and Ch. V of Ref. [40]. Practically all of it is contained in a wealth of literature about the group $SO^{\uparrow}(1,2)$ which is the most elementary of noncompact semisimple Lie groups. The readers of the present paper will probably find it convenient to have the required properties assembled in an appendix here, too.

The essential classical paper on the group $SO^{\uparrow}(1,2)$ and its irreducible unitary repesentations is (still!) that of Bargmann [257]. In the meantime there are a number of monographs (and reviews) which deal with the group $SO^{\uparrow}(1,2)$, its covering groups and their representations [258–263, 265–268]. As these textbooks contain many references to the original literature we mention only the most essential ones for our purposes.

B.1 The group and some covering groups

B.1.1 The groups $SU(1,1) \cong SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$

In order to see the homomorphism between $SO^{\uparrow}(1,2)$ and its isomorphic twofold covering groups $SU(1,1), SL(2,\mathbb{R})$ and the symplectic group $Sp(2,\mathbb{R})$ in 2 dimensions, it is convenient to start from the action of the group $SL(2,\mathbb{C})$ – the twofold covering group of the proper orthochronous Lorentz group $SO^{\uparrow}(1,3)$ – on Minkowski space M^4 with the scalar product $x \cdot x = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$:

Define the hermitean matrix

$$X = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} = \sum_{j=0}^{3} x^j \sigma_j \,, \tag{B.1}$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \,,\ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,,\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \,,\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,, \tag{B.2}$$

are the Pauli matrices.

As

$$\det X = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \,, \tag{B.3}$$

the proper orthochronous Lorentz transformations

$$x^j \to \hat{x}^j = \sum_{k=0}^{3} \Lambda^j{}_k x^k \,, \;\; \det(\Lambda^j{}_k) = 1 \,, \;\; \Lambda^0{}_0 > 0 \,, \tag{B.4}$$

may be implemented as follows:

If $A \in SL(2,\mathbb{C}) \subset GL(2,\mathbb{C})$, $\det A = 1$, then

$$X \to \hat{X} = A \cdot X \cdot A^+ = \sum_{j=0}^{3} \hat{x}^j \, \sigma_j \,, \quad \det\hat{X} = \det X \,. \tag{B.5}$$

Here A^+ means the hermitean conjugate of the matrix A and $\hat{X}$ the matrix B.1 with x^j replaced by $\hat{x}^j$.

The well-known properties of the Pauli matrices allow to express the parameters $\Lambda^j{}_k$ in terms of the matrices A as follows

$$\Lambda^j{}_k(A) = \frac{1}{2} \, \mathrm{tr}(\sigma_j \cdot A \cdot \sigma_k \cdot A^+) \,. \tag{B.6}$$

Those subgroups $SO^\uparrow(1,2)$ which interest us here may be obtained by looking for the transformations B.5 which leave one of the coordinates x^j, $j = 1, 2$ or 3 fixed:

The transformations with the property

$$A \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot A^+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.7}$$

leave the coordinates x^3 invariant and represent the subgroup $SU(1,1) = \{g_0\} \subset SL(2,\mathbb{C})$:

$$g_0 = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \;\; , \;\; \det g_0 = |\alpha|^2 - |\beta|^2 = 1 \;. \tag{B.8}$$

$\bar{\alpha}$: complex conjugate of α. The group elements g_0 act on a 2-dimensional complex vector space $\mathbb{C}^2$ as

$$g_0 \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \end{pmatrix} \;, \quad \text{with } |\hat{z}_1|^2 - |\hat{z}_2|^2 = |z_1|^2 - |z_2|^2 \;. \tag{B.9}$$

If $|z_2| > |z_1|$ and $z = z_1/z_2$ then $SU(1,1)$ maps the interior

$$\mathbb{D}_1 = \{z; |z| < 1\} \tag{B.10}$$

of the unit disc in the complex z-plane (transitively) onto itself:

$$z \in \mathbb{D}_1 \to \hat{z} = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}} \in \mathbb{D}_1 \;. \tag{B.11}$$

This property is important for the construction of the irreducible unitary representations of $SU(1,1)$ (see Sec. B.3 below).

Notice that the group elments g_0 and $-g_0$ yield the same transformation B.11. Thus, the transformation group is actually $SO^\uparrow(1,2) = SU(1,1)/Z_2$ where $Z_2 = \{e, -e\}$, e : unit group element, is the (discrete) center of $SU(1,1)$.

The subgroup of $SL(2,\mathbb{C})$ with the property

$$A \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot A^{+} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{B.12}$$

leaves the coordinates x^2 invariant. It constitutes the group $SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$:

$$C = g_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\ ,\ a_{jk} \in \mathbb{R}\,,\ \ \det g_1 = 1\,. \tag{B.13}$$

As

$$g_1^T \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot g_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ , \tag{B.14}$$

the group $SL(2,\mathbb{R})$ is identical with the *real symplectic group* $Sp(2,\mathbb{R})$ in 2 dimensions.
(I follow the convention to denote the real symplectic group of a $2n$-dimensional vector space by $Sp(2n,\mathbb{R})$. In many papers the convention $Sp(n,\mathbb{R})$ is being used instead. The number n coincides with the rank of the group.)

The unitary matrix

$$C_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\ ,\ \mathrm{det}C_0 = 1\,,\ C_0^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = C_0^{+}\ , \tag{B.15}$$

has the property

$$C_0 \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot C_0^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.16}$$

and therefore implements an isomorphism between $SU(1,1)$ and $SL(2,\mathbb{R})$:

$$C_0 \cdot g_1 \cdot C_0^{-1} = g_0\ . \tag{B.17}$$

It is obvious from B.5 that the isomorphic groups $SU(1,1), SL(2,\mathbb{R})$ and $Sp(2,\mathbb{R})$ are twofold covering groups of $SO^{\uparrow}(1,2)$, because A and $-A$ induce the same Lorentz transformation!

The group $SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$ maps the ("Siegel") complex upper half plane

$$\mathbb{S}_1 = \{z = x + iy,\ y > 0\} \tag{B.18}$$

transitively onto itself:

$$z \in \mathbb{S}_1 \to \hat{z} = \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}}\ ,\quad \Im(\hat{z}) = \frac{y}{(a_{22} + a_{21}x)^2 + a_{21}^2 y^2}\ > 0. \tag{B.19}$$

This feature corresponds to the property B.11 of the group $SU(1,1)$.

Again, the group elements g_1 and $-g_1$ give the same transformations B.19. They, too, are important for the explicit construction of irreducible unitary representations (see Sec. B.3).

Convenient for several of our purposes is the *Iwasawa* decomposition [265,269] of the groups $G_1 \equiv SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$ and $G_0 \equiv SU(1,1)$: $G_1 \equiv R_1 \cdot A_1 \cdot N_1$, $G_0 = R_0 \cdot A_0 \cdot N_0$, where R is the maximal compact subgroup, A a maximally abelian noncompact subgroup and N a nilpotent group.

For G_1 this decomposition is

$$R_1: \qquad r_1 = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}\ ,\ \theta \in (-2\pi, +2\pi]\ , \tag{B.20}$$

$$A_1: \qquad a_1 = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}\ ,\ t \in \mathbb{R}\ , \tag{B.21}$$

$$N_1: \qquad n_1 = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}\ ,\ \xi \in \mathbb{R}\ \ . \tag{B.22}$$

Each element g_1 has a unique decomposition $g_1 = k_1 \cdot a_1 \cdot n_1$.

The isomorphism B.17 gives the corresponding decomposition of G_0:

$$R_0: \quad r_0 = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}, \ \theta \in (-2\pi, +2\pi], \tag{B.23}$$

$$A_0: \quad a_0 = \begin{pmatrix} \cosh(t/2) & i\sinh(t/2) \\ -i\sinh(t/2) & \cosh(t/2) \end{pmatrix}, \ t \in \mathbb{R}, \tag{B.24}$$

$$N_0: \quad n_0 = \begin{pmatrix} 1 + i\xi/2 & \xi/2 \\ \xi/2 & 1 - i\xi/2 \end{pmatrix}, \ \xi \in \mathbb{R} \ . \tag{B.25}$$

In addition to the above subgroups the following two ones are of interest to us:

$$\mathrm{B}_1: \quad b_1 = \begin{pmatrix} \cosh(s/2) & \sinh(s/2) \\ \sinh(s/2) & \cosh(s/2) \end{pmatrix}, \ s \in \mathbb{R}, \tag{B.26}$$

$$\mathrm{B}_0: \quad b_0 = C_0 \cdot b_1 \cdot C_0^{-1} = \begin{pmatrix} \cosh(s/2) & \sinh(s/2) \\ \sinh(s/2) & \cosh(s/2) \end{pmatrix}, \tag{B.27}$$

$$\bar{N}_1: \quad \bar{n}_1 = \begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix}, \ \zeta \in \mathbb{R}, \tag{B.28}$$

$$\bar{N}_0: \quad \bar{n}_0 = \begin{pmatrix} 1 - i\zeta/2 & \zeta/2 \\ \zeta/2 & 1 + i\zeta/2 \end{pmatrix}. \tag{B.29}$$

Two more decompositions of $SL(2,\mathbb{R})$ or $SU(1,1)$ are important for the construction of their unitary representations:

Cartan (or "polar") decomposition [265, 269]:

Each element of $SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$ can be written as

$$g_1 = r_1(\theta_2) \cdot a_1(t) \cdot r_1(\theta_1), \tag{B.30}$$

where $a_1(t)$ is determined uniquely and $r_1(\theta_1), r_1(\theta_2)$ up to a relative sign, that is, up to the center Z_2 of $SL(2,\mathbb{R})$.

Bruhat decomposition [269–271]:

From

$$r_1(\theta) \cdot a_1(t) \cdot r_1(-\theta) = \tag{B.31}$$
$$= \begin{pmatrix} \cos^2(\theta/2)e^{t/2} + \sin^2(\theta/2)e^{-t/2} & \sin(\theta/2)\cos(\theta/2)(e^{-t/2} - e^{t/2}) \\ \sin(\theta/2)\cos(\theta/2)(e^{-t/2} - e^{t/2}) & \cos^2(\theta/2)e^{-t/2} + \sin^2(\theta/2)e^{t/2} \end{pmatrix}$$

one sees that

$$r_1(\theta) \cdot a_1(t) \cdot r_1(-\theta) = a_1(t) \text{ for } \theta = 0, 2\pi, \tag{B.32}$$

$$r_1(\theta) \cdot a_1(t) \cdot r_1(-\theta) \subset A_1 \text{ for } \theta = 0, \pm\pi, 2\pi, \tag{B.33}$$

which means that the centralizer $C_{R_1}(A_1)$ and normalizer $N_{R_1}(A_1)$ of A_1 in R_1 are given by

$$C_{R_1}(A_1) = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \mathsf{M}, \tag{B.34}$$

$$N_{R_1}(A_1) = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} = \mathsf{M}^* . \tag{B.35}$$

The quotient group

$$W = \mathsf{M}^*/\mathsf{M} \cong Z_2 \tag{B.36}$$

is called the *Weyl* group of $SL(2,\mathbb{R})$. The associated Bruhat decomposition of $SL(2,\mathbb{R})$ is [272]

$$G_1 = \mathsf{M}\cdot A_1\cdot N_1 \cup N_1\cdot w\cdot \mathsf{M}\cdot A_1\cdot N_1\ ,\ w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ . \tag{B.37}$$

Here $\mathsf{M}\cdot A_1$ is the group

$$D_1 = \mathsf{M}\cdot A_1 = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}\ ,\ c\in\mathbb{R}-\{0\} \right\}\ . \tag{B.38}$$

The relation B.37 means that each element of $SL(2,\mathbb{R})$ is either an element of the "parabolic" subgroup $P_1 = D_1\cdot N_1$ or an element of $N_1\cdot w\cdot P_1$.

The Bruhat decomposition of $SL(2,\mathbb{R})$ plays a central role in Sally's construction [259] of the irreducible unitary representations of the universal covering group $\widetilde{SL(2,\mathbb{R})}$.

B.1.2 The universal covering group of $SO^{\uparrow}(1,2)$

As the compact subgroups $SO(2)\simeq S^1\subset SO^{\uparrow}(1,2)$ and R_1 or $R_0\ \simeq S^1$ are infinitely connected, the groups $SO^{\uparrow}(1,2)$, $SL(2,\mathbb{R})$ and $SU(1,1)$ have an infinitely sheeted universal covering group which, according to Bargmann [257], may be parametrized as follows:

Starting from $SU(1,1)$ one defines

$$\gamma = \beta/\alpha\ ,\ |\alpha|^2-|\beta|^2=1\ (\Rightarrow|\gamma|<1);\quad \omega=\arg(\alpha)\ ; \tag{B.39}$$

$$\alpha = e^{i\omega}(1-|\gamma|^2)^{-1/2}\ ,\ |\gamma|<1\ ,\ \beta = e^{i\omega}\gamma(1-|\gamma|^2)^{-1/2}\ . \tag{B.40}$$

Then

$$SU(1,1) = \{g_0=(\omega,\gamma),\ \omega\in(-\pi,\pi],\ |\gamma|<1\}\ , \tag{B.41}$$

$$\tilde{G}\equiv\widetilde{SU(1,1)} = \widetilde{Sp(2,\mathbb{R})} = \{\tilde{g}=(\omega,\gamma),\ \omega\in\mathbb{R},\ |\gamma|<1\}\ . \tag{B.42}$$

The group composition law for $\tilde{g}_3=\tilde{g}_2\cdot\tilde{g}_1$ is given by

$$\gamma_3 = (\gamma_1+\gamma_2 e^{-2i\omega_1})(1+\bar{\gamma}_1\gamma_2 e^{-2i\omega_1})^{-1}\ , \tag{B.43}$$

$$\omega_3 = \omega_1+\omega_2+\frac{1}{2i}\ln[(1+\bar{\gamma}_1\gamma_2 e^{-2i\omega_1})(1+\gamma_1\bar{\gamma}_2 e^{2i\omega_1})^{-1}]\ . \tag{B.44}$$

For the universal covering group the transformations A.52 and A.53 take the form

$$\hat{I} = \rho(\tilde{g},\varphi)\,I\ ,\ \rho(\tilde{g},\varphi)=|1+e^{i\varphi}\gamma|^2\,(1-|\gamma|^2)^{-1}\ , \tag{B.45}$$

$$e^{i\hat{\varphi}} = e^{-2i\omega}\,\frac{e^{i\varphi}+\bar{\gamma}}{1+e^{i\varphi}\gamma}\ . \tag{B.46}$$

As $\partial\hat{\varphi}/\partial\varphi = 1/\rho(\tilde{g},\varphi)$, the equality A.55 holds again.

With the elements of the group $SU(1,1)$ given by the restriction $-\pi<\omega\le+\pi$, $\alpha=\exp(i\omega)(1-|\gamma|^2)^{-1/2}$, $\beta=\gamma\,\alpha$, the homomorphisms

$$h^{\#}\ :\quad \widetilde{SU(1,1)}\to SU(1,1)\cong Sp(2,\mathbb{R})\,, \tag{B.47}$$

$$h^{0}\ :\quad SU(1,1)\to SO^{\uparrow}(1,2)\,, \tag{B.48}$$

have the kernels $\ker(h^{\#}) = 2\pi\,\mathbb{Z}$, $\ker(h^0) = Z_2$, respectively, and the composite homomorphism $h^0 \circ h^{\#}$ has the kernel $\pi\mathbb{Z}$.

As the space $\mathcal{S}^2_{\varphi,I}$, Eq. A.42, is homeomorphic to $\mathbb{R}^2 - \{0\} = \mathbb{C} - \{0\}$, its universal covering space is given by $\varphi \in \mathbb{R}$, $I \in \mathbb{R}^+$, which is the infinitely sheeted Riemann surface of the logarithm.

The transformations B.45 and B.46 may be interpreted as acting transitively and effectively on that universal covering space.

B.1.3 The group $Sp_c(2) \cong Sp(2,\mathbb{R}) \cong SU(1,1)$

For the interpretation of the crucial role the groups $SO^{\uparrow}(1,2)$ etc. play for our approach to the quantization of the symplectic space A.42 the following isomorphic version of $Sp(2,\mathbb{R})$ is of interest:

Consider $x = (q,p)^T \in \mathbb{R}^2$. Then $g_1 \in Sp(2,\mathbb{R})$ transforms x as

$$x \to \hat{x} = \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} = g_1 \cdot x = g_1 \cdot \begin{pmatrix} q \\ p \end{pmatrix} , \tag{B.49}$$

with the property

$$d\hat{q} \wedge d\hat{p} = dq \wedge dp\,. \tag{B.50}$$

If we define

$$b = \begin{pmatrix} a = \frac{1}{\sqrt{2}}(q + ip) = |a|e^{-i\varphi} \\ \bar{a} = \frac{1}{\sqrt{2}}(q - ip) = |a|e^{i\varphi} \end{pmatrix} , \tag{B.51}$$

then

$$b = C_1 \cdot x\,,\ C_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} ,\quad C_1^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = C_1^{+}\,, \tag{B.52}$$

with

$$dq \wedge dp = i\,da \wedge d\bar{a}\,. \tag{B.53}$$

The matrix C_1 is unitary with $\det C_1 = -i$.
The transformations B.49 of x imply for those of b:

$$b \to \hat{b} = g_c \cdot b\,,\quad g_c = C_1 \cdot g_1 \cdot C_1^{-1} \in Sp_c(2)\,. \tag{B.54}$$

The group $Sp_c(2)$ is obviously isomorphic to the groups $Sp(2,\mathbb{R})$ and $SU(1,1)$. With respect to the latter we have

$$g_c = C_2 \cdot g_0 \cdot C_2^{-1}\,,\ C_2 = C_1 \cdot C_0^{-1} = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix} . \tag{B.55}$$

The transformations B.54 have the property

$$d\hat{a} \wedge d\bar{\hat{a}} = da \wedge d\bar{a}\,. \tag{B.56}$$

For the group $Sp_c(2)$ the subgroups B.20-B.22 and B.26 have the form

$$R_c: \qquad r_c = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} ,\ \theta \in (-2\pi, +2\pi] , \tag{B.57}$$

$$A_c: \qquad a_c = \begin{pmatrix} \cosh(t/2) & \sinh(t/2) \\ \sinh(t/2) & \cosh(t/2) \end{pmatrix} ,\ t \in \mathbb{R} , \tag{B.58}$$

$$N_c: \qquad n_c = \begin{pmatrix} 1 - i\xi/2 & i\xi/2 \\ \xi/2 & 1 + i\xi/2 \end{pmatrix} ,\ \xi \in \mathbb{R} , \tag{B.59}$$

$$B_c: \qquad b_c = \begin{pmatrix} \cosh(s/2) & i\,\sinh(s/2) \\ -i\,\sinh(s/2) & \cosh(s/2) \end{pmatrix} ,\ s \in \mathbb{R}. \tag{B.60}$$

B.2 Lie algebra

As the structure of the 3-dimensional Lie algebra $\mathfrak{so}(1,2) = \{l\}$ of $SO^{\uparrow}(1,2)$ is the same as that of all its covering groups we may calculate it by using any of them.

For $SL(2,\mathbb{R})$ we get from B.20-B.22, B.26 and B.28:

$$l_{R_1} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ l_{A_1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ l_{B_1} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \tag{B.61}$$

$$l_{N_1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ ,\quad l_{\bar{N}_1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\ , \tag{B.62}$$

which are not independent (in the following I skip the indices "1", because the structure relations are independent of them):

$$l_N + l_{\bar{N}} = 2\, l_B\ ,\quad l_N - l_{\bar{N}} = 2\, l_R\ . \tag{B.63}$$

We have the commutation relations

$$\begin{aligned}
[l_R, l_A] &= -l_B\ ,\quad [l_R, l_B] = l_A\ ,\quad [l_A, l_B] = l_R\ , && \text{(B.64)}\\
&\quad [l_R, l_N] = l_A\ ,\quad [l_R, l_{\bar{N}}] = l_A\ , && \text{(B.65)}\\
&\quad [l_A, l_N] = l_N\ ,\quad [l_A, l_{\bar{N}}] = -l_{\bar{N}}\ , && \text{(B.66)}\\
&\quad [l_B, l_N] = -l_A\ ,\quad [l_B, l_{\bar{N}}] = l_A\ , && \text{(B.67)}\\
&\quad [l_N, l_{\bar{N}}] = 2\, l_A\ . && \text{(B.68)}
\end{aligned}$$

The relations B.64 show that the algebra is simple, B.66 that A and N combined form a 2-dimensional subgroup, as do A and $\bar{N}$.

B.3 Irreducible unitary representations of the positive discrete series

As the group $SO^{\uparrow}(1,2)$ is noncompact, its irreducible unitary representations are infinite-dimensional. Their structure can be seen already from its Lie algebra: In unitary representations the elements $-il_R, -il_A, -il_B$ of the Lie algebra correspond to self-adjoint operators K_0, K_1, K_2 which obey die commutation relations

$$[K_0, K_1] = iK_2\ ,\quad [K_0, K_2] = -iK_1\ ,\quad [K_1, K_2] = -iK_0\ , \tag{B.69}$$

or, with the definitions

$$K_+ = K_1 + iK_2\,,\quad K_- = K_1 - iK_2\,, \tag{B.70}$$

$$[K_0, K_+] = K_+\ ,\quad [K_0, K_-] = -K_-\ ,\quad [K_+, K_-] = -2K_0\ . \tag{B.71}$$

The relations B.69 are invariant under the replacement $K_1 \to -K_1, K_2 \to -K_2$ and the relations B.71 invariant under $K_+ \to \omega K_+, K_- \to \bar{\omega} K_-, |\omega| = 1$. These relations are in addition invariant under the transformations $K_+ \leftrightarrow K_-, K_0 \to -K_0$.

In irreducible unitary representations with a scalar product (f_1, f_2) the operator K_- is the adjoint operator of K_+ : $(f_1, K_+ f_2) = (K_- f_1, f_2)$, and vice versa, where it is assumed that f_1, f_2 belong to the domains of definition of K_+ and K_-.

The Casimir operator L of a representation is defined by

$$L = K_1^2 + K_2^2 - K_0^2 \tag{B.72}$$

and we have the relations

$$K_+K_- = L + K_0(K_0 - 1) \ , \quad K_-K_+ = L + K_0(K_0 + 1) \ . \tag{B.73}$$

All unitary representations make use of the fact that K_0 is the generator of a compact group and that its eigenfunctions g_m are normalizable elements of the associated Hilbert space $\mathcal{H}$.

The relations B.71 imply

$$\begin{aligned} K_0\, g_m &= m\, g_m \ , & \text{(B.74)} \\ K_0\, K_+ g_m &= (m+1) K_+ g_m \ , & \text{(B.75)} \\ K_0\, K_- g_m &= (m-1) K_- g_m \ , & \text{(B.76)} \end{aligned}$$

which, combined with B.73, lead to

$$\begin{aligned} (g_m, K_+K_-g_m) &= (K_-g_m, K_-g_m) = l + m(m-1) \geq 0 \ , & \text{(B.77)} \\ (g_m, K_-K_+g_m) &= l + m(m+1) \geq 0 \ , \ l = (g_m, Lg_m), & \text{(B.78)} \end{aligned}$$

implying

$$(K_+g_m, K_+g_m) = 2m + (K_-g_m, K_-g_m) \geq 0. \tag{B.79}$$

In the following we assume that we have an irreducible representation for which the functions g_m are eigenfunctions of the Casimir operator L, too: $Lg_m = l\, g_m$.

The relations B.74-B.76 show that the eigenvalues m of K_0 in principle can be any real number, where, however, different eigenvalues differ by an integer.

For the "principle" and the "complementary" series the spectrum of K_0 is unbounded from below and above [273]. As K_0 corresponds to the classical positive definite quantity I, these unitary representations are of no interest here.

Important for us is the positive discrete series D_+ of irreducible unitary representations. It is characterized by the property that there exists a lowest eigenvalue $m = k$ such that

$$K_-\, g_k = 0 \ , \quad K_0\, g_k = k\, g_k \ . \tag{B.80}$$

Then the relations B.77-B.79 imply

$$l = k(1-k) \ , \quad k > 0 \ , \quad m = k + n, \ n = 0, 1, 2, \ldots . \tag{B.81}$$

The relations B.74-B.76 now take the form

$$\begin{aligned} K_0 g_{k,n} &= (k+n)\, g_{k,n} \ , & \text{(B.82)} \\ K_+ g_{k,n} &= \omega_n \left[(2k+n)(n+1)\right]^{1/2} g_{k,n+1} \ , \quad |\omega_n| = 1 \ , & \text{(B.83)} \\ K_- g_{k,n} &= \frac{1}{\omega_{n-1}} \left[(2k+n-1)n\right]^{1/2} g_{k,n-1} \ . & \text{(B.84)} \end{aligned}$$

The phases ω_n guarantee that $(f_1, K_+f_2) = (K_-f_1, f_2)$. In most cases ω_n is independent of n. Then one can absorb it into the definition of $K_\pm$ and forget the phases ω_n!

Up to now we have allowed for any value of $k > 0$. It turns out [257,259,274] that this is so for the irreducible representations of the universal covering group $\widetilde{SO^\uparrow(1,2)}$. These representations may be realized for $k \geq 1/2$ in the Hilbert space of holomorphic functions on the unit disc $\mathbb{D}_1 = \{z, |z| < 1\}$ with the scalar product

$$(f, g)_k = \frac{2k-1}{\pi} \int_{\mathbb{D}_1} \bar{f}(z) g(z) (1 - |z|^2)^{2k-2} dxdy \ . \tag{B.85}$$

The unitary operators representing the universal covering group are given by

$$[U(\tilde{g},k)f](z) = e^{2ik\omega}(1-|\gamma|^2)^k(1+\bar{\gamma}z)^{-2k} f\left(\frac{\alpha z+\beta}{\bar{\beta}z+\bar{\alpha}}\right) , \tag{B.86}$$

$$\tilde{g} = (\omega,\gamma), \quad \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = h^{\#}(\tilde{g}) \in SU(1,1) . \tag{B.87}$$

Because $|\gamma z| < 1$, the function $(1+\bar{\gamma}z)^{-2k}$ is, for $k > 0$, defined in terms of a series expansion.

For $SU(1,1)$ we have $\omega \in \mathbb{R} \bmod 2\pi$. Uniqueness of the phase factor then requires $k = 1/2, 1, 3/2, \cdots$.

For $SO^{\uparrow}(1,2)$ itself we have $\omega \in \mathbb{R} \bmod \pi$ which implies $k = 1, 2, \cdots$.

B.3.1 Hilbert space of holomorphic functions inside the unit disc $\mathbb{D}_1$

One of the more important Hilbert spaces for the explicit construction of irreducible unitary representations of the group $SU(1,1)$ is the (Bargmann) Hilbert space $\mathcal{H}_{\mathbb{D}_1,k}$ of holomorphic functions in the unit disc $\mathbb{D}_1 = \{z = x+iy,\ |z| < 1\}$, with the scalar product

$$(f,g)_{\mathbb{D}_1,k} = \frac{2k-1}{\pi}\int_{\mathbb{D}_1} \bar{f}(z)g(z)\,(1-|z|^2)^{2k-2}dxdy . \tag{B.88}$$

It can be used for any real $k > 1/2$ and also in the limiting case $k \to 1/2$. As

$$(z^{n_1}, z^{n_2})_{\mathbb{D}_1,k} = \frac{n_1!}{(2k)_{n_1}}\,\delta_{n_1 n_2} , \ (2k)_n = \frac{\Gamma(2k+n)}{\Gamma(2k)} , \tag{B.89}$$

and since any holomorphic function in $\mathbb{D}_1$ can be expanded in powers of z, the functions

$$e_{k,n}(z) = \sqrt{\frac{(2k)_n}{n!}}\,z^n \quad , \quad n \in \mathbb{N}_0 = \{n = 0,1,2,\ldots\} , \tag{B.90}$$

form an orthonormal basis of $\mathcal{H}_{\mathbb{D}_1,k}$.

It follows from the the unitary transformation B.86 that the operators $K_j, j = 0,1,2$, here have the explicit forms

$$K_0 = k + z\frac{d}{dz} , \tag{B.91}$$

$$K_+ \equiv K_1 + i\,K_2 = 2kz + z^2\frac{d}{dz} , \tag{B.92}$$

$$K_- \equiv K_1 - iK_2 = \frac{d}{dz} . \tag{B.93}$$

The basis functions B.90 are the eigenfunctions of K_0 with eigenvalues $k+n$, the operators K_+ and K_- being raising and lowering operators:

$$K_0\,e_{k,n} = (k+n)\,e_{k,n} \quad , \tag{B.94}$$

$$K_+\,e_{k,n} = [(2k+n)(n+1)]^{1/2}\,e_{k,n+1} \quad , \tag{B.95}$$

$$K_-\,e_{k,n} = [(2k+n-1)n]^{1/2}\,e_{k,n-1} \quad . \tag{B.96}$$

If we have on $\mathbb{D}_1$ the holomorphic functions

$$f(z) = \sum_{n=0}^{\infty} a_n\,z^n \ , \quad g(z) = \sum_{n=0}^{\infty} b_n\,z^n \ , \tag{B.97}$$

then, according to Eq. B.89, their scalar product $(f,g)_{\mathbb{D}_1,k}$ is given by

$$(f,g)_{\mathbb{D}_1,k} = \sum_{n=0}^{\infty} \frac{n!}{(2k)_n} \bar{a}_n \, b_n \ . \tag{B.98}$$

This series can be used as a scalar product to extend the definition of the Hilbert spaces $\mathcal{H}_{\mathbb{D},k}$ to all real $k > 0$ [266, 275].

B.3.2 Unitary representations in the Hilbert space of holomorphic functions on the upper half plane

The unit disc $\mathbb{D}_1$ and its associated Hilbert space with the scalar product B.85 is especially suited for the construction of unitary representations of $SU(1,1)$ because that group acts transitively on $\mathbb{D}_1$. Similarly, the group $SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$, isomorphic to $SU(1,1)$, acts transitively on the upper complex half plane $\mathbb{S}_1 = \{w = u + iv,\ v > 0\}$. The mapping

$$w = \frac{1-iz}{z-i} = \frac{2x + (1-x^2-y^2)\,i}{x^2+(y-1)^2} \ , \tag{B.99}$$

$$z = \frac{iw+1}{w+i} \, , \quad |z|^2 = \frac{u^2+(v-1)^2}{u^2+(v+1)^2} \, , \tag{B.100}$$

provides a holomorphic diffeomeorphism from $\mathbb{D}_1$ onto $\mathbb{S}_1$ and back.

Because of

$$\frac{dudv}{4v^2} = \frac{dxdy}{(1-|z|^2)^2} \, , \quad 1-|z|^2 = \frac{2^2\, v}{(w+i)(\bar{w}-i)} \, , \tag{B.101}$$

we have for $k = 1/2, 1, 3/2, 2, \ldots$ the following isomorphism:

$$(f,g)_{\mathbb{D}_1,k} = (\tilde{f},\tilde{g})_{\mathbb{S}_1,k} = \frac{1}{\Gamma(2k-1)} \int_{\mathbb{S}_1} \overline{\tilde{f}} \ \tilde{g}\, v^{2k-2} du dv \, , \tag{B.102}$$

where

$$E_k : \qquad \tilde{f}(w) = \sqrt{\frac{\Gamma(2k)}{\pi}}\, 2^{2k-1} (w+i)^{-2k} f\left(z = \frac{1+iw}{i+w}\right) \ , \tag{B.103}$$

$$E_k^{-1} : \qquad f(z) = 2\sqrt{\frac{\pi}{\Gamma(2k)}}\, (z-i)^{-2k} \tilde{f}\left(w = \frac{1-iz}{z-i}\right) \ . \tag{B.104}$$

The (unitary) transformation E_k maps the basis B.89 of $\mathcal{H}_{\mathbb{D}_1,k}$ onto the basis

$$\tilde{e}_{k,n}(w) = \sqrt{\frac{\Gamma(2k+n)}{\pi\, n!}}\ 2^{2k-1}\ i^n\ (w-i)^n\ (w+i)^{-2k-n} \ , n \in \mathbb{N}_0 \ , \tag{B.105}$$

of $\mathcal{H}_{\mathbb{S}_1,k}$. One can, of course, discard the phase factor i^n.

On this Hilbert space the irreducible unitary representations T_k^+ of the positive discrete series of $SL(2,\mathbb{R})$ are given by

$$[T^+(g_1,k)\tilde{f}](w) = (a_{12}w + a_{22})^{-2k}\, \tilde{f}\left(\frac{a_{11}w+a_{21}}{a_{12}w+a_{22}}\right) \ , \tag{B.106}$$

$$g_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SL(2,\mathbb{R}) \, , \tag{B.107}$$

which is defined for $k = 1/2, 1, 3/2, 2, \ldots$ only.

The subgroups

$$\mathrm{R}_1: \qquad r_1 = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \tag{B.108}$$

$$\mathrm{A}_1: \qquad a_1 = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \tag{B.109}$$

$$\mathrm{B}_1: \qquad b_1 = \begin{pmatrix} \cosh(s/2) & \sinh(s/2) \\ \sinh(s/2) & \cosh(s/2) \end{pmatrix}, \tag{B.110}$$

are associated with the following generators of their unitary representations (the sign of $\tilde{K}_0$ is chosen such that its spectrum is positive):

$$\tilde{K}_0 = \frac{1}{i}(k\,w + \frac{1}{2}(w^2+1)\frac{d}{dw}), \tag{B.111}$$

$$\tilde{K}_\pm = \pm k\,(w \mp i) \pm \frac{1}{2}(w \mp i)^2 \frac{d}{dw}. \tag{B.112}$$

Their action on the basis B.105 is given by

$$\tilde{K}_0\,\tilde{e}_{k,n} = (k+n)\,\tilde{e}_{k,n}\ , \tag{B.113}$$

$$\tilde{K}_+\,\tilde{e}_{k,n} = i[(2k+n)(n+1)]^{1/2}\,\tilde{e}_{k,n+1}\,, \tag{B.114}$$

$$\tilde{K}_-\tilde{e}_{k,n} = \frac{1}{i}[(2k+n-1)n)]^{1/2}\tilde{e}_{k,n-1}\,. \tag{B.115}$$

For the limiting case $k \to 1/2$ the Hilbert space with the scalar product B.102 now can be replaced by the "Hardy space $H^2(\mathbb{R}, du)$ of the upper half plane" [31, 127], the elements of which are the functions $\tilde{f}(u)$, limits for $\Im(w) = v \to 0$ of the previous holomorphic functions $\tilde{f}(w)$ on the upper half plane and the Hilbert space of which has the scalar product

$$(\tilde{f}_1, \tilde{f}_2) = \int_{-\infty}^{\infty} du\,\overline{\tilde{f}}_1(u)\,\tilde{f}_2(u)\ . \tag{B.116}$$

Applications of this space are discussed in Subsec. 4.4, where further details can be found.

Appendix C

The symplectic group $Sp(4,\mathbb{R})$ and the positive discrete series of its irreducible unitary representations

A reader might wonder why I include a rather long appendix on the symplectic group in four dimensions. But we have seen in Ch. 8 that the canonical group for the 4-dimensional phase space 8.15 of interference phenomena is the group $Sp(4,\mathbb{R})$. The crucial point here is again that the necessary deletion of the origin of that phase space requires a quantization procedure which is different from the conventional one! Therefore one needs the appropriate "observables" associated with the group $Sp(4,\mathbb{R})$ and with the quotient group $SO^{\uparrow}(2,3) = Sp(4,\mathbb{R})/Z_2$ and one needs the irreducible unitary representations of the positive discrete series of these groups.

For that reason the present appendix collects - incompletely - some properties of $Sp(4,\mathbb{R})$ and $SO^{\uparrow}(2,3)$ and tries to point out some of the essential References.

C.1 Properties of $Sp(4,\mathbb{R})$

The main References for this and the next Section are [276–279].

Let

$$\langle y, x\rangle = y^T \Omega\, x = y_1 x_3 - y_3 x_1 + y_2 x_4 - y_4 x_2 \ , \tag{C.1}$$

where

$$\Omega = \begin{pmatrix} 0_2 & E_2 \\ -E_2 & 0_2 \end{pmatrix} \ , \ E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \ 0_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \ , \tag{C.2}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \ , \ y^T = (y_1, y_2, y_3, y_4)\,,$$

be a skew symmetric bilinear form on $\mathbb{R}^4$.

The matrix Ω has the property

$$\Omega^2 = -1\,, \ \Omega^{-1} = -\Omega = \Omega^T \ . \tag{C.3}$$

If $e_j, j = 1, ..., 4$, is the cartesian basis of $\mathbb{R}^4$, then

$$\langle e_1, e_3\rangle = -\langle e_3, e_1\rangle = 1\,, \ \langle e_2, e_4\rangle = -\langle e_4, e_2\rangle = 1\,, \ \langle e_j, e_k\rangle = 0 \text{ else}\,. \tag{C.4}$$

In the following we in general will identify the vector x as

$$x = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \tag{C.5}$$

The group $Sp(4,\mathbb{R})$ consists of all 4×4 real matrices (mappings) w which leave the bilinear form C.1 invariant:

$$\langle w\cdot y, w\cdot x\rangle = \langle y, x\rangle \ . \tag{C.6}$$

This implies for the matrices w

$$w^T\Omega\, w = \Omega \ , \ (w^{-1})^T\Omega w^{-1} = \Omega \ , \text{ or } w^{-1} = \Omega w^T\Omega^{-1} \ . \tag{C.7}$$

The relations C.4 show that the group $Sp(4,\mathbb{R})$ may also be defined as the transformations which leave the sum of exterior products [280, 281]

$$e_1\wedge e_3 + e_2\wedge e_4 \quad (\text{or} \quad dq_1\wedge dp_1 + dq_2\wedge dp_2\,) \tag{C.8}$$

invariant.
If we write w in block form in terms of 2×2-matrices,

$$w = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \ , \tag{C.9}$$

then the conditions C.7 imply for the submatrices:

$$A_{11}^T A_{22} - A_{21}^T A_{12} = E_2 \ , \tag{C.10}$$

$$A_{11}^T A_{21} = A_{21}^T A_{11} \ , \qquad A_{12}^T A_{22} = A_{22}^T A_{12} \ , \tag{C.11}$$

$$w^{-1} = \begin{pmatrix} A_{22}^T & -A_{12}^T \\ -A_{21}T & A_{11}^T \end{pmatrix} \ . \tag{C.12}$$

It follows from the last of the relations C.7 that $w^T \in Sp(4,\mathbb{R})$ if $w\in Sp(4,\mathbb{R})$ and one can show that $\det w = 1$.

Important is the identification of the maximal compact subgroup $K = U(2)$ of $Sp(4,\mathbb{R})$: Any real non-singular 4×4 matrix A may be uniquely written as $A = O\cdot P$, where $P = +(A^TA)^{1/2}$ is a positive definite matrix and $O = A\cdot P^{-1}$ an orthogonal one, $O^{-1} = O^T$. If $A\in Sp(4,\mathbb{R})$, then $P, O \in Sp(4,\mathbb{R})$. As the orthogonal group $O(4)$ is compact, the maximal compact subgroup K of $Sp(4,\mathbb{R})$ is given by $Sp(4,\mathbb{R})\cap O(4)$. By identifying $\mathbb{R}^4$ with $\mathbb{C}^2$, one can see that $K = U(2)$: If $u\in U(2)$, then the corresponding element in $Sp(4,\mathbb{R})$ is given by

$$w(K) = \begin{pmatrix} \Re(u) & -\Im(u) \\ \Im(u) & \Re(u) \end{pmatrix} \ , \tag{C.13}$$

where the unitarity relations $u^+u = u\,u^+ = E_2$, when rewritten for $\Re(u)$ and $\Im(u)$, are just the relations C.10 and C.11! Notice that $\Omega\, w(K) = w(K)\,\Omega$.

Any element $u\in U(2)\cong U(1)\times SU(2)$ may be parametrized as

$$u = e^{i\beta/2}\,\breve{u}\ , \ \beta\in(-2\pi, 2\pi]\,, \tag{C.14}$$

$$\breve{u}(\alpha,\vec{n}) = \cos\frac{\alpha}{2}\,E_2 - i\sin\frac{\alpha}{2}\sum_{j=1}^{3} n_j\sigma_j \ , \ \vec{n}^2 = 1 \ , \ \alpha\in(-2\pi,2\pi]\,, \tag{C.15}$$

$$\sigma_j, j = 1,2,3 : \text{Pauli's matrices}\,;$$

$$\det\breve{u} = 1\,, \tag{C.16}$$

$$\det u = e^{i\beta} \ . \tag{C.17}$$

For the abelian subgroup $U(1) = \{e^{i\beta/2}\}$ we have

$$w(e^{i\beta/2}) = \begin{pmatrix} \cos\frac{\beta}{2} & 0 & \sin\frac{\beta}{2} & 0 \\ 0 & \cos\frac{\beta}{2} & 0 & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & 0 & \cos\frac{\beta}{2} & 0 \\ 0 & -\sin\frac{\beta}{2} & 0 & \cos\frac{\beta}{2} \end{pmatrix} , \tag{C.18}$$

and for the 1-parameter subgroup $\{\check{u}(\alpha, \vec{n} = (0,0,1))\}$ of $SU(2)$ we get

$$w[\check{u}(\alpha, \vec{n} = (0,0,1))] = \begin{pmatrix} \cos\frac{\alpha}{2} & 0 & \sin\frac{\alpha}{2} & 0 \\ 0 & \cos\frac{\alpha}{2} & 0 & -\sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & 0 & \cos\frac{\alpha}{2} & 0 \\ 0 & \sin\frac{\alpha}{2} & 0 & \cos\frac{\alpha}{2} \end{pmatrix} , \tag{C.19}$$

Another important subgroup is the 2-parameter abelian subgroup

$$\begin{aligned} A =& \{a_1(t_1), a_2(t_2)\} , \\ w(A) =& w(a_1(t_1)) \cdot w(a_2(t_2)) = \begin{pmatrix} e^{t_1/2} & 0 & 0 & 0 \\ 0 & e^{t_2/2} & 0 & 0 \\ 0 & 0 & e^{-t_1/2} & 0 \\ 0 & 0 & 0 & e^{-t_2/2} \end{pmatrix} , \\ & t_j \in \mathbb{R},\ j = 1,2 . \end{aligned} \tag{C.20}$$

This group constitutes the maximal abelian non-compact subgroup of an Iwasawa decomposition

$$K \cdot A \cdot N \tag{C.21}$$

of the group $Sp(4,\mathbb{R})$, the group K being the maximal compact subgroup. The remaining nilpotent subgroup

$$N = \{n_1(\xi_1), n_2(\xi_2), n_3(\xi_3), n_4(\xi_4)\} \tag{C.22}$$

of that decomposition is generated by four 1-parameter subgroups which have the following elements

$$n_1(\xi_1) = \begin{pmatrix} 1 & 0 & \xi_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \qquad n_2(\xi_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \xi_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{C.23}$$

$$n_3(\xi_3) = \begin{pmatrix} 1 & 0 & 0 & \xi_3 \\ 0 & 1 & \xi_3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \qquad n_4(\xi_4) = \begin{pmatrix} 1 & \xi_4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\xi_4 & 1 \end{pmatrix} , \tag{C.24}$$

$$\xi_j \in \mathbb{R},\ j = 1,2,3,4 .$$

The proof for this will be indicated below in connection with the Lie algebra of the group.

Notice that $N_3 = \{n_1(\xi_1), n_2(\xi_2), n_3(\xi_3)\}$ forms a 3-parameter abelian subgroup of N.

Another subgroup of interest is the (commuting) product $Sp(2,\mathbb{R})_1 \otimes Sp(2,\mathbb{R})_2$, where the first factor acts on the (q_1, p_1)- subspace as described in appendix A and the second factor on the (q_2, p_2)- subspace correspondingly. The matrices related to the first factor are (see the formulae B.20, B.21 and B.22):

$$\begin{pmatrix} \cos(\frac{\theta_1}{2}) & 0 & \sin(\frac{\theta_1}{2}) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\frac{\theta_1}{2}) & 0 & \cos(\frac{\theta_1}{2}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \begin{pmatrix} e^{t_1/2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-t_1/2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \begin{pmatrix} 1 & 0 & \xi_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{C.25}$$

and those for the second factor

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\frac{\theta_2}{2}) & 0 & \sin(\frac{\theta_2}{2}) \\ 0 & 0 & 1 & 0 \\ 0 & -\sin(\frac{\theta_2}{2}) & 0 & \cos(\frac{\theta_2}{2}) \end{pmatrix} , \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{t_2/2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-t_2/2} \end{pmatrix} , \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \xi_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{C.26}$$

In Appendix B.1 we have seen that the group $Sp(2,\mathbb{R}) = SL(2,\mathbb{R})$ is isomorphic to the complex group $SU(1,1)$. A corresponding property holds for the general case $Sp(n,\mathbb{R})$: This group is isomorphic to a subgroup $SU(n,n)_{sp}$ of the complex group $SU(n,n)$ acting on $\mathbb{C}^{2n}$, the elements of which leave the quadratic form

$$\bar{z}_1 z_1 + \cdots + \bar{z}_n z_n - \bar{z}_{n+1} z_{n+1} - \cdots - \bar{z}_{2n} z_{2n} \tag{C.27}$$

invariant and have determinant = 1.

The subgroup $SU(n,n)_{sp}$ is defined as

$$SU(n,n)_{sp} = \{w_c\} : \; w_c \in SU(n,n) \text{ and } w_c^T \Omega \, w_c = \Omega = \begin{pmatrix} 0_n & E_n \\ -E_n & 0_n \end{pmatrix} . \tag{C.28}$$

Next I specialize to $n = 2$: We identify the vector x in C.1 according to C.5, define the complex numbers

$$a_j = \frac{1}{\sqrt{2}}(q_j + i p_j) = |a_j| \, e^{-i\varphi_j} \, , \; j = 1, 2 \, , \tag{C.29}$$

and the complex vectors $\in \mathbb{C}^4$

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \bar{a}_1 \\ \bar{a}_2 \end{pmatrix} . \tag{C.30}$$

The column vectors x and a are related by a unitary transformation:

$$a = C \cdot x \, , \; C = \frac{1}{\sqrt{2}} \begin{pmatrix} E_2 & iE_2 \\ E_2 & -iE_2 \end{pmatrix} , \; C^{-1} = C^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} E_2 & E_2 \\ -iE_2 & iE_2 \end{pmatrix} . \tag{C.31}$$

The elements $w_c \in SU(2,2)_{sp} \equiv Sp_c(4)$ are then given by

$$w_c = C \cdot w \cdot C^{-1} \, . \tag{C.32}$$

The first of the conditions C.7 now becomes

$$w_c^+ \Delta \, w_c = \Delta \, , \; \Delta = \begin{pmatrix} E_2 & 0_2 \\ 0_2 & -E_2 \end{pmatrix} . \tag{C.33}$$

This says that w_c is an element of $SU(2,2)$.
A general element w_c has the form

$$w_c = \begin{pmatrix} B_1 & B_2 \\ \bar{B}_2 & \bar{B}_1 \end{pmatrix} , \; B_1^+ \cdot B_1 - B_2^T \cdot \bar{B}_2 = E_2 \, , \; B_2^+ \cdot B_1 = B_1^T \cdot \bar{B}_2 \, . \tag{C.34}$$

For the compact subgroup C.13 we get

$$w_c(K) = \begin{pmatrix} u & 0_2 \\ 0_2 & \bar{u} \end{pmatrix} , \tag{C.35}$$

where u is the unitary matrix C.14.

The maximal abelian non-compact subgroup C.20 now takes the form

$$w_c(A) = \begin{pmatrix} \cosh(t_1/2) & 0 & \sinh(t_1/2) & 0 \\ 0 & \cosh(t_2/2) & 0 & \sinh(t_2/2) \\ \sinh(t_1/2) & 0 & \cosh(t_1/2) & 0 \\ 0 & \sinh(t_2/2) & 0 & \cosh(t_2/2) \end{pmatrix} , \tag{C.36}$$

and for the nilpotent group C.22 one gets

$$w_c(N_3) = \begin{pmatrix} E_2 - i\,\Xi_3/2 & i\,\Xi_3/2 \\ -i\,\Xi_3/2 & E_2 + i\,\Xi_3/2 \end{pmatrix} , \; \Xi_3 = \begin{pmatrix} \xi_1 & \xi_3 \\ \xi_3 & \xi_2 \end{pmatrix} , \tag{C.37}$$

and

$$w_c[n_4(\xi_4)] = \begin{pmatrix} 1 & \xi_4/2 & 0 & \xi_4/2 \\ -\xi_4/2 & 1 & \xi_4/2 & 0 \\ 0 & \xi_4/2 & 1 & \xi_4/2 \\ \xi_4/2 & 0 & -\xi_4/2 & 1 \end{pmatrix} . \tag{C.38}$$

C.2 The Lie algebra $\mathfrak{sp}(4,\mathbb{R})$

In the neighbourhoods of the unit element E_4 of $Sp(4,\mathbb{R})$ the group elements w may be approximated as

$$w = E_4 + \hat{w}\,\epsilon\,, \; |\epsilon| \ll 1\,, \tag{C.39}$$

where the matrix $\hat{w}$ is an element of the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$ of the group $Sp(4,\mathbb{R})$. It follows from the first of the relations C.7 that $\hat{w}$ has to obey the condition

$$\hat{w}^T\Omega + \Omega\,\hat{w} = 0\,. \tag{C.40}$$

For the ansatz

$$\hat{w} = \begin{pmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \end{pmatrix} \tag{C.41}$$

the condition C.40 implies

$$\hat{B}_{12}^T = \hat{B}_{12}\,, \; \hat{B}_{21}^T = \hat{B}_{21}\,, \; \hat{B}_{22} = -\hat{B}_{11}^T\,. \tag{C.42}$$

If $g(t)$ is a 1-parameter group we shall denote the generating Lie algebra element by $\hat{g}, g(t) = \exp(\hat{g}\,t)$, in the following.

A basis of the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$ is easily obtained from the subgroups C.14, C.20 and C.22. For the Lie algebra $\mathfrak{k}$ of the compact subgroup C.13 we get from C.14 the 4 basis elements:

$$\hat{u}_0 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{C.43}$$

for the $U(1)$ subgroup C.18 and

$$\hat{u}_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} , \quad \hat{u}_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \tag{C.44}$$

$$\hat{u}_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} , \tag{C.45}$$

for the group $SU(2)$. The latter obey the usual commutation relations

$$[\hat{u}_j, \hat{u}_k] = \epsilon_{jkl}\, \hat{u}_l \, . \tag{C.46}$$

For the 2 generators of the abelian subgroup C.20 we get

$$\hat{a}_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \hat{a}_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{C.47}$$

The algebra $\mathfrak{a}$ is maximal abelian, non-compact in both dimensions and constitutes one of the 4 Cartan subalgebras of $\mathfrak{sp}(4,\mathbb{R})$ [282]. Its general elements have the form

$$\hat{a} = 2\omega_1\, \hat{a}_1 + 2\omega_2\, \hat{a}_2 \, , \, \omega_j \in \mathbb{R} . \tag{C.48}$$

(The factor 2 is convention. As to the following see, e.g. the Refs. [283–285])

As all the elements of $\mathfrak{a}$ commute with each other, they have a common set of eigenvectors in the adjoint representation which here are to be understood as 4×4 matrices E_λ satisfying

$$[\hat{a}, E_\lambda] = \lambda E_\lambda \, . \tag{C.49}$$

The eigenvalues λ are called "roots". In our case there are 8 different eigenvectors with 8 different eigenvalues

$$\pm\, 2\omega_1 \, , \, \pm 2\omega_2 \, , \, \pm(\omega_1 + \omega_2) \, , \, \pm(\omega_1 - \omega_2) \, . \tag{C.50}$$

Choosing $\omega_1 > \omega_2 > 0$ we have 4 positive and 4 negative roots. In the following it is convenient to introduce the matrices

$$E_{jk} \, = \, \begin{pmatrix} \text{number 1 at the crossing of the } j\text{-th row} \\ \text{with the } k\text{-th column, all other elements} = 0 \end{pmatrix} . \tag{C.51}$$

We then have the eigenvectors

$$E_{2\omega_1} = E_{13} \, , \, E_{2\omega_2} = E_{24} \, , \, E_{\omega_1+\omega_2} = E_{14} + E_{23} \, , \, E_{\omega_1-\omega_2} = E_{12} - E_{43} \, , \tag{C.52}$$

of the four positive roots and the eigenvectors of the corresponding negative roots are

$$E_{-2\omega_1} = E_{31} \, , \, E_{-2\omega_2} = E_{42} \, , \, E_{-\omega_1-\omega_2} = E_{32} + E_{41} \, , \, E_{-\omega_1+\omega_2} = E_{21} - E_{34} \, . \tag{C.53}$$

The eigenvectors C.52 form a basis of the Lie algebra $\mathfrak{n}$ of the nilpotent group C.22, i.e. we have the relations

$$E_{2\omega_1} = \hat{n}_1 \, , \, E_{2\omega_2} = \hat{n}_2 \, , \, E_{\omega_1+\omega_2} = \hat{n}_3 \, , \, E_{\omega_1-\omega_2} = \hat{n}_4 \, , \tag{C.54}$$

which are exactly the generators of the four 1-parameter groups C.23.
The associated commutation relations are

$$[\hat{n}_1, \hat{n}_2] = 0\,, \quad [\hat{n}_1, \hat{n}_3] = 0\,, \qquad [\hat{n}_1, \hat{n}_4] = 0\,, \tag{C.55}$$

$$[\hat{n}_2, \hat{n}_3] = 0\,, \quad [\hat{n}_2, \hat{n}_4] = -\hat{n}_3\,, \quad [\hat{n}_3, \hat{n}_4] = -2\hat{n}_1\,. \tag{C.56}$$

They prove that $\mathfrak{n}$ is indeed a (nilpotent) subalgebra and its construction shows that $\mathfrak{n}$ and $\mathfrak{a}$ combined form a subalgebra, too.

The eigenvectors C.53 of the negative roots constitute another 4-dimensional nilpotent algebra $\bar{\mathfrak{n}}$ of a 4-dimensional nilpotent group $\bar{N}$ with obvious 1-parameter subgroups.

The Lie algebra of the subgroup C.25 is generated by

$$l_0^{(1)} = \frac{1}{2}(\hat{u}_0 + \hat{u}_3)\,,\, l_A^{(1)} = \hat{a}_1\,,\, l_N^{(1)} = \hat{n}_1\,, \tag{C.57}$$

and that of the subgroup C.26 by

$$l_0^{(2)} = \frac{1}{2}(\hat{u}_0 - \hat{u}_3)\,,\, l_A^{(2)} = \hat{a}_2\,,\, l_N^{(2)} = \hat{n}_2\,, \tag{C.58}$$

where the relations C.43, C.45, C.47 and C.54 have been used.

In Ch. 8 we encountered the ten Hamiltonian functions $g_0, \ldots, g_9$ on the phase space 8.15 which generate the Lie algebra $\mathfrak{sp}(4, \mathbb{R})$ via their Poisson brackets. If we denote the corresponding 4×4 matrices by $\hat{g}_0, \ldots, \hat{g}_9$ we get the following relations the validity of which will be demonstrated below:

$$\hat{g}_0 = \hat{u}_0\,,\ \hat{g}_1 = \hat{u}_1\,,\ \hat{g}_2 = \hat{u}_2\,,\ \hat{g}_3 = \hat{u}_3\,,\ \hat{g}_4 = \hat{n}_3 - \hat{u}_1\,,\ \hat{g}_5 = \hat{n}_4 - \hat{u}_2\,, \tag{C.59}$$

$$\hat{g}_6 = \hat{a}_1 + \hat{a}_2\,,\ \hat{g}_7 = \hat{n}_1 + \hat{n}_2 - \hat{u}_0\,,\ \hat{g}_8 = \hat{n}_1 - \hat{n}_2 - \hat{u}_3\,,\ \hat{g}_9 = \hat{a}_2 - \hat{a}_1\,.$$

C.3 On the isomorphism between the groups $Sp(4, \mathbb{R})$ and Spin$[SO^{\uparrow}(2,3)]$

(Here again I closely follow Bargmann's paper [278] to a large extent.)

There is a very interesting and helpful isomorphism between the group $Sp(4, \mathbb{R})$ and the double covering - i.e. the spinor variant - of the identity component $SO^{\uparrow}(2,3)$ of the group which leaves the quadratic form

$$u^T \cdot \eta \cdot u = u^j \eta_{jk} u^k = -(u^1)^2 - (u^2)^2 - (u^3)^2 + (u^4)^2 + (u^5)^5\,,\ u = \begin{pmatrix} u^1 \\ \vdots \\ u^5 \end{pmatrix}\,, \tag{C.60}$$

invariant. The matrix

$$\eta = \eta^{-1} = (\eta_{jk}) = (\eta^{jk}) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{C.61}$$

defines the metric. It may be used for lowering and raising indices, e.g. $u_j = \eta_{jk} u^k$ or $v^j = \eta^{jk} v_k$. Here and in the following Einstein's summation convention is assumed: summation over equal indices.

A transformation

$$u \to v = B \cdot u \,,\; v^j = b^j{}_k u^k \,,\; B = (b^j{}_k) \,, \tag{C.62}$$

leaves the quadratic form C.60 invariant iff

$$B^T \cdot \eta \cdot B = \eta \;\; \text{or} \;\; B \cdot \eta \cdot B^T = \eta \,. \tag{C.63}$$

The transformations C.62 form a group $O(2,3)$ with 4 components (unconnected pieces). The identity component $SO^{\uparrow}(2,3)$ (i.e. the component which contains the unit element) is given by the conditions

$$\det B = 1 \,,\; b^4{}_4\, b^5{}_5 - b^4{}_5\, b^5{}_4 > 0 \,. \tag{C.64}$$

In the early days the group has been called the "2+3 de Sitter group" because it is the group of motions for the de Sitter space

$$-(u^1)^2 - (u^2)^2 - (u^3)^2 + (u^4)^2 + (u^5)^2 = -a^2 \,. \tag{C.65}$$

There is another, different de Sitter group, the "1+4 de Sitter group" $O(1,4)$, which leaves a corresponding quadratic form of a "1+4 de Sitter space" invariant [286]

As the global geometries of the two de Sitter spaces are quite different it has become customary in gravity, field and string theory [287, 288] to call the 4-dimensional space C.65 "anti–de Sitter space", AdS_4, and the transformation group $O(2,3)$ the AdS_4 group!

The isomorphismen between the Lie algebras $\mathfrak{sp}(4,\mathbb{R})$ and $\mathfrak{so}(2,3)$ was first shown by E. Cartan [289]. As both groups are "infinitely" connected (they contain the subgroups $U(1)$ or $O(2)$ respectively) the situation is more subtle on the group level: The group $Sp(4,\mathbb{R})$ is actually isomorphic to a double covering (i.e. a spinor version) of the group $SO^{\uparrow}(2,3)$.

Siegel in his famous paper on symplectic geometry [290] gave 2 independent proofs of this isomorphism. One by using a complex version of the quadratic form C.60 and its transformations and by using properties of a so-called "Siegel domain" (see Sec. C.4 below). The second one by exploiting the fact that the Clifford algebra associated with the group $SO(2,3)$ can be realized by purely real 4×4 matrices which in turn "support" an associated spinor group isomorphic to $Sp(4,\mathbb{R})$.

Dirac, at the end of his paper [291] on two "remarkable" repesentations of the $2+3$ de Sitter group, points out the general relationship between the two groups after having learnt about it from Res Jost. Dirac's arguments are in the spirit of Siegel's first proof, but he was not aware of Siegel's work and he calls the two groups "equivalent", without detailing the isomorphism.

Bargmann in his paper [278] gives a detailed proof of the isomorphism which is completely equivalent to Siegel's second proof. He also was not aware of Siegel's paper.

I shall indicate the idea of the proof for the isomorphism mentioned above without giving any details. These may be found in Bargmann's paper (or Siegel's).

The procedure is well-known from the use of Dirac matrices for the construction of the spinor version $SL(2,\mathbb{C})$ of the homogeneous Lorentz group as required in the context of the Dirac equation (see Appendix B.1).

The Clifford algebra associated with the quadratic form C.60 is defined by

$$\{\gamma_j, \gamma_k\} = -2\,\eta_{jk}\, E_4 \,,\;\; j,k = 1,\ldots,5 \,. \tag{C.66}$$

These relations may be satisfied by 5 *real* 4×4 matrices which have the property

$$\gamma_j^T = \Omega \gamma_j \Omega^{-1} \,, \tag{C.67}$$

where Ω is the matrix from C.1.

Using the irreducibility of a given set of matrices γ_j satisfying the relations C.66, observing

that any other set α_j obeying them has to be related to the given one by an equivalence transformation and that $\alpha_j = b_j^{\ k}\gamma_k$, $b_j^{\ k} = \eta_{jl}\,\eta^{km}\,b^l_{\ m}$, obeys C.66, gives

$$w \cdot \gamma_j \cdot w^{-1} = b^k_{\ j}\gamma_k\,, \quad w^T \cdot \Omega \cdot w = \Omega\,, \ (b^k_{\ j}) \in SO^{\uparrow}(2,3)\,. \tag{C.68}$$

The relations C.68 show that $w \in Sp(4,\mathbb{R})$ and that w and $-w$ correspond to the same $B \in SO^{\uparrow}(2,3)$. Thus, $Sp(4,\mathbb{R})$ is isomorphic to the spinor group (double covering) of $SO^{\uparrow}(2,3)$.

Using the properties C.66 of the γ-matrices and the relation C.68 one may express the coefficients $b^j_{\ k}$ in terms of w as

$$b^j_{\ k} = -\frac{1}{4}\mathrm{tr}(\gamma^j \cdot w \cdot \gamma_k \cdot w^{-1})\,. \tag{C.69}$$

There is a close relationship between the Lie algebra generators $\hat{g}_0, \ldots, \hat{g}_9$ of Eqs. C.59 and the Lie algebra generators $\hat{m}_{jk} = -\hat{m}_{kj} \in \mathfrak{so}(2,3)$ of the rotations or special Lorentz transformations in the $u^j - u^k$–planes which will be demonstrated below. Here we just give the result:

$$\begin{aligned} &\hat{m}_{12} = \hat{g}_3\,, \quad \hat{m}_{23} = \hat{g}_1\,, \ \hat{m}_{31} = \hat{g}_2\,, \ \hat{m}_{41} = \hat{g}_7\,, \quad \hat{m}_{42} = \hat{g}_9\,, \\ &\hat{m}_{43} = -\hat{g}_5\,, \ \hat{m}_{51} = \hat{g}_8\,, \ \hat{m}_{52} = \hat{g}_6\,, \ \hat{m}_{53} = -\hat{g}_4\,, \ \hat{m}_{54} = \hat{g}_0\,. \end{aligned} \tag{C.70}$$

These relations make the isomorphism of the Lie algebras obvious.

One can also verify the $\mathfrak{so}(2,3)$ commutation relations

$$[\hat{m}_{jk}, \hat{m}_{lm}] = \eta_{kl}\,\hat{m}_{jm} + \eta_{jm}\,\hat{m}_{kl} - \eta_{jl}\,\hat{m}_{km} - \eta_{km}\,\hat{m}_{jl} \tag{C.71}$$

by means of the commutation relations of the $\hat{g}_0, \ldots, \hat{g}_9$.

C.4 Transformations induced by $Sp(4,\mathbb{R})$ on certain spaces

In the following I briefly discuss the actions of the group $Sp(4,\mathbb{R})$ on the spaces

$$\mathcal{S}^4_{q,p} = \{x = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \in \mathbb{R}^4\}\,, \tag{C.72}$$

$$\mathcal{S}^4_{q,p;0} = \mathcal{S}^4_{q,p} - \{x = 0\}\,, \tag{C.73}$$

$$\check{\mathcal{S}}^4_{q,p} = \mathcal{S}^4_{q,p}/Z_2\,, \tag{C.74}$$

(which are generalizations of the 2-dimensional spaces A.91, A.94 and A.99), on

$$\mathcal{S}^4_{\varphi,I} = \{\varphi_1 \in \mathbb{R} \bmod 2\pi, I_1 > 0, \varphi_2 \in \mathbb{R} \bmod 2\pi, I_2 > 0\}\,, \tag{C.75}$$

a generalization of A.42, and on the Siegel domains $\mathbb{S}_2$ and $\mathbb{D}_2$, generalizations of the spaces B.18 and B.10.

A discussion of the group action on the spaces C.72-C.74 and C.75 essentially amounts to a generalizing repetition of the arguments in section A.3. So I shall be very brief here and stress only a few important points:

The orbifold C.74 is not just the topological product of the orbifold A.99 with itself. The non-trivial center elements of the subgroups C.25 and C.26 are

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{C.76}$$

They do *not* belong to the center of $Sp(4,\mathbb{R})$! According to C.69 they correspond to the element

$$\begin{pmatrix} -1 & 0 & 0 & 0 & \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{C.77}$$

of $SO^{\uparrow}(2,3)$.

Eq. C.74 means that the canonical 2-dimensional subspaces are "conified" simultaneously. The conic space C.75 now may be represented by the orbifold C.74 (with the Z_2 fix point $x = 0$ deleted) on which the group $Sp(4,\mathbb{R})$ acts in the same way as the group $Sp(2,\mathbb{R})$ on the orbifold A.99. The effective transformation group here is $SO^{\uparrow}(2,3) \cong Sp(4,\mathbb{R})/Z_2$.

For the explicit construction of the irreducible unitary representations of the positive discrete series of $Sp(4,\mathbb{R})$ the following generalizations of the "Siegel upper half plane" B.18 and of the "Siegel unit disc" B.10 which got their name from Siegel's paper [276] but where already introduced by E. Cartan [292]. (See Appendix B.3 for the important role $\mathbb{S}_1$ and $\mathbb{D}_1$ play for the discrete series of $Sp(2,\mathbb{R})$ and $SU(1,1)$.)

In our present case the generalized Siegel upper plane is given by symmetric complex 2×2 matrices with positive definite imaginary part:

$$W = U + iV = \begin{pmatrix} u_{11} & u_{12} \\ u_{12} & u_{22} \end{pmatrix} + i \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix} , \tag{C.78}$$

$$W^T = W \, , \, u_{jk}, v_{jk} \in \mathbb{R} \, , \tag{C.79}$$

$$V > 0 \Leftrightarrow v_{11} > 0 \, , \, v_{11}v_{22} - (v_{12})^2 > 0 \, . \tag{C.80}$$

W has 3 complex and 6 real dimensions.

The group $Sp(4,\mathbb{R})$ with its elements w (see C.9) acts on W as follows

$$W \to \hat{W} = (A_{11} \cdot W + A_{12}) \cdot (A_{12} \cdot W + A_{22})^{-1} \, . \tag{C.81}$$

The group action is transitive, but only almost effecive because w and $-w$ give the same transformation on $\mathbb{S}_2$, i.e. here again the effective group is $SO^{\uparrow}(2,3)$!

The stability group of the points of $\mathbb{S}_2$ is the compact subgroup K (see C.14). Thus, $\mathbb{S}_2$ is the homogeneous space

$$\mathbb{S}_2 = Sp(4,\mathbb{R})/K \, . \tag{C.82}$$

The invariant positive definite metric on $\mathbb{S}_2$ is

$$\mathrm{tr}(V^{-1} \cdot dW \cdot V^{-1} \cdot d\bar{W}) \tag{C.83}$$

and the associated volume element is

$$d\omega = 2\,(\det V)^{-3}\, d^3u\, d^3v = 2\, d^3u\, d^3t \, , \tag{C.84}$$

$$d^3u = du_{11}du_{12}du_{22} \, ,$$

$$V^{-1} = \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix} \, , \, d^3t = dt_{11}dt_{12}dt_{22} \, . \tag{C.85}$$

The Cayley transform

$$W \to Z = (E_2 + iW) \cdot (E_2 - iW)^{-1} = Z^T = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix} \, , \, z_{jk} \in \mathbb{C} \, , \tag{C.86}$$

maps $\mathbb{S}_2$ one-to-one onto the generalized Siegel disc

$$\mathbb{D}_2 = \{ Z : \, Z = Z^T \, , E_2 - Z \cdot \bar{Z} > 0 \} \, . \tag{C.87}$$

The group $Sp_c(4)$ with its elements C.34 acts transitively on $\mathbb{D}_2$:

$$Z \to \hat{Z} = (B_1 \cdot Z + B_2) \cdot (\bar{B}_2 \cdot Z + \bar{B}_1)^{-1} \, . \tag{C.88}$$

Like $\mathbb{S}_2$ the disc $\mathbb{D}_2$ is a homogeneous space, namely

$$\mathbb{D}_2 = Sp_c(4)/K_c \, . \tag{C.89}$$

For more details as to this section see the Refs. [276, 277, 279].

C.5 Irreducible unitary repesentations of the positive discrete series for $Sp(4, \mathbb{R})$

Like in the case of the group $SL(2, \mathbb{R})$ we are mainly interested in the positive discrete series of the irreducible unitary representations of the group $Sp(4, \mathbb{R})$ or $SO^{\uparrow}(2, 3)$, because in general one wants the Hamilton operator H of a physical system to be bounded from below! In Ch. 8 we saw that in the case of 2 classical oscillators the Hamilton function is given by $2g_0$ where g_0 is the function which corresponds to the generator of the abelian subgroup $U(1)$ of the maximal compact subgroup $U(2)$ of $Sp(4, \mathbb{R})$ (see the relations 8.24 and 8.25 and on the quantum level 8.141; see also the relation $\hat{g}_0 = \hat{m}_{45}$: generator of the $O(2)$ subgroup of $SO^{\uparrow}(2, 3)$), Eq. C.71.

If denote by G_0 the self-adjoint operator which represents $\hat{g}_0$ in an irreducible unitary representation then G_0 has only positive (discrete) eigenvalues for the positive discrete series! Its relation to the Hamilton operator H is

$$H = 2 \, G_0 \, , \tag{C.90}$$

i.e. the energy levels are *twice* the eigenvalues of G_0!!

Before I discuss more details let me make some historical remarks:

C.5.1 A few historical remarks

If one tries to trace the work on irreducible unitary representations of the group $Sp(4, \mathbb{R})$ or (and) $SO^{\uparrow}(2, 3)$ in the literature one makes the surprising discovery that most authors are not aware of important previous work pertaining to their own. As there is no comprehensive monograph or review article on the subject - at least to my knowledge - one has to collect bits and pieces from many (more or less) original papers!

I shall try to give a brief guide to the more important papers on the subject - as far as I found and judged them! I also certainly missed essential ones. My selection is primarily motivated by the interests of a physicist, not by those of a mathematician.

Let me begin with an exceptional case: There is a beautiful and comprehensive paper by Bargmann [278] on those unitary representations of $Sp(2n; \mathbb{R})$ the Lie algebra of which is

generated by the self-adjoint operator versions $G_j, j = 0, \ldots, 9$, of the functions $g_j, j = 0, \ldots, 9$ encountered in Ch. 8. The associate representations of the group are actually those of a double covering of $Sp(4, \mathbb{R})$, named "metaplectic" bei the mathematician Weil [293] and denoted by $M_p(2n, \mathbb{R})$ for general $Sp(2n, \mathbb{R})$.

Bargmann's paper is hardly quoted by authors of later papers on the subject. It is preceded by an important similar paper by Itzykson [294].

Bargmann's results were extended by Folland in Ch. 4 of his very recommendable book [279] which is also not well-known.

Let me now turn to earlier times: This is characterized by the feature that the isomorphism between $Sp(4, \mathbb{R})$ and the spin version of $SO^{\uparrow}(2, 3)$ is not taken into account.

The story of the irreducible unitary representations of the latter appears to begin with the 1954 Princeton thesis by Ehrman (supervised by Bargmann), excerpts of which are published in Ref. [295]. Ehrman's aim was to determine all irreducible unitary representations of the universal covering group. Following Harish-Chandra he used the Iwasawa decomposition C.21 and the Cartan subgroup of the compact group K generated by $\hat{m}_{12}$ and $\hat{m}_{45}$ (see Eqs. C.70) as well as the Casimir operators

$$L_2 = \frac{1}{2}\hat{m}_{jk}\hat{m}^{jk}\,, \tag{C.91}$$

$$L_4 = \hat{w}_j\hat{w}^j\,,\ \hat{w}_j = \frac{1}{8}\epsilon_{jklmn}\hat{m}^{kl}\hat{m}^{mn}\,, \tag{C.92}$$

$$\epsilon_{jklmn} : \text{ completely antisymmetric in } j, k, \ldots\,,$$

$$\epsilon_{12345} = 1\,,$$

for classifying the irreducible representations.

Ehrman dealt only briefly at the end of his paper with the positive discrete series, where he coined the term "singleton" which has been used frequently afterwards:

We have already seen at the end of Ch. 8 and will again discuss it below that the irreducible unitary representations of the discrete series may be characterized by a pair (ϵ_0, j_0), where ϵ_0 denotes the lowest eigenvalue of the self-adjoint operator $J_{45} = G_0$ which represents $\hat{m}_{45} = \hat{g}_0$.

j_0 is the largest eigenvalue of the angular momentum operator J_{12} associated with that irreducible representation of $SU(2)$ which has the smallest dimension of all the $SU(2)$ representations contained in the infinite dimensional representation of the full group $SO^{\uparrow}(2, 3)$. If $j_0 > 0$ this means the ground state is degenerate with a degeneracy $2j_0 + 1$. Higher levels with $\epsilon_n = \epsilon_0 + n$, $n = 1, 2, \ldots$, also carry angular momenta $j > j_0$. There may be several of them. If there is just one, Ehrman called the corresponding representation of the full group a "singleton"!

In 1963 Dirac discovered and discussed [291] the two unitary representations of the positive discrete series which may be generated by the operator version of the functions $g_0, \ldots, g_9$, of Ch. 8. They are the "singletons" $(1/2, 0)$ and $(1, 1/2)$.

In 1965 Fronsdal gave a brief characterization of irreducible unitary representations of $SO^{\uparrow}(2, 3)$ - especially those of the discrete series - in the appendix of his paper [296] on the possible role of the anti-de Sitter space in particle physics.

In 1966 an important paper by Evans appeared [297] in which he gave a complete clear classification of all irreducible unitary representations of the universal covering group of $SO^{\uparrow}(2, 3)$ in terms of the pairs (ϵ_0, l_0). I shall come back to this paper below. It was almost never quoted by later papers on the subject. Evans did not give *explicit* constructions of the representations he classified.

The explicit constructions of the positive discrete series of $Sp(2n, \mathbb{R})$, inspired by the work of Bargmann for the group $SU(1, 1)$, had already been given by the mathematician Godement

in 1958 [298], but they remained unnoticed for a long time.

In 1966 non-compact "dynamical groups" became popular in particle physics including the use of Majorana-type infinite component wave equations [299]. In this context irreducible unitary representations of $SO^{\uparrow}(2,3)$ were employed by several research groups [300–303], mainly using Dirac's (singleton) representations from 1963.

Later Howe discussed some of the related mathematics [304].

In 1968 Goshen and Lipkin extended their previous work on applications of $Sp(2,\mathbb{R})$ [167] to questions of nuclear structure to the group $Sp(4,\mathbb{R})$ [305].

In 1971 Moshinsky and Quesne discussed [306] unitary representations of symplectic groups (without knowning the results of Itzykson and Bargmann).

In 1972 Kirillov in his textbook [307] briefly discussed unitary spinor representation of the symplectic group.

In 1973 Gelbart published a paper [308] on the explicit construction of the discrete series for $Sp(2n,\mathbb{R})$ in terms of Weil's representation [293], using Godement's previous work.

In 1977 Rosensteel and Rowe used Godement's results as to the positive discrete series [298] for their description of nuclear models in terms of unitary representations of symplectic groups [309].

In 1983 a (not easy to read) paper by Angelopoulos with a complete (?) list of the irreducible unitary representations of $SO^{\uparrow}(2,3)$ appeared [310].

In 1984 the first paper on possible applications of the group $Sp(4,\mathbb{R})$ in quantum optics was published, by Milburn [58]. Later several papers on that subject followed [311].

From 1985 on Mukunda, Simon, Sudarshan et al. published a series [312] of interesting papers on the structure and possible physical (quantum optical) applications of the groups $Sp(4,\mathbb{R})$ and $SO^{\uparrow}(2,3)$.

There are also a number of papers on the isomorphism between the Lie algebras of these groups and a number of their subalgebras [313].

On the more mathematical side there is, of course, the important monograph by Guillemin and Sternberg [21].

In the textbook by Knapp [314] the group $Sp(4,\mathbb{R})$ serves as an example in many chapters.

In an important memoire from 1989 on a certain (Howe) duality between the groups $O(2,2)$ and $Sp(4,\mathbb{R})$ Przebinda [315] also gives a complete list of the irreducible unitary representations of $Sp(4,\mathbb{R})$, including those of the discrete series. The article is not introductory, but a research report with many cross-references as to technical details. It mainly appeals to the experts and is not easy to read for somebody who is not!

What is urgently needed is an introductory monograph which combines the results of Godement, Evans and Przebinda, written in the style of Bargmann!

C.5.2 About the positive discrete series of $Sp(4,\mathbb{R})$

I shall start with the two representations of the Lie algebra $\mathfrak{so}(2,3)$ found by Dirac [291] and thoroughly analysed by Bargmann on the group level [278]. Then I shall mention some elements of the classification scheme used by Evans [297], with a few remarks as to the results of Przebinda [315]. And finally I shall briefly describe the principle of the explicit construction of the positive discrete series due to Godement [298]. I want to stress that all this is very fragmentary and merely intended to give a superficial overview as to the problems involved and to be dealt with more systematically in the future, with the aim to apply the results to quantum optical and other physical phenomena!

The two Dirac-Bargmann representations

These representations have already been mentioned at the end of Ch. 8.2 and in Sec. C.5.1 above. Their Lie algebra corresponds to the classical Hamiltonian functions $g_j, j = 0, \ldots, 9$, discussed in Ch. 8 in connection with interference patterns. As these functions constitute the "observables" associated with the canonical group $Sp(4, \mathbb{R})$ of the phase space C.73 and it appears to be worthwhile to collect them here once more:

$$
\begin{aligned}
g_0 &= \frac{1}{4}(q_1^2 + p_1^2 + q_2^2 + p_2^2)\,, & g_1 &= \frac{1}{2}(q_1 q_2 + p_1 p_2)\,, \\
g_2 &= \frac{1}{2}(q_1 p_2 - q_2 p_1)\,, & g_3 &= \frac{1}{4}(q_1^2 + p_1^2 - q_2^2 - p_2^2)\,, \\
g_4 &= \frac{1}{2}(q_1 q_2 - p_1 p_2)\,, & g_5 &= \frac{1}{2}(q_1 p_2 + q_2 p_1)\,, \\
g_6 &= -\frac{1}{2}(q_1 p_2 + q_2 p_1)\,, & g_7 &= \frac{1}{2}(q_1 p_1 - q_2 p_2)\,, \\
g_8 &= \frac{1}{4}(q_1^2 - p_1^2 - q_2^2 + p_2^2)\,, & g_9 &= \frac{1}{4}(q_1^2 - p_1^2 + q_2^2 - p_2^2)\,.
\end{aligned}
\tag{C.93}
$$

We have seen that these functions fulfill the Lie algebra $\mathfrak{so}(2,3)$ (or $\mathfrak{sp}(4, \mathbb{R})$) in terms of their Poisson brackets.

Dirac discussed [291] the operator version of C.93 and two related representations, Bargmann [278] the irreducible unitary representations of the associated (metaplectic) group.

The self-adjoint operators $G_j, j = 0, \ldots, 9$, or $J_{jk} = -J_{kj}$, corresponding to the functions C.93 or the $\hat{m}_{jk}$ of C.70 may conveniently expressed by the associated creation and annihilation operators:

$$
\begin{aligned}
G_0 &= J_{54} = \frac{1}{2}(a_1^+ a_1 + a_2^+ a_2 + 1)\,, & G_1 &= J_{23} = \frac{1}{2}(a_1^+ a_2 + a_1 a_2^+)\,, \\
G_2 &= J_{31} = \frac{i}{2}(a_1 a_2^+ - a_1^+ a_2)\,, & G_3 &= J_{12} = \frac{1}{2}(a_1^+ a_1 - a_2^+ a_2)\,, \\
G_4 &= -J_{53} = \frac{1}{2}(a_1^+ a_2^+ + a_1 a_2)\,, & G_5 &= -J_{43} = \frac{i}{2}(a_1^+ a_2^+ - a_1 a_2)\,, \\
G_6 &= J_{52} = \frac{i}{4}(a_1^2 - (a_1^+)^2 + a_2^2 - (a_2^+)^2)\,, \\
G_7 &= J_{41} = \frac{i}{4}((a_1^+)^2 - a_1^2 + a_2^2 - (a_2^+)^2)\,. \\
G_8 &= J_{51} = \frac{1}{4}(a_1^2 + (a_1^+)^2 - a_2^2 - (a_2^+)^2)\,, \\
G_9 &= J_{42} = \frac{1}{4}(a_1^2 + (a_1^+)^2 + a_2^2 + (a_2^+)^2)\,.
\end{aligned}
\tag{C.94}
$$

These operators act in a Hilbert space $\mathcal{H}_1^{osc} \otimes \mathcal{H}_2^{osc} = \mathcal{H}_+ \oplus \mathcal{H}_-$, the tensor product of two harmonic oscillator Hilbert spaces, with the even and odd states

$$
\mathcal{H}_+ \ni |n_1, n_2\rangle_+ \equiv |n_1\rangle_1 \otimes |n_2\rangle_2\,,\ n_1 + n_2 \text{ even}\,, \tag{C.95}
$$

$$
\mathcal{H}_- \ni |n_1, n_2\rangle_+ \equiv |n_1\rangle_1 \otimes |n_2\rangle_2\,,\ n_1 + n_2 \text{ odd}\,. \tag{C.96}
$$

For the "even" ground state we have

$$
G_0 |0,0\rangle_+ = \frac{1}{2}|0,0\rangle_+\,, \tag{C.97}
$$

$$
J_3 |0,0\rangle_+ = 0\,, \quad J_3 \equiv J_{12}\,, \tag{C.98}
$$

and for a state with $n_1 + n_2 = 2n$

$$G_0|n_1, n_2; n_1 + n_2 = 2n\rangle_+ = \epsilon_n^{(+)}|n_1, n_2; n_1 + n_2 = 2n\rangle_+ \,, \tag{C.99}$$

$$\epsilon_n^{(+)} = n + \frac{1}{2} \,,\; n = 0, 1, \ldots . \tag{C.100}$$

The state C.99 is $(2n+1)$-fold degenerate. The corresponding subspace carries an irreducible unitary representation of the group $SU(2)$ with angular momentum $j = n$. The 3 generators of that represenation are G_1, G_2 and $G_3 = J_3$. The eigenstate of J_3 with eigenvalue $j = n$ is given by $n_1 = 2n, n_2 = 0$.

The ground state of $\mathcal{H}_-$ is degenerate with respect to $G_0 = J_{54}$, too:

$$G_0|1, 0\rangle_- = |1, 0\rangle_- \,, \tag{C.101}$$

$$G_0|0, 1\rangle_- = |0, 1\rangle_- \,, \,. \tag{C.102}$$

If $n_1 + n_2 = 2n + 1$ then

$$G_0|n_1, n_2; n_1 + n_2 = 2n + 1\rangle_- = \epsilon_n^{(-)}|n_1, n_2; n_1 + n_2 = 2n + 1\rangle_- \,, \tag{C.103}$$

$$\epsilon_n^{(-)} = n + 1 \,,\; n = 0, 1, \ldots . \tag{C.104}$$

Here the degeneracy of the eigen-subspace is $(2n+2)$-fold. It carries an irreducible unitary $SU(2)$ representation with $j = n + 1/2$.

Notice that in the "even" case C.95 the eigenvalues of G_0 are half-integer, whereas the values j of the angular momentum are integer. For the "odd" case C.96 it is the other way round. This is what Dirac found "remarkable" [291]. But notice also (see C.94) that G_0 is just half the Hamiltonian H of the 2-dimensional harmonic oscillator!

Sketch of the classification scheme

The nucleus of the classification scheme for the irreducible unitary representations of the positive discrete series of $Sp(4, \mathbb{R})$ is already visible in the case of the Dirac-Bargmann representations.

I mainly follow Evans [297] in this brief sketch:

Crucial is the existence of the compact Cartan subalgebra spanned by the two commuting Lie algebra elements $\hat{g}_0 = \hat{m}_{54}$ and $\hat{g}_3 = \hat{m}_{12}$. They are two of the four generators of the maximal compact group $U(2)$ of $Sp(4, \mathbb{R})$. It is then possible to characterize the representations in question uniquely by the lowest eigenvalue ϵ_0 of $G_0 = J_{54}$ and the angular momentum value j_0 of the lowest-dimensional unitary representation of $SU(2)$ contained in the representation of $Sp(4, \mathbb{R})$. In general this $SU(2)$ representation will characterize the degeneracy of the ground state.

Thus, any irreducible unitary representation of the positive discrete series is characterized by a pair (ϵ_0, j_0). The $2j_0+1$ associated eigenvalues m_0 of $G_3 = J_{12}$ are $m_0 = j_0, j_0-1, \ldots, -j_0+1, -j_0$.

One may also use the eigenvalues of the two Casimir operators C.91, for a classification, but that appears here to be much more cumbersome than using the pair (ϵ_0, j_0) (see below).

Notice that ϵ_0 might be interpreted as half the energy of the ground state, at least this is so for the two Dirac-Bargmann representations.

The operator $G_0 = J_{54}$ is the generator of the representations of the $U(1)$ subgroup of $U(2)$. It commutes with the $SU(2)$ generators G_1, G_2 and G_3 of the representations.

Like in the case of the representations of $Sp(2, \mathbb{R}) \cong SU(1, 1)$ (see Appendix B) where the lowest eigenvalue k of the generator K_0 for the corresponding subgroup $U(1)$ is determined by

the choice of the covering group of $U(1)$ (there are infinitely many of them!), so does the value of ϵ_0 depend on that covering group.

In the case of the Dirac-Bargmann representations we have met the values $\epsilon_0 = 1/2$ and $= 1$. For the universal covering group of $Sp(4,\mathbb{R})$ (see Ref. [278]) ϵ_0 may take any value in the interval $(0,1]$.

Given the lowest eigenvalue ϵ_0 the higher eigenvalues of G_0 in a given representation are

$$G_0|n;(\epsilon_0,j_0)\rangle = \epsilon_n|n;(\epsilon_0,j_0)\rangle\,, \tag{C.105}$$

$$\epsilon_n = n + \epsilon_0\,,\ n = 0,1,\ldots\,, \tag{C.106}$$

The eigenstates C.105 are degenerate. The carry at least one irreducible representation of $SU(2)$ with angular momentum j, $j = 0, 1/2, 1, 3/2, \ldots$, where G_3 has the eigenvalues $m = j, j-1, \ldots, -j+1, -j$. The whole infinite dimensional Hilbert space may be built up from such subspaces.

Important restrictions on j and m for a given ϵ_n follow from the following observation: According to the Eqs. C.57 and C.58 the operators $(G_0+G_3)/2$ and $(G_0-G_3)/2$ are the generators $K_0^{(1)}$ and $K_0^{(2)}$ of two positive discrete series sub-representations of two independent $SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$ subgroups of $Sp(4,\mathbb{R})$. As both $K_0^{(j)}, j = 1,2$, are positive definite we have the important inequlities

$$\frac{1}{2}(\epsilon_n + m) > 0\,,\quad \frac{1}{2}(\epsilon_n - m) > 0\,,\ m = j, j-1, \ldots, -j+1, -j\,. \tag{C.107}$$

This implies

$$|m| < \epsilon_n\,,\quad \text{especially } j < \epsilon_n\,. \tag{C.108}$$

These inequalities applied to the ground state(s) give

$$|m_0| < \epsilon_0\,,\quad j_0 < \epsilon_0\,. \tag{C.109}$$

The last conditions are obviously fulfilled for the Dirac-Bargmann representations.

The above remarks in connection with the two subalgebras C.57 and C.58 lead to another important consequence: If we denote the Bargmann indices of the corresponding irreducible unitary representations of the discrete series by k_1 and k_2, then a decomposition of the irreducible representation (ϵ_0, j_0) of $Sp(4,\mathbb{R})$ with respect to sub-representations k_1 and k_2 of $Sp(2,\mathbb{R})$ leads to the $2j_0+1$ possible values

$$k_1 = \frac{1}{2}(\epsilon_0 + m_0)\,,\quad k_2 = \frac{1}{2}(\epsilon_0 - m_0)\,. \tag{C.110}$$

For $(\epsilon_0, j_0) = (1/2, 0)$ we have $k_j = 1/4, j = 1,2$, and for $(1, 1/2)$ we get $k_j = 1/4, 3/4$, which are just the metaplectic representations we encountered in Ch. 6.2.

As to the list of possible irreducible unitary representations of the positive discrete series I refer to Evan's paper [297]. Notice that his list applies to the universal covering group of $Sp(4,\mathbb{R})$, not to $Sp(4,\mathbb{R})$.

For illustration I give Evan's expressions for the eigenvalues l_2 and l_4 of the Casimir operators C.91:

$$l_2 = -[\epsilon_0(\epsilon_0 - 3) + j_0(j_0+1)]\,,\ l_4 = -j_0(j_0+1)(\epsilon_0 - 1)(\epsilon_0 - 2)\,. \tag{C.111}$$

The list of Przebinda [316] is more special because he is only interested in true representations of $Sp(4,\mathbb{R})$ itself. He parametrizes the representations by integers $m = \epsilon_0 + j_0$ and $n = \epsilon_0 - j_0$ which excludes the Dirac-Bargmann representations.

Sketch of Godement's construction

Godement's explicit construction of the positive discrete series of the irreducible unitary representations of the groups $Sp(2n,\mathbb{R})$ is a generalization of the corresponding construction for $Sp(2,\mathbb{R}) = SL(2,\mathbb{R})$ on the Siegel upper half plane as discussed in Appendix B.3.2 above, Eq. B.106.

In order to briefly describe Godement's construction for $Sp(4,\mathbb{R})$ we need some facts from the theory of certain finite dimensional representations of the groups $GL_+(2,\mathbb{C}$ or $\mathbb{R})$ and $U(2)$ on complex vector spaces with an hermitian scalar product.
These vector spaces are constructed [317–319] from a 2-dimenional one

$$V^2 = \{b = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, z_j \in \mathbb{C}\,,\ j = 1,2\}\,,\ \langle b, b\rangle = b^+ \cdot b\,. \tag{C.112}$$

The homogeneous polynomials of degree n,

$$P_k^{(n)}(b) = z_1^k z_2^{n-k}\,,\ k = 0,1,\ldots,n\,, \tag{C.113}$$

span a $(n+1)$-dimenional complex vector space V^{n+1}.
The group elements

$$g = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \in GL(2,\mathbb{C}) \tag{C.114}$$

act on the basis C.113 as

$$P_k^{(n)}(b) \to (g \cdot P_k^{(n)})(b) = P_k^{(n)}(b^T \cdot g) = \sum_{l=0}^{n} \tilde{D}_{kl}(g)\, P_k^{(n)}(b)\,, \tag{C.115}$$

and thus induce a $(n+1)$ dimensional representation of $GL(2,\mathbb{C})$ on V^{n+1} in terms of the matrices $\tilde{D}(g) = (\tilde{D}_{kl}(g))$.

For $GL(2,\mathbb{C}) = U(2)$ the representations are equivalent to unitary ones: If one introduces the normalized basis

$$e_k(b) = \frac{P_k^{(n)}(b)}{\sqrt{k!(n-k)!}}\,,\ k = 0,1,\ldots,n, \tag{C.116}$$

and replaces the basis $P_k^{(n)}(b)$ in C.115 by $e_k(b)$ then one gets representation matrices

$$D(g) = (D_{kl}(g))\,,\ k,l = 0,1,\ldots,n, \tag{C.117}$$

instead of $\tilde{D}(g)$. If $g = g_u \in U(2)$, then the matrices $S(g_u) \equiv D(g_u)$ are unitary. This procedure is well-known for $SU(2)$ where $n = 2j$, j: angular momentum.

If we put

$$v_j = \sum_{k=0}^{n} c_k(j)\, e_k(b)\,,\ j = 1,2\,, \tag{C.118}$$

then we have the hermitian scalar product

$$\langle v_2, v_1\rangle = \sum_{k=0}^{n} \bar{c}_k(2) c_k(1)\,. \tag{C.119}$$

The representations C.115 are irreducible and the matrices $D(g)$ are polynomials in the matrix elements c_{jk} of the 2×2 matrices C.114. Of special interest are the following cases: Let g be the positive definite diagonal matrix

$$g = g_a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\,,\ a_j > 0\,,\ j = 1,2\,, \tag{C.120}$$

then $D(g_a)$ is diagonal and positive definite, too:

$$D(g_a) = \text{ diag}\,(a_2^n, a_1 a_2^{n-1}, \ldots, a_1^k a_2^{n-k}, \ldots, a_1^n)\,, \tag{C.121}$$

with

$$\det D(g_a) = (\det g_a)^{n(n+1)/2}\,. \tag{C.122}$$

Relations like C.121 and C.122 hold for arbitrary diagonal matrices $g_d = \text{ diag}\,(\lambda_1, \lambda_2)$.

It follows from C.121 that

$$D(g_a^s) = D^s(g_a)\,,\ s \in \mathbb{R}\,. \tag{C.123}$$

One can also show that $D(g_h)$ is hermitian if g_h is hermitian. In our context the following special case of C.123 is of interest:

$$g_a = y = g^T \cdot g\,,\ g \in GL(2,\mathbb{R})\,,\ \det g > 0\,. \tag{C.124}$$

We then have from the hermiticity of $D(y)$ and C.123 that

$$\langle D(y^{1/2})\, v, D(y^{1/2} v\rangle = \langle v, D(y = g^T \cdot g)\, v\rangle = \langle D(g)v, D(g)v\rangle\,. \tag{C.125}$$

I am now ready to sketch Godement's construction (see also Ref. [309]):

In Sec. C.4 the symmetric 2×2 matrices of the Siegel upper half plane $\mathbb{S}_2$ were denoted by $W = U + iV$, $V > 0$, and those of the Siegel unit disc $\mathbb{D}_2$ by $Z = X + iY$. As we shall not discuss the disc here (see, however, Ref. [320]), I switch the notation and use $Z = X + iY \in \mathbb{S}_2\,,\ , Y > 0$ instead of W. The notation for the invariant volume element C.84 has to be changed accordingly.

Let $f(Z)$ be a matrix-valued function on $\mathbb{S}_2$ with values in the complex vector space V^{n+1} from above. Then a Hilbert space $\mathcal{H}(D)$ may be defined in terms of the scalar product

$$(f,f) \equiv \int_{\mathbb{S}_2} d^3x\, d^3y\, (\det Y)^{-3} \langle D(Y^{1/2}) \cdot f(Z), D(Y^{1/2}) \cdot f(Z)\rangle < \infty\,. \tag{C.126}$$

On this space the group elements (see C.9-C.12)

$$w = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{C.127}$$

act as

$$f(Z) \to D(A_{21} \cdot Z + A_{22})^{-1} \cdot f[(A_{11} \cdot Z + A_{12}) \cdot (A_{21} \cdot Z + A_{22})^{-1}]\,. \tag{C.128}$$

Godement shows that these transformations are irreducible and unitary.

In general the representation matrix $D(g)$ will be of the type

$$D(g) = (\det g)^{\alpha_2}\, \hat{D}(g)\,, \tag{C.129}$$

where $\hat{D}(g)$ is again a polynomial in the matrix elements c_{jk} of C.114. If α_2 is not an integer, then one needs additional discussions [298].

Appendix D

Estimates and asymptotic expansions of some functions

D.1 Proof that $I_{\nu+1}(x)/I_\nu(x) < 1$

In Sec. 3.1.2 of Ch. 3 the ratio 3.78,

$$\rho_k(|z|) = I_{2k}(2|z|)/I_{2k-1}(2|z|)\,, \tag{D.1}$$

plays a major role. I shall prove now that this ratio is smaller than 1 for all finite $|z|$.

It follows from the relation [321]

$$x\,\frac{dI_\nu}{dx}(x) = \nu\, I_\nu(x) + x\, I_{\nu+1} \tag{D.2}$$

that

$$I_{\nu+1}(x)/I_\nu(x) = \frac{d}{dx}\ln(I_\nu(x)/x^\nu)\,. \tag{D.3}$$

As [322]

$$I_\nu(x) = \frac{x^\nu}{2^\nu\,\sqrt{\pi}\,\Gamma(\nu+1/2)}\int_0^\pi d\theta\, e^{x\,\cos\theta}\,\sin^{2\nu}\theta\,, \tag{D.4}$$

we get for D.3

$$I_{\nu+1}(x)/I_\nu(x) = \frac{\int_0^\pi d\theta\,(\cos\theta)\,e^{x\,\cos\theta}\,\sin^{2\nu}\theta}{\int_0^\pi d\theta\, e^{x\,\cos\theta}\,\sin^{2\nu}\theta} < 1\,. \tag{D.5}$$

This proves the assertion[1]!

D.2 Asymptotic expansion of an integral

Next I want to prove the asymptotic expansion of the integral in Eq. 3.112,

$$l_k(|z|^2) = \frac{1}{\sqrt{\pi}}\int_0^\infty dt\, t^{-1/2}e^{-2kt}g_k(|z|^2e^{-t})\,, \tag{D.6}$$

for large $|z|$ which leads to the expansion 3.114.

[1] The inequality (D.5) is valid only for $k \geq 1/4$. In the interval $k \in (0,1/4)$ the ratio (D.2) can be larger than 1 !

From Eq. 3.6 we have

$$g_k(|z|^2) \asymp \frac{\Gamma(2k)}{2\sqrt{\pi}}|z|^{1/2-2k}\, e^{2|z|}\,[1 - a_{-1}/|z| + O(|z|^{-2})] \text{ for } |z| \to \infty\,. \tag{D.7}$$

Inserting this expression into the integral D.6 and making the change of variables

$$1 - u = e^{-t/2} \tag{D.8}$$

leads to

$$l_k(|z|^2) \asymp \frac{\Gamma(2k)}{\sqrt{2}\pi}|z|^{1/2-2k} e^{2|z|} \int_0^1 du\, u^{-1/2}[-\frac{1}{u}\ln(1-u)]^{-1/2}(1-u)^{2k-1/2}\times$$
$$\times\, e^{-2|z|u}\,[1 - \frac{a_{-1}}{|z|}(1-u)^{-1} + O(|z|^{-2})]\,. \tag{D.9}$$

For a large $|z|$ expansion the value $u = 0$ is the critical point under the integral!

Expanding

$$[-\frac{1}{u}\ln(1-u)]^{-1/2} = 1 - \frac{1}{4}\,u + O(u^2)\,,\ (1-u)^{2k-1/2} = 1 - (2k-1/2)\,u + O(u^2)\,, \tag{D.10}$$

and introducing $v = 2|z|u$ gives, up to next-to-leading order,

$$l_k(|z|^2) \asymp \frac{\Gamma(2k)}{2\pi}|z|^{1/2-2k} e^{2|z|}(1 - a_{-1}/|z|)\times \tag{D.11}$$
$$\times\, |z|^{-1/2} \int_0^{2|z|} dv\, e^{-v} v^{-1/2}[1 - (2k-1/4)\frac{v}{2|z|}]\,. \tag{D.12}$$

Letting the upper limit $2|z|$ of the integral go to ∞ and recalling that [323]

$$\int_0^\infty dv\, e^{-v}\, v^{\mu-1} = \Gamma(\mu)\,,\ \Gamma(\frac{1}{2}) = \sqrt{\pi}\,,\ \Gamma(\frac{3}{2}) = \Gamma(\frac{1}{2}+1) = \sqrt{\pi}/2\,, \tag{D.13}$$

we obtain

$$l_k(|z|^2) \asymp \frac{\Gamma(2k)}{2\sqrt{\pi}}|z|^{1/2-2k} e^{2|z|}(1 - a_{-1}/|z|)\,|z|^{-1/2}\,[1 - (2k-1/4)\frac{1}{4|z|}]\,. \tag{D.14}$$

Combining this with the relation D.7 finally gives the expansion 3.114

D.3 Asymptotic expansions of certain series

In Sec. 3.3.2 we encountered (Eqs. 3.241 and 3.245) the functions

$$h_1^{(k)}(|\alpha|) = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \sqrt{n+2k}\,\frac{|\alpha|^{2n}}{n!}\,, \tag{D.15}$$

$$h_2^{(k)}(|\alpha|) = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \sqrt{(2k+n)(2k+n+1)}\,\frac{|\alpha|^{2n}}{n!}\,, \tag{D.16}$$

and the need for their asymptotic expansions if $|\alpha|$ becomes large. I used those expansions in Sec. 3.3.2 without justifying them. This will be done now.

The asymptotic expensions of the functions D.15 and D.16 may be reduced to that of the series

$$F_{a,s}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!\,(n+a)^s}\,,\ x, a, s \in \mathbb{R}\,,\ > 0\,. \tag{D.17}$$

How that reduction is to be done will be indicated below.

The asymptotic expansion of the series D.17 was derived about simultaneously by Hardy [324] and Barnes [325] around 1905. Both discussed the series for complex values of x, a and s, too, but we do not need the more general case here. Their nice result is

$$F_{a,s}(x) \asymp \frac{x^{-s}\,e^x}{\Gamma(s)}[\sum_{n=0}^{N} c_n\,\frac{\Gamma(s+n)}{x^n} + O(x^{-N-1})]\,, \tag{D.18}$$

where c_n is the coefficient of u^n in the Taylor expansion at $u = 0$ of the function

$$f(u) = (1-u)^{a-1}[-\frac{1}{u}\ln(1-u)]^{s-1} = \sum_{n=0}^{\infty} c_n\,u^n\,. \tag{D.19}$$

For the first three c_n one gets

$$c_0 = 1\,, \tag{D.20}$$

$$c_1 = \frac{1}{2}(s+1) - a\,, \tag{D.21}$$

$$c_2 = \frac{1}{8}(s-1)(s-4a+14/3) + \frac{1}{2}(a-1)(a-2)\,. \tag{D.22}$$

For higher terms in the expansion of $f(u)$ it is helpful to know that

$$[-\frac{1}{u}\ln(1-u)]^{\alpha} = 1 + \alpha\sum_{n=0}^{\infty}\psi_n(\alpha+n)\,u^{n+1}\,, \tag{D.23}$$

where the $\psi_n(y)$ are Stirling's polynomials [326]. The first three of them are [327]

$$\psi_0(y) = \frac{1}{2}\,, \tag{D.24}$$

$$\psi_1(y) = \frac{1}{4!}(3y+2)\,, \tag{D.25}$$

$$\psi_2(y) = \frac{y(y+1)}{4!\,2}\,. \tag{D.26}$$

(See also Ref. [328].)

For the derivation of D.20-D.22 one needs only ψ_0 and ψ_1.

Let us denote the sum in Eq. D.15 by $\hat{h}_1^{(k)}(|\alpha|^2)$. It may be rewritten as

$$\hat{h}_1^{(k)}(|\alpha|^2) = \sum_{n=0}^{\infty}(2k+n)\frac{|\alpha|^{2n}}{n!\,\sqrt{2k+n}}\,. \tag{D.27}$$

Expressed in terms of the function D.17 this means

$$\hat{h}_1^{(k)}(|\alpha|^2) = 2k\,F_{2k,1/2}(|\alpha|^2) + |\alpha|^2\,F_{2k+1,1/2}(|\alpha|^2)\,. \tag{D.28}$$

The asymptotic expansion 3.255 of $h_1^{(k)}(|\alpha|)$ follows from the relations D.28 and D.18.

If we denote the sum in Eq. D.16 by $\hat{h}_2^{(k)}(|\alpha|^2)$, it can be written as

$$\hat{h}_2^{(k)}(|\alpha|^2) = \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \frac{(2k+n)(2k+n+1)}{\sqrt{2k+n}\sqrt{2k+n+1}} \,. \tag{D.29}$$

The asymptotic expansion 3.257 of the series D.29 can be derived by using a generalization of the above procedure which is also due to Barnes [329]:

For n large enough we have

$$(1+2k+n)^{-1/2} = (2k+n)^{-1/2} \, (1+\frac{1}{2k+n})^{-1/2} \,, \tag{D.30}$$

$$(1+\frac{1}{2k+n})^{-1/2} \approx 1 - \frac{1}{2}\frac{1}{2k+n} + \frac{3}{8}\frac{1}{(2k+n)^2} + O[(2k+n)^{-3}] \,. \tag{D.31}$$

Inserting this into the series D.29 yields

$$\hat{h}_2^{(k)}(|\alpha|^2) \approx e^{|\alpha|^2}(|\alpha|^2 + 2k + 1/2) - \frac{1}{8}\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!}[\frac{1}{2k+n} + O(1/(2k+n)^2)] \,. \tag{D.32}$$

The asymptotic behaviour of the sum can again be deduced with the help of D.18, so that we finally obtain

$$\hat{h}_2^{(k)}(|\alpha|^2) \asymp e^{|\alpha|^2}|\alpha|^2[1 + (2k+1/2)|\alpha|^{-2} - \frac{1}{8}|\alpha|^{-4} + O(|\alpha|^{-6}] \,. \tag{D.33}$$

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