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Simulations in the ϵ-Regime of Chiral Perturbation Theory *

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We discuss the potential of Ginsparg-Wilson fermion simulations in the ϵ-regime of chiral perturbation theory, regarding the determination of the leading low energy constants of the effective chiral Lagrangian. It turns out to be very hard to measure observables in the topologically trivial sector. There a huge statistics would be required, due to the frequent occurrence of very small eigenvalues. Moreover, contact with chiral perturbation theory is only established if the physical volume of the system is sufficiently large.

QCD at low energy can be described by an effective Lagrangian $\mathcal{L}_{\rm eff}$ for the quasi-Goldstone bosons of chiral symmetry breaking. $\mathcal{L}_{\rm eff}$ may then be evaluated by chiral perturbation theory (χPT). The case of a finite volume and very light quarks — so that the pion Compton wave length clearly exceeds the box length — is called the *ϵ-regime*. There the zero-mode is treated by collective variables, while the excitations (which do fit into the box) can be evaluated perturbatively in the ϵ-expansion [1]. In this framework, observables strongly depend on the topology of the gauge background [2], hence they should be measured in fixed topological sectors.

$\mathcal{L}_{\rm eff}$ involves coupling constants which appear as free parameters in χPT. In principle they can be determined by comparison with numerical data from lattice QCD in the ϵ-regime. The same values are also relevant for the physical situation in a large volume (the standard χPT), hence their knowledge is very important in view of experiments and further numerical studies, especially for the chiral extrapolation. Their numerical evaluation, however, may be easier in the ϵ-regime, due to the usage of a small volume.

Such simulations in the ϵ-regime are now conceivable by using Ginsparg-Wilson fermions, which avoid additive mass renormalization and define the topological charge through the Index Theorem [3]. Here we discuss the question how one could extract results for the low energy constants which occur in the leading order of $\mathcal{L}_{\rm eff}$, the pion decay constant F_π and the scalar condensate Σ.

We performed quenched QCD simulations with the Wilson gauge action and the overlap Dirac operator [4]

$$D_{\rm ov} = \Big(1 - \frac{m}{2\mu}\Big) D_{ov}^{(0)} + m \ , \qquad (1)$$

$$D_{\rm ov}^{(0)} = \mu(1 + A/\sqrt{A^\dagger A}) \ , \quad A = D_W - \mu \ ,$$

where m is the quark mass and D_W the Wilson operator. At $\beta = 6$ and $\beta = 5.85$ we set $\mu = 1.4$ resp. $\mu = 1.6$. We approximated the inverse square root by Chebyshev polynomials to an accuracy of 10^{-12}. To identify the low lying eigenvalues (EVs) of $D_{\rm ov}^{(0)\,\dagger} D_{\rm ov}^{(0)}$ we applied alternatively the Ritz functional method and the Arnoldi algorithm. The latter allows for the determination of many EVs ($O(100)$). The very lowest EVs determine the index ν: there is usually a gap by several orders of magnitude between the numerical values of the zero EVs and the rest of the spectrum. The chirality of the zero modes reveals the sign of ν. Two typical examples are given in Table 1.

On a $10^3 \times 24$ lattice at $\beta = 6$ we also compared the index with the topological charge determined from (standard) cooling. In this respect, we investigated 51 configurations: for 41 of them the index of $D_{\rm ov}$ and the charge determined from cooling coincided, and in the remaining 10 cases it differed by 1.

Next we mention that Random Matrix Theory (RMT) has been applied to QCD and it yields predictions for the probability distributions of the

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EVs of $D_{ov}^{(0)\,\dagger} D_{ov}^{(0)}$	chirality
example for charge $\nu = 0$	
$6.23013e-3$	0.3608201
$6.23014e-3$	-0.3608198
$9.90174e-3$	-0.7622260
$9.90177e-3$	0.7622260
$2.69086e-2$	-0.7528350
example for charge $\nu = -2$	
$6.03761e-16$	-1.0000000
$9.43764e-11$	-0.9999999
$3.68203e-3$	-0.9240370
$3.73623e-3$	0.9205659
$7.04176e-3$	-0.7016518

Table 1
Typical examples for the lowest eigenvalues of $D_{\rm ov}^{(0)\,\dagger} D_{\rm ov}^{(0)}$ *on a* $10^3 \times 24$ *lattice at* $\beta = 6$.

low lying eigenvalues [5]. For the first non-zero EV these predictions are confirmed if the physical volume is sufficiently large, $V \gtrsim (1.2\ \mathrm{fm})^4$ [6]. In particular, we know from these probability distributions that in the topologically trivial sector there is a significant density of *very small* (non-zero) EVs. As an example, we found a configuration on the $10^3 \times 24$ lattice at $\beta = 6$ where the lowest EVs of $D_{\rm ov}^{(0)\,\dagger} D_{\rm ov}^{(0)}$ read: $7.32e-5$, $7.33e-5$, $1.04e-4$ etc. The corresponding chiralities -0.127, 0.127, 0.826 show clearly that this configuration has index 0. This is also confirmed from cooling, but the form of the cooled configuration suggests that charge 0 arises from the cancellation of a pair of topological objects with charges ± 1, see Fig. 1. The index never involved such a cancellation. However, in the case of small non-zero EVs there seems to be a trend towards a cancellation picture from cooling, as in the example discussed here.

We now address the *meson correlators* as physical observables, from which one can try to extract F_π and Σ. The corresponding formulae for quenched χPT have been worked out in Ref. [7]. For a first comparison to numerical data we refer to Refs. [8]. The axial current correlation function on an $L^3 \times T$ lattice,

$$\langle A_\mu(t)\, A_\mu(0)\rangle_\nu = \frac{F_\pi^2}{T} + 2m\,\Sigma_{|\nu|}(z)\,T\cdot h_1(\tau) \quad (2)$$
$$2h_1(\tau) = \tau^2 - \tau + 1/6\ , \quad \tau = t/T$$

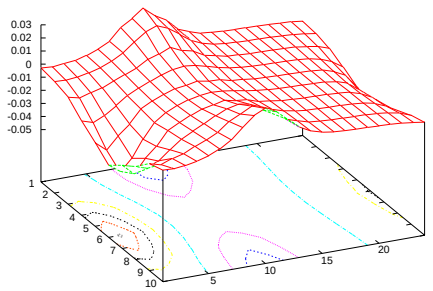

Figure 1. *A 2-d cut of the configuration discussed in the text after 200 cooling sweeps. The vertical axis shows the naive topological charge density, which extends over both signs, suggesting something like an instanton anti-instanton pair.*

$$\frac{\Sigma_\nu(z)}{\Sigma} = z\Big[I_\nu(z)K_\nu(z) + I_{\nu+1}(z)K_{\nu-1}(z)\Big] + \frac{\nu}{z}$$

is most suitable for our purpose, because it only involves F_π and Σ as free parameters. The determination of F_π is relatively easy in the minimum at $\tau = 1/2$. Σ is related to the curvature and far more difficult to extract, since the sensitivity of this curvature to a change of Σ is hardly visible over a wide range [2]. However, we saw also here that the physical volume must exceed the lower limit given before in the context of RMT, otherwise one does not obtain the curvature required by any acceptable value of Σ. Once we are in the right regime, $\nu = 0$ may look like the simplest case from eq. (2), but from the numerical point of view this is a nightmare. The problems in this sector are related to the danger of very small EVs. Ref. [9] pointed out before that they may be a serious problem, and our observations confirm this in a striking way. As an example, we show histories for the measurement of $\langle A_0(t)\, A_0(0)\rangle_0$ and $\langle A_0(t)\, A_0(0)\rangle_1$ in Fig. 2. In the topologically neutral sector there are strong spikes at those configurations which have very small EVs. Their height is maximal at very small quark mass, as Fig. 3 illustrates.

Obviously the occurrence of such spikes makes it very hard to measure expectation values — they stabilize only with a huge statistics. To get

[2] This situation is somehow complementary to the RMT comparison of the EVs, where Σ is easier to determine [6].

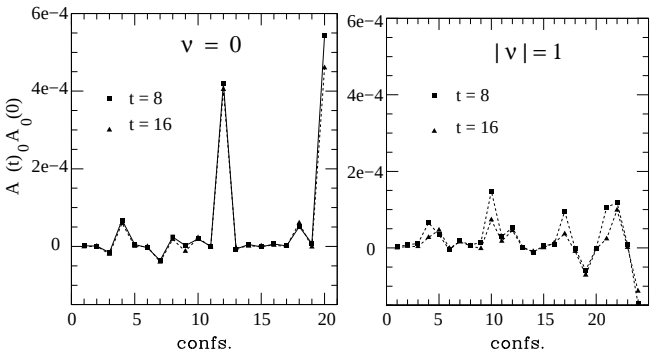

Figure 2. *Histories of the axial correlation function $A_0(t)A_0(0)$ at $t = 8$ and at $t = 16$, with $m = 21.3$ MeV. For some $\nu = 0$ configurations (on the left) we recognize pronounced spikes. At $|\nu| = 1$ (on the right) the history is much smoother.*

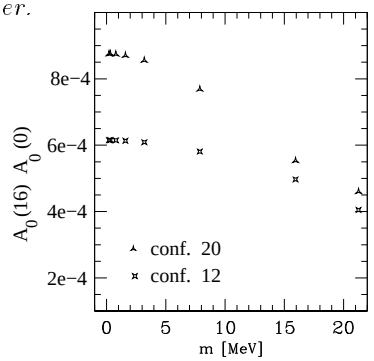

Figure 3. *The height of the two spikes of $A_0(16)A_0(0)$ in the $\nu = 0$ history of Fig. 2, at different quark masses m.*

an estimate for this effect, we consider the contribution of the smallest (non-zero) EV alone to the scalar condensate. This contribution reads

$$\Sigma_{\rm min}^{(\nu)} = \frac{1}{V}\int_0^\infty dz\, P_\nu(z)\frac{2m}{m^2+(z/\Sigma V)^2}\ , \qquad (3)$$

where we insert the probabilities $P_\nu(z)$ provided by RMT. We now generated fake values for the lowest EV with probability $P_\nu(z)$ and computed $\Sigma_{\rm min}^{(\nu)}$ from them. The result can only be trusted when the standard deviation becomes practically constant. Fig. 4 shows the evolution of this standard deviation as the statistics is enhanced. We see that the sector $\nu = 0$ would require a tremendous statistics of $O(10^4)$ configurations, but the situation is clearly better for $\nu \neq 0$.

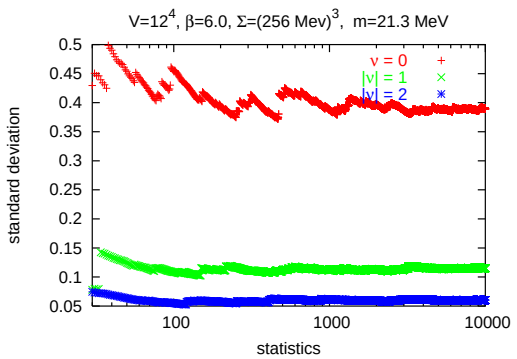

Figure 4. *The standard deviation in the (fake) measurement of $\Sigma_{\rm min}^{(\nu)}$, the contribution of the lowest non-zero EV to the scalar condensate.*

We conclude that there are several conditions for running conclusive simulations with Ginsparg-Wilson fermions in the ϵ-regime. The volume should obey $V \gtrsim (1.2\ \text{fm})^4$, and measurements have to be performed in a sector of *non-trivial topology*. The quark mass should be small for conceptual reasons, but taking it too small causes technical problems in the measurements.

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