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Random Matrix Theory and the Spectra of Overlap Fermions *

S. Shcheredin [a], W. Bietenholz [a], T. Chiarappa [b], K. Jansen [b] and K.-I. Nagai [b]

[a] Institut für Physik, Humboldt Universität zu Berlin, Newtonstr. 15, D-12489 Berlin, Germany
[b] NIC/DESY Zeuthen, Platanenallee 6, D-15738 Zeuthen, Germany

The application of Random Matrix Theory to the Dirac operator of QCD yields predictions for the probability distributions of the lowest eigenvalues. We measured Dirac operator spectra using massless overlap fermions in quenched QCD at topological charge $\nu = 0$, ± 1 and ± 2, and found agreement with those predictions — at least for the first non-zero eigenvalue — if the volume exceeds about $(1.2 \text{ fm})^4$.

In QCD with N_f massless quark flavors, chiral symmetry is assumed to be broken spontaneously as $SU(N_f)_R \otimes SU(N_f)_L \to SU(N_f)_{R+L}$, which generates $N_f^2 - 1$ Goldstone bosons in the coset space $SU(N_f)$. They pick up a mass — so they can be identified with light mesons — in the case of small quark masses. Their dynamics is described by chiral perturbation theory as a low energy effective model of QCD. To the lowest order in the momenta and masses, the effective Lagrangian reads

$$\mathcal{L}[U] = \frac{F_\pi^2}{4} \mathrm{Tr}\,[\partial_\mu U \partial_\mu U] - \frac{\Sigma}{2} \mathrm{Tr}\left[\mathcal{M}(U + U^\dagger)\right] \quad (1)$$

where $U(x) \in SU(N_f)$ and $\mathcal{M}$ is the (diagonal) quark mass matrix. The low energy constants F_π and Σ appear here as free parameters. In a box of size $V = L^4$ the relations $\Lambda_{QCD}^{-1} \ll L \ll m_\pi^{-1}$ characterize the *ϵ-regime* [1].

In that regime we consider the eigenvalues (EVs) $i\lambda_n$ of the Dirac operator, resp. the dimensionless variable $z_n = \lambda_n V \Sigma$. The spectral density is defined by $\rho(\lambda) = \langle \sum_n \delta(\lambda - \lambda_n) \rangle$, and in particular the *microscopic spectral density* ρ_s is given by

$$\rho_s(z) = \lim_{V \to \infty} \frac{1}{\Sigma V} \rho\left(\frac{z}{\Sigma V}\right) , \quad z = \lambda V \Sigma \ . \quad (2)$$

In the sectors of topological charge $\pm\nu$ it can be decomposed as

$$\rho_s^{(\nu)}(z) = \sum_{k \geq 1} p_k^{(\nu)}(z) \ , \quad (3)$$

where $p_k^{(\nu)}(z)$ is the probability distribution of the k-th EV (excluding the zero EV). The application of Random Matrix Theory (RMT) to QCD provides explicit predictions for the functions $p_k^{(\nu)}(z)$ in the ϵ-regime [2].

To test these predictions on the lattice, we simulated quenched QCD with the Wilson gauge action and the Neuberger overlap operator D_{ov} at zero quark mass [3]. At $\beta = 6$ resp. 5.85 the mass of the Wilson kernel, $-\mu$, was fixed by $\mu = 1.4$ resp. 1.6. In contrast to the continuum, where the EVs are imaginary, the EVs of D_{ov} lie on a circle in $\mathbb{C}$ with center and radius μ. For comparison to the continuum predictions we mapped this circle stereographically onto the imaginary axis.

The functions $p_k^{(\nu)}(z)$ depend significantly on $|\nu|$; for instance, the peak of $p_1^{(\nu)}(z)$ moves to larger values of z as $|\nu|$ increases. This effect could not be seen convincingly, however, in simulations with staggered fermions (at least for the couplings investigated so far) [4]. On the other hand, previous studies with Ginsparg-Wilson fermions were consistent with the RMT predictions in QED_2 [5], QED_4 [6] and in QCD on a 4^4 lattice [7]. QCD studies on larger lattices were presented in Refs. [3,8].

Usually the test of RMT predictions was done by comparing the functions $p_k^{(\nu)}(z)$ to the histograms of lattice data. To avoid the arbitrariness of the bin size in that procedure, we compared instead the cumulative density

$$\rho_{k,c}^{(\nu)}(z) = \frac{\int_0^z \rho_k^{(\nu)}(z')\, dz'}{\int_0^\infty \rho_k^{(\nu)}(z')\, dz'} \ . \quad (4)$$

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In Fig. 1 we show the RMT curves for $\rho^{(\nu)}_{1,c}(z)$, $|\nu| = 0$, 1 and 2, compared to data from the following lattices: 10^4 at $\beta = 5.85$, 12^4 at $\beta = 6$ and 8^4 at $\beta = 5.85$. The corresponding physical volumes are $\approx (1.23\ \mathrm{fm})^4$, $(1.12\ \mathrm{fm})^4$ and $(0.98\ \mathrm{fm})^4$. We see that the $k = 1$ EV tends to agree quite well with RMT, *if* the volume exceeds about $(1.2\ \mathrm{fm})^4$. It disagrees, however, if the physical volume is too small; then the data do not distinguish any more the different topological sectors, which is consistent with Ref. [7].

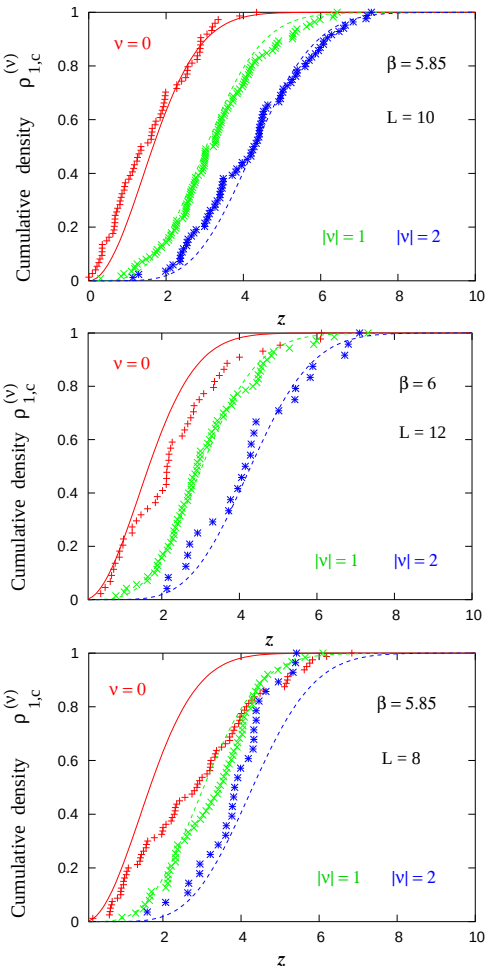

Figure 1. *The cumulative density of the first non-zero EV in the sectors* $|\nu| = 0, 1$ *and* 2. *We compare the RMT predictions (curves) to the lattice data on volumes* $(1.23\ \mathrm{fm})^4$ *(top),* $(1.12\ \mathrm{fm})^4$ *(center) and* $(0.98\ \mathrm{fm})^4$ *(bottom). The quality of agreement increases with the volume.*

This comparison involves Σ as the only free parameter, which was optimized for agreement with RMT. In $(1.23\ \mathrm{fm})^4$ and $(1.12\ \mathrm{fm})^4$ we obtained very similar values, $\Sigma = (253\ \mathrm{MeV})^3$ resp. $(256\ \mathrm{MeV})^3$. These values are consistent with the literature, although Σ is expected to diverge logarithmically at large V in quenched QCD [9]. Further volumes and higher statistics would be required to verify this behavior; our statistics was too small to associate error bars with the above values for Σ.

The range of z where the predictions are compatible with the RMT curves raises gradually as the volume is enlarged. For $V < (1\ \mathrm{fm})^4$ this range is so short that it doesn't even capture the peak of the first non-zero EV, hence the RMT prediction is not applicable in that regime. However, our largest volume is still not sufficient to capture well the $k = 2$ EV; we show results for our larger two volumes in Fig. 2.

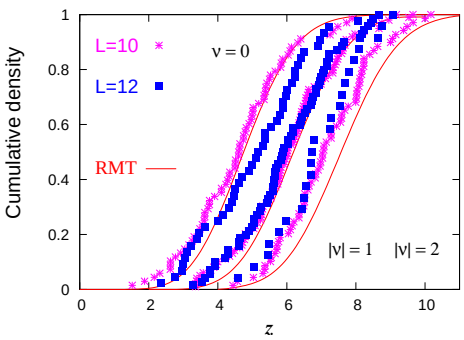

Figure 2. *The cumulative density of the second non-zero EV at* $|\nu| = 0, 1, 2$. *We compare the RMT predictions (curves) to the lattice data in the volumes* $(1.23\ \mathrm{fm})^4$ *(stars) and* $(1.12\ \mathrm{fm})^4$ *(squares). The agreement is worse than in Fig. 1, since the relevant values of* z *are larger.*

Next we present results for the full spectral density $\rho^{(\nu)}_s(z)$ of eq. (2) (summed over k). Fig. 3 shows a histogram at $|\nu| = 1$ on the 12^4 lattice, which roughly follows the RMT prediction up to the second peak, before turning into the bulk behavior $\rho^{(\nu)}_s(z) = c_0 + c_1 z^3$.

We also studied the "unfolded level spacing distribution" which was considered earlier on a 4^4 lattice [10]. There we found good agreement with the behavior specific for $SU(3)$ in the fundamen-

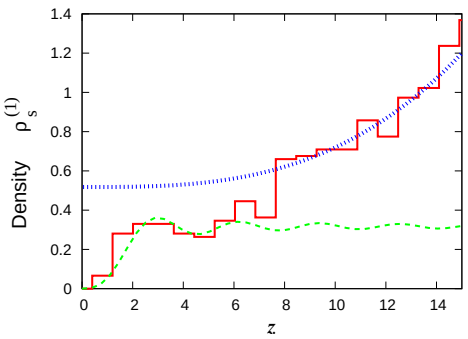

Figure 3. *The spectral density of the low lying EV on a lattice of volume* $(1.12\ \mathrm{fm})^4$. *The histogram of the lattice data roughly follows the RMT prediction (dashed line) up to the second peak, before turning into the cubic bulk behavior (dotted line).*

tal representation that we were using. In this case, the results agree well with the prediction on all the three lattice sizes that we considered; the lower limit for the volume does not seem to affect this quantity, see Fig. 4. However, this type of "unfolding" only keeps track of the relative order of the EV in the ensemble, hence it is not sensitive to the energy scale and not directly physical.

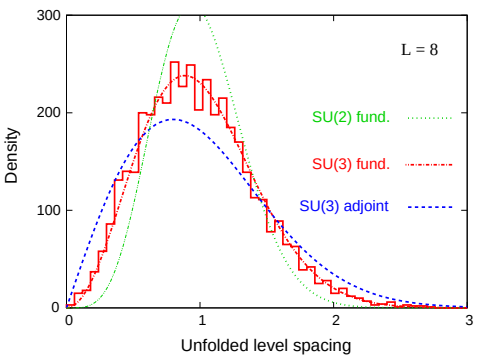

Figure 4. *The unfolded level spacing distribution on a* 8^4 *lattice at* $\beta = 5.85$. *The histogram follows closely the curve predicted for the gauge group* $SU(3)$ *in the fundamental representation.*

In view of the future numerical exploration of the ϵ-regime by means of Ginsparg-Wilson fermions, our main conclusions are

- The volume should be larger than about $(1.2\ \mathrm{fm})^4$, which might just reflect that chiral perturbation theory is only applicable in the confinement phase. This observation can also be viewed as an empirical determination of the so-called Thouless energy.
- Once we are in the right regime, the RMT predictions for the low lying EVs are confirmed in some range $z = 0 \ldots z_{\mathrm{crit}}$, where z_{crit} grows with the volume.
- The latter means that particularly in the topologically neutral sector there is a non-negligible density of very small non-zero EVs. Such EVs are dangerous for the measurement of physical observables; they lead to the requirement of a huge statistics [11].

In the ϵ-regime, observables tend to be strongly $|\nu|$ dependent [12], hence they should be measured at fixed $|\nu|$. The last observation now implies that such measurements should beware of $\nu = 0$.

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