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Meson Correlation Functions in the ϵ-Regime *

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We present a numerical pilot study of the meson correlation functions in the ϵ-regime of chiral perturbation theory (χPT). Based on simulations with overlap fermions we measured the axial and pseudo-scalar correlation functions, and we discuss the implications for the leading low energy constants in the chiral Lagrangian.

1. THEORETICAL BACKGROUND

Ginsparg-Wilson fermions [1,2] obey an exact, lattice modified chiral symmetry [3]. Therefore they have exact zero modes, which provides together with the Index Theorem a conceptually clean definition of the topological charge [2].

In particular the overlap operator [4] represents a relatively simple solution to the Ginsparg-Wilson relation. It allows us to tackle now a number of physically interesting issues, which were inaccessible to numerical studies before. In particular, there is hope for simulations to penetrate the regime of light mesons, which is described by χPT. The latter involves low energy constants as free parameters, which can in principle be determined by QCD simulations in that regime. To the lowest order, the effective chiral Lagrangian of χPT takes the form

$$\begin{aligned} \mathcal{L}_{\rm eff}[U] &= \frac{F_\pi^2}{4}{\rm Tr}\,[\partial_\mu U \partial_\mu U] \\ &- \frac{\Sigma}{2}{\rm Tr}\left[\mathcal{M}(e^{i\theta/N_f}U + e^{-i\theta/N_f}U^\dagger)\right] \end{aligned} \quad (1)$$

where $U(x) \in SU(N_f)$, N_f is the number of flavors, $\mathcal{M}$ is the (diagonal) quark mass matrix and θ the vacuum angle. The pion decay constant F_π and the scalar condensate Σ appear here as the leading low energy constants.

From the practical point of view, the ϵ-regime of χPT looks particularly attractive because it deals with small volumes [5]. This regime is characterized by the hierarchy relation

$$\Lambda_{\rm QCD}^{-1} \ll L \ll m_\pi^{-1} \quad (2)$$

so that the pion correlation length m_π^{-1} is much larger than the linear size of the system L. Moreover, $\Lambda_{\rm QCD}$ can be related to the cutoff of the effective theory, $\Lambda_{\rm QCD} \sim 4\pi F_\pi$. One identifies the matrix field as $U = \exp(i\sqrt{2}\xi/F_\pi)$, where the field $\xi(x)$ describes the light mesons. The ϵ-expansion then relies on a non-perturbative treatment of the zero-mode, whereas the excitations (the non-zero modes of $\xi(x)$) do fit into the volume and can be evaluated perturbatively. In the ϵ-regime, the topological charge ν of the gauge field plays an important rôle [6]; observables should be measured at fixed values of $|\nu|$. Note that the values of F_π and Σ in the ϵ-regime are the same as in the physical situation of an infinite volume.

Hence the main virtues of Ginsparg-Wilson fermions — no additive mass renormalization and a clean definition of the topological charge — render this lattice fermion formulation ideally suited for simulations in the ϵ-regime. However, they are numerically so demanding that for the time being only quenched QCD simulations are feasible. We performed such simulations with overlap fermions and the Wilson gauge action at $\beta = 6$ on lattices of size $10^3 \times 24$ and 12^4. The mass of the Wilson kernel in the overlap operator was set to -1.4, which is optimal for locality [7].

Let us mention that Random Matrix Theory applied to QCD yields predictions for the low lying eigenvalues (EVs) $i\lambda$ of the Dirac operator [8]. Simulations in the ϵ-regime confirm these predictions, at least for the first non-zero EV, if the physical volume is not too small, $L \gtrsim 1.2$ fm [9].[2]

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[2] A lower bound for L has also been obtained from the condition for the ϵ parameter to be sufficiently small [10].

Fig. 1 shows the probability distribution $\rho_1^{(\nu)}$ of this first EV in various topological sectors, as a function of the dimensionless variable $\mathbf{z} = \lambda \Sigma V$. In particular we see that in the neutral sector there is a significant probability $\rho_1^{(0)}$ for very small EVs. In turned out that this property makes measurements very hard, i.e. they would require a huge statistics [11]. However, the situation is clearly better at non-trivial topology. Therefore we concentrate here on the sector $|\nu| = 1$.

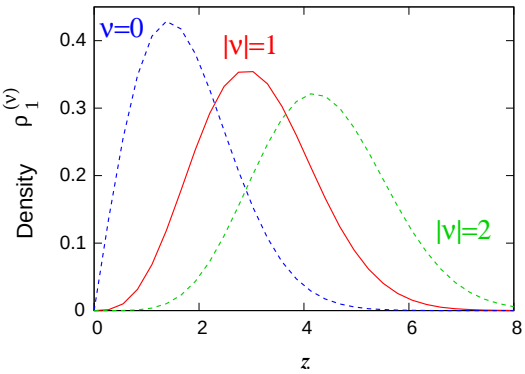

Figure 1. *The probability distribution for the first non-zero EV of the massless Dirac operator, according to Random Matrix Theory.*

We measured various types of *meson correlation functions* at momentum $\vec{p} = \vec{0}$ and Euclidean time t. The corresponding formulae predicted by quenched χPT in a volume $L^3 \times T$ are given in Refs. [12]. They were obtained consistently from two methods denoted as "replica" [13] and "supersymmetric" [14]. The vector correlation function vanishes identically. The simplest non-trivial case is the bare axial correlation function,

$$\langle A_\mu(t)\, A_\mu(0)\rangle_\nu = \frac{F_\pi^2}{T} + 2m\,\Sigma_{|\nu|}(z)\,T\cdot h_1(\tau) \quad (3)$$

$$2h_1(\tau) = \tau^2 - \tau + 1/6\ ,\ \ \tau = t/T\ ,\ \ z = m\Sigma V\ ,$$

$$\frac{\Sigma_\nu(z)}{\Sigma} = z\Big[I_\nu(z)K_\nu(z) + I_{\nu+1}(z)K_{\nu-1}(z)\Big] + \frac{\nu}{z},$$

where I_ν and K_ν are modified Bessel functions and m is the bare quark mass.

To the first order the pseudo-scalar correlation function reads

$$\langle P(t)\, P(0)\rangle_\nu = \frac{\Sigma_{|\nu|}^{1-\mathrm{loop}}(z)}{2mT}L^3 - \frac{\Sigma^2}{2F_\pi^2}h_1(\tau)\times$$

$$\Big[\frac{\alpha}{3}c_+^{(|\nu|)}(z) - b_+^{(\nu)}(z)\Big] - \frac{\Sigma^2}{6F_\pi^2}h_2(\tau)m_0^2 c_+^{(|\nu|)}(z)\ ,$$

$$h_2(\tau) = -\frac{T^3}{24}\Big(\tau^4 - 2\tau^3 + \tau^2 - \frac{1}{30}\Big)\ ,$$

$$\Sigma_\nu^{1-\mathrm{loop}}(z) = \Sigma_\nu(z)\frac{z'}{z}\ ,\quad z' = m\,\Sigma_{\mathrm{eff}}(V)\,V\ , \quad (4)$$

In this case, and also in the scalar correlation function, new parameters of the effective chiral Lagrangian have to be taken into account, which are specific for the quenched approximation: m_0 and α are a scalar mass and a kinetic coupling, respectively. For details and for the form of Σ_{eff}, $c_+^{(\nu)}$ and $b_+^{(\nu)}(z)$ we refer to Refs. [12].

2. NUMERICAL RESULTS

We first consider the axial correlation function because it only involves the constants Σ and F_π. On the $10^3 \times 24$ lattice the spatial extent is below 1 fm, and our data show clearly that in such a small volume χPT is not applicable, as expected.

Therefore we now present results from the 12^4 lattice, which corresponds to a volume of $(1.12\ \mathrm{fm})^4$, i.e. close to the point where the applicability of χPT should set in. Our quark mass amounts to $m = 21.3$ MeV, and our statistics involves 78 configurations with $\nu = \pm 1$.

Fig. 2 (on the left) shows our data for the axial correlator. The curve corresponding to eq. (3) can be fitted well over some interval that excludes the points near the boundary. This fixes F_π, which enters in an additive constant. On the right of Fig. 2 we plot the resulting F_π as a function of the number t_f of t values (around $T/2$) that we include in the fit. We find a decent plateau, which suggests — up to renormalization — a value of

$$F_\pi = (86.7 \pm 4.0)\ \mathrm{MeV}\ . \quad (5)$$

On the other hand, Σ cannot be easily extracted from these fits. It is related to the curvature in the minimum, but if we vary Σ over a wide range the fit function is hardly affected. In fact, Fig. 2 (left) shows fits to $t_f = 7$ points for Σ varying from 0 to $(250\ \mathrm{MeV})^3$, but those curves can hardly be distinguished.

As an attempt to proceed also in that respect, we show in Fig. 3 (above) the results for the axial and pseudo-scalar correlation functions, which

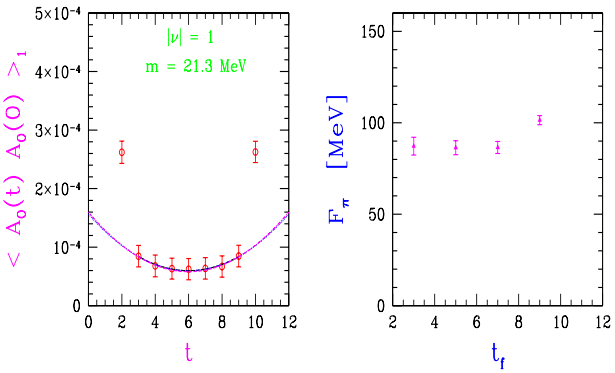

Figure 2. *The axial correlation function (left) and the values of F_π obtained from fits over t_f points around the center $t = 6$ (right).*

are fitted simultaneously. Here the fits involve the parameters α, m_0 and $\Sigma_1^{1-\mathrm{loop}}$ (introduced in eq. (4)), in addition to F_π and Σ. These five parameter fits lead to decent plateaux simultaneously for F_π and $\Sigma_1^{1-\mathrm{loop}}$, as Fig. 3 (below) illustrates. $\Sigma_1^{1-\mathrm{loop}}$ depends on the other four parameters, and it represents a one loop approximation to Σ.

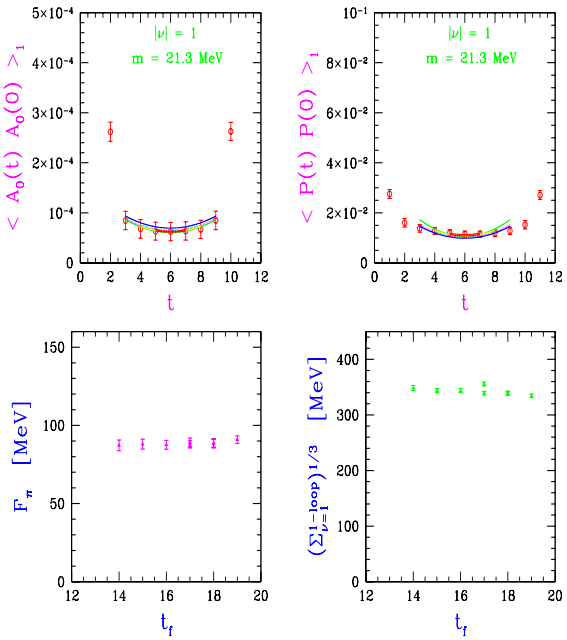

Figure 3. *Above: measured data and fits for the axial and pseudo-scalar correlation functions. Below: The values F_π and $\Sigma_1^{1-\mathrm{loop}}$ obtained from five parameter fits. Here t_f is the sum of points considered in the simultaneous fits for both correlation functions.*

3. CONCLUSIONS

We presented results of a pilot study of meson correlation functions in the ϵ-regime, based on quenched QCD with overlap fermions. To avoid trouble with very low EVs we worked in the sector $|\nu| = 1$. Although the statistics is still rather modest, we recognize some qualitative features. If the physical volume is sufficiently large, the data can be fitted to the functions predicted by quenched χPT. These fits allow for a good determination of F_π, whereas the evaluation of Σ is far more difficult.

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