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On the exact open-closed vertex in plane-wave light-cone string field theory

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Abstract

The open-closed vertex in the maximally supersymmetric type IIB plane-wave light-cone string field theory is considered and an explicit solution for the bosonic part of the vertex is derived, valid for all values of the mass parameter, μ. This vertex is of relevance to IIB plane-wave orientifolds, as well as IIB plane-wave strings in the presence of D-branes and their gauge theory duals. Methods of complex analysis are used to develop a systematic procedure for obtaining the solution. This procedure is first applied to the vertex in flat space and then extended to the plane-wave case. The plane-wave solution for the vertex requires introducing certain "μ-deformed Gamma functions", which are generalizations of the ordinary Gamma function. The behaviour of the Neumann matrices is graphically illustrated and their large-μ asymptotics are analysed.

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1. Introduction

The plane-wave limit of the AdS/CFT correspondence has been a fertile and active area of research. In the most prominent variant, this limit relates string theory in the plane-wave background obtained as a Penrose-limit of $AdS_5 \times S^5$ [1,2] with a particular sector of $\mathcal{N} = 4$, $d = 4$ Super Yang-Mills theory [3]. This sector of the SYM theory is known as the BMN sector. In this limit both the string theory and the gauge theory are perturbative and thus are independently accessible to direct computations.

Understanding this duality in the presence of interactions is an important problem, as it will provide more underpinning evidence for the correspondence. In order to study interactions on the string theory side, one has to resort to light-cone superstring field theory, as developed initially in [4,5] for flat space and extended to the plane-wave string theory in [6,7,8,9]. When considering plane-waves with D-branes [10,11,12,13,14,15,16,17,18,19,20] or orientifolds [21,22,23], open strings are naturally present and their interactions need to be taken into account. This open-closed string theory is captured by seven basic interactions that can occur. In addition to the free closed and open string Hamiltonians, H_{cc} and H_{oo}, these open-closed interactions enter the Hamiltonian in the following manner

$$H = H_{cc}+H_{oo}+\sqrt{g_s}(H_{o\leftrightarrow oo}+H_{o\leftrightarrow c})+g_s(H_{c\leftrightarrow cc}+H_{o\leftrightarrow oc}+H_{oo\leftrightarrow oo}+H_{o\leftrightarrow o}+H_{c\leftrightarrow c})\,, \quad (1.1)$$

where $\sqrt{g_s}$ and g_s, denote the open and closed string coupling constants, respectively. The first of the $O(\sqrt{g_s})$ terms represents the cubic open string interaction, while the second represents the open to closed transition. The $H_{o\leftrightarrow oo}$ interaction was studied in [24,25] and the large-μ limit of the open-closed interaction $H_{o\leftrightarrow c}$ and its gauge theory implications was the focus of [26].

Each term in the Hamiltonian can be computed in two steps. Firstly, one imposes the geometrical continuity conditions on the coordinates and conjugate momenta, *i.e.* the kinematical constraints. Secondly, one imposes that the Hamiltonian satisfies the supersymmetry algebra. The first step of the calculation involves calculating the so-called Neumann matrices, which follow from solving the continuity conditions written in terms of the string modes. These Neumann matrices thus relate the various string modes and form a crucial ingredient in determining the correction to the Hamiltonian. The second step involves determining the so-called prefactor by imposing the supersymmetry algebra. The prefactor is a polynomial in creation operators, which implements the dynamical

constraints. Determining the prefactor depends on certain decomposition theorems, which require the knowledge of the Neumann matrices.

In the plane-wave background the explicit determination of these Neumann matrices is highly non-trivial, in particular due to their dependence on the background constant Ramond-Ramond five-form flux μ. As the large-μ limit of the plane-wave string theory is conjectured to correspond to the BMN sector of the $\mathcal{N} = 4$ SYM theory, the Neumann matrices in this limit have been of foremost interest and the only example so far of a solution known for all values of μ has been the cubic closed string vertex [6,27,28].

In this paper, we will explicitly construct the Neumann matrices for the open-closed transition vertex, which are valid for all μ. In particular, this allows to rigorously obtain both the large-μ asymptotics as well as to reproduce the correct flat space limit. The corresponding vertex in flat-space has been discussed in [4,29,30,31,32,33].

The open-closed vertex in the large-μ limit and its relation to the gauge theory was discussed in [26]. Our analysis will determine the solution to the vertex equations for all values of μ and then study the asymptotics for large μ. This stands in contrast to taking the large-μ approximation of the vertex equations before solving them, which is what has been proposed in [26]. The expressions obtained in this paper will yield, compared to the naive approximation, the same large-μ asymptotics for one of the matrices, but renormalized results for the other two. So, one has to treat such naive approximations with a grain of salt and has to carefully analyse whether they are mathematically justified, which generically they are not. This point shall be elaborated upon in due course.

In flat space the open-closed Neumann matrices [4] are constructed out of certain functions u_m, which are defined as

$$u_m = \frac{\Gamma(m+1/2)}{\sqrt{\pi}\Gamma(m+1)}\,, \tag{1.2}$$

where m represents the mode number. It will turn out that in the plane-wave background these functions are replaced by certain "μ-deformed" generalizations; this will in particular require the definition of two generalizations of the Gamma function, which we shall refer to as *μ-deformed Gamma functions*.

In deriving the Neumann matrices, we will use methods of complex analysis in order to rewrite certain infinite sums in terms of contour integrals on the complex plane, where the complex variable will represent the mode number that is being summed over. The pole and zero structure of the Neumann matrices will be motivated using this integral

representation of the sums. We illustrate this method in the flat space case. However, in the plane-wave case certain subtleties in the method arise, which are due to the presence of the mode numbers $\omega_n = \mathrm{sgn}(n)\sqrt{n^2+\mu^2}$ and the thereby resulting square root branch cuts. These points will be addressed in detail.

We shall determine the Neumann matrices for both Dirichlet and Neumann boundary conditions of the open string. In flat space, these matrices are related by T-duality. In the plane-wave background, statements about T-duality are more obscure. In our case it will turn out that the Neumann matrices for Dirichlet and Neumann boundary conditions differ by a μ-dependent factor, which goes to unity in the $\mu = 0$ limit. In this limit, our solutions are precisely equal to the flat space result. The implications of this need further investigation.

The paper is organized as follows. In section 2, we shall derive the continuity conditions to be imposed on the open-closed vertex and thereby derive the equations for the Neumann matrices. In section 3, we will illustrate a procedure using methods of contour integration to solve these constraints explicitly. Using this procedure we will re-derive the known flat space solutions [4]. We elaborate on the subtleties arising from branch cuts and branch point singularities in the plane-wave background and motivate the pole and zero structure for the Neumann matrices. In section 4, we will solve the equations for the Neumann matrices for all values of μ in the case of Neumann as well as Dirichlet boundary conditions. This involves the definition of new μ-deformed Gamma-functions. In sections 5 and 6, we will analyse the large-μ asymptotics of the solutions and their behaviour will be illustrated graphically. We conclude with discussions and open problems in section 6. Appendix A summarizes known identities relevant to the flat space analysis and an example using the contour method to sum a series is provided. In appendix B, several key properties and the asymptotics of the newly defined μ-deformed Gamma functions are discussed.

2. The open-closed vertex

In this section, we will set up the notation and conventions to be used in the rest of the paper. We will derive the continuity conditions to be imposed on the bosonic vertex and the resulting constraint on the Neumann matrices.

2.1. Neumann boundary conditions

We start with the calculation in the plane-wave of the open-closed vertex, with Neumann boundary conditions on the open string. The corresponding discussion for the flat space superstring can be found in [4]. The bosonic part of the world sheet action is

$$\int_0^{\pi|\alpha|} d^2\sigma(\partial X\cdot\partial X+\mu^2 X^2)\,, \tag{2.1}$$

where we have suppressed the spacetime vector index for convenience. The length of the world-sheet is parametrized by $|\alpha|=|2p^+\alpha'|$, which without loss of generality, we will take to be unity for the purpose of this paper[1]. The equations of motion read

$$(-\partial_\tau^2+\partial_\sigma^2-\mu^2)X=0\,. \tag{2.2}$$

The mode expansions for the closed string and open string with Neumann boundary conditions, which satisfy the above equations of motion are

$$X^I_{closed}(\sigma,\tau)=x^I_c\cos\mu\tau+p^I_c\frac{\sin\mu\tau}{\mu}+i\sum_{m\neq 0}\frac{1}{\omega_{2\,m}}\left(\alpha^I_m e^{-i(\omega_{2m}\tau+2m\sigma)}+\tilde\alpha^I_m e^{-i(\omega_{2m}\tau-2m\sigma)}\right)\,. \tag{2.4}$$

$$X^I_{open}(\sigma,\tau)=x^I_0\cos\mu\tau+p^I_o\frac{\sin\mu\tau}{\mu}+i\sum_{m\neq 0}\frac{1}{\omega_m}\beta^I_m e^{-i\omega_m\tau}\cos m\sigma\,, \tag{2.3}$$

which at $\tau=0$ become

$$\begin{aligned} X^I_{closed}(\sigma)&=x^I_c+i\sum_{m\neq 0}\frac{1}{\omega_{2\,m}}\left(\alpha^I_m e^{-2im\sigma}+\tilde\alpha^I_m e^{2im\sigma}\right)\\ X^I_{open}(\sigma)&=x^I_o+i\sum_{m=1}^{\infty}\frac{1}{\omega_m}(\beta^I_m-\beta^I_{-m})\cos m\sigma\,. \end{aligned} \tag{2.5}$$

We also define $P^I=\frac{\partial}{\partial\tau}X^I$ so that at $\tau=0$,

$$\begin{aligned} P^I_{closed}(\sigma)&=p^I_c+\sum_{m\neq 0}\left(\alpha^I_m e^{-2im\sigma}+\tilde\alpha^I_m e^{2im\sigma}\right)\\ P^I_{open}(\sigma)&=p^I_o+\sum_{m=1}^{\infty}(\beta^I_m+\beta^I_{-m})\cos m\sigma\,. \end{aligned} \tag{2.6}$$

[1] This is also the convention used in [26].

The non-trivial commutation relations are

$$\begin{aligned}
[\beta_m\,,\beta_n] &= \omega_m \delta_{m,-n} \\
[\alpha_m\,,\alpha_n] &= \frac{\omega_{2\,m}}{2}\delta_{m,-n} \\
[\tilde{\alpha}_m\,,\tilde{\alpha}_n] &= \frac{\omega_{2\,m}}{2}\delta_{m,-n}\,.
\end{aligned} \tag{2.7}$$

The conventions have been chosen such that

$$\omega_n = \mathrm{sgn}(n)\sqrt{n^2+\mu^2}\,, \tag{2.8}$$

where μ is the RR-field strength and we have parametrized the string world-sheet to be of length π. Note that the mode expansion and the commutation relations have the expected flat space limit and coincide with the expressions given in [4]. For the open-closed vertex, the following relation has to be satisfied by the fields at $\tau = 0$

$$X^I(\sigma)_{open} = X^I(\sigma)_{closed}\,, \qquad P^I(\sigma)_{open} + P^I(\sigma)_{closed} = 0\,, \tag{2.9}$$

which results in terms of the modes in

$$\frac{1}{\omega_{2\,n}}(\alpha_n - \tilde{\alpha}_{-n}) - \sum_{m=1}^{\infty}\frac{1}{\omega_m}c_{nm}(\beta_m - \beta_{-m}) = 0 \tag{2.10}$$

$$(\alpha_n + \tilde{\alpha}_{-n}) + \sum_{m=1}^{\infty} c_{nm}(\beta_m + \beta_{-m}) = 0\,. \tag{2.11}$$

These equations are understood as holding upon the vertex $|V\rangle$ and the constants c_{nm} are given by

$$c_{nm} = \frac{1}{\pi}\int_0^{\pi} d\sigma\, e^{2in\sigma}\cos(m\sigma) \;=\; \begin{cases} \dfrac{1}{2}(\delta_{m,-2\,n} + \delta_{m,2\,n}) & m \text{ even}, \\[2ex] \dfrac{4in}{\pi(4n^2 - m^2)} & m \text{ odd}\,. \end{cases} \tag{2.12}$$

The relations (2.10) and (2.11) are equivalent to the following conditions (derived by combining the n and $-n$ versions of the equations)

$$\alpha_n + \tilde{\alpha}_n + \beta_{-2\,n} = 0\,, \tag{2.13}$$

and

$$\alpha_n - \tilde{\alpha}_n - \frac{in}{\pi}\sum_{m=0}^{\infty}\frac{1}{(m+1/2)^2 - n^2}\left(\left(1 - \frac{\omega_{2\,n}}{\omega_{2\,m+1}}\right)\beta_{2\,m+1} + \left(1 + \frac{\omega_{2\,n}}{\omega_{2\,m+1}}\right)\beta_{-2\,m-1}\right) = 0\,. \tag{2.14}$$

These equations reduce precisely to the relations in [4] for $\mu = 0$[2]. (2.13) implies that the vertex $|V\rangle = \exp(\Delta)|\Omega\rangle$ has to have a decomposition $\Delta = \Delta_1 + \Delta_2$, where

$$\Delta_1 = -\sum_{m=1}^{\infty} \frac{\sqrt{2}}{\omega_{2\,m}} \beta_{-2\,m} \alpha^I_{-m} \,, \tag{2.15}$$

and where we defined $\sqrt{2}\alpha^{I/II} = \alpha \pm \tilde{\alpha}$, so that $\alpha^{II}_0 = 0$. In order to solve (2.14), we further make the ansatz

$$\Delta_2 = \sum_{m,n=0}^{\infty} A_{mn} \beta_{-2\,m-1} \alpha^{II}_{-n} + \frac{1}{2} B_{mn} \beta_{-2\,m-1} \beta_{-2\,n-1} + \frac{1}{2} C_{mn} \alpha^{II}_{-m} \alpha^{II}_{-n} \,. \tag{2.16}$$

The resulting generalizations of the equations (7.15)-(7.18) in [4][3] are

$$-\frac{2\sqrt{2}in}{\pi} \sum_{m=0}^{\infty} \frac{1}{\omega_{2\,m+1} - \omega_{2\,n}} A_{mk} = \delta_{n,k} \tag{2.17}$$

$$-\sum_{m=0}^{\infty} \frac{B_{mk}}{\omega_{2\,m+1} - \omega_{2\,n}} = \frac{1}{(\omega_{2\,n} + \omega_{2\,k+1})\omega_{2\,k+1}} \tag{2.18}$$

$$\frac{4\sqrt{2}in}{\pi\omega_{2\,n}} \sum_{m=0}^{\infty} \frac{1}{\omega_{2\,m+1} + \omega_{2\,n}} A_{mp} = C_{np} \tag{2.19}$$

$$\frac{4\sqrt{2}in}{\pi\omega_{2\,n}} \left(\sum_{m=0}^{\infty} \frac{1}{\omega_{2\,m+1} + \omega_{2\,n}} B_{mp} + \frac{1}{(\omega_{2p+1} - \omega_{2\,n})\omega_{2p+1}} \right) = A_{pn} \,. \tag{2.20}$$

The fact that the open string must join smoothly at its end points imposes the additional condition for A that

$$\sum_{m=0}^{\infty} A_{mk} = 0 \,, \tag{2.21}$$

and another condition for B

$$\sum_{m=0}^{\infty} B_{mk} = \frac{1}{\omega_{2\,k+1}} \,. \tag{2.22}$$

For $\mu \to 0$ these reduce to the correct flat space relations. When we consider the limit $\mu \to \infty$, naively from equation (2.17) it seems that A must be of order $O(1/\mu)$ and from equations (2.18), (2.19) that B, C must be $O(1/\mu^3)$. Considering equation (2.20) and

[2] Note that there is a slight convention mismatch in [4], in that the mode expansion in (2.11) in [4] together with the definition of the $c_{n\,m}$ do not give rise to (7.10,11). This can be remedied by choosing the opposite sign for σ in the exponentials in (2.11).

[3] Note that there is a factor of 2 missing on the LHS of equation (7.15) in [4].

neglecting the B term, we see that the large-μ asymptotics of A by this naive analysis is given by

$$A_{pn} = \frac{1}{\pi\mu}\frac{8in\sqrt{2}}{(2p+1)^2-(2n)^2} + O\left(\frac{1}{\mu^3}\right) . \tag{2.23}$$

As we will see in the section on the large-μ asymptotics, the naive approximations differ for B and C from the actual results.

2.2. Dirichlet boundary conditions

In the case of Dirichlet boundary conditions, the open string mode expansion is

$$X^I_{open}(\sigma,\tau) = (\text{zero modes}) + \sum_{m\neq 0}\frac{1}{\omega_m}\beta^I_m e^{-i\omega_m\tau}\sin m\sigma\,. \tag{2.24}$$

The presence of zero-modes is dependent on what type of D-brane is being considered. In particular, for class I branes, the zero modes vanish, however for class II and oblique branes, the zero modes are in fact σ-dependent [16,18,20,17][4]. For class I branes the vertex equations take the form

$$i\frac{1}{\omega_{2n}}(\alpha_n - \tilde{\alpha}_{-n}) - \sum_{m=1}^{\infty}\frac{1}{\omega_m}\check{c}_{nm}(\beta_m + \beta_{-m}) = 0 \tag{2.25}$$

$$i(\alpha_n + \tilde{\alpha}_{-n}) + \sum_{m=1}^{\infty}\check{c}_{nm}(\beta_m - \beta_{-m}) = 0\,, \tag{2.26}$$

where

$$\check{c}_{nm} = \frac{1}{\pi}\int_0^{\pi} d\sigma\, e^{2in\sigma}\sin(m\sigma) \;=\; \begin{cases} \dfrac{i}{2}(\delta_{m,2n} - \delta_{m,-2n}) & m \text{ even}, \\ -\dfrac{1}{\pi}\dfrac{2m}{(4n^2-m^2)} & m \text{ odd}\,, \end{cases} \tag{2.27}$$

which correspond to the Fourier modes of $\sin(m\sigma)$ that appear in the Dirichlet open string. The resulting vertex equations are then

$$\alpha_n - \tilde{\alpha}_n - \beta_{-2n} = 0 \tag{2.28}$$

$$\alpha_n + \tilde{\alpha}_n - i\frac{2}{\pi}\sum_{m=0}^{\infty}\frac{(2m+1)}{\omega_{2m+1}}\left(\frac{\beta_{2m+1}}{(\omega_{2n}+\omega_{2m+1})} + \frac{\beta_{-2m-1}}{(\omega_{2n}-\omega_{2m+1})}\right) = 0\,. \tag{2.29}$$

[4] In particular, for the class I $D7$-brane, considered in [26], there are no zero-modes for the open string.

In general, the zero modes for the Dirichlet open strings, for instance the D-instanton of [18,17], are σ-dependent. This leads to a subtlety in the continuity condition since the Fourier modes of these terms will be non-vanishing for the non-zero modes. One can take this into account by redefining α^I and α^{II} in the following manner

$$\left(\alpha_n^I\right)_{new} = \left(\alpha_n^I\right)_{old} + \frac{i}{2\sqrt{2}}\omega_{2n}(f_n + f_{-n}) \tag{2.30}$$

$$\left(\alpha_n^{II}\right)_{new} = \left(\alpha_n^{II}\right)_{old} + \frac{i}{2\sqrt{2}}\omega_{2n}(f_n - f_{-n})\,, \tag{2.31}$$

where

$$f_n = \frac{1}{\pi}\int_0^\pi d\sigma\,(\text{zero modes})\,e^{2in\sigma}\,. \tag{2.32}$$

With this redefinition, one would need to define the vacuum in terms of these new oscillators[5]. The analysis will now be the same as that for the case of vanishing zero modes.

The key point is again to solve for the part of the vertex involving odd open string modes. The ansatz for the vertex is

$$\check{\Delta}_2 = \sum_{m=0,n=1}^{\infty} \check{A}_{mn}\beta_{-2m-1}\alpha^I_{-n} + \frac{1}{2}\sum_{m,n=0}^{\infty}\check{B}_{mn}\beta_{-2m-1}\beta_{-2n-1} + \frac{1}{2}\sum_{m,n=1}^{\infty}\check{C}_{mn}\alpha^I_{-m}\alpha^I_{-n}\,, \tag{2.33}$$

for which (2.29) imposes the following conditions

$$i\frac{\sqrt{2}}{\pi}\sum_{m=0}^{\infty}\frac{(2m+1)}{\omega_{2m+1}-\omega_{2n}}\check{A}_{mk} = \delta_{n,k} \tag{2.34}$$

$$\sum_{m=0}^{\infty}\frac{(2m+1)}{\omega_{2m+1}-\omega_{2n}}\check{B}_{mk} = \frac{(2k+1)}{(\omega_{2n}+\omega_{2k+1})\,\omega_{2k+1}} \tag{2.35}$$

$$i\frac{2\sqrt{2}}{\pi\omega_{2n}}\sum_{m=0}^{\infty}\frac{(2m+1)}{\omega_{2m+1}+\omega_{2n}}\check{A}_{mk} = \check{C}_{nk} \tag{2.36}$$

$$i\frac{2\sqrt{2}}{\pi\omega_{2n}}\left(\sum_{m=0}^{\infty}\frac{(2m+1)}{\omega_{2m+1}+\omega_{2n}}\check{B}_{mk} - \frac{(2k+1)}{(\omega_{2k+1}-\omega_{2n})\omega_{2k+1}}\right) = \check{A}_{kn}\,. \tag{2.37}$$

$(X(0)-X(\pi))|V\rangle = 0$ is autmatic for the non-zero modes and so does not impose additional constraints. For $\mu \to 0$ these reproduce the flat space equations of [33]. Again, one can study a naive approximation of the solutions as $\mu \to \infty$, which yields, *e.g.*,

$$\check{A}_{kn} = -\frac{4i\sqrt{2}(2k+1)}{\pi((2k+1)^2-(2n)^2)\mu} + O\left(\frac{1}{\mu^3}\right)\,. \tag{2.38}$$

We shall determine the solutions and in particular this will show that they are closely related by a μ-dependent factor to the solutions in the Neumann case.

[5] Explicitly, if $(\alpha_n)_{new} = (\alpha_n)_{old} + c_n$, where c_n are c-numbers, then $|\Omega\rangle_{new} = \exp\left(-\sum_1^\infty 2\frac{c_n}{\omega_{2n}}(\alpha_{-n})_{old}\right)|\Omega\rangle_{old}$. Clearly then $(\alpha_n)_{new}|\Omega\rangle_{new} = 0$ for $n > 0$.

3. Summation technique for solving the vertex equations

3.1. Contour integration method

Let us consider the system of equations (2.17)-(2.22) in flat space, *i.e.* when $\mu = 0$. In order to try and solve these we employ the following technique. Let $f(z)$ be analytic except for possibly poles, which will be at positions z_k. Suppose $f(z)$ is zero, when z is a negative integer. Then by Cauchy's theorem we have

$$\sum_{n=0}^{\infty} f(n) + \sum_{k} Res_{z=z_k} \pi \cot(\pi z) f(z) = \lim_{R\to\infty} \oint_{C_R} \frac{dz}{2\pi i} \pi \cot(\pi z) f(z)\,, \tag{3.1}$$

where C_R is the contour given by a circle of radius R centred on the origin, which does not intersect any poles of the integrand (so in particular $R \neq 1, 2, 3...$). The contour is depicted in figure 1.

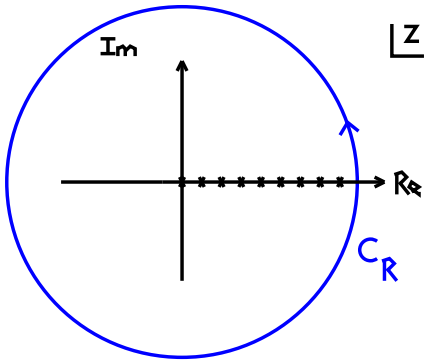

Figure 1 Contour C_R.

In order for our technique to be useful we require that the RHS of the above equation vanishes, which leaves us with an expression for $\sum_0^\infty f(n)$. This will impose a condition on the behaviour of $f(z)$ at infinity, which we will now deduce. Fortunately this turns out to be easy since $|\cot(\pi z)|$ is bounded on C_R as $R \to \infty$. Therefore it is sufficient to require that $|zf(z)| \to 0$ on C_R as $R \to \infty$. Thus the problem can be reformulated as: given $\sum_0^\infty f(n)$ find an $f(z)$ which has the correct asymptotic behaviour. This will then provide us with a solution for $f(n)$. At this stage it is not obvious under what conditions the solutions are unique and we shall return to this point below. It is important to note that if $f(z)$ were to have no poles, it would be analytic everywhere and thus due to the asymptotic property would be bounded; hence by Liouville's theorem a constant, which must equal zero. Thus we conclude that $f(z)$ must have poles. An explicit example of this method is provided in appendix A.2 for proving a well known identity.

Note that given the behaviour at infinity, the pole and zero structure (*i.e.* positions and multiplicities of all the poles and zeroes) may be sufficient to determine $f(z)$ uniquely. To do this we employ deeper results on holomorphic functions, namely, if $h(z)$ is analytic everywhere in $\mathbb{C}$, has no zeroes and is of *order* zero, then it is a constant. An analytic function is of order zero, if $\log M(R) = O(R^\epsilon)$ for $\epsilon > 0$, where $M(R)$ is the maximum of $|h(z)|$ on circles of radius R (centred on the origin). We may now apply this to our case. Suppose we have two possibilities for $f(z)$, call them $f_1(z)$ and $f_2(z)$, with the same pole and zero structure. Consider $f_1(z)/f_2(z)$. This is analytic everywhere with no zeroes. If $f_2(z) \sim z^{-\beta}$ where $\beta > 1$ then $f_1(z)/f_2(z) = O(z^k)$ where $k \geq \beta - 1$. Thus we see that for f_1/f_2 we have $\log M(R) = O(\log R)$ and hence it is of order zero. We deduce that $f_1/f_2 = A$, a constant. Techniques of complex analysis similar to these used above can be found, *e.g.*, in [34].

3.2. Flat space case

As a warm-up for the plane-wave case, we shall use the above contour method to explicitly construct the known solutions to the flat space open-closed vertex equations. These were obtained in [4] by using the identities given in appendix A. Firstly, consider the flat space limit $\mu = 0$ of the equations (2.17)-(2.22) for A and C

$$\sum_{m=0}^{\infty} A_{mn} = 0, \tag{3.2}$$

$$\sum_{m=0}^{\infty} \frac{A_{mk}}{2m+1-2n} = -\frac{\pi\delta_{nk}}{2\sqrt{2}in}\,, \tag{3.3}$$

$$\frac{i\sqrt{2}}{\pi} \sum_{m=0}^{\infty} \frac{A_{mp}}{m+1/2+n} = C_{np}\,. \tag{3.4}$$

In order to solve these equations, we first convert the sums into contour integrals using the method of the previous section, *i.e.*, by introducing the complex function $A(m,k)$ of $m \in \mathbb{C}$ such that it coincides with A_{mn}, when m is a non-negative integer. The idea will be to determine the poles and zeroes of A and reconstruct the function using this information. For example the contour integral for the LHS of (3.3) takes the form

$$\oint_C dm\, \cos(\pi m)\Gamma(-m)\Gamma(m+1)\frac{A(m,k)}{m+1/2-n}\,, \tag{3.5}$$

where C is the contour which encloses only the positive integers. We will send the contour to infinity assuming that A has the asymptotic behaviour, such that the integrand tends to

zero at infinity. From (3.2) we can deduce that $A(m,k)$ will have at most simple poles at half integer values of m (since they will cancel with the zeroes of $\cos(\pi m)$), if we assume no relative cancellations between residues of $A(m,k)$ [6]. Also we see that $A(m,k)$ must have zeroes at negative integers (again assuming no relative cancellations occur) since otherwise $\Gamma(m+1)$ would give non-zero contributions. Further, setting $n=k$ in (3.3) implies that there must be a simple pole at $m=k-1/2$. Thus $A(m,k)$ must be of the form

$$A(m,k)=\frac{f(m,k)}{m+1/2-k}\,, \tag{3.6}$$

where $f(m,k)$ is inversely proportional to $\Gamma(m+1)$. Now consider equation (3.4). If $f(m,k)$ had no further poles, then following the line of reasoning above, C_{np} would vanish, which is ruled out by physical considerations. Thus $f(m,k)$ must have further poles at $m=-n-1/2$, for each $n=0,1,\cdots$. Equation (3.2) immediately tells us that these should be simple poles. Thus $f(m,k)$ must be proportional to $\prod_{n=0}^{\infty}1/(m+n+1/2)$. The product is divergent as it stands, but can be made convergent by using the Weierstrass form of the Gamma function as in appendix A. Thus $f(m,k)$ is proportional to $\Gamma(m+1/2)/\Gamma(m+1)$, which in the notation of [4] is defined as $\sqrt{\pi}u_m$. As there cannot be further zeroes or poles in $A(m,k)$, we have determined $A(m,k)$ up to an unknown function in m, with neither zeroes nor poles. Using the theorem in the previous section we can show that if this function is holomorphic and has the required asymptotic behaviour for the contour method to work, then it must be a constant. The constant of proportionality can now be easily determined by considering equation (3.3) with $n=k$. The complete answer for A_{mk} is thus

$$A_{mk}=i\sqrt{2}\frac{u_m u_k}{2m+1-2k}\,. \tag{3.7}$$

We should emphasise that we have proven that the above expression for A_{mk} is the only meromorphic function in m with the given poles and zeroes, which solves (3.2) and (3.3) with the forementioned asymptotic property. Using this solution in equation (3.4), we can now easily derive the solution for C_{nk} using the contour integration technique. The non-trivial residues of relevance in the integration are at $m=-n-1/2$. The solution works out to be

$$C_{nk}=\frac{u_n u_k}{n+k}\,. \tag{3.8}$$

[6] In fact it seems that this must be the case, for A_{mk} must satisfy two separate sums and thus if the residues cancelled in one they would not in the other.

In a manner that is identical to deriving the solution for A_{mk} as illustrated above, it can be shown that the solution for B_{mk} is given by

$$B_{mk} = \frac{u_m u_k}{2m + 2n + 2} \,. \tag{3.9}$$

The so-derived solutions for A, B, C are precisely as given in the literature [4].

3.3. Subtleties in the plane-wave case

We now turn to the plane-wave open-closed vertex equations and their solutions. This section provides a discussion of various subtleties, which arise in generalizing the contour method to the plane-wave case. The reader interested only in the solution to the vertex equations may therefore turn directly to the next section, where the full solution will be presented and proven.

The main strategy in solving (2.17)-(2.22) will be to proceed as in the flat space case, *i.e.*, to analyze the structure of poles and zeroes. Again, we assume that there are no relative cancellations of residues. Consider for definiteness the equation for B_{mk}, (2.18). The relevant contour integral is

$$\oint dm \, \cot(\pi m) \frac{B(m,k)}{\omega_{2\,m+1} - \omega_{2\,n}} \,. \tag{3.10}$$

In order to satisfy (2.18) for all values of n, $B(m,k)$ cannot have poles in m at positive half-integers. This can be seen as follows. (2.22) implies that B has to have a pole at one negative integer value. The integral for (2.18) obtains precisely a residue at this integer as well (since the only extra factor in the integrand compared to (2.22) has a pole at $m = n - 1/2$) and the value of the residue is precisely the LHS of (2.18). Now, assume B had a pole for a positive half-integer, $p+1/2$. If the denominator term $1/(\omega_{2\,m+1} - \omega_{2\,n})$ has no other poles at $p + 1/2$, then due to the cot-factor the corresponding residue vanishes. However, since (2.18) has to hold for all n, for $p = n$, there would be an additional non-zero residue. So we conclude that B cannot have poles at positive half-integer values. However, B can have poles at negative half-integers, as these are again cancelled by the $\cot(\pi m)$ factor; in fact it must due to (2.20) . Further B has to have zeroes in m at negative integers (cancelling the poles of the $\cot(\pi m)$), except for $m = -k - 1$, where there has to be a non-trivial residue of the above integral, which gives rise to the RHS of (2.18). This yields the following ansatz

$$B(m,k) = \frac{g(m,k)}{(\omega_{2m+1} + \omega_{2k+1})} \,, \tag{3.11}$$

where $g(m,k)$ has poles at negative half-integers and zeroes at negative integers. From these constraints alone one may be led to choose $g(m,k) = u_m u_k$, however at this point the following subtlety, characteristic for the plane-wave case, presents itself.

The key problem arises through contributions to the contour integral from the square root branch cut of ω_{2m+1}. The branch points are located at $m_\pm = -\frac{1}{2} \pm i\frac{\mu}{2}$. Writing $\omega_{2m+1} = \sqrt{2m+1-i\mu}\sqrt{2m+1+i\mu}$, and choosing the cuts for the square root factors to extend from m_- to $i\infty$ and m_+ to $i\infty$, respectively, we obtain a branch line extending from m_- to m_+. This choice of cut is suitable, as it ensures that when restricted to the integers, $sgn(n)$ is automatically incorporated in ω_n, *i.e.*, to the right of the cut the phase of the square root is chosen to be $+1$ and to the left (in particular for all negative integers) it is -1. The important point to note now, is that in the contour method, one has to take the contributions from the integral around the cut (depicted blue in figure 2) into account. These come from the two line integrals along L_1 and L_2, as well as the integrals around the branch points, $K_{1,2}$. We have also set $\mu_\pm = \pm i\mu/2$.

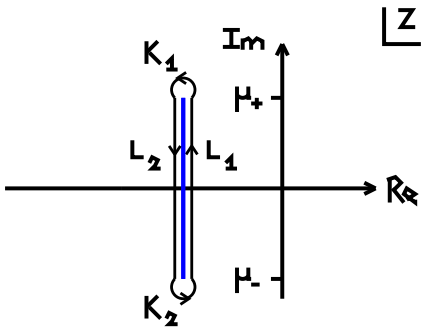

Figure 2 Contribution from the branch cut.

In crossing the branch cut, the square root picks up the following phases. Along L_1 the argument of the $\sqrt{2m+1+i\mu}$-factor is $\pi/4$ and of the $\sqrt{2m+1-i\mu}$ is $-\pi/4$, whereas along L_2 the arguments are $\pi/4$ and $3\pi/4$. Thus, by crossing from L_1 to L_2, ω_{2m+1} picks up a minus sign. In particular, a term

$$\int_{L_1} dm[...] \frac{1}{(\omega_{2m+1}+\omega_{2n})(\omega_{2m+1}+\omega_{2k})} ,$$

will go over into

$$\int_{L_2} dm[...] \frac{1}{(-\omega_{2m+1}+\omega_{2n})(-\omega_{2m+1}+\omega_{2k})} .$$

The most natural way to circumvent this problem is to demand that each of the line integrals above vanish individually. In order for this to happen, note that $\cot\pi\left(-\frac{1}{2}+i\frac{y}{2}\right)$

is an odd function in $y \in \mathbb{R}$, thus if in the remaining part of the integrand, the dependence on m led to an even function in $\mathrm{Im}(m)$ then the integral would vanish along each branch. This is possible if the μ and m dependence was packaged together in the form $\omega_{2\,m+1}$. This is natural in light of the plane-wave mode expansions. One can also anticipate that whenever an index couples to a closed string mode α_k, the k and μ dependence is packaged into ω_{2k}. With $m = -1/2 + iy/2$, we see that ω_{2m+1} becomes $\pm\sqrt{-y^2+\mu^2}$, which is even in y. In addition one has to ensure that the contour integrals around the two branch points, $m_\pm$, denoted by $K_{1,2}$ in figure 2, vanish as well. The solutions that we shall present in the next section will be shown to satisfy both these requirements.

4. Solution for the plane-wave open-closed vertex

4.1. Neumann boundary conditions

In order to solve the plane-wave vertex equations, the following generalized Gamma-functions will be of key importance. We define [7] the *μ-deformed Gamma-functions of the first and second kind* [8]

$$\Gamma^I_\mu(z) = e^{-\gamma\omega_{2z}/2}\left(\frac{1}{z}\right)\prod_{n=1}^{\infty}\left(\frac{\omega_{2n}}{\omega_{2\,z}+\omega_{2n}}\,e^{\omega_{2z}/2n}\right) \tag{4.1}$$

$$\Gamma^{II}_\mu(z) = e^{-\gamma(\omega_{2z-1}+1)/2}\left(\frac{2}{\omega_{2\,z-1}+\omega_1}\right)\prod_{n=1}^{\infty}\left(\frac{\omega_{2\,n}}{\omega_{2\,z-1}+\omega_{2\,n+1}}\,e^{(\omega_{2z-1}+1)/2n}\right), \tag{4.2}$$

which we shall abbreviate by Γ^I, Γ^{II}, if this does not cause any ambiguities. By comparison with the Weierstrass form of the standard Gamma function (see appendix B), these satisfy

$$\Gamma^I_{\mu=0}(z) = \Gamma(z)\,, \qquad \Gamma^{II}_{\mu=0}(z) = \Gamma(z)\,. \tag{4.3}$$

Various properties of these μ-deformed Gamma-functions are discussed in appendix B. These modified Gamma-functions satisfy generalized reflection identities

$$\Gamma^I_\mu(z)\Gamma^I_\mu(-z) = -\frac{\alpha}{z\sin(\pi z)} \tag{4.4}$$

$$\Gamma^{II}_\mu(1+z)\Gamma^{II}_\mu(-z) = -\frac{\alpha}{\sin(\pi z)}\,, \tag{4.5}$$

[7] In version 1 and 2 of the preprint, the "zero-mode" part of Γ^I in the case of Neumann boundary conditions was erroneous, as it would have resulted in the non-vanishing contributions from the branch cuts.

[8] To the best of our knowledge, these Gamma functions are not related to the q-deformed Gamma functions and have not been previously investigated in the literature.

where $\alpha = 2\sinh(\pi\mu/2)/\mu$ and $\alpha \to \pi$ as $\mu \to 0$.

Further we define the generalizations of the functions

$$u_m = \frac{\Gamma(m+1/2)}{\sqrt{\pi}\,\Gamma(m+1)}\,, \tag{4.6}$$

which appear in the flat space solutions of the vertex. Let

$$v^I_m = \frac{(2m+1)}{\omega_{2\,m+1}}\;\frac{\Gamma^I(m+1/2)}{\sqrt{\pi}\;\Gamma^{II}(m+1)} \tag{4.7}$$

$$v^{II}_m = \frac{2}{\omega_{2\,m}}\;\frac{\Gamma^{II}(m+1/2)}{\sqrt{\pi}\;\Gamma^{I}(m)}\,, \tag{4.8}$$

which both reduce to u_m in the flat space limit $\mu \to 0$. Note that v^I_z has branch points at $-1/2 \pm i\mu/2$, whereas v^{II}_z has branch points at $\pm i\mu/2$. Invoking the reflection formulae for the modified Gamma functions, we compute

$$Res_{\,m=-n-1/2}\; v^I_m = v^{II}_n/\pi\,. \tag{4.9}$$

Note that the μ-dependent constant α cancels.

First, we summarize the solutions to the vertex equations with Neumann boundary conditions, (2.17)-(2.22), and then provide the proofs thereof. The coefficients for the open-closed vertex are given by

$$A_{mk} = i\sqrt{2}\frac{v^I_m v^{II}_k}{(\omega_{2\,m+1} - \omega_{2\,k})} \tag{4.10}$$

$$B_{mk} = \frac{v^I_m v^I_k}{(\omega_{2\,m+1} + \omega_{2\,k+1})} \tag{4.11}$$

$$C_{mk} = 2\frac{v^{II}_m v^{II}_k}{(\omega_{2\,m} + \omega_{2\,k})}\,. \tag{4.12}$$

These solutions can be motivated by noting that the new functions v^I and v^{II} have the same pole and zero structure as the u functions in flat space, to which they further reduce in the $\mu = 0$ limit. Moreover, according to the observation in the previous section, there are no contributions from the line integrals around the branch cut. This relies on the fact that $v^I_{-1/2+iy}$ is an even function in y. We will also demonstrate that the branch point singularities in v^I and v^{II} will not affect the calculation.

In order to use the contour method discussed in section 3.1, it is also crucial to show that A, B, C have the correct asymptotics. Using the results derived in appendix B.3, we

see that A, B, C are all $O(1/z^{3/2})$ which is the same as in flat space. Thus we are indeed justified to use the contour method. Note also, that by comparison with [4] these solutions have the correct flat space behaviour.

To prove the assertions, we first consider B_{mk}. Using the summation technique described previously, we easily see that,

$$\sum_{m=0}^{\infty} \frac{B_{mk}}{\omega_{2\,m+1} - \omega_{2\,n}} = -Res_{m=-k-1} \frac{\pi \cot(\pi m) B_{mk}}{\omega_{2\,m+1} - \omega_{2\,n}}. \tag{4.13}$$

A direct computation yields

$$\begin{aligned} & Res_{m=-k-1} \frac{\pi \cot(\pi m) B_{mk}}{\omega_{2\,m+1} - \omega_{2\,n}} \\ & = -\frac{\pi v_k^I}{\omega_{2\,k+1} + \omega_{2\,n}} \lim_{m \to -k-1} \left(\cot(\pi m) v_m^I\right) \; Res_{m=-k-1} \left(\frac{1}{\omega_{2\,k+1} + \omega_{2\,m+1}}\right) \\ & = \frac{1}{(\omega_{2\,k+1} + \omega_{2\,n})\omega_{2\,k+1}}, \end{aligned} \tag{4.14}$$

where the last equality follows after using the reflection identities for both types of Gamma function, one of which is required to evaluate the limit $\lim_{m \to -k-1}[\cot(\pi m) v_m^I]$.

Using the solution for B_{mk} as well as (2.20), the solution for A_{mk} can now be determined. First, note that

$$\sum_{m=0}^{\infty} \frac{B_{mk}}{\omega_{2\,m+1} + \omega_{2\,n}} = -Res_{m=-k-1} \frac{\pi \cot(\pi m) B_{mk}}{\omega_{2\,m+1} + \omega_{2\,n}} - Res_{m=-n-1/2} \frac{\pi \cot(\pi m) B_{mk}}{\omega_{2\,m+1} + \omega_{2\,n}}\,. \tag{4.15}$$

We see that the first term on the RHS is essentially identical to the previous calculation and thus we have

$$Res_{m=-k-1} \frac{\pi \cot(\pi m) B_{mk}}{\omega_{2\,m+1} + \omega_{2\,n}} = \frac{1}{(\omega_{2\,k+1} - \omega_{2\,n})\omega_{2\,k+1}}\,. \tag{4.16}$$

The second term can be computed as follows

$$\begin{aligned} Res_{m=-n-1/2} \frac{\pi \cot(\pi m) B_{mk}}{\omega_{2\,m+1} + \omega_{2\,n}} &= \frac{v_k^I \pi}{\omega_{2\,k+1} - \omega_{2\,n}} \, Res_{m=-n-1/2} \frac{\cot(\pi m) v_m^I}{\omega_{2\,m+1} + \omega_{2\,n}} \\ &= \left(\frac{v_k^I \, \pi}{\omega_{2\,k+1} - \omega_{2\,n}}\right) \left(\frac{-\pi \omega_{2\,n}}{4n}\right) \; Res_{m=-n-1/2} \, v_m^I \\ &= \left(\frac{v_k^I \, v_n^{II}}{\omega_{2\,k+1} - \omega_{2\,n}}\right) \left(\frac{-\pi \omega_{2\,n}}{4n}\right)\,. \end{aligned} \tag{4.17}$$

Thus, defining

$$A_{mk} = \sqrt{2}i \left(\frac{v^I_m \, v^{II}_k}{\omega_{2\,m+1} - \omega_{2\,k}} \right) , \tag{4.18}$$

the equation (2.20) is satisfied. Now we check the remaining equations satisfied by A_{mk}. Since, $\cot(\pi m) A_{mk}$ only has poles at $m = 0, 1, 2, \cdots$, we see immediately that

$$\sum_{m=0}^{\infty} A_{mk} = 0. \tag{4.19}$$

To check the other equation we note that,

$$\sum_{m=0}^{\infty} \frac{A_{mk}}{\omega_{2\,m+1} - \omega_{2\,n}} = -\delta_{nk} \, Res_{m=n-1/2} \frac{\pi \cot(m\pi) A_{mn}}{\omega_{2\,m+1} - \omega_{2\,n}}. \tag{4.20}$$

The residue is computed to be

$$\begin{aligned} &Res_{m=n-1/2} \frac{\pi \cot(m\pi) A_{mn}}{\omega_{2\,m+1} - \omega_{2\,n}} \\ &= i\sqrt{2}\pi v^{II}_n v^I_{n-1/2} \lim_{m \to n-1/2} \frac{\cot(\pi m)}{\omega_{2\,m+1} - \omega_{2\,n}} \, Res_{m=n-1/2} \frac{1}{\omega_{2\,m+1} - \omega_{2\,n}} = -\frac{i\pi}{2\sqrt{2}n} , \end{aligned} \tag{4.21}$$

and therefore we have verified (2.17).

Finally we may compute C_{mk} using (2.19). We are led to

$$\begin{aligned} Res_{m=-n-1/2} \frac{A_{mk} \pi \cot(\pi m)}{\omega_{2\,m+1} + \omega_{2\,n}} &= \frac{i\sqrt{2} v^{II}_k \, \pi}{\omega_{2\,n} + \omega_{2\,k}} \, Res_{m=-n-1/2} \frac{v^I_m \cot(m\pi)}{\omega_{2\,m+1} + \omega_{2\,n}} \\ &= \left(\frac{i\sqrt{2} v^{II}_k v^{II}_n}{\omega_{2\,n} + \omega_{2\,k}} \right) \frac{\pi \omega_{2\,n}}{4n} . \end{aligned} \tag{4.22}$$

Thus,

$$C_{mk} = 2 \left(\frac{v^{II}_k v^{II}_m}{\omega_{2\,m} + \omega_{2\,k}} \right) , \tag{4.23}$$

will ensure (2.19) as well as having the correct flat space limit.

As pointed out in section 3.2, there can be contributions from the integrals around the branch points $m_\pm$, which were depicted as K_1 and K_2 in figure 2. Thus, we need to show that these integrals do in fact vanish in order for the contour argument to hold. Consider A_{mk} for example. Let $m = -1/2 + i\mu/2 + \epsilon e^{i\theta}/2$; this implies that $\omega_{2\,m+1} = O(\epsilon^{1/2})$ and thus $A_{mk} = O(\epsilon^{-1/2})$. Therefore,

$$\oint_{K_1, K_2} dm \, A_{mk} = O(\epsilon^{1/2}) , \tag{4.24}$$

where the extra factor of ϵ comes from the integration measure of course. Hence as $\epsilon \to 0$ we see that the integral around the branch point does indeed vanish as required. Proofs for the other sum with A_{mk} and the ones with B_{mk} are entirely analogous.

This completes the proof, that (4.10)-(4.12) solve the plane-wave open-closed vertex equations (2.17)-(2.22).

4.2. Dirichlet boundary conditions

Using the above line of thought one can also determine the solution for the vertex with Dirichlet boundary conditions on the open string. To this effect we introduce the μ-deformed Gamma-function

$$\check{\Gamma}^I_\mu(z) = e^{-\gamma\omega_{2z}/2}\left(\frac{2}{\omega_{2\,z}+\mu}\right)\prod_{n=1}^{\infty}\left(\frac{\omega_{2\,n}}{\omega_{2\,z}+\omega_{2\,n}}\,e^{\omega_{2z}/2\,n}\right)\,, \tag{4.25}$$

which differs from $\Gamma^I_\mu(z)$ only in the "zero-mode" part. This Gamma-function has again the key property that it satisfies a reflection identity

$$\check{\Gamma}^I_\mu(z)\check{\Gamma}^I_\mu(-z) = -\frac{\alpha}{z\sin(\pi z)}\,, \tag{4.26}$$

which is proven in the same manner as provided in Appendix B for $\Gamma^I_\mu(z)$. Further we introduce generalizations of the functions u_m

$$\check{v}^I_m = \frac{(2m+1)}{\omega_{2\,m+1}}\,\frac{\check{\Gamma}^I(m+1/2)}{\sqrt{\pi}\,\Gamma^{II}(m+1)}\,,\qquad \check{v}^{II}_m = \frac{2}{\omega_{2\,m}}\,\frac{\Gamma^{II}(m+1/2)}{\sqrt{\pi}\,\check{\Gamma}^I(m)}\,. \tag{4.27}$$

Note that $\check{v}^I_{-1/2+iy}$ is an odd function of y, which in the Dirichlet case is needed for the integrals along the cuts to vanish. The functions $\check{v}^{I,II}$ are related to $v^{I,II}$ by a μ-dependent factor, which tends to 1 as $\mu\to 0$. The coefficients for the open-closed vertex in this case are

$$\check{A}_{m\,k} = -i\sqrt{2}\frac{\check{v}^I_m\check{v}^{II}_k}{(\omega_{2\,m+1}-\omega_{2\,k})} \tag{4.28}$$

$$\check{B}_{m\,k} = \frac{\check{v}^I_m\check{v}^I_k}{(\omega_{2\,m+1}+\omega_{2\,k+1})} \tag{4.29}$$

$$\check{C}_{m\,k} = 2\frac{\check{v}^{II}_m\check{v}^{II}_k}{(\omega_{2\,m}+\omega_{2\,k})}\,. \tag{4.30}$$

These can be verified in the same manner as in the Neumann case. Consider *e.g.* the equation for $\check{A}$, (2.34). The contour method implies

$$\sum_{m=0}^{\infty}\check{A}_{m\,k}\frac{(2m+1)}{\omega_{2\,m+1}-\omega_{2\,n}} = -\delta_{n\,k}\;Res_{m=n-1/2}\frac{\pi\cot(m\pi)(2m+1)\,\check{A}_{m\,n}}{\omega_{2\,m+1}-\omega_{2\,n}}\,, \tag{4.31}$$

where the residue is evaluated as

$$\begin{aligned} &Res_{m=n-1/2}(2m+1)\frac{\pi\cot(m\pi)\check{A}_{m\,n}}{\omega_{2\,m+1}-\omega_{2\,n}} \\ &= -i2\sqrt{2}n\,\pi\,\check{v}^{II}_n\check{v}^I_{n-1/2}\lim_{m\to n-1/2}\frac{\cot(\pi m)}{\omega_{2\,m+1}-\omega_{2\,n}}\;Res_{m=n-1/2}\frac{1}{\omega_{2\,m+1}-\omega_{2\,n}} = i\frac{\pi}{\sqrt{2}}\,, \end{aligned} \tag{4.32}$$

and therefore we have verified (2.34). The other equations can be proven to hold in a similar manner.

5. Large-μ asymptotics

5.1. Neumann boundary conditions

In this section we will analyse the large-μ asymptotics of the solutions that we have determined. We shall consider $\mu > 0$. One can find the bulk of the details in Appendix B, where in particular we give the asymptotics of v_m^I and v_m^{II}. Consider equation (4.10) now

$$A_{mk} = i\sqrt{2}\frac{v_m^I v_k^{II}}{(\omega_{2\,m+1} - \omega_{2\,k})} \,. \tag{5.1}$$

We find using the asymptotic formulas that

$$A_{mk} = \frac{i\sqrt{2}}{\pi}\frac{4k(\omega_{2\,m+1}+\omega_1)}{\omega_{2\,m+1}\omega_{2\,k}(\omega_{2m+1}-\omega_{2\,k})(\omega_{2\,k}+\omega_1)}\left(\frac{\omega_{2\,k}+\mu}{\omega_{2\,m+1}+\mu}\right)^{1/2} + O\left(e^{-\mu}\right) \,. \tag{5.2}$$

From the above we see that the leading asymptotic behaviour of A_{mk} is

$$A_{mk} \sim \frac{1}{\pi\mu}\frac{4i\sqrt{2}(2k)}{(2m+1)^2-(2k)^2} \,, \tag{5.3}$$

which agrees with the naive approximation in (2.23). For completeness we also give

$$\begin{aligned} B_{mk} &= \frac{e^{\gamma+2\,a_1}(\omega_{2\,m+1}+\omega_1)(\omega_{2\,k+1}+\omega_1)}{\pi\omega_{2\,m+1}\omega_{2\,k+1}(\omega_{2\,m+1}+\omega_{2\,k+1})(\omega_{2\,m+1}+\mu)^{1/2}(\omega_{2\,k+1}+\mu)^{1/2}} + O\left(e^{-\mu}\right) \,, \\ C_{mk} &= 2\frac{e^{-\gamma-2\,a_1}(16mk)(\omega_{2\,m}+\mu)^{1/2}(\omega_{2\,k}+\mu)^{1/2}}{\pi\omega_{2\,m}\omega_{2\,k}(\omega_{2\,m}+\omega_{2\,k})(\omega_{2\,m}+\omega_1)(\omega_{2\,k}+\omega_1)} + O\left(e^{-\mu}\right) \,. \end{aligned} \tag{5.4}$$

We deduce that

$$B_{mk} \sim \frac{e^{\gamma+2\,a_1}}{\pi\mu^2} \,, \tag{5.5}$$

as well as

$$C_{mk} \sim \frac{8mke^{-\gamma-2\,a_1}}{\pi\mu^4} \,. \tag{5.6}$$

For the reader's convenience we give A_{mk} to $O(1/\mu^5)$ more explicitly,

$$A_{mk} = \frac{8i\sqrt{2}k}{\pi(-(2k)^2+(2m+1)^2)}\left(\frac{1}{\mu} - \frac{(12k^2+(2m+1)^2)}{8\mu^3}\right) + O\left(\frac{1}{\mu^5}\right) \,. \tag{5.7}$$

As we have emphasised the naive approximation leads one to conclude that B, C behave as $O(1/\mu^3)$, whereas the analysis above disagrees with this. One can think of this discrepancy as arising from a renormalisation due to the higher modes [8]. We have thus determined the Neumann matrices A, B, C up to $O(e^{-\mu})$. One should note the similarity of these expression to the ones for the large μ Neumann matrices in [27].

5.2. Dirichlet boundary conditions

Similarly, one obtains the asymptotics of the Neumann coefficients in the case of Dirichlet boundary conditions. They are given by

$$\begin{aligned}
\check{A}_{mk} &= -\frac{i2\sqrt{2}}{\pi}\frac{(2m+1)(\omega_{2\,m+1}+\omega_1)(\omega_{2\,k}+\mu)^{3/2}}{\omega_{2\,m+1}\omega_{2\,k}(\omega_{2\,k}+\omega_1)(\omega_{2\,m+1}+\mu)^{3/2}(\omega_{2\,m+1}-\omega_{2\,k})} + O\left(e^{-\mu}\right) \\
\check{B}_{mk} &= \frac{e^{\gamma+2a_1}(2m+1)(2k+1)(\omega_{2\,m+1}+\omega_1)(\omega_{2\,k}+\omega_1)}{\pi\omega_{2\,m+1}\omega_{2\,k+1}(\omega_{2\,m+1}+\mu)^{3/2}(\omega_{2\,k+1}+\mu)^{3/2}(\omega_{2\,m+1}+\omega_{2\,k+1})} + O\left(e^{-\mu}\right)\,, \\
\check{C}_{mk} &= \frac{8e^{-\gamma-2a_1}(\omega_{2\,m}+\mu)^{3/2}(\omega_{2\,k}+\mu)^{3/2}}{\pi\omega_{2\,m}\omega_{2\,k}(\omega_{2\,m}+\omega_1)(\omega_{2\,k}+\omega_1)(\omega_{2\,m}+\omega_{2\,k})} + O\left(e^{-\mu}\right)\,.
\end{aligned} \tag{5.8}$$

In particular, the first terms in the asymptotic expansion are thus

$$\check{B}_{mk} \sim \frac{e^{\gamma+2a_1}(2m+1)(2k+1)}{4\pi\mu^4}\,, \tag{5.9}$$

as well as

$$\check{C}_{mk} \sim \frac{8e^{-\gamma-2a_1}}{\pi\mu^2}\,, \tag{5.10}$$

and $\check{A}_{mk}$ to $O(1/\mu^5)$ is given by

$$\check{A}_{mk} = -\frac{4i\sqrt{2}(2m+1)}{\pi((2m+1)^2-(2k)^2)}\left(\frac{1}{\mu}-\frac{3(2m+1)^2+(2k)^2}{8\mu^3}+O\left(\frac{1}{\mu^5}\right)\right)\,. \tag{5.11}$$

The first order term in this expansion is again in agreement with the gauge theory analysis in [26]. However, the asymptotics for $\check{B}$ and $\check{C}$ differ again from the naive approximation.

6. Discussion

In this paper, we constructed the solutions for the Neumann matrices of the open-closed vertex in the plane-wave light-cone string field theory for all values of μ. Complex analytic methods were invoked in order to derive these solutions and along the way, we were led to define a set of new, μ-deformed Gamma functions. In summary, the exact bosonic vertex for Neumann boundary conditions was shown to be

$$|V\rangle = \exp(\Delta_1+\Delta_2)|\Omega\rangle\,, \tag{6.1}$$

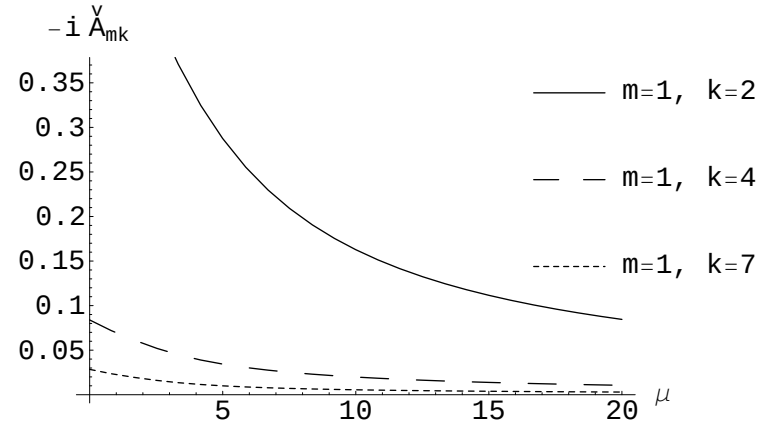

Figure 3 Graph of $\check{A}_{mk}(\mu)$.

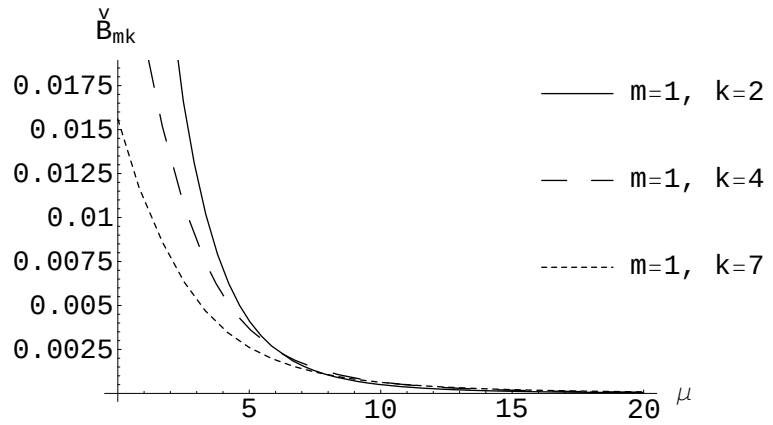

Figure 4 Graph of $\check{B}_{mk}(\mu)$.

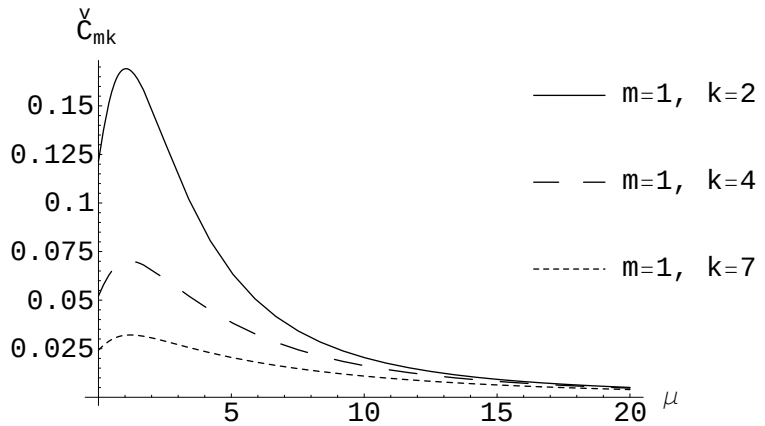

Figure 5 Graph of $\check{C}_{mk}(\mu)$.

where

$$\Delta_1 = -\sum_{m=1}^{\infty} \frac{\sqrt{2}}{\omega_{2\,m}} \beta_{-2\,m} \alpha^{I}_{-m} \,, \tag{6.2}$$

and

$$\Delta_2 = \sum_{m,n} A_{mn} \beta_{-2\,m-1} \alpha^{II}_{-n} + \frac{1}{2} B_{mn} \beta_{-2\,m-1} \beta_{-2\,n-1} + \frac{1}{2} C_{mn} \alpha^{II}_{-m} \alpha^{II}_{-n} \,. \tag{6.3}$$

The Neumann matrices A, B, C were determined to be

$$\begin{aligned} A_{mk} &= i\sqrt{2} \frac{v^I_m v^{II}_k}{(\omega_{2\,m+1} - \omega_{2\,k})} \\ B_{mk} &= \frac{v^I_m v^I_k}{(\omega_{2\,m+1} + \omega_{2\,k+1})} \\ C_{mk} &= 2 \frac{v^{II}_m v^{II}_k}{(\omega_{2\,m} + \omega_{2\,k})} \,, \end{aligned} \tag{6.4}$$

where v^I_m and v^{II}_m are the μ-deformed generalizations of the corresponding functions in flat space, u_m. A similar expression for the Dirichlet case was obtained. In contrast to flat space, the Neumann and Dirichlet solutions differ. It would be interesting to understand this from the point of view of T-duality in plane-wave backgrounds [35,36].

Figures 3, 4 and 5 show the behaviour of the Neumann coefficients for Dirichlet boundary conditions, $\check{A}_{mk}, \check{B}_{mk}$ and $\check{C}_{mk}$, for fixed m, k as a function of μ. It is clear from the behaviour of $\check{C}_{mk}$ that there appears to be a maximum for a particular value of μ, the physical significance of which still needs to be elucidated. It is also clear from the asymptotic behaviour of the graphs that $\check{A}$ falls off the slowest followed by $\check{C}$ and then $\check{B}$. This is consistent with the large-μ asymptotics analysed in the previous section.

In the light of the BMN correspondence, we discussed the large-μ asymptotics of the Neumann matrices and found the results for A and $\check{A}$ to agree in both the Dirichlet and Neumann case with [26], where the corresponding result was computed in the gauge theory. Note however that our results for B and C in both Neumann and Dirichlet case differ from the naive approximation. It would be very interesting to check the next to leading order corrections on the gauge theory side to confirm the accuracy of our results.

Using the exact Neumann matrices for the bosonic part of the vertex, it should now be straightforward to determine the prefactor exactly and extend our results to the complete superstring vertex. It would be interesting to study the μ-dependence of the scattering amplitudes that can be exactly computed.

Finally, in view of the analytic methods applied in this paper, it may be possible to simplify the derivation of the exact Neumann coefficients for the cubic vertex [27].

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Appendix A. Collection of useful formulae

In this appendix we collect some useful formulae (*cf.* [37], [4]). Stirling approximation

$$\Gamma(z) \quad \overset{z\to\infty}{\sim} \quad \sqrt{2\pi}\, z^{z-1/2}\, e^{-z}\,, \qquad |\arg z| < \pi\,. \tag{A.1}$$

The reflection identity is given by

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}\,. \tag{A.2}$$

Some useful identities relevant to the flat space section are collected here. Defining

$$u_n = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)}\,, \tag{A.3}$$

it can be shown that the u_n's satisfy the following identity which can be readily verified using the contour integration technique explained in the paper.

$$\sum_{m=0}^{\infty} \frac{u_m}{m+a+1} = \frac{\sqrt{\pi}\Gamma(a+1)}{\Gamma(a+\frac{3}{2})}\,. \tag{A.4}$$

Using this it can be shown that

$$\begin{aligned}
\sum_{m=0}^{\infty} \frac{u_m}{m-n+\frac{1}{2}} &= 0, \quad n = 1,2,\cdots \\
\sum_{m=0}^{\infty} \frac{u_m u_n}{m+n+1} &= \frac{1}{n+\frac{1}{2}}, \quad n = 0,1,2,\cdots \\
\sum_{m=0}^{\infty} \frac{u_m}{m+n+\frac{1}{2}} &= \pi u_n, \quad n = 0,1,2,\cdots \\
\sum_{m=0}^{\infty} \frac{u_m u_p}{(m-n+\frac{1}{2})(m-p+\frac{1}{2})} &= \frac{\pi}{n}\delta_{np}, \quad n,p = 1,2,\cdots\,.
\end{aligned} \tag{A.5}$$

A.1. Sample application of the contour method

As a warm-up and to illustrate the contour method of section 3, we shall provide a proof of (D.3) in [4]

$$\sum_{m=0}^{\infty} \frac{u_m}{m-n+1/2} = 0\,, \tag{A.6}$$

using the above method. Consider thus the following integral

$$\oint_{C_1} dq\, \cos(\pi q)\, \Gamma(-q)\Gamma(q+1/2)\, \frac{1}{q-n+1/2}\,, \tag{A.7}$$

which has residues at $q \in \mathbb{N}$ and $q = n - 1/2$. The contour is depicted in figure 6.

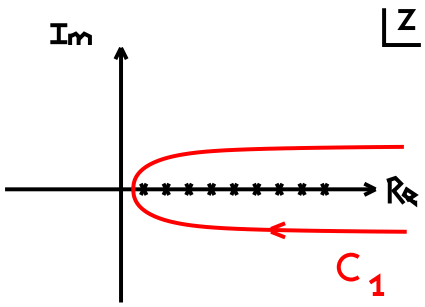

Figure 6 Contour C_1.

The first type of residues, *i.e.*, $q \in \mathbb{N}$, will reproduce the sum in (A.6). The latter residue vanishes for all $n \in \mathbb{N}$. Thus we are left with the evaluation of the contour integral in order to show (A.6). Using Stirling's formula, the integrand behaves like $q^{-3/2}$ for large modulus of q. Thus, the contour can be deformed to infinity, *i.e.*, $\mathcal{C}_R$ and this integral can be shown to vanish. This completes the proof.

Appendix B. μ-deformed Gamma functions

B.1. Definitions and identities

Here we define two functions, each of which reduce to the Gamma function as $\mu \to 0$. First recall that the standard Gamma function may be defined by its Weierstrass product

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}\,, \tag{B.1}$$

where γ is the Euler constant. Now let us define the following μ-deformed Gamma-function

$$\frac{1}{\Gamma^I_\mu(z)} = z e^{\gamma\omega_{2z}/2} \prod_{n=1}^{\infty} \left(\frac{\omega_{2\,n} + \omega_{2\,z}}{\omega_{2\,n}}\right) e^{-\omega_{2z}/2n} \,. \tag{B.2}$$

We will define this to be the *μ-deformed Gamma function of the first kind*. Note that $\omega_z = \sqrt{z^2+\mu^2}$, where we choose the finite branch cut for the square root so that $\omega_{-z} = -\omega_z$. This implies $\omega_n = \mathrm{sgn}(n)\sqrt{n^2+\mu^2}$, for $n \in \mathbb{Z}$. In order to show that $\Gamma^I_\mu(z)$ is indeed well-defined, one can use similar arguments as those for $\Gamma(z)$ (see [38] p. 235-236). For completeness we give an argument. Observe that the factors in the infinite product go as $1 + O(1/n^2)$ for sufficiently large n; this implies the infinite product converges absolutely and uniformly [9]. Notice that $\Gamma^I_\mu(z)$ has simple poles at $z = -1, -2, \cdots$, and branch points at $z = \pm i\mu/2$ and a branch cut on $[i\mu/2, -i\mu/2]$.

Next we will derive a generalisation of the reflection identity (A.2). Note that

$$\begin{aligned}
&\frac{1}{\Gamma^I_\mu(z)\Gamma^I_\mu(-z)} \\
&= z e^{\gamma\omega_{2z}/2} \prod_{n=1}^{\infty} \left(1 + \frac{\omega_{2\,z}}{\omega_{2\,n}}\right) e^{-\omega_{2z}/2n} (-z)\, e^{-\gamma\omega_{2z}/2} \prod_{n=1}^{\infty} \left(1 - \frac{\omega_{2\,z}}{\omega_{2\,n}}\right) e^{\omega_{2z}/2n} \\
&= -z^2 \prod_{n=1}^{\infty} \left(\frac{(2n)^2 - (2z)^2}{(2n)^2 + \mu^2}\right) = -z\sin(\pi z)\frac{\mu}{2\sinh(\pi\mu/2)} \,,
\end{aligned} \tag{B.3}$$

where the nice formula $\sin(\pi z) = \pi z \prod_{n=1}^{\infty}(1 - \frac{z^2}{n^2})$ has been used. Thus the reflection identity takes the form

$$\Gamma^I_\mu(z)\Gamma^I_\mu(-z) = -\frac{\alpha}{z\sin(\pi z)} \,, \tag{B.4}$$

where we have defined $\alpha = 2\sinh(\pi\mu/2)/\mu$. It is easy to see that it reduces to the usual identity when $\mu \to 0$. One can use this identity to calculate the residues of this new function. For $n \neq 0$

$$\begin{aligned}
Res_{z=-n}\Gamma^I_\mu(z) &= \lim_{z\to -n} (z+n)\Gamma^I_\mu(z) \\
&= \lim_{z\to -n} \frac{\alpha(z+n)}{-z\sin(\pi z)\Gamma^I_\mu(-z)} \\
&= \frac{\alpha}{\pi}\frac{(-1)^n}{n\Gamma^I_\mu(n)} \,.
\end{aligned} \tag{B.5}$$

[9] Recall that $\prod_n (1+a_n)$ converges absolutely iff $\sum_n a_n$ converges absolutely.

Finally, one more crucial property of this function needs to be mentioned. $\Gamma^I_\mu(iy)$ is an odd function of y, on both sides of the branch cut, albeit not the same function on either side!

For the solution of the plane-wave open-closed vertex, we shall need an additional kind of μ-deformed Gamma-function. Define the *μ-deformed Gamma function of the second kind* as follows

$$\frac{1}{\Gamma^{II}_\mu(z)} = \left(\frac{\omega_{2\,z-1}+\omega_1}{2}\right) e^{\gamma(\omega_{2z-1}+1)/2} \prod_{n=1}^{\infty} \left(\frac{\omega_{2\,z-1}+\omega_{2\,n+1}}{\omega_{2\,n}}\right) e^{-(\omega_{2z-1}+1)/2n}\,, \tag{B.6}$$

which obviously satisfies $\Gamma^{II}_{\mu=0}(z) = \Gamma(z)$. Note that in this definition it is not obvious that the infinite product converges. However we can prove that it does, although convergence relies crucially on the exponential factors. The argument runs as follows; for large n

$$\begin{aligned}
&\left(\frac{\omega_{2\,z-1}+\omega_{2n+1}}{\omega_{2\,n}}\right) e^{-(\omega_{2z-1}+1)/2n} = \frac{\omega_{2\,n+1}}{\omega_{2\,n}}\left(1+\frac{\omega_{2\,z-1}}{\omega_{2\,n+1}}\right)\left(1-\frac{\omega_{2\,z-1}+1}{2n}+O\left(\frac{1}{n^2}\right)\right)\\
&=\left(1+\frac{1}{2n}\right)\left(1+O\left(\frac{1}{n^2}\right)\right)\left(1+\frac{\omega_{2\,z-1}}{2n+1}+O\left(\frac{1}{n^2}\right)\right)\left(1-\frac{\omega_{2\,z-1}+1}{2n}+O\left(\frac{1}{n^2}\right)\right)\\
&=\left(1+\frac{1}{2n}\right)\left(1-\frac{1}{2n}-\frac{\omega_{2\,z-1}}{2n(2n+1)}+O\left(\frac{1}{n^2}\right)\right)=1+O\left(\frac{1}{n^2}\right)\,,
\end{aligned} \tag{B.7}$$

and hence the infinite product converges absolutely and uniformly. The function $\Gamma^{II}_\mu(z)$ has simple poles at $z=0,-1,-2,\cdots$, branch points at $1/2\pm i\mu/2$ and a branch cut on $[1/2-i\mu/2, 1/2+i\mu/2]$. Interestingly a reflection identity exists for this function as well

$$\Gamma^{II}_\mu(1+z)\Gamma^{II}_\mu(-z) = -\frac{\alpha}{\sin(\pi z)}\,. \tag{B.8}$$

Finally we should emphasise that $\Gamma^{II}_\mu(1/2+iy)$ is even in y on both sides of the branch cut.

For the solution to the vertex with Dirichlet boundary conditions we further introduce a μ-deformed Gamma function

$$\frac{1}{\check{\Gamma}^I_\mu(z)} = \left(\frac{\omega_{2\,z}+\mu}{2}\right) e^{\gamma\omega_{2z}/2} \prod_{n=1}^{\infty} \left(\frac{\omega_{2n}+\omega_{2\,z}}{\omega_{2\,n}}\right) e^{-\omega_{2z}/2n}\,, \tag{B.9}$$

which differs from $\Gamma^I_\mu(z)$ only in the "zero-mode" term. It is straightforward to prove that this satisfies the same reflection identitiy as $\Gamma^I_\mu(z)$. Note however, that $\check{\Gamma}^I_\mu(iy)$ is an even function of y on both sides of the branch cut. This is the key difference to Γ^I_μ, and ensures that in the derivation of the Neumann coefficients, the integrals along each of the branch cuts vanish for the case of Dirichlet boundary conditions.

Note that generalisations of the identity $\Gamma(z+1) = z\Gamma(z)$ have not been found for any of the μ-deformed Gamma-functions.

B.2. Large-μ asymptotics

In this section we will develop the necessary tools to calculate the large-μ asymptotics of certain combinations of our new functions. Namely we are interested in the behaviour of v_m^I and v_m^{II}, which are defined as

$$\begin{aligned} v_m^I &= \frac{(2m+1)}{\omega_{2\,m+1}} \, \frac{\Gamma^I(m+1/2)}{\sqrt{\pi}\,\Gamma^{II}(m+1)} \\ v_m^{II} &= \frac{2}{\omega_{2\,m}} \, \frac{\Gamma^{II}(m+1/2)}{\sqrt{\pi}\,\Gamma^I(m)} \, . \end{aligned} \tag{B.10}$$

Using the definitions of our Gamma functions we can be more explicit

$$\begin{aligned} v_m^I &= \frac{e^{\gamma/2}}{\sqrt{\pi}} \frac{\omega_{2\,m+1}+\omega_1}{\omega_{2\,m+1}} \prod_{n=1}^{\infty} \left(\frac{\omega_{2\,m+1}+\omega_{2\,n+1}}{\omega_{2\,m+1}+\omega_{2\,n}} \right) e^{-1/2\,n} \\ v_m^{II} &= \frac{2}{\omega_{2\,m}} \frac{e^{-\gamma/2}}{\sqrt{\pi}} \frac{2m}{\omega_{2\,m}+\omega_1} \prod_{n=1}^{\infty} \left(\frac{\omega_{2\,m}+\omega_{2\,n}}{\omega_{2\,m}+\omega_{2\,n+1}} \right) e^{1/2\,n} \, , \end{aligned} \tag{B.11}$$

and we see that it is sufficient to study the following infinite product

$$e^{S_z} \equiv \prod_{n=1}^{\infty} \left(\frac{\omega_z+\omega_{2\,n+1}}{\omega_z+\omega_{2\,n}} \right) e^{-1/2\,n} \, . \tag{B.12}$$

Taking the logarithmic derivative of this infinite product with respect to μ we get

$$\frac{\partial S_z}{\partial \mu} = \frac{\mu}{\omega_z} \sum_{n=1}^{\infty} \frac{1}{\omega_{2\,n+1}} - \frac{1}{\omega_{2\,n}} \, . \tag{B.13}$$

Let

$$R = \sum_{n=1}^{\infty} \frac{1}{\omega_{2\,n+1}} - \frac{1}{\omega_{2\,n}} \, , \tag{B.14}$$

and differentiating with respect to μ, we get

$$\frac{\partial R}{\partial \mu} = \sum_{n=1}^{\infty} -\frac{\mu}{\omega_{2\,n+1}^3} + \frac{\mu}{\omega_{2\,n}^3} \, , \tag{B.15}$$

and thus we see that all we need now is the asymptotic behaviour of

$$\sum_{n=1}^{\infty} \frac{1}{\omega_{2\,n}^3} \quad , \quad \sum_{n=1}^{\infty} \frac{1}{\omega_{2\,n+1}^3} \, , \tag{B.16}$$

which can be worked out as follows. One may obtain an integral representation of the first sum using the integral definition of $\Gamma(3/2)$, *i.e.*, making use of

$$\Gamma(z) = x^z \int_0^\infty e^{-xt} t^{z-1} dt \,, \tag{B.17}$$

which holds for $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(x) > 0$. Thus we can write

$$\sum_{n=1}^\infty \frac{1}{\omega_{2\,n}^3} = \frac{2}{\sqrt{\pi}} \int_0^\infty dt\; t^{1/2} e^{-\mu^2 t} \sum_{n=1}^\infty e^{-4n^2 t} \,, \tag{B.18}$$

and we see that we have something related to a theta function in the integrand. In fact we have $\sum_{n=1}^\infty e^{-4n^2 t} = (\psi(4t/\pi) - 1)/2$ where $\psi(t) = \sum_{n=-\infty}^\infty e^{-n^2 \pi t}$, which has the nice transformation law $\psi(t) = \frac{1}{\sqrt{t}} \psi(1/t)$. Now if we change variables to $s = \mu^2 t$ it is easy to deduce

$$\begin{aligned} \sum_{n=1}^\infty \frac{1}{\omega_{2\,n}^3} &= -\frac{1}{2\mu^3} + \frac{1}{\mu^3 \sqrt{\pi}} \int_0^\infty ds\; s^{1/2} e^{-s} \psi(4s/(\pi\mu^2)) \\ &= -\frac{1}{2\mu^3} + \frac{1}{2\mu^2} \int_0^\infty ds\; e^{-s} \psi(\mu^2 \pi/4s) \,, \end{aligned} \tag{B.19}$$

where we have used the transformation law for $\psi(t)$ in the second equality. Now since $\lim_{t\to\infty} \psi(t) = 1$ we deduce that

$$\sum_{n=1}^\infty \frac{1}{\omega_{2\,n}^3} \sim \frac{1}{2\mu^2} \,, \tag{B.20}$$

for $\mu \to \infty$. In fact we have a much stronger result. This is easily derived from our integral representation as follows. We have[10]

$$\begin{aligned} \sum_{n=1}^\infty \frac{1}{\omega_{2\,n}^3} &= -\frac{1}{2\mu^3} + \frac{1}{2\mu^2} \int_0^\infty ds\; e^{-s} \psi(\mu^2 \pi/4s) \\ &= \frac{1}{2\mu^2} - \frac{1}{2\mu^3} + \frac{1}{2\mu^2} \int_0^\infty ds\; e^{-s} (\psi(\mu^2 \pi/4s) - 1) \\ &= \frac{1}{2\mu^2} - \frac{1}{2\mu^3} + \frac{1}{2\mu^{2+N}} \int_0^\infty dr\; e^{-r/\mu^N} (\psi(\mu^{2+N} \pi/4r) - 1) \\ &= \frac{1}{2\mu^2} - \frac{1}{2\mu^3} + O\left(\frac{1}{\mu^{2+N}}\right) \\ &= \frac{1}{2\mu^2} - \frac{1}{2\mu^3} + O\left(e^{-\mu}\right) \,, \end{aligned} \tag{B.21}$$

[10] Related asymptotics have been discussed in [27].

where we have used the change of variables $r = \mu^N s$ in the third equality and N can be any positive integer, in particular as large as we like [11]. In a similar manner the integral representation of $\Gamma(3/2)$ can be used to deal with the second sum too. We easily show that

$$\sum_{n=1}^{\infty} \frac{1}{\omega_{2n+1}^3} = -\frac{1}{\mu^3} + \frac{1}{\sqrt{\pi}} \int_0^{\infty} dt\, t^{1/2} e^{-\mu^2 t} \theta_2(4it/\pi)\,, \tag{B.22}$$

where $\theta_2(\tau) = 2\sum_0^{\infty} e^{i\pi\tau(n+1/2)^2}$ is one of the theta functions. Using the modular transformation property $\sqrt{-i\tau}\theta_2(\tau) = \theta_4(-1/\tau)$, it then follows that

$$\sum_{n=1}^{\infty} \frac{1}{\omega_{2n+1}^3} = \frac{1}{2\mu^2} - \frac{1}{\mu^3} + \frac{1}{2\mu^2} \int_0^{\infty} ds\, e^{-s} (\theta_4(-\mu^2\pi/(4is)) - 1)\,, \tag{B.23}$$

where $\theta_4(\tau) = \sum_{-\infty}^{\infty} (-1)^n e^{i\pi\tau n^2}$. If we make the change of variables in the integral on the RHS $r = \mu^N s$, it is easy to see that this term is $O(1/\mu^{N+2})$ for any positive integer N; we have thus derived the useful result

$$\sum_{n=1}^{\infty} \frac{1}{\omega_{2n+1}^3} = \frac{1}{2\mu^2} - \frac{1}{\mu^3} + O\left(e^{-\mu}\right)\,. \tag{B.24}$$

Using this together with (B.21) allows us to deduce that

$$\frac{\partial S_z}{\partial \mu} = -\frac{1}{2\omega_z} + O\left(e^{-\mu}\right)\,, \tag{B.25}$$

and therefore

$$e^{S_z} = \frac{e^{a_1}}{\sqrt{\mu + \omega_z}} + O\left(e^{-\mu}\right)\,, \tag{B.26}$$

where a_1 is a constant. A priori it seems that a_1 could depend on z, however it can be proven it does not. To see this take the logarithmic derivative with respect to z of (B.26) and then the limit $\mu \to \infty$ and compare with the exact expression for $\frac{\partial S_z}{\partial z}$. The hard part is now done! We can now straightforwardly find the expansions for v_m^I and v_m^{II}. We get

$$\begin{aligned} v_m^I &= \frac{e^{\gamma/2 + a_1}}{\sqrt{\pi}\,\omega_{2\,m+1}} \frac{\omega_{2\,m+1} + \omega_1}{(\omega_{2\,m+1} + \mu)^{1/2}} + O\left(e^{-\mu}\right) \\ v_m^{II} &= \frac{e^{-\gamma/2 - a_1}}{\sqrt{\pi}\,\omega_{2\,m}} \frac{4m(\omega_{2\,m} + \mu)^{1/2}}{\omega_{2\,m} + \omega_1} + O\left(e^{-\mu}\right)\,. \end{aligned} \tag{B.27}$$

[11] Recall that $f(x) = O(g(x))$ if there exists a constant C such that $|f(x)| < C|g(x)|$ for all x greater than some x_0.

Similarly the expansions for the $\check{v}_m^I$ and $\check{v}_m^{II}$ as defined in (4.27) are computed to be

$$\begin{aligned}\check{v}_m^I &= \frac{e^{\gamma/2+a_1}(2m+1)}{\sqrt{\pi}\,\omega_{2\,m+1}}\frac{\omega_{2m+1}+\omega_1}{(\omega_{2\,m+1}+\mu)^{3/2}} + O\left(e^{-\mu}\right)\\ \check{v}_m^{II} &= \frac{2e^{-\gamma/2-a_1}}{\sqrt{\pi}\,\omega_{2m}}\frac{(\omega_{2\,m}+\mu)^{3/2}}{\omega_{2m}+\omega_1} + O\left(e^{-\mu}\right)\ .\end{aligned} \tag{B.28}$$

B.3. More asymptotics

In this section we provide the asymptotics, which will ensure that the integrals of the circle at infinity, that arise in the contour method, do indeed vanish. Consider e^{S_z}. Taking the logarithmic derivative with respect to z we get

$$\frac{\partial S_z}{\partial z} = \frac{z}{\omega_z}\sum_{n=1}^{\infty}\left(\frac{1}{\omega_z+\omega_{2\,n+1}} - \frac{1}{\omega_z+\omega_{2\,n}}\right)\ . \tag{B.29}$$

Let us define

$$K_z = \sum_{n=1}^{\infty}\left(\frac{1}{\omega_z+\omega_{2\,n+1}} - \frac{1}{\omega_z+\omega_{2\,n}}\right)\ , \tag{B.30}$$

so that

$$\frac{\partial K_z}{\partial z} = \frac{z}{\omega_z}\sum_{n=1}^{\infty}\frac{1}{(\omega_z+\omega_{2\,n})^2} - \frac{1}{(\omega_z+\omega_{2\,n+1})^2}\ . \tag{B.31}$$

We will now find an integral representation for each of the sums on the RHS. We will denote

$$\begin{aligned}S_1(a) &= \sum_{n=1}^{\infty}\frac{1}{(a+\omega_{2\,n})^2}\ ,\\ S_2(a) &= \sum_{n=1}^{\infty}\frac{1}{(a+\omega_{2\,n+1})^2}\ .\end{aligned} \tag{B.32}$$

First consider evaluating $S_1(a)$, for which we will again use complex methods. We have (if $\mathrm{Re}(a) > 0$ [12])

$$S_1(a) = -\oint_C \frac{dz}{2\pi i}\,\frac{\pi\cot(\pi z)}{(a+\omega_{2\,z})^2}\ , \tag{B.33}$$

where C is the contour depicted in figure 7, which runs along the imaginary axis (avoiding the pole in $\cot(\pi z)$ to the right) just to the right of the finite branch cut due to the $\omega_{2\,z}$ and closes to the right in a semi-circle of radius R enclosing the whole of the right hand

[12] For $\mathrm{Re}(a) < 0$ we close in the left hand side of the plane and run along the left hand side of the branch cut. This will give the same answer.

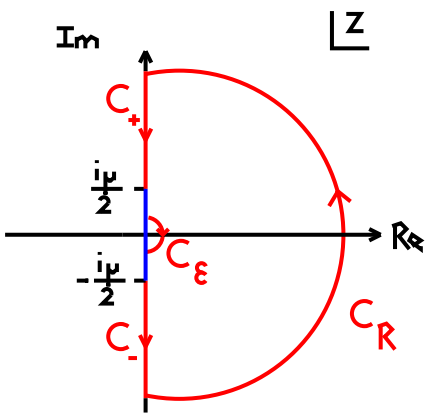

Figure 7 Contour C.

plane as $R \to \infty$. We split the contour up C as comprising of $C_\pm = (\pm i\mu/2, \pm i\infty)$, $C_\epsilon = \{\epsilon e^{i\theta} | \ -\pi/2 \leq \theta \leq \pi/2\}$ and $C_R = \{Re^{i\theta} | \ -\pi/2 \leq \theta \leq \pi/2\}$ and of course the line integrals along the branch cut (depicted in blue in figure 4), however these vanish due to the integrand being odd.

Thus the relevant path is $C_+ \cup C_R \cup C_- \cup C_\epsilon$. We traverse the contour in a clockwise direction, hence the minus in the integral above. It is easy to see that the integral along C_R tends to zero as $R \to \infty$. Also we have

$$\lim_{\epsilon \to 0} \oint_{C_\epsilon} \frac{dz}{2\pi i} \frac{\pi \cot(\pi z)}{(a + \omega_{2\,z})^2} = \frac{1}{2(a+\mu)^2} \,. \tag{B.34}$$

Finally noting that $\omega_{2\,z}$ is $\sqrt{\mu^2 - 4y^2}$ along C_+ and $-\sqrt{\mu^2 - 4y^2}$ along C_-, we arrive at,

$$\oint_{C_+ \cup C_-} \frac{dz}{2\pi i} \frac{\pi \cot(\pi z)}{(a + \omega_{2\,z})^2} = -2a \int_{\mu/2}^{\infty} dy \, \frac{\coth(\pi y)\sqrt{4y^2 - \mu^2}}{(a^2 - \mu^2 + 4y^2)^2}. \tag{B.35}$$

All this means is that

$$S_1(a) = -\frac{1}{2(a+\mu)^2} + 2a \int_{\mu/2}^{\infty} dy \, \frac{\coth(\pi y)\sqrt{4y^2 - \mu^2}}{(a^2 - \mu^2 + 4y^2)^2}. \tag{B.36}$$

By a completely analogous method one may show that

$$S_2(a) = -\frac{1}{(a+\omega_1)^2} + 2a \int_{\mu/2}^{\infty} dy \, \frac{\tanh(\pi y)\sqrt{4y^2 - \mu^2}}{(a^2 - \mu^2 + 4y^2)^2} \,. \tag{B.37}$$

By writing $\coth(\pi y) = 1 + 2/(e^{2\,\pi y} - 1)$ and $\tanh(\pi y) = 1 - 2/(e^{2\,\pi y} + 1)$ one may convince oneself that the remaining integrals in our expression for $S_1(a) - S_2(a)$ are $O(1/a^3)$ and hence that

$$S_2(a) - S_1(a) = -\frac{1}{2a^2} + O\left(\frac{1}{a^3}\right) \,. \tag{B.38}$$

Therefore we get

$$K_z = -\frac{1}{2z} + O\left(\frac{1}{z^2}\right) , \tag{B.39}$$

since the integration constant must be zero. Finally we have

$$S_z = -\frac{1}{2}\log z + c_3 + O\left(\frac{1}{z}\right) , \tag{B.40}$$

thus proving that v_m^I and v_m^{II} have the same asymptotics as u_m which is of course crucial in order to justify the contour method.

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