arXiv:hep-lat/0309015 v1 4 Sep 2003

Vector and Axial-Vector Propagators in the ϵ-Regime of QCD

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Using quenched and unquenched chiral perturbation theory we compute vector and axial current two-point functions at finite volume and fixed gauge field topology, in the so-called ϵ-regime of QCD. A comparison of these results with finite volume lattice calculations allows to determine the parameters of the corresponding chiral Lagrangians.

Consider QCD in a toroidal volume V with $L = V^{1/4}$, and assume that V is large with respect to the QCD scale, *i.e.*, $FL \gg 1$, where F is the pion decay constant. Then take the chiral limit of vanishing quark mass, $m \to 0$, so that $m_\pi \ll 1/L$. This is the ϵ-regime of QCD [1,2].

As in an infinite volume the lightest degrees of freedom are the Goldstone bosons of chiral symmetry breaking, described by a chiral Lagrangian,

$$\mathcal{L}_{\chi PT} = \mathrm{Re\,Tr}\left[\frac{F^2}{4}\partial_\mu U \partial_\mu U^\dagger - \Sigma U M e^{i\theta/N_f}\right] ,$$

where Σ, θ, N_f are the chiral condensate, the vacuum angle, and the number of light flavours.

In the quenched theory, the flavour singlet field $\Phi_0 \sim \ln\det U$ does not decouple. We thus add, to leading order [3],

$$\delta\mathcal{L}_{\chi PT} = \frac{m_0^2}{2N_c}\Phi_0^2 + \frac{\alpha}{2N_c}\partial_\mu\Phi_0\partial_\mu\Phi_0 \ .$$

We have performed our calculations in both the supersymmetric [3] and replica formulations [4,5] of quenched chiral perturbation theory, and shown that they agree. More details can be found in ref. [6].

The ϵ-regime requires an exact evaluation of the zero momentum mode integrals. We split up ($\xi(x)$ are non-zero momentum modes):

$$U(x) \ = \ \exp\left[\frac{2i\xi(x)}{F}\right] U_0 \ , \qquad (1)$$

and perform the U_0 integration exactly.

At the quark level the vector and axial vector currents are

$$V_\mu^a(x) \ \equiv \ \bar\psi(x) i\gamma_\mu T^a_{N_v}\psi(x) \ ,$$

$$A_\mu^a(x) \ \equiv \ \bar\psi(x) i\gamma_\mu\gamma_5 T^a_{N_v}\psi(x) \ , \qquad (2)$$

and we add sources to extract them from the effective theory,

$$\partial_\mu U \to \partial_\mu U + i(v_\mu^a - a_\mu^a)T^a_{N_v}U - iUT^a_{N_v}(v_\mu^a + a_\mu^a).$$

Correlation functions are computed by taking derivatives of the partition function w.r.t. these sources in the chiral effective theory. We have determined the relevant 2-point functions up to and including next-to-leading order in the ϵ-expansion, in the sense of refs. [1,5]. The gauge field topological charge is fixed to be ν by Fourier transformation w.r.t. θ of the partition function.

While we present results here for the vector and axial currents correlators only, similar computations are well motivated and have been carried out for other observables as well, such as 3-point functions containing weak operators [7,8]. Various numerical techniques for the ϵ-regime have recently been discussed in [7]. Preliminary measurements of correlation functions in quenched QCD have already been reported [9,10,11].

Predictions for full QCD:

It turns out that all zero-mode integrals can be expressed in terms of the finite-volume chiral condensate

$$\Sigma_\nu(\mu) \ = \ \frac{\Sigma}{N_f}\langle \mathrm{Re\ Tr}[U_0]\rangle_{\nu,U_0} \ , \qquad (3)$$

and derivatives thereof. Here $\mu \equiv m\Sigma V$ and we also define, to one loop,

$$\mu' \ \equiv \ \mu\left(1 + \frac{N_f^2-1}{N_f}\frac{\beta_1}{F^2\sqrt{V}}\right) \ , \qquad (4)$$

where β_1 is a shape-dependent (but universal)

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constant [1,12]. We get

$$\int \mathrm{d}^3\vec{x}\,\langle V_0^a(x)V_0^a(0)\rangle_\nu \;=\; -\frac{F^2}{2T}\left\{\mathcal{J}_- + \frac{N_f}{F^2}\left(\frac{\beta_1}{\sqrt{V}}\mathcal{J}_- - \frac{T^2}{V}k_{00}\mathcal{J}_+\right)\right\}, \quad (5)$$

$$\int \mathrm{d}^3\vec{x}\,\langle A_0^a(x)A_0^a(0)\rangle_\nu \;=\; -\frac{F^2}{2T}\left\{\mathcal{J}_+ + \frac{N_f}{F^2}\left(\frac{\beta_1}{\sqrt{V}}\mathcal{J}_+ - \frac{T^2}{V}k_{00}\mathcal{J}_-\right) + \frac{4\mu}{N_f F^2}\frac{T^2}{V}h_1(\tau)\langle \mathrm{Re\,Tr}[U_0]\rangle_{\nu,U_0}\right\}, \quad (6)$$

where $h_1(\tau) \equiv [(|\tau| - 1/2)^2 - 1/12]/2$, $\tau \equiv x_0/T$, T is the extent of the lattice in the time direction, and k_{00} is a numerical factor [12,13]. Analogous results but without projecting onto fixed topological charge ν were first computed by Hansen [13]. The expectation values

$$\mathcal{J}_+ \;=\; \frac{1}{N_f^2-1}\left(N_f^2 - 2 + \langle \mathrm{Tr}[U_0]\mathrm{Tr}[U_0^\dagger]\rangle_{\nu,U_0}\right)$$
$$\mathcal{J}_- \;=\; \frac{1}{N_f^2-1}\left(N_f^2 + \langle \mathrm{Tr}[U_0]\mathrm{Tr}[U_0^\dagger]\rangle_{\nu,U_0}\right) \quad (7)$$

are known analytically [5],

$$\langle \mathrm{Tr}[U_0]\mathrm{Tr}[U_0^\dagger]\rangle_{\nu,U_0} \;=\; N_f\left[\frac{\Sigma'_\nu(\mu')}{\Sigma} + N_f\left(\frac{\Sigma_\nu(\mu')}{\Sigma}\right)^2 + \frac{1}{\mu'}\frac{\Sigma_\nu(\mu')}{\Sigma} - \frac{\nu^2 N_f}{\mu'^2}\right], \quad (8)$$

where $\Sigma_\nu(\mu)$ can be expressed explicitly in terms of the modified Bessel functions $I_n(x)$.

Predictions for quenched QCD:

We now consider N_v valence quarks embedded in a theory of N_f quarks in total, and then take the replica limit $N_f \to 0$ [4,5] (results agree with what one obtains by the supersymmetric formulation [3]). Remarkably, all contributions from the famous double-pole propagator of quenched χPT *cancel* exactly at both leading and next-to-leading order. We thus only need the usual massless pion propagator,

$$\bar{\Delta}(x) \equiv \frac{1}{V}\sum_{p\neq 0}\frac{e^{ipx}}{p^2}\,. \quad (9)$$

Let $\mathcal{O}_\mu^{a,-}(x) \equiv V_\mu^a(x)$ and $\mathcal{O}_\mu^{a,+}(x) \equiv A_\mu^a(x)$, and define $t_\pm^a \equiv T_{N_v}^a \pm U_0 T_{N_v}^a U_0^{-1}$. Then

$$\left\langle \mathcal{O}_\mu^{a,\sigma}(x)\;\mathcal{O}_\zeta^{b,\tau}(0)\right\rangle_\nu \;=\; -\frac{F^2}{2}\sigma\tau\left\langle \mathrm{Tr}[t_\sigma^a t_\tau^b]\right\rangle_{\nu,U_0}\partial_\mu\partial_\zeta\bar{\Delta}(x) - \frac{m\Sigma}{4}\sigma\tau\left\langle \mathrm{Tr}[\{t_\sigma^a, t_\tau^b\}(U_0 + U_0^{-1})]\right\rangle_{\nu,U_0} \times \int \mathrm{d}^4z\;\partial_\mu\bar{\Delta}(z-x)\partial_\zeta\bar{\Delta}(z)\,. \quad (10)$$

As in full QCD, the required zero-mode integrals are known in closed analytical form, and the result can be expressed in terms of [14]

$$\frac{\Sigma_\nu(\mu)}{\Sigma} \equiv \mu\Big[I_\nu(\mu)K_\nu(\mu) + I_{\nu+1}(\mu)K_{\nu-1}(\mu)\Big] + \frac{\nu}{\mu},$$

where also $K_n(x)$ is a modified Bessel function. As will become clear below, we will not need μ' and the one-loop corrected condensate [15] (with its problematical finite volume logarithm) in the quenched case, since the correction is of one order higher than what we are computing.

Including next-to-leading order, and making use of the exact results for the quenched zero-momentum mode integrals of ref. [16], we find

$$\langle V_0^a(x)\;V_0^a(0)\rangle_\nu \;=\; 0, \quad (11)$$

$$\int \mathrm{d}^3\vec{x}\,\langle A_0^a(x)\;A_0^a(0)\rangle_\nu \;=\; -\frac{F^2}{T}\left[1 + \frac{2\,m\Sigma_\nu(\mu)T^2}{F^2}h_1\left(\frac{x_0}{T}\right)\right]. \quad (12)$$

The vector-vector correlator of quenched QCD thus vanishes identically up to and including next-to-leading order. Examples of these correlation functions are plotted in Figure 1 along with their unquenched counterparts of Eqs. 5 and 6.

Measuring the axial-vector–axial-vector correlation in this ϵ-regime of QCD directly gives the pion decay constant F to leading order. At next-to-leading order also the infinite-volume chiral condensate Σ can be extracted.

An argument to all orders:

The quenched $\langle V^a(x)V^b(y)\rangle$ correlation function must vanish to all orders. This follows from the following argument. There are two ways to

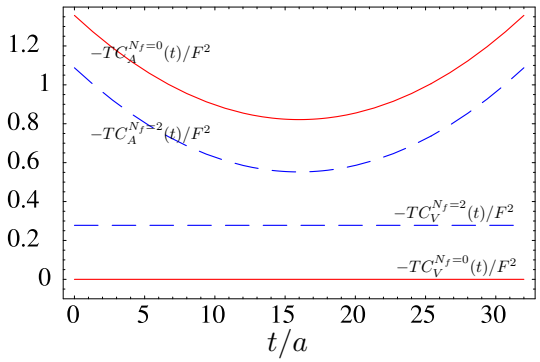

Figure 1. *Quenched and $N_f = 2$ correlation functions of Eqs. 5 and 6 and Eqs. 5 and 6 for $\nu = 0$ plotted vs t/a. Here $a = 2\,\mathrm{GeV}$, $L = 24a$, $T = 32a$, $m = 0.1\,\mathrm{MeV}$, $F = 93\,\mathrm{MeV}$ and $\Sigma = (250\,\mathrm{MeV})^3$.*

contract external quark lines to generate the 2-point functions. One is the "connected" contraction, where the quarks flow from point x to point y. The other is the "disconnected" contraction, where the quarks flow back to the starting points x or y. In the quenched approximation no other quark flow topologies are possible.

Now consider the singlet correlator in the full theory, in the replica limit $N_f \to 0$. It is then easy to see that the disconnected piece vanishes in this limit. Therefore the quenched *flavor non-singlet* correlator, which also only gets contributions from the connected piece, can, up to an overall factor, be computed in the *singlet sector* of the full theory. But the singlet vector current vanishes identically because the corresponding source does not couple to the pion field, and thus the non-singlet quenched vector correlator is also zero.

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