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Supersymmetric WZW models and twisted K–theory of SO(3)

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Abstract

We present an encompassing treatment of D–brane charges in supersymmetric $SO(3)$ WZW models. There are two distinct supersymmetric CFTs at each even level: the standard bosonic $SO(3)$ modular invariant tensored with free fermions, as well as a novel twisted model. We calculate the relevant twisted K–theories and find complete agreement with the CFT analysis of D–brane charges. The K–theoretical computation in particular elucidates some important aspects of $\mathcal{N} = 1$ supersymmetric WZW models on non-simply connected Lie groups.

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1 Introduction

Much evidence has been provided recently in support of the conjecture that charges of D–branes in string theory are measured by K–theory [1, 2]. Exactly solvable conformal field theory (CFT) backgrounds such as Wess–Zumino–Witten (WZW) models and cosets thereof have proven to be particularly fruitful grounds for testing this claim.

The CFT description of D–branes is relatively well under control and the charges for $\mathcal{N} = 1$ supersymmetric WZW models on simply-connected Lie groups have been computed in [3, 4, 5] and for $\mathcal{N} = 2$ coset models in [6, 7]. Due to the non-trivial NSNS 3-form flux in these backgrounds, the main actors on the other side of the conjecture are twisted K–theories. For simply-connected group manifolds these have been obtained in [3, 8, 9] and for $\mathcal{N} = 2$ coset models in [10, 11], and shown to be in perfect agreement with the CFT prediction of the charge groups — as far as these are accessible. One of the most useful tools in determining the twisted K–groups, which will also feature prominently in the present paper, is the seminal work by Freed, Hopkins and Teleman (FHT) [12, 13, 14, 15] relating twisted equivariant K–theory to the representation theory of loop groups (see also [16, 17]). This allows to reduce many of the K–theoretical computations to algebraic problems.

The presence of at least $\mathcal{N} = 1$ supersymmetry is vital for the comparison with K–theory. The importance of fermions should not be surprising as K–theory has deep ties with spinors and the Dirac operator, and in string theory it is general lore that there are no conserved D–brane charges in the bosonic string. The most transparent justification[1] of this point is as follows: from a boundary field theory point of view the charges are determined by boundary conditions modulo RG-flows. Thus, in order to obtain non-trivial charges or equivalently non-trivial path components of the boundary theory, it is necessary to project out the unit operator [18].

Based on the charge relations derived by Fredenhagen and Schomerus [3], recently, Gaberdiel and Gannon [19] determined the charges of D–branes in WZW models on non-simply connected group manifolds. The purpose of the present paper is to compute the corresponding K–theories for the simplest such group, $SO_3 \stackrel{\text{def}}{=} SO(3)$.

There is one key subtlety in the case of non-simply connected groups, that makes the computations slightly more cumbersome (and thus more interesting) compared to the simply-connected case. K–theoretically this can be phrased as follows: in addition to the standard twisting in $H^3(G;\mathbb{Z})$ there is an additional possibility to twist with an element in $H^1(G;\mathbb{Z}_2)$. In the case of interest to us, $H^1(SO_3;\mathbb{Z}_2) = \mathbb{Z}_2$, which can be interpreted as an additional grading of the twisted K–theories. This additional choice has a precise counterpart in the world-sheet description, where it corresponds to different spin-structures for the fermions. In fact, this interpretation is most apparent using the identification proven by Atiyah and Hopkins [20] of $H^1(X;\mathbb{Z}_2)$–twisted K–theory with the Hopkins K–theory $K_\pm(X)$, which made its first appearances in the context of D–brane charges in $(-1)^F$ orbifolds, where F is the (left-moving) space-time fermion number, see e.g. [2, 21].

In summary, we obtain the following picture: let G be a non-simply connected group, with universal cover $\widetilde{G}$ such that $G = \widetilde{G}/\mathbb{Z}_2$. Then one has in general two $\mathcal{N} = 1$ supersymmetric WZW models for G, corresponding to the choices of twistings

[1] We thank G. Moore for pointing this out.

in $H^1(G;\mathbb{Z}_2) = \mathbb{Z}_2$. Equivalently, these choices distinguish two modular invariants corresponding to the WZW model on G, in the precise sense that they are obtained as simple current extensions from the supersymmetric WZW model on $\widetilde{G}$, which differ by the action of a $\mathbb{Z}_2$ simple current on the free fermion theory. In case the latter is trivial the resulting model is simply the tensor product of the bosonic WZW model on G as of [22, 23, 24] with free fermions. This is the kind of model that is relevant for the discussion in [19]. On the other hand if the action on the fermions is non-trivial, the resulting modular invariant does not factor into bosonic and fermionic parts, and has not been discussed in the literature. We shall refer to these models as $(-)$–*twisted* and $(+)$–*twisted* supersymmetric WZW models on G, respectively. Clearly, it would be interesting to systematically explore these models further. This construction has also interesting applications in finding new symmetry-breaking boundary conditions, which we shall comment upon in our concluding remarks.

The outline of this paper is as follows. Section 2 gives an overview of the conformal field-theoretical aspects of the supersymmetric WZW models on SO_3, in particular giving a detailed exposition of the two different choices of spin structures, and the charge groups in either model. The K–theory computation comprises the main body of the paper, starting with a purely topological computation in section 3. This is then refined using FHT-like methods in chapter 4, where we provide a complete derivation of the twisted K–theories for both types of twists in $H^1(SO_3;\mathbb{Z}_2)$. We conclude in section 5 and discuss various directions in which the present work can be extended.

2 Supersymmetric WZW models on SO_3

2.1 The level manifesto

Let us begin by addressing the technical and subtle, but very crucial issue of the level or equivalently, the twisting or equivalently, the NSNS flux. Although we are really only interested in the SO_3 supersymmetric WZW model we are about to encounter various auxiliary WZW models. In addition, in view of the K–theory computation, we wish to use a meaningful notation for the levels, where precisely the positive integers are allowed. We will denote them as follows:

- k The level of the bosonic WZW model on SO_3, i.e. $k = 0$ is the model with only one primary field, $k = 1$ is the next smallest model and so on. This is the integer that classifies the LSO_3 central extension.
- κ The $H^3(SO_3;\mathbb{Z}) = \mathbb{Z}$ twist in the corresponding K–theory, equivalently the NSNS background flux.

The difference between the level and the flux is a constant called the adjoint shift, in our case (see section 4.1)

$$\kappa = k + 1 \tag{1}$$

Now we are really interested only in the supersymmetric WZW model, where

κ The level of the $\mathcal{N} = 1$ supersymmetric WZW model on SO_3, i.e. the central element in the super Kac–Moody algebra.

As we will discuss in more detail later, this is always a $\mathbb{Z}_2$ orbifold of the $\mathcal{N} = 1$ supersymmetric WZW model on SU_2 at level 2κ. As is well-known [25], the latter model is isomorphic to a level-shifted bosonic SU_2 WZW model together with a free fermion theory where

$2\kappa - 2 = 2k$ The level of this bosonic SU_2 WZW model.

Our use of k vs. κ is the standard notation to distinguish between the supersymmetric and bosonic levels, respectively. For the comparison with the CFT computation we should also comment upon the relation of our conventions to the ones chosen in [19, 26], where the authors study only the bosonic SO_3 WZW model and denote its level by $k \in 2\mathbb{Z}$. (i.e. with a spurious factor of 2). Denote their k by k_{GG}, then the following conversion rules apply:

$$k = \frac{k_{\mathrm{GG}}}{2}\,, \qquad \kappa = \frac{k_{\mathrm{GG}}}{2} + 1\,. \tag{2}$$

2.2 Supersymmetric WZW models

Our present objective is to study D–brane charges in $\mathcal{N} = 1$ supersymmetric WZW models on SO_3. The key ingredient for the construction is to observe that $SO_3 = SU_2/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ acts as the antipodal map. The bosonic SO_3 WZW model can therefore be constructed as a simple-current extension of the diagonal $\widehat{\mathfrak{su}}(2)_{2k}$ theory [22, 23, 24], where the order 2 simple current acts on the integrable highest weights $\Lambda = [2k - \lambda, \lambda]$ with $\lambda \in 0 \ldots 2k$, as

$$J: \qquad [2k - \lambda, \lambda] \to [\lambda, 2k - \lambda]\,. \tag{3}$$

The thereby resulting state space for the WZW model on SO_3 is (see [19]) for k odd and even, respectively,

$$\begin{aligned} k \in 2\mathbb{Z}_{\geq} + 1: \quad \mathcal{H}_{SO_3} &= \bigoplus_{n=0}^{k} \mathcal{H}_{2n} \otimes \bar{\mathcal{H}}_{2n} \oplus \bigoplus_{n=1}^{k} \mathcal{H}_{2n-1} \otimes \bar{\mathcal{H}}_{2k-2n+1} \\ k \in 2\mathbb{Z}_{\geq}: \quad \mathcal{H}_{SO_3} &= \bigoplus_{n=0}^{k/2-1} (\mathcal{H}_{2n} \oplus \mathcal{H}_{2k-2n}) \otimes (\bar{\mathcal{H}}_{2n} \oplus \bar{\mathcal{H}}_{2k-2n}) \\ &\qquad \oplus 2\,(\mathcal{H}_k \otimes \bar{\mathcal{H}}_k)\,. \end{aligned} \tag{4}$$

But the bosonic theory does not have any conserved D–brane charges and is not interesting for our purposes. We want to study the supersymmetric version hereof.

The supersymmetric $\widehat{\mathfrak{su}}(2)$ model at level 2κ has a description in terms of the chiral algebra

$$\mathcal{A} = \widehat{\mathfrak{su}}(2)_{2k} \oplus \widehat{\mathfrak{so}}(3)_1 \,, \tag{5}$$

where $\kappa = k + 1$. The diagonal modular invariant for eq. (5) is

$$\begin{aligned} \mathcal{H}_{\text{diag}} &= \left(\bigoplus_{\lambda} \mathcal{H}_\lambda \otimes \bar{\mathcal{H}}_\lambda \right) \otimes \left(\bigoplus_{l=0,1,2} \mathcal{H}_l \otimes \bar{\mathcal{H}}_l \right) \\ &= \mathcal{H}_{su(2)_{2k}} \otimes \mathcal{H}_F \,. \end{aligned} \tag{6}$$

In particular, one obtains a supersymmetric WZW model on SO_3 by tensoring eq. (4) with the state space $\mathcal{H}_F$ of the $\widehat{\mathfrak{so}}(3)_1$ free fermion theory. This is the model studied in [19].

In the above construction, it was assumed that the simple current acts only on the $\widehat{\mathfrak{su}}(2)_{2k}$ part. However, one could also contemplate the following construction of a supersymmetric SO_3 WZW model: Extend by the order 2 simple current $\langle J \oplus j \rangle$, where the simple current j acts on the $\widehat{\mathfrak{so}}(3)_1$ weights by

$$j: \qquad [2-l, l] \to [l, 2-l] \,, \qquad l = 0, 1, 2 \,. \tag{7}$$

The currents J and j generate a simple current group $\mathcal{G} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ for the theory eq. (5). We shall be interested in the following $\mathbb{Z}_2$ subgroups of $\mathcal{G}$:

$$\begin{aligned} (-) \text{ twist:} \qquad & \mathcal{G}_{(-)} = \langle J \oplus \mathrm{Id} \rangle \\ (+) \text{ twist:} \qquad & \mathcal{G}_{(+)} = \langle J \oplus j \rangle \,. \end{aligned} \tag{8}$$

The corresponding simple current extensions of eq. (6) will be denoted by $(-)$–*twisted model* and $(+)$–*twisted model*, respectively (this notation will be explained in section 4.1). In particular, the $(-)$–twisted model is the one discussed in [19].

The state space for the $(-)$–twisted model is given by

$$\mathcal{H}_{SO_3,(-)} = \mathcal{H}_{SO_3} \otimes \mathcal{H}_F \,. \tag{9}$$

The state space for the $(+)$–twisted model is straight forward to obtain using simple-current techniques. As this will not be of main concern to our discussion, we shall leave this for future work.

Note that if one was to extend the theory with all of $\mathcal{G}$, non-trivial discrete torsion [27, 28] is allowed as $H^2\big(\mathbb{Z}_2 \times \mathbb{Z}_2, U_1\big) = \mathbb{Z}_2$. We will not pursue these here since they do not correspond to a bona fide (non-orbifolded) WZW model on SO_3.

2.3 D–brane charges

In [19] the charge groups for the D–branes in the $(-)$–twisted model were computed, and obtained to be

$$\mathcal{K}_{SO_3,(-)} = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & k_{GG} \equiv 0 \mod 4 \\ \mathbb{Z}_4 & k_{GG} \equiv 2 \mod 4\,. \end{cases} \tag{10}$$

The NIM-reps $\mathcal{N}_{\mu\lambda}{}^{\nu}$ for the $(+)$–twisted model follow straight forwardly, thanks to known simple-current technology. The derivation of the charges necessitates a generalization of [3] to supersymmetric CFT, where the fermions do not necessarily factor out and therefore the NIM–reps do not separate into an affine and a free fermion part. We leave this for future discussions, see also [29].

We shall for the present paper content ourselves with the following heuristic derivation of the charge groups. Geometrically, the two choices of twist correspond to the following identifications in the $\widehat{\mathfrak{su}}(2)$ WZW model. The $(-)$–twisted case corresponds to the superposition of the brane with charge q_λ with its image under the antipodal map, i.e. the brane of charge $q_{2k-\lambda}$. So in this model one superposes the brane with its anti-brane (see also [19]) resulting in

$$\begin{aligned} (-) \text{ twist}: \qquad q_{(-),\lambda} &= q_\lambda + q_{2k-\lambda} = (\lambda + 1 + 2k - \lambda + 1)q_0 = \\ &= (2\kappa)q_0 = 0\,, \end{aligned} \tag{11}$$

using $q_\lambda = (\lambda+1)q_0$. Thus these branes do not carry any non-trivial charges. If $2|k$ then there is a brane invariant under the antipodal map, yielding a $\mathbb{Z}_2$ charge.

The $(+)$–twist on the other hand corresponds to superposing the q_λ-charged brane with the anti-brane of the brane with weight $2k-\lambda$, wherefore

$$\begin{aligned} (+) \text{ twist}: \qquad q_{(+),\lambda} &= q_\lambda - q_{2k-\lambda} = (2\lambda - 2k)q_0 \\ &= (2\lambda + 2)q_0\,, \end{aligned} \tag{12}$$

which implies that the corresponding charge group is $\mathbb{Z}_{k+1} = \mathbb{Z}_\kappa$. Furthermore, the brane with label $\lambda = k$ carries charge: identifying it with its image under the antipodal map results in an unoriented world-volume, thus allowing for at most 2-torsion charges. The K–theory computation below will confirm this. Clearly there are no space-filling D3–branes [30]. We should stress that a proper derivation of the charge relations in supersymmetric theories should confirm this.

3 Pure topology

3.1 Quick review of twisted cohomology

The archetypical example of a twisted cohomology theory is Čech cohomology for a nontrivial $\mathbb{Z}$ bundle, that is instead of taking constant coefficients we take them to be only locally constant but with a monodromy around some noncontractible loop in our space X. This obviously changes the cohomology groups, for example $H^0(X;\mathbb{Z}) = \mathbb{Z}$ for any connected space (given by the constants, i.e. sections of the trivial $\mathbb{Z}$ bundle) whereas the twisted cohomology group is ${}^tH^0(X;\mathbb{Z}) = 0$: There are no sections in a nontrivial $\mathbb{Z}$ bundle except the zero section.

Clearly, the possible twists in ordinary cohomology are defined by specifying the monodromies around noncontractible loops, so by a map $\pi_1(X) \to GL_1(\mathbb{Z}) = \mathbb{Z}_2$. But since the target is abelian such a group homomorphism must factor through the abelianization $\pi_1(X)/[-,-] = H_1(X;\mathbb{Z})$. So the twist represents an element of the dual of homology, i.e. of $H^1(X;\mathbb{Z}_2)$. Technically the choice of twist always depends on the representative of the cohomology class, but different representatives of the twist class lead to isomorphic (albeit not canonically) twisted cohomology theories. We will ignore this subtlety usually.

Since we will use them shortly let us compute the (twisted) cohomology groups of $\mathbb{R}\mathrm{P}^3$, say using CW-cohomology. The real projective 3–space has a cell decomposition into a single cell c_i in dimensions $i = 0$ to 3, and each cell c_i is attached such that two points of ∂c_i are identified with one point in the lower dimensional skeleton Σ_{i-1}. So the attaching maps would be degree 2 if it were not for the orientation: for example the 2 endpoints of the interval ∂c_1 map to $\Sigma_0 = c_0$ but with opposite orientation, so they cancel. So the cohomology is the homology of the cochain complex:

$$H^i(\mathbb{R}\mathrm{P}^3;\mathbb{Z}) = H_i\Big(0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0 \Big) = \begin{cases} \mathbb{Z} & i = 3 \\ \mathbb{Z}_2 & i = 2 \\ 0 & i = 1 \\ \mathbb{Z} & i = 0\,. \end{cases} \tag{13}$$

Now since $H^1(\mathbb{R}\mathrm{P}^3;\mathbb{Z}_2) = \mathbb{Z}_2$ there is also a twisted cohomology, which we will denote ${}^-H(\mathbb{R}\mathrm{P}^3;\mathbb{Z})$. The twisting effects the orientations in the boundary maps $\partial c_i \to \Sigma_{i-1}$: Where the two contributions in the untwisted case added up, they now cancel and vice versa. So the twisted cohomology is

$${}^-H^i(\mathbb{R}\mathrm{P}^3;\mathbb{Z}) = H_i\Big(0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} 0 \Big) = \begin{cases} \mathbb{Z}_2 & i = 3 \\ 0 & i = 2 \\ \mathbb{Z}_2 & i = 1 \\ 0 & i = 0\,. \end{cases} \tag{14}$$

Now let us turn towards K–theory. Here it turns out that (some of) the possible twists are representing a class in $H^1(X;\mathbb{Z}_2) \oplus H^3(X;\mathbb{Z})$. The effect of the twist in $H^1(X;\mathbb{Z}_2)$ is again a twisted identification as one goes around a noncontractible loop: If $[E]-[F]$ is an element in the twisted K–theory then the bundles E, F are exchanged as one goes around a "twist" loop.

The $H^3(X;\mathbb{Z})$ part of the twist class can be understood from the transition function point of view (see e.g. [31]). By a standard argument this corresponds to a U_1 valued function $\varphi_{ijk} : U_i \cap U_j \cap U_k \to U_1$ on each triple overlap. The transition functions $g_{ij} : U_i \cap U_j \to GL$ of a twisted bundle then do not quite fit together, but up to a phase factor:

$$g_{ij}g_{jk}g_{ki} = \varphi_{ijk} \qquad \text{on } U_i \cap U_j \cap U_k \,. \tag{15}$$

Again there is a subtlety here in that for non torsion twist classes one cannot use finite dimensional bundles, but this technical problem can be dealt with so we will ignore it in the following.

3.2 Twisted K–theory

For any generalized cohomology theory there is some Atiyah–Hirzebruch–Whitehead spectral sequence relating it to ordinary cohomology. For the case at hand this is the following:

Theorem 1 (Generalized Rosenberg spectral sequence). *Fix a closed manifold X and let $t_1 \oplus t_3$ be a cocycle in $H^1(X;\mathbb{Z}_2) \oplus H^3(X;\mathbb{Z})$. Then there is a $\mathbb{Z} \oplus \mathbb{Z}_2$ graded spectral sequence with*

$$E_2^{p,q} = {}^{t_1}H^p\Big(X;\ K^q(X)\Big)\,, \tag{16}$$

converging to the twisted K–theory ${}^{t_1\oplus t_3}K^*(X)$. The spectral sequence is bounded in p and moreover the first differential is $d_3 = \mathrm{Sq}_3 + t_3\cup$.

Proof. The only novelty is the $t_1 \in H^1(X;\mathbb{Z}_2)$ twist, everything else can be found in [32]. Again let X^n be the n–skeleton of a cell decomposition of X. Then there is a spectral sequence with

$$E_1^{p,q} = {}^{t_1\oplus t_3}K^{p+q}\Big(X^p, X^{p-1}\Big) \simeq K^{p+q}\Big(X^p, X^{p-1}\Big) \simeq K^q\big(\{\mathrm{pt.}\}\big) \tag{17}$$

The only novelty is the differential d_1, which is now the t_1–twisted coboundary operator. Hence the E_2 tableau is eq. (16). □

We are interested in $SO(3) \simeq \mathbb{R}\mathrm{P}^3$ with the possible[2] twists

$$(\pm,\kappa) \ \in\ \mathbb{Z}_2 \oplus \mathbb{Z}_{>} \ \subset\ H^1(\mathbb{R}\mathrm{P}^3;\mathbb{Z}_2) \oplus H^3(\mathbb{R}\mathrm{P}^3;\mathbb{Z})\,. \tag{18}$$

[2]Of course we are only interested in positive levels.

In the $(+)$ case we find (2–periodic in q)

$$E_2^{p,q} = \begin{array}{c|cccc} q=2 & \mathbb{Z} & 0 & \mathbb{Z}_2 & \mathbb{Z} \\ q=1 & 0 & 0 & 0 & 0 \\ q=0 & \mathbb{Z} & 0 & \mathbb{Z}_2 & \mathbb{Z} \\ \hline & p=0 & p=1 & p=2 & p=3 \end{array} \quad d_3=\kappa \quad ; \quad E_3^{p,q} = E_\infty^{p,q} = \begin{array}{c|cccc} q=2 & 0 & 0 & \mathbb{Z}_2 & \mathbb{Z}_\kappa \\ q=1 & 0 & 0 & 0 & 0 \\ q=0 & 0 & 0 & \mathbb{Z}_2 & \mathbb{Z}_\kappa \\ \hline & p=0 & p=1 & p=2 & p=3 \end{array} \tag{19}$$

$$\Rightarrow \qquad {}^{(+,\kappa)}K^1(\mathbb{R}\mathrm{P}^3) = \mathbb{Z}_\kappa, \quad {}^{(+,\kappa)}K^0(\mathbb{R}\mathrm{P}^3) = \mathbb{Z}_2\,,$$

whereas in the $(-)$–twisted case we obtain

$$E_2^{p,q} = E_\infty^{p,q} = \begin{array}{c|cccc} q=2 & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 \\ q=1 & 0 & 0 & 0 & 0 \\ q=0 & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 \\ \hline & p=0 & p=1 & p=2 & p=3 \end{array} \tag{20}$$

$$\Rightarrow \qquad {}^{(-,\kappa)}K^1(\mathbb{R}\mathrm{P}^3) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \text{ or } \mathbb{Z}_4, \quad {}^{(-,\kappa)}K^0(\mathbb{R}\mathrm{P}^3) = 0\,.$$

Almost everything is determined directly from our knowledge of Rosenberg's spectral sequence. We are left only with one tiny ambiguity, we cannot decide the group law on the order 4 charge group ${}^{(-,\kappa)}K^1(\mathbb{R}\mathrm{P}^3)$.

In general we expect for each possible twisted K–theory some CFT or string compactification unless there is some physical reason why this particular choice of discrete torsion is forbidden. So we should expect there to be different WZW models for every choice of twist $(\pm,\kappa) \in \mathbb{Z}_2 \oplus \mathbb{Z}_>$. Especially we should not be too surprised if the order of the charge group is independent of the level.

It remains of course to decide the final ambiguity, but this is surprisingly hard compared to how easily we found almost the complete answer. The actual computation will be in section 4 and is quite lengthy, but has the redeeming feature that it makes contact with CFT methods.

For now let us have a closer look at the $(-,0)$–twisted K–theory, where we can use a simple trick to discern between the two possibilities for ${}^{(-,0)}K^1(\mathbb{R}\mathrm{P}^3)$. The idea (see also [21] for a similar use) is that this K–theory is the $K_\pm(S^3)$ of [20], where S^3 comes with the antipodal involution. Then this K–group can be computed as the ordinary K–theory of $L \stackrel{\text{def}}{=} \big(S^{(4,0)} \times R^{(1,1)}\big)/\mathbb{Z}_2$, the nontrivial real line bundle over $\mathbb{R}\mathrm{P}^3$. Now it happens that the one point compactification of L is smooth, and in fact $\mathbb{R}\mathrm{P}^4$. Hence

we find

$$
\begin{aligned}
{}^{(-,0)}K^i(\mathbb{R}\mathrm{P}^3) = K^i_{\pm}(S^3) = K^{i+1}(L) = K^{i+1}(\mathbb{R}\mathrm{P}^4 - \{\mathrm{pt.}\}) = \\
= \widetilde{K}^{i+1}(\mathbb{R}\mathrm{P}^4) = \begin{cases} \mathbb{Z}_4 & i = 1 \\ 0 & i = 0\,. \end{cases}
\end{aligned} \tag{21}
$$

Unfortunately this trick is not easily extended to twistings $(-,\kappa)$ with $\kappa > 0$, but it is already tantalizing to see that the twisted K–groups are indeed different from the twisted cohomology ${}^{-}H^*(\mathbb{R}\mathrm{P}^3) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

4 FHT computation for SO_3

So far we only used purely topological methods to find the relevant K–groups. This seems to yield the correct result, although we are unable to resolve the remaining $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ vs. $\mathbb{Z}_4$ ambiguity. It would be nice if we could resolve this, and even more interestingly, if we could draw a parallel between the representation theoretic argument on the CFT side and the K–theory computation.

We achieve this in the by now familiar way (see [10]): Use some basic tricks to rewrite the desired K–group ${}^tK(SO_3)$ as equivariant K–theory of some product space, and then use the equivariant Künneth theorem [33] to relate that to tensor products involving ${}^tK_G(G^{\mathrm{Ad}})$. The latter is — by the FHT theorem [12, 14, 15] — the Verlinde algebra, also known as the fusion ring. The computation of the twisted K–theory thus boils down to simple algebra involving fusion rings. We will not make use of the FHT theorem directly but determine all necessary rings in the following directly.

In particular we use

$$
{}^tK^*\left(SO_3\right) = {}^tK^*_{SU_2}\left(SO_3^{\mathrm{Ad}} \times SU_2^{\mathrm{L}}\right), \tag{22}
$$

where the superscripts Ad, L denote the SU_2 group[3] action: **Ad**joint and **L**eft multiplication. It turns out that the twist class is only on the first factor, so the Cartesian product really is a product, even considering the twist. Then we can apply the equivariant Künneth theorem to the effect that we get a spectral sequence

$$
\begin{aligned}
E_2^{*,*} = \mathrm{Tor}^*_{R(SU_2)}\Big({}^tK^*_{SU_2}(SO_3^{\mathrm{Ad}}),\ \mathbb{Z}\Big) = \\
= \mathrm{Tor}^*_{R(SU_2)}\Big({}^tK^*_{SU_2}(SO_3^{\mathrm{Ad}}),\ {}^tK^*_{SU_2}(SU_2^{\mathrm{L}})\Big) \\
\Rightarrow\ {}^tK^*_{SU_2}\Big(SO_3^{\mathrm{Ad}} \times SU_2^{\mathrm{L}}\Big) = {}^tK^*\left(SO_3\right)\,.
\end{aligned} \tag{23}
$$

[3]It is important to use SU_2 (as opposed to SO_3) equivariant K–theory since the Künneth theorem would not hold in the latter case: SO_3 is not a Hodgkin group.

So now we first have to compute the twisted equivariant K–group ${}^tK_{SU_2}(SO_3)$, which will occupy sections 4.1 to 4.5 (Note that this is similar, but not quite the same as ${}^tK_{SO_3}(SO_3)$, which is computed in [14]). Then we will evaluate the Künneth spectral sequence in sections 4.6 and 4.7. Finally we compare our result with the CFT analysis in section 4.8.

4.1 Poincaré duality and adjoint shift

We want to compute the D–brane charge group of the $\mathcal{N} = 1$ supersymmetric WZW model at level κ, so what is the correct level for the corresponding bosonic WZW model? A partial answer is well-known for simply connected Lie groups, where the level of the auxiliary bosonic WZW model is $\kappa - h^\vee$. But this shift (by the dual Coxeter number in the simply connected case) is not the whole story. Really we have to shift by the twist class induced via the adjoint representation $Ad : G \to SO(\mathfrak{g})$ from the element (cf. [14])

$$(-, 1, -) \in H^1_{SO}(SO; \mathbb{Z}_2) \oplus H^3_{SO}(SO; \mathbb{Z}) \simeq \mathbb{Z}_2 \oplus \Big(\mathbb{Z} \oplus \mathbb{Z}_2\Big) . \tag{24}$$

In our case we want to use $G = SO_3$ and then the double cover $\widetilde{SO}_3 = SU_2$, i.e. pull back

$$H^*_{SO}(SO) \longrightarrow H^*_{SO_3}\big(SO_3\big) \longrightarrow H^*_{SU_2}\big(SO_3\big) . \tag{25}$$

It is easy to see that the final adjoint shift is

$$1 \in H^3_{SU_2}\big(SO_3; \mathbb{Z}\big) \simeq \mathbb{Z} \quad \text{and} \quad - \in H^1_{SU_2}\big(SO_3; \mathbb{Z}_2\big) \simeq \mathbb{Z}_2 . \tag{26}$$

An important point is that we also have to flip the sign, so the straightforward bosonic SO_3 WZW model tensored with free fermions corresponds to the $- \in H^1(SO_3; \mathbb{Z}_2)$ twisted K–theory.

Furthermore it will be more convenient to calculate the twisted equivariant K–groups in K–homology in the following. Poincaré duality (see [14]) relates this back to the K–cohomology as of

$${}^{(\epsilon,\kappa)}K^i_G(X) = {}^{(\epsilon,\kappa)}K^G_{\dim(G)-i}(X) . \tag{27}$$

4.2 Mayer–Vietoris sequence

SU_2 acts by conjugation on $SO_3 = SU_2/\pm$. There are three kinds of orbits:

- The orbit of $1 \in SU_2/\pm$, a fixed point.
  The stabilizer is the whole SU_2.

- The orbit of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in SU_2/\pm$, which is topologically $\mathbb{R}\mathrm{P}^2$.
  The stabilizer is $\mathbb{Z}_2 \ltimes U_1 \subset SU_2$.
- The orbit of a generic point is an S^2.
  The stabilizer is a $U_1 \subset SU_2$.

This suggests the following cell decomposition of $SO_3 \simeq \mathbb{R}\mathrm{P}^3$:

U The complement of the $\mathbb{R}\mathrm{P}^2$.

V The complement of $1 \in SU_2/\pm$.

such that

U is contractible to the fixed point.

V is contractible to the special $\mathbb{R}\mathrm{P}^2$ orbit.

$U \cap V$ is contractible to a generic S^2 orbit.

The Mayer–Vietoris sequence in K–homology for the cover U, V of SO_3 is then the following 6 term cyclic exact sequence:

$$\begin{array}{ccccc} {}^tK_1^{SU_2}(SO_3^{\mathrm{Ad}}) & \longleftarrow & {}^tK_1^{SU_2}(\mathbb{R}\mathrm{P}^2) & \longleftarrow & 0 \\ \downarrow & & & & \uparrow \\ R(U_1) & \xrightarrow{\mathrm{Ind}} & R(SU_2) \oplus {}^tK_0^{SU_2}(\mathbb{R}\mathrm{P}^2) & \longrightarrow & {}^tK_0^{SU_2}(SO_3^{\mathrm{Ad}}) \,. \end{array} \tag{28}$$

The inclusion $i : \mathbb{R}\mathrm{P}^2 \hookrightarrow \mathbb{R}\mathrm{P}^3$ identifies the cohomology groups

$$i^* : H^1(\mathbb{R}\mathrm{P}^3; \mathbb{Z}_2) \xrightarrow{\sim} H^1(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2) \,, \tag{29}$$

so we can take the cocycle's support disjoint from U in the cyclic exact sequence.

Now concerning the K–groups of $\mathbb{R}\mathrm{P}^2$, they are again the representation ring of the stabilizer, as SU_2 acts transitively. But there is a subtlety as this cell might come with a nontrivial twist class. Even more delicately, the identification of the K–homology groups with the representation ring uses Poincaré duality, and this flips the sign of the twist as $\mathbb{R}\mathrm{P}^2$ is not orientable:

$${}^tK_0^{SU_2}\left(\mathbb{R}\mathrm{P}^2\right) = {}^{-t}K_{SU_2}^0\left(\mathbb{R}\mathrm{P}^2\right) = {}^{-t}R\left(\mathbb{Z}_2 \ltimes U_1\right) \,. \tag{30}$$

4.3 The (twisted) representation rings

Let us review the representation rings that occur in our discussion to fix notation. The most important one is for SU_2, since everything in eq. (28) is an $R(SU_2)$ module:

$$R(SU_2) = \mathbb{Z}[\Lambda]\,, \tag{31}$$

generated by the fundamental (2 dimensional) representation Λ. Instead of taking powers of Λ there is a different $\mathbb{Z}$ basis that is very useful in practice. This basis are the irreducible representations of SU_2, which are all symmetric powers of Λ. They are given recursively as

$$\begin{aligned} \mathrm{Sym}^{-1}(\Lambda) &= 0 \\ \mathrm{Sym}^{0}(\Lambda) &= 1 \\ \Lambda\,\mathrm{Sym}^{n}\Lambda &= \mathrm{Sym}^{n+1}(\Lambda) + \mathrm{Sym}^{n-1}(\Lambda)\,. \end{aligned} \tag{32}$$

Next we have the representations of U_1, those are

$$R(U_1) = \mathbb{Z}[\alpha, \alpha^{-1}]\,, \tag{33}$$

with $R(SU_2)$ module structure $\mu : R(SU_2) \times R(U_1) \to R(U_1)$ induced by the embedding $U_1 \subset SU_2$. Explicitly the $R(SU_2)$ action is given by

$$\mu(\Lambda, x) = (\alpha + \alpha^{-1})x \quad \forall x \in R(U_1)\,. \tag{34}$$

The representation theory of the semidirect product $\mathbb{Z}_2 \ltimes U_1$ is more complicated. This is abstractly the group

$$\mathbb{Z}_2 \ltimes U_1 = \big\{(s,\phi) : s \in \mathbb{Z}_2,\ \phi \in \mathbb{R}/2\pi\mathbb{Z}\big\}, \quad (s_1,\phi_1)\cdot(s_2,\phi_2) = (s_1 s_2, \phi_1 + s_1\phi_2)\,, \tag{35}$$

which is the same as O_2, but more naturally we should think of it as the double cover of O_2. There are two obvious one dimensional representations, the trivial and the sign representation σ. In addition to those we also have the 2 dimensional representation from the embedding $\mathbb{Z}_2 \ltimes U_1 \subset SU_2$, which we call again Λ (this notation makes sense, since by definition then the $R(SU_2)$ module structure is just multiplication)

$$\Lambda(+1,\phi) = \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \quad \Lambda(-1,\psi) = \begin{pmatrix} 0 & -e^{i\psi} \\ e^{i\psi} & 0 \end{pmatrix}. \tag{36}$$

One can easily check that $\Lambda \otimes \sigma$ is conjugate to $\Lambda \otimes 1 = \Lambda$, while σ is of course not conjugate to 1. Hence the representation ring is

$${}^{+}R(\mathbb{Z}_2 \ltimes U_1) \stackrel{\text{def}}{=} R(\mathbb{Z}_2 \ltimes U_1) = \mathbb{Z}[\Lambda,\sigma] \Big/ \big\langle \sigma^2 - 1,\ \Lambda(\sigma - 1) \big\rangle\,. \tag{37}$$

Finally, there is the possibility to twist the $\mathbb{Z}_2 \ltimes U_1$ representations, and we get the corresponding twisted representation rings. Really those are defined as the twisted equivariant K–groups of a point, and for the case at hand are (see [13]):

$$
\begin{aligned}
{}^{-}K^1_{\mathbb{Z}_2 \ltimes U_1}\big(\{\text{pt.}\}\big) &\stackrel{\text{def}}{=} {}^{-}R^1\big(\mathbb{Z}_2 \ltimes U_1\big) = \langle \sigma - 1 \rangle_{R(\mathbb{Z}_2 \ltimes U_1)} && \text{(38a)}\\
{}^{-}K^0_{\mathbb{Z}_2 \ltimes U_1}\big(\{\text{pt.}\}\big) &\stackrel{\text{def}}{=} {}^{-}R\big(\mathbb{Z}_2 \ltimes U_1\big) = \langle \sigma + 1 \rangle_{R(\mathbb{Z}_2 \ltimes U_1)} && \text{(38b)}\\
&\simeq R(SU_2) = \mathbb{Z}[\Lambda] \text{ as } R(SU_2) \text{ module}\,.
\end{aligned}
$$

4.4 Dirac induction

The essential part of the whole computation is to identify the map dubbed Ind in eq. (28). The pushforward in K–homology is actually Dirac induction, a version of Borel–Weil induction that does not require complex structures (see [15])

$$
\mathrm{Ind} : R(U_1) \to R(SU_2) \oplus {}^{\pm}R(\mathbb{Z}_2 \ltimes U_1)\,. \tag{39}
$$

The first component is just the usual induction, precomposed with multiplication by α^κ which is the effect of the twist class $\kappa = k + 1 \in H^3_{SU_2}(SO_3)$:

$$
\pi_1 \circ \mathrm{Ind}(\alpha^n) = \mathrm{Sym}^{n+\kappa}\Lambda\,. \tag{40}
$$

Concerning the second component we have to distinguish between the possible $\pm$ twists (for representation rings it makes also sense to call this a grading), we will come to that shortly.

Having identified the induction map and assuming that $\kappa > 0$ (as we will always do) it is then easy to see that the total Ind is injective, so we can indeed determine all the unknowns in the exact sequence eq. (28):

$$
\begin{aligned}
{}^{t}K^{SU_2}_1\big(SO_3\big) &= {}^{-t}R^1\big(\mathbb{Z}_2 \ltimes U_1\big) && \text{(41a)}\\
{}^{t}K^{SU_2}_0\big(SO_3\big) &= \Big(R(SU_2) \oplus {}^{-t}R\big(\mathbb{Z}_2 \ltimes U_1\big)\Big) \Big/ \mathrm{Ind}\,\big(R(U_1)\big)\,. && \text{(41b)}
\end{aligned}
$$

Moreover the first component turns out to be surjective, so we can write ${}^{t}K^{SU_2}_0\big(SO_3\big)$ as a quotient of ${}^{-t}R\big(\mathbb{Z}_2 \ltimes U_1\big)$ only.

4.4.1 Twisted Poincaré duality

It remains to identify the second component of the induction map eq. (39). This is almost, but not quite, the Dirac induction

$$
\mathrm{Ind}^{\pm} : R(U_1) \to {}^{\pm}R(\mathbb{Z}_2 \ltimes U_1)\,. \tag{42}
$$

An important subtlety here is that in the twisted Poincaré duality ${}^{\pm}K^0_{SU_2}\left(\mathbb{R}\mathrm{P}^2\right) = {}^{\mp}K_0^{SU_2}\left(\mathbb{R}\mathrm{P}^2\right)$ we had to pick a fundamental class, or dually a class in ${}^{-}K^0_{SU_2}\left(\mathbb{R}\mathrm{P}^2\right)$. But the generator is $1 \in {}^{-}R(\mathbb{Z}_2 \ltimes U_1)$ which is a nontrivial[4] twisted representation.

To compensate for this we have to make sure that our K-homology pushforward $\pi_2 \circ \mathrm{Ind}(1)$ is again the fundamental class. From that we can identify[5]

$$\pi_2 \circ \mathrm{Ind}(-) = \mathrm{Ind}^{\pm}(\alpha \cdot -)\,, \tag{43}$$

using the results on $\mathrm{Ind}^{\pm}$ from the remainder of this section.

4.4.2 Ungraded Induction

First, let us look at the induction involving only untwisted representation rings (this computes then the $(-)$–twisted K–theory). We find

$$\pi_2 \circ \mathrm{Ind}(1) = \mathrm{Ind}^{+}(\alpha) = \Lambda \tag{44a}$$

$$\pi_2 \circ \mathrm{Ind}(\alpha^{-1}) = \mathrm{Ind}^{+}(1) = 1 + \sigma\,. \tag{44b}$$

Let us pause to explain the latter eq. (44b), which might be less obvious. This is a rather degenerate case of Dirac induction as the quotient $(\mathbb{Z}_2 \ltimes U_1)/U_1 \simeq \mathbb{Z}_2$ is 0–dimensional.

Recall the usual Dirac induction for a subgroup $H \subset G$, see e.g. [34, 35]: Given a representation $\rho : H \to V$ we can construct a G representation on $\Gamma(G \times_H V)$. The problem is that the latter (the space of sections) will in general not be finite dimensional. The solution is to define an elliptic operator $D : \Gamma(G \times_H V_1) \to \Gamma(G \times_H V_2)$, then the G equivariant index $\mathrm{Index}_G(D) \in R(G)$ yields a finite dimensional (virtual) representation.

But in the case at hand $G/H \simeq \mathbb{Z}_2$ is just two points, so $\Gamma(G \times_H V)$ is $2\dim(V)$ dimensional. Especially for $V = \mathbb{C}$ the trivial representation we see that $\Gamma(G \times_H \mathbb{C}) = L^2(\mathbb{Z}_2) = \mathrm{span}_{\mathbb{C}}(1, \sigma)$ is generated by the trivial and the sign representation, this explains eq. (44b).

4.4.3 Graded Induction

The induction to ${}^{-}R\big(\mathbb{Z}_2 \ltimes U_1\big)$ (which necessary for the $(+)$–twisted K–theory) is related to the untwisted restriction and induction via the following diagram with

[4] In other words, the trivial representation is not "$-$" twisted.

[5] Or $\mathrm{Ind}^{\pm}(\alpha^{-1} \cdot -)$, depending on the chosen orientation.

exact rows

$$\begin{array}{ccccc} R(\mathbb{Z}_2 \ltimes U_1) & \xrightarrow{\mathrm{Res}^+} & R(U_1) & \xrightarrow{\mathrm{Ind}^-} & {}^-R(\mathbb{Z}_2 \ltimes U_1) \\ \| & & \| & & \\ R(\mathbb{Z}_2 \ltimes U_1) & \xleftarrow{\mathrm{Ind}^+} & R(U_1) & \xleftarrow{\mathrm{Res}^-} & {}^-R(\mathbb{Z}_2 \ltimes U_1)\,, \end{array} \tag{45}$$

and moreover induction and restriction are adjoint functors, i.e.

$$\mathrm{Hom}_{R(U_1)}(\mathrm{Res}^{\pm}\ -\,,\ -\,) = \mathrm{Hom}_{\pm R(\mathbb{Z}_2 \ltimes U_1)}(\ -\,, \mathrm{Ind}^{\pm}\ -\,)\,. \tag{46}$$

Furthermore all maps are $R(\mathbb{Z}_2 \ltimes U_1)$–module maps using the module structure discussed in section 4.3, so to specify the maps we just have to write down the image of generators in the respective presentations eqns. (37),(38b),(33):

$$\begin{aligned} \mathrm{Res}^+(1) &= 1 \\ \mathrm{Res}^-(1) &= \alpha - \alpha^{-1}\,. \end{aligned} \tag{47}$$

The ordinary restriction Res^+ is obvious. For the twisted restriction Res^- note that $\alpha - \alpha^{-1}$ generates the kernel of Ind^+, and since the horizontal lines in eq. (45) are exact this already fixes the restriction[6] (up to an irrelevant overall sign).

Now Ind^- is right adjoint to Res^-, so e.g.

$$\begin{aligned} \mathbb{C} \simeq \mathrm{Hom}_{R(U_1)}\Big(\alpha - \alpha^{-1},\ \alpha\Big) &= \mathrm{Hom}_{R(U_1)}\Big(\mathrm{Res}^-(1),\ \alpha\Big) = \\ &= \mathrm{Hom}_{-R(\mathbb{Z}_2 \ltimes U_1)}\Big(1,\ \mathrm{Ind}^-(\alpha)\Big)\,. \end{aligned} \tag{48}$$

Together with exactness of the top row in eq. (45) this determines

$$\begin{aligned} \pi_2 \circ \mathrm{Ind}(1) &= \mathrm{Ind}^-(\alpha) = 1 \\ \pi_2 \circ \mathrm{Ind}(\alpha^{-1}) &= \mathrm{Ind}^-(1) = 0\,. \end{aligned} \tag{49}$$

4.5 Determining the quotient

The representation ring $R(U_1)$ is a free $R(SU_2)$ module, generated by α^n and α^{n+1} (i.e. any two consecutive powers of α are $R(SU_2)$–linearly independent and generate all of $R(U_1)$). Their image under the pushforward then generates $\mathrm{Ind}\left(R(U_1)\right)$ as an $R(SU_2)$ module. Taking $n = -\kappa - 1$, the pushforward has the form

$$\begin{aligned} \mathrm{Ind}(\alpha^{-\kappa}) &= \mathrm{Sym}^0(\Lambda) \oplus \left(\cdots\right) = 1 \oplus \left(\cdots\right) \\ \mathrm{Ind}(\alpha^{-\kappa-1}) &= \mathrm{Sym}^{-1}(\Lambda) \oplus \left(\cdots\right) = 0 \oplus \left(\pi_2 \circ \mathrm{Ind}(\alpha^{-\kappa-1})\right)\,. \end{aligned} \tag{50}$$

[6] One can make this more precise using the description as super-representations.

So using the first relation we can write every equivalence class in the quotient eq. (41b) uniquely as $0 \oplus$ (something). The second relation keeps that choice of representative, so

$$ {}^{t}K_0^{SU_2}(SO_3) = {}^{-t}R\big(\mathbb{Z}_2 \ltimes U_1\big) \Big/ \Big(\big(\pi_2 \circ \mathrm{Ind}(\alpha^{-\kappa-1})\big) \cdot R(SU_2) \Big) \,. \tag{51} $$

In the $(+)$ case we found

$$ \pi_2 \circ \mathrm{Ind}(\alpha^n) = \mathrm{Sym}^n(\Lambda) \quad \in {}^{-}R\big(\mathbb{Z}_2 \ltimes U_1\big) \,, \tag{52} $$

in the same way as for the first component of Ind. Applying eq. (32) we find that

$$ \pi_2 \circ \mathrm{Ind}(\alpha^{-\kappa-1}) = -\mathrm{Sym}^{\kappa-1}(\Lambda) = -\Lambda^{\kappa-1} + \cdots \quad \in {}^{-}R\big(\mathbb{Z}_2 \ltimes U_1\big) \,, \tag{53} $$

generates the relation.

In the $(-)$ case it is not quite so easy to write down a formula, however we can find $\mathrm{Ind}(\alpha^n)$ recursively using the $R(SU_2)$ module structure

$$ \Lambda\, \mathrm{Ind}(\alpha^n) = \mathrm{Ind}\left(\Lambda \alpha^n\right) = \mathrm{Ind}\left(\big(\alpha + \alpha^{-1}\big)\alpha^n\right) = \mathrm{Ind}(\alpha^{n+1}) + \mathrm{Ind}(\alpha^{n-1}) \,, \tag{54} $$

and eq. (44a),(44b). The result is that

$$ \pi_2 \circ \mathrm{Ind}(\alpha^{-n-1}) = \begin{cases} p_n(\Lambda) & \forall n \text{ even} \\ p_n(\Lambda) + (-1)^{\frac{n}{2}}(1+\sigma) & \forall n \text{ odd} \end{cases} \quad \in {}^{+}R\big(\mathbb{Z}_2 \ltimes U_1\big) \,, \tag{55} $$

where p_n is a polynomial of degree $|n|$ without constant part. Its value at 2 will be important in the following, by straightforward induction one can show that

$$ p_n(2) = \begin{cases} 2 & \forall\, n \in 2\mathbb{Z}+1 \\ 4 & \forall\, n \in 4\mathbb{Z}+2 \\ 0 & \forall\, n \in 4\mathbb{Z} \,. \end{cases} \tag{56} $$

Putting everything together we find[7]

$$ \begin{aligned} {}^{(+,\kappa)}K_1^{SU_2}(SO_3) &= \mathbb{Z}[\Lambda]/\Lambda \\ {}^{(+,\kappa)}K_0^{SU_2}(SO_3) &= \mathbb{Z}[\Lambda] \Big/ \big\langle \mathrm{Sym}^{\kappa-1}(\Lambda) \big\rangle \\ {}^{(-,\kappa)}K_1^{SU_2}(SO_3) &= 0 \\ {}^{(-,\kappa \text{ odd})}K_0^{SU_2}(SO_3) &= \mathbb{Z}[\Lambda,\sigma] \Big/ \big\langle \Lambda(\sigma-1),\ \sigma^2-1,\ p_\kappa(\Lambda) \big\rangle \\ {}^{(-,\kappa \text{ even})}K_0^{SU_2}(SO_3) &= \mathbb{Z}[\Lambda,\sigma] \Big/ \big\langle \Lambda(\sigma-1),\ \sigma^2-1,\ p_\kappa(\Lambda) + (-1)^{\frac{\kappa}{2}}(1+\sigma) \big\rangle \end{aligned} \tag{57} $$

as $R(SU_2) = \mathbb{Z}[\Lambda]$ modules, the corresponding cohomology groups are then determined by Poincaré duality eq. (27).

[7] In the $(-)$–twisted case here the $\langle\cdot\rangle$ means: Act with all of $\mathbb{Z}[\Lambda,\sigma]$. But we really only want to mod out the $R(SU_2) = \mathbb{Z}[\Lambda]$ image of $\mathrm{Ind}(\alpha^{-\kappa-1})$. However, this is the same thing since $\sigma p_n(\Lambda) = p_n(\Lambda)$, because p_n does not have a constant part.

4.6 The tensor product

Now that we determined the twisted equivariant K–groups we can apply the equivariant Künneth theorem and determine the corresponding non–equivariant K–theory. The result is a spectral sequence with

$$E_2^{*,*} = \mathrm{Tor}^*_{R(SU_2)}\Big({}^tK^*_{SU_2}(SO_3),\ \mathbb{Z}\Big)\,, \tag{58}$$

where $R(SU_2) = \mathbb{Z}[\Lambda]$ acts on $\mathbb{Z}$ by multiplication with the dimension of the representation. With other words, $-\otimes_{R(SU_2)}\mathbb{Z}$ is evaluation at $\Lambda = 2$:

$$\begin{aligned}
{}^{(+,\kappa)}K_1^{SU_2}\big(SO_3\big)\otimes_{R(SU_2)}\mathbb{Z} &= \mathbb{Z}_2\\
{}^{(+,\kappa)}K_0^{SU_2}\big(SO_3\big)\otimes_{R(SU_2)}\mathbb{Z} &= \mathbb{Z}_\kappa\\
{}^{(-,\kappa)}K_1^{SU_2}\big(SO_3\big)\otimes_{R(SU_2)}\mathbb{Z} &= 0\\
{}^{(-,\kappa\ \text{odd})}K_0^{SU_2}\big(SO_3\big)\otimes_{R(SU_2)}\mathbb{Z} &= \mathbb{Z}[\sigma]\Big/\big\langle 2(\sigma-1),\ \sigma^2-1,\ 2\big\rangle =\\
&= \mathbb{Z}_2[\sigma]\Big/\big\langle\sigma^2-1\big\rangle = \mathbb{Z}_2\oplus\mathbb{Z}_2\\
{}^{(-,\kappa\in 4\mathbb{Z}>)}K_0^{SU_2}\big(SO_3\big)\otimes_{R(SU_2)}\mathbb{Z} &= \mathbb{Z}[\sigma]\Big/\big\langle 2(\sigma-1),\ \sigma^2-1,\ 1+\sigma\big\rangle = \mathbb{Z}_4\\
{}^{(-,\kappa\in 4\mathbb{Z}_{\geq}+2)}K_0^{SU_2}\big(SO_3\big)\otimes_{R(SU_2)}\mathbb{Z} &= \mathbb{Z}[\sigma]\Big/\big\langle 2(\sigma-1),\ \sigma^2-1,\ 3-\sigma)\big\rangle =\\
&= \mathbb{Z}\big/\gcd(4,8) = \mathbb{Z}_4\,.
\end{aligned} \tag{59}$$

4.7 Higher Tor

The CFT charge equation [3]

$$\dim(\lambda)q_a = \sum \mathcal{N}_{\lambda a}{}^b q_b\,, \tag{60}$$

really tells you that the charge group is the tensor product

$$\mathcal{N}\otimes_{RG}\mathbb{Z}\,, \tag{61}$$

where $\mathcal{N}$ is the algebra of the q_a with structure constants $\mathcal{N}_{ab}{}^c$.

But the derivation of the charge equation is by no means mathematically strict. Indeed we know examples where the twisted K–theory and hence the charge group is strictly bigger than eq. (61), for example most[8] WZW models on compact simply connected simple Lie groups (see [9]). But by a generalized nonsense argument involving the Künneth spectral sequence we know that the tensor product eq. (61) is a

[8] With the exception of SU_2 at arbitrary level and other Lie groups at special levels where the K–groups vanish.

subgroup of the K–group, i.e. there are no additional relations between the charges in eq. (60).

But to find the whole charge group we must determine the whole $\mathrm{Tor}(-,-)$, not just its degree zero piece $-\otimes-$. Since the second argument $\mathbb{Z}=\mathbb{Z}[\Lambda]/\langle\Lambda-2\rangle$ has only one relation as $R(SU_2)$ module, only $\mathrm{Tor}^0=\otimes$ and Tor^1 can be nonvanishing.

A quick way to argue that Tor^1 always vanishes is the following: There cannot be any nontrivial differential after E_2 in the Künneth spectral sequence, so any nonvanishing Tor^1 would increase the order of the charge groups. But we know from comparison with the generalized Rosenberg spectral sequence already that Tor^0 accounts for all elements of the K–group.

Nevertheless it would be nice to see directly that Tor^1 has to vanish. This will be the topic of the remainder of this section. In the simpler $(+)$–twisted case we can straightforwardly determine the derived tensor product (as in [9]) and find

$$\mathrm{Tor}^1_{R(SU_2)}\Big({}^{(+,\kappa)}K^1_{SU_2}(SO_3),\ \mathbb{Z}\Big)=0\,,\qquad \mathrm{Tor}^1_{R(SU_2)}\Big({}^{(+,\kappa)}K^0_{SU_2}(SO_3),\ \mathbb{Z}\Big)=0\,. \tag{62}$$

The $(-)$–twisted case is more complicated since there are additional relations, see eq. (57). We again have to distinguish odd and even κ.

If κ is even, then the K–groups fit into a short exact sequence (recall that the polynomials p_n have no constant term):

$$0\longrightarrow\mathbb{Z}[\Lambda]\Big/\Big\langle\tfrac{p_\kappa(\Lambda)}{\Lambda}\Big\rangle\xrightarrow{\cdot\Lambda}{}^{(-,\kappa\ \mathrm{odd})}K^1_{SU_2}(SO_3)\xrightarrow{\Lambda\mapsto0}\mathbb{Z}[\sigma]\Big/\langle\sigma^2-1\rangle\longrightarrow0\ . \tag{63}$$

The long exact sequence for Tor then yields the desired result.

Finally, if κ is even then we can use the relation

$$p_\kappa(\Lambda)+(-1)^{\frac{\kappa}{2}}(1+\sigma)=0\quad\Leftrightarrow\quad\sigma=-(-1)^{\frac{\kappa}{2}}p_\kappa(\Lambda)-1\,, \tag{64}$$

to eliminate σ and write

$${}^{(-,\kappa\ \mathrm{even})}K^1_{SU_2}(SO_3)=\mathbb{Z}[\Lambda]\Big/\langle\tilde p_\kappa(\Lambda)\rangle,\qquad\tilde p_\kappa(2)\neq0\,, \tag{65}$$

and again we see that the Tor^1 vanishes.

For reference, we get

$$\begin{aligned}&{}^{(+,\kappa)}K^1(SO_3)=\mathbb{Z}_\kappa && {}^{(+,\kappa)}K^0(SO_3)=\mathbb{Z}_2\\ &{}^{(-,\kappa)}K^1(SO_3)=\begin{cases}\mathbb{Z}_2\oplus\mathbb{Z}_2 & \kappa\text{ odd}\\ \mathbb{Z}_4 & \kappa\text{ even}\end{cases} && {}^{(-,\kappa)}K^0(SO_3)=0\,.\end{aligned} \tag{66}$$

Note that since $\kappa=k+1=k_{\mathrm{GG}}/2+1$, this agrees precisely with the result of [19].

4.8 Comparison with the CFT computation

How does all this relate to the CFT charge computation? We actually did something very similar. First, note that the twisted equivariant K–group

$$ {}^{(-,\kappa)}K_0^{SU_2}\big(SO_3\big) = \Big(R(SU_2) \oplus {}^{+}R\big(\mathbb{Z}_2 \ltimes U_1\big)\Big) \Big/ \operatorname{Ind}\big(R(U_1)\big)\,, \tag{67} $$

is the same $R(SU_2)$ module as the charges $q_0, \ldots, q_{\kappa-2}, q_+, q_-$ of Gaberdiel and Gannon, see [19] eq. (2.23) — of course up to our more rational labeling of the level, i.e. their $n_{\mathrm{GG}} = \frac{k_{\mathrm{GG}}}{2} + 2 = \kappa + 1$. To see this define

$$ \begin{aligned} q_\ell &\stackrel{\text{def}}{=} \mathrm{Sym}^\ell(\Lambda) \oplus 0 \qquad \forall 0 \leq \ell \leq \kappa - 2 \\ q_+ &\stackrel{\text{def}}{=} 0 \oplus (-1) \qquad q_- \stackrel{\text{def}}{=} 0 \oplus (-\sigma)\,, \end{aligned} \tag{68} $$

and take the following $R(SU_2)$ generators for the image of the Dirac induction:

$$ \operatorname{Ind}(\alpha^{-1}) = \mathrm{Sym}^{\kappa-1}(\Lambda) \oplus (1+\sigma) \tag{69a} $$

$$ \operatorname{Ind}(\alpha^{-2}) = \mathrm{Sym}^{\kappa-2}(\Lambda) \oplus \Lambda\,. \tag{69b} $$

Then clearly $R(SU_2)$ acts as follows on the q generators:

$$ \begin{aligned} \Lambda q_0 &= \Lambda\Big(1 \oplus 0\Big) = q_1 \\ \Lambda q_\ell &= \Lambda \mathrm{Sym}^\ell(\Lambda) \oplus 0 = \Big(\mathrm{Sym}^{\ell-1}(\Lambda) + \mathrm{Sym}^{\ell+1}(\Lambda)\Big) \oplus 0 = \\ &= q_{\ell-1} + q_{\ell+1} \qquad \forall 1 \leq \ell \leq \kappa - 3 \\ \Lambda q_{\kappa-2} &= \Big(\mathrm{Sym}^{\kappa-3}(\Lambda) + \mathrm{Sym}^{\kappa-1}(\Lambda)\Big) \oplus 0 = \\ &= q_{\kappa-3} - \Big(0 \oplus (1+\sigma)\Big) = q_{\kappa-3} + q_+ + q_- \\ \Lambda q_+ &= 0 \oplus (-\Lambda) = \mathrm{Sym}^{\kappa-2}(\Lambda) \oplus 0 = q_{\kappa-2} \\ \Lambda q_- &= 0 \oplus (-\Lambda) = \mathrm{Sym}^{\kappa-2}(\Lambda) \oplus 0 = q_{\kappa-2}\,. \end{aligned} \tag{70} $$

Since Gaberdiel and Gannon were then computing the tensor product of the q with $\mathbb{Z}$ it is no wonder that they obtain the same result as we did. However the presentation of the module we are working with, eq. (57), is more useful for the computation of the tensor product, which is now just one line.

5 Conclusions and outlook

In this paper we computed the charges of D–branes on SO_3 as twisted K–theories and found perfect agreement with the CFT results of [19]. Furthermore, the twisted K–

theory point of view elucidated certain aspects of supersymmetric[9] WZW models on non-simply connected groups, in particular it forces us to study two inequivalent such theories, which we call $(+)$ and $(-)$–twisted SO_3 WZW model. The latter is simply the well-known bosonic SO_3 WZW model tensored with free fermions, whereas the former is a novel theory, that to our knowledge has not been discussed in the literature so far.

There are two key points in our analysis, which we should emphasize. Firstly, the twisted K–theories for SO_3 come in two guises, distinguished by a sign $\pm \in H^1(SO_3;\mathbb{Z}_2) = \mathbb{Z}_2$, by which the K–theory is twisted in addition to the standard twist class $\kappa \in H^3(SO_3;\mathbb{Z})$. A very subtle point in the supersymmetric CFT construction is that the conceptually simpler $(-)$–twisted WZW model corresponds to the $- \in H^1(SO_3;\mathbb{Z}_2)$ twisted K–theory. Be that as it may, by applying similar technology as in the work of Freed, Hopkins and Teleman [12, 13, 14, 15] we are able to determine the relevant K–groups as in [9, 10, 11].

Secondly, the additional twisting in the K–theory enforces that there should be two inequivalent $\mathcal{N} = 1$ supersymmetric WZW models on SO_3, which are precisely distinguished by the sign in $H^1(SO_3;\mathbb{Z}_2)$. This motivates us to define the $(+)$ and $(-)$–twisted models as a simple current extension of the supersymmetric SU_2 WZW model: there are two simple currents, which have the same action on the bosonic $\widehat{\mathfrak{su}}(2)$ part but differ in their action on the free fermions in $\widehat{\mathfrak{so}}(3)_1$. The non-trivial action on the fermions essentially amounts to including $(-1)^F$ in the orbifold action. This is in particular in accord with the identification of $H^1(G;\mathbb{Z}_2)$ twisted K–theory with Hopkins' $K_\pm$ [20].

There are several avenues in which to extend the present work. Clearly, it would be very interesting to discuss the novel $(+)$–twisted SO_3 WZW model in more detail, both from the bulk and boundary CFT point of view. In fact, the construction of the $(+)$–twisted model obviously applies to the more general setting of any non-simply connected Lie group G with $|\pi_1(G)|$ even. The inequivalent supersymmetric WZW models for G can be obtained by an analogous simple current construction.

The generalized CFT construction suggested above should of course be complemented by the corresponding twisted K–theory calculation. We have seen a beautiful match between the two sides in the SO_3 case, which should generalize to all compact Lie groups. We shall address this question elsewhere. Extensions to orientifolds are also conceivable, e.g. the D–branes for SO_3 have been computed in [36] and the relevant K–theory would be a suitably twisted version of Real K–theory.

The key point of this paper is that we cannot simply analyze supersymmetric WZW models by looking at the bosonic part and completely ignoring the fermions. A possibly very interesting application of this would be to revisit the construction

[9] K–theory would not be relevant for boundary states in the purely bosonic SO_3 WZW model, as explained in the introduction.

of symmetry breaking boundary conditions in WZW models. Let us sketch the idea: the analysis of boundary states in WZW models and the corresponding charge computation has so far essentially neglected the fermions. In particular, the symmetry-breaking boundary conditions that have been constructed so far only break parts of the bosonic chiral algebra. It is however conceivable that new boundary states arise if the free fermion part of the chiral algebra is partially broken as well, e.g. by implementing a simple-current action of $j = (-1)^F$ in addition to the simple currents of the bosonic chiral algebra.

Recently [37] there has been some progress for $SU(n)$ WZW–models in understanding the long-standing mismatch between twisted K–theory and WZW D–brane charges for higher rank simply-connected groups. The construction we suggest above gives rise to a set of new boundary states and it would be important to see whether these yield additional charges. The proof of completeness and linear-independence of charges is certainly one of the main unresolved issue in the CFT analysis of D–brane charges, a rigorous treatment of which may be inspired by K–theory.

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