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Simple Space-Time Symmetries: Generalizing Conformal Field Theory

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Abstract We study simple space-time symmetry groups G which act on a space-time manifold $\mathbb{M} = {}^{G}\!/_{H}$ which admits a G-invariant global causal structure. We classify pairs $(G, \mathbb{M})$ which share the following additional properties of conformal field theory: 1) The stability subgroup H of $\mathbf{o} \in \mathbb{M}$ is the identity component of a parabolic subgroup of G, implying factorization $H = MAN^-$, where M generalizes Lorentz transformations, A dilatations, and N^- special conformal transformations. 2) special conformal transformations $\xi \in N^-$ act trivially on tangent vectors $v \in T_{\mathbf{o}}\mathbb{M}$. The allowed simple Lie groups G are the universal coverings of $SU(m,m), SO(2,D), Sp(l,\mathbb{R}), SO^*(4n)$ and $E_{7(-25)}$ and H are particular maximal parabolic subgroups. They coincide with the groups of fractional linear transformations of Euklidean Jordan algebras whose use as generalizations of Minkowski space time was advocated by Günaydin. All these groups G admit positive energy representations. It will also be shown that the classical conformal groups $SO(2,D)$ are the only allowed groups which possess a time reflection automorphism T; in all other cases space-time has an intrinsic chiral structure.

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1 Introduction

Although the Poincaré group is not semi-simple, it is of interest to study also quantum field theories (QFT) with semi-simple space-time symmetry groups G. It is necessary for the existence of QFT satisfying conventional principles that G acts on a space-time manifold $\mathbb{M}$ which admits a G-invariant global causal structure, implying a notion of spacelike, positive or negative timelike

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tangent vectors, and $\mathbb{M}$ should be globally hyperbolic. If the existence of a ground state is demanded, G should admit positive energy representations.

We are only interested in cases where G acts transitively on $\mathbb{M}$, hence

$$\mathbb{M} = {}^{G}\!/_{H} \ ,$$

where H is the stability subgroup of some point $\mathbf{o} \in \mathbb{M}$.

Conformal field theories in arbitrary dimension D are the classical examples. It had been believed for a long time that conformal symmetry had causality problems because special conformal transformations in Minkowski space could interchange spacelike and timelike, but the issue was clarified by Lüscher and Mack [1]. Causality (in the sense of global hyperbolicity) is satisfied when one considers the action of the universal covering G of the conformal group $SO(2, D)$ on an ∞-sheeted covering $\mathbb{M}$ of compactified Minkowski space.

In the conformal case the pair $(G, \mathbb{M})$ has the following two additional properties, which can be formulated for general semi-simple Lie groups.

Assumption 1 *The stability subgroup H of $\mathbf{o} \in \mathbb{M}$ is the identity component of a parabolic subgroup of G, implying factorization $H = MAN^-$, where elements of M generalize Lorentz transformations, A dilatations, and N^- special conformal transformations.*

Assumption 2 *Special conformal transformations $\xi \in N^-$ act trivially on tangent vectors to $\mathbb{M}$ at $\mathbf{o}$.*

It is essential that one divides by the identity component of the parabolic subgroup and not by the whole group. Otherwise causality will be destroyed. By their definition, parabolic subgroups contain the whole center of G which is infinite in our case, implying that the parabolic subgroup has infinitely many connected components. If one were to divide by the center, space-time $\mathbb{M}$ would become compact – compactified Minkowski space in the conformal case - and therefore not globally hyperbolic.

We wish to classify the pairs $(G, \mathbb{M})$ such that $\mathbb{M}$ admits a global causal structure, and the two properties assumption 1 and assumption 2 just mentioned are true, with appropriate groups M, A, N^-. The definition of a parabolic subgroup implies that A is non-compact abelian, the Lie algebra of N^- is nilpotent, and M is a reductive Lie algebra whose elements commute with those of A.

The results which are to be proven in this paper are stated in section 2. The classification is in theorem 4 and table 1.

It turns out that under the stated assumptions global hyperbolicity of $\mathbb{M}$ implies that G possesses positive energy representations, see theorem 3 below and the subsection following it.

We also examine the question which of the groups G admit a time reflection automorphism.

2 Summary of Results

2.1 First Implications of Assumptions 1 and 2

The assumptions 1 and 2 go a long way towards satisfying causality requirements, if G is connected and simply connected. For groups which are not simply connected, problems like closed timelike paths may appear. They are eliminated by passing to the universal covering G of the group.

To state the results precisely, we must recall some structure theory.

The Lie algebra $\mathfrak{g}$ of a non-compact Lie group G admits a Cartan decomposition,

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} \ .$$

The center Z of G is contained in the subgroup K of G with Lie algebra $\mathfrak{k}$ and K/Z is the maximal compact subgroup of G/Z. We will call $\mathfrak{k}$ the maximal compact subalgebra of $\mathfrak{g}$ for short. But note that K is not compact if the center Z is infinite.

If $H = MAN^-$ is the identity component of a parabolic subgroup of G, there exists an inner involutive automorphism w of G which carries M to M, A to A and maps N^- to an isomorphic subgroup N^+ such that the Lie algebra $\mathfrak{g}$ admits a direct sum decomposition

$$\mathfrak{g} = \mathfrak{n}^+ \oplus (\mathfrak{a} \oplus \mathfrak{m}) \oplus \mathfrak{n}^- \tag{1}$$

where $\mathfrak{n}^+$ is the Lie algebra of N^+ etc. (generalized Bruhat decomposition). $\mathfrak{a}$ is abelian, $\mathfrak{m}$ is reductive, $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent, $\mathfrak{a} \subset \mathfrak{p}$, and the elements of $\mathfrak{a}$ and of $\mathfrak{m}$ commute. Furthermore,

$$\big[\mathfrak{a},\mathfrak{n}^+\big] \subseteq \mathfrak{n}^+ \ , \quad \big[\mathfrak{m},\mathfrak{n}^+\big] \subseteq \mathfrak{n}^+ \ , \tag{2}$$

and similarly for $\mathfrak{n}^-$.

Theorem 1 *It follows from the assumptions 1 and 2 that the corresponding generalized Bruhat decomposition (1) of* $\mathfrak{g}$ *satisfies*

$$\left[\mathfrak{n}^+,\mathfrak{n}^-\right] \subseteq \mathfrak{a} \oplus \mathfrak{m} \ . \tag{3}$$

Conversely, suppose that H *is the identity component of a parabolic subgroup. Then (3) implies that* $\mathfrak{n}^-$ *acts trivially on tangent space* $T_{\mathfrak{o}}\mathbb{M}$.

In fact, more is true:

Theorem 2 *Assuming* $\mathfrak{g}$ *is simple, the commutation relations (3) imply that* $\mathfrak{n}^+$ *and* $\mathfrak{n}^-$ *are commutative and* $\mathfrak{a}$ *is 1-dimensional.*

In this sense the groups G are very similar to the conformal group, apart from the generalized Lorentz group. In particular, a generalized Poincaré group MN^+ exists which is the semi-direct product of an abelian group N^+ and a generalized Lorentz group M.

2.2 Causal Structure

Now we are ready to formulate the first main result:

Theorem 3 *Suppose that* G *is a non-compact simple real Lie group with Lie algebra* $\mathfrak{g}$ *and assumptions 1 and 2 are satisfied. Then*

i) Assuming the complexification $(\mathfrak{g})_{\mathbb{C}}$ *of* $\mathfrak{g}$ *is simple and* M *is semi-simple or trivial,* $\mathbb{M}$ *admits an* G*-invariant infinitesimal causal structure if and only if the tangent space* $T_{\mathfrak{o}}\mathbb{M}$ *to* $\mathbb{M}$ *at* $\mathfrak{o}$ *possesses a non-zero vector which is invariant under the rotation group* $M \cap K$.

ii) $\mathbb{M}$ *is globally hyperbolic if and only if the maximal compact Lie subalgebra* $\mathfrak{k}$ *of* $\mathfrak{g}$ *has nontrivial center* $\mathfrak{z}_{\mathfrak{k}}$, $\mathfrak{z}_{\mathfrak{k}}$ *is not contained in the Lie algebra* $\mathfrak{m}$ *of* M*, and the corresponding Lie group* $\mathsf{Z}_{\mathfrak{k}} \simeq \mathbb{R}$.

By definition, global hyperbolicity implies the existence of a G-invariant infinitesimal causal structure (i.e. of light cones with appropriate properties) see section 5.3. It will follow from the final classification that the assumption in the if part of ii) that $\mathfrak{z}_{\mathfrak{k}}$ is not contained in $\mathfrak{m}$ is automaticaly fulfilled.

Let us explicitly state that the universal covering group G of $SO(1,2) = Sl(2,\mathbb{R})/\mathbb{Z}_2 = SU(1,1)/\mathbb{Z}_2$ is covered by the theorem. In this case $\mathfrak{m}$ is trivial,

and $\mathbb{M} = \mathbb{R}$. The invariant cones defining the infinitesimal causal structure are half lines $\mathbb{R}_{\pm}$.

Let us agree for the sequel of this paper to count trivial $\mathfrak{m}$ as semi-simple.

Let us also note that $\mathfrak{g}$ simple implies that $(\mathfrak{g})_{\mathbb{C}}$ is simple, unless $\mathfrak{g}$ is a complex Lie algebra, and in the latter case, $\mathfrak{k}$ does not have nontrivial center.

2.3 Positive Energy Representations

Positive energy representation means that there exists a generator iH_0 in the Lie algebra $\mathfrak{g}$ of G such that H_0 has positive spectrum ($H_0 \geq 0$) in the representation. The non-compact simple Lie groups G possessing positive energy representations have been classified long ago [2]. Let Z be the center of G and K/Z the maximal compact subgroup of G/Z. G possesses positive energy representations if K/Z has a $U(1)$-factor, i.e. $\mathfrak{k}$ has nontrivial center $\mathfrak{z}_{\mathfrak{k}}$. If $\mathfrak{g}$ is simple, $\mathfrak{z}_{\mathfrak{k}}$ is 1-dimensional and $H_0 \geq 0$ in positive energy representations, for a generator $iH_0 \in \mathfrak{z}_{\mathfrak{k}}$.

By the second part of theorem 3, global hyperbolicity of $\mathbb{M}$ implies that this criterion for the existence of positive energy representations is satisfied.

It follows that the center Z of the simply connected group G is infinite, containing a factor $\mathbb{Z}$. The simple groups G with positive energy representations are coverings of

$$SU(m,n),\ SO(2,D),\ Sp(l,\mathbb{R}),\ SO^*(2l),\ E_{6(-14)},\ E_{7(-25)} \tag{4}$$

In the conformal case, the rotation invariant generator P_0 of Poincaré-translations is also positive, because it is limit of elements conjugate to H_0 under dilatations, and conversely, $H_0 = P_0 + w(P_0)$. This result generalizes. In particular, $iH_0 \in (\mathfrak{n}^+ \oplus \mathfrak{n}^-) \cap \mathfrak{k}$, and it lies in the commutant of $\mathfrak{m} \cap \mathfrak{k}$.

2.4 Classification

A classification is obtained from theorem 1 by inspection of root systems as will be described below. The result is as follows.

Theorem 4 *The pairs $(G, \mathbb{M})$ such that G is a simple connected simply connected Lie group and $\mathbb{M} = G/H$ where $H = MAN^-$ is identity component of a parabolic subgroup of G satisfying assumption 1 and 2 and such that $\mathbb{M}$ carries a G-invariant global causal structure are as follows.*

G is the universal covering of one of the groups $SU(m,m)$, $SO(2,D)$, $Sp(l,\mathbb{R})$, $SO^*(4n)$, and $E_{7(-25)}$, with $m \geq 1, D \geq 3, l \geq 2, n \geq 2$, and H is a maximal parabolic subgroup. H are uniquely determined by G up to conjugation. In all these groups, iH_0 is never a generator of M.

The following table 1 lists the Lie algebras $\mathfrak{g}$, together with the Lorentz subalgebra $\mathfrak{m}$ and the semi-simple part $\mathfrak{k}_s$ of the Lie algebra $\mathfrak{k} = \mathfrak{k}_s \oplus \mathfrak{z}_\mathfrak{k}$ of the maximal compact subgroup K/Z of G/Z. r is the split rank of $\mathfrak{g}$, i.e. the maximal number of non-compact generators in a Cartan subalgebra. The remaining entries determine the parabolic subgroup and H_0, as follows.

It is recalled in section 4 how the parabolic subgroups of G are classified by proper subsets Θ of the set of simple restricted roots. For maximal parabolic subgroups, Θ lacks a single simple restricted root λ_e, which is restriction of a root α_e. The entry λ_e of the table gives the corresponding node in the Dynkin diagram of the restricted root system Σ of $\mathfrak{g}$, table 2. Nodes are enumerated $1, 2, ...$ from top to bottom and from left to right. The last column of the table contains information about H_0 and will be explained at the beginning of section 7.

Type	r	λ_e	$\mathfrak{m}$	$\dim \mathbb{M}$	$\mathfrak{k}_s$	Node
$\mathfrak{su}(m,m)$	m	m	$\mathfrak{sl}(m,\mathbb{C})$	m^2	$\mathfrak{su}(m) \oplus \mathfrak{su}(m)$	m
$\mathfrak{so}(2,D)$	2	2	$\mathfrak{so}(1,D-1)$	D	$\mathfrak{so}(D)$	2
$\mathfrak{sp}(l,\mathbb{R})$	l	1	$\mathfrak{sl}(l,\mathbb{R})$	$\frac{1}{2}l(l+1)$	$\mathfrak{su}(l)$	1
$\mathfrak{so}^*(4n)$	n	1	$\mathfrak{su}^*(2n)$	$2n(n-1)$	$\mathfrak{su}(2n)$	1
$\mathfrak{e}_{7(-25)}$	3	3	$\mathfrak{e}_{6(-26)}$	27	$\mathfrak{e}_6$	7

Table 1: The Lie algebras with positive energy representations which admit parabolic subalgebras satisfying assumptions 1 and 2. $m \geq 1$, $D \geq 3$, $l \geq 2$, $n \geq 2$. Their split rank r, the Lorentz subalgebra $\mathfrak{m}$ and semi-simple part $\mathfrak{k}_s$ of the maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}$, and the dimension of $\mathbb{M}$ are also listed. The remaining entries determine the parabolic subalgebra and H_0, see text.

2.5 Interesting Observations

It is an important observation that $\mathfrak{m}$ and $\mathfrak{k}_s$ have isomorphic complexification, and therefore also $\mathfrak{m} \oplus \mathfrak{a}$ and $\mathfrak{k} = \mathfrak{k}_s \oplus \mathfrak{z}_\mathfrak{k}$. Their finite dimensional representations can therefore be identified, by Weyls unitary trick. This permits to relate lowest weight representations of $\mathfrak{g}$, which are determined by the lowest weight of a finite dimensional representation of $\mathfrak{k}$, with induced representations of G on $\mathbb{M}$, which are determined by finite dimensional representations of $\mathfrak{m} \oplus \mathfrak{a}$, see the Outlook. This fact and its implications were noted before by Günaydin [3, 4]. Lowest weight representations of $\mathfrak{g}$ are positive energy representations. Such representations have been studied long ago by Harish-Chandra [5, 6].

There are similarities among the Lorentz groups. With the remarkable exception of conformal theories in odd dimensions D, nonzero $\mathfrak{m}$ are real Lie algebras which owe their existence to a symmetry of their Dynkin diagrams. They possess no compact Cartan subalgebra, as can be seen from the fact that their maximal compact subalgebras $\mathfrak{u} = \mathfrak{m} \cap \mathfrak{k}$ have lower rank. For instance, $\mathfrak{g} = \mathfrak{e}_{7(-25)}$ has $\mathfrak{u} = \mathfrak{f}_4$. The Lie algebra $\mathfrak{su}(1,1)$ is a degenerate case with $\mathfrak{m} = 0$.

Among the listed groups G there is one non-compact real form $E_{7(-25)}$ of an exceptional group. Space-time $\mathbb{M}$ has dimension 27 in this case. From the point of view of string theory this is one dimension too much for a bosonic theory. But then, the 11 dimensions of 11-dimensional supergravity are also one too much for a supersymmetric theory.

The exceptional group $E_{6(-14)}$ also possesses positive energy representations, but it has no parabolic subgroup which fulfills our requirements.

Our assumptions amount to requiring that $\mathbb{M}$ is a generalization of (an ∞-sheeted covering of compactified) Minkowski space. In the outlook we will mention some further quantum field theoretic motivation for this assumption.

In a series of papers starting in 1975, Günaydin proposed the idea of using Jordan algebras, and more generally Jordan triple systems, to define generalized space times and corresponding symmetry groups [7, 8, 9, 10, 3, 4]. The list of groups which satisfy our selection criteria, which include global causality, coincides with the groups of conformal transformations (fractional linear transformations) of Euklidean Jordan algebras [10, 11]. They generalize Minkowski space. This means that the Minkowski subspace of $\mathbb{M}$ can be made into a commutative nonassociative algebra. It does not admit a global causal structure nor is it a homogeneous space for G. The universal cover $\mathbb{M}$ of its compactification has both properties but is not an algebra.

All the simple groups with positive energy representations, including $E_{6(-14)}$ and $SU(m,n)$ with $m \neq n$ are conformal groups of Hermitean Jordan triple systems [10].

2.6 Time Reflection

We will derive one further result: It is a prerequisite for time reflection symmetry of a G-invariant theory that time reflections T can act on $\mathbb{M}$, and this induces a time reflection automorphism of G.

Global hyperbolicity of $\mathbb{M}$ implies that it is homeomorphic to $\mathbb{R} \times \Sigma$, and interpretation of T as a time reflection is understood to require that it reflects $\mathbb{R}$ and acts trivially on Σ for appropriate choice of Σ ("space"). .

Theorem 5 *Of all the groups listed in theorem 4, only the conformal groups $SO(2, D)$ admit a time reversal automorphism.*

The covering of the group $SU(2, 1)$ is the only other group with positive energy representations which admits a time reversal automorphism.

A PT-automorphism always exists. Therefore this means that in the other cases, space-time has an intrinsic chiral structure.

3 Action of H on the tangent space $T_{\mathbf{o}}\mathbb{M}$

Denote the differentiable action of $g \in G$ on $\mathbb{M} = {}^{G}\!/_{H}$ by $\rho(g)$,

$$\rho(g)[\mathbf{x}]_H = [g\mathbf{x}]_H \quad \text{for } \mathbf{x} \in G \, . \tag{5}$$

Since ρ determines a map of smooth curves, it induces a map σ of tangent spaces, known as the derivative of ρ. The action at $x \in \mathbb{M}$ will be denoted by σ^x,

$$\sigma^x(g) : T_x\mathbb{M} \mapsto T_{\rho(x)}\mathbb{M}$$

Given that $\mathbb{M}$ is the homogeneous space ${}^{G}\!/_{H}$, the tangent space $T_{\mathbf{o}}\mathbb{M}$ at $\mathbf{o} = [\mathbf{e}]_H$ can be identified with the quotient vector space of the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ of G and H,

$$T_{\mathbf{o}}\mathbb{M} = {}^{\mathfrak{g}}\!/_{\mathfrak{h}} \, . \tag{6}$$

The action (5) of g on $\mathbb{M}$ restricts to an action of $h \in H$ on $\mathbb{M}$, $\rho(h)[\mathbf{x}]_H = [h\mathbf{x}h^{-1}]_H$. It follows from this and from the identification (6) that the induced action of H on $T_{\mathbf{o}}\mathbb{M}$ is given by

$$\sigma^{\mathbf{o}}(h)([X]_{\mathfrak{h}}) = [\mathrm{Ad}_G(h)X]_{\mathfrak{h}} \text{ , where } h \in H, \ X \in \mathfrak{g} \, , \tag{7}$$

and Ad_G is the adjoint representation of G acting on $\mathfrak{g}$.

Since $\mathfrak{h}$ is an invariant subspace of $\mathfrak{g}$ under the restriction of the adjoint representation Ad_G of G to H, we see that $T_o\mathbb{M} = \mathfrak{g}/\mathfrak{h}$ carries the quotient representation.

This result can be phrased in another way which is a generalization of a result used in the theory of causal symmetric spaces [12].

Lemma 1 *Choose any subspace $\mathfrak{q}$ of $\mathfrak{g}$ such that $\mathfrak{g}$ is a direct sum, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. Let $P_\mathfrak{q}$ be the corresponding restriction to $\mathfrak{q}$, viz. $P_\mathfrak{q}(Y + Q) = Q$ for $Q \in \mathfrak{q}, Y \in \mathfrak{h}$. The map*

$$\sigma_\mathfrak{q} : h \mapsto P_\mathfrak{q} \circ \mathrm{Ad}_G(h)$$

defines a representation of H on $\mathfrak{q}$ which is equivalent to its representation σ^o on the tangent space $T_o\mathbb{M}$

In favorable cases, $\mathrm{Ad}_G(h), h \in H$ will map $\mathfrak{q}$ into itself. In this case the projector can be omitted. We shall also have occasion to consider cases where the projector cannot be omitted.

Proof of theorem 1:
According to the generalized Bruhat decomposition (1), $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h}$, i.e. we may choose $\mathfrak{q} = \mathfrak{n}^+$ in lemma 1. It follows from (7) that trivial action of $\mathfrak{n}^-$ on $T_o\mathbb{M} \simeq \mathfrak{q}$ is equivalent to $[\mathfrak{n}^-,\mathfrak{n}^+] \subseteq \mathfrak{h}$. Applying the involutive automorphism w to both sides it follows that $[\mathfrak{n}^+,\mathfrak{n}^-] \subseteq w(\mathfrak{h})$. Since $\mathfrak{h} \cap w(\mathfrak{h}) = \mathfrak{a} \oplus \mathfrak{m}$, theorem 1 follows. *End of Proof*

Choosing $\mathfrak{q} = \mathfrak{n}^+$ in lemma 1, it simplifies because of properties (2).

Lemma 2 *Assuming trivial action of N^- on tangent space $T_o\mathbb{M} = \mathfrak{g}/\mathfrak{h} \simeq \mathfrak{q}$, the assertion of lemma 1 simplifies for the choice $\mathfrak{q} = \mathfrak{n}^+$ to*

$$\sigma_{\mathfrak{n}^+}(nl) = \mathrm{Ad}_G(l)$$

for $n \in N^-, l \in MA$.

4 Parabolic subalgebras and subgroups

Parabolic subgroups $\mathsf{B} = \mathsf{M}AN^-$ of G play a central role in the representation theory and harmonic analysis on simple non-compact Lie groups G [13]. They are used to construct induced representations of G on G/B, of which $\mathbb{M}$ is a covering space. These representations are induced by finite dimensional representations of B which are trivial on N^-. It is known that all irreducible representations of G are subrepresentations of such induced representations. The

unitary positive energy representations of the conformal group in 4 dimensions were constructed in this way in [14]. In cases of interest here, neither M nor B are connected. We will need their identity components M and $H = MAN^-$, and their Lie algebras.

Nonisomorphic parabolic subgroups will be labelled by Θ as explained below. Our assumptions will require a special choice of Θ, and the parabolic subgroup mentioned in section 2 correspond to this choice.

4.1 Minimal Parabolic Subalgebras

Remember the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the Lie algebra of G. Let $\mathfrak{a}_\mathfrak{p}$ be a subspace of $\mathfrak{p}$ which is maximal subject to the condition that its elements commute. $\mathfrak{a}_\mathfrak{p}$ may be extended to a Cartan subalgebra $\mathfrak{a}_\mathfrak{k} \oplus \mathfrak{a}_\mathfrak{p}$ of $\mathfrak{g}$ with $\mathfrak{a}_\mathfrak{k} \subset \mathfrak{k}$. This is called a maximally non-compact Cartan subalgebra. Consider roots $\alpha \in \Phi$ for this Cartan subalgebra. They are linear maps from the Cartan subalgebra $\mathfrak{a}_\mathfrak{k} \oplus \mathfrak{a}_\mathfrak{p}$ to $\mathbb{C}$ and as such may be restricted to $\mathfrak{a}_\mathfrak{p}$. The resulting linear maps λ on $\mathfrak{a}_\mathfrak{p}$ are called restricted roots and form the restricted root system Σ.

The root spaces $\mathfrak{g}^\alpha$ for roots $\alpha \in \Phi$ are one-dimensional subspaces of the complexification $(\mathfrak{g})_\mathbb{C}$ of $\mathfrak{g}$ and obey

$$\left[\mathfrak{g}^\alpha, \mathfrak{g}^\beta\right] = \mathfrak{g}^{\alpha+\beta} \tag{8}$$

as an equality ([15], Lemma 3.5.10), with $\mathfrak{g}^{\alpha+\beta} = 0$ if $\alpha + \beta$ is not a root.

The root spaces may be composed to restricted root spaces $(\mathfrak{g}^\lambda)_\mathbb{C} = \sum \mathfrak{g}^\alpha$, where the sum is over all $\alpha \in \Phi$ which restrict to λ on $\mathfrak{a}_\mathfrak{p}$. Consider their real subspaces $\mathfrak{g}^\lambda = (\mathfrak{g}^\lambda)_\mathbb{C} \cap \mathfrak{g}$. It follows from (8) that

$$\left[\mathfrak{g}^\lambda, \mathfrak{g}^\mu\right] \neq 0$$

if λ, μ and $\lambda + \mu$ are restricted roots.

The restricted roots may be divided into positive and negative roots, $\Sigma = \Sigma^+ \cup \Sigma^-$, and among the positive restricted roots a set of simple restricted roots λ_i, $i = 0, ...l$ is singled out in the usual way. They may be regarded as restrictions of the simple roots α_i of Φ.

Parabolic subgroups are conjugate to standard parabolic subgroups; what is standard depends on the choice of $\mathfrak{a}_\mathfrak{p}$ and the division into positive and negative roots. The standard parabolic subgroups B_Θ are classified by proper subsets Θ of the set of simple restricted roots λ_i. Their defining feature is that they all contain the minimal standard parabolic subgroup B_0 , which corresponds to the

empty set $\Theta = 0$. Its Lie algebra $\mathfrak{b}_0$ is as follows. One defines $\mathfrak{n}_0^- = \sum_{\lambda\in\Sigma^-} \mathfrak{g}^\lambda$. This nilpotent subalgebra appears as a summand in the Iwasawa decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_\mathfrak{p} + \mathfrak{n}_0^- \tag{9}$$

One defines $\mathfrak{m}_0$ as the commutant of $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{k}$. Then the minimal parabolic subalgebra is defined by

$$\mathfrak{b}_0 = \mathfrak{m}_0 + \mathfrak{a}_\mathfrak{p} + \mathfrak{n}_0^-$$

and appears as a summand in the Bruhat decomposition

$$\mathfrak{g} = \mathfrak{n}_0^+ + \mathfrak{m}_0 + \mathfrak{a}_\mathfrak{p} + \mathfrak{n}_0^- \ ,$$

with $\mathfrak{n}_0^+ = \sum_{\lambda\in\Sigma^+} \mathfrak{g}^\lambda$.

4.2 Gradings and Standard Parabolic Subalgebras

It will be convenient for our purposes to introduce the other standard parabolic subgroups together with gradations of $\mathfrak{g}$.

First recall the following definitions (e.g. [16, 17]): A *gradation* of a Lie algebra $\mathfrak{g}$ by an abelian semi-group S is a vector space decomposition

$$\mathfrak{g} = \bigoplus_{j\in S} \mathfrak{g}^{(j)} \quad \text{such that} \quad \left[\mathfrak{g}^{(j)},\mathfrak{g}^{(k)}\right] = \mathfrak{g}^{(j+k)} \ .$$

For symmetric spaces $\mathbb{Z}_2$-gradations play an important role. They are in one-to-one correspondence with involutive authomorphisms of $\mathfrak{g}$. A $\mathbb{Z}$-gradation with the additional property $\mathfrak{g}^{(j)} = \{0\}$ for $|j| > n$ is called a $(2n+1)$-*grading* if n is the smallest such integer – the number of subspaces $\mathfrak{g}^{(j)}$ which are non-zero is $(2n+1)$.

For simplicity we assume temporarily that the zero vector is included in Σ, and the root space $\mathfrak{g}^{(0)}$ corresponding to it is the centralizer $\mathfrak{m}_0 \oplus \mathfrak{a}_\mathfrak{p}$ of $\mathfrak{a}_\mathfrak{p}$.

Given the subset Θ of all simple restricted roots, a $(2n+1)$-grading is obtained as follows.

Fix an ordering of the simple restricted roots such that $\Theta = \{\lambda_{m+1}, \ldots, \lambda_l\}$ while the remaining simple restricted roots are $\lambda_1, \ldots, \lambda_m$. For a restricted root $\lambda = \sum_{i=1}^l \alpha^i \lambda_i$ define the *level* to be $\ell_\Theta(\lambda) = \sum_{i=1}^m \alpha^i$ which is an integer. A $(2n+1)$-grading of $\mathfrak{g}$ is now defined by

$$\mathfrak{g}^{(j)} = \bigoplus_{\ell_\Theta(\lambda)=j} \mathfrak{g}^\lambda \ .$$

From $\left[\mathfrak{g}^{\lambda},\mathfrak{g}^{\mu}\right] \subset \mathfrak{g}^{\lambda+\mu}$ and $\ell_\Theta(\lambda+\mu) = \ell_\Theta(\lambda) + \ell_\Theta(\mu)$ it is obvious that this really defines an $(2n+1)$-grading for some n.

Introducing the subalgebras

$$\mathfrak{n}_\Theta^{\pm} = \bigoplus_{j=\pm 1}^{\pm n} \mathfrak{g}^{(j)} , \qquad \mathfrak{l}_\Theta = \mathfrak{g}^{(0)} . \tag{10}$$

the parabolic subalgebra is $\mathfrak{b}_\Theta = \mathfrak{l}_\Theta + \mathfrak{n}_\Theta^-$.

We state a further decomposition. The socalled Levi subalgebra $\mathfrak{l}_\Theta$ contains the abelian subalgebra $\mathfrak{a}_\Theta$. It is the subspace of $\mathfrak{a}_\mathfrak{p}$ on which all elements of Θ are zero. The Levi subalgebra $\mathfrak{l}_\Theta$ possesses an alternative characterization as the commutant of $\mathfrak{a}_\Theta$ in $\mathfrak{g}$. It may be further decomposed as orthogonal sum

$$\mathfrak{l}_\Theta = \mathfrak{a}_\Theta \oplus \mathfrak{m}_\Theta ,$$

orthogonal with respect to the Killing form $B(\cdot,\cdot)$. The root system of the complexification of $\mathfrak{m}_\Theta$ consists of the roots α of $\mathfrak{g}$ with $\ell_\Theta(r(\alpha)) = 0$, where r is the restriction of roots.

In total, the parabolic subalgebra becomes

$$\mathfrak{b}_\Theta = \mathfrak{m}_\Theta \oplus \mathfrak{a}_\Theta \oplus \mathfrak{n}_\Theta^-$$

The Bruhat decomposition generalizes to $\mathfrak{g} = \mathfrak{n}_\Theta^+ \oplus \mathfrak{m}_\Theta \oplus \mathfrak{a}_\Theta \oplus \mathfrak{n}_\Theta^-$. For any Θ, a maximally non-compact Cartan subalgebra of $\mathfrak{l}_\Theta$ is also a maximaly non-compact Cartan subalgebra of $\mathfrak{g}$.

4.3 A Three-grading

To prove proposition 1 below, we need some properties of the gradation (10). We extend the definition of the level function $\ell_\Theta(\cdot)$ on restricted roots to roots by $\ell_\Theta(\alpha) = \ell_\Theta(r(\alpha))$ where r is the restriction map.

Lemma 3 *All the subspaces* $\mathfrak{g}^{(j)} \neq 0$, *for* $j = -n...+n$ *and* $\left[\mathfrak{g}^{(j)},\mathfrak{g}^{(-1)}\right] \neq 0$ *for* $j = 1...n$

Proof: For each root $\alpha \in \Phi$ the root space $\mathfrak{g}^\alpha$ is nonzero by definition. For any positive root α there is a simple root α_i such that $\alpha - \alpha_i$ is a positive root or zero. Now there are two cases. If $\alpha_i \in \Theta$, then $\ell_\Theta(\alpha - \alpha_i) = \ell_\Theta(\alpha)$, and otherwise $\ell_\Theta(\alpha - \alpha_i) = \ell_\Theta(\alpha) - 1$. In the first case there is another simple root α_j such that $\alpha - \alpha_i - \alpha_j$ is a simple root or zero, and the same two cases are

possible for $\alpha - \alpha_i - \alpha_j$. This procedure is repeated until a positive or zero root β is found such that

$$\ell_\Theta(\beta) = \ell_\Theta(\alpha) - 1 \qquad \text{and} \qquad \ell_\Theta(\beta + \alpha_k) = \ell_\Theta(\alpha)$$

where α_k is the last simple root found. It obeys $\ell_\Theta(\alpha_k) = 1$ This happens in a finite number of steps because the positive roots are linear combinations of simple roots with non-negative coefficients, and $\ell_\Theta(0) = 0$. Thus for each positive root α there is such a root β.

Suppose $\mathfrak{g}^\alpha \subset \mathfrak{g}^{(j)}$ with $j > 0$. Then, by (8), $0 \neq \mathfrak{g}^\beta = \left[\mathfrak{g}^{\beta+\alpha_k}, \mathfrak{g}^{\alpha_k}\right] \subset \left[\mathfrak{g}^{(j)}, \mathfrak{g}^{(-1)}\right]$. This proves the second statement of the lemma and shows at the same time that $\mathfrak{g}^{(j)} \neq 0$ implies $\mathfrak{g}^{(j-1)} \neq 0$, hence $\mathfrak{g}^{(j)} \neq 0$ for $j = n...1$. With any root α, there is also a root $-\alpha$ of level $-\ell_\Theta(\alpha)$. Therefore $\mathfrak{g}^{(-j)} \neq 0$ if $\mathfrak{g}^{(j)} \neq 0$. This completes the proof of the lemma. *End of Proof*

In each restricted root system there is a unique *highest* root $\lambda_{\max}$. It is defined by the property that $\lambda_{\max} + \lambda_i$ is not a root for any simple restricted root λ_i. The coefficients a_i in the expansion $\lambda_{\max} = \sum_{i=1}^{l} a_i \lambda_i$ are called *Coxeter labels*.

Proposition 1 *If $\mathfrak{g}$ is simple, Θ non-trivial, and $\left[\mathfrak{n}_\Theta^+, \mathfrak{n}_\Theta^-\right] \subseteq \mathfrak{l}_\Theta$ then*

1. *the generalized Bruhat decomposition $\mathfrak{g} = \mathfrak{n}_\Theta^+ \oplus \mathfrak{l}_\Theta \oplus \mathfrak{n}_\Theta^-$ is a three-grading of $\mathfrak{g}$ with grading given by the level function $\ell_\Theta(\cdot)$.*
2. *$\mathfrak{n}_\Theta^+$ and $\mathfrak{n}_\Theta^-$ are abelian*
3. *There is exactly one restricted root λ_e not in Θ. Its Coxeter label is 1. If the complexification $(\mathfrak{g})_{\mathbb{C}}$ of $\mathfrak{g}$ is simple, there is exactly one simple root α_e of level $\ell_\Theta(\alpha_e) = 1$; its restriction is λ_e*
4. *$\mathfrak{a}_\Theta$ is 1-dimensional.*

Proof: 1. The level function $\ell_\Theta(\cdot)$ furnishes a $(2n+1)$-grading for some n. Since $\mathfrak{n}_\Theta^\pm$ are not empty, $n > 0$. Since $\mathfrak{g}^{(j)} \neq 0$ for $-n \leq j \leq n$, $n = 1$ follows from $\mathfrak{g}^{(2)} = 0$.

Suppose $\mathfrak{g}^{(2)} \neq 0$. Then $\mathfrak{g}^{(2)} \subset \mathfrak{n}_\Theta^+$ and $0 \neq \mathfrak{g}^{(-1)} \subset \mathfrak{n}_\Theta^-$, therefore $\left[\mathfrak{n}_\Theta^+, \mathfrak{n}_\Theta^-\right] \supset \left[\mathfrak{g}^{(2)}, \mathfrak{g}^{(-1)}\right]$. But $\left[\mathfrak{g}^{(2)}, \mathfrak{g}^{(-1)}\right] \neq 0$ by lemma 3. On the other hand, $\left[\mathfrak{g}^{(2)}, \mathfrak{g}^{(-1)}\right] \subset \mathfrak{g}^{(1)}$ by the defining property of a gradation. This contradicts the assumption $\left[\mathfrak{n}_\Theta^+, \mathfrak{n}_\Theta^-\right] \subset \mathfrak{l}_\Theta = \mathfrak{g}^{(0)}$. Therefore $\mathfrak{g}^{(2)} = 0$ and $\mathfrak{g} = \mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ is a three-grading.

2. By the three-grading property, $\left[\mathfrak{g}^{(1)},\mathfrak{g}^{(1)}\right] = 0$ and $\mathfrak{n}^+_\Theta = \mathfrak{g}^{(1)}$, implying that $\mathfrak{n}^+_\Theta$ is abelian, and similarly for $\mathfrak{n}^-_\Theta$.

3.As mentioned above, the Coxeter labels a_i are the coefficients in the decomposition of the highest restricted root in simple restricted roots λ_i, viz. $\lambda_{\max} = \sum_{i=1}^{l} a_i\lambda_i$. Order the restricted roots as described above, so that $\lambda_1,\ldots,\lambda_m$ are not in Θ. Then $\ell_\Theta(\lambda_{\max}) = \sum_{i=1}^{m} a_i$. Since the restricted root system is irreducible, the Coxeter labels a_i are non-zero positive integers. Also, $m > 0$ because Θ is a proper subset of the set of all simple restricted roots. Therefore, $|\ell_\Theta(\lambda_{\max})| \leq 1$ is only possible when $m = 1$ and λ_1 is the only simple restricted root not in Θ, and it has Coxeter label $a_1 = 1$.

If $(\mathfrak{g})_{\mathbb{C}}$ is simple, it has a connected Dynkin diagram. It follows that $\beta = \sum \alpha_i$ (sum over all simple roots of $\mathfrak{g}_{\mathbb{C}}$) is a root. (This follows from property ii) in the proof of lemma 4). If there were more than one simple root of non-zero level, β would have level greater one, contradicting the 3-grading.

4. As a real vector space, $\mathfrak{a}_\mathfrak{p}$ is the dual of the real linear space spanned by the simple restricted roots. By 3), $\mathfrak{a}_\Theta \subset \mathfrak{a}_\mathfrak{p}$ is the subspace on which all but one of the simple restricted roots vanish. Therefore it is 1-dimensional.

End of Proof

Proof of theorem 2: Theorem 2 is merely a restatement of assertions 2. and 4. of the above proposition.

End of Proof

4.4 An Irreducible Representation

Let A_Θ be the connected subgroup of G with Lie algebra $\mathfrak{a}_\Theta$. For the analysis of the existence of an infinitesimal causal structure in section 5.1 the following result is needed.

Proposition 2 *If the complexification $(\mathfrak{g})_{\mathbb{C}}$ of $\mathfrak{g}$ is simple and $\left[\mathfrak{n}^+_\Theta,\mathfrak{n}^-_\Theta\right] \subseteq \mathfrak{l}_\Theta$ then $\mathfrak{n}^+_\Theta$ carries an irreducible representation of $\mathfrak{m}_\Theta$ and elements of A_Θ act on $\mathfrak{n}^+_\Theta$ as multiplication by positive constants.*

Proof:

$\mathfrak{n}^+_\Theta$ carries a representation of $\mathfrak{l}_\Theta$ as a consequence of the 3-grading. Because $\mathfrak{l}_\Theta$ is reductive, this representation is fully reducible. By Schurs lemma, the assertion of the lemma is equivalent to the statement that this representation is irreducible. Elements of A_Θ act as multiplication with *real* constants because $\mathrm{ad}(X)$ is hermitean for elements $X \in \mathfrak{p}$. The real constants are *positive* because A_Θ is connected.

The maximally non-compact Cartan subalgebra $\mathfrak{a}_\mathfrak{k} \oplus \mathfrak{a}_\mathfrak{p}$ of $\mathfrak{g}$ is also a Cartan subalgebra of $\mathfrak{l}_\Theta$. The weights of the representation of $\mathfrak{l}_\Theta$ on $\mathfrak{n}_\Theta^+$ are therefore the roots α of $\mathfrak{g}$ whose root spaces span $\mathfrak{n}_\Theta^+$, i.e. the roots of level one. (This can be seen more explicitly in [18].) Because root spaces are one-dimensional, all these weights have multiplicity one.

Every irreducible subrepresentation has a lowest weight vector and is uniquely determined by it. Because the weights have multiplicity 1, two different lowest weight vectors have different weight. Irreducibility is proven by showing that the lowest weight μ is unique. As a lowest weight, it has the property that $\mu - \alpha_i$ is not a weight for any simple root α_i of $\mathfrak{m}_\Theta$, and therefore not a root of $\mathfrak{g}$. By definition, the roots α_i of $\mathfrak{m}_\Theta$ are the roots of $\mathfrak{g}$ of level 0.

By proposition 1, there is a unique simple root α_e of $\mathfrak{g}$ of level 1. It is a lowest weight by the properties of simple roots. By lemma 4 below, any other root μ of level 1 can be obtained from α_e by adding simple roots of $\mathfrak{g}$ one by one, in such a way that a root is obtained at each stage. Simple roots of level > 0 cannot be added in this process, because there are no roots of level ≥ 2, by the three-grading. Therefore the added roots must be roots α_i of $\mathfrak{m}_\Theta$. It follows that $\mu - \alpha_i$ is a root for some simple root α_i of $\mathfrak{m}_\Theta$, except for $\mu = \alpha_e$. This proves uniqueness of the lowest weight, hence irreducibility. *End of Proof*

Lemma 4 *Let $\alpha = \sum \alpha^i \alpha_i$ be the expansion of a root α into simple roots. All positive roots α in Φ with the property that the coefficient α^k is non-zero can be constructed by successively adding simple roots starting with α_k such that every intermediate sum is a root.*

Proof: From the well known construction of the root systems [19, 18] it is known that each positive root can be obtained by successively adding simple roots such that each intermediate sum is also a positive root. It remains to be shown that this construction can be started with α_k.

For a positive root α define T_α to be the subdiagram of the Dynkin diagram of Φ consisting of the simple roots α_j for which $\alpha^j \neq 0$ in the expansion. The following three facts are needed:

(i) The subdiagram T_α for a positive root α is a connected Dynkin diagram.

(ii) If α is a positive root and α_i a simple root for which $\alpha^i = 0$ and α_i is connected to the subdiagram T_α within the Dynkin diagram of Φ then $\alpha + \alpha_i$ is a positive root.

(iii) Assume that in the construction a sequence like $(\alpha + \alpha_i) + \alpha_k$ appears, where α has coefficients $\alpha^i \neq 0$, $\alpha^k = 0$, and $\alpha + \alpha_i$ is a root. Then $\alpha + \alpha_k$ is also a root, and the construction may be reorderd $(\alpha + \alpha_k) + \alpha_i$.

We will first show that the statement of lemma 4 follows from (i), (ii), and (iii): When using (iii) repeatedly to reorder a given construction of some positive root α, in the resulting construction the root $\alpha' = \sum 1\ \alpha_i$ appears, where summation is only over simple roots α_i in T_α. Because of (i) and (ii) this root α' may be constructed starting from α_k and adding all the other simple roots in T_α by moving along the connected graph.

To show (i), (ii), and (iii) the following fact is needed: Given any two roots α_i and α the *α_i-string of roots through α* is defined to be the set of all roots of the form $\alpha + k\alpha_i$. Theorem 13.5.IX in [19] states that $\alpha + k\alpha_i$ is a root for all $-p \leq k \leq q$ for some positive or zero integers p and q. For the α_i-string of roots through α these integers are constrained by

$$q = p - A_{\alpha_i,\alpha}\ .$$

where $A_{\alpha_i,\alpha}$ are Cartan integers. Statement (ii) follows since $p = 0$ and $A_{\alpha_i,\alpha}$ is negative in that situation. If on the other hand α_i is not connected to T_α, then $A_{\alpha_i,\alpha}$ is zero and consequently $\alpha + \alpha_i$ is no root. This justifies (i). Statement (iii) can be seen because $\alpha + \alpha_k$ is a root, which follows from (ii).

End of Proof

4.5 Space-time as a homogeneous space for K

For the analysis of global causal structure, another decomposition of the Lie algebras of parabolic subgroups will be needed.

Remember that $K\!/\!\mathsf{Z}$ is the maximal compact subgroup of $G\!/\!\mathsf{Z}$, where Z is the center of G which is infinite for groups G which are of interest to us.

We begin by writing down the group version of the above Lie algebra decompositions. Let M_Θ, A_Θ, and N_Θ^- be the connected subgroups of G whose Lie algebras are $\mathfrak{m}_\Theta$, $\mathfrak{a}_\Theta$, and $\mathfrak{n}_\Theta^-$, respectively. Let $\mathsf{M}_\Theta(K)$ be the centralizer of $\mathfrak{a}_\Theta$ in K. With the definition $\mathsf{M}_\Theta = \mathsf{M}_\Theta(K)M_\Theta$, the parabolic subgroup is $\mathsf{B}_\Theta = \mathsf{M}_\Theta A_\Theta N_\Theta^-$. By its definition, M_Θ contains the entire center Z of G. In cases of interest to us, it follows that M_Θ has infinitely many connected components, while M_Θ is the identity component of M_Θ. The identity component $H = B_\Theta$ of the parabolic subgroup is

$$H = B_\Theta = M_\Theta A_\Theta N_\Theta^-\ . \tag{11}$$

We will be interested in a particular Θ. The groups introduced in the introduction are $A = A_\Theta$, $N^- = N_\Theta^-$ and $M = M_\Theta$ for this particular Θ, and M_Θ is the generalized Lorentz group. It is connected, by its definition. Its center is finite if it is semi-simple.

Consider the Iwasawa decomposition of the generalized Lorentz group

$$M_\Theta = M_\Theta(K)A(\Theta)N^-(\Theta), \quad M_\Theta(K) = M_\Theta \cap K \ .$$

For the conformal group $SO(2,4)$, $M_\Theta(K)$ is the group of rotations, $A(\Theta)$ consists of the Lorentz boosts in z-direction, and $N^-(\Theta)$ is a two dimensional abelian subgroup of the little group of a light-like vector pointing in z-direction. It holds $A_\mathfrak{p} = A(\Theta)A_\Theta$, $N_0^\pm = N^\pm(\Theta)N_\Theta^\pm$.

It follows from the definitions that the groups

$$A_\mathfrak{p} = A(\Theta)A_\Theta \text{ and } N_0^- = N^-(\Theta)N_\Theta^- \tag{12}$$

agree with the subgroups $A_\mathfrak{p}$ and N_0^- in the Iwasawa decomposition $G = KA_\mathfrak{p}N_0^-$ of G, whose Lie algebra version is (9). Comparing with the definition (11) of H, we find that

$$\mathbb{M} = {}^{G}\!/_{H} = {}^{K}\!/_{M_\Theta(K)} \ . \tag{13}$$

This is the desired exposition of $\mathbb{M}$ as a homogeneous space for K. It is noncompact because of the infinite center Z of K.

Using this result, we obtain another realization of the tangent space of $\mathbb{M}$ at $\mathbf{o}$. Let $\mathfrak{m}_\Theta(K)$ be the Lie algebra of $M_\Theta(K)$. Given a decomposition of $\mathfrak{k}$ as a direct sum $\mathfrak{k} = \mathfrak{m}_\Theta(K) \oplus \mathfrak{q}_\mathfrak{k}$,

$$T_\mathbf{o}\mathbb{M} = {}^{\mathfrak{k}}\!/_{\mathfrak{m}_\Theta(K)} \simeq \mathfrak{q}_\mathfrak{k} \ , \tag{14}$$

Inserting the decomposition of $\mathfrak{k}$ in the Iwasawa decomposition (9), and noting the Lie algebra versions of decompositions (12), we see that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}_\mathfrak{k}$, where $\mathfrak{h}$ is the Lie algebra of $H = B_\Theta$. Therefore $\mathfrak{q}_\mathfrak{k}$ can be used as a choice of $\mathfrak{q}$ in lemma 1, which asserts that $\mathfrak{q}_\mathfrak{k}$ carries a representation of H. It is equivalent to the representation on $\mathfrak{n}^+$ and on $T_\mathbf{o}\mathbb{M}$. This will be used in the discussion of the causal structure.

Comparing with the generalized Bruhat decomposition in proposition 1, we see that a possible choice for $\mathfrak{q}_\mathfrak{k}$ is

$$\mathfrak{q}_\mathfrak{k} = \big(\mathfrak{n}^-(\Theta) \oplus \mathfrak{n}^+(\Theta)\big) \cap \mathfrak{k} \ .$$

Lemma 5 *If the generator iH_0 of the center $\mathfrak{z}_\mathfrak{k}$ of $\mathfrak{k}$ is not in $\mathfrak{m}_\Theta(K)$ then $\mathfrak{n}_\Theta^+$ contains a vector which is invariant under elements of $M_\Theta(K)$ ("rotations").*

Proof: $\mathfrak{n}_\Theta^+$ and $\mathfrak{q}_\mathfrak{k}$ carry equivalent representations of $\mathfrak{l}_\Theta$. But $\mathfrak{q}_\mathfrak{k}$ can be chosen such that $\mathfrak{z}_\mathfrak{k} \subset \mathfrak{q}_\mathfrak{k}$ if the generator iH_0 of $\mathfrak{z}_\mathfrak{k}$ is not in $\mathfrak{m}_\Theta(K)$. It commutes with $\mathfrak{m}_\Theta(K)$ because it is in the center of $\mathfrak{k}$. By lemma 1, it therefore defines a $M_\Theta(K)$-invariant vector in $\mathfrak{q}_\mathfrak{k}$. *End of Proof*

5 Causal Structure

The global causal structure needed can be described infinitesimally by a specification of tangent vectors which non-spacelike curves are allowed to have. These lie in cones which are non-trivial in a certain way described below and invariant under the assumed symmetry. We will first develop criteria for the existence of such an infinitesimal causal structure.

5.1 Infinitesimal Causal Structure

We will summarize a few definitions and facts about cones and cone-fields in homogenous spaces [20, 12]

A *cone* C is a subset of a vector space which contains λv for all $v \in C$ and $\lambda > 0$. A cone is *proper* if $C \cap -C = \{0\}$ and *generating* if its linear span is V. A convex, proper, generating, and closed cone is called *regular* cone.

An *infinitesimal causal structure* or *cone-field* $\mathcal{C}$ on a d-dimensional manifold $\mathbb{M}$ is an assignment of a regular cone $\mathcal{C}(x)$ in $T_x\mathbb{M}$ to each $x \in \mathbb{M}$ such that there are an open covering $\{U_i\}_{i\in I}$ of $\mathbb{M}$, a cone $C \in \mathbb{R}^d$, and (smooth or analytic) maps $\varphi_i : U_i \times \mathbb{R}^d \to T\mathbb{M}$ with $\varphi_i(x, C) = \mathcal{C}(x)$.

As before, let the Lie group G act differentiably on $\mathbb{M}$ and call the action ρ. Let the derivative of $\rho(g)$ at x be $\sigma^x(g) = T_x(\rho(g)) : T_x\mathbb{M} \to T_{\rho(g)x}\mathbb{M}$.

A causal structure is *G-invariant* if for each $g \in G$ and $x \in \mathbb{M}$ the derivative $\sigma^x(g)$ maps the cone-field into a itself:

$$\sigma^x(g)\mathcal{C}(x) \subset \mathcal{C}(\rho(g)x)\,.$$

If the action of G is transitive, so that $\mathbb{M} = {}^{G}\!/_{H}$, there is a bijection from the set of regular H-invariant cones in $T_{\mathbf{o}}\mathbb{M}$ with $\mathbf{o} = [\mathbf{e}]_H$ onto the set of G-invariant infinitesimal causal structures on $\mathbb{M}$ ([12]):

$$C \mapsto (\mathcal{C} : [g]_H \mapsto \sigma^{\mathbf{o}}(g)C)\,. \tag{15}$$

Therefore the action $\sigma^{\mathbf{o}}$ of H on $T_{\mathbf{o}}\mathbb{M}$ and the cones invariant under it have to be studied. We will drop the "$\mathbf{o}$" in the notation were appropriate: $\sigma(g) = \sigma^{\mathbf{o}}(g)$.

If H is trivial, i.e. if $\mathbb{M} \cong G$, then the H-invariant regular cones in $T_{\mathbf{e}}G$ are just all regular cones. Therefore, there always exist G-invariant infinitesimal causal structures on G.

Some general results on invariant cones in arbitrary vector spaces follow. They can be found in [21, 22, 20] in a slightly different form.

Let M be any connected semi-simple real Lie group and V a real finite-dimensional M-module. Define U to be a maximal subgroup of M such that the image in $\mathrm{Gl}(V)$ is compact. Note that this is not necessarily the compact factor of a Cartan decomposition of M. This notation is fixed for the following three lemmata.

Lemma 6 *There is a U-stable $v_U \in V$, $v_U \neq 0$ if and only if there exists a M-invariant closed convex cone C in V with $C \neq C \cap -C$. In this case there will always be a proper cone of this type.*

Let us explicitly state that the lemma holds true if $V = \mathbb{R}$ and M and U are trivial. In this case, the half lines $\mathbb{R}_{\pm}$ are invariant cones.

Proof: [21] For simplicity set M and U equal to their images in $\mathrm{Gl}(V)$. Let C be a closed convex cone in V. The following is a standard result about convexity (\$17.1 and \$16.3 in [23]). Because $C \neq C \cap -C$ there exists a linear functional f in V which is non-negative and non-trivial on C. Select $v \in C$ with $f(v) > 0$, then

$$v_U = \int_U k \cdot v \; d\mu(k)$$

is U-invariant and $f(v_U) > 0$. It follows from linearity that $v_U \neq 0$.

Let on the other hand v_U be U-invariant. Choose a scalar product $(,)$ in the complexification $(V)_{\mathbb{C}}$ of V which is left invariant by the compact form of the complexification $(M)_{\mathbb{C}}$ of M. Let $\mathfrak{m} = \mathfrak{u} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra $\mathfrak{m}$ of M. $X \in \mathfrak{p}$ is hermitian and e^X positive-definite hermitian. Because of the invariance of v_U and the Cartan decomposition $M = PU$, the M-orbit of v_U is equal to the P-orbit. For u, w in this orbit we have $(u, w) = (e^X v_U, e^Y v_U) = (e^Y e^X v_U, v_U)$. With the Cartan decomposition $e^Y e^X = e^Z k$, $k \in U$ and the positive-definiteness of e^Z it is $(u, w) > 0$. This extends by linearity to the convex cone generated by this orbit. This convex cone is M-invariant, and by continuity we have $(u, w) \geq 0$ for u, w in the closure C. If v and $-v$ are in C, $(v, -v) \geq 0$ i.e. $v = 0$, thus C is proper. *End of Proof*

Irreducibility will be needed for the cones to be regular:

Lemma 7 *If V is an irreducible M-module and contains a non-trivial M-invariant convex closed cone C, C is regular.*

Proof: Both the linear span of C and the subspace $C \cap -C$ are invariant linear subspaces of V which can only be V or $\{0\}$. If the linear span of C is $\{0\}$ then C is trivial. If $C \cap -C$ is V then C is V. *End of Proof*

The irreducibility will be important for global causality, because the following

result of [22] will be needed:

Lemma 8 *If V is irreducible and contains a non-zero U-stable vector v_U then v_U is unique up to scalar multiplication with respect to this property.*

5.2 Generalized Conformal Symmetry

Referring to our standing assumptions 1 and 2 we will now specialize to the case where H is a parabolic subgroup of $\mathfrak{g}$, and the associated generalized Bruhat decomposition (1) satisfies eq. (3). We are interested in simple $\mathfrak{g}$, and we may assume that $(\mathfrak{g})_{\mathbb{C}}$ is simple, because this will be implied for Lie algebras whose maximal compact subalgebra has nontrivial center.

We wish to apply the results of the previous subsection to the action of the generalized Lorentz group $M = M_\Theta$ on $V = \mathfrak{n}^+_\Theta \simeq T_o\mathbb{M}$ where Θ are particular subsets of the set of restricted roots such that the hypotheses of proposition 2 are fulfilled. Therefore the result of this proposition becomes available, and we may conclude that any M-invariant cone in V will also be invariant under $H = MAN^-$, because N^- acts trivially by hypothesis, and elements of A act by multiplication with positive constants, and such multiplication carries any cone into itself by definition.

Using lemma 6 and lemma 7 to ensure M-invariance, we obtain the following corollary of proposition 2.

Corollary 1 *Under the same hypotheses, the following holds true. If M is semi-simple there is a M-invariant regular cone in $\mathfrak{n}^+_\Theta$ if and only if there is a U-stable vector, where $U = M \cap K = M_\Theta(K)$. If and only if this is the case, there exists an G-invariant infinitesimal causal structure on $\mathbb{M}$.*

Proof of first part of theorem 3: This follows from corollary 1 and theorem 1 which asserts that $\mathfrak{n}^+_\Theta \simeq T_o\mathbb{M}$ for Θ such that $H = M_\Theta A_\Theta N^-_\Theta$. *End of Proof*

The question arises what is the U-stable vector and whether this infinitesimal causal structure is a global causal structure in one of the senses defined in [24]. In the next section it will be shown that the diffeomorphism $\mathbb{M} = \Sigma \times \mathbb{R}$ which is required by global hyperbolicity will require that $\mathfrak{k}$ has nontrivial center $\mathfrak{z}_\mathfrak{k}$ not contained in the Lie algebra $\mathfrak{m}$ of M and the Lie subgroup of G associated with $\mathfrak{z}_\mathfrak{k}$ must be isomorphic to $\mathbb{R}$. Conversely, this property yields a U-stable vector and implies global hyperbolicity.

5.3 Global Causality

We recall a few definitions: A differentiable curve $\lambda: I \to \mathbb{M}$, where $I \subset \mathbb{R}$ is an interval, is (future-directed) *non-spacelike* with respect to an infinitesimal causal structure $\mathcal{C}$ if the tangent vector at $\lambda(\tau)$ is in the cone $\mathcal{C}(\lambda(\tau)) \setminus \{0\}$ for all $\tau \in I$. For $p \in \mathbb{M}$ the *causal future* resp. *past* $J^{\pm}(p)$ of p is the set of points $q \in \mathbb{M}$ with a non-spacelike curve λ such that $\lambda(0) = p$ and $\lambda(\pm\tau) = q$ with $\tau \geq 0$.

An infinitesimal causal structure $p \mapsto \mathcal{C}(p)$ is *strongly causal*, if every neighbourhood of a point $p \in \mathbb{M}$ contains a neighbourhood of p which no non-spacelike curve intersects more than once, i.e. the neighbourhood is mapped to an interval by λ^{-1}. An infinitesimal causal structure is *globally hyperbolic* if it is strongly causal and if for $p, q \in \mathbb{M}$ the set $J^+(p) \cap J^-(q)$ is compact. A homogeneous space $\mathbb{M}$ of G will be called globally hyperbolic if it admits a G-invariant infinitesimal causal structure which is globally hyperbolic.

A set $S \in \mathbb{M}$ is called *acausal*, if for $p \in S$ no $q \in S$, $p \neq q$ is in $J^+(p)$.

The *edge* of an acausal set $S \in \mathbb{M}$ is the set of all $p \in \overline{S}$ such that in every neighbourhood of p there are points q in the future and r in the past of p which can be joined by a timelike curve within the neighbourhood without intersecting S. If S is a submanifold without boundary, the edge is empty, because the boundary clearly contains the edge.

A set $\Sigma \in \mathbb{M}$ is called a *Cauchy surface* for $\mathbb{M}$, if it is acausal, has no edge, and each inextendible non-spacelike curve in $\mathbb{M}$ intersects Σ. A criterion for global hyperbolicity is given by the

Proposition 3 *[24]* $\mathbb{M}$ *is globally hyperbolic if and only if it has a Cauchy surface. Then* $\mathbb{M}$ *is homeomorphic to* $\mathbb{R} \times \Sigma$*, where* Σ *is a manifold of codimension 1 and each* $\{a\} \times \Sigma$*,* $a \in \mathbb{R}$ *is a Cauchy surface for* $\mathbb{M}$*.*

Now we turn to sufficient conditions for the existence of a global causal structure. The proof of the only if part of part ii) of theorem 3 will follow from theorem 6 below by showing that its hypothesis are fulfilled. To prepare for theorem 6 we formulate and prove three lemmas.

Lemma 9

$$\mathbb{M} \simeq \mathbb{R} \times \Sigma \tag{16}$$

if and only if $\mathfrak{k}$ *has nontrivial center* $\mathfrak{z}_{\mathfrak{k}}$ *not contained in the Lie algebra* $\mathfrak{m}$ *of* $M = M_\Theta(K)$ *and the connected Lie subgroup* $Z_{\mathfrak{k}}$ *of* G *with Lie algebra* $\mathfrak{z}_{\mathfrak{k}}$ *is isomorphic to* $\mathbb{R}$*.*

In this case, $K = K_s \times Z_{\mathfrak{k}}$ *and*

$$\mathbb{M} = \Sigma \times Z_{\mathfrak{k}}, \qquad \Sigma = {}^{K_s}\!/_{M_\Theta(K)}\,. \tag{17}$$

Proof: Consider the homogenous space $\mathbb{M} = {}^{K}\!/_{M_\Theta(K)}$ from equation (13). If K is compact $\mathbb{M}$ cannot have the topology described above. However, in a Cartan decomposition of a simple Lie group $G = KP$ the "compact" factor K is non-compact or has a noncompact covering if and only if the Lie algebra $\mathfrak{k}$ of K has a non-trivial center $\mathfrak{z}_{\mathfrak{k}}$. Using several facts which can be found in [16] this can be seen as follows: Generally, $\mathfrak{k}$ is reductive and $\mathfrak{z}_{\mathfrak{k}}$ is at most one-dimensional. I.e. we have $\mathfrak{k} = \mathfrak{k}_s \oplus \mathfrak{z}_{\mathfrak{k}}$ where $\mathfrak{k}_s$ is semi-simple. The connected Lie subgroup K_s of G corresponding to $\mathfrak{k}_s$ has to be compact. However, the connected Lie subgroup $Z_{\mathfrak{k}}$ corresponding to $\mathfrak{z}_{\mathfrak{k}}$ may be non-compact and its universal covering is isomorphic to $\mathbb{R}$. If $\mathfrak{z}_{\mathfrak{k}}$ were contained in the Lie algebra $\mathfrak{m}$ of M then the noncompact factor $Z_{\mathfrak{k}}$ would cancel out in ${}^{K}\!/_{M_\Theta(K)}$ and K would be compact. *End of Proof*

Given eq.(17) the next task is to show that the quotient $\Sigma = {}^{K_s}\!/_{M_\Theta(K)}$ is a Cauchy surface.This is addressed by the following lemma.

Lemma 10 *Assume the existence of an infinitesimal causal structure with cones* $\mathcal{C}(p)$*, let* $t : \Sigma \times Z_{\mathfrak{k}} \to Z_{\mathfrak{k}}$ *be the component map. For any* $a \in Z_{\mathfrak{k}} \cong \mathbb{R}$ *the submanifold* $\Sigma \times \{a\}$ *is a Cauchy surface if the derivative of* t *along each vector in the cone* $\mathcal{C}(p)$ *for all* $p \in \mathbb{M}$ *is strictly positive (or negative for all vectors).*

Proof: Let $\lambda : I \to \mathbb{M}$ be any non-spacelike curve and choose any $\tau \in I$. Because the tangent vector $T_\tau\lambda(1)$ of λ at τ lies in the cone $\mathcal{C}(\lambda(\tau))$, it follows by the assumption in the lemma that $\frac{d}{d\tau} t \circ \lambda(\tau) > 0$ and thus $t \circ \lambda$ is a strongly monotonous function.

Thus, λ has exactly one intersection with $\Sigma \times \{t \circ \lambda(\tau)\}$. Since this is true for all non-spacelike curves, any $\Sigma \times \{a\}$ is acausal. It has no edge because it is a submanifold.

An inextendible non-spacelike curve λ has to intersect each $\Sigma \times \{a\}$ because otherwise it would approach one $\Sigma \times \{a\}$ arbitrarily closely and not have a limit point. This would mean that its tangent vector would approach the tangent bundel of $\Sigma \times \{a\}$. Since the $\mathcal{C}(p)$ are closed cones which contain this vector, the tangent space of $\Sigma \times \{a\}$ would actually intersect some $\mathcal{C}(p)$. A curve with tangent vectors in this intersection would clearly have zero derivative of t. A contradiction. *End of Proof*

Suppose that $\mathfrak{k} = \mathfrak{k}_s \oplus \mathfrak{z}_{\mathfrak{k}}$ (sum of ideals) and $\mathfrak{m}_\Theta(K) \subset \mathfrak{k}_s$. The suitably chosen

subspace $\mathfrak{q}_\mathfrak{k}$ introduced before (14), which was defined to satisfy $\mathfrak{k} = \mathfrak{m}_\Theta(K) \oplus \mathfrak{q}_\mathfrak{k}$ may then be chosen to contain $\mathfrak{z}_\mathfrak{k}$, so that $\mathfrak{k}_s = \mathfrak{m}_\Theta(K) \oplus \mathfrak{q}_s$, and $\mathfrak{q}_\mathfrak{k} = \mathfrak{q}_s \oplus \mathfrak{z}_\mathfrak{k}$.

We finally end up with the following Lie algebraic criterion:

Lemma 11 $\mathbb{M}$ *carries a G-invariant globally hyperbolic causal structure, if there is an H-invariant regular cone C in $\mathfrak{q}_\mathfrak{k}$ which has intersection $\{0\}$ with $\mathfrak{q}_s$.*

Proof: Since $\mathfrak{q}_s$ has codimension 1 in $\mathfrak{q}_\mathfrak{k}$, C lies in one halfspace with boundary $\mathfrak{q}_s$. The decomposition $\mathfrak{q}_\mathfrak{k} = \mathfrak{q}_s \oplus \mathfrak{z}_\mathfrak{k}$ corresponds via the equivalence established in lemma 1 to the decomposition of the tangent space $T_o\mathbb{M} = T_o\Sigma \oplus T_o Z_\mathfrak{k}$.

From C we obtain a G-invariant infinitesimal causal ordering of $\mathbb{M}$ via the bijection (15) between H-invariant regular cones in $T_o\mathbb{M}$ and the infinitesimal causal structures in $\mathbb{M}$. For $\mathbb{M} \ni p = [g]$ we can choose the representative g to be in K. Then $\sigma(g)$ maps $T_o Z_\mathfrak{k}$ to $T_p Z_\mathfrak{k}$ and $T_o\Sigma$ to $T_p\Sigma$. C is now mapped to a regular cone in a halfspace of $T_p\mathbb{M}$ with boundary $T_p\Sigma$ which intersects $T_p\Sigma$ only at zero. Identify $Z_\mathfrak{k}$ – and with it all $T_p Z_\mathfrak{k}$ – with $\mathbb{R}$ such that the $T_p Z_\mathfrak{k}$-component of all elements in $\sigma(g)(C)$ is positive via this identification. This is then by continuity and connectedness of G the case for all $\sigma(g)(C)$ for $g \in K$. The derivative of the function t in direction of a vector with positive $T_p Z_\mathfrak{k}$-component is clearly positive. Therefore the hypothesis of Lemma 10 is satisfied and Σ is a Cauchy surface. By proposition 3 this implies global hyperbolicity.

End of Proof

We are now in a position to formulate the criteria for the existence of a G-invariant infinitesimal causal structure on $\mathbb{M}$ that turns $\mathbb{M}$ into a globally hyperbolic manifold:

Theorem 6 *Let G be a connected simple Lie group and B_Θ the identity component of a parabolic subgroup satisfying assumptions 1 and 2. $\mathbb{M} = {}^{G}\!/_{B_\Theta}$ carries an infinitesimal causal structure such that $\mathbb{M}$ is globally hyperbolic if*

(i) *$\mathfrak{k}$ has non-trivial center $\mathfrak{z}_\mathfrak{k}$ and $Z_\mathfrak{k}$ is non-compact,*

(ii) *$\mathfrak{z}_\mathfrak{k}$ is not contained in $\mathfrak{m}_\Theta(K)$, and*

(iii) *$\mathfrak{m}_\Theta$ is semi-simple.*

If $\mathbb{M}$ is globally hyperbolic (i) and (ii) always hold.

Proof: First we show that (i) and (ii) imply that $\mathbb{M}$ carries an infinitesimal causal structure.

The conditions (i) and (ii) also lead to the existence of a $M_\Theta(K)$-stable vector in $\mathfrak{q}_\mathfrak{k} = \mathfrak{q}_s \oplus \mathfrak{z}_\mathfrak{k}$: Any $iH_0 \in \{0\} \oplus \mathfrak{z}_\mathfrak{k}$ is such a vector. Fix iH_0 in the following and choose the identification of $Z_\mathfrak{k}$ with $\mathbb{R}$ such that the derivative of t in direction of iH_0 is positive.

With (iii) we know from corollary 1 that an M_Θ-invariant regular cone exists in $\mathfrak{n}_\Theta^+$ and therefore there also exists an invariant cone C in the equivalent module $\mathfrak{q}_\mathfrak{k}$. Lemma 2 and proposition 2 show that it is invariant under the action of B_Θ. C determines a G-invariant infinitesimal causal structure on $\mathbb{M}$ via the bijection (15).

Next we use Lemma 11 to show that the infinitesimal causal structure obtained from C turns $\mathbb{M}$ into a globally hyperbolic manifold: If the intersection of $\mathfrak{q}_s$ and C were not $\{0\}$, this intersection would be a M_Θ-invariant regular cone in $\mathfrak{q}_s$. With (iii) and lemma 6 there would be an $M_\Theta(K)$-stable non-zero vector in $\mathfrak{q}_s$ – clearly not proportional to iH_0. This contradicts lemma 8.

Finally, the last assertion of theorem 6 follows from Lemma 9. *End of Proof*

Proof part ii) of theorem 3:

The *only if* part is Lemma 9.

if part: we must show that the hypotheses i), ii) and iii) of theorem 6 are true. i) is true by hypothesis, and so is ii) (Actually ii) follows from i) as we shall see in a moment).

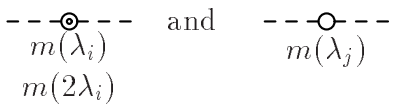
The proof of (iii) will rely on the classification which will be performed in the next section. It will rely on assumptions 1 and 2 and condition (i) in theorem 6. It turns out that for all cases M_Θ will be simple (zero in the well known case of $G = Sl(2,\mathbb{R})$), and that iH_0 will lie outside of $\mathfrak{m}_\Theta(K)$. It will also follow from this classification that condition ii) of theorem 6 follows from assumptions 1 and 2 and from condition i).

Since condition (i) is equivalent to the existence of positive energy representations, it will be a complete classification. *End of Proof*

6 Classification

Non-compact real forms of simple complex Lie algebras can be classified by Satake diagrams. This particular classification is useful for us, because it makes reference to the maximally non-compact Cartan subalgebra $\mathfrak{a}_\mathfrak{p} \oplus \mathfrak{a}_\mathfrak{k}$ which in turn is the basis for the classification of parabolic subgroups.

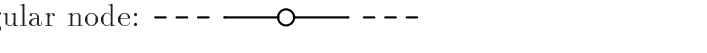
Recall the definition of the restricted root system Σ in section 4. It is the set of non-zero restrictions of the roots in the root system Φ of $\mathfrak{g}$ to $\mathfrak{a}_{\mathfrak{p}}$. The number of roots in Φ which is projected on $\lambda \in \Sigma$ is called the multiplicity $m(\lambda)$ of λ. Σ may be described by an ordinary Dynkin diagram with additional information about the multiplicities of λ_i and $2\lambda_i$ for each simple restricted root λ_i. Let λ_i and λ_j be simple restricted roots such that $2\lambda_i$ is a root while $2\lambda_j$ is not. Then these simple restriced roots will be denoted by



The real Lie algebra is completely described by the *Satake diagram*. It is the Dynkin diagram of Φ with two additional elements:

Type 1 Simple roots in Φ which are restricted to zero in Σ are denoted by a filled node in the Satake diagram:

Type 2 Two simple roots in Φ which are restricted to the same element in Σ are connected by an arrow:

Type 3 All simple roots which are not of type 1 or type 2 are denoted by a regular node: 

The Satake diagrams for the simple real Lie algebras with positive energy representations mentioned in section 2.3 are listed in table 2. They are shown together with the Dynkin diagrams of the restricted root systems. The Coxeter labels of the simple restricted roots are indicated above the corresponding nodes.

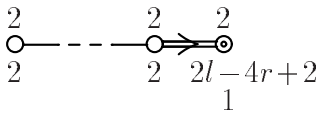
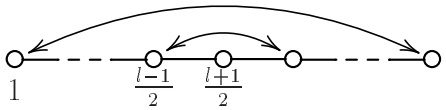
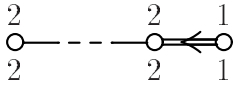
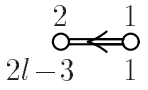
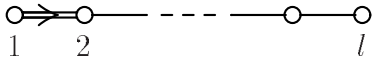
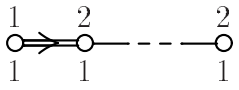
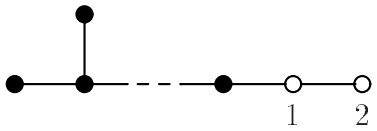
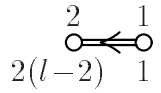
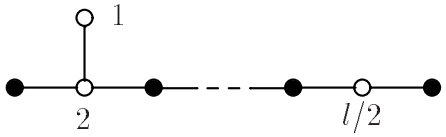
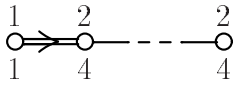
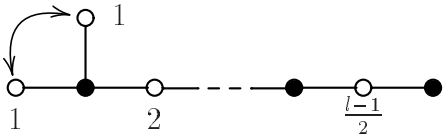
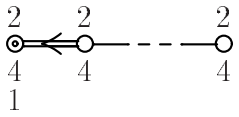
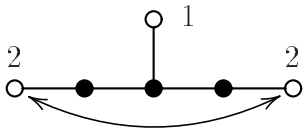
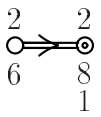
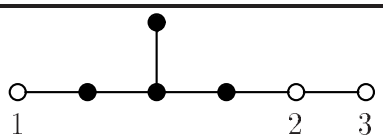
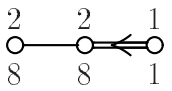
Type	Φ	Σ
$\mathfrak{su}(m,n)$ $(n{=}r, m{+}n{=}l{+}1)$		
$\mathfrak{su}(\mathfrak{m},\mathfrak{m})$ $(m{=}\frac{l+1}{2})$		
$\mathfrak{so}(2,D)$		
$\mathfrak{sp}(l,\mathbb{R})$		
$\mathfrak{so}(2,D)$		
$\mathfrak{so}^*(2l)$		
$\mathfrak{so}^*(2l)$		
$\mathfrak{e}_{6(-14)}$		
$\mathfrak{e}_{7(-25)}$		

Table 2: All Satake diagrams and corresponding Dynkin diagrams of the restricted root systems of simple Lie algebras whose maximal compact Lie subalgebra has non-trivial center. The Coxeter labels of restricted roots are indicated above the node.

Group	iH_0
$\mathfrak{su}(m,m)$	$f_{\alpha_m} + f_{\alpha_{m-1}+\alpha_m+\alpha_{m+1}} + \cdots + f_{\alpha_1+\cdots+\alpha_l}$
$\mathfrak{so}(2,D)$ B_l	$f_{\alpha_1} + f_{\alpha_{\max}}$
$\mathfrak{sp}(l,\mathbb{R})$	$f_{\alpha_1} - f_{\alpha_1+2\alpha_2} + f_{\alpha_1+2\alpha_2+2\alpha_3} + \cdots \pm f_{\alpha_{\max}}$
$\mathfrak{so}(2,D)$ D_l	$f_{\alpha_l} + f_{\alpha_{\max}}$
$\mathfrak{so}^*(4n)$	$f_{\alpha_1} + f_{\alpha_1+\alpha_2+2\alpha_3+\alpha_4} + f_{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6} + f_{\alpha_{\max}}$
$\mathfrak{e}_{7(-25)}$	$f_{\alpha_7} + f_{\alpha_1+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7} + f_{\alpha_{\max}}$

Table 3: iH_0 in the Cartan-Weyl basis with respect to the maximally non-compact Cartan subalgebra, where $f_\alpha = e_\alpha + e_{-\alpha}$.

Proof of theorem 4: Theorem 1 and proposition 1 requires that we should look for a restricted root λ_e with Coxeter label 1. Inspection shows that there is at most one such. The Lie algebras of $SU(n,m)$ with $n < m$ and of $E_{6(-14)}$ have no restricted root with Coxeter label 1. Therefore no space-time manifolds $\mathbb{M}$ exist for them which satisfy our assumptions.

For the remaining Lie algebras, the parabolic subalgebra is uniquely determined and has as Θ the set of all simple restricted roots other than λ_e. The generalized Lorentz subalgebra $\mathfrak{m} = \mathfrak{m}_\Theta$ is generated by the simple roots which do not restrict to λ_e. The resulting $\mathfrak{m}$ are listed in table 1 and it can be seen that $\mathfrak{m}$ is simple in all cases except that $\mathfrak{m}$ is trivial in the well known case $\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{su}(1,1)$.

Condition (ii) of theorem 6 has been checked with the help of the computer algebra package LambdaLie [25] by computing H_0 explicitly in the Cartan Weyl basis based on the maximally non-compact Cartan subalgebra $\mathfrak{a}_\mathfrak{p} \oplus \mathfrak{a}_\mathfrak{k}$. Details can be found in [18]. The resulting vectors iH_0 can be found in table 3. In all cases iH_0 is a linear combination containing $f_{\alpha_{\max}}$ which is never in $\mathfrak{m}$. Therefore (ii) is satisfied for all cases.

End of Proof

For general Θ, the structure of $\mathfrak{m}_\Theta$ can also be obtained from the Satake diagram and Θ: The subdiagram of the Satake diagram which consists of all nodes of type 1 and the nodes of type 2 and 3 which correspond to an element of Θ describes the semi-simple part of $\mathfrak{m}_\Theta$. The number of pairs of type 2 which do not correspond to any element of Θ is equal to the dimension of the (possibly empty) compact Abelian ideal of $\mathfrak{m}_\Theta$ – it is part of $\mathfrak{a}_\mathfrak{k}$.

The dimension of $\mathbb{M}$ equals the dimension of $\mathfrak{n}^+$. Because of the generalized Bruhat decomposition

$$\dim \mathbb{M} = \dim \mathfrak{n}^{\pm}_{\Theta} = \tfrac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{m}_{\Theta} - \dim \mathfrak{a}_{\Theta})$$

The results are collected in table 1, together with the split rank of $\mathfrak{g}$, i.e. the maximal number of noncompact generators in a Cartan subalgebra. The split rank of the generalized Lorentz algebra is one less, because $\mathfrak{a}_{\Theta}$ is 1-dimensional.

Note that the generalized Lorentz group $E_{6(-26)}$ of $E_{7(-25)}$ has split rank 2, i.e. two commuting boosts, whereas the Lorentz groups proper has only one. This suggests a more restrictive symmetry.

7 The Generator H_0 and Time Reflections

Let us remember from the introduction that the existence of positive energy representations of a simple Lie algebra $\mathfrak{g}$ requires that its maximal compact subalgebra $\mathfrak{k}$ has a 1-dimensional center $\mathfrak{z}_{\mathfrak{k}}$, hence $\mathfrak{k} = \mathfrak{k}_s \oplus \mathfrak{z}_{\mathfrak{k}}$, where $\mathfrak{k}_s$ is the semi-simple part of $\mathfrak{k}$. It follows that $\mathfrak{g}$ has a compact Cartan subalgebra $\mathfrak{h}$, and that the simple roots $\{\alpha_1, ..., \alpha_l\}$ with respect to this Cartan subalgebra are the simple roots of $\mathfrak{k}$ plus one simple root α_c with Coxeter label 1. Read the above Satake diagrams as Dynkin diagrams of the complex Lie algebra $(\mathfrak{g})_{\mathbb{C}}$, with nodes corresponding to the simple roots $\{\alpha_1, ..., \alpha_l\}$. The node corresponding to α_c is called the marked node. We order the nodes $1, \ldots, l$ from top to bottom and from left to right. The table 1 gives the number of the marked node under the heading "Node". The Dynkin diagram of $\mathfrak{k}_s$ is obtained by removing the marked node.

We adopt the following standard notation. Given a linear combination $\Lambda = \sum_i m^i \alpha_i$ of simple roots, let h_Λ be the corresponding element of $\mathfrak{h}$ which obeys $\Lambda(h) = B(h_\Lambda, h)$ for all $h \in \mathfrak{h}$, where $B(\cdot,\cdot)$ is the Killing form. Its commutator with ladder operators e_{α_j} of simple roots is $\left[h_\Lambda, e_{\alpha_j}\right] = \langle \Lambda, \alpha_j \rangle e_{\alpha_j}$. The generator iH_0 of the center $\mathfrak{z}_{\mathfrak{k}}$ must commute with the ladder operators of simple roots of $\mathfrak{k}_s$, therefore, up to normalization

$$H_0 = i h_{\Lambda^c} \tag{18}$$

where Λ^c is the fundamental weight which is orthogonal to all simple roots of $\mathfrak{g}$ except α_c.

The fundamental weights are well known. This yields the coefficients in the expansion of iH_0 in the expansion in h_{α_j} as given in the table below.

Name	Index	γH_0
$\mathfrak{su}(m,n)$	$1-(m-n)^2$	$((1)n,\ldots,(m-1)n,mn,m(n-1),\ldots,m(1))$
$\mathfrak{so}(2,d)$, B_l	$-2l^2+7l-4$	$(1,\ldots,1)$
$\mathfrak{sp}(l,\mathbb{R})$	l	$(\frac{1}{2}l,l-1,\ldots,2,1)$
$\mathfrak{so}(2,d)$, D_l	$-2l^2+9l-8$	$(\frac{1}{2},\frac{1}{2},1,\ldots,1)$
$\mathfrak{so}^*(2l)$	$-l$	$(\frac{1}{2}l,\frac{1}{2}l-1,l-2,\ldots,1)$
$\mathfrak{e}_{6(-14)}$	-14	$(3,2,4,6,5,4)$
$\mathfrak{e}_{7(-25)}$	-25	$(3,2,4,6,5,4,3)$

Table 4: Killing index and Time translation generator for the simple Lie groups with pos. energ. representations in the basis corresponding to a maximally compact Cartan subalgebra. γ is some imaginary number.

7.1 Time reflection

Let us assume a time reversal T exists. It acts on $\mathbb{M}=\mathbb{R}\times\Sigma$ in such a way that this induces an automorphism of G of the form $\rho(g)\mapsto T\rho(g)T^{-1}$ and an automorphism $\mathrm{Ad}(T)$ of $\mathfrak{g}$ with the following properties: Since T reflects time, $\mathrm{Ad}(T)H_0=-H_0$. Since T should act trivially on space Σ, it has to be $\mathrm{Ad}(T)X=X$ for all $X\in\mathfrak{k}_s$.

Recall from (18) that $H_0=ih_{\Lambda^i}$, where i is the number of the marked node. This clearly is a linear combination of the h_{α_j} with pure imaginary coefficients.

Lemma 12 *Write the generator of time translations in the form*

$$H_0=r\left(h_{\alpha_i}+\tfrac{1}{2}\sum_{j\neq i}\mu_j h_{\alpha_j}\right)\ . \tag{19}$$

with pure imaginary r.

An automorphism $\mathrm{Ad}(T)$ of $\mathfrak{g}$ such that $\mathrm{Ad}(T)H_0=-H_0$ and $\mathrm{Ad}(T)h_{\alpha_j}=h_{\alpha_j}$ for $j\neq i$ can only exist if μ_j are integers.

Proof: The automorphism $\mathrm{Ad}(T)$ of $\mathfrak{g}$ maps the Cartan subalgebra $\mathfrak{h}$ into itself and must therefore define an automorphism of the root system. We will also

call it T. In particular, $T(\alpha_j)$ must be a root, $T(\Lambda^i) = -\Lambda^i$, because of (18), and $T(\alpha_j) = \alpha_j$ for $j \neq i$.

From (19) it follows that

$$T(\alpha_i) = \left(-\alpha_i - \sum_{j \neq i} \mu_j \alpha_j\right).$$

Roots are sums of integer multiples of simple roots, therefore $T(\alpha_i)$ is not a root if any of the μ_j are non-integer. *End of Proof*

Using this result and table 4, necessary conditions for the existence of a time reflection automorphism are obtained. It is well known and compatible with these necessary conditions that $\mathfrak{so}(2,d)$ admits a time reflection automorphism. Apart from these, only the two Lie algebras $\mathfrak{su}(2,1)$ and $\mathfrak{sp}(4,\mathbb{R})$ satisfy the necessary conditions. Using the computer algebra package LambdaLie it was found that $\mathfrak{su}(2,1)$ admits a time reflection automorphism, but $\mathfrak{sp}(4,\mathbb{R})$ does not. Details are as follows.

Proof of theorem 5: Examine the necessary conditions. For $\mathfrak{su}(m,n)$ the following have to be integer: $\frac{2p}{m}$ for $0 < p < m$ and $\frac{2q}{n}$ for $0 < q < n$. Therefore $m, n \leq 2$. The only case not covered by an isomorphic conformal group is $\mathfrak{su}(1,2)$. For $\mathfrak{sp}(l,\mathbb{R})$, $\frac{4p}{l}$ has to be integer for $0 < p < l$, which is true only for $l = 1, 2, 4$ where only the last case is not isomorphic to a conformal group. For $\mathfrak{so}^*(2l)$ $\frac{2l-4}{l}$ and $\frac{4p}{l}$ for $0 < p < l-1$ have to be integer. This is true for $l = 2, 4$, but $\mathfrak{so}^*(4)$ is not simple and $\mathfrak{so}^*(8)$ again isomorphic to a conformal group. For $\mathfrak{e}_{6(-14)}$ and $\mathfrak{e}_{7(-25)}$ we have $\mu = (\frac{3}{2}, 1, 2, 3, \frac{5}{2}, 1)$ and $\mu = (2, \frac{2}{3}, \frac{4}{3}, 2, \frac{5}{3}, \frac{4}{3}, 1)$ respectively. Thus, both do not allow a time reversal.

This leaves the two cases $\mathfrak{su}(2,1)$ and $\mathfrak{sp}(4,\mathbb{R})$ which are settled by computer algebra as explained in [18]. *End of Proof*

8 Outlook

In conformal field theory in 4 dimensions, further results were obtained which it would be interesting to generalize.

1. The irreducible positive energy representations of G can be constructed as induced representations on $\mathbb{M}$ [14]. They are induced by finite dimensional representations of the parabolic stability group $H = MAN^-$ which are trivial on N^-, and are labelled (l, δ) where l specifies a finite dimensional representation of M, and the dimension δ specifies the

representation of A. Since in an irreducible representation the center is represented by a multiple of the identity, the functions $\Psi_\alpha(x)$ in the representation space actually live on $\mathbb{M}/\mathbb{Z}$, where $\mathbb{Z}$ is the ∞ factor of the center. The scalar product is furnished by the Kunze Stein formula [26] and the only nontrivial task is to determine when it is positive, and to treat some degenerate cases.

It is essential for this construction that H is (the identity component of) a parabolic subgroup.

2. One introduces field operators $\phi_\alpha(x), x \in \mathbb{M}$ with corresponding transformation law labelled by (l, δ) and such that $\phi_\alpha(x)|\Omega\rangle$, Ω = vacuum span irreducible positive energy representation spaces isomorphic to the above. The two point function $\langle\Omega|\phi^*_\beta(y)\phi_\alpha(x)|\Omega\rangle$ equals the above scalar product.

3. One shows that the 3-point functions $\langle\Omega|\phi_\alpha(x)\Phi_\beta(y)\phi_\gamma(z)|\Omega\rangle$ are determined by symmetry up to some arbitrary (coupling) constants [27].

4. Suitably summed up operator product expansions $\phi_\alpha(x)\phi_\beta(y)|\Omega\rangle = \ldots$ converge on the vacuum Ω if they are valid as short distance expansions, because they amount to partial wave expansions on the conformal group [28].

5. Given that this is so, one can construct n-point functions $\langle\Omega|\phi_{\alpha_1}(x_1)\ldots$ $\ldots\phi_{\alpha_n}(x_n)|\Omega\rangle$ from the two and three point functions. The will satisfy all Wightman axioms [29] except local commutativity

$$[\phi_\alpha(x),\phi_\beta(y)]_- = 0 \tag{20}$$

of observable fields such as the stress energy tensor, for relatively spacelike x, y. They depend on the afore mentioned a priori arbitrary coupling constants.

6. To construct a local theory, the remaining nontrivial task is then to satisfy the commutation relations (20). If it should turn out that for some groups G also the four point functions are determined by symmetry (up to some normalization constants), as is the case in 2-dimensional conformal field theory, this task would be very much simplified.

7. The relation between conformal field theory in the universal covering of compactified Minkowski space and in Euklidean space was established by Lüscher and Mack [1, 2]. It involves analytic continuation through a holomorphic semi-group which acts contractively on the Hilbert space of physical states. The method involves Euklidean time reversal as an

involutive automorphism. Similar methods are employed in the Gel'fand-Gindikin program [30], which aims at the realization of families of similar unitary representations of Lie groups in a unified geometric way, and was applied to causal symmetric spaces [31, 32].

We expect that in general the relation of G with a Euklidean symmetry will be more subtle than in the conformal case.

8. It is interesting to study supersymmetric versions of the pairs $(G, \mathbb{M})$ as generalizations of super conformal field theory. Jordan super algebras and their associated super conformal groups were classified by Günaydin [33, 34]. He also presented a general way to construct unitary positive energy representations, for both the Lie algebras and super Lie algebras, by using multiplets of step operators for harmonic oscillators [35, 36, 37].

The identification of lowest weigt representations with induced representations for all our groups G was already accomplished by Günaydin, using the isomorphism mentioned in section 2.5 He used this to illuminate AdS/CFT correspondences from a representation theoretic point of view.

A group theoretical study of dimensional reduction would also be of interest. In the present context, space is always compact (a sphere in the conformal case). In Minkowski space, decompactification is associated with the breaking of conformal symmetry. In particular, mass generation effectively removes the point at ∞.

The exceptional group $E_{7(-25)}$ is particularly interesting. It contains a maximal subgroup $G^4 \times \tilde{U}$ where $G^4 = SO(2,4)$ may be interpreted as conformal space-time symmetry in 4 dimensions, and $\tilde{U} = SU(4) \times SL(2,\mathbb{R})$ which one is tempted to interpret as an internal symmetry. The strange feature is the appearance of a noncompact internal symmetry group $SL(2,\mathbb{R})$, reminiscent of hidden symmetries in supergravity [38, 39, 40].

It would appear at first sight that we will not quite get the $SU(3) \times U(1) \times SU(2)$-symmetry group of the standard model, because there is a factor $SL(2,\mathbb{R})$ in place of $SU(2)$

However, the internal symmetry of the dimensionally reduced theory is more subtle, because

i) The internal symmetry which acts nontrivially may be smaller than $\tilde{U}$.

ii) There is no difference between compact and non-compact real forms for internal symmetries which act on finite-dimensional vector spaces (e.g. act on indices of fields), because of a corollary of Weyl's unitary trick which asserts

not only equivalence of representations but also of invariants (see Appendix A). Every semi-simple complex group $U_{\mathbb{C}}$ possesses a compact real form.

It is also of interest to consider space-time manifolds $\mathbb{M} = {}^{G}\!/_{H}$ with semi-simple or reductive stability group H. G-invariant causal structures on symmetric spaces ${}^{G}\!/_{H}$ were investigated in the mathematical literature [41, 12]. This includes anti de Sitter space. We did not deal with this case in the body of the paper, but we list some examples with semi-simple G which obviously satisfy the requirement of causality.

Let G_1 be a simple connected simply connected Lie group which possesses positive energy representations, i.e. the universal covering of one of the groups in (4). From arguments of Lüscher [2] it follows that G_1 possesses an $\mathrm{Ad}(G_1)$-invariant causal structure. Set $\mathbb{M} = G_1$, acted upon by elements $(g_L, g_R) \in G = G_1 \times G_1$ according to $m \mapsto g_L m g_R^{-1}$. Then H is the diagonal subgroup of $G_1 \times G_1$ and its elements (g, g) act on $\mathbb{M}$ according to $m \mapsto \mathrm{Ad}(g)m$. It follows from elementary results in the body of this paper that a $\mathrm{Ad}(G_1)$ invariant causal structure on $\mathbb{M}$ is also G-invariant.

The universal covering of the central extension of the diffeomorphism group G_1 of the circle admits positive energy representations. The Lie algebra of $G = G_1 \times G_1$ consists of two copies of the Virasoro algebra. Assuming that Lüschers arguments extend to this ∞-dimensional example, we would conclude that the ∞-dimensional space-time manifold $\mathbb{M} = G_1$ admits a G-invariant causal structure. This suggests that quantum field theories on ∞-dimensional space-time manifolds may exist.

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Appendix A: Weyl's unitary trick

Proposition 4 *Let G' and G be two real forms of a complex Lie group $G_{\mathbb{C}}$, with Lie algebras $\mathfrak{g}' = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \subset \mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_1 \subset \mathfrak{g}_{\mathbb{C}}$. Let $V^0, V^1, \ldots, V^n$ be finite dimensional complex representation spaces carrying representations $\tau^1, ..., \tau^n$ of G'. Then*

i. The representation operators τ^i qua functions of $g \in G'$ extend to holomorphic functions on $G_{\mathbb{C}}$, making V^i into representation spaces for $G_{\mathbb{C}}$, hence of G.

ii. Every intertwiner

$$C : V^1 \otimes ... \otimes V^n \mapsto V^0$$

between representations of G' extends to an intertwiner of representations of $G_{\mathbb{C}}$, hence of G.

Invariants are intertwiners to the trivial representation.

Proof: part i. is known as Weyl's unitary trick.

The intertwining property reads

$$C\big(\tau^1(g) \otimes ... \otimes \tau^n(g)\big) = \tau^0(g)C$$

for $g \in G'$. By i., both sides extend to holomorphic functions on $G_{\mathbb{C}}$ which agree on the real neighborhood G'. They are therefore equal. *End of Proof*

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