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Generalized parton distributions for the pion in chiral perturbation theory

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Abstract

Generalized parton distributions provide a unified parameterization of hadron structure and allow one to combine information from many different observables. Lattice QCD calculations already provide important input to determine these distributions and hold the promise to become even more important in the future. To this end, a reliable extrapolation of lattice calculations to the physical quark and pion masses is needed. We present an analysis for the moments of generalized parton distributions of the pion in one-loop order of chiral perturbation theory.

1 Introduction

Generalized parton distributions (GPDs) provide general parameterizations for non-forward hadronic matrix elements, see e.g. the original work [1, 2, 3] and the recent reviews [4, 5, 6]. They contain the usual parton distribution functions and elastic form factors, and they are closely related with light-cone wave functions and distribution amplitudes. GPDs can be analyzed using standard operator product expansion techniques [1, 7], and factorization theorems have been proven for many different processes. In addition, evolution equations have been derived to next-to-leading order accuracy, and generally speaking the GPD formalism has reached a similar level of stringency as the QCD description of inclusive deep inelastic scattering. Finally, it has become clear that GPDs contain information which cannot be deduced directly from any experiment, most notably the orbital angular momentum carried by partons [2] and the transverse structure of hadrons in the impact parameter plane [8]. On the experimental side, data which is especially suited to constrain the form of GPDs has been obtained at DESY [9] and Jefferson Lab [10].

An important problem hampering the practical realization of the physics potential of GPDs is that, typically, measurements are only sensitive to convolutions of GPDs, such that the assumed functional form enters crucially into the extraction of the distributions themselves. The situation is further complicated by potentially large higher-twist contributions in some processes. Therefore, the possibility to obtain independent information from lattice QCD, which allows one to directly calculate x-moments of GPDs, is especially interesting. Pioneering lattice calculations for nucleon GPDs have already been performed some time ago [11]. These calculations are being improved and their scope is being systematically widened [12], also in the direction of analyzing other hadrons, in particular the pion. Calculations for the pion are being performed by the QCDSF collaboration, using dynamical $N_F = 2$ improved Wilson fermions, but have to be made at unphysically large quark masses to save on computing time. The reliable extrapolation to the chiral limit, i.e. to physical quark masses, requires precise knowledge on the functional form of the quark mass dependence of moments of GPDs. Close to the chiral limit, this knowledge is provided by chiral perturbation theory (ChPT), and in recent years the combination of ChPT and lattice QCD has in fact proven to be very powerful in obtaining precise lattice predictions. Obviously the ChPT analysis should be performed in parallel to the lattice calculations, and this is the objective of our paper.

In this Letter we present an analysis for the moments of pion GPDs in one-loop order of ChPT. The paper is organized as follows. In Section 2 we recall the definition of the pion GPDs and the expression of their moments in terms of the matrix elements of local twist-two operators. In Sections 3 and 4 we construct the relevant twist-two operators in the first two orders of ChPT, and in Section 5 we calculate the one-loop contributions to their matrix elements in the pion. We summarize our findings in Section 6.

2 Definitions and properties of pion GPDs

The pion GPDs can be introduced as the pion matrix elements of nonlocal operators. As is done in lattice QCD calculations, we assume isospin symmetry to be exact, neglecting the difference between u- and d-quark masses. We shall study the isoscalar and isovector quark GPDs, which are respectively defined as

$$\delta^{ab}\, H^{I=0}(x,\xi,t) \;=\; \frac{1}{2}\int \frac{d\lambda}{2\pi}\, e^{ix\lambda(Pu)}\, \langle \pi^b(p')|\, \bar{q}(-\tfrac{1}{2}\lambda u)\, u^\mu \gamma_\mu\, q(\tfrac{1}{2}\lambda u)\, |\pi^a(p)\rangle\,, \tag{1}$$

$$i\epsilon^{abc}\, H^{I=1}(x,\xi,t) \;=\; \frac{1}{2}\int \frac{d\lambda}{2\pi}\, e^{ix\lambda(Pu)}\, \langle \pi^b(p')|\, \bar{q}(-\tfrac{1}{2}\lambda u)\, u^\mu \gamma_\mu\, \tau^c\, q(\tfrac{1}{2}\lambda u)\, |\pi^a(p)\rangle\,, \tag{2}$$

where q denotes the doublet of u- and d-quark fields and τ^a the Pauli matrices in isospin space. u is a lightlike auxiliary vector, and we use the standard notations for the kinematical variables

$$P = \frac{p+p'}{2}, \qquad \Delta = p' - p, \qquad t = \Delta^2, \qquad \xi = -\frac{\Delta u}{2Pu}\,. \tag{3}$$

In terms of the distributions for individual quark flavors defined in [5] we have $H^{I=0} = H^u_\pi + H^d_\pi$ and $H^{I=1} = H^u_\pi - H^d_\pi$. As usual, Wilson lines between the quark fields are to be inserted in (1) and (2) if one is not working in the light-cone gauge $u^\mu A_\mu = 0$.

Time reversal and charge conjugation symmetry result in the following properties for the pion GPDs:

$$H^I(x,\xi,t) = H^I(x,-\xi,t)\,, \tag{4}$$

$$H^I(x,\xi,t) = (-1)^{1-I} H^I(-x,\xi,t)\,. \tag{5}$$

The distributions can be analytically continued to positive t and are real-valued for $t < 4m_\pi^2$, as discussed in [13, 14].

The x-moments of the GPDs are related to the matrix elements of the local isoscalar and isovector operators

$$\mathcal{O}^s_{\mu_1\mu_2\ldots\mu_n} = \mathbf{S}\,\frac{1}{2}\bar{q}\gamma_{\mu_1} i\overleftrightarrow{D}_{\mu_2}\ldots i\overleftrightarrow{D}_{\mu_n}\,q\,, \qquad \mathcal{O}^a_{\mu_1\mu_2\ldots\mu_n} = \mathbf{S}\,\frac{1}{2}\bar{q}\gamma_{\mu_1} i\overleftrightarrow{D}_{\mu_2}\ldots i\overleftrightarrow{D}_{\mu_n}\tau^a\,q\,, \tag{6}$$

where $\overleftrightarrow{D}_\mu = \frac{1}{2}(\overrightarrow{D}_\mu - \overleftarrow{D}_\mu) = \frac{1}{2}(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu) - igA_\mu$ and all fields are to be taken at the same space-time coordinate. $\mathbf{S}$ denotes symmetrization and subtraction of trace terms for all uncontracted Lorentz indices. Defining form factors

$$\langle\pi^b(p')|\,\mathcal{O}^s_{\mu_1\mu_2\ldots\mu_n}|\pi^a(p)\rangle = \delta^{ab}\sum_{k=0,2,\ldots}^{n} A^{I=0}_{n,k}(t)\,\mathbf{S}\,\Delta_{\mu_1}\ldots\Delta_{\mu_k}P_{\mu_{k+1}}\ldots P_{\mu_n}\,, \tag{7}$$

$$\langle\pi^b(p')|\,\mathcal{O}^c_{\mu_1\mu_2\ldots\mu_n}|\pi^a(p)\rangle = i\epsilon^{abc}\sum_{k=0,2,\ldots}^{n} A^{I=1}_{n,k}(t)\,\mathbf{S}\,\Delta_{\mu_1}\ldots\Delta_{\mu_k}P_{\mu_{k+1}}\ldots P_{\mu_n}\,, \tag{8}$$

where the operators are to be taken at zero space-time position and where time reversal invariance limits k to even values, one finds

$$\int_{-1}^{1} dx\, x^{n-1}\, H^I(x,\xi,t) = \sum_{k=0,2,\ldots}^{n} (2\xi)^k\, A^I_{n,k}(t)\,. \tag{9}$$

Note that because of the symmetry relation (5) the moments with odd n in (9) vanish for $I=0$ and those with even n vanish for $I=1$. The form factors $A^I_{n,k}(t)$ are real-valued in the region $t < 4m_\pi^2$. Information on the lowest moments

$$\int_{-1}^{1} dx\, H^{I=1}(x,\xi,t) = A^{I=1}_{1,0}(t)\,, \tag{10}$$

$$\int_{-1}^{1} dx\, x\, H^{I=0}(x,\xi,t) = A^{I=0}_{2,0}(t) + 4\xi^2 A^{I=0}_{2,2}(t) \tag{11}$$

will soon be accessible from the lattice for pion masses down to values $m_\pi \sim 500$ MeV [15]. Note that the first moment of $H^{I=1}$ is related to the pion electromagnetic form factor as $A^{I=1}_{1,0}(t) = 2F^{em}_\pi(t)$ and the second moment of $H^{I=0}$ with the form factors in the parameterization [16]

$$\langle\pi^b(p')|\,T^Q_{\mu\nu}\,|\pi^a(p)\rangle = \delta^{ab}\left[2P_\mu P_\nu\,\theta_2(t) + \frac{1}{2}\left(g_{\mu\nu}\Delta^2 - \Delta_\mu\Delta_\nu\right)\theta_1(t)\right], \tag{12}$$

where $T^Q_{\mu\nu} = \frac{1}{2}\bar{q}\,(\gamma_\mu\, i\overleftrightarrow{D}_\nu + \gamma_\nu\, i\overleftrightarrow{D}_\mu)\,q$ is the quark part of the energy-momentum tensor. Comparison with (7) gives $\theta_2(t) = A^{I=0}_{2,0}(t)$ and $\theta_1(t) = -4A^{I=0}_{2,2}(t)$.

An analogous discussion can be given for the gluon GPD, which is defined by

$$\delta^{ab}\, H^g_\pi(x,\xi,t) = \frac{1}{Pu}\int \frac{d\lambda}{2\pi}\, e^{ix\lambda(Pu)}\, \langle \pi^b(p')|\, u_\alpha G^{\alpha\mu}(-\tfrac{1}{2}\lambda u)\, u_\beta G_\mu{}^\beta(\tfrac{1}{2}\lambda u)\, |\pi^a(p)\rangle \tag{13}$$

in the convention of [5]. It has the same symmetry properties as $xH^{I=0}(x,\xi,t)$, and its moments $\int dx\, x^{n-2} H^g_\pi(x,\xi,t)$ are analogs of the moments $\int dx\, x^{n-1} H^{I=0}(x,\xi,t)$, with which they mix under renormalization scale evolution. For ease of notation we will in the following concentrate on the quark sector, bearing in mind that our results for the isoscalar form factors $A^{I=0}_{n,k}(t)$ readily generalize to their gluonic counterparts.

3 Tensor operators in ChPT

We use the standard $\mathcal{O}(p^n)$ power-counting of ChPT, where p denotes a generic pion energy or momentum. The Goldstone boson fields are collected in the matrix-valued field U, and the leading-order Lagrangian, which is of order $\mathcal{O}(p^2)$, reads [17]

$$\mathcal{L}^{(2)}_{\pi\pi} \;=\; \frac{1}{4}F^2\,\mathrm{tr}\left(D_\mu U D^\mu U^\dagger + \chi^\dagger U + \chi U^\dagger\right) \tag{14}$$

for two light flavors, with the covariant derivative D_μ and the external field tensor χ defined as

$$D_\mu U \;=\; \partial_\mu U - i r_\mu\, U + i U\, l_\mu\,, \tag{15}$$

$$\chi \;=\; 2B_0\,(s + i\,p)\,. \tag{16}$$

Here s and p denote external scalar and pseudoscalar fields, which count as quantities of order $\mathcal{O}(p^2)$. Furthermore, r_μ and l_μ are external right- and left-handed vector fields with intrinsic chiral power $\mathcal{O}(p)$. The two leading-order parameters appearing in (14) are the pion decay constant F (normalized to $F \approx 92$ MeV) and the two-flavor chiral condensate $B_0 = -\langle 0|\bar{q}q|0\rangle/F^2$, both evaluated in the chiral limit [17]. Throughout this work we use the non-linear representation for the pion fields π^a,

$$U = \exp\{i\tau^a\pi^a/F\}\,. \tag{17}$$

Left and right chiral rotations of the nonlinear pion field

$$U \to V_R\, U\, V_L^\dagger \tag{18}$$

induce the following transformation of the external fields:

$$\begin{aligned} r_\mu &\to V_R\, r_\mu\, V_R^\dagger + i\, V_R \partial_\mu V_R^\dagger\,, \qquad & \chi &\to V_R\, \chi\, V_L^\dagger\,, \\ l_\mu &\to V_L\, l_\mu\, V_L^\dagger + i\, V_L\, \partial_\mu V_L^\dagger\,. \end{aligned} \tag{19}$$

One method to calculate the matrix elements of the operators $\mathcal{O}^s_{\mu_1\mu_2\ldots\mu_n}$ and $\mathcal{O}^a_{\mu_1\mu_2\ldots\mu_n}$ from (6) in ChPT would be to introduce corresponding tensor currents as external sources in the Lagrangian, in generalization of the vector currents r_μ and l_μ in (14). For tensors of higher rank, this becomes however increasingly awkward. Instead, we will directly match the operators in (6) onto operators with pion degrees of freedom. The expression of the isovector vector current is for instance well-known to be [17]

$$\mathcal{O}^a_\mu = \frac{1}{2}\bar{q}\gamma_\mu\tau^a\, q \;\;\simeq\;\; V^a_\mu = -\frac{i}{4}F^2\,\mathrm{tr}\left\{\tau^a\left(U^\dagger\partial_\mu U + U\partial_\mu U^\dagger\right)\right\} \tag{20}$$

to leading order in the chiral expansion. Despite the progress in deriving chiral dynamics directly from QCD (see e.g. [18]) this matching problem is far from being solved in general. The so-called low-energy constants appearing in ChPT can in principle be determined by fits to either experimental or lattice data, but only a limited number of them are known in practice. What we can however do is to construct for a given quark or gluon operator in QCD all operators in ChPT with the same symmetry. Their matching coefficients typically are linear combinations of low-energy constants and can be fitted directly to lattice data at different unphysical pion masses, provided they are sufficiently close to the chiral limit. In Section 5 we will determine the functional dependence on m_π^2 of the form factors $A^I_{n,k}(t)$ required for such a fit at one-loop level in ChPT.

For the discussion of chiral symmetry it is useful to consider the left- and right-handed tensor operators

$$(\mathcal{O}^L_{ij})_{\mu_1\ldots\mu_n} = \mathbf{S}\,\bar{q}_j\gamma_{\mu_1}\frac{1-\gamma_5}{2}\,i\overleftrightarrow{D}_{\mu_2}\ldots i\overleftrightarrow{D}_{\mu_n}\,q_i, \tag{21}$$

$$(\mathcal{O}^R_{ij})_{\mu_1\ldots\mu_n} = \mathbf{S}\,\bar{q}_j\gamma_{\mu_1}\frac{1+\gamma_5}{2}\,i\overleftrightarrow{D}_{\mu_2}\ldots i\overleftrightarrow{D}_{\mu_n}\,q_i, \tag{22}$$

which take values in two-dimensional isospin space. The operators in (6) are then given by

$$\mathcal{O}^s_{\mu_1\ldots\mu_n} = \frac{1}{2}\mathrm{tr}\left\{\mathcal{O}^L_{\mu_1\ldots\mu_n}+\mathcal{O}^R_{\mu_1\ldots\mu_n}\right\}, \qquad \mathcal{O}^a_{\mu_1\ldots\mu_n} = \frac{1}{2}\mathrm{tr}\left\{\tau^a\left(\mathcal{O}^L_{\mu_1\ldots\mu_n}+\mathcal{O}^R_{\mu_1\ldots\mu_n}\right)\right\}. \tag{23}$$

Note that for ease of notation we shall use the same symbol for the quark operators in QCD and for the operators representing them in ChPT. The left- and right-handed operators behave like

$$\mathcal{O}^L_{\mu_1\ldots\mu_n} \to V_L\mathcal{O}^L_{\mu_1\ldots\mu_n}V_L^\dagger, \qquad \mathcal{O}^R_{\mu_1\ldots\mu_n} \to V_R\mathcal{O}^R_{\mu_1\ldots\mu_n}V_R^\dagger \tag{24}$$

under chiral rotations and are related as $\mathcal{O}^R_{\mu_1\ldots\mu_n} = P\mathcal{O}^L_{\mu_1\ldots\mu_n}P^{-1}$ by the parity transformation P. To construct a linearly independent set of operators for the expansion of $\mathcal{O}^L_{\mu_1\ldots\mu_n}$, let us inspect the building blocks at hand. For the purpose of calculating matrix elements between pion states we can set the external fields p, r_μ and l_μ to zero, and keep the scalar field s as a nonzero constant field implementing the explicit breaking of chiral symmetry by the quark masses. Since we assume exact isospin symmetry, this simply gives $\chi = m^2\,\mathbb{1}$, where m is the bare pion mass in ChPT. For constructing the operator this leaves us with the objects U, χ and ∂_μ, with the additional condition $\partial_\mu\chi = 0$. It is convenient to introduce the following quantities

$$L_\mu = U^\dagger\partial_\mu U, \qquad \chi^L_\pm = \chi^\dagger U \pm U^\dagger\chi\,, \tag{25}$$

which transform covariantly under left rotations and are invariant under right rotations. The product of some number of L_μ and $\chi^L_\pm$ with any number of additional derivatives acting on them transforms covariantly under left rotations. Since $\partial_\mu\chi^L_\pm = \frac{1}{2}[\chi^L_\pm, L_\mu] + \frac{1}{2}\{\chi^L_\mp, L_\mu\}$ for $\partial_\mu\chi = 0$, one can eliminate all derivatives acting on $\chi^L_\pm$. Similarly, taking into account the identities

$$\partial_\mu L_\nu - \partial_\nu L_\mu = [L_\mu, L_\nu], \qquad \partial_\mu\partial^\mu L_\nu = \partial_\nu\partial^\mu L_\mu + [\partial^\mu L_\nu, L_\mu] + [L_\nu, \partial^\mu L_\mu] \tag{26}$$

and the equation of motion for the field U,

$$\partial^\mu L_\mu = -\frac{1}{4}\Big(2\chi^L_- - \mathbb{1}\,\mathrm{tr}\,\chi^L_-\Big)\,, \tag{27}$$

one can trade the tensors $\partial_{\nu_1}\ldots\partial_{\nu_{n-1}}L_{\nu_n}$ for the symmetric and traceless ones,

$$L_{\nu_1\ldots\nu_n} = \mathbf{S}\,\partial_{\nu_1}\ldots\partial_{\nu_{n-1}}L_{\nu_n}\,. \tag{28}$$

A linearly independent set of operators transforming like $\mathcal{O}^L_{\mu_1\ldots\mu_n}$ under chiral rotations can hence be constructed from products of the fields $\chi^L_\pm$ and the tensors $L_{\nu_1\ldots\nu_k}$. In the ChPT expansion of $\mathcal{O}^L_{\mu_1\ldots\mu_n}$ they can be multiplied by chirally invariant operators, which can in turn be constructed from the traces of products of $\chi^L_\pm$ and $L_{\nu_1\ldots\nu_k}$.[1] In each term of the expansion, some Lorentz indices of the tensors $L_{\nu_1\ldots\nu_k}$ are equal to $\mu_1,\ldots,\mu_n$ and all others are contracted. Since ∂_μ and L_μ are of order $\mathcal{O}(p)$, each contraction of the indices, as well as the insertion of a field $\chi^L_\pm$ increases the chiral counting by a factor $\mathcal{O}(p^2)$.

At leading order of ChPT the operator $\mathcal{O}^L_{\mu_1\ldots\mu_n}$ can hence be represented as the sum of products of L tensors and their traces, with no contracted Lorentz indices. It is convenient to contract free indices with the light-line vector u, introducing the notation

$$\mathcal{O}^L_n(u) = \mathcal{O}^L_{\mu_1\ldots\mu_n} u^{\mu_1}\ldots u^{\mu_n}, \qquad\qquad L_k(u) = L_{\mu_1\ldots\mu_k} u^{\mu_1}\ldots u^{\mu_k}. \tag{29}$$

The operator $\mathcal{O}^L_n(u)$ is then expanded on terms of the form

$$L_{k_1}(u)\ldots L_{k_{j_1}}(u)\ \mathrm{tr}\left\{L_{k_{j_1+1}}(u)\ldots L_{k_{j_2}}(u)\right\}\mathrm{tr}\left\{L_{k_{j_2+1}}(u)\ldots L_{k_{j_3}}(u)\right\}\ldots \tag{30}$$

and explicitly reads

$$\mathcal{O}^L_n(u) = c_n\, L_n(u) + \sum_{k=1}^{n-1} c_{n,k}\, L_k(u)\, L_{n-k}(u) + \ldots, \tag{31}$$

where we used that $\mathrm{tr}\, L_\mu = 0$ and have omitted terms containing three or more L tensors. Such terms do in fact not contribute to the two-pion matrix elements in leading order and next-to-leading order of ChPT. To see this, notice that the expansion of L_μ into pion fields starts as

$$L_\mu = i\tau^a\left(\partial_\mu\pi^a/F - \epsilon^{abc}\pi^b\partial_\mu\pi^c/F^2 + \tfrac{2}{3}\pi^b\,(\pi^a\partial_\mu\pi^b - \pi^b\partial_\mu\pi^a)/F^3 + \mathcal{O}(\pi^3)\,\partial_\mu\pi\right). \tag{32}$$

Terms with three or more L tensors hence do not contribute to two-pion matrix elements at Born level. At one loop, they contribute only through the tadpole diagram of Fig. 1a, where three or four pion fields from the inserted operator then carry a derivative contracted with the light-like vector u. The numerator in the corresponding Feynman integral thus has the form $(uk)^i$, where $i \geq 1$ and k is the loop momentum. For odd i the integral vanishes because the integrand is antisymmetric in k, whereas for even i the integral is proportional to $(u^2)^{i/2}$ by Lorentz invariance, and hence zero as well.

Repeating our discussion for the right-handed operators $\mathcal{O}^R_{\mu_1\ldots\mu_n}$, with

$$R_\mu = U\partial_\mu U^\dagger, \qquad\qquad \chi^R_\pm = \chi U^\dagger \pm U\chi^\dagger \tag{33}$$

and definitions analogous to (28) and (29), the tensor operator $\mathcal{O}_n(u) = \mathcal{O}^L_n(u) + \mathcal{O}^R_n(u)$ reads at order $\mathcal{O}(p^0)$ in ChPT

$$\mathcal{O}_n(u) = c_n\left(L_n(u) + R_n(u)\right) + \sum_{k=1}^{n-1} c_{n,k}\left(L_k(u)\, L_{n-k}(u) + R_k(u)\, R_{n-k}(u)\right) + \ldots, \tag{34}$$

where the dots denote terms which do not contribute to two-pion matrix elements to $\mathcal{O}(p^2)$ accuracy. The coefficients of the L and R terms must be equal because of parity invariance. Using the hermiticity

[1] Equivalently, one may take the traces of products of $\chi^R_\pm$ and $R_{\nu_1\ldots\nu_k}$ introduced below.

and time reversal properties of $\mathcal{O}_n(u)$ following from (21), (22) and the corresponding properties of $L_n(u)$ and $R_n(u)$, one finds

$$(c_n)^* = -c_n\,, \qquad\qquad (c_n)^* = (-1)^n\, c_n\,, \tag{35}$$

$$(c_{n,k})^* = c_{n,n-k}\,, \qquad\qquad (c_{n,k})^* = (-1)^n\, c_{n,k}\,. \tag{36}$$

Neglecting terms which do not contribute to two-pion matrix elements one derives for the isoscalar and isovector operators at order $\mathcal{O}(p^0)$

$$\mathcal{O}_n^s(u) \;=\; \sum_{k=1}^{n/2} a_{n,k} \Big(L_k^a(u)\, L_{n-k}^a(u) + R_k^a(u)\, R_{n-k}^a(u) \Big)\,, \qquad (n \text{ even}) \tag{37}$$

$$\begin{aligned} \mathcal{O}_n^a(u) \;&=\; ib_{n,0} \Big(L_n^a(u) + R_n^a(u) \Big) \\ &+\; \varepsilon^{abc} \sum_{k=1}^{(n-1)/2} b_{n,k} \Big(L_k^b(u)\, L_{n-k}^c(u) + R_k^b(u)\, R_{n-k}^c(u) \Big)\,, \qquad (n \text{ odd}) \end{aligned} \tag{38}$$

where $L_k^a(u)$ is defined by $L_k(u) = \tau^a L_k^a(u)$ and $R_k^a(u)$ in analogy. The expansion coefficients $a_{n,k}$ and $b_{n,k}$ are linear combinations of the coefficients in (34),

$$a_{n,k} \;=\; c_{n,k} + c_{n,n-k} \qquad \text{for } k \geq 1, \tag{39}$$

$$b_{n,k} \;=\; i(c_{n,k} - c_{n,n-k}) \qquad \text{for } k \geq 1, \qquad b_{n,0} = -ic_n\ . \tag{40}$$

Because of the symmetry relations (35), (36) one has $a_{n,k} = 0$ for odd n and $b_{n,k} = 0$ for even n, and the nonzero coefficients $a_{n,k}$ and $b_{n,k}$ are real. To obtain simple expressions for the form factors $A_{n,k}^I(t)$ in (9), we rearrange derivatives and obtain for the isosinglet operator

$$\mathcal{O}_n^s(u) = F^2 \sum_{k=0,2,\ldots}^{n-2} \tilde{a}_{n,k}\, (iu\partial)^k \Big(L^a(u)\, (2iu\overleftrightarrow{\partial})^{n-k-2} L^a(u) + R^a(u)\, (2iu\overleftrightarrow{\partial})^{n-k-2} R^a(u) \Big) \tag{41}$$

with $\overleftrightarrow{\partial} = \frac{1}{2}(\overrightarrow{\partial} - \overleftarrow{\partial}\,)$. Here we used $L_k^a(u) = (u\partial)^{k-1} L_1^a(u)$ and the abbreviation $L^a(u) = L_1^a(u) = u^\mu L_\mu^a$ and their analogs for the right-handed fields. The coefficients $\tilde{a}_{n,k}$ are linear combinations of the $a_{n,k}$. Calculating pion matrix elements one gets for the isoscalar form factors $A_{n,k}^{I=0}(t)$ at order $\mathcal{O}(p^0)$

$$A_{n,k}^{(0)} \;=\; 2^{n-k} \Big[\, \tilde{a}_{n,k-2} - \tilde{a}_{n,k} \,\Big], \qquad (n \text{ and } k \text{ even},\, k \leq n) \tag{42}$$

where we set $\tilde{a}_{n,n} = \tilde{a}_{n,-2} = 0$. To ease the notation we have omitted the isospin label, implying here and in the following $I = 0$ if the first index n of a form factor is even, and $I = 1$ if it is odd. The inverse of (42) reads

$$\tilde{a}_{n,k} = -\sum_{m=0}^{k} 2^{m-n} A_{n,m}^{(0)} = \sum_{m=k+2}^{n} 2^{m-n} A_{n,m}^{(0)}\,. \qquad (n,\, m \text{ and } k \text{ even},\, k \leq n-2) \tag{43}$$

From (42) we also see that at order $\mathcal{O}(p^0)$ the sum of the isoscalar form factors with given n vanishes,

$$\sum_{k=0,2,\ldots}^{n} 2^k A_{n,k}^{(0)} = 0\,, \qquad (n \text{ even}) \tag{44}$$

which according to (9) gives the soft pion theorem $H^{I=0}(x,\xi\to 1,0)=0$ first derived in [14]. Recalling our discussion at the end of Section 2, we emphasize at this point that the ChPT equivalent of the twist-two gluon operators has the same form as (41), of course with different matching coefficients. Analogs of (42) to (44) therefore hold for the appropriate moments of the gluon GPD $H^g_\pi(x,\xi,t)$.

Rewriting (38) in a form similar to (41) gives for the isovector operator

$$\begin{aligned}\mathcal{O}^a_n(u) &= \tilde b_{n,n-1}\,(iu\partial)^{n-1}V^a(u)\\ &+ iF^2\epsilon^{abc}\sum_{k=0,2}^{n-3}\tilde b_{n,k}\,(iu\partial)^k\Big(L^b(u)\,(2iu\overleftrightarrow{\partial})^{n-k-2}L^c(u)+R^b(u)\,(2iu\overleftrightarrow{\partial})^{n-k-2}R^c(u)\Big)\,,\end{aligned}\tag{45}$$

where the $\tilde b_{n,k}$ are linear combinations of the $b_{n,k}$ in (38). Here

$$V^a(u)=u^\mu V^a_\mu=-\frac{i}{2}F^2\Big(L^a(u)+R^a(u)\Big)\tag{46}$$

is the isovector vector current already given in (20), which also implies $\tilde b_{1,0}=1$. For the isovector form factors $A^{I=1}_{n,k}(t)$ one derives at $\mathcal{O}(p^0)$

$$A^{(0)}_{n,k}=2^{n-k}\Big[\tilde b_{n,k}-\tilde b_{n,k-2}\Big]\,,\qquad \tilde b_{n,k}=\sum_{m=0,2,\ldots}^{k}2^{m-n}A^{(0)}_{n,m}\,,\qquad (n\text{ odd},\,k\text{ even},\,k\le n-1)\tag{47}$$

where we set $\tilde b_{n,-2}=0$. Note that the isovector analog of (44),

$$\sum_{k=0,2,\ldots}^{n-1}2^k A^{(0)}_{n,k}=2^n\,\tilde b_{n,n-1}\,,\qquad (n\text{ odd})\tag{48}$$

does not have a vanishing right-hand side. The coefficient $\tilde b_{n,n-1}$ is however related to another observable, namely the one-pion matrix element of the isovector axial twist-two operator

$$\mathcal{O}^a_{5,n}(u)=\frac{1}{2}\,\mathrm{tr}\left\{\tau^a\left(\mathcal{O}^R_n(u)-\mathcal{O}^L_n(u)\right)\right\}=\frac{1}{2}\bar q\,u^\mu\gamma_\mu\gamma_5\,(iu\overleftrightarrow{D})^{n-1}\,\tau^a\,q\,,\tag{49}$$

which to leading order of ChPT has the form

$$\mathcal{O}^a_{5,n}(u)=\tilde b_{n,n-1}(iu\partial)^{n-1}A^a(u)+\ldots,\tag{50}$$

where $A^a(u)=u^\mu A^a_\mu=-\frac{1}{2}iF^2\,[R^a(u)-L^a(u)]$ is the isovector axial current and the dots denote terms which do not contribute to the one-pion matrix element to $\mathcal{O}(p^2)$ accuracy for reasons analogous to those we discussed after (32). One readily finds $\langle\pi^b(p)|\,\mathcal{O}^a_{5,n}(u)\,|0\rangle=-i\delta^{ab}(pu)^nF\,\tilde b_{n,n-1}$ at order $\mathcal{O}(p^0)$. The pion distribution amplitude is defined as

$$\frac{1}{2}\int\frac{d\lambda}{2\pi}\,e^{ix\lambda(pu)/2}\,\langle\pi^b(p)|\,\bar q(-\tfrac{1}{2}\lambda u)\,u^\mu\gamma_\mu\gamma_5\,\tau^a q(\tfrac{1}{2}\lambda u)\,|0\rangle=-i\delta^{ab}F_\pi\,\phi_\pi(x)\,,\tag{51}$$

where $\frac{1}{2}(1+x)$ is the momentum fraction of the quark in the pion and $\frac{1}{2}\int_{-1}^1 dx\,\phi_\pi(x)=1$. Here F_π is the physical value of the pion decay constant, given in one-loop ChPT as [17]

$$F_\pi=F\left(1-\frac{m^2}{16\pi^2F^2}\,\log\frac{m^2}{\mu^2}+\frac{m^2}{F^2}\,l^r_4(\mu)+\mathcal{O}(p^4)\right),\tag{52}$$

where μ denotes the renormalization scale and $l_4^r(\mu)$ is a renormalized low-energy constant. Taking x-moments of the distribution amplitude, one obtains the matrix elements of $\mathcal{O}^a_{5,n}(u)$, so that to order $\mathcal{O}(p^0)$ one has

$$B_n \stackrel{\text{def}}{=} \frac{1}{2^n}\int_{-1}^1 dx\, x^{n-1}\phi_\pi(x) = \tilde b_{n,n-1}\,. \qquad (n \text{ odd}) \tag{53}$$

Combining this with (48) and going from moments to x-space, we obtain the soft pion theorem $H^{I=1}(x,\xi\to 1,0) = \phi_\pi(x)$ from [14]. Let us now consider the order $\mathcal{O}(p^2)$ corrections to the one-pion matrix elements. A nonanalytic dependence on the pion mass is generated by one-loop graphs with the insertion of the leading-order operator (50), which according to (32) contains terms with three pion fields. Since for different n the operators in (50) only differ by their coefficients $\tilde b_{n,n-1}$ and the number of overall derivatives $u\partial$, the one-loop corrections of the one-pion matrix elements for all n are universal and hence the same as the corrections to the isovector axial current, given by (52). We can therefore write

$$\langle\pi^b(p)|\,\mathcal{O}^a_{5,n}(u)\,|0\rangle = -i\delta^{ab}(pu)^n F_\pi\,\tilde b_{n,n-1} + \mathcal{O}(m^2) \tag{54}$$

to order $\mathcal{O}(p^2)$ accuracy, where the $\mathcal{O}(m^2)$ terms are generated by tree-level insertions of higher-order terms in the ChPT expansion of $\mathcal{O}^a_{5,n}(u)$, which can be constructed in analogy to those in Section 4. As a consequence, the relation (53) does not receive nonanalytic corrections in the pion mass at next-to-leading order in ChPT, and one has $B_n = \tilde b_{n,n-1} + \mathcal{O}(m^2)$ up to terms of order $\mathcal{O}(p^4)$. Together with the $\mathcal{O}(p^2)$ corrections of the form factors $A^{I=1}_{n,k}(t)$, this will allow us to discuss the one-loop corrections to the soft pion theorem in Section 5. We note that to the order given, the bare pion mass m in (52) and (54) can be replaced with its renormalized value m_π, given by [17]

$$m_\pi^2 = m^2\Big(1 + \frac{m^2}{32\pi^2F^2}\,\log\frac{m^2}{\mu^2} + \frac{2m^2}{F^2}\,l_3^r(\mu) + \mathcal{O}(p^4)\Big)\,. \tag{55}$$

Let us finally show that our representations (41) and (45) of the operators $\mathcal{O}^s_n(u)$ and $\mathcal{O}^a_n(u)$ in ChPT are equivalent to those given in [19] without derivation. There, the matching was done for the nonlocal operators appearing in the definition of GPDs. For the left-handed current we find in our notation

$$\begin{aligned}&\bar q_j(-\tfrac12\lambda u)\,u^\mu\gamma_\mu\frac{1-\gamma_5}{2}\,q_i(\tfrac12\lambda u)\\ &\quad\simeq\ \frac12F^2\int_{-1}^1 d\beta\int_{-1+|\beta|}^{1-|\beta|}d\alpha\,F(\beta,\alpha)\,U^\dagger_{ik}(\lambda_1 u)\,(iu\overleftrightarrow{\partial})\,U_{kj}(\lambda_2 u)+\dots\,,\end{aligned} \tag{56}$$

where $\lambda_1 = -\frac12(\beta+\alpha)\lambda$ and $\lambda_2 = \frac12(\beta-\alpha)\lambda$, and where the dots denote terms not contributing to two-pion matrix elements at order $\mathcal{O}(p^2)$. Due to time reversal invariance the function $F(\beta,\alpha)$ is even in α. Taylor expansion gives the local operators

$$\begin{aligned}&\bar q_j u^\mu\gamma_\mu\frac{1-\gamma_5}{2}\,(iu\overleftrightarrow{D})^{n-1}\,q_i \\ &\quad\simeq\ \frac12F^2\sum_{m=0,2,\dots}^{n-1}\binom{m}{n-1}\int_{-1}^1 d\beta\int_{-1+|\beta|}^{1-|\beta|}d\alpha\,\beta^{n-m-1}\alpha^m F(\beta,\alpha)\,(\tfrac12 iu\partial)^m\Big[U^\dagger_{ik}\,(iu\overleftrightarrow{\partial})^{n-m}U_{kj}\Big]+\dots\,,\end{aligned} \tag{57}$$

where all fields are taken at space-time position zero and the dots have the same meaning as before. By repeated use of the identity $\overleftrightarrow{\partial}U = U\overleftrightarrow{\partial} + UL$ we can rewrite

$$U^\dagger\,(u\overleftrightarrow{\partial})^n\,U = \begin{cases}\displaystyle\sum_{m=0,2,\dots}^{n-2}(\tfrac12u\partial)^m\Big[L\,(u\overleftrightarrow{\partial})^{n-m-2}\,L\Big] + \mathcal{O}(L^3) & (n\text{ even})\\ \displaystyle\sum_{m=0,2,\dots}^{n-3}(\tfrac12u\partial)^m\Big[L\,(u\overleftrightarrow{\partial})^{n-m-2}\,L\Big] + (\tfrac12u\partial)^{n-1}L + \mathcal{O}(L^3) & (n\text{ odd})\end{cases} \tag{58}$$

and then project out the isoscalar part for n even and the isovector part for n odd. Repeating the same arguments for the right-handed operators, we recognize precisely the operators that appear in (41) and (45).

4 Tensor operators at order $\mathcal{O}(p^2)$

The number of independent terms in the ChPT expansion of operators rapidly grows with the order. Henceforth we shall consider only the terms which contribute to two-pion matrix elements at tree-level. Taking into account the discussion in the previous section, one finds that the corrections of order $\mathcal{O}(p^2)$ to the operators can be cast into the form

$$\begin{aligned}\Delta\mathcal{O}^s_n(u) &= \frac{1}{2}\,a^\chi_n\,\mathrm{tr}\left\{\chi^L_-\,L_n(u)+\chi^R_-\,R_n(u)\right\}+\frac{1}{2}\sum_{k=1}^{n/2}a^\chi_{n,k}\;\mathrm{tr}\left\{L_k(u)\,L_{n-k}(u)+R_k(u)\,R_{n-k}(u)\right\}\mathrm{tr}\,\chi^L_+\\ &+\frac{1}{2}\sum_{k=0}^{n/2}\Delta a_{n,k}\,\mathrm{tr}\left\{L_{\rho,k}(u)\,L^\rho_{n-k}(u)+R_{\rho,k}(u)\,R^\rho_{n-k}(u)\right\}\end{aligned}\tag{59}$$

for $\mathcal{O}^s_n(u)$ and

$$\begin{aligned}\Delta\mathcal{O}^a_n(u) &= ib^\chi_{n,0}\left(L^a_n(u)+R^a_n(u)\right)\mathrm{tr}\,\chi^L_+ +\varepsilon^{abc}\sum_{k=1}^{(n-1)/2}b^\chi_{n,k}\;\left(L^b_k(u)\,L^c_{n-k}(u)+R^b_k(u)\,R^c_{n-k}(u)\right)\mathrm{tr}\,\chi^L_+\\ &+\varepsilon^{abc}\sum_{k=0}^{(n-1)/2}\Delta b_{n,k}\;\left(L^b_{\rho,k}(u)\,L^{\rho,c}_{n-k}(u)+R^b_{\rho,k}(u)\,R^{\rho,c}_{n-k}(u)\right)\end{aligned}\tag{60}$$

for $\mathcal{O}^a_n(u)$, where we have introduced $L_{\rho,k}(u)=L_{\rho\mu_1\ldots\mu_k}\,u^{\mu_1}\ldots u^{\mu_k}$ and its right-handed counterpart. Using hermiticity and time reversal invariance one can deduce that the coefficients a^χ_n, b^χ_n, $a^\chi_{n,k}$, $b^\chi_{n,k}$ and $\Delta a_{n,k}$, $\Delta b_{n,k}$ are real. The order $\mathcal{O}(p^2)$ part of a twist-two operator which contributes to two-pion matrix elements is hence parameterized by $2\,([n/2]+1)$ real constants. Note that the number of the form factors $A^I_{n,k}(t)$ in (9) is $[n/2]+1$. At order $\mathcal{O}(p^2)$ the form factors thus read

$$A^I_{n,k}(t)=A^{(0)}_{n,k}+A^{(1,m)}_{n,k}\,m^2+A^{(1,t)}_{n,k}\,t+\text{loop contributions},\tag{61}$$

where all parameters $A^{(1,m)}_{n,k}$, $A^{(1,t)}_{n,k}$ are independent and can be expressed in terms of the coefficients a^χ_n, b^χ_n etc. in (59) and (60).

5 One-loop contributions

To calculate the two-pion matrix element of the operator $\mathcal{O}^s_n(u)+\Delta\mathcal{O}^s_n(u)$ or $\mathcal{O}^a_n(u)+\Delta\mathcal{O}^a_n(u)$ one has to take into account loop contributions. The diagrams to be calculated are drawn in Fig. 1. The technique of such calculations in ChPT is well elaborated, so we omit the details and directly give our results. Introducing the notation $L^a(\lambda u)=u^\mu L^a_\mu(\lambda u)$ and its analog for the right-handed fields one gets

$$\begin{aligned}\langle\pi^b(p')|\,L^c(\lambda_1 u)\,L^c(\lambda_2 u)\,|\pi^a(p)\rangle^{1\text{ loop}} &= -\,\delta^{ab}\,\frac{m^2-2t}{32\pi^2F^4}\,\Gamma(\epsilon)\\ &\times\xi^2(Pu)^2\,e^{i\xi(Pu)\,(\lambda_1+\lambda_2)}\int_{-1}^{1}d\alpha\,e^{i\alpha\xi(Pu)\,(\lambda_1-\lambda_2)}\,(1-\alpha^2)\left[\frac{4m^2-(1-\alpha^2)\,t}{16\pi\mu^2}\right]^{-\epsilon}\end{aligned}\tag{62}$$

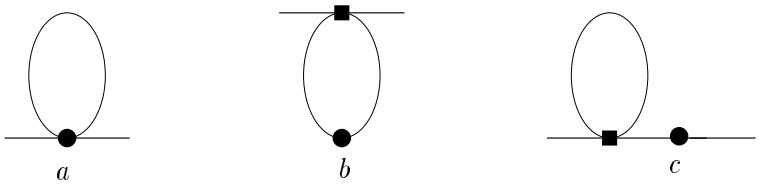

Figure 1: The one-loop graphs contributing to the two-pion matrix elements of the twist-two operators $\mathcal{O}_s(u)$ and $\mathcal{O}_a(u)$. The black blobs denote the operator insertion and the black boxes the π^4 vertex.

in $d = 4 - 2\epsilon$ dimensions, where μ is the renormalization scale. Taylor expanding in λ_1 and λ_2 as we did in going from (56) to (57), one can read off the one-loop corrections for the local operators in (41). Taking into account (9), one concludes that the form factors $A_{n,k}^{I=0}(t)$ with $k < n$ do not receive additional contributions at one-loop order and hence depend linearly on m^2 and t,

$$A_{n,k}^{I=0}(t) = A_{n,k}^{(0)} + A_{n,k}^{(1,m)}\, m^2 + A_{n,k}^{(1,t)}\, t \qquad\qquad (k \le n-2) \tag{63}$$

up to corrections of $\mathcal{O}(p^4)$, where as in Section 3 we omit isospin labels in $A_{n,k}^{(0)}$, $A_{n,k}^{(1,m)}$ and $A_{n,k}^{(1,t)}$. For the form factor $A_{n,n}^{I=0}(t)$ one finds after some algebra

$$\begin{aligned} A_{n,n}^{I=0}(t) &= A_{n,n}^{(0)} + A_{n,n}^{(1,m)}(\mu)\, m^2 + A_{n,n}^{(1,t)}(\mu)\, t + \frac{m^2-2t}{64\pi^2F^2} \\ &\times \sum_{k=2,4,\ldots}^{n} 2^{k-n} A_{n,k}^{(0)} \int_{-1}^{1} d\alpha\, (1-\alpha^k)\alpha^{n-k} \left(\log\left[\frac{4m^2-(1-\alpha^2)\,t}{4\mu^2}\right] + 1 \right), \end{aligned} \tag{64}$$

where $A_{n,k}^{(0)}$ is given in (42). Here $A_{n,n}^{(1,m)}(\mu)$ and $A_{n,n}^{(1,t)}(\mu)$ are renormalized using the subtraction scheme of [17], where together with a $1/\epsilon$ pole one subtracts a finite constant in the combination $1/\epsilon + \psi(2) + \log(4\pi)$. In particular, we find for $n = 2, 4$

$$\begin{aligned} A_{2,2}^{I=0}(t) &= A_{2,2}^{(0)} \left(1 + \frac{m^2-2t}{48\pi^2F^2} \left[\log\frac{m^2}{\mu^2} + \frac{4}{3} - \frac{t+2m^2}{t}\, J(t) \right] \right) \\ &+ A_{2,2}^{(1,m)}(\mu)\, m^2 + A_{2,2}^{(1,t)}(\mu)\, t\,, \end{aligned} \tag{65}$$

$$\begin{aligned} A_{4,4}^{I=0}(t) &= A_{4,4}^{(0)} \left(1 + \frac{m^2-2t}{40\pi^2F^2} \left[\log\frac{m^2}{\mu^2} + \frac{11}{10} + \frac{1}{6}\sigma^2 + \frac{1}{4}(\sigma^4-5)\, J(t) \right] \right) \\ &+ A_{4,2}^{(0)}\, \frac{m^2-2t}{960\pi^2F^2} \left[\log\frac{m^2}{\mu^2} - \frac{1}{15} + \sigma^2 + \frac{\sigma^2}{2}(3\sigma^2-5)\, J(t) \right] \\ &+ A_{4,4}^{(1,m)}(\mu)\, m^2 + A_{4,4}^{(1,t)}(\mu)\, t\,, \end{aligned} \tag{66}$$

where

$$J(t) = 2 + \sigma \log\frac{\sigma-1}{\sigma+1}\,, \qquad\qquad \sigma = \sqrt{1-\frac{4m^2}{t}}\,. \tag{67}$$

For $|t| < 4m^2$ one can expand $J(t) = -2/(3\sigma^2) - 2/(5\sigma^4) - \mathcal{O}(1/\sigma^6) = t/(6m^2) + t^2/(60m^4) - \mathcal{O}(t^3/m^6)$. Note that to the order we are working at, one may replace the bare quantities F^2 and m^2 in the formulae of the present section with their one-loop expressions, given in (52) and (55).

The order $\mathcal{O}(p^2)$ correction to the matrix element of the isovector vector current (20) reads [17] [2]

$$\langle\pi^b(p')|\,V_\mu^c\,|\pi^a(p)\rangle = 2i\epsilon^{abc}\,P_\mu\left(1+F_V(t,m^2)-l_6^r(\mu)\,t/F^2\right), \tag{68}$$

where the renormalized low-energy constant $l_6^r(\mu)$ is defined in [17] and

$$\begin{aligned} F_V(t,m^2) &= \frac{1}{128\pi^2F^2}\int_{-1}^1 d\alpha\left[4m^2-(1-\alpha^2)\,t\right]\log\left[\frac{4m^2-(1-\alpha^2)\,t}{4\mu^2}\right]-\frac{m^2}{16\pi^2F^2}\log\frac{m^2}{\mu^2} \\ &= \frac{1}{96\pi^2F^2}\left[(t-4m^2)\,J(t)-t\left(\frac{1}{3}+\log\frac{m^2}{\mu^2}\right)\right]. \end{aligned} \tag{69}$$

To complete the calculation of the isovector current at order $\mathcal{O}(p^2)$ it is sufficient to compute the one-loop corrections to the pion matrix element of the nonlocal operator $\mathcal{O}^a(\lambda_1,\lambda_2)=\epsilon^{abc}L^b(\lambda_1 u)L^c(\lambda_2 u)$. We obtain

$$\begin{aligned} \langle\pi^b(p')|\,\mathcal{O}^c(\lambda_1,\lambda_2)\,|\pi^a(p)\rangle^{1\text{ loop}} &= -\frac{m^2}{16\pi^2F^2}\,\Gamma(-1+\epsilon)\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon}\langle\pi^b(p')|\,\mathcal{O}^c(\lambda_1,\lambda_2)\,|\pi^a(p)\rangle^{\text{tree}} \\ &- \epsilon^{abc}\frac{1}{64\pi^2F^4}\,\Gamma(-1+\epsilon)\,\xi\,(Pu)^2\,e^{i\xi(Pu)\,(\lambda_1+\lambda_2)} \\ &\times\int_{-1}^1 d\alpha\left[4m^2-(1-\alpha^2)\,t\right]\left[\frac{4m^2-(1-\alpha^2)\,t}{16\pi\mu^2}\right]^{-\epsilon}\frac{\partial}{\partial\alpha}\left[e^{i\alpha\xi(Pu)\,(\lambda_1-\lambda_2)}\,(1-\alpha^2)\right]. \end{aligned} \tag{70}$$

Making use of (45) and (47) one can represent the form factors $A_{n,k}^{I=1}(t)$ with $k<n-1$ as

$$A_{n,k}^{I=1}(t)=A_{n,k}^{(0)}\left(1-\frac{m^2}{16\pi^2F^2}\log\frac{m^2}{\mu^2}\right)+A_{n,k}^{(1,m)}(\mu)\,m^2+A_{n,k}^{(1,t)}\,t\qquad(k\le n-3) \tag{71}$$

up to corrections of order $\mathcal{O}(p^4)$. For the form factor $A_{n,n-1}^{I=1}(t)$ one finds after some algebra

$$\begin{aligned} A_{n,n-1}^{I=1}(t) &= A_{n,n-1}^{(0)}\left(1-\frac{m^2}{16\pi^2F^2}\log\frac{m^2}{\mu^2}\right)+A_{n,n-1}^{(1,m)}(\mu)\,m^2+A_{n,n-1}^{(1,t)}(\mu)\,t \\ &+ \frac{1}{64\pi^2F^2}\sum_{k=0,2,\ldots}^{n-1}2^{k-n}\,(n-k)\,A_{n,k}^{(0)}\int_{-1}^1 d\alpha\,\alpha^{n-k-1}\left[4m^2-(1-\alpha^2)\,t\right]\log\left[\frac{4m^2-(1-\alpha^2)\,t}{4\mu^2}\right]. \end{aligned} \tag{72}$$

For $n=1$ this gives

$$A_{1,0}^{I=1}(t)=2\left(1+F_V(t,m^2)\right)+A_{1,0}^{(1,t)}(\mu)\,t\,, \tag{73}$$

where $A_{1,0}^{(1,t)}(\mu)=-2l_6^r(\mu)/F^2$ according to (68). Note that at $t=0$ this form factor is fixed by current conservation, so that $A_{1,0}^{(1,m)}(\mu)=0$. For $n=3$ we find

$$\begin{aligned} A_{3,2}^{I=1}(t) &= A_{3,2}^{(0)}\left(1+F_V(t,m^2)\right)+A_{3,2}^{(1,m)}(\mu)\,m^2+A_{3,2}^{(1,t)}(\mu)\,t \\ &- A_{3,0}^{(0)}\,\frac{t}{1280\pi^2F^2}\left[(5\sigma^2-3)\log\frac{m^2}{\mu^2}+\frac{6}{5}-\frac{4}{3}\sigma^2-2\sigma^4J(t)\right]. \end{aligned} \tag{74}$$

[2] The corresponding form factor is known to two-loop accuracy now [20].

Our results (71) and (72) agree with Eq. (27) in [19], and our results (63) and (64) agree with Eq. (26) there if a misprint is corrected.[3] Furthermore, our expression (65) for $A^{I=0}_{2,2}(t)$ agrees with [16].

From (63) and (64) we can readily generalize to order $\mathcal{O}(p^2)$ the relation (44) for the sum of isoscalar form factors of given n. In particular we find for $t = 0$

$$\sum_{k=0,2,\ldots}^{n} 2^k A^{I=0}_{n,k}(0) \;=\; \frac{m_\pi^2}{64\pi^2 F_\pi^2}\,\log\frac{m_\pi^2}{\mu^2} \sum_{k=2,4,\ldots}^{n} 2^k A^{I=0}_{n,k}(0)\,\frac{2k}{(n+1)(n-k+1)} + \mathcal{O}(m_\pi^2), \tag{75}$$

where on the right-hand side we have replaced the tree-level quantities F, m, $A^{(0)}_{n,k}$ by the full ones. We have a relation between observables, which may for instance be tested in lattice calculations as a consistency test for the applicability of one-loop ChPT for a given range of unphysical pion masses. Using (71) and (72) and our discussion following (54), we obtain a relation at order $\mathcal{O}(p^2)$ between the sum of isovector form factors of given n and the moments B_n of the pion distribution amplitude. For $t = 0$ the logarithms in m^2 cancel and we have

$$\sum_{k=0,2,\ldots}^{n-1} 2^k A^{I=1}_{n,k}(0) = 2^n B_n + \mathcal{O}(m_\pi^2)\,. \tag{76}$$

For $n = 1$ this relation is trivial since $A^{I=1}_{1,0}(0) = 2$ and $B_1 = 1$ are exact, but for $n \geq 3$ it presents again an interesting test in lattice QCD, where both the form factors $A^I_{n,k}(t)$ and the moments B_n are calculable. Translated to x-space, one finds that the soft-pion theorem $H^{I=1}(x, \xi \to 1, 0) = \phi_\pi(x) + \mathcal{O}(m_\pi^2)$ does not contain nonanalytic terms in the pion mass at one-loop order, as was pointed out in [19].[4]

6 Summary and conclusions

We have calculated the moments of the generalized parton distributions of the pion at order $\mathcal{O}(p^2)$ of ChPT. Our main results are given in (63), (64) and (71), (72) for the isoscalar and isovector matrix elements, respectively. We find that at one-loop order only the form factors accompanying the highest power of ξ at given n have a nontrivial dependence on t and m_π^2. All others depend linearly on t. The isoscalar form factors are linear functions of m_π^2 as well, whereas the isovector ones receive universal logarithmic corrections in m_π^2. At order $\mathcal{O}(p^2)$ the $[n/2]+1$ form factors $A^I_{n,k}$ at given n are determined by $3\,([n/2]+1)$ real parameters, which in the isoscalar case are subject to the restriction (44). One can check that for a fixed ratio t/m_π^2 the form factors in (64) and (72) deviate from a linear behavior in m_π^2 only for pion masses close to the physical value, and that this deviation is rather small.

The possibility to use the formulae derived here to extrapolate lattice data obtained for pion masses around 500 MeV to the physical pion mass obviously depends crucially on the size of the higher-order corrections. Taking into account that ChPT works quite well in the kaon sector [22] one may hope that the two-loop corrections will not be large for $m_\pi^2 \sim 500$ MeV. In any case a strong deviation of the lattice data from a linear dependence on m_π^2 would indicate the need for higher-order ChPT fits to describe the lattice data. The comparison of the ChPT predictions with lattice data for the form factor $A^{I=1}_{1,0}(t) = 2F_\pi^{em}(t)$, where all $\mathcal{O}(p^2)$ parameters are known [17, 20], will provide a crucial convergence test. The same holds for relations which involve only lattice observables, namely the relation (75) between isoscalar form factors and the relation (76) between isovector form factors and the moments of the pion distribution amplitude, as well as their generalizations for finite t.

[3] The one-loop term in Eq. (26) of [19] needs to be multiplied with a factor 1/2 [21].

[4] To be precise, in [19] this relation was given with the pion distribution amplitude taken in the chiral limit, without discussing its one-loop properties in ChPT.

At two-loop order six new parameters appear for each form factor. Three of them are the coefficients of the analytic terms m_π^4, tm_π^2, t^2, and the other three arise from terms in the expansion of operators which do not contribute to the two-pion matrix element at one-loop order ($m_\pi^4 \log^2 m_\pi^2$, $m_\pi^4 \log m_\pi^2$, and $tm_\pi^2 \log m_\pi^2$). Obviously the lattice data, even if it covers several different volumes and many pion masses, can hardly be hoped to be accurate enough to obtain a stable simultaneous fit of all these parameters. However, the same parameters should enter the m_π and volume dependence of other lattice observables, such that these will provide complementary information. A detailed analysis of these relationships remains, however, to be worked out.

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