DESY 05-151
WUE-ITP-2005-009
SFB/CPP-05-40

arXiv:hep-ph/0508212 v1 19 Aug 2005

DESY 05-151
WUE-ITP-2005-009
SFB/CPP-05-40

August 2005

Harmonic polylogarithms for massive Bhabha scattering

M. Czakon[ab], J. Gluza[bc], T. Riemann[c]

[a]Institut für Theoretische Physik und Astrophysik, Universität Würzburg, Am Hubland, D-97074 Würzburg, Germany

[b]Institute of Physics, Univ. of Silesia, Universytecka 4, 40007 Katowice, Poland

[c]DESY, Platanenallee 6, 15738 Zeuthen, Germany

One- and two-dimensional harmonic polylogarithms, HPLs and GPLs, appear in calculations of multi-loop integrals. We discuss them in the context of analytical solutions for two-loop master integrals in the case of massive Bhabha scattering in QED. For the GPLs we discuss analytical representations, conformal transformations, and also their transformations corresponding to relations between master integrals in the s- and t-channel.

1. Introduction

Multiloop calculations may be treated with numerical or analytical approaches. The latter is limited to issues with few, typically at most three different scales. But it has well-known attractive features, notably the knowledge of the explicit structure of the singularities and a good control of numerical stability.

The analytical approach to be discussed here relies on differential equations [1,2]. Iterative solutions in powers of $\epsilon = (4-D)/2$ are inherently connected with certain classes of special functions. For two-scale problems (e. g. QED self-energies and vertices), the harmonic polylogarithms $H(\{a\},x)$ (HPLs) have been introduced [3]; the index vector $\{a\}$ has elements $1, 0, -1$. We just mention that HPLs with some generalized arguments are introduced in [4]. A generalization of HPLs for three-scale problems (e.g. QED boxes) are the two-dimensional harmonic polylogarithms $G(\{b\},x)$ (GPLs); the index vector $\{b\}$ has elements $1, 0, -1$, but now also those depending on a second kinematic variable y. This was observed in [5] and systematically worked out in [6] for a planar massless problem. There one may find analytical expressions for the GPLs until weight 3 with indexes $0, 1, z, 1-z$. Another generalization is performed in [7]. Algebraic non-linear factors are introduced as being motivated by the physical nature of the problem treated there.

Let us shortly review some facts on HPLs. HPLs up to weight 4 are expressed, with few exceptions, in terms of Nielsen Polylogarithms [3,8,9]. The exceptions are of weight 4 and may be expressed by few integrals [9]:

$$I_1(x) = \int_0^x \frac{dy}{1+y}\mathrm{Li}_3(y), \tag{1}$$

$$I_2(x) = \int_0^x \frac{dy}{1+y}\mathrm{Li}_2(y)\ln(1-y), \tag{2}$$

$$I_3(x) = \int_0^x \frac{dy}{1+y}\ln(y)\ln^2(1-y), \tag{3}$$

$$I_4(x) = \int_0^x \frac{dy}{1+y}\ln^2(y)\ln(1-y), \tag{4}$$

where $I_1(x) = H[-1,0,0,1,x]$. However, it cannot be excluded that some relations among them exist. In fact, a relation between I_3 and I_4 exists [10]. In fact, exploiting that $\int dy \ln^3(y/(1-y))$ is known to Mathematica (after substituting $y/(1-y) \to z$) and is related to $I_3 - I_4$, one may derive

$$\begin{aligned}
I_4(x) &= I_3(x) + \frac{1}{3}\Bigg[\ln^3(1-x)\ln\left(\frac{1+x}{2}\right) \\
&- \ln^3(x)\ln(1+x) + \ln^3\left(\frac{x}{1-x}\right)\ln(1+x)\Bigg] \\
&+ \ln^2(1-x)\mathrm{Li}_2\left(\frac{1-x}{2}\right) - \ln^2(x)\mathrm{Li}_2(-x) \\
&- 2\ln(1-x)\mathrm{Li}_3\left(\frac{1-x}{2}\right) + 2\ln(x)\mathrm{Li}_3(-x) \\
&+ \ln\left(\frac{x}{1-x}\right)\Bigg[\ln\left(\frac{x}{1-x}\right) \\
&\times \left(\mathrm{Li}_2\left(\frac{2x}{x-1}\right) - \mathrm{Li}_2\left(\frac{x}{x-1}\right)\right) \\
&+ 2\left(\mathrm{Li}_3\left(\frac{x}{x-1}\right) - \mathrm{Li}_3\left(\frac{2x}{x-1}\right)\right)\Bigg] \\
&+ 2\Bigg[\mathrm{Li}_4\left(\frac{1-x}{2}\right) - \mathrm{Li}_4(1/2) - \mathrm{Li}_4(-x) \\
&- \mathrm{Li}_4\left(\frac{x}{x-1}\right) + \mathrm{Li}_4\left(\frac{2x}{x-1}\right)\Bigg]. \qquad (5)
\end{aligned}$$

As far as numerical solutions are concerned, the problem is solved for any weight of HPLs. The numerical evaluation of HPLs with indexes $\{0,1,-1\}$ (and the analytic continuation) is described in [11] and may be performed with the Fortran program `hplog`. These HPLs have been programmed also in Mathematica [12]. The systematic numerical treatment of GPLs with indexes $\{0,1,1-z,-z\}$ is given in [13] and performed with Fortran program `tdhpl`.[1] Please notice a change of notation compared to [6] as described in the Appendix of [13].

In this paper, we will discuss features of the GPLs which we are exploiting for a study of massive QED two-loop boxes. These GPLs were used in e.g. [14,15] and [9,16,17]. In the next section we describe the general idea of solving differential equations for Feynman integrals with GPLs, then we sketch a simple one loop case, and what finally follows is a section on some algebraic relations connected with GPLs as we are using them for Bhabha scattering.

[1] This is in fact not sufficient to cover our physical cases; we need indexes $\{0,1,-1,1-z,-z\}$).

2. Differential equations

Master integrals (MIs) M may be determined as solutions of appropriate differential equations. The general idea is simple.[2] Let M fulfill some differential equation with respect to x,

$$\frac{d}{dx}M(x) = A(x)M(x) + B(x), \qquad (6)$$

where x is usually the external scale like s (or t) or a conformal counterpart x (or y); for definitions and details see [9]. The inhomogeneity B is a linear combination of MIs of lower complexity and assumed to be known from an earlier stage of iteration. For simplicity we assume here M to depend only on x. If $H(x)$ is a solution of the homogeneous equation,

$$\frac{d}{dx}H(x) = A(x)H(x), \qquad (7)$$

then the full solution is given by

$$M(x) = H(x)\left[Const + \int^x \frac{dx'}{H(x')}B(x')\right]. \qquad (8)$$

The solutions in arbitrary dimension D are generally combinations of generalized hypergeometric functions which are difficult (or at least tedious) to find and to expand in powers of ϵ. This statement is true already for the massive QED one-loop box [19], where Appell hypergeometric functions and a Kampé de Fériet function appear. An alternative idea is to expand (8) in ϵ:

$$M = \sum_{i=-\alpha}^{\beta} m^i \epsilon^i, \qquad (9)$$

$$A = \sum_{i=0}^{\alpha+\beta} a^i \epsilon^i, \quad B = \sum_{i=-\alpha}^{\beta} b^i \epsilon^i, \qquad (10)$$

where α is fixed by the physical problem and β chosen reasonably. Then one may try to solve

[2] A nice pedagogical introduction is [18].

iteratively the system of equations for the $m^i(x)$:

$$\frac{d}{dx}m^i(x) = \sum_{j=0}^{\alpha-i} a^j(x)\ m^{i-j}(x) + b^i(x). \qquad (11)$$

Let us mention that A should not be singular in ϵ ($i \geq 0$ in (10)). Otherwise some $m^i(x)$ on the LHS of (11) would depend on a higher order component $m^{i-j}(x)$ of M sitting on the RHS, and the recursion could not be solved. With this assumption the solution is of the form

$$m^i(x) = H(x)\left(Const + \int^x \frac{dx'}{H(x')}\left[\sum_{j=1}^{\alpha-i} a^j(x) m^{i-j}(x) + b^i(x)\right]\right). \qquad (12)$$

$H(x)$ is the solution of the homogeneous equation (7) and is the same for all orders in ϵ.

In general, some sets of MIs will fulfill a system of linear differential equations. For massive Bhabha scattering so far only one such system `B5l4m`, the two-loop boxes with 5 lines, 4 of them massive, with two MIs has been fully solved analytically in the language of HPls and GPLs [9,15]. Another analytically fully solved 4-point MI is `B5l2m1` [16]. We will comment on the more complex system of 4 MIs `B5l2m3` in the context of GPLs in Section 4.

3. The Bhabha one-loop box: An illustration

A differential equation for the one-loop box `B4l2m` is [3]

$$s\frac{\partial}{\partial s}\texttt{B4l2m[s,t]} = a\ \texttt{B4l2m[s,t]} + b\ \texttt{SE2l2m[s]} + c\ \texttt{T1l1m} + d\ \texttt{SE2l0m[t]} + e\ \texttt{V3l1m[s]}. \qquad (13)$$

The tadpole `T`, self energies `SE`, and vertex `V` are known. The factor a of the homogeneous part is

$$a = \frac{8 + s^2 - 2t + s(-6 + t + \epsilon t)}{(-4+s)(-4+s+t)},$$

After a change to conformal variables

$$x(y) = \frac{\sqrt{-s(t)+4} - \sqrt{-s(t)}}{\sqrt{-s(t)+4} + \sqrt{-s(t)}}, \qquad (14)$$

[3] For notations see [9].

the a has a simple rational denominator:

$$a = \frac{1}{(1+x)^2}\frac{1}{(x+y)}\frac{1}{(1+xy)} \times \{\cdots\}.$$

Such an expression can now be decomposed into terms with monomial denominators depending on x only,

$$f(0,x) = \frac{1}{x}, \qquad f(1,x) = \frac{1}{1-x},$$
$$f(-1,x) = \frac{1}{1+x}, \qquad (15)$$

and those which depend both on x and y:

$$g(-y,x) = \frac{1}{x+y}, \quad g(-1/y,x) = \frac{y}{1+xy}. \qquad (16)$$

Factors b, c, d, e in (13) follow the same structure.

The monomials become the kernels for two specific classes of GPLs:

$$G(b,\{a\},x) = \int_0^x dx' g(b,x') G(\{a\},x'), \qquad (17)$$
$$b = -y, -1/y.$$

Analogous relations define the HPLs, now with the index vector $\{0,1,-1\}$. If a simpler master was expressed by a structure like (17), then m^i will be easily determined by iterating the GPLs.

In this way, after performing the ϵ expansion of the objects in (13), the MI `B4l2m` can be solved systematically step by step to the desired order in ϵ. For more details on the specific example we refer to [15].

4. Algebra of GPLs

4.1. Analytical representation of GPLs

Up to weight two, it makes not much effort to find analytical representations for GPLs in terms of Nielsen polylogarithms. If we consider only the index vectors $\{-y, -1/y\}$ and $\{0,-1,1\}$, we have e.g. at weight two 25 GPLs (see Table 1), of which we can choose freely 5 as the irreducible integrals.

We put some results on GPLs up to weight four in file `GPL.m` at [20]. With conventions we follow [15,16,17,9]. Starting from weight three some of the GPLs in `GPL.m` are written as numerical integrals, e.g.

$$G[-y,-y,-1,x] = \int_0^x G[-y-1,z]/(y+z)dz. \qquad (18)$$

Similarly to the case of HPLs discussed in the Introduction, some of these integrals can be surely solved further to their analytical form.

Table 1
Number of GPLs for the index vectors $\{-y, -1/y\}$ and $\{0, -1, 1\}$. The second row counts HPLs with indexes $\{0, -1, 1\}$.

weight 1	weight 2	weight 3	weight 4
2 GPLs	16 GPLs	98 GPLs	544 GPLs
3 HPLs	9 HPLs	27 HPLs	81 HPLs

4.2. Interchange of arguments in GPLs

Sometimes in the course of solving MIs we want to use the knowledge of differential equations in a fixed channel, but in both s- and t-operators (or equivalently in x and y). As an example may serve the MI `B5l2m3d3`. The homogeneous part of its equation, if derived with the s-operator, vanishes, so that its solution is a constant. Then, the solution of the inhomogeneous equation is easy but it is not possible to find the constant term (a function $C_y(y)$) in the usual way by exploiting the knowledge of analyticity. However, we can alternatively determine the same function from an equation derived with the t-operator, having now another constant term (a function $C_x(x)$). An easy way to determine $C_y(y)$ is to compare the functional dependences of both solutions, e.g. in the limit $x = 1$, and to end up with an unknown true constant $C_x(1)$. It depends on polynomials in $\zeta_2, \zeta_3, \ldots$ The $C_x(1)$, in turn, can be numerically fitted by knowledge of `B5l2m3d3` at some Euclidean kinematic point using the sector decomposition method [21,22]. In this way, `B5l2m3d3` has been determined up to ϵ^0 (yet unpublished). Sometimes it may appear from the very beginning more convenient to solve some differential equations in the t-operator (where e.g. functions $G[-x, ..., y]$ appear) and rewrite this solution later for the s-operator where other MIs are to be solved (and where functions $G[-y, ..., x]$ appear). The interchange of arguments can also be used to find limits of GPLs when e.g. $x = 1$ or $y = 1$.

Useful relations for the case of massive Bhabha scattering are tabulated up to weight four in the file `GPLtransf.m` at [20]. All of them are based on the type of identities described in [6]. For instance, the first item of weight two in `GPLtransf.m`,

$$\begin{aligned} G[-y,-1,x] &= H[-1,x]H[1,y] \\ &- H[0,x]H[1,y] \\ &+ H[-1,x]H[-x,y] \\ &+ H[0,-1,x] \\ &+ H[1,0,y] - G[1,-x,y] \end{aligned} \tag{19}$$

can be solved using the relation

$$\begin{aligned} H[\vec{m}(z);y] &= H[\vec{m}(z=0);y] \\ &+ \int_0^z dz' \frac{d}{dz'} H[\vec{m}(z');y] \end{aligned} \tag{20}$$

after interchanging the differential and integration operations, combined with a decomposition into basic polynomials.

4.3. Conformal transformations

There are two different definitions in use for conformal transformations x (y) of s (t) in massive QED. These are (14), being used e.g. in [9,15], and

$$x'(y') = \frac{1}{\sqrt{1 - 4/s(t)}}, \tag{21}$$

being used e.g. in [23,24]. However, x and x' are connected by a relation which is also used for a transformation of variables in HPLs [3]:

$$x = \frac{1-x'}{1+x'}, \quad y = \frac{1-y'}{1+y'}. \tag{22}$$

So, if we have to transform MIs defined in terms of x to MIs defined in terms of x', we have to know the corresponding relations between GPLs.[4] Here is an example for this:

$$G(-y,0,x) = H[-y,0,1] + \int_1^x dz \frac{H[0,z]}{z + \frac{1-y'}{1+y'}}$$

[4] Results given in [23,24] are written directly using Nielsen Polylogarithms.

$$
\begin{aligned}
&= H\left[-\frac{1-y'}{1+y'},0,1\right] \\
&+ \int_0^{x'} dz[H(1,z)+H(-1,z)] \\
&\times \left[\frac{1}{1+z}+\frac{y'}{1-zy'}\right]. \qquad (23)
\end{aligned}
$$

Interchanging arguments,

$$H\,[-y,0,1] = -\zeta_2 + H[0,-1,y] - H[0,0,y],$$

using conformal transformations for HPLs [3], and integrating the expression in (23) we finally get

$$
\begin{aligned}
G[-y,0,x] &= -\zeta_2 - H[-1,1,y'] - H[1,1,y'] \\
&- H[-1,1](H[-1,y'] + H[1,y']) \\
&+ H[0,-1,1] - H[0,0,1] \\
&+ H[-1,-1,x'] + H[-1,1,x'] \\
&+ G[1/y',-1,x'] + G[1/y',1,x']. \qquad (24)
\end{aligned}
$$

Let us note that the conformal transformation (22) extends the original set of GPLs by those with additional arguments $\{+y,+1/y\}$. Again, results are given in a file `GPLconf.m` at [20].

In conclusion, we have discussed some basic features of GPLs which are used in an ongoing determination of fully analytical results in massive Bhabha scattering in QED. For other interesting properties of GPLs we have to refer to the literature; e.g. analytical continuation of GPLs is explored in [11]. A complete GPL package for general use in Mathematica will be given elsewhere.

Acknowledgements

We would like to thank for partial support by Sonderforschungsbereich/Transregio 9–03 of DFG 'Computergestützte Theoretische Teilchenphysik', by the Sofja Kovalevskaja Award of the Alexander von Humboldt Foundation sponsored by the German Federal Ministry of Education and Research, and by the Polish State Committee for Scientific Research (KBN) for the research project in 2004–2005.

REFERENCES

1. A.V. Kotikov, Phys. Lett. B259 (1991) 314.
2. E. Remiddi, Nuovo Cim. A110 (1997) 1435, hep-th/9711188.
3. E. Remiddi and J. Vermaseren, Int. J. Mod. Phys. A15 (2000) 725, hep-ph/9905237.
4. U. Aglietti and R. Bonciani, Nucl. Phys. B668 (2003) 3, hep-ph/0304028.
5. T. Gehrmann and E. Remiddi, Nucl. Phys. B580 (2000) 485, hep-ph/9912329.
6. T. Gehrmann and E. Remiddi, Nucl. Phys. B601 (2001) 248, hep-ph/0008287.
7. T. Birthwright, N. Glover and P. Marquard, JHEP 09 (2004) 042, hep-ph/0407343.
8. S. Moch and J. Vermaseren, Nucl. Phys. B573 (2000) 853, hep-ph/9912355.
9. M. Czakon, J. Gluza and T. Riemann, Phys. Rev. D71 (2005) 073009, hep-ph/0412164.
10. A. Davydychev, private communication.
11. T. Gehrmann and E. Remiddi, Comput. Phys. Commun. 141 (2001) 296, hep-ph/0107173.
12. D. Maitre, HPL, a Mathematica implementation of the harmonic polylogarithms, hep-ph/0507152.
13. T. Gehrmann and E. Remiddi, Comput. Phys. Commun. 144 (2002) 200, hep-ph/0111255.
14. R. Bonciani, P. Mastrolia and E. Remiddi, Nucl. Phys. B661 (2003) 289, hep-ph/0301170.
15. R. Bonciani et al., Nucl. Phys. B681 (2004) 261, hep-ph/0310333.
16. M. Czakon, J. Gluza and T. Riemann, Nucl. Phys. (Proc. Suppl.) B135 (2004) 83, hep-ph/0406203.
17. M. Czakon, J. Gluza and T. Riemann, On master integrals for two loop Bhabha scattering, hep-ph/0409017.
18. U. Aglietti, The evaluation of loop amplitudes via differential equations, hep-ph/0408014.
19. J. Fleischer, T. Riemann and O. Tarasov, Acta Phys. Polon. B34 (2003) 5345, hep-ph/0508194.
20. M. Czakon, J. Gluza and T. Riemann, http://www-zeuthen.desy.de/theory/

research/bhabha/.
21. T. Binoth and G. Heinrich, Nucl. Phys. B585 (2000) 741, hep-ph/0004013.
22. T. Binoth and G. Heinrich, Nucl. Phys. B680 (2004) 375, hep-ph/0305234.
23. V. Smirnov, Phys. Lett. B524 (2002) 129, hep-ph/0111160.
24. G. Heinrich and V. Smirnov, Phys. Lett. B598 (2004) 55, hep-ph/0406053.