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Explicit results for the anomalous three point function and non-renormalization theorems

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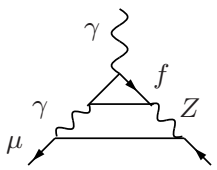
Abstract

Two-loop corrections for the $\langle VVA\rangle$ correlator of the singlet axial and vector currents in QCD are calculated in the chiral limit for arbitrary momenta. Explicit calculations confirm the non-renormalization theorems derived recently by Vainshtein [Phys. Lett. B **569** (2003) 187] and Knecht et al. [JHEP **0403** (2004) 035]. We find that as in the one-loop case also at two loops the $\langle VVA\rangle$ correlator has only three independent form-factors instead of four. From the explicit results we observe that the two-loop correction to the correlator is equal to the one-loop result times the constant factor $C_2(R)\alpha_s/\pi$ in the $\overline{\rm MS}$ scheme. This holds for the full correlator, for the anomalous longitudinal as well as for the non-anomalous transversal amplitudes. The finite overall α_s dependent constant has to be normalized away by renormalizing the axial current according to Witten's algebraic/geometrical constraint on the anomalous Ward identity [$\langle VV\partial A\rangle$ correlator]. Our observations, together with known facts, suggest that in perturbation theory the $\langle VVA\rangle$ correlator is proportional to the one-loop term to all orders and that the non-renormalization theorem of the Adler-Bell-Jackiw anomaly carries over to the full correlator.

1 Introduction

The Adler-Bell-Jackiw [1, 2] triangle anomaly in the divergence of the axial vector current is well known to play a crucial role at several places in elementary particle physics. Its nature is controlled by the Adler-Bardeen non-renormalization theorem [3], by the Wess-Zumino integrability condition and the Wess-Zumino effective action [4], by Witten's algebraic/geometrical interpretation which requires the axial current to be normalized to an integer [5] and by the t' Hooft quark-hadron duality matching conditions [6]. Phenomenologically, it plays a key role in the prediction of $\pi^0 \to \gamma\gamma$, and in the solution of the $U(1)$ problem. Last but not least, renormalizability of the electroweak Standard Model requires the anomaly cancellation which dictates the lepton-quark family structure.
More recently Vainshtein [7] found an important new relation $w_T(q^2) = \frac{1}{2}\, w_L(q^2)$ (see below) matching to all orders in perturbation theory, in some kinematical limit, the transversal part to the anomalous longitudinal amplitude which is subject to the Adler-Bardeen non-renormalization theorem. Later Knecht et al. [8] were able to generalize this kind of non-renormalization theorems. These recent investigations came up in connection with problems in calculating the leading hadronic effects in the electroweak two–loop contributions to the muon anomalous magnetic moment a_μ [9, 10, 11, 12].
The first electroweak two–loop calculations [13] for a_μ revealed that triangle fermion–loops may give rise to unexpectedly large radiative corrections. The diagrams which yield the leading corrections are those including a VVA triangular fermion–loop ($VVA \neq 0$ while $VVV = 0$) associated with a Z boson exchange



and a fermion of flavor f gives a contribution, up to UV singular terms which will cancel,

$$a_\mu^{(4)\,\mathrm{EW}}([f]) \simeq \frac{\sqrt{2} G_\mu m_\mu^2}{16\pi^2} \frac{\alpha}{\pi}\, 2 T_f N_{cf} Q_f^2 \left[3 \ln \frac{M_Z^2}{m_{f'}^2} + C_f \right]$$

where α is the fine structure constant, G_μ the Fermi constant, T_{3f} the 3rd component of the weak isospin, Q_f the charge and N_{cf} the color factor, 1 for leptons, 3 for quarks. The mass $m_{f'}$ is m_μ if $m_f < m_\mu$ and m_f if $m_f > m_\mu$. C_f denotes constant terms. Since, as granted in the Standard Model of elementary particles, anomaly cancellation by lepton–quark duality $\sum_f N_{cf} Q_f^2 T_{3f} = 0$ is at work, only the sums over complete lepton–quark families yield meaningful results relevant to physics. In any case the quark contributions have to be taken into account. In fact treating the quarks like free fermions (quark parton model QPM) the first family yields

$$a_\mu^{(4)\,\mathrm{EW}}([e,u,d])_{\mathrm{QPM}} \simeq -\frac{\sqrt{2} G_\mu\, m_\mu^2}{16\pi^2}\, \frac{\alpha}{\pi}\, \left[\ln \frac{m_u^8}{m_\mu^6 m_d^2} + \frac{17}{2} \right]$$

which demonstrates that the leading large logs $\sim \ln M_Z$ have dropped! However, the quark masses which appear here are illdefined constituent quark masses, which can hardly account reliably for the strong interaction effects.
Since we are interested in a static low energy quantity $a_\mu = \frac{1}{2}\,(g-2)_\mu = F_{\mathrm{M}}(0)$, given by the Pauli form factor at zero momentum transfer, the perturbative QCD (pQCD) calculation of the light quark

contributions seems more than questionable. However, there is a large scale in the game, namely the Z boson mass $M_Z \sim 91.19$ GeV, which makes an estimate by the quark parton model as a first step not completely unreasonable. Indeed, in the relevant kinematical region, the leading strong interaction effects may be parametrized by two VVA amplitudes, a longitudinal $w_L(Q^2)$ and a transversal $w_T(Q^2)$ one, which contribute as [11, 12]

$$\Delta a_\mu^{(4)\,\mathrm{EW}}([f])_{\mathrm{VVA}} \;\simeq\; \frac{\sqrt{2}G_\mu\, m_\mu^2}{16\pi^2}\,\frac{\alpha}{\pi}\int_{m_\mu^2}^{\Lambda^2} dQ^2\,\left(w_L(Q^2)+\frac{M_Z^2}{M_Z^2+Q^2}\,w_T(Q^2)\right)\,, \tag{1.1}$$

where Λ is a cutoff to be taken to ∞ at the end. For a perturbative fermion loop to leading order [14]

$$\begin{aligned} w_L^{1-\mathrm{loop}}(Q^2) = 2w_T^{1-\mathrm{loop}}(Q^2) \quad &= \quad \sum_f 4T_f N_{cf} Q_f^2 \int_0^1 \frac{dx\; x\;(1-x)}{x\;(1-x)\,Q^2+m_f^2} \\ &\overset{m_f^2\ll Q^2}{=} \quad \sum_f 4T_f N_{cf} Q_f^2 \left[\frac{1}{Q^2}-\frac{2m_f^2}{Q^4}\,\ln\frac{Q^2}{m_f^2}+O(\frac{1}{Q^6})\right]\,. \end{aligned} \tag{1.2}$$

Vainshtein [7] has shown that in the chiral limit the relation

$$w_T(Q^2)_{\mathrm{pQCD}}\Big|_{m=0} = \frac{1}{2}\;w_L(Q^2)\Big|_{m=0} \tag{1.3}$$

is valid actually to all orders of perturbative QCD. Thus the non-renormalization theorem valid beyond pQCD for the anomalous amplitude w_L (considering the quarks $q = u, d, s, c, b, t$ only):

$$w_L(Q^2)\Big|_{m=0} = w_L^{1-\mathrm{loop}}(Q^2)\Big|_{m=0} = \sum_q (2T_q Q_q)\,\frac{2N_c}{Q^2} \tag{1.4}$$

carries over to the perturbative part of the transversal amplitude. Thus in the chiral limit the perturbative QPM result for w_T is exact. This may be somewhat puzzling, since in low energy effective QCD, which encodes the non-perturbative strong interaction effects, this kind of term seems to be absent.

We observe that the contributions from w_L for individual fermions is logarithmically divergent, but it completely drops for a complete family due to the vanishing anomaly cancellation coefficient. The contribution from w_T is convergent for individual fermions due to the damping by the Z propagator. In fact it is the leading $1/Q^2$ term of the w_T amplitude which produces the $\ln\frac{M_Z}{m}$ terms. However, the coefficient is the same as for the anomalous term and thus for each complete family also the $\ln M_Z$ terms must drop out.

Low energy QCD is characterized in the chiral limit of massless light quarks u, d, s, by spontaneous chiral symmetry breaking (SχSB) of the chiral group $SU(3)_V \otimes SU(3)_A$, which in particular implies the existence of the pseudoscalar octet of pions and kaons as Goldstone bosons. The light quark condensates are essential features in this situation and lead to non-perturbative effects completely absent in a perturbative approach. Thus such low energy QCD effects are intrinsically non–perturbative and controlled by chiral perturbation theory (χPT), the systematic QCD low energy expansion, which accounts for the SχSB and the chiral symmetry breaking by quark masses in a systematic manner. The low energy effective theory evaluation of the hadronic contributions related to the light quarks u, d, s was worked out in [9, 11] and later in [12]. It was shown that in the operator product expansion

(OPE) the leading non-perturbative (NP) term in the chiral limit is due to the u, d, s quark condensate $\langle\bar{\psi}\psi\rangle \neq 0$

$$w_T(Q^2)_{\rm NP} \simeq \frac{16}{9}\pi^2\,\frac{1}{M_\rho^2}\,\frac{\alpha_s}{\pi}\frac{\langle\bar{\psi}\psi\rangle^2}{Q^6} \quad \text{at} \quad Q^2 \quad \text{large} \tag{1.5}$$

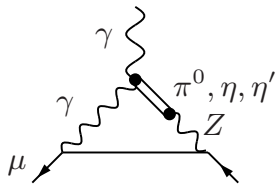
which breaks the degeneracy $w_T(Q^2) = \frac{1}{2}\,w_L(Q^2)$ found in perturbation theory[1]. M_ρ is the ρ mass. Relevant effective couplings are the neutral current part

$$\mathcal{L}^{(2)} = -\frac{e}{2\sin\Theta_W\cos\Theta_W} f_\pi \partial_\mu \left(\pi^0 + \frac{1}{\sqrt{3}}\,\eta_8 - \frac{1}{\sqrt{6}}\,\eta_0\right) Z^\mu \ ,$$

and the Wess-Zumino Lagrangian

$$\mathcal{L}_{\rm WZ} = \frac{\alpha}{\pi}\frac{N_c}{24 f_\pi}\,\left(\pi^0 + \frac{1}{\sqrt{3}}\,\eta_8 + 2\sqrt{\frac{2}{3}}\,\eta_0\right)\,\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} \ ,$$

which reproduces the ABJ anomaly via the PCAC relation. e is the positron charge, $\sin^2\Theta_W$ is the weak mixing parameter, f_π the pion decay constant and π^0 is the neural pion field. The pseudoscalars η_8, η_0 are mixing to η, η'. The $[u, d, s]$ contribution evaluated this way is given by the diagram



which together with its crossed version in the unitary gauge and in the chiral limit and completed for the first two fermion families yields [9, 11]

$$a_\mu^{(4)\,\rm EW}([e,u,d;\mu,c,s])_{\chi\rm PT} \;=\; \frac{\sqrt{2}G_\mu\,m_\mu^2}{16\pi^2}\,\frac{\alpha}{\pi}\left[-\frac{14}{3}\ln\frac{M_Z^2}{m_\mu^2} + 4\ln\frac{M_Z^2}{m_c^2} - \frac{35}{3} + \frac{8}{9}\pi^2\right] \ .$$

The residual M_Z–dependence in effective theory was controversial[2], and is contradicting Vainshtein's non-renormalization theorem [7, 12]. In fact, there is not really a contradiction. The point is that the low energy effective theory does not apply up to the M_Z mass scale as assumed in obtaining the above result. One rather has to apply a cut–off $\Lambda_{\rm matching}$ of the order of the proton mass m_p, say, and above that scale of course the QPM gives the correct answer. The leading large log terms proportional to $\ln M_Z$ then again cancel and the M_Z–dependence gets replaced by a $\Lambda_{\rm matching}$–dependence (with coefficient -14/3 changed to -2/3 and -35/3 to -47/3) in the above result.
As the extensions of the Adler-Bardeen non-renormalization theorem for the anomalous Ward identity $\langle VV\partial A\rangle$ turn out to play an important role in new phenomenological applications, we will study in the following such possible generalizations by an explicit calculation of the leading QCD corrections to the $\gamma\gamma Z$ triangle.

[1]The OPE only provides information on w_T for Q^2 large. At low Q^2 we only know that $w_T(0) = 128\pi^2\,C_{22}^W$ where C_{22}^W is one of the unknown χPT constants in the $O(p^6)$ parity odd part of the chiral Lagrangian [15]. The low energy effective theory does not yield a $1/Q^2$–pole as required by the non-renormalization theorem. In the OPE $1/Q^4$ terms are due to explicite chiral symmetry breaking which yields $\Delta w_T(Q^2)_{\rm NP} \simeq -\frac{4}{9}\,\frac{1}{M_\rho^2}\,\frac{(4m_u - m_d - m_s)\langle\bar{\psi}\psi\rangle}{Q^4}$.

[2]Numerical estimates presented in [11] and [12] agree well within errors and the discrepancy in question is of conceptual nature and not relevant for the interpretation of the present $g-2$ experiment.

2 Definitions

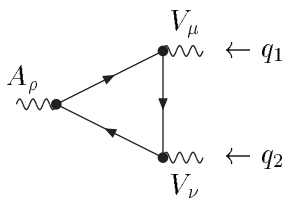
Let us consider the VVA three point function

$$\mathcal{W}_{\mu\nu\rho}(q_1,q_2) = i\int d^4x_1 d^4x_2\, e^{i(q_1\cdot x_1+q_2\cdot x_2)}\,\times\,\langle\,0\,|\,\mathrm{T}\{V_\mu(x_1)V_\nu(x_2)A_\rho(0)\}\,|\,0\,\rangle \tag{2.6}$$

of the flavor and color diagonal fermion currents

$$V_\mu \,=\, \overline{\psi}\gamma_\mu\,\psi \quad , \quad A_\mu \,=\, \overline{\psi}\gamma_\mu\gamma_5\,\psi \tag{2.7}$$

where ψ is a quark field. The vector currents are strictly conserved $\partial_\mu V^\mu = 0$, while the axial vector current satisfies a PCAC relation plus the anomaly $\partial_\mu A^\mu = 2im_0\bar{\psi}\gamma_5\psi + \frac{\alpha_0}{4\pi}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}(x)F^{\rho\sigma}(x)$. We will be mainly interested in the properties of strongly interacting quark flavor currents in perturbative QCD. To leading order the correlator of interest is associated with the one–loop triangle diagram



plus its crossed $(q_1,\mu\leftrightarrow q_2,\nu)$ partner. In the following we will closely follow the notation of [8]. The Ward identities restrict the general covariant decomposition of $\mathcal{W}_{\mu\nu\rho}(q_1,q_2)$ into invariant functions to four terms

$$\begin{aligned}\mathcal{W}_{\mu\nu\rho}(q_1,q_2) \;&=\; -\frac{1}{8\pi^2}\Big\{-w_L\left(q_1^2,q_2^2,q_3^2\right)\,(q_1+q_2)_\rho\,\varepsilon_{\mu\nu\alpha\beta}\,q_1^\alpha q_2^\beta \\ &+ w_T^{(+)}\left(q_1^2,q_2^2,q_3^2\right)\,t^{(+)}_{\mu\nu\rho}(q_1,q_2) + w_T^{(-)}\left(q_1^2,q_2^2,q_3^2\right)\,t^{(-)}_{\mu\nu\rho}(q_1,q_2) + \widetilde{w}_T^{(-)}\left(q_1^2,q_2^2,q_3^2\right)\,\widetilde{t}^{(-)}_{\mu\nu\rho}(q_1,q_2)\Big\}\,,\end{aligned} \tag{2.8}$$

with the transverse tensors given by

$$\begin{aligned} t^{(+)}_{\mu\nu\rho}(q_1,q_2) \;&=\; q_{1\nu}\,\varepsilon_{\mu\rho\alpha\beta}\,q_1^\alpha q_2^\beta \,-\, q_{2\mu}\,\varepsilon_{\nu\rho\alpha\beta}\,q_1^\alpha q_2^\beta \,-\, (q_1\cdot q_2)\,\varepsilon_{\mu\nu\rho\alpha}\,(q_1-q_2)^\alpha \\ &\qquad + \frac{q_1^2+q_2^2-q_3^2}{q_3^2}\,\varepsilon_{\mu\nu\alpha\beta}\,q_1^\alpha q_2^\beta(q_1+q_2)_\rho\,, \\ t^{(-)}_{\mu\nu\rho}(q_1,q_2) \;&=\; \left[(q_1-q_2)_\rho \,-\, \frac{q_1^2-q_2^2}{(q_1+q_2)^2}\,(q_1+q_2)_\rho\right]\,\varepsilon_{\mu\nu\alpha\beta}\,q_1^\alpha q_2^\beta \\ \widetilde{t}^{(-)}_{\mu\nu\rho}(q_1,q_2) \;&=\; q_{1\nu}\,\varepsilon_{\mu\rho\alpha\beta}\,q_1^\alpha q_2^\beta \,+\, q_{2\mu}\,\varepsilon_{\nu\rho\alpha\beta}\,q_1^\alpha q_2^\beta \,-\, (q_1\cdot q_2)\,\varepsilon_{\mu\nu\rho\alpha}\,(q_1+q_2)^\alpha\,. \end{aligned} \tag{2.9}$$

Bose symmetry $(q_1,\mu\leftrightarrow q_2,\nu)$ entails

$$\begin{aligned} w_T^{(+)}\left(q_2^2,q_1^2,q_3^2\right) \;&=\; +w_T^{(+)}\left(q_1^2,q_2^2,q_3^2\right)\,, \\ w_T^{(-)}\left(q_2^2,q_1^2,q_3^2\right) \;&=\; -w_T^{(-)}\left(q_1^2,q_2^2,q_3^2\right)\,, \quad \widetilde{w}_T^{(-)}\left(q_2^2,q_1^2,q_3^2\right) = -\widetilde{w}_T^{(-)}\left(q_1^2,q_2^2,q_3^2\right)\,. \end{aligned} \tag{2.10}$$

The longitudinal part is entirely fixed by the anomaly,

$$w_L\left(q_1^2,q_2^2,q_3^2\right) \,=\, -\frac{2N_c}{q_3^2} \tag{2.11}$$

which is exact to all orders of perturbation theory, the famous Adler-Bardeen non-renormalization theorem.
In [8] the following three chiral symmetry relations between amplitudes were derived in pQCD:

$$\left\{\left[w_T^{(+)} + w_T^{(-)}\right]\left(q_1^2,q_2^2,q_3^2\right) - \left[w_T^{(+)} + w_T^{(-)}\right]\left(q_3^2,q_2^2,q_1^2\right)\right\}_{\rm pQCD} = 0$$

$$\left\{\left[\widetilde{w}_T^{(-)} + w_T^{(-)}\right]\left(q_1^2,q_2^2,q_3^2\right) + \left[\widetilde{w}_T^{(-)} + w_T^{(-)}\right]\left(q_3^2,q_2^2,q_1^2\right)\right\}_{\rm pQCD} = 0$$

and

$$\left\{\left[w_T^{(+)} + \widetilde{w}_T^{(-)}\right]\left(q_1^2,q_2^2,q_3^2\right) + \left[w_T^{(+)} + \widetilde{w}_T^{(-)}\right]\left(q_3^2,q_2^2,q_1^2\right)\right\}_{\rm pQCD} - w_L\left(q_3^2,q_2^2,q_1^2\right)$$
$$= -\left\{\frac{2\,(q_2^2+q_1\cdot q_2)}{q_1^2}\,w_T^{(+)}\left(q_3^2,q_2^2,q_1^2\right) - 2\,\frac{q_1\cdot q_2}{q_1^2}\,w_T^{(-)}\left(q_3^2,q_2^2,q_1^2\right)\right\}_{\rm pQCD}, \qquad (2.12)$$

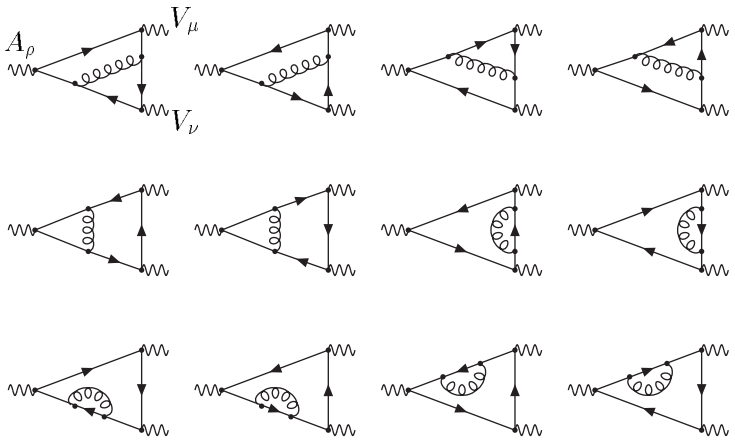
involving the transverse part of the $\langle VVA\rangle$ correlator $\mathcal{W}_{\mu\nu\rho}(q_1,q_2)$, and which hold for all values of the momentum transfers q_1^2, q_2^2 and q_3^2. In the kinematical configuration relevant for g-2 calculations, $q_1 = k \pm q$, $q_2 = -k$, expanding to linear order in k and noting that $\widetilde{t}^{(-)}_{\mu\nu\rho}(q_1,q_2) \sim t^{(+)}_{\mu\nu\rho}(q_1,q_2) = q^2\varepsilon_{\mu\nu\rho\sigma}k^\sigma - q_\mu\varepsilon_{\nu\rho\alpha\beta}q^\alpha k^\beta - q_\rho\varepsilon_{\mu\nu\alpha\beta}q^\alpha k^\beta + O(k^2)$ and $t^{(-)}_{\mu\nu\rho}(q_1,q_2) = O(k^2)$, these relations imply the non-renormalization theorem (1.3) obtained in Refs. [7, 12] upon identifying

$$w_L(Q^2) \equiv w_L(-Q^2,0,-Q^2)\ , \quad w_T(Q^2) \equiv w_T^{(+)}(-Q^2,0,-Q^2) + \widetilde{w}_T^{(-)}(-Q^2,0,-Q^2)\ , \qquad (2.13)$$

with $Q^2 = -q^2$.

3 Calculations

We perform the calculation with conventional dimensional regularization [17] and use a linear covariant gauge with arbitrary gauge parameter ξ throughout the calculation.



Two-loop QCD diagrams contributing to $\langle VVA\rangle$ correlator

Our procedure of treating γ_5 is similar to the one used in [16]. We write down all fermion loops starting with the axial-vector vertex, and then perform Feynman integrals and Dirac algebra without assuming any property of γ_5 at all. In this way all diagrams will be expressed in terms of traces of 10 combinations of γ matrices:

$$\begin{array}{lll} 4iA_1 = \gamma_\rho\gamma_5\gamma_\mu\gamma_\nu\hat{q}_2, & 4iA_2 = \gamma_\rho\gamma_5\hat{q}_1\gamma_\mu\gamma_\nu, & 4iA_3 = q_2^\mu\gamma_\rho\gamma_5\hat{q}_1\gamma_\nu\hat{q}_2, \\ 4iA_4 = q_1^\mu\gamma_\rho\gamma_5\hat{q}_1\gamma_\nu\hat{q}_2, & 4iA_5 = -q_2^\nu\gamma_\rho\gamma_5\hat{q}_1\gamma_\mu\hat{q}_2, & 4iA_6 = -q_1^\nu\gamma_\rho\gamma_5\hat{q}_1\gamma_\mu\hat{q}_2, \\ 4iA_7 = \gamma_\rho\gamma_5\hat{q}_1, & 4iA_8 = \gamma_\rho\gamma_5\hat{q}_2, & 4iA_9 = \gamma_\rho\gamma_5\gamma_\mu, \\ 4iA_{10} = \gamma_\rho\gamma_5\gamma_\nu & & \end{array} \tag{3.14}$$

with $\hat{q} \equiv q_\mu\gamma^\mu$. The prescription is sufficient to enable us to arrive at expressions in front of $A_1, \ldots A_{10}$ which have finite limits as $d \to 4$. After this the usual formulae

$$\mathrm{Tr}[\gamma_5\gamma_\alpha\gamma_\beta\gamma_\mu\gamma_\nu] = 4i\varepsilon_{\alpha\beta\mu\nu} \ , \ \ \mathrm{Tr}[\gamma_5\gamma_\alpha\gamma_\beta] = 0 \tag{3.15}$$

valid in $d = 4$ dimensions were used. In our convention $\varepsilon_{0123} = +1$ and $(1-\gamma_5)/2$ projects to left–handed fermion fields.

Tensor integrals were expressed in terms of integrals with different shifts of the space-time dimension [18]. All scalar integrals were reduced to 6 master integrals by using the Gröbner basis technique proposed in [19]. The expressions for the individual diagrams are sums over 15 terms which are combinations of the 6 basis integrals

$$\begin{aligned} I_2^{(d)}(q_j^2) &= \int \frac{d^dk_1}{[i\pi^{d/2}]}\frac{1}{k_1^2(k_1-q_j)^2}, \\ I_3^{(d)}(q_1^2,q_2^2,q_3^2) &= \int \frac{d^dk_1}{[i\pi^{d/2}]}\frac{1}{k_1^2(k_1-q_1)^2(k_1-q_2)^2} \\ J_3^{(d)}(q_j^2) &= \int\int \frac{d^dk_1d^dk_2}{[i\pi^{d/2}]^2}\frac{1}{k_1^2(k_1-k_2)^2(k_2-q_j)^2}, \\ R_1(q_1^2,q_2^2,q_3^2) &= \int\int \frac{d^dk_1d^dk_2}{[i\pi^{d/2}]^2}\frac{1}{k_1^2(k_1-k_2)^2(k_2-q_1)^2(k_2+q_2)^2} \\ R_2(q_1^2,q_2^2,q_3^2) &= \int\int \frac{d^dk_1d^dk_2}{[i\pi^{d/2}]^2}\frac{1}{k_1^4(k_1-k_2)^2(k_2-q_1)^2(k_2+q_2)^2}, \\ P_5(q_1^2,q_2^2,q_3^2) &= \int\int \frac{d^dk_1d^dk_2}{[i\pi^{d/2}]^2}\frac{1}{k_1^2k_2^2(k_1-k_2)^2(k_1-q_1)^2(k_2+q_2)^2}. \end{aligned} \tag{3.16}$$

and multiplied by ratios of polynomials in momenta and d:

$$D_j = \sum_{k=1}^{15} M_k \frac{P_k(q_r^2,d)}{Q_k(q_s^2,d)} \ . \tag{3.17}$$

The momentum dependence of the denominators turns out to be rather simple:

$$Q_k(q_s^2,d) = Q(d)(q_1^2)^{a_k}(q_2^2)^{b_k}(q_3^2)^{c_k}\Delta^{e_k} \tag{3.18}$$

where a_k, b_k, c_k, e_k, are some numbers, $Q(d)$ is a polynomial in d and

$$\Delta = q_1^4 + q_2^4 + q_3^4 - 2q_1^2q_2^2 - 2q_1^2q_3^2 - 2q_2^2q_3^2 \ . \tag{3.19}$$

The integrals (3.16) form a complete set of master integrals needed for the calculation of massless vertex diagrams with planar topology.
The sum of all diagrams turns out to be gauge parameter independent.
In the Feynman gauge at $q_3 = 0$ and for arbitrary d, the results of our calculation are in agreement with the ones presented in [16] diagram by diagram.
By applying the prescription outlined above for the evaluation of individual diagrams, the tensor structures

$$N_1 = q_1^\rho \varepsilon_{\alpha\beta\mu\nu} \; , N_2 = q_2^\rho \varepsilon_{\alpha\beta\mu\nu} \; , \tag{3.20}$$

which appear in the Ansatz (2.8) are actually absent. In place of N_1, N_2, two other tensor structures are present which we eliminate by means of the Schouten identities:

$$\begin{aligned} q_1^\alpha q_2^\beta q_2^\nu \varepsilon_{\alpha\beta\mu\rho} &= +q_1^\alpha q_2^\beta q_2^\rho \varepsilon_{\alpha\beta\mu\nu} + q_1^\alpha q_2^\beta q_2^\mu \varepsilon_{\alpha\beta\nu\rho} - q_2^2 q_1^\alpha \varepsilon_{\alpha\mu\nu\rho} + q_2^\alpha q_1 q_2 \varepsilon_{\alpha\mu\nu\rho}, \\ q_1^\alpha q_2^\beta q_1^\mu \varepsilon_{\alpha\beta\nu\rho} &= -q_1^\alpha q_2^\beta q_{1\rho} \varepsilon_{\alpha\beta\mu\nu} + q_1^\alpha q_2^\beta q_1^\nu \varepsilon_{\alpha\beta\mu\rho} + q_1^\alpha q_1 q_2 \varepsilon_{\alpha\mu\nu\rho} - q_2^\alpha q_1^2 \varepsilon_{\alpha\mu\nu\rho} \; . \end{aligned} \tag{3.21}$$

Reshuffling terms in this way allows us to express each diagram in terms of the tensor structures introduced in (2.8) exhibiting manifestly the vector current conservation.

4 Results and Discussion

Including one– and two–loop contributions, we may represent the form-factors in the form

$$\begin{aligned} w_T^{(\pm)}(q_1^2, q_2^2, q_3^2) &= n_f \, N_c \, w_{1,T}^{(\pm)}(q_1^2, q_2^2, q_3^2) \; + \; a \; n_f \, N_c \, C_2(R) \, w_{2,T}^{(\pm)}(q_1^2, q_2^2, q_3^2) \\ w_L(q_1^2, q_2^2, q_3^2) &= n_f \, N_c \, w_{1,L}(q_1^2, q_2^2, q_3^2) \; + \; a \; n_f \, N_c \, C_2(R) \, w_{2,L}(q_1^2, q_2^2, q_3^2) \end{aligned} \tag{4.22}$$

where

$$a = \frac{\alpha_s}{4\pi} = \frac{g^2}{16\pi^2}. \tag{4.23}$$

includes the QCD coupling α_s, g as usual is the gauge coupling, n_f is the number of flavors and N_c the number of colors. The quarks are in the fundamental representation R and the corresponding group theory factor is given by

$$C_2(R) I = R^a R^a \quad , \quad C_2(R) = 4/3 \;\; \text{for QCD} \; . \tag{4.24}$$

We have been working in the $\overline{\rm MS}$ renormalization scheme. The singlet axial current $J_\rho^5 = A_\rho$ is non-trivially renormalized because of the axial anomaly. It is known [23] that in addition to the standard ultraviolet renormalization constant $Z_{\overline{\rm MS}}$ which reads $Z_{\overline{\rm MS}} = 1$ in our case (as $Z_{\overline{\rm MS}} - 1 = O(a^2)$), one has to apply a finite renormalization constant Z_5 such that renormalized and bare currents are related as:

$$(J_\rho^5)_r = Z_5 Z_{\overline{\rm MS}} \, (J_\rho^5)_0. \tag{4.25}$$

The counterterms coming from the wave function renormalization of quarks and ultraviolet renormalization of the axial and vector currents cancel. The finite renormalization constant is known at the three-loop level [24]. For our calculations we take

$$Z_5 = 1 - 4 C_2(R) \, a \; . \tag{4.26}$$

The result of the two–loop calculation after adding all diagrams is surprisingly simple and, normalized according to (4.22), is given by

$$
\begin{aligned}
q_3^2 w_{2,L}(q_1^2,q_2^2,q_3^2) &= -8 && (4.27)\\
\widetilde{w}_{2,T}^{(-)}(q_1^2,q_2^2,q_3^2) &= -w_{2,T}^{(-)}(q_1^2,q_2^2,q_3^2), && (4.28)\\
q_3^2\Delta^2 w_{2,T}^{(-)}(q_1^2,q_2^2,q_3^2) &= 8(x-y)\Delta + 8(x-y)(6xy+\Delta)\Phi^{(1)}(x,y)\\
&- 4[18xy+6x^2-6x+(1+x+y)\Delta)]L_x\\
&+ 4[18xy+6y^2-6y+(1+x+y)\Delta)]L_y && (4.29)
\end{aligned}
$$

$$
\begin{aligned}
q_3^2\Delta^2 w_{2,T}^{(+)}(q_1^2,q_2^2,q_3^2) &= 8[6xy+(x+y)\Delta]\Phi^{(1)}(x,y)+8\Delta\\
&- 4[6x+\Delta](x-y-1)L_x\\
&+ 4[6y+\Delta](x-y+1)L_y && (4.30)
\end{aligned}
$$

with

$$
L_x = \ln x, \quad L_y = \ln y, \quad x = \frac{q_1^2}{q_3^2} \quad y = \frac{q_2^2}{q_3^2} \ . \tag{4.31}
$$

The explicit expression for $\Phi^{(1)}$ may be found in [25]:

$$
\Phi^{(1)}(x,y) = \frac{1}{\lambda}\left\{ 2\left(\mathrm{Li}_2\left(-\rho x\right) + \mathrm{Li}_2\left(-\rho y\right)\right) + \ln\frac{y}{x}\ln\frac{1+\rho y}{1+\rho x} + \ln(\rho x)\ln(\rho y) + \frac{\pi^2}{3}\right\}, \tag{4.32}
$$

where

$$
\lambda(x,y) \equiv \sqrt{\Delta} \quad , \quad \rho(x,y) \equiv 2\ (1-x-y+\lambda)^{-1}, \quad \Delta = (1-x-y)^2 - 4xy \ . \tag{4.33}
$$

The comparison with the results of the one-loop calculation reveals that

$$
\mathcal{W}_{\mu\nu\rho}(q_1,q_2)\mid_{two-loop} = 4C_2(R)a\mathcal{W}_{\mu\nu\rho}(q_1,q_2)\mid_{one-loop} \tag{4.34}
$$

Multiplying the sum of one- and two-loop terms by the finite factor Z_5 we arrive at

$$
\mathcal{W}_{\mu\nu\rho}(q_1,q_2) = \mathcal{W}_{\mu\nu\rho}(q_1,q_2)\mid_{one-loop} \tag{4.35}
$$

This is the non-renormalization theorem for the full off shell correlator at two–loops. While our calculation confirms the relations (2.12) derived in [8] and the non-renormalization theorem (1.3) found in [7, 12], these findings are not sufficient to explain our result valid for generic momenta.
Taking into account the rather non-trivial momentum dependence of the form-factors it is very tempting to suggest that it could hold to all orders of perturbation theory because of the topological nature of the anomaly, for example.
We would like to stress that the surprising relation could be discovered only by keeping the general non-trivial momentum dependence. The anomalous three point correlator exhibits an unusually simple structure, while contributions from individual diagrams are very unwieldy. One can expect similar effects for other anomalous correlators. Since at the order considered the QCD calculation is essentially a QED calculation, it is highly non-trivial whether this carries over to higher orders. For the $\langle VV\partial A\rangle$ anomalous correlator a large number of two–loop calculations have been performed ([26]-[32]), mainly in QED and we refer to the comprehensive review by Adler [33] and the references therein.

In electroweak SM calculations one would a priori expect that remormalizing parameters and fields would be sufficient for renormalizing the SM. Our calculation shows that on top of the standard renormalization, it is mandatory to renormalize the anomalous currents J^5_ρ by the finite renormalization factor Z_5 because the lepton currents and the quark currents pick different Z–factors and if they are not renormalized away the anomaly cancellation and hence renormalizability obviously would get spoiled. As pointed out by Adler and many others [33] the point is there exists a renormalization scheme for which the one–loop anomaly is exact. Only in this scheme anomaly cancellation and thus renormalizability will carry over to higher orders in the SM. Our result shows that due to the necessity of renormalizing away possible higher order contributions from the anomaly also the non-anomalous transversal contributions are affected. We have shown that at least at two–loops the entire contribution gets renormalized away in the zero mass limit.

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