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Path Integral Approach for Spaces of Non-constant Curvature in Three Dimensions

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Abstract

In this contribution I show that it is possible to construct three-dimensional spaces of non-constant curvature, i.e. three-dimensional Darboux-spaces. Two-dimensional Darboux spaces have been introduced by Kalnins et al., with a path integral approach by the present author. In comparison to two dimensions, in three dimensions it is necessary to add a curvature term in the Lagrangian in order that the quantum motion can be properly defined. Once this is done, it turns out that in the two three-dimensional Darboux spaces, which are discussed in this paper, the quantum motion is similar to the two-dimensional case. In $D_{3d-\mathrm{I}}$ we find seven coordinate systems which separate the Schrödinger equation. For the second space, $D_{3d-\mathrm{II}}$, all coordinate systems of flat three-dimensional Euclidean space which separate the Schrödinger equation also separate the Schrödinger equation in $D_{3d-\mathrm{II}}$. I solve the path integral on $D_{3d-\mathrm{I}}$ in the (u, v, w)-system, and on $D_{3d-\mathrm{II}}$ in the (u, v, w)-system and in spherical coordinates.

1 Introduction

In this paper the quantum motion on three-dimensional spaces of non-constant curvature is studied. In [12, 13] two-dimensional spaces of non-constant curvature, called Darboux spaces, were introduced. Particular emphasis was put on separation of variables and to find all coordinate systems which separate the Schrödinger equation (respectively the Helmholtz equation) and the path integral. Another important issue was to find all potentials in these spaces which are superintegrable. These potentials have the property that there are additional constants of motion and that the corresponding Schrödinger equation separates in more that one coordinate system. Actually, in two dimensions these systems have three constants of motion.

In [8] the path integral method [3, 11, 14, 16] was applied to study the free motion on the four Darboux spaces $D_{\rm I}$–$D_{\rm IV}$, and a study of superintegrable potentials was completed in [10].

In [12, 13] the two-dimensional Darboux were introduced as follows (we also insert for the coordinates $x = u + {\rm i}v, y = u - {\rm i}v$, and (u, v) will be called the (u, v)-system):

$$\text{(I)} \qquad {\rm d}s^2 = (x+y){\rm d}x{\rm d}y = 2u({\rm d}u^2 + {\rm d}v^2) \tag{1.1}$$

$$\text{(II)} \qquad {\rm d}s^2 = \left(\frac{a}{(x-y)^2} + b\right){\rm d}x{\rm d}y = \frac{bu^2 - a}{u^2}({\rm d}u^2 + {\rm d}v^2) \tag{1.2}$$

$$\text{(III)} \qquad {\rm d}s^2 = (a\ {\rm e}^{-(x+y)/2} + b\ {\rm e}^{-x-y}){\rm d}x{\rm d}y = {\rm e}^{-2u}(b + a\ {\rm e}^{u})({\rm d}u^2 + {\rm d}v^2) \tag{1.3}$$

$$\text{(IV)} \qquad {\rm d}s^2 = -\frac{a({\rm e}^{(x-y)/2} + {\rm e}^{(y-x)/2}) + b}{({\rm e}^{(x-y)/2} - {\rm e}^{(y-x)/2})^2}{\rm d}x{\rm d}y = \left(\frac{a_+}{\sin^2 u} + \frac{a_-}{\cos^2 u}\right)({\rm d}u^2 + {\rm d}v^2)\ . \tag{1.4}$$

a and b are additional (real) parameters ($a_\pm = (a \pm 2b)/4$). $D_{\rm II}$ has the property that for $a = 0$, $b = 1$, we recover the two-dimensional Euclidean plane, and all four coordinate systems on the two-dimensional Euclidean plane are also separable coordinate systems on $D_{\rm II}$ for the Schrödinger, respectively the Helmholtz equation. For $b = 0$, $a = -1$ the two-dimensional hyperboloid is contained as a second special case.

Let us consider three-dimensional generalization of the Darboux spaces $D_{\rm I}, D_{\rm II}$, respectively, with the following line elements

$$\text{(I)} \qquad {\rm d}s^2 = 2u({\rm d}u^2 + {\rm d}v^2 + {\rm d}w^2)\ , \tag{1.5}$$

$$\text{(II)} \qquad {\rm d}s^2 = \frac{bu^2 - a}{u^2}({\rm d}u^2 + {\rm d}v^2 + {\rm d}w^2)\ , \tag{1.6}$$

and w is the new variable. The present cases of (1.5,1.6), which we call three-dimensional Darboux space I and II, for short $D_{3d-{\rm I}}, D_{3d-{\rm II}}$, respectively, are studied in this contribution. In comparison to their two-dimensional analogies new features appear. If we consider the Laplace-Beltrami operator, $\Delta_{LB} = g^{-1/2}\partial_{q^a} g^{ab} g^{1/2}\partial_{q^b}$, we see that $g^{ab}g^{1/2} \neq \mathbb{1}$, as it is always the case in two dimensions if the metric tensor is proportional to the unit tensor. We obtain an additional term $\propto (g^{ab}\Gamma_a + g^{ab}_{,a})\partial_{q^b}$, where $\Gamma_a = \partial_{q^a} \ln\sqrt{\det(g_{ab})}$. This has the consequence that curvature terms $\propto \hbar^2$ appear in the quantum Hamiltonian which must be dealt with. We will see that we must add such curvature terms in the metrics (1.5,1.6) which cancel these $\propto \hbar^2$-terms in the quantization procedure. If this is done, proper quantum systems can be established.

In the following, we study the cases of (1.5,1.6) within the path integral approach, first for (1.5) and second for (1.6). In both cases we find the coordinate systems which separate the

Schrödinger equation, respectively the path integral. We solve the path integral in each space in the (u, v, w)-system, and the path integral corresponding to (1.6) also in spherical coordinates. The extension to this study to find the coordinate systems in the two other three-dimensional Darboux-spaces and to find the corresponding path integral solutions will be subject to a future publication. The last section contains a summary of the achieved results and an outlook.

2 The Path Integral Solution on the Three-Dimensional Darboux Space $D_{3d-\mathrm{I}}$

We start with the three-dimensional Darboux space $D_{3d-\mathrm{I}}$ and consider the metric:

$$\mathrm{d}s^2 = 2u(\mathrm{d}u^2 + \mathrm{d}v^2 + \mathrm{d}w^2) \ . \tag{2.1}$$

The proper definition of the range of the variables (u, v, w) depends on the proper definition of the space we in fact consider. As it is known from the two-dimensional case [13], an embedding in a three-dimensional Euclidean space yields $a > \frac{1}{2}$, whereas an embedding in a three-dimensional Minkowskian space yields $a > 0$. We assume in the following that $u > a$, where $a > 0$ and that there is no restriction on the variables v, w. They can be cyclic or range within the entire real line. According to the general theory we have $g = \det(g_{ab}) = (2u)^3$, therefore $\Gamma_u = 3/2u, \Gamma_v = \Gamma_w = 0$. The Laplace-Beltrami operator has the form

$$\Delta_{LB} = \frac{1}{2u}\Big(\frac{\partial^2}{\partial u^2} - \frac{1}{2u}\frac{\partial}{\partial u} + \frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial w^2}\Big), \qquad p_u = \frac{\hbar}{\mathrm{i}}\Big(\frac{\partial}{\partial u} + \frac{3}{4u}\Big) \ , \tag{2.2}$$

and p_u are the corresponding momentum operator for the coordinate u. Of course, $p_v = \frac{\hbar}{\mathrm{i}}\partial_v, p_w = \frac{\hbar}{\mathrm{i}}\partial_w$. According to our theory we can calculate the corresponding quantum potential by means of [11]

$$\Delta V = \frac{\hbar^2}{8m}\frac{D-2}{f^4}\Big[(D-4)f'^2 + 2ff''\Big] \ , \tag{2.3}$$

provided the metric is proportional to the unit tensor $(g_{ab}) = f^2 \mathbb{1}_3$. Indeed $f = \sqrt{2u}$ and $D = 3$, which yields

$$\Delta V = -\frac{3\hbar^2}{64mu^3} \ . \tag{2.4}$$

This gives an effective Lagrangian in the corresponding path integral in the product form definition [11]

$$\mathcal{L}_{\text{eff}}(u, \dot{u}, v, \dot{v}, w, \dot{w}) = \frac{m}{2}(2u)(\dot{u}^2 + \dot{v}^2 + \dot{w}^2) + \frac{3\hbar^2}{64mu^3} \ . \tag{2.5}$$

The quantum potential (2.4) has actually the form of a Schwarzian derivative. Performing a space-time transformation in the path integral where $u^2 = 4r$ cancels ΔV, and produces in turn in the transformed Lagrangian $\mathcal{L}_E = \mathcal{L}_{\text{eff}} + E$ a potential $\propto 2E/\sqrt{r}$ (coupling constant metamorphosis). Potentials like this are called "conditionally solvable" [6]. However, in order that they are in fact conditionally solvable, requires that an additional potential of the form of ΔV is present. This is not the case here after the transformation into the new variable r and the corresponding time-transformation; we are left with an intractable path integral.

In order to obtain a proper quantum theory on $D_{3d-\mathrm{I}}$, we therefore *define* our quantum theory for the free motion on $D_{3d-\mathrm{I}}$ as follows:

$$\begin{aligned} H_{D_{3d-\mathrm{I}}} &= -\frac{\hbar^2}{2m}\frac{1}{2u}\Big(\frac{\partial^2}{\partial u^2}-\frac{1}{2u}\frac{\partial^2}{\partial u}+\frac{\partial^2}{\partial v^2}+\frac{\partial^2}{\partial w^2}\Big)+\frac{3\hbar^2}{64mu^3} \\ &= \frac{1}{2m}\frac{1}{\sqrt{2u}}(p_u^2+p_v^2+p_w^2)\frac{1}{\sqrt{2u}} \ . \end{aligned} \tag{2.6}$$

This gives in turn a proper definition of the Lagrangian on $D_{3d-\mathrm{I}}$

$$\mathcal{L}_{\mathrm{eff}}^{(D_{3d-\mathrm{I}})}(u,\dot{u},v,\dot{v},w,\dot{w}) := \frac{m}{2}(2u)(\dot{u}^2+\dot{v}^2+\dot{w}^2) \ , \tag{2.7}$$

and the path integral has the form

$$\begin{aligned} &K^{(D_{3d-\mathrm{I}})}(u'',u',v'',v',w'',w';T) \\ &:= \lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\epsilon\hbar}\right)^N\prod_{j=1}^{N-1}\int(2u_j)^{3/2}\mathrm{d}u_j\mathrm{d}v_j\mathrm{d}w_j\exp\left[\frac{\mathrm{i}m}{\hbar}\sum_{j=1}^{N}\widehat{u_j}(\Delta^2u_j+\Delta^2v_j+\Delta^2w_j)\right] \\ &= \int_{u(t')=u'}^{u(t'')=u''}\mathcal{D}u(t)\int_{v(t')=v'}^{v(t'')=v''}\mathcal{D}v(t)\int_{w(t')=w'}^{w(t'')=w''}\mathcal{D}w(t)(2u)^{3/2}\exp\left[\frac{\mathrm{i}m}{\hbar}\int_0^T u(\dot{u}^2+\dot{v}^2+\dot{w}^2)\mathrm{d}t\right]. \end{aligned} \tag{2.8}$$

($u_j = u(t_j)$, $\Delta u_j = u_j - u_{j-1}$, $\epsilon = T/N$, $\widehat{u_j} = \sqrt{u_{j-1}u_j}$). One may now ask, why we subtract in the definition of our quantum theory on $D_{3d-\mathrm{I}}$ a quantum potential of order $\hbar^2$, which is usually *absolutely necessary to incorporate* [11]. It is well-known that other lattice definitions lead to other quantum potentials which in turn correspond to different ordering prescriptions in the quantum Hamiltonian [11]. This problem can be addresses as follows: If we perform a time-transformation in the path integral (2.8), it is inevitable that we switch in this procedure to a lattice as given in (2.8) [14]. Had we set up our path integral in a different lattice as in (2.8) corresponding to another ordering prescription for the quantum Hamiltonian $H_{D_{3d-\mathrm{I}}}$, say the midpoint prescription and Weyl-ordering, respectively, we must switch to the lattice in (2.8) in order to perform the time-transformation properly. This changing of the lattice in turn would produce additional quantum terms of order $\hbar^2$ which then would lead back to our definition (2.8).

From Table 1 we can determine the coordinate systems of three dimensional Euclidean space which separate the Schrödinger equation for the quantum motion in $D_{3d-\mathrm{I}}$, respectively the path integral (2.8). We find the Cartesian, the three circular systems, the parabolic and the paraboloidal systems (we take $u = z$ and $(v,w) = (y,z)$), and in addition the rotated (r,q)-system from [13] with the additional variable w. This gives seven coordinate systems for $D_{3d-\mathrm{I}}$. We discuss only the first, the (u,v,w)-system: A rotated system is very similar to the (u,v,w)-system; the three circular systems are contained in the (v,w)-coordinates as subsystems; for the parabolic and the paraboloidal systems which separate the Schrödinger equation, however, we encounter intractable power-terms similar as in [8].

In the path integral (2.8) we perform a time transformation according to $\Delta t_{(j)} = 2\widehat{u_j}\Delta s_{(j)}$, i.e. with time-transformation function $f^2(u) = 2u$, and we get:

Table 1: Coordinates in Three-Dimensional Euclidean Space

Coordinate System	Coordinates
I. Cartesian	$x = x'$, $y = y'$, $z = z'$
II. Circular Polar	$x = \varrho\cos\varphi$, $y = \varrho\sin\varphi$, $z = z'$
III. Circular Elliptic	$x = d\cosh\mu\cos\nu$, $y = d\sinh\mu\sin\nu$, $z = z'$
IV. Circular Parabolic	$x = \frac{1}{2}(\eta^2 - \xi^2)$, $y = \xi\eta$, $z = z'$
V. Sphero-Conical	$x = r\mathrm{sn}(\alpha,k)\mathrm{dn}(\beta,k')$, $y = r\mathrm{cn}(\alpha,k)\mathrm{cn}(\beta,k')$ $z = r\mathrm{dn}(\alpha,k)\mathrm{sn}(\beta,k')$
VI. Spherical	$x = r\sin\vartheta\cos\varphi$, $y = r\sin\vartheta\sin\varphi$, $z = r\cos\vartheta$
VII. Parabolic	$x = \xi\eta\cos\varphi$, $y = \xi\eta\sin\varphi$, $z = \frac{1}{2}(\eta^2 - \xi^2)$
VIII. Prolate Spheroidal	$x = d\sinh\mu\sin\nu\cos\varphi$, $y = d\sinh\mu\sin\nu\sin\varphi$ $z = d\cosh\mu\cos\nu$
IX. Oblate Spheroidal	$x = d\cosh\mu\sin\nu\sin\varphi$, $y = d\cosh\mu\sin\nu\sin\varphi$ $z = d\sinh\mu\cos\nu$
X. Ellipsoidal	$x = k^2\sqrt{a^2 - c^2}\mathrm{sn}\alpha\mathrm{sn}\beta\mathrm{sn}\gamma$ $y = -(k^2/k')\sqrt{a^2 - c^2}\mathrm{cn}\alpha\mathrm{cn}\beta\mathrm{cn}\gamma$ $z = (\mathrm{i}/k')\sqrt{a^2 - c^2}\mathrm{dn}\alpha\mathrm{dn}\beta\mathrm{dn}\gamma$
XI. Paraboloidal	$x = 2d\cosh\alpha\cos\beta\sinh\gamma$, $y = 2d\sinh\alpha\sin\beta\cosh\gamma$ $z = d(\cosh^2\alpha + \cos^2\beta - \cosh^2\gamma)$

$$
\begin{aligned}
&K^{(D\,3d-1)}(u'', u', v'', v', w'', w'; T) \\
&= (4u'u'')^{-1/2}\int_{-\infty}^{\infty}\frac{\mathrm{d}E}{2\pi\hbar}\,\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^{\infty}\mathrm{d}s''K^{(D\,3d-1)}(u'', u', v'', v', w'', w'; s'') \qquad (2.9)
\end{aligned}
$$

with $K(s'')$ given by:

$$
\begin{aligned}
&K^{(D\,3d-1)}(u'', u', v'', v', w'', w'; s'') \\
&= \int_{u(0)=u'}^{u(s'')=u''}\mathcal{D}u(s)\int_{v(0)=v'}^{v(s'')=v''}\mathcal{D}v(s)\int_{w(0)=w'}^{w(s'')=w''}\mathcal{D}w(s)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}(\dot{u}^2 + \dot{v}^2 + \dot{w}^2) + 2uE\right]\mathrm{d}s\right\} \\
&= \sum_{l_v, l_w=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}l_v(v''-v')+\mathrm{i}l_w(w''-w')}}{(2\pi)^2}\exp\left(-\frac{\mathrm{i}}{\hbar}\frac{\hbar^2}{2m}(l_v^2 + l_w^2)s''\right) \\
&\qquad\times\int_{u(0)=u'}^{u(s'')=u''}\mathcal{D}u(s)\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}\dot{u}^2 + 2uE\right)\mathrm{d}s\right]\quad . \qquad (2.10)
\end{aligned}
$$

I have separated the (v, w)-dependent parts of the path integral in circular waves. Depending on the boundary conditions, also plane waves can be possible [13]. The remaining path integral in the variable u is a path integral for the linear potential. In a similar way as in [8] we obtain

for the kernel $K(T)$:

$$K^{(D_{3d-\mathrm{I}})}(u'',u',v'',v',w'',w';T)=\int_{-\infty}^{\infty}\frac{\mathrm{d}E}{2\pi\hbar}\,\mathrm{e}^{-\mathrm{i}ET/\hbar}$$
$$\times\sum_{l_v,l_w=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}l_v(v''-v')+\mathrm{i}l_w(w''-w')}}{(2\pi)^2}\,G_u^{(D_{3d-\mathrm{I}})}\left(E;u'',u';-\frac{\hbar^2}{2m}\mathbf{L}^2\right)\ , \qquad (2.11)$$

and we have abbreviated $\mathbf{L}^2=l_v^2+l_w^2$. For the complete solution we must know the kernel $G_u(u'',u';\mathcal{E})$, which is obtained from the Green function for the linear potential $V(x)=kx$, and is given by [11]

$$G^{(k)}(x'',x';\mathcal{E}) \;=\; \frac{4m}{3\hbar}\left[\left(x'-\frac{\mathcal{E}}{k}\right)\left(x''-\frac{\mathcal{E}}{k}\right)\right]^{1/2}$$
$$\times I_{1/3}\left[\frac{\sqrt{8mk}}{3\hbar}\left(x_<-\frac{\mathcal{E}}{k}\right)^{3/2}\right]K_{1/3}\left[\frac{\sqrt{8mk}}{3\hbar}\left(x_>-\frac{\mathcal{E}}{k}\right)^{3/2}\right]\ . \qquad (2.12)$$

I_ν and K_ν are modified Bessel-functions [4], and $x_<$ and $x_>$ denote the smaller and larger of x' and x'', respectively. We have to identify $\mathcal{E}=-\mathbf{L}^2\hbar^2/2m$, $k=-2E$, and $x=u$. In addition, we have to recall that the motion in u takes place only in the half-space $u>a$. In order to construct the Green function in the half-space $x>a$ we have to put Dirichlet boundary-conditions at $x=a$ [5]. Therefore we obtain finally:

$$G^{(D_{3d-\mathrm{I}})}(u'',u',v'',v',w'',w';E)$$
$$=\sum_{l_v,l_w=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}l_v(v''-v')+\mathrm{i}l_w(w''-w')}}{(2\pi)^2}\frac{4m}{3\hbar}(4u'u'')^{-1/2}\left[\left(u'-\frac{\mathbf{L}^2\hbar^2}{4mE}\right)\left(u''-\frac{\mathbf{L}^2\hbar^2}{4mE}\right)\right]^{1/2}$$
$$\times\left[\tilde{I}_{1/3}\left(u_<-\frac{\mathbf{L}^2\hbar^2}{4mE}\right)\tilde{K}_{1/3}\left(u_>-\frac{\mathbf{L}^2\hbar^2}{4mE}\right)\right.$$
$$\left.-\frac{\tilde{I}_{1/3}\left(a-\dfrac{\mathbf{L}^2\hbar^2}{4mE}\right)}{\tilde{K}_{1/3}\left(a-\dfrac{\mathbf{L}^2\hbar^2}{4mE}\right)}\tilde{K}_{1/3}\left(u'-\frac{\mathbf{L}^2\hbar^2}{4mE}\right)\tilde{K}_{1/3}\left(u''-\frac{\mathbf{L}^2\hbar^2}{4mE}\right)\right]\ . \qquad (2.13)$$

$\tilde{I}_\nu(z)$ denotes $\tilde{I}_\nu(z)=I_\nu\left(\frac{4\sqrt{-mE}}{3\hbar}z^{3/2}\right)$, with $\tilde{K}_\nu(z)$ similarly. Due to the relation to the Airy-function [1] $K_{\pm1/3}(\zeta)=\pi\sqrt{3/z}\,\mathrm{Ai}(z)$, $z=(3\zeta/2)^{2/3}$, and the observation that for $E<0$ the argument of $\mathrm{Ai}(z)$ is always greater than zero, there are no bound states. Let us note that we can replace in (2.13) the expansion of the circular- (respectively plane-) waves of the (v,w)-subsystem by the appropriate expansion of the remaining three-circular systems of Table 1, i.e. circular polar coordinates with a Bessel-function times circular waves, circular parabolic coordinates with a product of two parabolic cylinder functions and circular elliptic coordinates with Mathieu-functions [7]. This concludes the discussion of $D_{3d-\mathrm{I}}$.

3 The Path Integral Solution on the Three-Dimensional Darboux Space $D_{3d-\mathrm{II}}$

3.1 The (u,v,w)- and the Cylindrical Systems

For the second three-dimensional Darboux space, we consider the metric

$$\mathrm{d}s^2 = \frac{bu^2-a}{u^2}(\mathrm{d}u^2+\mathrm{d}v^2+\mathrm{d}w^2), \qquad p_u = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial u}+\frac{\Gamma_u}{2}\right) \ , \tag{3.1}$$

and w is the new variable. We can write the metric tensor according to $(g_{ab}) = f^2 \mathbb{1}_3$ with $f = h/u$, and $h = \sqrt{bu^2-a}$. The general theory yields $g = (h/u)^6$, $\Gamma_u = 3h'/h - 3/u$, and

$$\begin{aligned} \Delta V &= \Delta V_1 + \Delta V_2 \\ \Delta V_1 &= \frac{\hbar^2}{8mh^6}\Big(2ab(u^2-1) - 3b^2u^4\Big), \qquad \Delta V_2 = \frac{3\hbar^2}{8mf^2u^2} \ . \end{aligned} \tag{3.2}$$

The quantum potential ΔV_2 is necessary in order to obtain the correct energy spectrum, the quantum potential ΔV_1 is interpreted as a curvature term which we add in the metric for our proper quantum theory on $D_{3d-\mathrm{II}}$. Therefore similar as for $D_{3d-\mathrm{I}}$:

$$\mathcal{L}_{\mathrm{eff}}^{(D_{3d-\mathrm{II}})}(u,\dot{u},v,\dot{v},w,\dot{w}) := \frac{m}{2}\frac{bu^2-a}{u^2}(\dot{u}^2+\dot{v}^2+\dot{w}^2) + \Delta V_1 \ , \tag{3.3}$$

and the quantum Hamiltonian has the form

$$H^{(D_{3d-\mathrm{II}})} := -\frac{\hbar^2}{2m}\frac{u^2}{bu^2-a}\left[\frac{\partial^2}{\partial u^2} + \left(\frac{b}{f}-\frac{1}{u}\right)\frac{\partial}{\partial u} + \frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial w^2}\right] - \Delta V_1 \tag{3.4}$$

$$= \frac{1}{2m}\frac{u}{\sqrt{bu^2-a}}(p_u^2+p_v^2+p_w^2)\frac{u}{\sqrt{bu^2-a}} + \frac{3\hbar^2}{8mf^2u^2} \ . \tag{3.5}$$

(p_v, p_w as in $D_{3d-\mathrm{I}}$). The special form of the incorporation of a curvature term in H can be justified in the same way as in the case for $D_{3d-\mathrm{I}}$. We consider the path integral on $D_{3d-\mathrm{II}}$ and obtain by performing a time-transformation in the usual way:

$$\begin{aligned} &K^{(D_{3d-\mathrm{II}})}(u'',u',v'',v',w'',w';T) \\ &= \int_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t) \int_{v(t')=v'}^{v(t'')=v''} \mathcal{D}v(t) \int_{w(0)=w'}^{w(s'')=w''} \mathcal{D}w(s) \\ &\quad \times \left(\frac{bu^2-a}{u^2}\right)^{3/2} \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T \left[\frac{m}{2}\frac{bu^2-a}{u^2}(\dot{u}^2+\dot{v}^2+\dot{w}^2) - \frac{3\hbar^2}{8mf^2u^2}\right]\mathrm{d}t\right\} \qquad (3.6) \\ &= \int_{-\infty}^{\infty}\frac{\mathrm{d}E}{2\pi\hbar}\,\mathrm{e}^{-\mathrm{i}ET/\hbar}[f(u')f(u'')]^{-1/4}\int_0^{\infty}\mathrm{d}s'' \\ &\quad \times \int_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s) \int_{v(0)=v'}^{v(s'')=v''} \mathcal{D}v(s) \int_{w(0)=w'}^{w(s'')=w''} \mathcal{D}w(s) \\ &\quad \times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}(\dot{u}^2+\dot{v}^2+\dot{w}^2) - \frac{\hbar^2}{2m}\frac{2maE/\hbar^2+3/4}{u^2}\right]\mathrm{d}s + \frac{\mathrm{i}}{\hbar}bEs''\right\}. \qquad (3.7) \end{aligned}$$

A look on Table 1 shows that all eleven coordinate systems can be used to separate variables in the path integral (3.6). The path integral in (3.7) in the variable u can be seen as the special case of V_1 (singular oscillator) from [9] where $\omega = 0, k_{2,3} = \pm\frac{1}{2}$. An explicit evaluation is possible in the (u,v,w)-system, in spherical, in circular-polar, circular-parabolic and in parabolic coordinates. Let us first consider the (u,v,w)-system. We continue in (3.7) in the same way as in [8], we set $\lambda^2 = 1 - 2m|a|E$, where we assume that $a < 0$, and we get due to the fact that the path integral (3.7) is of the radial $1/u^2$-type [11]:

$$\begin{aligned} & K^{(D_{3d-\mathrm{II}})}(u'',u',v'',v',w'',w';T) \\ &= \frac{1}{[f(u')f(u')]^{1/4}} \frac{m\sqrt{u'u''}}{\mathrm{i}\hbar} \int_{-\infty}^{\infty} \frac{\mathrm{d}E}{2\pi\hbar}\, \mathrm{e}^{-\mathrm{i}ET/\hbar} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{k} \frac{\mathrm{e}^{\mathrm{i}k_v(v''-v')+\mathrm{i}k_w(w''-w')}}{(2\pi)^2} \\ &\quad \times \int_0^{\infty} \frac{\mathrm{d}s''}{s''} \exp\left[\frac{\mathrm{i}}{\hbar}\left(bE - \frac{\hbar^2\mathbf{K}^2}{2m}\right)s'' + \frac{\mathrm{i}}{\hbar}\frac{m}{2s''}(u'^2+u''^2)\right] I_\lambda\left(\frac{mu'u''}{\mathrm{i}\hbar s''}\right) \ , \end{aligned} \tag{3.8}$$

where we have set $(\mathbf{K} = (k_v, k_w))$ and $\mathbf{K}^2 = k_v^2 + k_w^2$. The evaluation of the $\mathrm{d}s''$-integration integral in (3.8) yields for the Green function

$$\begin{aligned} & G^{(D_{3d-\mathrm{II}})}(u'',u',v'',v',w'',w';E) \\ &= \frac{1}{[f(u')f(u')]^{1/4}} \frac{\hbar}{\pi^2} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{k}\, \frac{\mathrm{e}^{\mathrm{i}k_v(v''-v')+\mathrm{i}k_w(w''-w')}}{(2\pi)^2} \\ &\quad \times \int_0^{\infty} \frac{2p\sinh\pi p \mathrm{d}p}{\frac{\hbar^2}{2m|a|}(p^2+1) - E} K_{\mathrm{i}p}\left(\sqrt{\mathbf{K}^2 - \frac{2mbE}{\hbar^2}}\, u'\right) K_{\mathrm{i}p}\left(\sqrt{\mathbf{K}^2 - \frac{2mbE}{\hbar^2}}\, u''\right) \ , \end{aligned} \tag{3.9}$$

with $\lambda = \sqrt{1 - 2m|a|E/\hbar^2} \equiv \mathrm{i}p$. The wave functions and the energy spectrum are read off:

$$\Psi(u,v) = \frac{\mathrm{e}^{\mathrm{i}k_v v + \mathrm{i}k_w w}}{2\pi f^{1/4}(u)} \cdot \frac{\sqrt{2p\sinh\pi p}}{\pi} K_{\mathrm{i}p}\left(\sqrt{\mathbf{K}^2 - \frac{2mbE}{\hbar^2}}\, u\right) \ , \tag{3.10}$$

$$E = \frac{\hbar^2}{2m|a|}(p^2+1) \ . \tag{3.11}$$

The case $b = 0$ gives the case of the quantum motion on the three-dimensional hyperboloid [7], as it should be. Let us note that we can replace in (3.9) the expansion of the circular- (respectively plane-) waves of the (v,w)-subsystem by the appropriate expansion of the remaining three circular systems of Table 1, i.e. circular-polar coordinates with a Bessel-function times circular waves, circular-parabolic coordinates with a product of two parabolic cylinder functions and circular-elliptic coordinates with Mathieu-functions [7].

3.2 The Spherical System

We consider the metric on $D_{3d-\mathrm{II}}$ in spherical coordinates

$$\mathrm{d}s^2 = \left(b - \frac{a}{r^2\cos^2\vartheta}\right)(\mathrm{d}r^2 + r^2\mathrm{d}\vartheta^2 + r^2\sin^2\vartheta\mathrm{d}\varphi^2) \tag{3.12}$$

from which the classical Lagrangian follows

$$\mathcal{L}(r,\dot{r},\vartheta,\dot{\vartheta},\varphi,\dot{\varphi}) = \frac{m}{2}\left(b - \frac{a}{r^2\cos^2\vartheta}\right)(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\sin^2\vartheta^2\dot{\varphi}^2) \ . \tag{3.13}$$

However, as has been pointed out in [8] this coordinate representation is not very well suited for our purposes, except that we recover for $a = 0$ polar coordinate in $\mathbb{R}^2$. We introduce $r = \mathrm{e}^{\tau_2}$ and $\cos\vartheta = 1/\cosh\tau_1$. We also have to take into account the curvature terms in a similar way as in the previous subsection which means that we obtain in the quantization procedure one term we subtract and one term we keep. This leads us to the following definition of the Lagrangian on $D_{3d-\mathrm{II}}$ for spherical coordinates in the transformed $(\tau_1, \tau_2, \varphi)$-system

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}}^{(D_{3d-\mathrm{II}})}(\tau_1, \dot\tau_1, \tau_2, \dot\tau_2, \varphi, \dot\varphi) &= \frac{m}{2}\left(\frac{b\ \mathrm{e}^{2\tau_2}}{\cosh^2\tau_1} - a\right)(\dot\tau_1^2 + \cosh^2\tau_1\dot\tau_2^2 + \sinh^2\tau_1\dot\varphi^2) \\ &\quad -\left(\frac{b\ \mathrm{e}^{2\tau_2}}{\cosh^2\tau_1} - a\right)^{-1}\frac{\hbar^2}{2m}\left(4 + \frac{1}{\cosh^2\tau_1} - \frac{1}{\sinh^2\tau_1}\right). \qquad (3.14)\end{aligned}$$

We obtain the following path integral representation $(f(\tau_1, \tau_2) = b\ \mathrm{e}^{2\tau_2}/\cosh^2\tau_1 - a)$

$$\begin{aligned}&K^{(D_{3d-\mathrm{II}})}(\tau_1'', \tau_1', \tau_2'', \tau_2', \varphi'', \varphi'; T) \\ &= \int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \mathcal{D}\tau_1(t) \int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''} \mathcal{D}\tau_2(t) \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t) \sinh\tau_1\cosh\tau_1\left(\frac{b\ \mathrm{e}^{2\tau_2}}{\cosh^2\tau_1} - a\right)^{3/2} \\ &\quad \times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T \left[\frac{m}{2}f(\tau_1, \tau_2)(\dot\tau_1^2 + \cosh^2\tau_1\dot\tau_2^2 + \sinh^2\tau_1\dot\varphi^2)\right.\right. \\ &\qquad \left.\left. -\frac{1}{f(\tau_1, \tau_2)}\frac{\hbar^2}{2m}\left(4 + \frac{1}{\cosh^2\tau_1} - \frac{1}{\sinh^2\tau_1}\right)\right]\mathrm{d}t\right\} \\ &= [f(\tau_1', \tau_2')f(\tau_1'', \tau_2'')]^{-1/4}\int_{-\infty}^{\infty}\frac{\mathrm{d}E}{2\pi\hbar}\ \mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty \mathrm{d}s'' K^{(D_{3d-\mathrm{II}})}(\tau_1'', \tau_1', \tau_2'', \tau_2', \varphi'', \varphi'; s'') \qquad (3.15)\end{aligned}$$

with the time-transformed path integral $K(s'')$ given by $(a < 0)$:

$$\begin{aligned}&K^{(D_{3d-\mathrm{II}})}(\tau_1'', \tau_1', \tau_2'', \tau_2', \varphi'', \varphi'; s'') \\ &= \int\limits_{\tau_1(0)=\tau_1'}^{\tau_1(s'')=\tau_1''} \mathcal{D}\tau_1(s) \int\limits_{\tau_2(0)=\tau_2'}^{\tau_2(s'')=\tau_2''} \mathcal{D}\tau_2(s) \int\limits_{\varphi(0)=\varphi'}^{\varphi(s'')=\varphi''} \mathcal{D}\varphi(s) \sinh\tau_1\cosh\tau_1 \\ &\quad \times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''} \left[\frac{m}{2}(\dot\tau_1^2 + \cosh^2\tau_1\dot\tau_2^2 + \sinh^2\tau_1\dot\varphi^2)\right.\right. \\ &\qquad \left.\left. +Eb\frac{\mathrm{e}^{2\tau_2}}{\cosh^2\tau_1} - aE - \frac{\hbar^2}{2m}\left(4 + \frac{1}{\cosh^2\tau_1} - \frac{1}{\sinh^2\tau_1}\right)\right]\mathrm{d}s\right\}\ . \qquad (3.16)\end{aligned}$$

This path integral in the variable φ can be easily evaluated, and in the variable τ_2 we have a path integral for Liouville quantum mechanics [11]. This yields

$$\begin{aligned}&K^{(D_{3d-\mathrm{II}})}(\tau_1'', \tau_1', \tau_2'', \tau_2', \varphi'', \varphi'; s'') \\ &= \sqrt{\cosh\tau_1'\cosh\tau_1'}\exp\left[\frac{\mathrm{i}}{\hbar}\left(|a|E - \frac{\hbar^2}{2m}\right)s''\right]\sum_{k_\varphi\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}k_\varphi(\varphi''-\varphi')}}{2\pi} \\ &\quad \times\frac{2}{\pi^2}\int_0^\infty \mathrm{d}k_{\tau_2}\, k_{\tau_2}\sinh\pi k_{\tau_2}K_{\mathrm{i}k_{\tau_2}}\left(\frac{\sqrt{-2mbE}}{\hbar}\ \mathrm{e}^{\tau_2'}\right)K_{\mathrm{i}k_{\tau_2}}\left(\frac{\sqrt{-2mbE}}{\hbar}\ \mathrm{e}^{\tau_2''}\right)\end{aligned}$$

$$
\times \int\limits_{\tau_1(0)=\tau_1'}^{\tau_1(s'')=\tau_1''} \mathcal{D}\tau_1(s) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{\tau}_1^2 - \frac{\hbar^2}{2m}\left(\frac{k_\varphi^2-\frac{1}{4}}{\sinh^2\tau_1} - \frac{-k_{\tau_2}^2-\frac{1}{4}}{\cosh^2\tau_1}\right]\mathrm{d}s\right\}
$$

$$
= \sqrt{\cosh\tau_1'\cosh\tau_1'}\exp\left[\frac{\mathrm{i}}{\hbar}\left(|a|E-\frac{\hbar^2}{2m}\right)s''\right]\sum_{k_\varphi\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}k_\varphi(\varphi''-\varphi')}}{2\pi}
$$

$$
\times\frac{2}{\pi^2}\int_0^\infty \mathrm{d}k_{\tau_2}\, k_{\tau_2}\sinh\pi k_{\tau_2} K_{\mathrm{i}k_{\tau_2}}\left(\frac{\sqrt{-2mbE}}{\hbar}\,\mathrm{e}^{\tau_2'}\right)K_{\mathrm{i}k_{\tau_2}}\left(\frac{\sqrt{-2mbE}}{\hbar}\,\mathrm{e}^{\tau_2''}\right)
$$

$$
\times\int_0^\infty \mathrm{d}s''\,\mathrm{e}^{-\mathrm{i}p^2\hbar s''/2m}\Psi^{(k_\varphi,\mathrm{i}k_{\tau_2})}(\tau_3'')\Psi^{(k_\varphi,\mathrm{i}k_{\tau_2})\,*}(\tau_3')\quad , \tag{3.17}
$$

and we have inserted in the last step the path integral solution for the modified Pöschl–Teller potential $V^{(mPT)}(r)$. The modified Pöschl–Teller functions $\Psi_p^{(\mu,\nu)}(\omega)$ for the continuous spectrum are given by [2, 15]

$$
\begin{aligned}
V^{(mPT)}(r) &= \frac{\hbar^2}{2m}\left(\frac{\eta^2-\frac{1}{4}}{\sinh^2 r}-\frac{\nu^2-\frac{1}{4}}{\cosh^2 r}\right)\\
\Psi_p^{(\eta,\nu)}(r) &= N_p^{(\eta,\nu)}(\cosh r)^{2k_1-\frac{1}{2}}(\sinh r)^{2k_2-\frac{1}{2}}\\
&\qquad\times {}_2F_1(k_1+k_2-\kappa,k_1+k_2+\kappa-1;2k_2;-\sinh^2 r) \qquad (3.18)\\
N_p^{(\eta,\nu)} &= \frac{1}{\Gamma(2k_2)}\sqrt{\frac{p\sinh\pi p}{2\pi^2}}\Big[\Gamma(k_1+k_2-\kappa)\Gamma(-k_1+k_2+\kappa)\\
&\qquad\times\Gamma(k_1+k_2+\kappa-1)\Gamma(-k_1+k_2-\kappa+1)\Big]^{1/2}\quad , \qquad (3.19)
\end{aligned}
$$

k_1, k_2 defined by: $k_1=\frac{1}{2}(1\pm\nu)$, $k_2=\frac{1}{2}(1\pm\eta)$, where the correct sign depends on the boundary-conditions for $r\to 0$ and $r\to\infty$, respectively. The number N_M denotes the maximal number of states with $0,1,\ldots,N_M<k_1-k_2-\frac{1}{2}$. $\kappa=k_1-k_2-n$ for the bound states and $\kappa=\frac{1}{2}(1+\mathrm{i}p)$ for the scattering states. ${}_2F_1(a,b;c;z)$ is the hypergeometric function [4, p.1057]. We omit the bound states, because they do not exist here.

Performing the s''-integration gives the energy-spectrum (3.11) with the Green function

$$
\begin{aligned}
&G^{(D_{3d-\mathrm{II}})}(\tau_1'',\tau_1',\tau_2'',\tau_2',\varphi'',\varphi';E)\\
&=\int_0^\infty \mathrm{d}p\int_0^\infty \mathrm{d}k_{\tau_2}\sum_{k_\varphi\in\mathbb{Z}}\frac{\Psi_{p,k_{\tau_2},k_\varphi}(\tau_1'',\tau_2'',\varphi'')\Psi^*_{p,k_{\tau_2},k_\varphi}(\tau_1'',\tau_2'',\varphi'')}{\frac{\hbar^2}{2m|a|}(p^2+1)-E}\quad , \qquad (3.20)
\end{aligned}
$$

and the wave-functions are given by

$$
\begin{aligned}
&\Psi_{p,k_{\tau_2},k_\varphi}(\tau_1,\tau_2,\varphi)\\
&=\frac{\sqrt{2\sinh\tau_1\cosh\tau_1}}{f(\tau_1,\tau_2)^{1/4}}\frac{\mathrm{e}^{\mathrm{i}k_\varphi(\varphi''-\varphi')}}{\sqrt{2\pi}}\frac{\sqrt{k_{\tau_2}\sinh\pi k_{\tau_2}}}{\pi}K_{\mathrm{i}k_{\tau_2}}\left(\mathrm{i}\sqrt{\frac{b}{|a|}(p^2+1)}\,\mathrm{e}^{\tau_2}\right)\Psi^{(k_\varphi,\mathrm{i}k_{\tau_2})}(\tau_3), \qquad (3.21)
\end{aligned}
$$

where the replacement $r=\mathrm{e}^{\tau_2}$ and $\cos\vartheta=1/\cosh\tau_1$ gives the wave-functions in the original spherical system.

Due to the $1/u^2$-term in the metric, it is possible to separate the path integral in conical coordinates, however, it cannot be evaluated.

4 Discussion and Summary

Our results are very satisfactory. It was possible to define a quantum theory on three-dimensional spaces of non-constant curvature and evaluate the path integral in the (u, v, w)-coordinate system in both spaces. In $D_{3d-\mathrm{II}}$ I also evaluated the path integral in spherical coordinates. A detailed investigation of the kernel and wave-functions depend on the particular choice of the boundary conditions (on $D_{3d-\mathrm{I}}$) and the parameters a, b (on $D_{3d-\mathrm{II}}$). In order to achieve the results we had to incorporate a curvature term in the definition of the quantum theory. Otherwise, a solution would not have been possible. The particular form of the additional term was determined by the method of (space-) time transformation. Of course, the solution of the Schrödinger equation by separation of variables is also only possible with this additional curvature term. This additional term can be cast in to the form $(\hbar^2/2m) \times (R/8)$, where R is the scalar curvature. In view of the fact that our world has three spatial dimensions and any theory of gravity requires spaces with (constant or non-constant) curvature, we find the important feature that such models require curvature terms in the corresponding Lagrangian in order to set up a proper and solvable quantum theory.

Therefore we found on $D_{3d-\mathrm{I}}$ seven separating coordinate systems, and on $D_{3d-\mathrm{II}}$ the eleven coordinate systems of three-dimensional Euclidean space. On $D_{3d-\mathrm{I}}$ the (u, v, w)-system is singled out because the cylindrical systems are contained as sub-systems, and the parabolic and paraboloidal systems cannot be evaluated further. The propagator on $D_{3d-\mathrm{II}}$ can be evaluated also in other coordinate systems, in the remaining three cylindrical systems (as subsystems), and in parabolic coordinates.

I have not discussed the other two three-dimensional extensiuns of the Darboux spaces as defined in [12]. Their corresponding generalization is more complicated and, furthermore, other coordinate systems which separate the Schrödinger equations come into play, which appear on the complex sphere and complex Euclidean space. This issue will be addresses in a future publication.

Having studied the free motion on these three-dimensional spaces, the next step is to search and investigate superintegrable potentials [9, 10, 17]. In particular, in three dimensions there is a great variety of such potentials. In total, there are five maximally superintegrable potentials [9], the first four of them also are superintegrable on $D_{3d-\mathrm{II}}$, including the singular harmonic oscillator, the Holt potential and the Coulomb potential. Studies along such lines are straightforward, many of the results from two dimensions can be also used in the corresponding three-dimensional cases; this will be investigated in a future publication.

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