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Real and Image Fields of a Relativistic Bunch

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Abstract

We derive analytical expressions for external fields of a charged relativistic bunch with a circular cross section. At distances far from the bunch, the field reduces to the relativistic modified Coulomb form and in the near region, reproduce the external fields of a continuous beam. If the bunch is surrounded by conducting surfaces, the bunch self-fields are modified. Image fields generated by a bunch between two parallel conducting planes are studied in detail. Exact summation of image fields by the direct method invented by Laslett allows the infinite series to be represented in terms of elementary trigonometric functions.

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1 Introduction

In an accelerator, the charged beam is influenced by an environment (beam pipe, accelerator gaps, magnets, collimators, etc.), and a high-intensity bunch induces surface charges or currents into this environment. This modifies the electric and magnetic fields around the bunch. There is a relatively simple method to account for the effect of the environment by introducing image charges and currents.

Over forty years ago Laslett [1] analyzed the influence of the transverse space-charge phenomena, due to image forces, on the instability of the coherent transverse motion of an intense beam. Methods of image fields summation are described in his paper [1] which presented some field coefficients calculated for infinite parallel plate vacuum chambers, magnetic poles and vacuum chambers with elliptical cross sections and variable aspect rations. The resulting image field was calculated only in the linear approximation and depends linearly on the deviations $\bar{x}$ and x of the bunch center and the position of a test particle, respectively, from the axis. They act therefore like a quadrupole causing a coherent tune shift. The approximation used is incorrect if the field observation point x is located far from the bunch or if the bunch center is close to a conducting wall.

In the present paper we consider the problem of the image field summation once again for a very simple geometry, namely, a relativistic bunch moving between infinitely wide parallel conducting planes. The problem is far from being a pure academic one [2]. In applications, in particular by study of the electron cloud effect [3] and the dynamics of photoelectrons in the beam transport system, it is important to know the distribution of electromagnetic fields not only in vicinity of the bunch, but in the whole gap. We did not find publications with attempts to sum up the series (17) in an approximation beyond the linear one. In Section 3 we present the exact solution of the problem.

To solve the problem formulated above, in Section 2 we first derive an equation for the external electromagnetic field generated by a cylindrical bunch of charged particles. The task is specified as follows.

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The external radial electric $\vec{E}_\perp$, and azimuthal magnet $\vec{B}_\phi$ self fields for a round unbunched relativistic beam of the radius b and a uniform charge density are [5]-[7]

$$E_\perp = \kappa \frac{2q\lambda}{r}, \tag{1}$$

$$B_\phi = \frac{\mu_0}{4\pi}\frac{2q\lambda}{r} c\beta, \tag{2}$$

where $\kappa = 1/4\pi\epsilon_0$, λ is the linear charge density, q is the charge, $\beta = v/c$ is a normalized velocity of the beam constituents and c is the velocity of light. In many applications, equations (1) and (2) are used to describe fields of an individual bunch too. However, in the form (1), (2) the bunch fields do not depend on the bunch energy and at large distances do not follow the Coulomb asymptotic. This contrasts sharply with the fields produced (at $t = 0$) by a rapidly moving single charge q

$$\vec{E} = \kappa \frac{q\,\gamma}{r^2}\Big[\frac{1-\beta^2}{1-\beta^2\, sin^2\,\theta}\Big]^{3/2}\frac{\vec{\mathbf{r}}}{r}, \qquad \vec{B} \sim \vec{\beta}\times\vec{E}, \tag{3}$$

where θ is the angle which the vector $\vec{\mathbf{r}}$ makes with the z-axis. Along the direction of motion the electric field is become weaker in γ^2 times, while in the transverse direction the electric field is enhanced by the factor γ

$$E_\perp = \kappa \frac{q\,\gamma}{r^2}. \tag{4}$$

Here, γ denotes the particle Lorentz factor.

In the next section we derive an expression for the transverse component of the bunch electric field, which the defects indicated above are rectified, and find the conditions at which the bunch fields are represented by (1) and (2).

2 Self-Fields of a Charged Finite Cylinder with a Circular Cross Section

Let us consider a bunch of charged particles uniformly distributed with a density ρ within a cylinder of length L and an elliptical cross section. The ellipsoid semi-axis in the x-y plane are a and b and the coordinate z-axis is along the bunch axis. Suppose that the bunch is moving along the z-axis with a relativistic velocity $\vec{v} = c\vec{\beta}$.

To compute the radial electric field of such a rapidly moving bunch, we have to sum up fields of the type (3), generated by the bunch constituents. In this way we get [4]

$$E_\perp(r,\xi,z) = \kappa\rho\gamma\{zI_1 + (L-z)I_2\} \tag{5}$$

with

$$I_1 = \int\int \frac{(r-\sigma\cos(\xi-\phi))\,\sigma d\sigma d\phi}{(r^2+\sigma^2-2r\sigma\cos(\xi-\phi))\sqrt{\gamma^2z^2+r^2+\sigma^2-2r\sigma\cos(\xi-\phi)}} \tag{6}$$

$$I_2 = \int\int \frac{(r-\sigma\cos(\xi-\phi))\,\sigma d\sigma d\phi}{(r^2+\sigma^2-2r\sigma\cos(\xi-\phi))\sqrt{\gamma^2(L-z)^2+r^2+\sigma^2-2r\sigma\cos(\xi-\phi)}} \tag{7}$$

where σ is the distance in the x-y plane from the z-axis to the elementary charged volume and

$$0 < \sigma < \frac{ab}{\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}}, \qquad 0 < \phi < 2\pi. \tag{8}$$

Equation (5) represents the radial electric field as observed at a distance r from the bunch axis, at an angle ξ relative to the x axis and at a distance z from the bunch tail.

The integrals I_1 and I_2 can be estimated only numerically [4], if integrands are taken as it is. However, the integrands are easy to simplify if the bunch is relativistic, $\gamma \gg 1$, and we would like calculate the field in vicinity of the bunch, $r \sim L$, but at distances much larger than the bunch radius, $b \ll r$.

To simplify, we make use of the notation

$$A = \frac{\sigma}{r}, \quad B = A\cos(\xi - \phi), \quad Y = A^2 - 2B, \quad C_1 = \Big[1 + \frac{\gamma^2 z^2}{r^2}\Big]^{-1}, \quad X = C_1 \cdot Y$$

and the integrand of I_1 can be written as

$$(r^2 + \gamma^2 z^2)^{-1/2} A(1-B)(1+Y)^{-1}(1+X)^{-1/2}. \tag{9}$$

Now we expand the above expression in a power series by using A as a small parameter and keeping only terms up to the power A^4 at each step. For the bunch shaped as a circular cylinder, $a = b$ and we may set $\xi = 0$. Due to the fact that

$$\int_0^{2\pi} \cos^{2k+1}\phi \, d\phi \; = \; 0, \tag{10}$$

all odd power of B vanish after integration in ϕ. This greatly simplifies the series generated from (9). After lengthy algebraic manipulations with (9), we get

$$(r^2 + \gamma^2 z^2)^{-1/2} A\Big[1 - (1 + \frac{1}{2}C_1)A^2 + (2 + C_1 + \frac{3}{2}C_1^2)B^2\Big]. \tag{11}$$

Substituting this expression in (6), we get

$$I_1 \; = \; \frac{\pi b^2}{r\sqrt{r^2 + \gamma^2 z^2}}\Big(1 + \frac{3}{8}C_1^2\frac{b^2}{r^2}\Big). \tag{12}$$

By changing z^2 to $(L-z)^2$ in (12), we obtain for I_2 the following result

$$I_2 \; = \; \frac{\pi b^2}{r\sqrt{r^2 + \gamma^2 (L-z)^2}}\Big(1 + \frac{3}{8}C_2^2\frac{b^2}{r^2}\Big), \tag{13}$$

where $C_2 = \Big[1 + \gamma^2(L-z)^2/r^2\Big]^{-1}$. Notice that for particles uniformly distributed in the bunch volume, $\rho = qN/\pi b^2 L$, where N is number particles per bunch. Substituting equations (12)-(13) in (5), finally we arrive to

$$E_\perp(r,z) \; = \; \kappa\frac{qN\gamma}{Lr}\Big\{\frac{z}{\sqrt{r^2 + \gamma^2 z^2}}\Big(1 + \frac{3}{8}\frac{b^2}{r^2}C_1^2\Big) + \frac{L-z}{\sqrt{r^2 + \gamma^2 (L-z)^2}}\Big(1 + \frac{3}{8}\frac{b^2}{r^2}C_2^2\Big)\Big\}. \tag{14}$$

This equation describe the electric field produced by a rapidly moving circular bunch.

The field of a relativistic bunch described by (14), has different behavior at distances far apart of the bunch and in the near region, $r \le L$. At very large distances, $r \gg \gamma z$ and $r \gg \gamma(L-z)$, equation (14) reduces to the Coulomb form (4). At the same time, in the near region and beyond the bunch tails, $\gamma z \approx \gamma(L-z) \gg r$ and equation (14) simplifies to

$$E_\perp \; = \; \kappa\frac{2qN}{L}\frac{1}{r}, \tag{15}$$

which coincide with the external field (1) of a continuous beam with $\lambda = N/L$.

Similarly we can show that the azimuthal magnetic field of the bunch is

$$B_\phi \; = \; \frac{\mu_0}{4\pi}\frac{\beta c}{\kappa}E_\perp(r,z). \tag{16}$$

3 Fields from Image Charges

Following Laslett [1] (see also [5]), we consider a relativistic bunch of the length L between infinitely wide conducting planes at $x = \pm h$. Suppose that constituents of the bunch are positively charged. For full generality, let the circular particle bunch be displaced in the horizontal plane by $\bar{x}$ from the midplane (0,y,z), and the observation point of the field be at $(x, 0, 0)$ between the conducting parallel planes. The end points of the bunch are at $z = \pm L/2$. The boundary condition for electric fields is $E_z(\pm h) = 0$ on the conducting plane and is satisfied if the image charges change sign from image to image. Suppose that the distance between planes is of the order L. Thus, the electric field of each image is described by (15). To calculate the image electric field $E_{\perp,image}(x)$ in front of the plate, we add the contributions from all image fields in the infinite series [1]

$$\begin{aligned} E_{\perp,image}(x,\bar{x}) \;=\; & \kappa\frac{2qN}{L}\cdot \\ & \Big\{ (2h-x_1)^{-1} - (2h+x_1)^{-1} - (4h-x_2)^{-1} + (4h+x_2)^{-1} \\ & + (6h-x_1)^{-1} - (6h+x_1)^{-1} - (8h-x_2)^{-1} + (8h+x_2)^{-1} \\ & + (10h-x_1)^{-1} - (10h+x_1)^{-1} - (12h-x_2)^{-1} + (12h+x_2)^{-1} + ... \Big\}, \end{aligned} \tag{17}$$

where $x_1 = x + \bar{x}$ and $x_2 = x - \bar{x}$. These image fields must be added to the direct field of the bunch (15) to meet the boundary condition that the electric field enters conducting surfaces perpendicularly.

In the original paper [1], the series (17) was summed up only in the linear approximation in x and $\bar{x}$,

$$E_{\perp,image}(x,\bar{x}) \;=\; \kappa\frac{4qN}{L}\frac{\epsilon_1}{h^2}(2\bar{x}+x). \tag{18}$$

The coefficient $\epsilon_1 = \pi^2/48$ is known as the Laslett coefficient (or form factor) for infinite parallel plate vacuum chambers and magnetic poles. The approximation used in (18) is incorrect if the deviation of the bunch center from the axis is large ($\bar{x} \sim h$) or if the field observation point x is located far off the bunch. Therefore, below we present the exact solution of the problem.

In Appendixes A and B we prove that the exact summation of the series (17) gives

$$E_{\perp,image}(x,\bar{x}) \;=\; \kappa\frac{4qN}{Lh}\Lambda(\delta,\bar{\delta}), \tag{19}$$

where the image field structure function Λ depends only on normalized variables $\delta = x/h$, $\bar{\delta} = \bar{x}/h$ in the form

$$\Lambda(\delta,\bar{\delta}) = \frac{1}{2}\Big[\frac{\pi}{2}\cdot\frac{\cos(\frac{\pi}{2}\bar{\delta})}{\sin(\frac{\pi}{2}\delta)-\sin(\frac{\pi}{2}\bar{\delta})} - \frac{1}{\delta-\bar{\delta}}\Big]. \tag{20}$$

In Appendix A it is shown that in the linear approximation equation (19) recovers the part (18) derived by Laslett.

We shall now estimate values of the function Λ in several particular points. If the observation point of the field is located at the plane, $x = h$, then $\delta = 1$ and the structure function depends only on the bunch center position between planes, $\bar{\delta}$. Thus, from (20) we get

$$\Lambda(1,\bar{\delta}) = \frac{1}{2}\Big\{\frac{\pi}{2}\frac{1+\sin(\frac{\pi}{2}\bar{\delta})}{\cos(\frac{\pi}{2}\bar{\delta})} - \frac{1}{1-\bar{\delta}}\Big\}. \tag{21}$$

Equation (21) is singular at $\bar{\delta} \to 1$ and shows that the conducting plane attracts the bunch with increasing force with the bunch displacement from the midplane. This phenomenon, involving the transverse movement of the bunch as a whole, arises from image forces and could lead to a transverse instability.

For a bunch in the midplane, $\bar{\delta} = 0$, the summed image field at the surface equals

$$E_{\perp,image}(h,0) \;=\; \kappa\frac{4qN}{Lh}\Lambda(1,0) \;=\; \kappa\frac{2qN}{Lh}(\frac{\pi}{2}-1). \tag{22}$$

The image field (19) must be added to the direct field of the bunch (15) to meet the boundary condition. Thus

$$E_{\perp,tot}(x,\bar{x}) \;=\; E_{\perp,bunch} + E_{\perp,image} = \kappa\frac{2qN}{Lh}\Big(\frac{1}{\delta} + 2\Lambda(\delta,\bar{\delta})\Big). \tag{23}$$

For a bunch in the midplane, $\bar{\delta} = 0$, we find from (23) the expression of the transverse component of electric field generated by a relativistic bunch moving between wide conducting parallel planes

$$E_{\perp,tot}(x,0) \;=\; \kappa\frac{2qN}{Lh}\cdot\frac{\pi/2}{\sin(\frac{\pi}{2}\delta)}. \tag{24}$$

That is, at the surface, $\delta = 1$, the field is enhanced by a factor $\pi/2$ due to the presence of the conducting planes.

Notice that in the linear approximation (18) the field gradient, $\partial E_\perp/\partial x$, is independent of position x. Thus the tune shift experienced by each particle in the bunch is the same (a coherent tune shift). However, the exact result (19) demonstrates that the coherence is violated and equation (19) allows us to estimate the accuracy of the linear approximation.

4 Magnetic Images

In the above, we have used electrostatic images. Magnetic images can be treated in much the same way. Let the ferromagnetic boundaries be represented by a pair of infinitely wide parallel surfaces at $x = \pm g$. The magnetic field lines must enter the magnetic pole faces perpendicularly. For magnetic image fields we distinguish between DC and AC image fields. The DC field penetrates the metallic vacuum chamber and reaches the ferromagnetic poles. In case of bunched beams the AC fields are of rather high frequency and we assume that they do not penetrate the thick metallic vacuum chamber.

The DC Fourier component of a bunched beam current is equal to twice the average beam current $qc\beta\lambda B$ [5], where B is the the Laslett bunching factor. Thus,

$$B_{y,image,DC}(x,\bar{x}) \;=\; -\frac{\mu_0}{4\pi}\frac{2qN\beta c}{L}B\cdot\frac{4}{g}\Lambda(\eta,\bar{\eta}), \tag{25}$$

where $\eta = x/g$ and $\bar{\eta} = \bar{x}/g$ and the function Λ is of the form (21).

The contribution of magnetic AC image field due to eddy currents in vacuum chamber walls is similar to electric image fields

$$B_{y,image,AC}(x,\bar{x}) \;=\; \frac{\mu_0}{4\pi}\frac{2qN\beta c}{L}(1-B)\cdot\frac{2}{h}\Lambda(\delta,\bar{\delta}), \tag{26}$$

where the factor $(1-B)$ accounts for the subtraction of the DC component.

The magnetic image fields must be added to the direct magnetic field (2) from the bunch to meet the boundary condition of normal components at ferromagnetic surfaces. That is, the summary magnetic field between the conducting planes is

$$\begin{aligned} B_{y,tot}(x,\bar{x}) &= B_y + B_{y,image,DC} + B_{y,image,AC} \\ &= \frac{\mu_0}{4\pi}\frac{2qN\beta c}{L}\Big\{\frac{1}{x} + (1-B)\frac{2}{h}\Lambda(\delta,\bar{\delta}) - B\frac{4}{g}\Lambda(\eta,\bar{\eta})\Big\}. \end{aligned} \tag{27}$$

5 Summary

We have derived an approximate expressions for electric (14) and magnetic (16) self-fields produced by a relativistic circular bunch with uniform charge density. They show that at distances far from the bunch the electromagnetic field coinsides with the field generated by a point-like charged paricle. At the same time, in the near region and beyond the bunch tails, the fields coincide with the external self-fields of a continuous beam (1)-(2).

We re-analyzed the problem of summing the image fields generated by a bunch of charged particles moving with a relativistic velocity between infinitely wide parallel conducting planes. The exact solution of the problem represented by the structure function of image fields Λ (20) depending only of the normalized variables.

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Appendices

A Image Fields in Vicinity of a Bunch

Here we derive the main formula (20).

Let split the contribution of all image fields (17) given in braces into two parts,

$$\begin{aligned} & (2h-x_1)^{-1} - (2h+x_1)^{-1} - (4h-x_2)^{-1} + (4h+x_2)^{-1} \\ + \; & (6h-x_1)^{-1} - (6h+x_1)^{-1} - (8h-x_2)^{-1} + (8h+x_2)^{-1} \\ + \; & (10h-x_1)^{-1} - (10h+x_1)^{-1} - (12h-x_2)^{-1} + (12h+x_2)^{-1} + ... \qquad (28) \\ = \; & \sum_k^{\infty}\Pi_k(x_1,h) - \sum_m^{\infty}\Pi_m(x_2,h), \qquad (29) \end{aligned}$$

where Π_k represents the contribution from the negative charged images and Π_m is the contribution from the positive charged images. Here and hereinafter, indexes k and m are possess odd, k=1,3,5,..., and even, m=2,4,6,... values.

An expansion of denominators of Π_k and Π_m into a power series of small parameters $\delta_1 = x_1/h < 1$ and $\delta_2 = x_2/h < 1$ gives

$$\Pi_k(x_1,h) = \frac{1}{2kh-x_1} - \frac{1}{2kh+x_1} = \frac{2x_1}{(2kh)^2-x_1^2} = \frac{2}{h}\sum_{n=1}^{\infty}\frac{\delta_1^{2n-1}}{(2k)^{2n}}, \tag{30}$$

$$\Pi_m(x_2,h) = \frac{1}{2mh-x_2} - \frac{1}{2mh+x_2} = \frac{2x_2}{(2mh)^2-x_2^2} = \frac{2}{h}\sum_{n=1}^{\infty}\frac{\delta_2^{2n-1}}{(2m)^{2n}}. \tag{31}$$

Now it is evident that the space structure of the image fields between planes is characterized by a specific function $\Lambda(\delta_1, \delta_2)$, we term it the structure function,

$$\sum_k^{\infty} \Pi_k - \sum_m^{\infty} \Pi_m = \frac{2}{h}\Lambda(\delta_1, \delta_2). \tag{32}$$

with

$$\Lambda(\delta_1, \delta_2) = \sum_k^{\infty} \Big[\frac{\delta_1}{(2k)^2} + \frac{\delta_1^3}{(2k)^4} + ...\Big] - \sum_m^{\infty} \Big[\frac{\delta_2}{(2m)^2} + \frac{\delta_2^3}{(2m)^4} + ...\Big]. \tag{33}$$

The structure function Λ depends only on the normalized variables.

To proceed further, let us define the following auxiliary quantities

$$M_j^{(-)} = \sum_k^{\infty} \frac{1}{(2k)^{2j}} - \sum_m^{\infty} \frac{1}{(2m)^{2j}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)^{2j}} = \frac{1}{2^{2j}} \cdot \frac{(2^{2j-1}-1)\pi^{2j}}{(2j)!}|B_{2j}|, \tag{34}$$

$$M_j^{(+)} = \sum_k^{\infty} \frac{1}{(2k)^{2j}} + \sum_m^{\infty} \frac{1}{(2m)^{2j}} = \sum_{n=1}^{\infty} \frac{1}{(2n)^{2j}} = \frac{1}{2^{2j}} \cdot \frac{2^{2j-1} \cdot \pi^{2j}}{(2j)!}|B_{2j}|, \tag{35}$$

where B_{2j} are Bernoulli numbers, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$ etc. By adding and subtracting the leftmost parts of (34) and (35), we express k and m numerical series of (33) in terms of $M_j^{(-)}$ and $M_j^{(+)}$. Therefore, we get from (33)

$$\Lambda(\delta_1, \delta_2) = \frac{1}{2}\sum_{n=1}^{\infty} \Big[(M_n^{(-)} + M_n^{(+)})\delta_1^{2n-1} + (M_n^{(-)} - M_n^{(+)})\delta_2^{2n-1}\Big] \tag{36}$$

or after substituting of (34)-(35) in (36), we find the following form of the structure function generated by the charged bunch,

$$\Lambda(\delta_1, \delta_2) = \frac{1}{2}\sum_{n=1}^{\infty} \Big[(2^{2n}-1)\delta_1^{2n-1} - \delta_2^{2n-1}\Big] \frac{\pi^{2n}}{2^{2n}(2n)!}|B_{2n}|\,. \tag{37}$$

Using only the linear terms we recover the part derived by Laslett [1] (see equation (18))

$$\Lambda(\bar{x}, x, h) = \frac{1}{h} \cdot \epsilon_1(2\bar{x} + x). \tag{38}$$

An inspection of (37) shows that the contributions of negative charged images are enhanced by the factor $2^{2n} - 1$, as compared with the contributions from the positive charged images. Equation (37) also shows that for x in the bunch center, $\delta_2 = 0$ and the contributions from the positive charged images are vanish.

At the final step, it is possible to rewrite the infinite series (37) in terms of elementary trigonometric functions. To do this, recall the relations between the Bernoulli numbers and the trigonometric functions [8],[9]

$$z\,\tan(z) = \sum_{n=1}^{\infty} \frac{(2^{2n}-1)(2z)^{2n}}{(2n)!}|B_{2n}|, \qquad z\,\cot(z) = 1 - \sum_{n=1}^{\infty} \frac{(2z)^{2n}}{(2n)!}|B_{2n}|. \tag{39}$$

After some algebraic manipulations and the use of (39), we get from (37) a new exact and compact expression of the structure function

$$\Lambda(\delta_1, \delta_2) = \frac{1}{2}\Big[\frac{\pi}{4}\tan(\frac{\pi}{4}\delta_1) + \frac{\pi}{4}\cot(\frac{\pi}{4}\delta_2) - \frac{1}{\delta_2}\Big]. \tag{40}$$

Now, if we recall that $\delta_1 = (x+\bar{x})/h = \delta + \bar{\delta}$ and $\delta_2 = (x-\bar{x})/h = \delta - \bar{\delta}$, we obtain

$$\Lambda(\delta,\bar{\delta}) = \frac{1}{2}\Big[\frac{\pi}{2}\cdot\frac{\cos(\frac{\pi}{2}\bar{\delta})}{\sin(\frac{\pi}{2}\delta)-\sin(\frac{\pi}{2}\bar{\delta})} - \frac{1}{\delta-\bar{\delta}}\Big]. \tag{41}$$

At a first glance, equation (40) or (41) is singular at $\delta_2 = 0$ or $\delta = \bar{\delta}$, respectively. However, as we already discussed right after (37), it is not the case. Starting once again from (37) with $\delta_2 = 0$ and account (39), we get formally

$$\Lambda(\bar{\delta},\bar{\delta}) = \frac{\pi}{8}\tan(\frac{\pi}{2}\bar{\delta}). \tag{42}$$

Equations (37), (40) and (41) were derived assuming $\delta < 1$ and $\bar{\delta} < 1$. Therefore, one cast doubts on validity of (41) at $\delta \sim 1$, near the conducting plane. For that reason in the next section we re-expand the series (28) into a power series of new small parameters.

B Image Fields in Vicinity of a Conducting Plane

A similar derivation is used to obtain the field structure near a conducting surface. For the case under consideration we have to choose new small parameters for the expansion. Each bracket in (28) we represent in the form $(1\pm\Delta)^{-1}$ and expand in series, recalling that at the plane $x \approx h$,

$$\Delta_1 = \frac{h-x_1}{h} \ll 1, \qquad \text{and} \qquad \Delta_2 = \frac{h-x_2}{h} \ll 1.$$

In this way,

$$(2kh-x_1)^{-1} = [(2k-1)h]^{-1}\Big[1+\frac{\Delta_1}{2k-1}\Big]^{-1} = \frac{1}{h}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\Delta_1^{n-1}}{(2k-1)^n}, \tag{43}$$

$$(2kh+x_1)^{-1} = [(2k+1)h]^{-1}\Big[1-\frac{\Delta_1}{2k+1}\Big]^{-1} = \frac{1}{h}\sum_{n=1}^{\infty}\frac{\Delta_1^{n-1}}{(2k+1)^n}, \tag{44}$$

$$(2mh-x_2)^{-1} = [(2m-1)h]^{-1}\Big[1+\frac{\Delta_2}{2m-1}\Big]^{-1} = \frac{1}{h}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\Delta_2^{n-1}}{(2m-1)^n}, \tag{45}$$

$$(2mh+x_2)^{-1} = [(2m+1)h]^{-1}\Big[1-\frac{\Delta_2}{2m+1}\Big]^{-1} = \frac{1}{h}\sum_{n=1}^{\infty}\frac{\Delta_2^{n-1}}{(2m+1)^n}. \tag{46}$$

Let us introduce the following auxiliary notations

$$L_j^{(\pm)} = \sum_k^{\infty}\Big[\frac{(-1)^{j-1}}{(2k-1)^j} - \frac{1}{(2k+1)^j}\Big] \pm \sum_m^{\infty}\Big[\frac{(-1)^{j-1}}{(2m-1)^j} - \frac{1}{(2m+1)^j}\Big], \tag{47}$$

$$L_{1,j}^{(-)} = \sum_k^{\infty}\frac{1}{(2k-1)^j} - \sum_m^{\infty}\frac{1}{(2m-1)^j} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^j}, \tag{48}$$

$$L_{2,j}^{(-)} = \sum_k^{\infty}\frac{1}{(2k+1)^j} - \sum_m^{\infty}\frac{1}{(2m+1)^j} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n+1)^j}, \tag{49}$$

$$L_{1,j}^{(+)} = \sum_k^{\infty}\frac{1}{(2k-1)^j} + \sum_m^{\infty}\frac{1}{(2m-1)^j} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^j}, \tag{50}$$

$$L_{2,j}^{(+)} = \sum_k^{\infty}\frac{1}{(2k+1)^j} + \sum_m^{\infty}\frac{1}{(2m+1)^j} = \sum_{n=1}^{\infty}\frac{1}{(2n+1)^j}. \tag{51}$$

By simple manipulations with seriess (48)-(49) and (50)-(51), it is easy to prove that

$$L^{(-)}_{2,j} = 1 - L^{(-)}_{1,j}, \qquad L^{(+)}_{2,j} = L^{(+)}_{1,j} - 1. \tag{52}$$

From (48)-(52) now easy to find

$$\begin{aligned} L^{(+)}_n + L^{(-)}_n &= [(-1)^{n-1} - 1]L^{(+)}_{1,n} + [(-1)^{n-1} + 1]L^{(-)}_{1,n}, & (53)\\ L^{(+)}_n - L^{(-)}_n &= [(-1)^{n-1} - 1]L^{(+)}_{1,n} - [(-1)^{n-1} + 1]L^{(-)}_{1,n} + 2. & (54) \end{aligned}$$

Let introduce the image field structure function Λ in the way similar to (32) and rewrite Λ in terms of notations (47)

$$\begin{aligned} \Lambda(\Delta_1, \Delta_2) &= \\ &= \frac{1}{2}\sum_{n=1}^{\infty}\Big\{\Delta_1^{n-1}\Big[\sum_k^{\infty}\Big(\frac{(-1)^{n-1}}{(2k-1)^n} - \frac{1}{(2k+1)^n}\Big)\Big] \\ &\qquad - \Delta_2^{n-1}\Big[\sum_m^{\infty}\Big(\frac{(-1)^{n-1}}{(2m-1)^n} - \frac{1}{(2m+1)^n}\Big)\Big]\Big\} \\ &= \frac{1}{2^2}\sum_{n=1}^{\infty}\Big\{\Delta_1^{n-1}(L^{(+)}_n + L^{(-)}_n) - \Delta_2^{n-1}(L^{(+)}_n - L^{(-)}_n)\Big\}. \end{aligned} \tag{55}$$

To perform the summation in (55), we have to split the series (55) into even, $n = 2i$, and odd, $n = 2i - 1$ parts and substitute (53), (54) in equation (55)

$$\Lambda(\Delta_1, \Delta_2) = \frac{1}{2}\sum_{n=0}^{\infty}\Big[(\Delta_1^{2n} + \Delta_2^{2n})L^{(-)}_{1,2n+1} - (\Delta_1^{2n+1} - \Delta_2^{2n+1})L^{(+)}_{1,2(n+1)} - \Delta_2^{2n} - \Delta_2^{2n+1}\Big]. \tag{56}$$

With the help of (48), (50) and [8] we find

$$L^{(-)}_{1,2n+1} = \frac{\pi^{2n+1}}{2^{2(n+1)}(2n)!}|E_{2n}|, \qquad L^{(+)}_{1,2(n+1)} = \frac{(2^{2(n+1)} - 1)\pi^{2(n+1)}}{2\cdot[2(n+1)]!}|B_{2(n+1)}|, \tag{57}$$

and

$$\sum_{n=0}^{\infty}(\Delta_2^{2n} + \Delta_2^{2n+1}) = \frac{1}{1-\Delta_2}, \tag{58}$$

where B_n and E_n are Bernoulli and Euler numbers, respectively. Thus,

$$\begin{aligned} \Lambda(\Delta_1, \Delta_2) &= \frac{1}{2}\Big\{2\cdot\frac{\pi}{4} - \frac{1}{1-\Delta_2} + \frac{\pi^2}{8}(\Delta_2 - \Delta_1) \\ &+ \sum_{n=1}^{\infty}\Big[(\Delta_1^{2n} + \Delta_2^{2n})L^{(-)}_{1,2n+1} - (\Delta_1^{2n+1} - \Delta_2^{2n+1})L^{(+)}_{1,2(n+1)}\Big]\Big\}. \end{aligned} \tag{59}$$

By using the results obtained in the previous section and the decomposition

$$\sec(x) = \sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}x^{2n}, \tag{60}$$

equation (59) can be finally expressed in terms of trigonometric functions,

$$\Lambda(\Delta_1, \Delta_2) = \frac{1}{2}\Big\{\frac{\pi}{4}\sec(\frac{\pi}{2}\Delta_1) - \frac{\pi}{4}\tan(\frac{\pi}{2}\Delta_1) + \frac{\pi}{4}\sec(\frac{\pi}{2}\Delta_2) + \frac{\pi}{4}\tan(\frac{\pi}{2}\Delta_2) - \frac{1}{1-\Delta_2}\Big\}. \tag{61}$$

The structure function $\Lambda(\Delta_1, \Delta_2)$, as written in (61), looks very different from (40). However, it is not difficult to check that by use of the relations $\Delta_1 = 1 - \delta_1$ and $\Delta_2 = 1 - \delta_2$, equation (61) transforms in (40) or (41).

In this way we ensure that equations (40), (41) and (61) are correct and represent exact summation of image fields generated by a charged bunch between infinitely wide conducting planes.

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