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Two–Loop Massive Operator Matrix Elements and Unpolarized Heavy Flavor Production at Asymptotic Values $Q^2 \gg m^2$ *

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Abstract

We calculate the $O(\alpha_s^2)$ massive operator matrix elements for the twist–2 operators, which contribute to the heavy flavor Wilson coefficients in unpolarized deeply inelastic scattering in the region $Q^2 \gg m^2$. The calculation has been performed using light–cone expansion techniques. We confirm an earlier result obtained in [1]. The calculation is carried out without using the integration-by-parts method and in Mellin space using harmonic sums, which lead to a significant compactification of the analytic results derived previously. The results allow to determine the heavy flavor Wilson coefficients for $F_2(x, Q^2)$ to $O(\alpha_s^2)$ and for $F_L(x, Q^2)$ to $O(\alpha_s^3)$ for all but the power suppressed terms $\propto (m^2/Q^2)^k, k \geq 1$.

*Dedicated to the Memory of W.L. van Neerven

1 Introduction

Deeply inelastic electron–nucleon scattering at large momentum transfer allows to measure the parton distribution functions of the nucleons together with the QCD scale $\Lambda_{\rm QCD}$. In the region of large hadronic masses $W^2 \simeq Q^2(1-x)/x$ the sea–quark distribution receives substantial contributions due to heavy flavor (charm and beauty) pair production. At the level of leading twist $\tau = 2$ their contribution to the deeply inelastic structure functions is described by heavy quark Wilson coefficients, which are convoluted with the light quark and gluon parton densities. Depending on the range of the Bjorken variable x and the gauge boson virtuality Q^2, these contributions can amount to 20–40% of the structure functions [2]. The unpolarized heavy flavor Wilson coefficients were calculated at leading order (LO) in Refs. [3]. The next-to-leading order (NLO) corrections were derived in semi-analytic form in Refs. [4] in x–space for the structure functions $F_2(x,Q^2)$ and $F_L(x,Q^2)$. A fast numerical implementation in Mellin space was given in [5]. For the asymptotic region $Q^2 \gg m^2$, an analytic result for the heavy flavor coefficient functions was calculated to $O(\alpha_s^2)$ in [1]. In the case of the structure function $F_L(x,Q^2)$, the asymptotic result to $O(\alpha_s^3)$ was derived in [6]. The leading order small-x resummation for $F_{2,L}^{Q\overline{Q}}(x,Q^2)$ was calculated in [7]. The heavy quark Wilson coefficients differ significantly from those of the light quarks even in the asymptotic region. Therefore, the scaling violations of the heavy flavor part in $F_{2,L}(x,Q^2)$ are different from those of the light flavor contributions. Both for the measurement of the QCD scale $\Lambda_{\rm QCD}$ and for the extraction of the light parton densities a correct description of the heavy flavor contributions is therefore required. As shown in Ref. [1], in case of the structure function $F_2(x,Q^2)$ the asymptotic heavy flavor terms describe the complete contributions very well already for scales $Q^2 \gtrsim 30\,\mathrm{GeV}^2$, whereas for $F_L(x,Q^2)$ this applies only at much higher scales, $Q^2 \gtrsim 800\,\mathrm{GeV}^2$.

In the present paper, we recalculate for the first time the asymptotic 2–loop corrections to the heavy flavor structure functions using a different method than in Ref. [1]. All logarithmic terms and the constant term of the heavy flavor Wilson coefficients are obtained due to a factorization of this quantity into the massive operator matrix elements and the light parton Wilson coefficients, which are known from the literature [8–10]. In [1] the massive operator matrix elements were derived in momentum-fraction (x)-space. The corresponding 2–loop integrals were simplified using the integration-by-parts method [11]. We will work in Mellin-space, accounting for the appropriate symmetry of the problem, and do thoroughly avoid the integration-by-parts method. This requires to solve more complicated Feynman-parameter integrals. However, in this way we are able to derive by far more compact results, even for the individual Feynman diagrams. In the direct calculation, we use Mellin–Barnes integrals [12,13] and representations through generalized hypergeometric functions [14]. A brief account on scalar 2–loop integrals to be derived in the present calculation was given in [15–17]. The final expressions obtained allow to represent the asymptotic heavy flavor contributions to the deep–inelastic structure functions in Mellin space in a completely analytic form. Precise representations of the analytic continuations of the harmonic sums w.r.t. the Mellin index N to complex variables are given in [18,19]. A fast single numerical inverse Mellin transformation yields the structure functions in x-space. This representation is well suited for fast numerical data analysis [20].

The paper is organized as follows. In Section 2, we describe the principal method applied to derive the 2–loop corrections in the asymptotic region $Q^2 \gg m^2$, covering all contributions but the power corrections $\propto (m^2/Q^2)^k, \quad k \geq 1$. In Section 3, the renormalization of the massive operator matrix elements is described. Section 4 gives a brief account of the 1–loop corrections. The 2–loop corrections to the operator matrix elements are derived in Section 5. Working in

$D = 4+\varepsilon$ space–time dimensions, the splitting functions, related to the problem, can be unfolded in leading and next-to-leading order, which provides a check for the calculation. Here we also discuss the mathematical structure of the results and compare to the result obtained in Ref. [1]. Section 6 contains the conclusions. The appendices summarize details of the calculation and different types of summation formulae used in the present calculation, which are of general interest for other higher order calculations.

2 The Method

In the twist–2 approximation, the deep–inelastic nucleon structure functions $F_i(x,Q^2),\ i = 2, L$, are described as Mellin convolutions between the parton densities $f_j(x,\mu^2)$ and the Wilson coefficients $\mathsf{C}_i^j(x,Q^2/\mu^2)$

$$F_i(x,Q^2) = \sum_j \mathsf{C}_i^j\left(x,\frac{Q^2}{\mu^2}\right) \otimes f_j(x,\mu^2) \tag{1}$$

to all orders in perturbation theory due to the factorization theorem. Here μ^2 denotes the factorization scale and the Mellin convolution is given by the integral

$$[A \otimes B](x) = \int_0^1 dx_1 \int_0^1 dx_2 \ \ \delta(x - x_1 x_2)\ A(x_1) B(x_2)\ . \tag{2}$$

Since the distributions f_j refer to massless partons, the heavy flavor effects are contained in the Wilson coefficients only. We will derive the massive contributions in the region $Q^2 \gg m^2$. These are the non–power corrections in m^2/Q^2, i.e. all logarithmic contributions and the constant term. We apply the collinear parton model, i.e. the parton 4–momentum is $p = xP$, with P the nucleon momentum. The massive Wilson coefficients itself can be viewed as a quasi cross section in pV^* scattering, where V^* denotes the exchanged virtual vector boson. In the limit $Q^2 \gg m^2$, the massive Wilson coefficients $H_{2,L,i}^{\rm S,NS}(Q^2/m^2, m^2/\mu^2, x)$ factorize [1] into Wilson coefficients $C_{2,L;k}^{S,NS}\left(Q^2/\mu^2, x\right)$ accounting for light flavors only and massive operator matrix elements $A_{k,i}^{\rm S,NS}\left(m^2/\mu^2, x\right)$.

$$H_{2,L;i}^{\rm S,NS}\left(\frac{Q^2}{m^2},\frac{m^2}{\mu^2},x\right) = C_{2,L;k}^{S,NS}\left(\frac{Q^2}{\mu^2},x\right) \otimes A_{k,i}^{\rm S,NS}\left(\frac{m^2}{\mu^2},x\right) \tag{3}$$

The latter take a similar role as the parton densities in (1). They are process independent but perturbatively calculable. The factorization (3) is a consequence of the renormalization group equation. The operator matrix elements $A_{k,i}^{\rm S,NS}$ obey the expansion

$$A_{k,i}^{\rm S,NS}\left(\frac{m^2}{\mu^2}\right) = \langle i|O_k|i\rangle = \delta_{k,i} + \sum_{l=1}^{\infty} a_s^l A_{k,i}^{\rm S,NS,(l)}, \qquad i = q, g \tag{4}$$

of the twist–2 quark singlet and non–singlet operators $O_k^{\rm S,NS}$ between partonic states $|i\rangle$, which are related by collinear factorization to the initial–state nucleon states $|N\rangle$. $a_s = \alpha_s(\mu^2)/(4\pi)$ denotes the strong coupling constant. The Feynman rules for the operator insertions are given in Figure 1. Since the operator matrix elements are process–independent quantities, the process dependence of $H_{2,L;i}^{\rm S,NS}$ is described by the associated light parton coefficient functions

$$C_{2,L;k}\left(\frac{Q^2}{\mu^2}\right) = \sum_{l=l_0}^{\infty} a_s^l C_{L,k}^{(l)}\left(\frac{Q^2}{\mu^2}\right), \qquad k = {\rm NS, S}, g\ . \tag{5}$$

The $\overline{\rm MS}$ coefficient (and splitting) functions, in the massless limit, corresponding to the heavy quarks only, are denoted by

$$\widehat{C}_{2,L;k}\left(\frac{Q^2}{\mu^2}\right) = C_{2,L;k}\left(\frac{Q^2}{\mu^2}, N_L + N_H\right) - C_{2,L;k}\left(\frac{Q^2}{\mu^2}, N_L\right) \ , \tag{6}$$

where N_H, N_L are the number of heavy and light flavors, respectively. In the following we will consider the case of a single heavy quark, i.e. $N_H = 1$. The formalism is easily generalized to more than one heavy quark species.

The massive operator matrix elements to $O(a_s^2)$ allow to calculate the heavy quark Wilson coefficients in the asymptotic region for $F_2(x,Q^2)$ to $O(a_s^2)$ [1] and for $F_L(x,Q^2)$ to $O(a_s^3)$ [6]. The general structure of the Wilson coefficients is

$$\begin{aligned}
H_{2,g}^{\rm S}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) &= a_s\left[\widehat{C}_{2,g}^{(1)}\left(\frac{Q^2}{\mu^2}\right) + A_{Qg}^{(1)}\left(\frac{\mu^2}{m^2}\right)\right] \\
&+ a_s^2\left[\widehat{C}_{2,g}^{(2)}\left(\frac{Q^2}{\mu^2}\right) + A_{Qg}^{(1)}\left(\frac{\mu^2}{m^2}\right) \otimes C_{2,q}^{(1)}\left(\frac{Q^2}{\mu^2}\right) + A_{Qg}^{(2)}\left(\frac{\mu^2}{m^2}\right)\right] \qquad (7) \\
H_{2,q}^{\rm PS}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) &= a_s^2\left[\widehat{C}_{2,q}^{\rm PS,(2)}\left(\frac{Q^2}{\mu^2}\right) + A_{Qq}^{\rm PS,(2)}\left(\frac{\mu^2}{m^2}\right)\right] \qquad (8) \\
H_{2,q}^{\rm NS}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) &= a_s^2\left[\widehat{C}_{2,q,Q}^{\rm NS,(2)}\left(\frac{Q^2}{\mu^2}\right) + A_{qq,Q}^{\rm NS,(2)}\left(\frac{\mu^2}{m^2}\right)\right] \qquad (9) \\
H_{L,g}^{\rm S}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) &= a_s\widehat{C}_{L,g}^{(1)}\left(\frac{Q^2}{\mu^2}\right) + a_s^2\left[A_{Qg}^{(1)}\left(\frac{\mu^2}{m^2}\right) \otimes C_{L,q}^{(1)}\left(\frac{Q^2}{\mu^2}\right) + \widehat{C}_{L,g}^{(2)}\left(\frac{Q^2}{\mu^2}\right)\right] \\
&+ a_s^3\left[A_{Qg}^{(2)}\left(\frac{\mu^2}{m^2}\right) \otimes C_{L,q}^{(1)}\left(\frac{Q^2}{\mu^2}\right) + A_{Qg}^{(1)}\left(\frac{\mu^2}{m^2}\right) \otimes C_{L,q}^{(2)}\left(\frac{Q^2}{\mu^2}\right) + \widehat{C}_{L,g}^{(3)}\left(\frac{Q^2}{\mu^2}\right)\right] \qquad (10) \\
H_{L,q}^{\rm PS}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) &= a_s^2\widehat{C}_{L,q}^{\rm PS,(2)}\left(\frac{Q^2}{\mu^2}\right) + a_s^3\left[A_{Qq}^{\rm PS,(2)}\left(\frac{\mu^2}{m^2}\right) \otimes C_{L,q}^{(1)}\left(\frac{Q^2}{\mu^2}\right) + \widehat{C}_{L,q}^{\rm PS,(3)}\left(\frac{Q^2}{\mu^2}\right)\right] \qquad (11) \\
H_{L,q}^{\rm NS}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) &= a_s^2\widehat{C}_{L,q}^{\rm NS,(2)}\left(\frac{Q^2}{\mu^2}\right) + a_s^3\left[A_{qq,Q}^{\rm NS,(2)}\left(\frac{\mu^2}{m^2}\right) \otimes C_{L,q}^{(1)}\left(\frac{Q^2}{\mu^2}\right) + \widehat{C}_{L,q}^{\rm NS,(3)}\left(\frac{Q^2}{\mu^2}\right)\right] . \qquad (12)
\end{aligned}$$

3 Renormalization

The calculation is performed in $D = 4 + \varepsilon$ dimensions in the $\overline{\rm MS}$–scheme. The massive operator matrix elements $A_{ij}(m^2/\mu^2)$, with μ being the renormalization scale, exhibit ultraviolet divergences which have to be removed by operator–, mass– and coupling constant renormalization. Furthermore, it contains collinear singularities, in the present case beginning with 2–loop order. We follow the notation of Ref. [1] and briefly summarize the renormalization procedure.

The external lines of the diagrams are treated on–shell after factorization. The scale for the process is set by the heavy quark mass m. The $Z_{O,ik}$–matrix performs the renormalization of the composite operator turning $\hat{A}_{ij}(m^2/\mu^2, a_s, \varepsilon)$ into $\tilde{A}_{ij}(m^2/\mu^2, a_s, \varepsilon)$,

$$\hat{A}_{ij}(m^2/\mu^2, a_s, \varepsilon) = Z_{O,jk}(a_s, \varepsilon) \otimes \tilde{A}_{kj}(m^2/\mu^2, a_s, \varepsilon) \ , \tag{13}$$

with

$$Z_{O,ij} = \sum_{k=0}^{\infty} a_s^k Z_{O,ij}^k, \tag{14}$$

$$\gamma_{ij}^N = -(Z_O(\mu))_{ik}^{-1} \frac{\partial}{\partial \mu} Z_{O,kj}(\mu) \; . \tag{15}$$

Here, γ_{ij}^N denote the singlet anomalous dimensions, which are related to the splitting functions by

$$\gamma_{ij}^N = -\int_0^1 dz z^{N-1} P_{ij}(z) \; . \tag{16}$$

The collinear singularities are removed by the matrices $\Gamma_{kj}(a_s, \varepsilon)$,

$$\tilde{A}_{ij}(m^2/\mu^2, a_s, \varepsilon) = \tilde{\tilde{A}}_{ik}(m^2/\mu^2, a_s, \varepsilon) \otimes \Gamma_{kj}(a_s, \varepsilon) \tag{17}$$

with

$$\Gamma_{ij} = \sum_{k=0}^{\infty} a_s^k \Gamma_{ij}^k \; . \tag{18}$$

To 2–loop order, one has

$$\begin{aligned} \Gamma_{ij} &= \delta_{ij} + a_s S_\varepsilon \frac{1}{\varepsilon} P_{ij}^{(0)} \\ &\quad + a_s^2 S_\varepsilon^2 \left[\frac{1}{\varepsilon^2} \left\{ \frac{1}{2} P_{ik}^{(0)} \otimes P_{kj}^{(0)} + \beta_0 P_{ij}^{(0)} \right\} + \frac{1}{2\varepsilon} P_{ij}^{(1)} \right] + O(a_s^3) \; . \end{aligned} \tag{19}$$

The factorization and renormalization scales are choosen to be equal, $\mu_R = \mu_F \equiv \mu$. The spherical factor S_ε collects all terms to be removed in the $\overline{\rm MS}$ scheme

$$S_\varepsilon = \exp\left\{ \frac{\varepsilon}{2} [\gamma_E - \ln(4\pi)] \right\} \; . \tag{20}$$

γ_E is the Euler–Mascheroni constant.

Finally, the mass and coupling constant renormalization has to be carried out. The bare coupling $\hat{a}_s$ and bare mass $\hat{m}$ are related to the renormalized quantities by

$$\hat{a}_s = Z_g^2 a_s = a_s(\mu^2) \left[1 + a_s(\mu^2) \cdot \delta a_s \right] + O(a_s^3) \; , \tag{21}$$

$$\hat{m} = Z_m m = a_s \delta m + O(a_s^2) \; . \tag{22}$$

We choose the on–mass–shell scheme for mass renormalization. Here,

$$Z_g = \frac{Z_1^l + Z_1^H}{(Z_3^l + Z_3^H)^{3/2}} = 1 + a_s S_\varepsilon \left\{ \frac{\beta_0}{\varepsilon} + \frac{\beta_{0,Q}}{\varepsilon} \left[1 + \frac{\zeta_2}{8} \varepsilon^2 \right] \sum_{N_H=4}^{6} \left(\frac{m_{N_H}^2}{\mu^2} \right)^{\varepsilon/2} \right\} \; , \tag{23}$$

$$Z_m = 1 + \hat{a}_s C_F S_\varepsilon \left(\frac{m^2}{\mu^2} \right)^{\varepsilon/2} \left[\frac{6}{\varepsilon} - 4 \right] \; , \tag{24}$$

in Feynman gauge [22–24]. N_H denotes the heavy quark species and the $SU(3)_c$ color factors are $C_F = (N_c^2-1)/(2N_c), C_A = N_c, T_R = 1/2, N_c = 3$. The Z–factors in (23, 24) read :

$$Z_1^l = 1 + a_s \frac{2}{\varepsilon} S_\varepsilon \left[-\frac{2}{3} C_A + \frac{4}{3} T_R N_f \right] \tag{25}$$

$$Z_3^l = 1 + a_s \frac{2}{\varepsilon} S_\varepsilon \left[-\frac{5}{3} C_A + \frac{4}{3} T_R N_f \right] \tag{26}$$

$$Z_1^H = Z_3^H = a_s \frac{S_\varepsilon}{\varepsilon} \frac{8}{3} T_R \left[1 + \frac{\zeta_2}{8} \varepsilon^2 \right] \sum_{N_H=4}^{6} \left(\frac{m_{N_H}^2}{\mu^2} \right)^{\varepsilon/2} . \tag{27}$$

The lowest expansion coefficients of the β-functions are

$$\beta_0 = \frac{1}{3} \left[11 C_A - 4 T_R N_f \right], \tag{28}$$

$$\beta_{0,Q} = -\frac{4}{3} T_R . \tag{29}$$

The bare and renormalized coupling and quark mass are related by

$$\delta a_s = S_\varepsilon \left[\frac{2\beta_0}{\varepsilon} + \sum_{N_H=4}^{6} \frac{2\beta_{0,Q}}{\varepsilon} \left(\frac{m_{N_H}^2}{\mu^2} \right)^{\varepsilon/2} \left(1 + \frac{1}{8} \varepsilon^2 \zeta_2 \right) \right] , \tag{30}$$

$$\delta m = C_F S_\varepsilon m \left(\frac{m^2}{\mu^2} \right)^{\varepsilon/2} \left\{ \frac{6}{\varepsilon} - 4 \right\} . \tag{31}$$

The renormalized operator matrix element A_{ij} is now given by

$$\begin{aligned} A_{ij} = {} & \delta_{ij} + a_s \left[\hat{A}_{ij}^{(1)} + Z_{O,ij}^{-1,(1)} + \Gamma_{ij}^{-1,(1)} \right] \\ & + a_s^2 \Big[\hat{A}_{ij}^{(2)} + \delta m \frac{d}{dm} A_{ij}^{(1)} + \delta a_s \hat{A}_{ij}^{(1)} + Z_{O,ik}^{-1,(1)} \otimes \hat{A}_{kj}^{(1)} + Z_{O,ij}^{-1,(2)} \\ & + \left\{ \hat{A}_{ik}^{(1)} + Z_{O,ik}^{-1,(1)} \right\} \otimes \Gamma_{kj}^{-1,(1)} + \Gamma_{kj}^{-1,(2)} \Big] + O(a_s^3) . \end{aligned} \tag{32}$$

In the following sections we first calculate the un-renormalized operator matrix elements $\hat{A}_{ij}$ from which A_{ij} is derived through (32). Due to (16, 19) the following leading and next-to-leading order splitting functions are needed. We will mainly work in Mellin space and therefore list these functions in this representation. The leading order splitting functions read [25]

$$P_{qq}^{(0)}(N) = 4 C_F \left[-2 S_1(N-1) + \frac{(N-1)(3N+2)}{2N(N+1)} \right] \tag{33}$$

$$P_{qg}^{(0)}(N) = 8 T_R N_F \frac{N^2+N+2}{N(N+1)(N+2)} \tag{34}$$

$$P_{gg}^{(0)}(N) = 8 C_A \left[-S_1(N-1) - \frac{N^3-3N-4}{(N-1)N(N+1)(N+2)} \right] + 2\beta_0 \tag{35}$$

$$P_{gq}^{(0)}(N) = 4 C_F \frac{N^2+N+2}{(N-1)N(N+1)} \tag{36}$$

Furthermore, the following next-to-leading order splitting functions contribute [21, 26]

$$\widehat{P}_{qq}^{\rm PS,(1)}(N) = 16 C_F T_R \frac{5N^5+32N^4+49N^3+38N^2+28N+8}{(N-1)N^3(N+1)^3(N+2)^2} \tag{37}$$

$$\begin{aligned} P_{qq,Q}^{\rm NS,(1)}(N) = \widehat{P}_{qq}^{\rm NS,(1)} &= C_F T_R \left\{ \frac{160}{9} S_1(N-1) - \frac{32}{3} S_2(N-1) \right. \\ &\left. - \frac{4}{9} \frac{(N-1)(3N+2)(N^2-11N-6)}{N^2(N+1)^2} \right\} \end{aligned} \tag{38}$$

$$\begin{aligned} \widehat{P}_{qg}^{(1)}(N) &= 8 C_F T_R \left\{ 2 \frac{N^2+N+2}{N(N+1)(N+2)} \left[S_1^2(N) - S_2(N) \right] - \frac{4}{N^2} S_1(N) \right. \\ &\left. + \frac{5N^6+15N^5+36N^4+51N^3+25N^2+8N+4}{N^3(N+1)^3(N+2)} \right\} \\ &+ 16 C_A T_R \left\{ - \frac{N^2+N+2}{N(N+1)(N+2)} \left[S_1^2(N) + S_2(N) - \zeta_2 - 2\beta'(N+1) \right] \right. \\ &\left. + 4 \frac{2N+3}{(N+1)^2(N+2)^2} S_1(N) + \frac{P_1(N)}{(N-1)N^3(N+1)^3(N+2)^3} \right\} , \end{aligned} \tag{39}$$

where

$$\begin{aligned} P_1(N) &= N^9 + 6N^8 + 15N^7 + 25N^6 + 36N^5 + 85N^4 + 128N^3 + 104N^2 \\ &+ 64N + 16 \ . \end{aligned} \tag{40}$$

Here, the harmonic sums [27, 28] are given by

$$S_1(N) = \sum_{l=1}^{N} \frac{1}{l} = \psi(N+1) + \gamma_E \tag{41}$$

$$S_k(N) = \sum_{l=1}^{N} \frac{1}{l^k} = \frac{(-1)^{k-1}}{(k-1)!} \psi^{(k-1)}(N+1) + \zeta_k, \quad k \geq 2 \tag{42}$$

$$\beta(N) = \frac{1}{2} \left[\psi\left(\frac{N+1}{2}\right) - \psi\left(\frac{N}{2}\right) \right] \tag{43}$$

$$S_{-1}(N) = (-1)^N \beta(N+1) - \ln(2) \tag{44}$$

$$S_{-k}(N) = \frac{(-1)^{N+k+1}}{(k-1)!} \beta^{k-1}(N+1) - \left(1 - \frac{1}{2^{k-1}}\right) \zeta_k, \quad k \geq 2 \ . \tag{45}$$

ζ_k denotes the Riemann ζ–function.

4 The one-loop massive operator matrix elements

At one–loop order, only gluonic terms contribute to heavy flavor production. The complete calculation of the differential scattering cross section $d^2\sigma(\gamma^* + N \to Q\overline{Q})/dxdQ^2$ was performed

in Refs. [3]. In the limit $Q^2 \gg m^2$, the heavy flavor Wilson coefficients are in the $\overline{\rm MS}$–scheme

$$H_{L,g}^{(1)}\left(\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) = \widehat{C}_{L,g}^{(1)}\left(\frac{Q^2}{\mu^2}\right) , \tag{46}$$

$$H_{2,g}^{(1)}\left(\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) = A_{Qg}^{(1)}\left(\frac{m^2}{\mu^2}\right) + \widehat{C}_{2,g}^{(1)}\left(\frac{Q^2}{\mu^2}\right) . \tag{47}$$

Here $\widehat{C}_{(2,L),g}^{(1)}(Q^2/\mu^2)$ denote the massless one-loop Wilson coefficients and $A_{Qg}^{(1)}(m^2/\mu^2)$ is the massive one–loop operator matrix element. Since $F_L(x,Q^2)$ is collinear finite at leading order, the Wilson coefficient $\widehat{C}_{L,g}^{(1)}(Q^2/\mu^2)$ is universal and independent of the scheme or method of calculation. $H_{L,g}^{(1)}(Q^2/m^2,m^2/\mu^2)$ does therefore not contain contributions due to operator matrix elements at this order. The logarithmic contributions to $F_L(x,Q^2)$ emerge as $(m^2/Q^2)\cdot\ln(Q^2/m^2)$ and vanish in the limit $Q^2 \gg m^2$. The massive operator matrix element $A_{Qg}^{(1)}(m^2/Q^2)$ is calculated from the diagrams in Figure 2. The symbol $\otimes$ denotes the operator insertion, cf. Figure 1. The massive operator matrix elements are obtained by contracting the diagrams with the projector

$$\hat{A}_{Qg}\left(\varepsilon,\frac{m^2}{\mu^2},a_s\right) = -\frac{1}{N_c^2-1}\frac{1}{D-2}g_{\mu\nu}\delta^{ab}(\Delta.p)^{-N}G_{Q,\mu\nu}^{ab} \tag{48}$$

where a and b are the outer color indices and μ and ν are the Lorentz-indices and G_{ij} is the respective Green's function. The diagrams yield

$$\begin{aligned} A_a^{Qg} &= -\frac{1+(-1)^N}{2}8a_sT_RS_\varepsilon\left(\frac{m^2}{\mu^2}\right)^{\varepsilon/2}\frac{1}{(2+\varepsilon)\varepsilon}\exp\left\{\sum_{l=2}^{\infty}\frac{\zeta_l}{l}\left(\frac{\varepsilon}{2}\right)^l\right\} \\ &\times\frac{2(N^2+3N+2)+\varepsilon(N^2+N+2)}{N(N+1)(N+2)} , \end{aligned} \tag{49}$$

$$A_b^{Qg} = 32\frac{1+(-1)^N}{2}a_sT_RS_\varepsilon\left(\frac{m^2}{\mu^2}\right)^{\varepsilon/2}\frac{1}{(2+\varepsilon)\varepsilon}\exp\left\{\sum_{l=2}^{\infty}\frac{\zeta_l}{l}\left(\frac{\varepsilon}{2}\right)^l\right\}\frac{1}{(N+1)(N+2)} . \tag{50}$$

For the unpolarized operator matrix elements only the even moments contribute due to the current crossing relations, see e.g. [29]. The analytic continuation is performed starting from the even moments. The matrix elements have to be expanded to $O(\varepsilon)$, since these terms are needed for the 2–loop result:

$$\begin{aligned} \hat{A}_{Qg}^{(1)} &= \frac{1}{a_s}\left[A_a^{Qg}+A_b^{Qg}\right] \\ &= -S_\varepsilon T_R\left(\frac{m^2}{\mu^2}\right)^{\varepsilon}\frac{1}{\varepsilon}\exp\left\{\sum_{l=2}^{\infty}\frac{\zeta_l}{l}\left(\frac{\varepsilon}{2}\right)^l\right\}\frac{8(N^2+N+2)}{N(N+1)(N+2)} \\ &= S_\varepsilon T_R\left(\frac{m^2}{\mu^2}\right)^{\varepsilon}\left(-\frac{1}{\varepsilon}-\frac{\zeta_2}{8}\varepsilon\right)\frac{8(N^2+N+2)}{N(N+1)(N+2)}+O(\varepsilon^2) . \end{aligned} \tag{51}$$

Note that the term $O(\varepsilon^0)$ vanishes. In x–space the result reads

$$\hat{A}_{Qg}^{(1)} = S_\varepsilon\left(\frac{m^2}{\mu^2}\right)^{\varepsilon/2}\left[-\frac{1}{\varepsilon}\widehat{P}_{qg}^{(0)}(x)+a_{Qg}^{(1)}+\varepsilon\overline{a}_{Qg}^{(1)}\right] , \tag{52}$$

where

$$a_{Qg}^{(1)} = 0 \ , \tag{53}$$

$$\overline{a}_{Qg}^{(1)} = -\frac{\zeta_2}{8}\widehat{P}_{qg}^{(0)}(x) \ , \tag{54}$$

and $\widehat{P}_{qg}^{(0)}(x)$ denotes the leading order splitting function for the transition $g \to q$ for one (heavy) flavor. At one–loop order, the renormalized operator matrix element $A_{Qg}^{(1)}$ is obtained by

$$A_{Qg}^{(1)}\left(\frac{m^2}{\mu^2}\right) = \hat{A}_{Qg}^{(1)} + Z_{O,Qg}^{-1,(1)} \ , \tag{55}$$

where

$$Z_{O,Qg}^{(1)} = S_\varepsilon \left[-\frac{1}{\varepsilon}P_{qg}^{(0)}\right] \ . \tag{56}$$

Here, $Z_{Qg}^{(1)}$ removes the ultraviolet singularities. At this order no collinear singularities are present. Choosing $\mu^2 = Q^2$ the massless Wilson 1–loop coefficients in the $\overline{\rm MS}$–scheme [8] are given by :

$$\hat{C}_{2,g}^{(1)}(z) = \left\{\frac{1}{2}\widehat{P}_{qg}^{(0)}(z)\left[\ln\left(\frac{1-z}{z}\right) - 4\right] + 12T_R\right\} \ , \tag{57}$$

$$\hat{C}_{L,g}^{(1)}(z) = 16T_R z(1-z) \ . \tag{58}$$

The asymptotic heavy quark Wilson coefficients (46,47) are

$$H_{L,g}^{(1)}\left(z, \frac{Q^2}{m^2}\right) = 16T_R z(1-z) \ , \tag{59}$$

$$H_{2,g}^{(1)}\left(z, \frac{Q^2}{m^2}\right) = T_R\left\{4(z^2+(1-z)^2)\left[\ln\left(\frac{1-z}{z}\right) + \ln\left(\frac{Q^2}{m^2}\right)\right] - 32z^2 + 32z - 4\right\} \ . \tag{60}$$

These are precisely the expressions obtained in the limit $m^2/Q^2 \to 0$ from [3] up to the constant term $(m^2/Q^2)^0$:

$$H_{L,g}^{\rm compl,(1)}\left(z, \frac{Q^2}{m^2}\right) = 16T_R\left[vz(1-z) + 2\frac{m^2}{Q^2}z^2\ln\left(\frac{1-v}{1+v}\right)\right] \ , \tag{61}$$

$$H_{2,g}^{\rm compl,(1)}\left(z, \frac{Q^2}{m^2}\right) = 8T_R\left\{v\left[-\frac{1}{2} + 4z - 4z^2 - 2\frac{m^2}{Q^2}z(1-z)\right] + \left[-\frac{1}{2} + z - z^2 + 2\frac{m^2}{Q^2}(3z^2 - z) + 4\frac{m^4}{Q^4}z^2\right]\ln\left(\frac{1-v}{1+v}\right)\right\} \ , \tag{62}$$

where v denotes the cms-velocity of the heavy quarks,

$$v = \sqrt{1 - \frac{4m^2}{Q^2}\frac{z}{1-z}} \ . \tag{63}$$

5 The two-loop massive operator matrix elements

There are three classes of 2–loop contributions to the massive operator matrix elements: the gluonic contributions (diagrams Figure 3), the pure-singlet contributions (diagrams Figure 4) and the non–singlet contributions (diagrams Figure 5). The diagrams are either one-loop insertions into one-loop diagrams or, in case of the gluonic contributions, also genuine two–loop diagrams. The calculation is performed using `FORM` [30] and `Maple` procedures. We calculate all diagrams directly, i.e. without decomposing them using the integration-by-parts method as done in [1]. The integrals to be performed are more involved. However, we avoid a large proliferation of terms in the results, in our case of nested sums of different kind, which add up to zero. Even in the case of individual diagrams, which are calculated in Feynman gauge, only a very small number of harmonic sums contributes finally. One of these, $S_{2,1}(N)$, does not emerge in the operator matrix elements. The results for the individual non-renormalized diagrams are given in Appendix A. It turns out, that some of the diagrams can be easily calculated to all orders in ε. The diagrams can be represented in terms of linear combinations of generalized hypergeometric functions [14]. Due to the given topologies the most complex function is ${}_3F_2(a_1, a_2, a_3; b_1, b_2; 1)$. In these representations, the conformal mapping of Feynman parameters is essential. The scalar integrals associated to the genuine two–loop diagrams have also been calculated using the Mellin–Barnes technique [12, 13], cf. Refs. [15, 16], and checked for fixed moments using the package `MB` [31]. The ε–expansion can be performed prior to the summation. It results into finite and infinite sums of various types, including harmonic sums attached with Euler Beta-functions and binomials. Here, we face a more general situation than in massless calculations, despite the fact that we work in the limit $Q^2 \gg m^2$. The sums are given in Appendix B. They were performed using suitable integral representations or using difference equations. Finally, the results depend only on harmonic sums, which are reduced further applying their algebraic relations [32].

After mass renormalization, the operator matrix element $\hat{A}_{Qg}^{(2)}\Big(m^2/\mu^2, \varepsilon\Big)$ is given by

$$
\begin{aligned}
\hat{A}_{Qg}^{(2)} &= S_\varepsilon^2 \Big(\frac{m^2}{\mu^2}\Big)^\varepsilon \Big[\frac{1}{\varepsilon^2}\Big\{\frac{1}{2}\hat{P}_{qg}^{(0)} \otimes (P_{qq}^{(0)} - P_{gg}^{(0)}) + \beta_0 \hat{P}_{qg}^{(0)}\Big\} + \frac{1}{\varepsilon}\Big\{-\frac{1}{2}\hat{P}_{qg}^{(1)}\Big\} + a_{Qg}^{(2)}\Big] \\
&\quad -\frac{2}{\varepsilon} S_\varepsilon \beta_{0,Q} \sum_{N_H=4}^{6} \Big(\frac{m_{N_H}^2}{\mu^2}\Big)^{\varepsilon/2} \Big(1 + \frac{\varepsilon^2}{8}\zeta_2\Big) \hat{A}_{Qg}^{(1)} \; . \\
&= S_\varepsilon^2 \left(\frac{m^2}{\mu^2}\right)^\varepsilon \left(\frac{1}{\varepsilon^2}\left\{8 T_R C_F \frac{N^2+N+2}{N(N+1)(N+2)}\left[-4 S_1(N) + \frac{3N^2+3N+2}{N(N+1)}\right]\right.\right. \\
&\quad \left.\left. +32 T_R C_A \frac{N^2+N+2}{N(N+1)(N+2)}\left[S_1(N) - 2\frac{N^2+N+1}{(N-1)N(N+1)(N+2)}\right]\right\}\right. \\
&\quad +\frac{1}{\varepsilon}\left\{4 T_R C_F \left[2\frac{N^2+N+2}{N(N+1)(N+2)}\Big(S_2(N) - S_1^2(N)\Big) + 4\frac{S_1(N)}{N^2}\right.\right. \\
&\quad \left.\left. -\frac{5N^6+15N^5+36N^4+51N^3+25N^2+8N+4}{N^3(N+1)^3(N+2)}\right]\right. \\
&\quad \left. +T_R C_A \left[8\frac{N^2+N+2}{N(N+1)(N+2)}\Big(-2\beta'(N+1) + S_2(N) + S_1^2(N) - \zeta_2\Big)\right.\right.
\end{aligned}
$$

$$-32\frac{2N+3}{(N+1)^2(N+2)^2}S_1(N) - 8\frac{\hat{P}_1(N)}{(N-1)N^3(N+1)^3(2+N)^3}\Bigg]\Bigg\}$$
$$+a_{Qg}^{(2)}(N)\Bigg) + S_\varepsilon\frac{8}{3\varepsilon}T_R\Big(1+\frac{\zeta_2}{8}\varepsilon^2\Big)\sum_{N_H=4}^{6}\left(\frac{m_{N_H}^2}{\mu^2}\right)^{\varepsilon/2}\hat{A}_{Qg}^{(1)}\ . \tag{64}$$

For the ghost contributions in Figure 3, the projector reads

$$\hat{A}_{Qg}^{(2),\rm ghost}\left(\varepsilon,\frac{m^2}{\mu^2},a_s\right) = \frac{1}{N_c^2-1}\frac{1}{D-2}\delta^{ab}\left(\Delta.p\right)^{-N}G_Q^{ab,(2),\rm ghost}\ . \tag{65}$$

From the terms $\propto 1/\varepsilon^2, 1/\varepsilon$, one may determine the respective QCD splitting functions to two–loop order (33–39), which are recalculated in this way.

The constant term in $\hat{A}_{Qg}^{(2)}\Big(m^2/\mu^2,\varepsilon\Big)$ reads :

$$\begin{aligned}
a_{Qg}^{(2)}(N) &= 4C_FT_R\Bigg\{\frac{N^2+N+2}{N\left(N+1\right)\left(N+2\right)}\Bigg[-\frac{1}{3}S_1^3(N-1)+\frac{4}{3}S_3(N-1)\\
&\quad -S_1(N-1)S_2(N-1)-2\zeta_2S_1(N-1)\Bigg]+\frac{2}{N(N+1)}S_1^2(N-1)\\
&\quad +\frac{N^4+16\,N^3+15\,N^2-8\,N-4}{N^2\left(N+1\right)^2\left(N+2\right)}S_2(N-1)\\
&\quad +\frac{3\,N^4+2\,N^3+3\,N^2-4\,N-4}{2N^2\left(N+1\right)^2\left(N+2\right)}\zeta_2\\
&\quad +\frac{N^4-N^3-16N^2+2N+4}{N^2\left(N+1\right)^2\left(N+2\right)}S_1(N-1)+\frac{\hat{P}_2(N)}{2N^4\left(N+1\right)^4\left(N+2\right)}\Bigg\}\\
&+4C_AT_R\Bigg\{\frac{N^2+N+2}{N(N+1)(N+2)}\Bigg[4\,\mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1+x}\right](N+1)+\frac{1}{3}S_1^3(N)+3S_2(N)S_1(N)\\
&\quad +\frac{8}{3}S_3(N)+\beta''(N+1)-4\beta'(N+1)S_1(N)-4\beta(N+1)\zeta_2+\zeta_3\Bigg]\\
&\quad -\frac{N^3+8\,N^2+11\,N+2}{N\left(N+1\right)^2\left(N+2\right)^2}S_1^2(N)-2\,\frac{N^4-2N^3+5N^2+2N+2}{(N-1)\,N^2\left(N+1\right)^2\left(N+2\right)}\zeta_2\\
&\quad -\frac{7N^5+21N^4+13N^3+21N^2+18N+16}{(N-1)N^2\left(N+1\right)^2\left(N+2\right)^2}S_2(N)\\
&\quad -\frac{N^6+8\,N^5+23\,N^4+54\,N^3+94\,N^2+72\,N+8}{N\left(N+1\right)^3\left(N+2\right)^3}S_1(N)\\
&\quad -4\,\frac{(N^2-N-4)}{\left(N+1\right)^2\left(N+2\right)^2}\beta'(N+1)+\frac{\hat{P}_3(N)}{(N-1)N^4(N+1)^4(N+2)^4}\Bigg\}\ .
\end{aligned} \tag{66}$$

The polynomials in Eqs. (64, 66) read

$$\hat{P}_1(N) = N^9+6N^8+15N^7+25N^6+36N^5+85N^4+128N^3+104N^2+64N+16 \ , \quad (67)$$
$$\hat{P}_2(N) = 12N^8+54N^7+136N^6+218N^5+221N^4+110N^3-3N^2-24N-4 \ , \quad (68)$$
$$\hat{P}_3(N) = 2\,N^{12}+20\,N^{11}+86\,N^{10}+192\,N^9+199\,N^8-N^7-297\,N^6-495\,N^5 -514\,N^4-488\,N^3-416\,N^2-176\,N-32 \ . \quad (69)$$

The pure–singlet $\hat{A}_{Qq}^{\rm PS,(2)}\Big(m^2/\mu^2,\varepsilon\Big)$ operator matrix element is given by

$$\hat{A}_{Qq}^{\rm PS,(2)}\Big(\frac{m^2}{\mu^2},\varepsilon\Big) = S_\varepsilon^2\Big(\frac{m^2}{\mu^2}\Big)^\varepsilon\Big[\frac{1}{\varepsilon^2}\Big\{-\frac{1}{2}\hat{P}_{qg}^{(0)}\otimes P_{gq}^{(0)}\Big\}+\frac{1}{\varepsilon}\Big\{-\frac{1}{2}\hat{P}_{qq}^{\rm PS,(1)}\Big\}+a_{Qq}^{\rm PS,(2)}\Big] \ .$$
$$= S_\varepsilon^2\left(\frac{m^2}{\mu^2}\right)^\varepsilon\left\{-\frac{1}{\varepsilon^2}T_RC_F\frac{16(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)}\right.$$
$$\left.-\frac{1}{\varepsilon}T_RC_F\frac{8(5N^5+32N^4+49N^3+38N^2+28N+8)}{(N-1)N^3(N+1)^3(N+2)^2}+a_{Qq}^{\rm PS,(2)}(N)\right\} . \quad (70)$$

In this case and the non–singlet case, the projector

$$\hat{A}_{Qq;(qq,Q)}^{(2)}\left(\varepsilon,\frac{m^2}{\mu^2},a_s\right) = \frac{1}{N_c}\delta^{ij}\frac{1}{4}\left(\Delta.p\right)^{-N}{\rm Tr}\left[\not{p}G_{Q,(q)}^{ij,(2)}\right] \ . \quad (71)$$

is applied. Here i and j denote the matrix-elements of the Gell-Mann matrices. The constant term is obtained by

$$a_{Qq}^{\rm PS,(2)}(N) = T_RC_F\left\{-4\frac{(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)}\left(2S_2(N)+\zeta_2\right)\right.$$
$$\left.+\frac{4\hat{P}_4(N)}{(N-1)N^4(N+1)^4(N+2)^3}\right\}, \quad (72)$$
$$\hat{P}_4(N) = N^{10}+8N^9+29N^8+49N^7-11N^6-131N^5-161N^4 -160N^3-168N^2-80N-16 \ . \quad (73)$$

Finally, the non-singlet operator matrix element $\hat{A}_{qq,Q}^{\rm NS,(2)}\Big(m^2/\mu^2,\varepsilon\Big)$ reads

$$\hat{A}_{qq,Q}^{\rm NS,(2)}\Big(\frac{m^2}{\mu^2},\varepsilon\Big) = S_\varepsilon^2\Big(\frac{m^2}{\mu^2}\Big)^\varepsilon\Big[\frac{1}{\varepsilon^2}\Big\{-\beta_{0,Q}P_{qq}^{(0)}\Big\}+\frac{1}{\varepsilon}\Big\{-\frac{1}{2}P_{qq,Q}^{\rm NS,(1)}\Big\}+a_{qq,Q}^{\rm NS,(2)}\Big] .$$
$$= S_\varepsilon^2\left(\frac{m^2}{\mu^2}\right)^\varepsilon\left\{\frac{1}{\varepsilon^2}T_RC_F\left[-\frac{32}{3}S_1(N)+8\frac{3N^2+3N+2}{3N(N+1)}\right]\right.$$
$$+\frac{1}{\varepsilon}T_RC_F\left[\frac{16}{3}S_2(N)-\frac{80}{9}S_1(N)+2\frac{3N^4+6N^3+47N^2+20N-12}{9N^2(N+1)^2}\right]$$
$$\left.+a_{qq,Q}^{\rm NS,(2)}(N)\right\} . \quad (74)$$

The constant term is given by

$$a_{qq,Q}^{\rm NS,(2)}(N) = C_F T_R \Biggl\{ -\frac{8}{3} S_3(N) - \frac{8}{3} \zeta_2 S_1(N) + \frac{40}{9} S_2(N) + 2 \frac{3N^2+3N+2}{3N(N+1)} \zeta_2 - \frac{224}{27} S_1(N) + \frac{219N^6+657N^5+1193N^4+763N^3-40N^2-48N+72}{54N^3(N+1)^3} \Biggr\}. \tag{75}$$

These results obtained in Mellin–space agree with those given in Ref. [1] in x–space, cf. [6].

The method applied here allowed to compactify the representation for the heavy flavor matrix elements and Wilson coefficients. As shown in Appendix A the individual Feynman diagrams depend on the harmonic sums $S_1(N), S_2(N), S_3(N), S_{-2}(N), S_{-3}(N), S_{2,1}(N), S_{-2,1}(N)$ only. In the final result the sum $S_{2,1}(N)$ drops out. The x-space representation in [1] contains the following 48 functions :

$$\begin{array}{lllll}
\delta(1-x) & 1 & \ln(x) & \ln^2(x) & \ln^3(x) \\
\ln(1-x) & \ln^2(1-x) & \ln^3(1-x) & \ln(x)\ln(1-x) & \ln(x)\ln^2(1-x) \\
\ln^2(x)\ln(1-x) & \ln(1+x) & \ln(x)\ln(1+x) & \ln^2(x)\ln(1+x) & {\rm Li}_2(1-x) \\
\ln(x){\rm Li}_2(1-x) & \ln(1-x){\rm Li}_2(1-x) & {\rm Li}_3(1-x) & S_{1,2}(1-x) & S_{1,2}(-x) \\
\frac{1}{1-x} & \frac{1}{1+x} & \frac{\ln(x)}{1-x} & \frac{\ln^2(x)}{1-x} & \frac{\ln^3(x)}{1-x} \\
\frac{\ln(x)}{1+x} & \frac{\ln^2(x)}{1+x} & \frac{\ln^3(x)}{1+x} & \frac{\ln(1+x)}{1+x} & \frac{\ln(x)\ln(1+x)}{1+x} \\
\frac{\ln(x)\ln^2(1+x)}{1+x} & \frac{\ln^2(x)\ln(1+x)}{1+x} & \frac{\ln(x)\ln(1-x)}{1-x} & \frac{\ln(x)\ln^2(1-x)}{1-x} & \frac{\ln(1-x){\rm Li}_2(x)}{1-x} \\
\frac{{\rm Li}_2(1-x)}{1-x} & \frac{\ln(x){\rm Li}_2(1-x)}{1-x} & \frac{\ln(x){\rm Li}_2(1-x)}{1+x} & \frac{\ln(1+x){\rm Li}_2(-x)}{1+x} & \ln(1+x){\rm Li}_2(-x) \\
{\rm Li}_2(-x) & \frac{{\rm Li}_2(-x)}{1+x} & \frac{\ln(x){\rm Li}_2(-x)}{1+x} & \frac{{\rm Li}_3(1-x)}{1-x} & \frac{{\rm Li}_3(-x)}{1+x} \\
\frac{S_{1,2}(1-x)}{1-x} & \frac{S_{1,2}(1-x)}{1+x} & \frac{S_{1,2}(-x)}{1+x} & &
\end{array}$$

As shown in [27], various of these functions have Mellin transforms which contain triple sums, which do not occur in our approach even on the level of individual diagrams.

In the Mellin–space representation, the sums listed in Table 1 contribute to the result of the individual diagrams. Note, that we express single harmonic sums with negative index in terms of β-functions and their derivatives, cf. [27]. They can be traced back to the single non-alternating harmonic sums, allowing for half-integer arguments. Therefore, all single harmonic sums form an equivalence class represented by $S_1(N)$, from which through differentiation and half-integer

relations the other single harmonic sums are easily derived. Further the equality,

$$\mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1+x}\right](N+1) - \zeta_2\beta(N+1) = (-1)^{N+1}\left[S_{-2,1}(N) + \frac{5}{8}\zeta_3\right] \tag{76}$$

holds. [†] Therefore, the operator matrix element $\hat{A}_{Qg}^{(2)}$ depends on one non-trivial basic function only [27]. The absence of harmonic sums containing $\{-1\}$ as index was noted before for all other classes of (space- and time-like) anomalous dimensions and Wilson coefficients, including those for other hard processes having been calculated so far, cf. [19, 33]. This can be seen if one represents the respective expressions in form of weighted harmonic sums, following an earlier suggestion of one of the authors. Linear representations do not allow this since they are non-minimal and contain algebraic redundancies.

Diagram	S_1	S_2	S_3	S_{-2}	S_{-3}	$S_{2,1}$	$S_{-2,1}$	# x-space fct.
A		+						8
B	+	+	+			+		10
C		+						4
D	+	+						5
E	+	+						9
F	+	+	+			+		24
G	+	+						6
H	+	+						7
I	+	+	+	+	+	+	+	20
J		+						7
K		+						7
L	+	+	+			+		13
M		+						7
N	+	+	+	+	+	+	+	38
O	+	+	+			+		13
P	+	+	+			+		14
S		+						7
T		+						7
PS_a		+						
PS_b		+						7
NS_a								
NS_b	+	+	+					5
Σ	+	+	+	+	+		+	48

Table 1: Harmonic sums contributing to the individual diagrams compared to the number of functions in x–space, Ref. [1].

The expressions for the renormalized two–loop operator matrix elements (32) are given by,

[†] We correct a typo in [6]. The argument of the Mellin-transform in (66) reads $N+1$, not N.

cf. [1],

$$A_{Qg}^{(2)}\left(\frac{m^2}{\mu^2},N\right) = \left\{\frac{1}{8}P_{qg}^{(0)}(N)\left[P_{qq}^{(0)}(N)-P_{gg}^{(0)}(N)\right]+\frac{1}{4}\beta_0 P_{qg}^{(0)}(N)\right\}\ln^2\left(\frac{m^2}{\mu^2}\right)$$
$$-\frac{1}{2}P_{qg}^{(1)}(N)\ln\left(\frac{m^2}{\mu^2}\right)+a_{Qg}^{(2)}(N)$$
$$+\overline{a}_{Qg}^{(1)}(N)\left\{2\beta_0+\left[P_{qq}^{(0)}(N)-P_{gg}^{(0)}(N)\right]\right\} \quad (77)$$

$$A_{Qq}^{\mathrm{PS},(2)}\left(\frac{m^2}{\mu^2},N\right) = \left\{-\frac{1}{8}P_{qg}^{(0)}(N)P_{gq}^{(0)}(N)\right\}\ln^2\left(\frac{m^2}{\mu^2}\right)+\left\{-\frac{1}{2}P_{qq}^{\mathrm{PS},(1)}(N)\right\}\ln\left(\frac{m^2}{\mu^2}\right)$$
$$+a_{Qq}^{\mathrm{PS},(2)}(N)-\overline{a}_{Qg}^{(1)}(N)P_{gq}^{(0)}(N) \quad (78)$$

$$A_{qq,Q}^{\mathrm{NS},(2)}\left(\frac{m^2}{\mu^2},N\right) = \left\{-\frac{1}{4}\beta_{0,Q}P_{qq}^{(0)}(N)\right\}\ln^2\left(\frac{m^2}{\mu^2}\right)+\left\{-\frac{1}{2}P_{qq,Q}^{\mathrm{NS},(1)}(N)\right\}\ln\left(\frac{m^2}{\mu^2}\right)$$
$$+a_{qq,Q}^{\mathrm{NS},(2)}(N)+\frac{1}{4}\beta_{0,Q}\zeta_2 P_{qq}^{(0)}(N)\;. \quad (79)$$

Besides the splitting functions up to next-to-leading order, the constant terms $a_{ij}^{A}(N)$ and $\overline{a}_{Qg}^{(1)}(N)$ determine the massive operator matrix elements (77–79).

The asymptotic heavy flavor Wilson coefficients $H_{2,L}(N,Q^2)$ are then given by (7–12). The corresponding expressions in x–space are obtained applying the inverse Mellin transform. Analytic continuations of the corresponding basic functions to complex values of N are given at high precision in [18, 19]. The inverse Mellin transform to obtain the respective contributions for the structure functions is performed by a single precise numeric contour integral around the singularities of the problem, after convoluting with the evolved parton densities [20].

6 Conclusions

We calculated the unpolarized massive 2–loop operator matrix elements, which are used to express the heavy flavor Wilson coefficients in the asymptotic region $Q^2 \gg m^2$ for $F_2(x,Q^2)$ to $O(a_s^2)$ and for $F_L(x,Q^2)$ to $O(a_s^3)$. We confirm the results obtained in Ref. [1]. The method applied in the present paper is widely different from the one used in [1]. We calculated the Feynman diagrams without applying the integration-by-parts method and worked in Mellin–space, to obey the natural symmetry of the problem. The calculation refers to nested sums in the first place, while in [1] the Feynman–parameter integrals were mapped to a single Mellin transform successively integrating Nielsen-type integrals. Furthermore we applied the algebraic relations between the harmonic sums to simplify the expressions further. The representation obtained for the individual Feynman diagrams was much more compact. Only a few harmonic sums contribute, which furthermore can be grouped into only two equivalence classes. This is to be compared to 48 functions in x–space, which were needed to express the result in [1]. The present problem exhibits a more involved nesting if compared to massless two-loop calculations, since the heavy quark mass connects Feynman parameters, although we work in the limit $Q^2 \gg m^2$. The representation of the Feynman-parameter integrals of the loop-diagrams in terms of higher transcendental functions, here generalized hypergeometric functions, before carrying out the ε–expansion, proved to be essential for the compactification. In the present calculation new types of finite and infinite sums beyond the case of multiple harmonic sums had to be performed. The final results could again be expressed by harmonic sums.

7 Appendix A: Results for the Individual 2–loop Diagrams

In the following, we list the results for the individual Feynman diagrams, in some cases to all orders in ε, to demonstrate the simplicity of their structure as obtained by the present method of direct calculation. An overall factor $\hat{a}_s^2 S_\varepsilon^2 (\hat{m}^2/\mu_0^2)^\varepsilon$ has been taken out. Here μ_0 denotes the initial scale to define the strong coupling constant.

The individual contributions to the operator matrix elements for the diagrams of Figure 3 are :

$$\begin{aligned}
A_a^{Qg} &= T_R C_F \frac{-\pi}{\sin((1+\varepsilon/2)\pi)} \exp\Big(\sum_{i=2}^{\infty} \frac{\zeta_i}{i} \varepsilon^i\Big) \frac{\Gamma(N-\varepsilon/2)\Gamma(N)}{\Gamma(N+2+\varepsilon/2)\Gamma(N+3-\varepsilon)} \frac{B(N,3)}{\varepsilon(\varepsilon+2)} \\
&\Big(16N(N+1)^2(N+2)^2 \\
&+8(N+1)(N+2)(3N^3-N^2-6N-4)\varepsilon \\
&+4N(9N^4+12N^3-9N^2-28N-20)\varepsilon^2 \\
&+(10N^5+8N^4+6N^3+24N^2+72N+64)\varepsilon^3 \\
&+(2N^5-10N^4-36N^3-24N^2+24N+16)\varepsilon^4 \\
&+(-4N^4-4N^3+2N^2+2N-12)\varepsilon^5 \\
&+(2N^3+4N^2+2N-4)\varepsilon^6\Big) \\
&= T_R C_F \left\{ \frac{1}{\varepsilon^2} \frac{16}{N^2(N+1)} + \frac{8}{\varepsilon} \frac{2N^3-N-2}{N^3(N+1)^2(N+2)} + \frac{8}{N^2(N+1)} S_2(N) \right. \\
&\left. + \frac{4}{N^2(N+1)} \zeta_2 + \frac{4P_1(N)}{N^4(N+1)^3(N+2)^2} \right\} + O(\varepsilon) \ , \qquad (7.1) \\
P_1(N) &= 7N^6+18N^5+18N^4-3N^3-21N^2-16N-4 \ . \\
A_b^{Qg} &= T_R C_F \left\{ \frac{1}{\varepsilon^2} \left[-\frac{32}{N} S_1(N) + \frac{32}{N} \right] + \frac{1}{\varepsilon} \left[\frac{24S_2(N) - 8S_1^2(N)}{N} \right. \right. \\
&\left. \left. +16 \frac{N^2+7N+2}{N(N+1)(N+2)} S_1(N) - 32 \frac{N^2+5N+2}{N(N+1)(N+2)} \right] - \frac{16}{N} S_{2,1}(N) + \frac{40}{3N} S_3(N) \right. \\
&-\frac{4}{N} S_1(N) S_2(N) - \frac{4}{3N} S_1^3(N) - \frac{8}{N} S_1(N) \zeta_2 + 4 \frac{N^2+7N+2}{N(N+1)(N+2)} S_1^2(N) \\
&+4 \frac{N^2-9N+2}{N(N+1)(N+2)} S_2(N) + \frac{8}{N} \zeta_2 - 16 \frac{N^3+9N^2+8N+4}{N^2(N+2)^2} S_1(N) \\
&\left. +32 \frac{N^5+10N^4+30N^3+37N^2+18N+4}{N(N+1)^3(N+2)^2} \right\} \ . \qquad (7.2) \\
A_c^{Qg} &= T_R C_F \left\{ -\frac{1}{\varepsilon^2} \frac{8}{N} + \frac{1}{\varepsilon} \frac{4(13N^4+82N^3+82N^2+N-6)}{N^2(N+1)(N+2)(N+3)} + \frac{20}{N} S_2(N) - \frac{2}{N} \zeta_2 \right. \\
&\left. - \frac{2P_2(N)}{N^3(N+1)^2(N+2)^2(N+3)} \right\} \ , \qquad (7.3)
\end{aligned}$$

$$P_2(N) = 16N^7 + 176N^6 + 520N^5 + 600N^4 + 257N^3 + 7N^2 + 16N + 12 \; .$$

$$\begin{aligned} A_d^{Qg} &= T_R C_F \left\{ -\frac{1}{\varepsilon^2}\frac{16}{N} + \frac{1}{\varepsilon}\left[-\frac{8}{N} S_1(N) + 8\frac{N^3+10N^2+59N+42}{N(N+1)(N+2)(N+3)} \right] - \frac{2}{N}\left[S_2(N) + S_1^2(N) \right] \right. \\ &\quad -\frac{4}{N}\zeta_2 + 4\frac{N^4+8N^3+43N^2+36N+12}{N^2(N+1)^2(N+2)} S_1(N) \\ &\quad \left. -\frac{8P_3(N)}{N(N+1)^3(N+2)^2(N+3)} \right\} , \end{aligned} \tag{7.4}$$

$$P_3(N) = N^6 + 10N^5 + 99N^4 + 350N^3 + 486N^2 + 274N + 60 \; .$$

$$\begin{aligned} A_e^{Qg} &= T_R \left[C_F - \frac{C_A}{2} \right] \left\{ \frac{1}{\varepsilon^2}\frac{16(N+3)}{(N+1)^2} + \frac{1}{\varepsilon}\left[-\frac{8(N+2)}{N(N+1)} S_1(N) \right. \right. \\ &\quad \left. -8\frac{3N^3+9N^2+12N+4}{N(N+1)^3(N+2)} \right] - 2\frac{9N^4+40N^3+71N^2-12N-36}{N(N+1)^2(N+2)(N+3)} S_2(N) \\ &\quad -2\frac{N^3-N^2-8N-36}{N(N+1)(N+2)(N+3)} S_1^2(N) + 4\frac{(N+3)}{(N+1)^2}\zeta_2 \\ &\quad +4\frac{4N^5+19N^4+31N^3-30N^2-44N-24}{N^2(N+1)^2(N+2)(N+3)} S_1(N) \\ &\quad \left. +\frac{4P_4(N)}{N^2(N+1)^4(N+2)^2(N+3)} \right\} , \end{aligned} \tag{7.5}$$

$$P_4(N) = 16N^7 + 111N^6 + 342N^5 + 561N^4 + 536N^3 + 354N^2 + 152N + 24 \; .$$

$$\begin{aligned} A_f^{Qg} &= T_R \left[C_F - \frac{C_A}{2} \right] \left\{ \frac{1}{\varepsilon^2}\left[\frac{64}{(N+1)(N+2)} S_1(N) - \frac{64}{(N+1)(N+2)} \right] \right. \\ &\quad +\frac{1}{\varepsilon}\left[-\frac{16}{N} S_2(N) + 16\frac{5N+2}{N^2(N+1)(N+2)} S_1(N) - \frac{32}{(N+1)^2(N+2)} \right] \\ &\quad +\frac{16}{N} S_{2,1}(N) - \frac{8}{N} S_3(N) + \frac{16}{(N+1)(N+2)} S_1(N)\zeta_2 \\ &\quad +4\frac{(9N+2)(2N-3)}{N^2(N+1)(N+2)} S_2(N) + 4\frac{2N^2-3N+2}{N^2(N+1)(N+2)} S_1^2(N) \\ &\quad -\frac{16}{(N+1)(N+2)}\zeta_2 - 8\frac{17N^2+32N+12}{N(N+1)^2(N+2)^2} S_1(N) \\ &\quad \left. +16\frac{2N^3+12N^2+23N+18}{(N+1)^3(N+2)^2} \right\} . \end{aligned} \tag{7.6}$$

$$\begin{aligned} A_g^{Qg} &= T_R C_F \left\{ \frac{1}{\varepsilon^2}\frac{32}{(N+1)(N+2)} + \frac{1}{\varepsilon}\left[\frac{8}{(N+1)(N+2)} S_1(N) \right. \right. \\ &\quad \left. -8\frac{17N^2+47N+28}{(N+1)^2(N+2)^2} \right] - \frac{38}{(N+1)(N+2)} S_2(N) + \frac{2}{(N+1)(N+2)} S_1^2(N) \end{aligned}$$

$$
\begin{aligned}
&+\frac{8}{(N+1)(N+2)}\zeta_2 - 4\frac{3N^3+31N^2+45N+8}{N(N+1)^2(N+2)^2}S_1(N)\\
&+36\frac{4N^4+26N^3+55N^2+43N+8}{(N+1)^3(N+2)^3}\Bigg\}\,. \qquad (7.7)\\
A_h^{Qg} &= T_R\left[C_F-\frac{C_A}{2}\right]\Bigg\{-\frac{1}{\varepsilon^2}\frac{32}{(N+1)(N+2)}+\frac{1}{\varepsilon}\Bigg[16\frac{N+3}{N(N+1)(N+2)}S_1(N)\\
&-8\frac{N^2+7N+8}{(N+1)^2(N+2)^2}\Bigg]-4\frac{N^2-18N+9}{N(N+1)(N+2)(N+3)}S_2(N)\\
&-4\frac{N^2-2N+9}{N(N+1)(N+2)(N+3)}S_1^2(N)-\frac{8}{(N+1)(N+2)}\zeta_2\\
&+4\frac{3N^4+N^3-27N^2-85N-84}{N(N+1)^2(N+2)^2(N+3)}S_1(N)\\
&-4\frac{(N-1)(14N^4+110N^3+337N^2+463N+240)}{(N+1)^3(N+2)^3(N+3)}\Bigg\}\,. \qquad (7.8)\\
A_i^{Qg} &= T_R C_A\Bigg\{\frac{1}{\varepsilon^2}\left[\frac{16}{(N+1)(N+2)}S_1(N)-\frac{16(N+4)}{(N+1)(N+2)^2}\right]+\frac{1}{\varepsilon}\Bigg[\frac{32}{(N+2)}S_{-2}(N)\\
&+\frac{4(4N+3)}{(N+1)(N+2)}S_2(N)-\frac{4}{(N+1)(N+2)}S_1^2(N)\\
&+8\frac{N^3+9N^2+17N+8}{N(N+1)^2(N+2)^2}S_1(N)-8\frac{2N^3+8N^2+19N+16}{(N+1)^2(N+2)^3}\Bigg]-\frac{32}{N+2}S_{-2,1}(N)\\
&-\frac{8(2N+1)}{(N+1)(N+2)}S_{2,1}(N)+\frac{16}{N+2}S_{-3}(N)+\frac{4(18N+17)}{3(N+1)(N+2)}S_3(N)\\
&+\frac{32}{N+2}S_{-2}(N)S_1(N)+\frac{2(8N+7)}{(N+1)(N+2)}S_2(N)S_1(N)-\frac{2}{3(N+1)(N+2)}S_1^3(N)\\
&+\frac{4}{(N+1)(N+2)}\zeta_2 S_1(N)-\frac{16(N^2-N-4)}{(N+1)(N+2)^2}S_{-2}(N)\\
&-2\frac{4N^4+N^3-7N^2+7N+8}{N(N+1)^2(N+2)^2}S_2(N)+2\frac{3N^3+7N^2-3N-8}{N(N+1)^2(N+2)^2}S_1^2(N)\\
&-\frac{4(N+4)}{(N+1)(N+2)^2}\zeta_2-4\frac{4N^5+36N^4+114N^3+174N^2+137N+48}{N(N+1)^3(N+2)^3}S_1(N)\\
&+4\frac{8N^5+68N^4+247N^3+449N^2+403N+144}{(N+1)^3(N+2)^4}\Bigg\}\\
&+T_R C_F\Bigg\{\frac{1}{\varepsilon}\Bigg[-\frac{16}{(N+1)(N+2)}S_2(N)+\frac{16}{(N+1)(N+2)}S_1^2(N)\\
&-\frac{64}{N(N+2)}S_1(N)+\frac{128}{(N+1)(N+2)}\Bigg]-\frac{32}{3(N+1)(N+2)}S_3(N)\\
&+\frac{8}{(N+1)(N+2)}S_2(N)S_1(N)+\frac{8}{3(N+1)(N+2)}S_1^3(N)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{8(3N+2)}{N(N+1)(N+2)}S_2(N) - \frac{8(3N-2)}{N(N+1)(N+2)}S_1^2(N)\\
&+\frac{16(5N^2+9N+6)}{N(N+1)^2(N+2)}S_1(N) - \frac{192}{(N+1)(N+2)}\Bigg\}. \qquad (7.9)\\
A_j^{Qg} &= -2T_R C_A \exp\left(\sum_{i=2}^{\infty}\frac{\zeta_i}{i}\varepsilon^i\right)\frac{\Gamma(N-\varepsilon/2)\Gamma(N)}{\Gamma(N+2+\varepsilon/2)\Gamma(N+3-\varepsilon)}\frac{B(1-\varepsilon/2,\varepsilon/2)}{\varepsilon(\varepsilon+2)}\\
&\Big(4(N+2)(4N^2+4N-5) - 4(11N^2+9N+9)\varepsilon - (4N^3-2N^2-27N-2)\varepsilon^2\\
&+(4N^2+2N+9)\varepsilon^3 - 2(2N-1)\varepsilon^4\Big)\\
&= T_R C_A\Bigg\{-\frac{1}{\varepsilon^2}\frac{8(4N^2+4N-5)}{N^2(N+1)^2} + \frac{1}{\varepsilon}\frac{4(4N^5+22N^4+11N^3+13N^2+35N+10)}{N^3(N+1)^3(N+2)}\\
&-4\frac{4N^2+4N-5}{N^2(N+1)^2}S_2(N) - 2\frac{4N^2+4N-5}{N^2(N+1)^2}\zeta_2 - \frac{2P_5(N)}{N^4(N+1)^4(N+2)^2}\Bigg\} + O(\varepsilon)\ , \qquad (7.10)\\
P_5(N) &= 20N^7+64N^6+120N^5+94N^4-140N^3-253N^2-100N-20\ .\\
\\
A_k^{Qg} &= 4T_R C_A \exp\left(\sum_{i=2}^{\infty}\frac{\zeta_i}{i}\varepsilon^i\right)\frac{\Gamma(N+1-\varepsilon/2)\Gamma(N-1)}{\Gamma(N+2+\varepsilon/2)\Gamma(N+3-\varepsilon)}\frac{B(1-\varepsilon/2,\varepsilon/2)}{\varepsilon(\varepsilon+2)}\Big(2(3N^2-23N-20)\\
&-(7N^2+9N+36)\varepsilon + 2(N^2+4N+1)\varepsilon^2 + (4N+9)\varepsilon^3 + 2\varepsilon^4\Big)\\
&= T_R C_A\Bigg\{\frac{1}{\varepsilon^2}\frac{8(3N^2-23N-20)}{(N-1)N(N+1)^2(N+2)} - \frac{1}{\varepsilon}\frac{4(10N^4+7N^3+51N^2+172N+112)}{(N-1)N(N+1)^3(N+2)^2}\\
&+4\frac{3N^2-23N-20}{(N-1)N(N+1)^2(N+2)}S_2(N) + 2\frac{3N^2-23N-20}{(N-1)N(N+1)^2(N+2)}\zeta_2\\
&+\frac{2P_6(N)}{(N-1)N(N+1)^4(N+2)^3}\Bigg\} + O(\varepsilon)\ , \qquad (7.11)\\
P_6(N) &= 14N^6+56N^5+153N^4+139N^3-414N^2-908N-448\ .\\
\\
A_l^{Qg} &= T_R C_A\Bigg\{\frac{1}{\varepsilon^2}\left[\frac{16}{N}S_1(N) + 8\frac{2N^3+5N^2+4N+2}{N^2(N+1)^2}\right] + \frac{1}{\varepsilon}\Bigg[\frac{4}{N}S_2(N) + \frac{4}{N}S_1^2(N)\\
&-\frac{16}{N(N+1)}S_1(N) - 4\frac{4N^6+30N^5+55N^4+38N^3+4N^2-10N-4}{N^3(N+1)^3(N+2)}\Bigg]\\
&+\frac{8}{N}S_{2,1}(N) + \frac{4}{3N}S_3(N) + \frac{2}{N}S_2(N)S_1(N) + \frac{2}{3N}S_1^3(N) + \frac{4}{N}S_1(N)\zeta_2\\
&-4\frac{2N^3+2N^2-N-2}{N^2(N+1)^2}S_2(N) - \frac{4}{N(N+1)}S_1^2(N) + 2\frac{2N^3+5N^2+4N+2}{N^2(N+1)^2}\zeta_2\\
&-4\frac{(N+2)(2N+1)}{N^2(N+1)^2}S_1(N) + 2\frac{P_7(N)}{N^4(N+1)^4(N+2)}\Bigg\}\ , \qquad (7.12)
\end{aligned}
$$

$$P_7(N) = 8N^8 + 68N^7 + 164N^6 + 171N^5 + 78N^4 + 12N^3 + 14N^2 + 14N + 4 \; .$$

$$\begin{aligned} A_m^{Qg} &= T_R C_A \left\{ \frac{1}{\varepsilon^2} \frac{8(N^2-2N-2)}{N^2(N+1)^2} - \frac{1}{\varepsilon} \frac{4(2N^5+11N^4+12N^3+2N^2+6N+4)}{N^3(N+1)^3(N+2)} \right. \\ &\left. + 4\frac{N^2-2N-2}{N^2(N+1)^2} S_2(N) + 2\frac{N^2-2N-2}{N^2(N+1)^2}\zeta_2 + \frac{2P_8(N)}{N^4(N+1)^4(N+2)} \right\} , \end{aligned} \tag{7.13}$$

$$P_8(N) = 2N^6 + 7N^5 + 12N^4 + 6N^3 - 8N^2 - 10N - 4 \; .$$

$$\begin{aligned} A_n^{Qg} &= T_R C_A \left\{ \frac{1}{\varepsilon^2} \left[8\frac{2N^2+3N+2}{N(N+1)(N+2)} S_1(N) - 8\frac{N(N+3)}{(N+1)^2(N+2)} \right] \right. \\ &+ \frac{1}{\varepsilon} \left[-16\frac{N-1}{N(N+1)} S_{-2}(N) - 2\frac{10N^2+21N+6}{N(N+1)(N+2)} S_2(N) \right. \\ &+ 2\frac{2N^2+3N+2}{N(N+1)(N+2)} S_1^2(N) - 4\frac{N^5+6N^4+4N^3-30N^2-40N-8}{N^2(N+1)^2(N+2)^2} S_1(N) \\ &\left. + 4\frac{2N^4+11N^3+15N^2+12N+8}{(N+1)^3(N+2)^2} \right] + 16\frac{N-1}{N(N+1)} S_{-2,1}(N) \\ &+ 4\frac{4N^2+5N-2}{N(N+1)(N+2)} S_{2,1}(N) - 8\frac{N-1}{N(N+1)} S_{-3}(N) \\ &- 2\frac{28N^2+45N-14}{3N(N+1)(N+2)} S_3(N) - 16\frac{N-1}{N(N+1)} S_{-2}(N) S_1(N) \\ &- \frac{6N^2+5N-18}{N(N+1)(N+2)} S_2(N) S_1(N) + \frac{2N^2+3N+2}{3N(N+1)(N+2)} S_1^3(N) \\ &+ 2\frac{2N^2+3N+2}{N(N+1)(N+2)} \zeta_2 S_1(N) + 16\frac{N^2-N-4}{(N+1)^2(N+2)} S_{-2}(N) \\ &+ \frac{7N^5+26N^4+16N^3-58N^2-88N-24}{N^2(N+1)^2(N+2)^2} S_2(N) \\ &- \frac{N^5+6N^4+4N^3-30N^2-40N-8}{N^2(N+1)^2(N+2)^2} S_1^2(N) - 2\frac{N(N+3)}{(N+1)^2(N+2)} \zeta_2 \\ &\left. + 2\frac{P_9(N)}{N(N+1)^3(N+2)^3} S_1(N) - 2\frac{P_{10}(N)}{(N+1)^4(N+2)^3} \right\} , \end{aligned} \tag{7.14}$$

$$P_9(N) = 2N^6 + 20N^5 + 40N^4 - 45N^3 - 170N^2 - 100N + 8 \; ,$$

$$P_{10}(N) = 4N^6 + 32N^5 + 91N^4 + 123N^3 + 62N^2 - 32N - 40 \; .$$

$$\begin{aligned} A_o^{Qg} &= T_R C_A \left\{ \frac{1}{\varepsilon^2} \left[-\frac{16}{N(N+2)} S_1(N) - 8\frac{N^2+7N+8}{(N+1)^2(N+2)^2} \right] \right. \\ &+ \frac{1}{\varepsilon} \left[-\frac{4}{N(N+2)} S_2(N) - \frac{4}{N(N+2)} S_1^2(N) + 4\frac{2N^2+9N+12}{N(N+1)(N+2)^2} S_1(N) \right. \\ &\left. \left. + 4\frac{(11N^3+56N^2+92N+49)N}{(N+1)^3(N+2)^3} \right] - \frac{8}{N(N+2)} S_{2,1}(N) \right. \end{aligned}$$

$$
\begin{aligned}
&-\frac{4}{3N(N+2)}S_3(N) - \frac{2}{N(N+2)}S_2(N)S_1(N) - \frac{2}{3N(N+2)}S_1^3(N)\\
&-\frac{4}{N(N+2)}S_1(N)\zeta_2 + \frac{10N^3+31N^2+41N+28}{N(N+1)^2(N+2)^2}S_2(N)\\
&+\frac{2N^2+9N+12}{N(N+1)(N+2)^2}S_1^2(N) - 2\frac{N^2+7N+8}{(N+1)^2(N+2)^2}\zeta_2\\
&+2\frac{4N^4+16N^3-4N^2-61N-48}{N(N+1)^2(N+2)^3}S_1(N) - 2\frac{P_{11}(N)}{(N+1)^4(N+2)^4}\Bigg\}\;, \qquad (7.15)
\end{aligned}
$$

$$
P_{11}(N) = 28N^6+222N^5+684N^4+1038N^3+811N^2+321N+64\;.
$$

$$
\begin{aligned}
A_p^{Qg} &= T_R C_A \Bigg\{ \frac{1}{\varepsilon^2}\left[-8\frac{(N-4)}{N(N+1)(N+2)}S_1(N) - 8\frac{N+4}{(N+1)(N+2)^2}\right]\\
&+\frac{1}{\varepsilon}\Bigg[2\frac{3N+4}{N(N+1)(N+2)}S_2(N) - 2\frac{N-4}{N(N+1)(N+2)}S_1^2(N)\\
&+4\frac{N^3-17N^2-41N-16}{N(N+1)^2(N+2)^2}S_1(N) + 4\frac{4N^3+26N^2+51N+32}{(N+1)^2(N+2)^3}\Bigg]\\
&-4\frac{N-4}{N(N+1)(N+2)}S_{2,1}(N) + \frac{2}{3}\frac{5N+4}{N(N+1)(N+2)}S_3(N)\\
&-\frac{1}{3}\frac{N-4}{N(N+1)(N+2)}S_1^3(N) - \frac{N-4}{N(N+1)(N+2)}S_1(N)S_2(N)\\
&-2\frac{N-4}{N(N+1)(N+2)}S_1(N)\zeta_2 - \frac{7N^3+17N^2+13N+16}{N(N+1)^2(N+2)^2}S_2(N)\\
&+\frac{N^3-17N^2-41N-16}{N(N+1)^2(N+2)^2}S_1^2(N) - 2\frac{N+4}{(N+1)(N+2)^2}\zeta_2\\
&+2\frac{2N^5+48N^4+174N^3+242N^2+161N+64}{N(N+1)^3(N+2)^3}S_1(N)\\
&-2\frac{10N^5+92N^4+329N^3+581N^2+507N+176}{(N+1)^3(N+2)^4}\Bigg\}\;. \qquad (7.16)
\end{aligned}
$$

$$
A_q^{Qg} = A_r^{Qg} = A_{r'}^{Qg} = 0 \qquad (7.17)
$$

$$
\begin{aligned}
A_s^{Qg} &= T_R C_A \Bigg\{ -\frac{1}{\varepsilon^2}\frac{8}{N^2(N+1)^2} + \frac{1}{\varepsilon}\frac{4(2N^3+N^2-3N-1)}{N^3(N+1)^3} - \frac{4}{N^2(N+1)^2}S_2(N)\\
&-\frac{2}{N^2(N+1)^2}\zeta_2 - \frac{2P_{12}(N)}{N^4(N+1)^4(N+2)}\Bigg\}\;, \qquad (7.18)
\end{aligned}
$$

$$
P_{12}(N) = 4N^6+4N^5-8N^4-2N^3+16N^2+9N+2\;.
$$

$$
\begin{aligned}
A_t^{Qg} &= T_R C_A \Bigg\{ \frac{1}{\varepsilon^2}\frac{8(N^2+3N+4)}{(N-1)N(N+1)^2(N+2)} - \frac{1}{\varepsilon}\frac{4(2N^4+5N^3-3N^2-20N-16)}{(N-1)N(N+1)^3(N+2)^2}\\
&+4\frac{N^2+3N+4}{(N-1)N(N+1)^2(N+2)}S_2(N) + 2\frac{N^2+3N+4}{(N-1)N(N+1)^2(N+2)}\zeta_2
\end{aligned}
$$

$$+\frac{2P_{13}(N)}{(N-1)N(N+1)^4(N+2)^3}\Bigg\}\,, \tag{7.19}$$

$$P_{13}(N) = 2N^6+4N^5-13N^4-35N^3+14N^2+92N+64\,.$$

The pure-singlet contributions read :

$$\begin{aligned} A_a^{Qq} &= T_R C_F\Bigg\{-\frac{1}{\varepsilon^2}\frac{16(N^2+N-2)}{N^2(N+1)^2}-\frac{1}{\varepsilon}\frac{8(5N^3-5N^2-16N-4)}{N^3(N+1)^3(N+2)} \\ &\quad -8\frac{N^2+N-2}{N^2(N+1)^2}S_2(N)-4\frac{N^2+N-2}{N^2(N+1)^2}\zeta(2)+\frac{4P_{14}(N)}{N^4(N+1)^4(N+2)^2}\Bigg\}\,, \end{aligned} \tag{7.20}$$

$$P_{14}(N) = N^8+7N^7+16N^6-9N^5-26N^4+61N^3+110N^2+44N+8\,.$$

$$\begin{aligned} A_b^{Qq} &= T_R C_F\Bigg\{-\frac{1}{\varepsilon^2}\frac{128}{(N-1)N(N+1)(N+2)}-\frac{1}{\varepsilon}\frac{128(3+2N)}{(N-1)N(N+1)^2(N+2)^2} \\ &\quad -\frac{64}{(N-1)N(N+1)(N+2)}S_2(N)-\frac{32}{(N-1)N(N+1)(N+2)}\zeta_2 \\ &\quad +\frac{32(N^4+6N^3+N^2-24N-24)}{(N-1)N(N+1)^3(N+2)^3}\Bigg\}\,. \end{aligned} \tag{7.21}$$

Finally the non-singlet contributions are :

$$\begin{aligned} A_a^{qq,Q} &= T_R C_F\Bigg\{-\frac{1}{\varepsilon^2}\frac{8(N^2-2+N)}{3N(N+1)}-\frac{1}{\varepsilon}\frac{8(N^4+2N^3-10N^2-5N+3)}{9N^2(N+1)^2} \\ &\quad -4\frac{11N^6+33N^5-34N^4-57N^3+5N^2+6N-9}{27N^3(N+1)^3}-2\frac{N^2-2+N}{3N(N+1)}\zeta_2\Bigg\}\,. \end{aligned} \tag{7.22}$$

$$\begin{aligned} A_b^{qq,Q} &= T_R C_F\Bigg\{\frac{1}{\varepsilon^2}\left[-\frac{32}{3}S_1(N)+\frac{32}{3}\right]+\frac{1}{\varepsilon}\left[\frac{16}{3}S_2(N)-\frac{80}{9}S_1(N)+\frac{32}{9}\right] \\ &\quad -\frac{8}{3}S_3(N)-\frac{8}{3}\zeta_2 S_1(N)+\frac{40}{9}S_2(N)+\frac{8}{3}\zeta_2-\frac{224}{27}S_1(N)+\frac{176}{27}\Bigg\}\,. \end{aligned} \tag{7.23}$$

$$A_c^{qq,Q} = T_R C_F\Bigg\{-\frac{2}{\varepsilon}-\frac{5}{6}\Bigg\}\,. \tag{7.24}$$

8 Appendix B: Finite and Infinite Sums

In the following, we list several classes of sums, which were used to derive the results in the present paper beyond well-known results for harmonic sums. Other relations can be found in [27,28,34–36] and were used in the present calculation. Here N, L, A denote arbitrary integers, a is a complex number, and $B(a,b)$ is Euler's Beta-function.

8.1 Sums involving Beta-Functions

$$\sum_{i=1}^{\infty} \frac{B(N,i)}{i} = \zeta_2 - S_2(N-1) \; , \tag{8.1}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i)}{i+1} = 1 + \Big[S_2(N-1) - \zeta_2\Big](N-1) \; , \tag{8.2}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i)}{i+2} = \frac{5-2N}{4} + \Big[\zeta_2 - S_2(N-1)\Big]\frac{(N-1)(N-2)}{2} \; , \tag{8.3}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i)}{i+3} = \frac{6N^2-33N+49}{36} + \Big[S_2(N-1) - \zeta_2\Big]\frac{(N-1)(N-2)(N-3)}{6} \; , \tag{8.4}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i)}{i^2} = \zeta_3 - \zeta_2 S_1(N-1) + S_{1,2}(N-1) \; , \tag{8.5}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i)}{i^3} = \frac{2}{5}\zeta_2^2 + \zeta_2 S_{1,1}(N-1) - S_{1,1,2}(N-1) - \zeta_3 S_1(N-1) \; , \tag{8.6}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i)}{N+i} = \frac{1}{N^2} \; , \tag{8.7}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i)}{1+N+i} = \frac{N^2+N+1}{N^2(N+1)^2} \; , \tag{8.8}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i)}{2+N+i} = \frac{N^4+4N^3+7N^2+6N+4}{N^2(N+1)^2(N+2)^2} \; , \tag{8.9}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i)}{3+N+i} = \frac{N^6+9N^5+34N^4+69N^3+85N^2+66N+36}{N^2(N+1)^2(N+2)^2(N+3)^2} \; , \tag{8.10}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i)}{4+N+i} = \frac{P(N)}{N^2(N+1)^2(N+2)^2(N+3)^2(N+4)^2} \; ,$$

$$P(N) = N^8 + 16N^7 + 110N^6 + 424N^5 + 1013N^4 + 1576N^3 + 1660N^2 + 1200N + 576 \ , \tag{8.11}$$

$$\sum_{i=1}^{\infty} \frac{B(N+1,i)}{N+i} = (-1)^N \Big[2S_{-2}(N) + \zeta_2\Big] \ , \tag{8.12}$$

$$\sum_{i=1}^{\infty} \frac{B(N+2,i)}{N+i} = (N+1)(-1)^N \Big[2S_{-2}(N) + \zeta_2\Big] - \frac{1}{N+1} \ , \tag{8.13}$$

$$\sum_{i=1}^{\infty} \frac{B(N+1,i)}{(N+i)^2} = (-1)^N \Big[\zeta_3 + S_1(N)\zeta_2 + 2S_{1,-2}(N) + S_{-3}(N)\Big] \ . \tag{8.14}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i)}{(N+i+1)^2} = \frac{(-1)^N}{N(N+1)} \Big[2S_{-2}(N) + \zeta_2\Big] + \frac{N-1}{N(N+1)^3} \ , \tag{8.15}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i)}{(2+N+i)^2} = 2(-1)^N \frac{2S_{-2}(N+2) + \zeta_2}{N(N+1)(N+2)} + \frac{N^2+N+1}{N(N+1)^2(N+2)^2} \ , \tag{8.16}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i)}{(N+i+1)^3} = \frac{(-1)^N}{N(N+1)} \Big[\zeta_3 + S_1(N+1)\zeta_2 - \zeta_2 + 2S_{1,-2}(N+1) - 2S_{-2}(N+1) + S_{-3}(N+1)\Big] \ , \tag{8.17}$$

8.2 Sums involving Beta-Functions and Harmonic Sums

$$\sum_{i=1}^{\infty} B(N,i) S_1(i) = \zeta_2 - S_2(N-2) \ , \tag{8.18}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i) S_1(i)}{i} = 2\zeta_3 + S_1(N-1)S_2(N-1) - \zeta_2 S_1(N-1) - S_{2,1}(N-1) \ , \tag{8.19}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i) S_1(i)}{i^2} = \frac{1}{2}\zeta_2^2 + \zeta_2 S_{1,1}(N-1) - 2\zeta_3 S_1(N-1) - S_{1,1,2}(N-1) + S_{1,3}(N-1) \ , \tag{8.20}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i) S_1(i)}{N+i} = \frac{\zeta_2 - S_2(N-1)}{N} \ , \tag{8.21}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i) S_1(i)}{N+i+1} = \frac{\zeta_2 - S_2(N-1)}{N+1} + \frac{1}{N^3(N+1)} \ , \tag{8.22}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i) S_1(i)}{N+i+2} = \frac{\zeta_2 - S_2(N+1)}{N+2} + \frac{2N^3+2N^2+3N+1}{N^3(N+1)^3} \ , \tag{8.23}$$

$$\sum_{i=1}^{\infty} \frac{B(N+1,i)S_1(i)}{N+i} = \frac{\zeta_2 - S_2(N)}{N} + (-1)^N \Big[\zeta_3 + S_{-3}(N) - 2\frac{S_{-2}(N)}{N} + 2S_{1,-2}(N) - \frac{\zeta_2}{N} + \zeta_2 S_1(N)\Big] , \tag{8.24}$$

$$\sum_{i=1}^{\infty} B(N,i)S_1(N+i) = \frac{S_1(N-1)}{N-1} + \frac{2N^2-2N+1}{N^2(N-1)^2} , \tag{8.25}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i)S_1(N+i)}{i} = 2\zeta_3 - 2S_3(N) + S_1(N)\Big[\zeta_2 - S_2(N)\Big] + \frac{S_1(N)}{N^2} + \frac{1}{N^3} , \tag{8.26}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i)S_1(N+i-1)}{i} = 2\zeta_3 - 2S_3(N-1) + S_1(N-1)\Big[\zeta_2 - S_2(N-1)\Big] , \tag{8.27}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i)S_1(N+i)}{N+i} = \frac{S_1(N-1)}{N^2} + \frac{2}{N^3} , \tag{8.28}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i)S_1(N+i)}{N+1+i} = \frac{N^2+N+1}{N^2(N+1)^2} S_1(N) - \frac{(-1)^N}{N(N+1)}\Big[2S_{-2}(N) + \zeta_2\Big] + \frac{1}{N^3} , \tag{8.29}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i)S_1(N+i-1)}{N+1+i} = \frac{N^2+N+1}{N^2(N+1)^2} S_1(N) - \frac{(-1)^N}{N(N+1)}\Big[2S_{-2}(N) + \zeta_2\Big] + \frac{2N+1}{N^3(N+1)^2} , \tag{8.30}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i)S_1(N+i)}{N+2+i} = \frac{N^3+2N^2+5N+2}{N^3(N+1)^2(2+N)} + \frac{N^4+4N^3+7N^2+6N+4}{N^2(N+1)^2(N+2)^2} S_1(N) - \frac{2(-1)^N}{N(N+1)(N+2)}\Big[2S_{-2}(N) + \zeta_2\Big] , \tag{8.31}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i)S_1(N+i-1)}{(N+1+i)^2} = \frac{(-1)^N}{N(N+1)}\Big[2S_{-2,1}(N+1) - 2S_{1,-2}(N+1) + 4S_{-2}(N+1) + 2\zeta_2 + \zeta_3 - \zeta_2 S_1(N+1)\Big] + \frac{S_1(N+1)}{N(N+1)^2} + \frac{1}{N(N+1)^3} , \tag{8.32}$$

$$\sum_{i=1}^{\infty} \frac{B(N+1,i)S_1(N+i)}{N+i} = (-1)^N \Big[2S_{-2,1}(N) + S_{-3}(N) + 2\zeta_3\Big] , \tag{8.33}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i) S_1(N+i)}{i^2} = \frac{1}{2}\zeta_2^2 + \zeta_3\Big[\frac{2}{N} - S_1(N)\Big] + \zeta_2\Big[\frac{S_1(N)}{N} - 2S_{1,1}(N)\Big] + S_{2,2}(N) + 2S_{1,3}(N) + S_1(N)S_{1,2}(N) - 2\frac{S_3(N)}{N} - \frac{S_1(N)S_2(N)}{N} \; , \tag{8.34}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i) S_1(N+i-1)}{i^2} = \frac{1}{2}\zeta_2^2 - \zeta_3 S_1(N-1) - 2\zeta_2 S_{1,1}(N-1) + S_{2,2}(N-1) + 2S_{1,3}(N-1) + S_1(N-1)S_{1,2}(N-1) \; , \tag{8.35}$$

$$\sum_{i=1}^{\infty} B(N,i) S_1(i)^2 = 3\zeta_3 - \zeta_2 S_1(N-2) + S_{1,2}(N-2) - 2S_3(N-2) \; , \tag{8.36}$$

$$\sum_{i=1}^{\infty} B(N,i) S_2(i) = -S_1(N-2)\zeta_2 + \zeta_3 + S_{1,2}(N-2) \; , \tag{8.37}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i) S_1^2(i)}{i} = +\frac{17}{10}\zeta_2^2 - 3\zeta_3 S_1(N-1) + \frac{1}{2}\zeta_2\Big[S_1^2(N-1) - S_2(N-1)\Big] - S_{1,1,2}(N-1) + S_{2,2}(N-1) + 2S_1(N-1)S_3(N-1) - 2S_{3,1}(N-1) \; , \tag{8.38}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i) S_2(i)}{i} = S_{2,2}(N-1) - S_{1,1,2}(N-1) + \zeta_2 S_{1,1}(N-1) - \zeta_2 S_2(N-1) - \zeta_3 S_1(N-1) + \frac{7}{10}\zeta_2^2 \; , \tag{8.39}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i) S_2(i)}{N+i} = \frac{-S_1(N-1)\zeta_2 + \zeta_3 + S_{1,2}(N-1)}{N} \; , \tag{8.40}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i) S_2(i)}{1+N+i} = \frac{-S_1(N)\zeta_2 + \zeta_3 + S_{1,2}(N)}{N+1} - \frac{S_2(N) - \zeta_2}{N^2} \; , \tag{8.41}$$

$$\sum_{i=1}^{\infty} B(N,i) S_{1,1}(N+i) = \frac{S_{1,1}(N-1)}{N-1} + \frac{S_1(N-1)}{(N-1)^2} + \frac{1}{(N-1)^3} + \frac{S_1(N-1)}{N^2} + \frac{2}{N^3} \; , \tag{8.42}$$

$$\sum_{i=1}^{\infty} B(N,i) S_{1,1}(N+i-1) = \frac{S_{1,1}(N-1)}{N-1} + \frac{S_1(N-1)}{(N-1)^2} + \frac{1}{(N-1)^3} \; , \tag{8.43}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i) S_{1,1}(N+i)}{i} = S_{1,1}(N)\Big[\zeta_2 - S_2(N)\Big] + \frac{6}{5}\zeta_2^2 - 3S_4(N) + 2S_1(N)\Big[\zeta_3 - S_3(N)\Big] + \frac{S_1(N)}{N^3} + \frac{S_{1,1}(N)}{N^2} + \frac{1}{N^4} \; , \tag{8.44}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i) S_{1,1}(N+i-1)}{i} = S_{1,1}(N-1)\Big[\zeta_2 - S_2(N-1)\Big] + \frac{6}{5}\zeta_2^2 - 3S_4(N-1)$$
$$+2S_1(N-1)\Big[\zeta_3 - S_3(N-1)\Big] \; , \tag{8.45}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i) S_{1,1}(N+i)}{N+i} = \frac{1}{N}\left[\frac{S_{1,1}(N)}{N} + \frac{S_1(N)}{N^2} + \frac{1}{N^3}\right] \; , \tag{8.46}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i) S_{1,1}(N+i)}{N+i+1} = \frac{S_1(N)}{N^3} + \frac{S_{1,1}(N)(N^2+N+1)}{N^2(N+1)^2} + \frac{1}{N^4}$$
$$-\frac{(-1)^N}{N(N+1)}\Big[2\zeta_3 + S_{-3}(N) + 2S_{-2,1}(N)\Big] \; , \tag{8.47}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i) S_{1,1}(N+i-1)}{N+i+1} = \frac{S_1(N-1)}{N^2(N+1)} + \frac{S_{1,1}(N-1)(N^2+N+1)}{N^2(N+1)^2}$$
$$-\frac{(-1)^N}{N(N+1)}\Big[\zeta_2 + 2\zeta_3 + 2S_{-2}(N-1)$$
$$+S_{-3}(N-1) + 2S_{-2,1}(N-1)\Big] \; , \tag{8.48}$$

$$\sum_{i=1}^{\infty} B(N,i) S_1(i) S_1(N+i) = S_1(N-1)\Big[\zeta_2 - S_2(N-1)\Big] + 2\Big[\zeta_3 - S_3(N-1)\Big]$$
$$+\frac{\zeta_2}{N} - \frac{S_2(N-1)}{N} + \frac{S_1(N-1)}{(N-1)^2} + \frac{2}{(N-1)^3} \; , \tag{8.49}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i) S_1(i) S_1(N+i)}{i} = \frac{17}{10}\zeta_2^2 + 2\frac{\zeta_3}{N} - \zeta_2\Big[\frac{1}{N^2} + \frac{S_1(N-1)}{N} + 2S_{1,1}(N-1)\Big]$$
$$+\frac{S_2(N-1)}{N^2} - \frac{S_3(N-1)}{N} + \frac{S_{1,2}(N-1)}{N}$$
$$+S_1(N-1)\Big(S_3(N-1) + S_{1,2}(N-1)\Big)$$
$$+S_2(N-1)^2 - S_{2,2}(N-1) - 2S_{3,1}(N-1) \; , \tag{8.50}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i) S_1(i) S_1(N+i)}{N+i} = \frac{1}{N}\Big\{S_1(N)\Big[\zeta_2 - S_2(N)\Big] + 2\Big[\zeta_3 - S_3(N)\Big]$$
$$+\frac{S_1(N)}{N^2} + \frac{2}{N^3}\Big\} \; , \tag{8.51}$$

$$\sum_{i=1}^{\infty} \frac{B(N,i) S_1(i) S_1(N+i)}{1+N+i} = \frac{(-1)^{N+1}}{N(N+1)}\left[\zeta_3 + \zeta_2 S_1(N) + S_{-3}(N) + 2S_{1,-2}(N)\right]$$
$$+\frac{\zeta_2 S_1(N)}{N+1} + S_1(N)\left[\frac{1}{N^3} - \frac{S_2(N)}{N+1}\right] + \frac{2\zeta_3}{N+1}$$
$$-\frac{2S_3(N)}{N+1} + \frac{2}{N^4} \; , \tag{8.52}$$

$$\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\frac{\Gamma(N+k)}{\Gamma(N)\Gamma(k+1)}\frac{B(k+i,N)}{k}\frac{S_1(i)}{i} = \frac{17}{10}\zeta_2^2 - 2\zeta_3 S_1(N-1) + S_{2,2}(N-1) + S_1(N-1)S_{2,1}(N-1) - 2S_{2,1,1}(N-1) \quad (8.53)$$

8.3 Sums involving Beta-Functions and Harmonic Sums with two Free Indexes

$$\sum_{i=1}^{\infty} B(N-a,i+a) = B(1+a,N-1-a)\ , \quad (8.54)$$

$$\sum_{i=1}^{\infty}\frac{B(N,i+k)}{i} = -\frac{d}{dN}B(k,N) = B(k,N)\Big[S_1(k+N-1) - S_1(N-1)\Big]\ , \quad (8.55)$$

$$\sum_{i=1}^{\infty}\frac{B(N,i+k)}{i}S_1(i+N+k-1) = B(k,N)\Big[2S_{1,1}(k+N-1) - S_2(N-1) - S_1(N+k-1)S_1(N-1)\Big]\ , \quad (8.56)$$

$$\sum_{i=1}^{\infty}\frac{B(N,i+k)}{i}S_1(i+k-1) = B(k,N)\Big[S_1(k+N-1)S_1(k-1) - S_2(N-1) - S_1(k-1)S_1(N-1) + \zeta_2\Big]\ . \quad (8.57)$$

8.4 Sums involving Binomials

$$\sum_{k=0}^{N-1}\binom{N-1}{k}(-1)^k\frac{1}{(k+A)^3} = \frac{B(N,A)}{2}\Bigg[\Big\{S_1(N+A-1) - S_1(A-1)\Big\}^2 - S_2(A-1) + S_2(N+A-1)\Bigg]\ , \quad (8.58)$$

$$\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{1}{k} = \lim_{\varepsilon\to 0}\Big\{B(N,\varepsilon) - \frac{1}{\varepsilon}\Big\} = -S_1(N-1)\ , \quad (8.59)$$

$$\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{1}{k^2} = -S_{1,1}(N-1)\ , \quad (8.60)$$

$$\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{1}{k^3} = -S_{1,1,1}(N-1)\ , \quad (8.61)$$

$$\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{1}{(k+1)^3} = \frac{S_{1,1}(N)}{N} - 1\ , \quad (8.62)$$

$$\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{1}{(k+2)^3} = \frac{S_{1,1}(N)}{N(N+1)} - \frac{S_1(N)}{(N+1)^2} - \frac{1}{(N+1)^3} - \frac{1}{8}\ , \tag{8.63}$$

$$\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{1}{(k+3)^3} = \frac{2S_{1,1}(N)}{N(N+1)(N+2)} - \frac{(5+3N)S_1(N)}{(N+1)^2(N+2)^2} + \frac{N^3-8N-9}{(N+1)^3(N+2)^3} - \frac{1}{27}\ , \tag{8.64}$$

$$\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{1}{(k+1)^4} = \frac{S_{1,1,1}(N)}{N} - 1\ , \tag{8.65}$$

$$\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{1}{(k+2)^4} = \frac{S_{1,1,1}(N)}{N(N+1)} - \frac{S_{1,1}(N)}{(N+1)^2} - \frac{S_1(N)}{(N+1)^3} - \frac{1}{(N+1)^4} - \frac{1}{16}\ , \tag{8.66}$$

$$\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{1}{(k+3)^4} = \frac{2S_{1,1,1}(N)}{N(N+1)(N+2)} - \frac{(5+3N)S_{1,1}(N)}{(N+1)^2(N+2)^2} + \frac{(N^3-8N-9)S_1(N)}{(N+1)^3(N+2)^3} + \frac{-17-21N-2N^2+6N^3+2N^4}{(N+1)^4(N+2)^4} - \frac{1}{81}\ . \tag{8.67}$$

8.5 Sums involving Binomials and Harmonic Sums

We set $S_A(0) := 0$. Therefore the sums below may begin at $k = 0$.

$$\sum_{k=0}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_1(k)}{k} = \lim_{\varepsilon\to 0}\Big\{B(N,\varepsilon)\Big[S_1(N+\varepsilon-1) - S_1(N-1)\Big]\Big\} = \zeta_2 - S_2(N-1)\ , \tag{8.68}$$

$$\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_1(k)}{k^2} = -S_{1,2}(N-1)\ , \tag{8.69}$$

$$\sum_{k=0}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_1(k+1)}{(k+1)^2} = \frac{S_2(N)}{N}\ , \tag{8.70}$$

$$\sum_{k=0}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_1(k+2)}{(k+2)^2} = \frac{S_2(N)}{N(N+1)} - \frac{1}{(N+1)^3}\ , \tag{8.71}$$

$$\sum_{k=0}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_1^2(k+1)}{k+1} = -\frac{S_1(N)}{N^2} + \frac{2}{N^3}\ , \tag{8.72}$$

$$\sum_{k=0}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_1^2(k+2)}{k+2} = -\frac{2N+1}{N^2(N+1)^2}S_1(N)+\frac{N^3+6N^2+6N+2}{N^3(N+1)^3}\;, \tag{8.73}$$

$$\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_2(k)}{k} = -S_{2,1}(N-1)\;, \tag{8.74}$$

$$\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_2(k)}{k+1} = -\frac{S_{1,1}(N)}{N}+\frac{S_1(N)}{N^2}\;, \tag{8.75}$$

$$\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_2(k)}{2+k} = -\frac{S_{1,1}(N)}{N(N+1)}+\frac{1-N}{N^2}S_1(N)+\frac{1}{N+1}\;, \tag{8.76}$$

$$\begin{aligned}\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_2(k)}{3+k} &= -\frac{2S_{1,1}(N)}{N(N+1)(N+2)}-\frac{N^2+4N-4}{2(N+2)N^2}S_1(N)\\ &\quad+\frac{N+11}{4(N+1)(N+2)}\;,\end{aligned} \tag{8.77}$$

$$\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_2(k)}{(1+k)^2} = \frac{S_{2,1}(N)}{N}-\frac{S_{1,1,1}(N)}{N}\;, \tag{8.78}$$

$$\begin{aligned}\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_2(k)}{(2+k)^2} &= \frac{S_{2,1}(N)}{N(N+1)}-\frac{S_{1,1,1}(N)}{N(N+1)}+\frac{S_{1,1}(N)}{(N+1)^2}-\frac{N+2}{N(N+1)}S_1(N)\\ &\quad+\frac{2N+3}{(N+1)^2}\;,\end{aligned} \tag{8.79}$$

$$\begin{aligned}\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_2(k)}{(3+k)^2} &= \frac{2S_{2,1}(N)}{N(N+1)(N+2)}-\frac{2S_{1,1,1}(N)}{N(N+1)(N+2)}\\ &\quad+\frac{(3N+5)S_{1,1}(N)}{(N+1)^2(N+2)^2}-\frac{40+38N+9N^2+N^3}{4N(N+1)(N+2)^2}S_1(N)\\ &\quad+\frac{59+66N+18N^2+N^3}{4(N+1)^2(N+2)^2}\;,\end{aligned} \tag{8.80}$$

$$\begin{aligned}\sum_{k=0}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_{1,1}(k+A)}{k+A} &= \frac{B(N,A)}{2}\Bigg[\Big\{S_1(N+A-1)-S_1(N-1)\Big\}^2\\ &\quad+S_2(N-1+A)-S_2(N-1)\Bigg]\;,\end{aligned} \tag{8.81}$$

$$\sum_{k=0}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_{1,1}(k)}{k} = \zeta_3-S_3(N-1)\;, \tag{8.82}$$

$$\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_3(k)}{k} = -S_{2,1,1}(N-1)\;, \tag{8.83}$$

$$\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_3(k)}{k+1} = -\frac{S_{1,1,1}(N-1)}{N}\;, \tag{8.84}$$

$$\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_3(k)}{2+k} = -\frac{S_{1,1,1}(N)}{N(N+1)}+\frac{1-N}{N^2}S_{1,1}(N)+\frac{S_1(N)}{N} -\frac{1}{N+1}\;, \tag{8.85}$$

$$\sum_{k=1}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_3(k)}{3+k} = -\frac{2S_{1,1,1}(N)}{N(N+1)(N+2)}-\frac{N^2+4N-4}{2(N+2)N^2}S_{1,1}(N) +\frac{S_1(N)(N+10)}{4N(N+2)}-\frac{N+19}{8(N+1)(N+2)}\;, \tag{8.86}$$

$$\sum_{k=0}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_{1,1,1}(k+A)}{k+A} = -\frac{1}{6}\frac{d^3}{dN^3}B(N,A)\;, \tag{8.87}$$

$$\sum_{k=0}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_{1,1,1}(k)}{k} = \frac{2}{5}\zeta_2^2-S_4(N-1)\;, \tag{8.88}$$

$$\sum_{k=0}^{N-1}\binom{N-1}{k}(-1)^k\frac{S_{2,1}(k)}{k} = \frac{7}{10}\zeta_2^2-S_{2,2}(N-1)\;, \tag{8.89}$$

$$\sum_{k=0}^{N-1}\binom{N-1}{k}(-1)^k\left\{-\frac{\zeta_3}{k}+2\frac{\zeta_2}{k^2}-2\frac{S_1(k)}{k^3}-\frac{S_{1,1}(k)}{k^2}\right\}$$
$$= -\frac{7}{10}\zeta_2^2+\zeta_3S_1(N-1)-2\zeta_2S_{1,1}(N-1)+S_{1,3}(N-1)+2S_{1,1,2}(N-1)\;. \tag{8.90}$$

8.6 Sums involving Binomials with Two Free Indexes

$$\sum_{k=0}^{L+1}\binom{L+1}{k}\frac{(-1)^k}{N-L+k} = B(N-L,L+2)\;, \tag{8.91}$$

$$\sum_{k=0}^{L+1}\binom{L+1}{k}\frac{(-1)^k}{(N-L+k)^2} = B(N-L,L+2)\Big[S_1(N+L) -S_1(N-L-1)\Big]\;, \tag{8.92}$$

$$\sum_{k=0}^{L+1}\binom{L+1}{k}(-1)^k\frac{S_1(N-L+k)}{N-L+k} = B(N-L,L+2)\Big[S_1(N+1)-S_1(L+1)\Big]\;, \tag{8.93}$$

$$\sum_{k=0}^{L+1}\binom{L+1}{k}(-1)^k\frac{S_1(N-L+k)}{1+N-L+k} = B(N-L+1,L+2)\Big[S_1(N-L)-S_1(L+1)\Big]$$
$$= -\frac{d}{dL}B(N-L+1,L+2)\;, \tag{8.94}$$

$$\sum_{k=1}^{\infty} \frac{B(k+\varepsilon/2, N+1)}{N+k} = (-1)^N \Big[2S_{-2}(N) + \zeta_2\Big]$$
$$+ \frac{\varepsilon}{2}(-1)^N \Big[-\zeta_3 + \zeta_2 S_1(N) + 2S_{1,-2}(N) - 2S_{-2,1}(N)\Big]$$
$$+ \frac{\varepsilon^2}{4}(-1)^N \Big[\frac{2}{5}\zeta_2^2 - \zeta_3 S_1(N) + \zeta_2 S_{1,1}(N) + 2\Big\{S_{1,1,-2}(N)$$
$$+ S_{-2,1,1}(N) - S_{1,-2,1}(N)\Big\}\Big] + O(\varepsilon^3) \ , \qquad (8.95)$$

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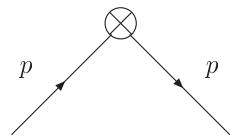
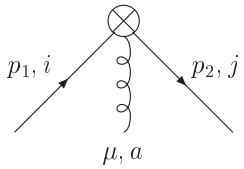
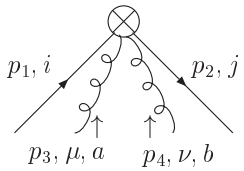
9 Figures



$$\not{\Delta}(\Delta\cdot p)^{N-1}\ , \quad N\geq 1$$



$$gt^a_{ji}\Delta^\mu \not{\Delta}\sum_{j=0}^{N-2}(\Delta\cdot p_1)^j(\Delta\cdot p_2)^{N-j-2}\ , \quad N\geq 2$$



$$\begin{aligned} g^2\Delta^\mu\Delta^\nu\not{\Delta}\sum_{0\leq j<l}^{N-2}\Big[&(\Delta p_1)^{N-l-2}(\Delta p_1+\Delta p_4)^{l-j-1}(\Delta p_2)^j(t^at^b)_{ji}\\ &+(\Delta p_1)^{N-l-2}(\Delta p_1+\Delta p_3)^{l-j-1}(\Delta p_2)^j(t^bt^a)_{ji}\Big]\ ,\\ &N\geq 3 \end{aligned}$$

Figure 1: Feynman rules for the operator insertion $\otimes$ to $O(a_s^2)$, cf. [21]. Δ denotes a light–like vector, $\Delta.\Delta = 0$.

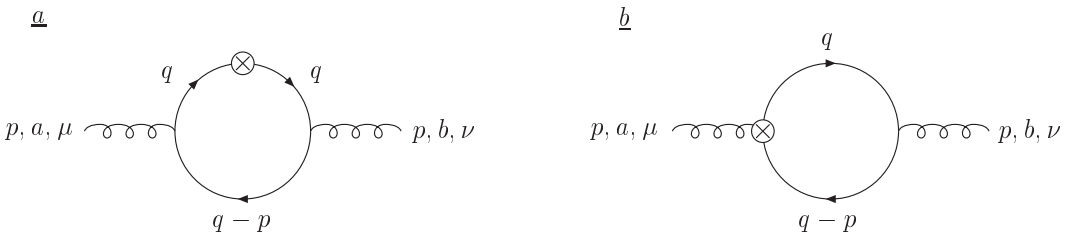


Figure 2: The Feynman diagrams contribution to the operator matrix element A_{Qg} at $O(a_s)$. Weavy lines denote gluons, and the full arrow lines are the heavy quark lines. The Feynman rules for the operator insertions are given in Figure 1.

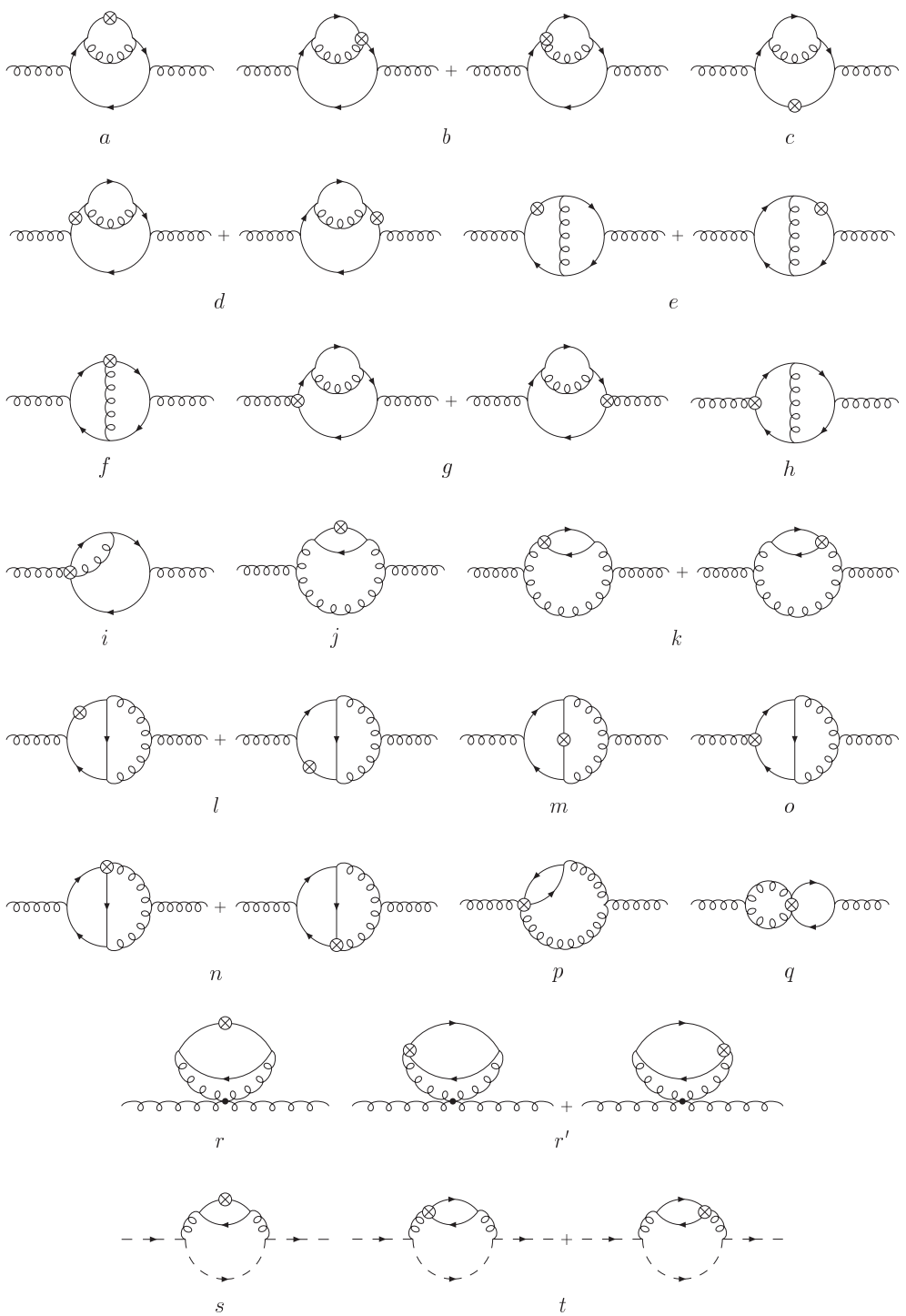


Figure 3: The diagrams contributing to the operator matrix element A_{Qg} at $O(a_s^2)$. Weavy lines denote gluons, dashed lines ghosts, and the full arrow lines are the heavy quark lines. The Feynman rules for the operator insertions are given in Figure 1.

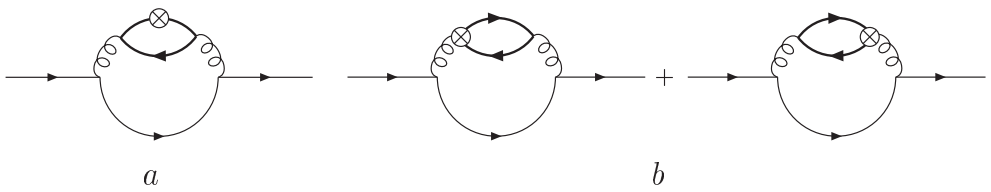


Figure 4: The diagrams contributing to the operator matrix element $A_{Qq}^{\rm PS}$ at $O(a_s^2)$. Weavy lines denote gluons, the thick full arrow lines are the heavy quark lines, and the thin full lines are light quark lines. The Feynman rules for the operator insertions are given in Figure 1.

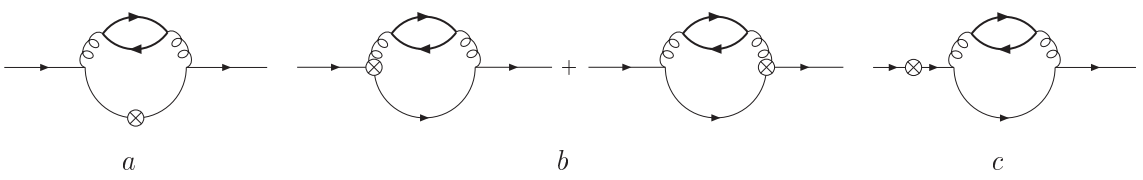


Figure 5: The diagrams contributing to the operator matrix element $A_{Qqq}^{\rm NS}$ at $O(a_s^2)$. Weavy lines denote gluons, the full arrow lines are the heavy quark lines, and the thin full lines are light quark lines. The Feynman rules for the operator insertions are given in Figure 1.