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Heavy-quark production in massless quark scattering at two loops in QCD

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Abstract

We present the two-loop virtual QCD corrections to the production of heavy quarks in the quark–anti-quark–annihilation channel in the limit when all kinematical invariants are large compared to the mass of the heavy quark. Our result is exact up to terms suppressed by powers of the heavy-quark mass. The derivation is based on a simple relation between massless and massive scattering amplitudes in gauge theories proposed recently by two of the authors as well as a direct calculation of the massive amplitude at two loops. The results presented here form an important part of the next-to-next-to-leading order QCD contributions to heavy-quark production in hadron-hadron collisions.

The hadro-production of heavy quarks, especially of top-quarks, is an important process at hadron colliders. Thus far, the Tevatron has provided us with a wealth of information on the top-quark, most prominently with a precise measurement of its mass. In the future, the LHC is expected to accumulate very high statistics for the production of $t\bar{t}$-pairs, approximately $8 \cdot 10^6$ events per year in the initial low luminosity run [1]. At the LHC the uses of $t\bar{t}$-pairs are e.g. for energy scale calibration, for background estimates and last but not least, for precision measurements of Standard Model parameters.

The present and in particular the anticipated future experimental precision on heavy-quark hadro-production requires theory predictions to include radiative corrections in Quantum Chromodynamics (QCD) beyond the next-to-leading order (NLO). For instance, at the LHC the total cross section for $t\bar{t}$-production at NLO in QCD is accurate to $O(15\%)$ only, which has to be contrasted with expected precision for the top-mass measurement of $O(1\text{GeV})$. Another important process is the production of b-quarks at moderate to large transverse momentum. NLO QCD corrections for heavy-quark hadro-production have been known since long, see e.g. Refs. [2–8]. Knowledge of the radiative QCD corrections at next-to-next-to-leading order (NNLO) will certainly improve the stability of theory predictions with respect to scale variations and provide a match with precise parton evolution at NNLO [9, 10].

In this letter, we present results for the virtual QCD corrections at two loops for the pair-production of heavy quarks in the $q\bar{q}$-annihilation channel. To be precise, we calculate the interference of the two-loop amplitude with the Born one and we work in the limit of fixed scattering angle and high energy, where all kinematic invariants are large compared to the heavy-quark mass. Thus, our result contains all logarithms in the heavy quark mass as well as all constant contributions (i.e. the mass-independent terms). Throughout this letter we neglect power corrections in the heavy-quark mass.

In our calculation we employ two different methods. On the one hand, we apply a generalization of the infrared factorization formula for massless QCD amplitudes [11, 12] to the case of massive partons [13]. In essence, it results in an extremely simple *universal* multiplicative relation between a massive QCD amplitude in the small-mass limit and its massless version [13]. In this way, we can largely use for our derivation the results of Ref. [14], where the NNLO QCD corrections to massless quark-quark scattering (i.e. $q\bar{q} \to q'\bar{q}'$) have been computed. On the other hand, we perform a direct calculation of the relevant Feynman diagrams in the massive case followed by a subsequent expansion in the small-mass limit. As an added benefit, this approach provides us with a non-trivial, albeit complicated, check of the massless results. Moreover, it makes it possible to systematically calculate power corrections in the mass, which can improve the convergence of the small-mass expansion. This is certainly relevant in the case of the top-quark pair-production at the LHC (less so, perhaps, for b-quark production).

We would like to emphasize that the agreement between our prediction based on the approach of Ref. [13] and our direct calculation constitutes the first non-trivial check of this factorization approach at two loops. The formalism of Ref. [13] has recently been also applied to the re-derivation of the two-loop QED corrections to Bhabha scattering [15] confirming earlier results on the two-

loop radiative photonic corrections [16–19] in that process.

Setting the stage

The pair-production of heavy quarks in the $q\bar{q}$-annihilation channel corresponds to the scattering process,

$$q(p_1) + \bar{q}(p_2) \;\to\; Q(p_3, m) + \bar{Q}(p_4, m)\,, \tag{1}$$

where p_i denote the quark momenta and m the mass of the heavy quark. Energy-momentum conservation implies

$$p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu\,. \tag{2}$$

Following the notation of Ref. [14] we consider the scattering amplitude $\mathcal{M}$ for the process (1) at fixed values of the external parton momenta p_i, thus $p_1^2 = p_2^2 = 0$ and $p_3^2 = p_4^2 = m^2$. It may be written as a series expansion in the strong coupling α_s,

$$|\mathcal{M}\rangle \;=\; 4\pi\alpha_s\Big[|\mathcal{M}^{(0)}\rangle + \Big(\frac{\alpha_s}{2\pi}\Big)|\mathcal{M}^{(1)}\rangle + \Big(\frac{\alpha_s}{2\pi}\Big)^2|\mathcal{M}^{(2)}\rangle + O(\alpha_s^3)\Big]\,, \tag{3}$$

where we define the expansion coefficients in powers of $\alpha_s(\mu^2)/(2\pi)$ with μ being the renormalization scale. We work in conventional dimensional regularization, $d = 4 - 2\varepsilon$, in the $\overline{\rm MS}$-scheme for the coupling constant renormalization. The heavy mass m on the other hand is always taken to be the pole mass.

We explicitly relate the bare (unrenormalized) coupling α_s^b to the renormalized coupling α_s by

$$\alpha_s^b S_\varepsilon = \alpha_s\Big[1 - \frac{\beta_0}{\varepsilon}\Big(\frac{\alpha_s}{2\pi}\Big) + \Big(\frac{\beta_0^2}{\varepsilon^2} - \frac{1}{2}\frac{\beta_1}{\varepsilon}\Big)\Big(\frac{\alpha_s}{2\pi}\Big)^2 + O(\alpha_s^3)\Big]\,, \tag{4}$$

where we put the factor $S_\varepsilon = (4\pi)^\varepsilon \exp(-\varepsilon\gamma_E) = 1$ for simplicity and β is the QCD β-function [20, 21]

$$\beta_0 = \frac{11}{6}C_A - \frac{2}{3}T_F n_f\,, \qquad \beta_1 = \frac{17}{6}{C_A}^2 - \frac{5}{3}C_A T_F n_f - C_F T_F n_f\,. \tag{5}$$

As we work in a general non-Abelian SU(N)-gauge theory we set $C_A = N$, $C_F = (N^2-1)/2N$ and $T_F = 1/2$. Throughout this letter, N denotes the number of colors and n_f the total number of flavors, which is the sum of n_l light and n_h heavy quarks.

The squared amplitude for the process (1) summed over spins and colors is a function of the Mandelstam variables s, t and u given by

$$s = (p_1 + p_2)^2\,, \qquad t = (p_1 - p_3)^2 - m^2\,, \qquad u = (p_1 - p_4)^2 - m^2\,. \tag{6}$$

Then it is convenient to define the function $\mathcal{A}(\varepsilon, m, s, t, \mu)$ for the spin and color averaged amplitudes as

$$\overline{\sum |\mathcal{M}(q + \bar{q} \to Q + \bar{Q})|^2} \;=\; \frac{1}{4N^2}\,\mathcal{A}(\varepsilon, m, s, t, \mu)\,, \tag{7}$$

which has a perturbative expansion similar to Eq. (3),

$$\mathcal{A}(\varepsilon,m,s,t,\mu) = 16\pi^2\alpha_s^2\left[\mathcal{A}^4 + \left(\frac{\alpha_s}{2\pi}\right)\mathcal{A}^6 + \left(\frac{\alpha_s}{2\pi}\right)^2\mathcal{A}^8 + O(\alpha_s^3)\right]. \tag{8}$$

In terms of the amplitudes the expansion coefficients in Eq. (8) may be expressed as

$$\mathcal{A}^4 = \langle\mathcal{M}^{(0)}|\mathcal{M}^{(0)}\rangle \equiv 2(N^2-1)\left(\frac{t^2+u^2}{s^2} - \varepsilon\right) + O(m), \tag{9}$$

$$\mathcal{A}^6 = \left(\langle\mathcal{M}^{(0)}|\mathcal{M}^{(1)}\rangle + \langle\mathcal{M}^{(1)}|\mathcal{M}^{(0)}\rangle\right), \tag{10}$$

$$\mathcal{A}^8 = \left(\langle\mathcal{M}^{(1)}|\mathcal{M}^{(1)}\rangle + \langle\mathcal{M}^{(0)}|\mathcal{M}^{(2)}\rangle + \langle\mathcal{M}^{(2)}|\mathcal{M}^{(0)}\rangle\right), \tag{11}$$

where we have neglected powers in the heavy-quark mass m in $\mathcal{A}^4$. Expressions for $\mathcal{A}^6$ with the complete heavy-quark mass dependence using dimensional regularization can be obtained e.g. from Ref. [7, 8]. The loop-by-loop contribution $\langle\mathcal{M}^{(1)}|\mathcal{M}^{(1)}\rangle$ in dimensional regularization in $\mathcal{A}^8$ and also with the full heavy-quark mass dependence can be computed with the help of Ref. [22]. In this letter, we provide for the first time the real part of $\langle\mathcal{M}^{(0)}|\mathcal{M}^{(2)}\rangle$ up to powers $O(m)$ in the heavy-quark mass m.

Massive amplitudes from QCD factorization

Let us briefly recall the key features of Ref. [13] to calculate loop amplitudes with massive partons from massless ones. The QCD factorization approach rests on the fact that a massive amplitude $\mathcal{M}^{[\mathrm{p}],(m)}$ for any given physical process shares essential properties in the small-mass limit with the corresponding massless amplitude $\mathcal{M}^{[\mathrm{p}],(m=0)}$. The latter one, $\mathcal{M}^{[\mathrm{p}],(m=0)}$, generally displays two types of singularities, soft and collinear, related to the emission of gluons with vanishing energy and to collinear parton radiation off massless hard partons, respectively. These appear explicitly as factorizing poles in ε in dimensional regularization after the usual ultraviolet renormalization is performed. In the former case, the soft singularities remain in $\mathcal{M}^{[\mathrm{p}],(m)}$ as single poles in ε while some of the collinear singularities are now screened by the mass m of the heavy fields, which gives rise to a logarithmic dependence on m, see e.g. Ref. [23].

Thus, in the small-mass limit the differences between a massless and a massive amplitude can be thought of as a mere change in the regularization scheme. As an upshot, QCD factorization provides a direct relation between $\mathcal{M}^{[\mathrm{p}],(m)}$ and $\mathcal{M}^{[\mathrm{p}],(m=0)}$ which can be cast in the remarkably simple and suggestive relation

$$\mathcal{M}^{[\mathrm{p}],(m)} = \prod_{i\in\{\text{all legs}\}}\left(Z_{[i]}^{(m|0)}\right)^{\frac{1}{2}} \times \mathcal{M}^{[\mathrm{p}],(m=0)}. \tag{12}$$

The function $Z^{(m|0)}$ is process independent and depends only on the external parton, i.e. quarks in the case at hand. For external massive quarks Q it is defined as the ratio of the on-shell heavy-quark form factor and the massless on-shell one, both being known [24–26] to sufficient orders in

α_s and powers of ε. An explicit expression for

$$Z_{[Q]}^{(m|0)} = 1 + \sum_{j=1}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^j Z^{(j)}, \tag{13}$$

up to two loops is given [1] in Ref. [13]. Exploiting the full predictive power of the relation Eq. (12) and applying it to the process Eq. (1) we get

$$2\mathrm{Re}\left\langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \right\rangle^{(m)} = 2\mathrm{Re}\left\langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \right\rangle^{(m=0)} + Z^{(1)} \mathcal{A}^{6,(m=0)} + 2Z^{(2)} \mathcal{A}^{4,(m=0)}, \tag{14}$$

which assumes the hierarchy of scales $m^2 \ll s,t,u$, i.e. we neglect terms $O(m)$. Eq. (14) predicts the complete real part of the squared amplitude $\left\langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \right\rangle$ except for those terms, which are linear in n_h, i.e. the number of heavy quarks. These two-loop contributions have been excluded explicitly from the definition [13] of $Z^{(m|0)}$, as one needs additional process dependent terms for their description. Their incorporation for the case of Bhabha scattering was presented in Ref. [15] in agreement with the direct calculation [27].

Direct calculation of the massive amplitude

An alternative method is the direct calculation of all necessary massive Feynman diagrams together with an expansion in the small mass. The advantage of this approach is an independent check of Eq. (14) as well as of the corresponding massless results. Moreover, it also allows for a relatively easy access to all heavy-quark loop corrections. Presently, these have not been obtained from the QCD factorization method, since they are related to the process dependent contributions.

We performed our computation using the DiaGen/IdSolver system of one of the authors (M.C.). After the diagram generation phase, all the integrals have been reduced to a set of 145 masters with the help of the Laporta algorithm [28] extended by topological symmetry properties, where, however, integrals which are related by a $t \leftrightarrow u$ exchange of the Mandelstam variables are considered as being independent. The evaluation of the masters proceeded similarly to the methodology developed in Ref. [29,30]. The main idea was to construct Mellin-Barnes [31,32] representations for all the integrals, followed by a subsequent analytic continuation with the MB package [33] and an expansion in the mass by closing contours. The resulting integrals have then been transformed into series representations, some two-fold, and resummed with the help of XSummer [34] in the cases of non-trivial dependence on the kinematic variables. Some constants, though, were not given by harmonic series and have been computed with the help of the PSLQ algorithm [35]. Needless to say that all of the above steps were performed fully automatically.

In our calculation, we have only been able to obtain the leading color term and the full dependence on the number of light and heavy fermion species. The reason is that the terms subleading in color contain contributions from non-planar graphs. Fig. 1 shows the bottleneck cases with 6- and 8-fold Mellin-Barnes representations, for the six and seven liners respectively. The representations

[1] Note the different normalization $\alpha_s/(4\pi)$ of the coupling used in that reference.

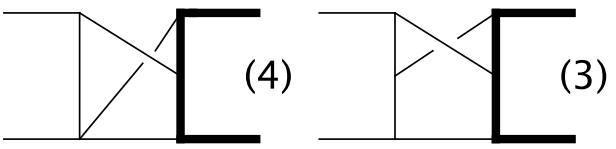

Figure 1: The most complicated non-planar topologies, together with the multiplicities of the respective master integrals for $q\bar{q} \to Q\bar{Q}$ scattering. The thick lines are massive.

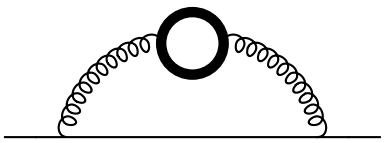

Figure 2: A heavy-quark loop contribution to the wave-function renormalization of the light quarks.

have been obtained similarly as in Ref. [32], i.e. from the Feynman-parametric representation of the two-loop graphs and not by integrating loop-by-loop, the strategy adopted for planars. In this way, the mass expansion generates integrals that are at worst as complex as the massless ones, plus some new terms that behave as $1/\sqrt{m^2}$ and must cancel in the complete result. The sheer amount of remaining difficult 4-fold integrals turned out to be presently intractable with the developed software.

Finally, as mentioned above, we renormalized the mass of the heavy quark and the external wave functions in the on-shell scheme (besides the $\overline{\rm MS}$ renormalization of the coupling constant). The former results are now known up to three-loop level from Ref. [36, 37]. A peculiarity of the light quark states is that they obtain contributions to the wave-function renormalization constants from heavy-quark loops, see Fig. 2. In fact, we have

$$Z_2 = 1 + \left(\frac{\alpha_s}{2\pi}\right)^2 C_F T_F n_h \left[\frac{1}{4\epsilon} - \frac{1}{2}\log\left(\frac{m^2}{\mu^2}\right) - \frac{5}{24}\right] . \tag{15}$$

Results

We are now in a position to present our result for $q\bar{q} \to Q\bar{Q}$ scattering for the interference of the two-loop and Born amplitude,

$$\begin{aligned} &2\mathrm{Re}\left\langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \right\rangle = \\ &\quad 2(N^2-1)\left(N^2 A + B + \frac{1}{N^2}C + N n_l D_l + N n_h D_h + \frac{n_l}{N}E_l + \frac{n_h}{N}E_h + (n_l+n_h)^2 F\right) , \end{aligned} \tag{16}$$

which we choose to express by grouping terms according to the power of the number of colors N and the numbers of n_l light and $n_h = 1$ heavy quarks with $n_f = n_l + n_h$ total flavors. As detailed above, the coefficients A, D_l, E_l and F have been computed with both methods, i.e. by employing our universal multiplicative relation (12) between the massive and massless amplitudes as well as a direct evaluation of the loop integrals in the small-mass expansion. The terms linear in n_h, D_h

and E_h could also be easily obtained from a direct Feynman diagram calculation, while for B and C the approach based on factorization proved more powerful.

As the only dimensionless kinematic variable in our problem, we choose $x = -t/s$ and keep the dependence on the renormalization scale, μ, explicit. We also introduce the following compact notation

$$\mathrm{L}_m = \log\left(\frac{m^2}{s}\right), \quad \mathrm{L}_s = \log\left(\frac{s}{\mu^2}\right), \quad \mathrm{L}_x = \log(x), \quad \mathrm{L}_y = \log(1-x). \tag{17}$$

The different components now read

$$\begin{aligned}
A &= \frac{1}{\varepsilon^4}\left\{\frac{x^2}{2}-\frac{x}{2}+\frac{1}{4}\right\}+\frac{1}{\varepsilon^3}\left\{\mathrm{L}_m\left[x^2-x+\frac{1}{2}\right]+\mathrm{L}_s\left[-x^2+x-\frac{1}{2}\right]+\frac{21x^2}{4}-\frac{21x}{4}+\mathrm{L}_x\left(-2x^2\right.\right.\\
&\left.\left.+2x-1\right)+\frac{19}{8}\right\}+\frac{1}{\varepsilon^2}\left\{\mathrm{L}_m\mathrm{L}_s\left[-2x^2+2x-1\right]+\mathrm{L}_s^2\left[x^2-x+\frac{1}{2}\right]+\mathrm{L}_m\left[\frac{29x^2}{6}-\frac{29x}{6}\right.\right.\\
&\left.+\mathrm{L}_x\left(-2x^2+2x-1\right)+\frac{23}{12}\right]+\mathrm{L}_s\left[-\frac{19x^2}{6}+\frac{19x}{6}+\mathrm{L}_x\left(4x^2-4x+2\right)-\frac{13}{12}\right]+\left(2x^2\right.\\
&\left.-\frac{5x}{2}+\frac{5}{4}\right)\mathrm{L}_x^2+\left(-\frac{26x^2}{3}+\frac{55x}{6}-\frac{23}{6}\right)\mathrm{L}_x+\frac{173x^2}{72}-\frac{173x}{72}+\pi^2\left(-\frac{x^2}{6}+\frac{x}{6}-\frac{1}{12}\right)\\
&\left.-\frac{205}{144}\right\}+\frac{1}{\varepsilon}\left\{\mathrm{L}_m^3\left[-\frac{x^2}{3}+\frac{x}{3}-\frac{1}{6}\right]+\mathrm{L}_m\mathrm{L}_s^2\left[2x^2-2x+1\right]+\mathrm{L}_s^3\left[-\frac{2x^2}{3}+\frac{2x}{3}-\frac{1}{3}\right]\right.\\
&+\mathrm{L}_m^2\left[-x^2+x+\mathrm{L}_x\left(x^2-x+\frac{1}{2}\right)-\frac{1}{2}\right]+\mathrm{L}_m\mathrm{L}_s\left[-\frac{7x^2}{3}+\frac{7x}{3}+\mathrm{L}_x\left(4x^2-4x+2\right)-\frac{1}{6}\right]\\
&+\mathrm{L}_s^2\left[-\frac{x^2}{2}+\frac{x}{2}+\mathrm{L}_x\left(-4x^2+4x-2\right)-\frac{3}{4}\right]+\mathrm{L}_m\left[\left(\frac{1}{4}-\frac{x}{2}\right)\mathrm{L}_x^2+\left(\frac{3x}{2}-x^2\right)\mathrm{L}_x-\frac{47x^2}{12}\right.\\
&\left.+\frac{47x}{12}-\frac{35}{8}\right]+\mathrm{L}_s\left[\left(-4x^2+5x-\frac{5}{2}\right)\mathrm{L}_x^2+\left(\frac{8x^2}{3}-\frac{11x}{3}+\frac{1}{3}\right)\mathrm{L}_x+\frac{487x^2}{36}-\frac{487x}{36}\right.\\
&\left.+\pi^2\left(\frac{x^2}{3}-\frac{x}{3}+\frac{1}{6}\right)+\frac{601}{72}\right]+\left(\frac{x^2}{3}+x-\frac{1}{2}\right)\mathrm{L}_x^3+\left(-\frac{5x}{2}+\mathrm{L}_y\left(-x^2+\frac{x}{2}-\frac{1}{4}\right)+\frac{3}{4}\right)\mathrm{L}_x^2\\
&+\mathrm{Li}_2(x)\left(-2x^2+x-\frac{1}{2}\right)\mathrm{L}_x+\left(\frac{43x^2}{3}-\frac{151x}{12}+\pi^2\left(\frac{4x^2}{3}-\frac{5x}{6}+\frac{5}{12}\right)+10\right)\mathrm{L}_x\\
&-\frac{9907x^2}{432}+\frac{9907x}{432}+\pi^2\left(-\frac{23x^2}{72}+\frac{5x}{72}+\frac{25}{144}\right)+\mathrm{Li}_3(x)\left(2x^2-x+\frac{1}{2}\right)+\left(-\frac{23x^2}{6}\right.\\
&\left.\left.+\frac{17x}{6}-\frac{17}{12}\right)\zeta_3-\frac{10945}{864}\right\}+\mathrm{L}_m^4\left[\frac{x^2}{4}-\frac{x}{4}+\frac{1}{8}\right]+\mathrm{L}_m^3\mathrm{L}_s\left[\frac{2x^2}{3}-\frac{2x}{3}+\frac{1}{3}\right]+\mathrm{L}_m\mathrm{L}_s^3\left[-\frac{4x^2}{3}\right.\\
&\left.+\frac{4x}{3}-\frac{2}{3}\right]+\mathrm{L}_s^4\left[\frac{x^2}{3}-\frac{x}{3}+\frac{1}{6}\right]+\mathrm{L}_m^3\left[-\frac{11x^2}{18}+\frac{11x}{18}+\mathrm{L}_x\left(-\frac{x^2}{3}+\frac{x}{3}-\frac{1}{6}\right)-\frac{5}{36}\right]\\
&+\mathrm{L}_m^2\mathrm{L}_s\left[-\frac{5x^2}{3}+\frac{5x}{3}+\mathrm{L}_x\left(-2x^2+2x-1\right)-\frac{5}{6}\right]+\mathrm{L}_m\mathrm{L}_s^2\left[-\frac{4x^2}{3}+\frac{4x}{3}+\mathrm{L}_x\left(-4x^2+4x-2\right)\right.\\
&\left.-\frac{5}{3}\right]+\mathrm{L}_s^3\left[\frac{14x^2}{9}-\frac{14x}{9}+\mathrm{L}_x\left(\frac{8x^2}{3}-\frac{8x}{3}+\frac{4}{3}\right)+\frac{10}{9}\right]+\mathrm{L}_m^2\left[\left(\frac{x}{4}-\frac{1}{8}\right)\mathrm{L}_x^2+\left(\frac{x^2}{2}-\frac{3x}{4}\right)\mathrm{L}_x\right.\\
&\left.+\frac{247x^2}{36}-\frac{247x}{36}+\frac{283}{72}\right]+\mathrm{L}_m\mathrm{L}_s\left[\left(x-\frac{1}{2}\right)\mathrm{L}_x^2+\left(2x^2-3x\right)\mathrm{L}_x+\frac{23x^2}{2}-\frac{23x}{2}+\frac{83}{12}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\mathrm{L}_s^2\left[\left(4x^2-5x+\frac{5}{2}\right)\mathrm{L}_x^2+\left(\frac{14x^2}{3}-\frac{11x}{3}+\frac{10}{3}\right)\mathrm{L}_x-\frac{37x^2}{4}+\frac{37x}{4}+\pi^2\left(-\frac{x^2}{3}+\frac{x}{3}-\frac{1}{6}\right)\right.\\
&\left.-\frac{35}{8}\right]+\mathrm{L}_m\left[\frac{x^2\mathrm{L}_x^3}{3}+\left(-\frac{x}{2}+\mathrm{L}_y\left(-x^2+\frac{x}{2}-\frac{1}{4}\right)+\frac{1}{4}\right)\mathrm{L}_x^2+\mathrm{Li}_2(x)\left(-2x^2+x-\frac{1}{2}\right)\mathrm{L}_x\right.\\
&+\left(-4x^2+\frac{19x}{4}+\pi^2\left(x^2-\frac{x}{2}+\frac{1}{4}\right)-2\right)\mathrm{L}_x-\frac{781x^2}{72}+\frac{781x}{72}+\pi^2\left(-\frac{7x^2}{12}+\frac{x}{3}-\frac{1}{24}\right)\\
&\left.+\mathrm{Li}_3(x)\left(2x^2-x+\frac{1}{2}\right)+\left(\frac{7x^2}{3}-\frac{10x}{3}+\frac{5}{3}\right)\zeta_3-\frac{499}{144}\right]+\mathrm{L}_s\left[\left(-\frac{2x^2}{3}-2x+1\right)\mathrm{L}_x^3\right.\\
&+\left(\frac{4x}{3}+\mathrm{L}_y\left(2x^2-x+\frac{1}{2}\right)+\frac{1}{3}\right)\mathrm{L}_x^2+\mathrm{Li}_2(x)\left(4x^2-2x+1\right)\mathrm{L}_x+\left(-\frac{86x^2}{3}+\frac{173x}{6}\right.\\
&\left.+\pi^2\left(-\frac{8x^2}{3}+\frac{5x}{3}-\frac{5}{6}\right)-\frac{49}{3}\right)\mathrm{L}_x+\frac{2003x^2}{216}-\frac{2003x}{216}+\mathrm{Li}_3(x)\left(-4x^2+2x-1\right)\\
&\left.+\pi^2\left(-\frac{43x^2}{36}+\frac{61x}{36}-\frac{91}{72}\right)+\left(\frac{23x^2}{3}-\frac{17x}{3}+\frac{17}{6}\right)\zeta_3-\frac{919}{432}\right]+\left(-x^2-\frac{x}{24}+\frac{1}{48}\right)\mathrm{L}_x^4\\
&+\left(-\frac{7x^2}{18}+\frac{13x}{12}+\mathrm{L}_y\left(\frac{10x^2}{3}-2x+1\right)-\frac{7}{12}\right)\mathrm{L}_x^3+\left(\left(-\frac{x^2}{2}-\frac{5x}{4}+\frac{11}{8}\right)\mathrm{L}_y^2+\left(\frac{7x^2}{6}\right.\right.\\
&\left.\left.+\frac{13x}{12}+\frac{13}{12}\right)\mathrm{L}_y+\frac{101x}{72}+\pi^2\left(-\frac{25x^2}{6}+\frac{25x}{12}-\frac{25}{24}\right)+\frac{3}{1-x}-\frac{617}{144}\right)\mathrm{L}_x^2+\mathrm{S}_{1,2}(x)\left(-2x^2\right.\\
&\left.-5x+\frac{11}{2}\right)\mathrm{L}_x+\left(\frac{260x^2}{9}-\frac{139x}{4}+\pi^2\left(-\frac{25x^2}{9}+\frac{53x}{36}-\frac{47}{18}\right)+\mathrm{L}_y\pi^2\left(-\frac{x^2}{3}+\frac{11x}{6}-\frac{17}{12}\right)\right.\\
&\left.+\left(\frac{52x^2}{3}-\frac{31x}{3}+\frac{31}{6}\right)\zeta_3+\frac{1027}{72}\right)\mathrm{L}_x+\frac{17845x^2}{2592}-\frac{17845x}{2592}+\pi^4\left(-\frac{11x^2}{80}-\frac{x}{240}+\frac{1}{480}\right)\\
&+\mathrm{Li}_4(x)\left(2x^2+3x-\frac{3}{2}\right)+\mathrm{S}_{2,2}(x)\left(2x^2+5x-\frac{11}{2}\right)+\pi^2\left(\frac{1009x^2}{432}-\frac{259x}{432}+\frac{403}{864}\right)\\
&+\mathrm{Li}_3(x)\left(-\frac{7x^2}{3}-\frac{13x}{6}+\mathrm{L}_x\left(-8x^2+2x-1\right)+\mathrm{L}_y\left(2x^2+5x-\frac{11}{2}\right)-\frac{13}{6}\right)+\mathrm{Li}_2(x)\left(\left(7x^2\right.\right.\\
&\left.\left.-\frac{7x}{2}+\frac{7}{4}\right)\mathrm{L}_x^2+\left(\frac{7x^2}{3}+\frac{13x}{6}+\mathrm{L}_y\left(-2x^2-5x+\frac{11}{2}\right)+\frac{13}{6}\right)\mathrm{L}_x+\pi^2\left(-\frac{x^2}{3}+\frac{11x}{6}-\frac{17}{12}\right)\right)\\
&+\mathrm{L}_y\left(-2x^2-5x+\frac{11}{2}\right)\zeta_3+\left(-\frac{14x^2}{9}+\frac{55x}{18}+\frac{149}{36}\right)\zeta_3+\frac{77287}{5184},
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
B \;=\;& \frac{1}{\varepsilon^4}\left\{-x^2+x-\frac{1}{2}\right\}+\frac{1}{\varepsilon^3}\left\{\mathrm{L}_m\left[-2x^2+2x-1\right]+\mathrm{L}_s\left[2x^2-2x+1\right]-\frac{31x^2}{4}+\frac{31x}{4}+\mathrm{L}_y\left(-4x^2\right.\right.\\
&\left.\left.+4x-2\right)+\mathrm{L}_x\left(6x^2-6x+3\right)-\frac{27}{8}\right\}+\frac{1}{\varepsilon^2}\left\{\mathrm{L}_m\mathrm{L}_s\left[4x^2-4x+2\right]+\mathrm{L}_s^2\left[-2x^2+2x-1\right]\right.\\
&+\mathrm{L}_m\left[-\frac{47x^2}{6}+\frac{47x}{6}+\mathrm{L}_y\left(-4x^2+4x-2\right)+\mathrm{L}_x\left(6x^2-6x+3\right)-\frac{35}{12}\right]+\mathrm{L}_s\left[\frac{49x^2}{6}-\frac{49x}{6}\right.\\
&\left.+\mathrm{L}_x\left(-12x^2+12x-6\right)+\mathrm{L}_y\left(8x^2-8x+4\right)+\frac{37}{12}\right]+\left(-6x^2+\frac{15x}{2}-\frac{15}{4}\right)\mathrm{L}_x^2+\left(\frac{67x^2}{3}\right.\\
&\left.-\frac{143x}{6}+\mathrm{L}_y\left(4x^2-4x+2\right)+\frac{29}{3}\right)\mathrm{L}_x-\frac{415x^2}{36}+\frac{415x}{36}+\mathrm{L}_y\left(-\frac{52x^2}{3}+\frac{49x}{3}-\frac{20}{3}\right)
\end{aligned}
$$

$$+\pi^2\left(\frac{17x^2}{12}-\frac{17x}{12}+\frac{17}{24}\right)+\mathrm{L}_y^2\left(2x^2-x+\frac{1}{2}\right)-\frac{17}{9}\Bigg\}+\frac{1}{\varepsilon}\Bigg\{\mathrm{L}_m^3\left[\frac{2x^2}{3}-\frac{2x}{3}+\frac{1}{3}\right]$$
$$+\mathrm{L}_m\mathrm{L}_s^2\left[-4x^2+4x-2\right]+\mathrm{L}_s^3\left[\frac{4x^2}{3}-\frac{4x}{3}+\frac{2}{3}\right]+\mathrm{L}_m^2\left[2x^2-2x+\mathrm{L}_x\left(-3x^2+3x-\frac{3}{2}\right)\right.$$
$$\left.+\mathrm{L}_y\left(2x^2-2x+1\right)+1\right]+\mathrm{L}_m\mathrm{L}_s\left[\frac{25x^2}{3}-\frac{25x}{3}+\mathrm{L}_x\left(-12x^2+12x-6\right)+\mathrm{L}_y\left(8x^2-8x+4\right)\right.$$
$$\left.+\frac{13}{6}\right]+\mathrm{L}_s^2\left[-\frac{9x^2}{2}+\frac{9x}{2}+\mathrm{L}_y\left(-8x^2+8x-4\right)+\mathrm{L}_x\left(12x^2-12x+6\right)-\frac{5}{4}\right]+\mathrm{L}_m\left[\left(\frac{3x}{2}\right.\right.$$
$$\left.-\frac{3}{4}\right)\mathrm{L}_x^2+\left(3x^2-\frac{9x}{2}\right)\mathrm{L}_x-\frac{16x^2}{3}+\mathrm{L}_y^2\left(x-\frac{1}{2}\right)+\frac{16x}{3}+\mathrm{L}_y\left(-2x^2+x+1\right)+\pi^2\left(\frac{7x^2}{6}\right.$$
$$\left.-\frac{7x}{6}+\frac{7}{12}\right)+\frac{5}{4}\Bigg]+\mathrm{L}_s\left[\left(12x^2-15x+\frac{15}{2}\right)\mathrm{L}_x^2+\left(-\frac{46x^2}{3}+\frac{55x}{3}+\mathrm{L}_y\left(-8x^2+8x-4\right)\right.\right.$$
$$\left.-\frac{14}{3}\right)\mathrm{L}_x+\frac{85x^2}{18}-\frac{85x}{18}+\mathrm{L}_y^2\left(-4x^2+2x-1\right)+\pi^2\left(-\frac{17x^2}{6}+\frac{17x}{6}-\frac{17}{12}\right)+\mathrm{L}_y\left(\frac{16x^2}{3}\right.$$
$$\left.-\frac{10x}{3}-\frac{4}{3}\right)-\frac{31}{18}\Bigg]+\left(-x^2-3x+\frac{3}{2}\right)\mathrm{L}_x^3+\left(\frac{15x}{2}+\mathrm{L}_y\left(3x^2-\frac{x}{2}+\frac{1}{4}\right)-\frac{9}{4}\right)\mathrm{L}_x^2$$
$$+\mathrm{Li}_2(x)\left(6x^2-3x+\frac{3}{2}\right)\mathrm{L}_x+\left(\left(\frac{1}{2}-x\right)\mathrm{L}_y^2-\mathrm{L}_y-\frac{50x^2}{3}+\frac{137x}{12}+\pi^2\left(-\frac{31x^2}{3}+\frac{41x}{6}-\frac{41}{12}\right)\right.$$
$$\left.-15\right)\mathrm{L}_x-\frac{419x^2}{108}+\frac{419x}{108}+\mathrm{L}_y^2\left(4x-\frac{5}{2}\right)+\mathrm{Li}_3(x)\left(-6x^2+3x-\frac{3}{2}\right)+\mathrm{S}_{1,2}(x)\left(-4x^2+6x-3\right)$$
$$+\mathrm{L}_y^3\left(\frac{2x^2}{3}-\frac{7x}{3}+\frac{7}{6}\right)+\pi^2\left(\frac{263x^2}{72}-\frac{101x}{72}-\frac{91}{144}\right)+\mathrm{L}_y\left(\frac{86x^2}{3}-\frac{193x}{6}+\pi^2\left(\frac{8x^2}{3}-\frac{11x}{3}\right.\right.$$
$$\left.+\frac{11}{6}\right)+\frac{47}{2}\Bigg)+\left(\frac{91x^2}{6}-\frac{73x}{6}+\frac{73}{12}\right)\zeta_3+\frac{413}{108}\Bigg\}+\mathrm{L}_m^4\left[-\frac{x^2}{2}+\frac{x}{2}-\frac{1}{4}\right]+\mathrm{L}_m^3\mathrm{L}_s\left[-\frac{4x^2}{3}+\frac{4x}{3}\right.$$
$$\left.-\frac{2}{3}\right]+\mathrm{L}_m\mathrm{L}_s^3\left[\frac{8x^2}{3}-\frac{8x}{3}+\frac{4}{3}\right]+\mathrm{L}_s^4\left[-\frac{2x^2}{3}+\frac{2x}{3}-\frac{1}{3}\right]+\mathrm{L}_m^3\left[\frac{11x^2}{18}-\frac{11x}{18}+\mathrm{L}_y\left(-\frac{2x^2}{3}+\frac{2x}{3}\right.\right.$$
$$\left.-\frac{1}{3}\right)+\mathrm{L}_x\left(x^2-x+\frac{1}{2}\right)-\frac{1}{36}\Bigg]+\mathrm{L}_m^2\mathrm{L}_s\left[-\frac{x^2}{3}+\frac{x}{3}+\mathrm{L}_y\left(-4x^2+4x-2\right)+\mathrm{L}_x\left(6x^2-6x+3\right)\right.$$
$$\left.-\frac{1}{6}\right]+\mathrm{L}_m\mathrm{L}_s^2\left[-\frac{14x^2}{3}+\frac{14x}{3}+\mathrm{L}_y\left(-8x^2+8x-4\right)+\mathrm{L}_x\left(12x^2-12x+6\right)-\frac{1}{3}\right]+\mathrm{L}_s^3\left[\frac{16x^2}{9}\right.$$
$$\left.-\frac{16x}{9}+\mathrm{L}_x\left(-8x^2+8x-4\right)+\mathrm{L}_y\left(\frac{16x^2}{3}-\frac{16x}{3}+\frac{8}{3}\right)+\frac{2}{9}\right]+\mathrm{L}_m^2\left[\left(\frac{3}{8}-\frac{3x}{4}\right)\mathrm{L}_x^2+\left(\frac{9x}{4}\right.\right.$$
$$\left.-\frac{3x^2}{2}\right)\mathrm{L}_x-\frac{157x^2}{36}+\mathrm{L}_y^2\left(\frac{1}{4}-\frac{x}{2}\right)+\frac{157x}{36}+\pi^2\left(-\frac{2x^2}{3}+\frac{2x}{3}-\frac{1}{3}\right)+\mathrm{L}_y\left(x^2-\frac{x}{2}-\frac{1}{2}\right)$$
$$\left.-\frac{229}{72}\right]+\mathrm{L}_m\mathrm{L}_s\left[\left(\frac{3}{2}-3x\right)\mathrm{L}_x^2+\left(9x-6x^2\right)\mathrm{L}_x+7x^2+\mathrm{L}_y^2(1-2x)-7x+\pi^2\left(-\frac{7x^2}{3}+\frac{7x}{3}\right.\right.$$
$$\left.-\frac{7}{6}\right)+\mathrm{L}_y\left(4x^2-2x-2\right)-\frac{2}{3}\Bigg]+\mathrm{L}_s^2\left[\left(-12x^2+15x-\frac{15}{2}\right)\mathrm{L}_x^2+\left(\frac{2x^2}{3}-\frac{11x}{3}+\mathrm{L}_y\left(8x^2-8x\right.\right.\right.$$
$$\left.\left.+4\right)-\frac{8}{3}\right)\mathrm{L}_x+\frac{40x^2}{9}-\frac{40x}{9}+\pi^2\left(\frac{17x^2}{6}-\frac{17x}{6}+\frac{17}{12}\right)+\mathrm{L}_y^2\left(4x^2-2x+1\right)+\mathrm{L}_y\left(\frac{28x^2}{3}-\frac{34x}{3}\right.$$
$$\left.+\frac{26}{3}\right)+\frac{161}{36}\Bigg]+\mathrm{L}_m\left[-x^2\mathrm{L}_x^3+\left(\frac{3x}{2}+\mathrm{L}_y\left(3x^2-\frac{3x}{2}+\frac{3}{4}\right)-\frac{3}{4}\right)\mathrm{L}_x^2+\mathrm{Li}_2(x)\left(6x^2-3x\right.\right.$$

$$+\frac{3}{2}\Big)\,\mathrm{L}_x+\left(12x^2-\frac{57x}{4}+\pi^2\left(-3x^2+\frac{3x}{2}-\frac{3}{4}\right)+6\right)\mathrm{L}_x-\frac{115x^2}{9}+\mathrm{L}_y^2\left(x-\frac{1}{2}\right)+\frac{115x}{9}$$
$$+\mathrm{Li}_3(x)\left(-6x^2+3x-\frac{3}{2}\right)+\mathrm{S}_{1,2}(x)\left(-4x^2+6x-3\right)+\mathrm{L}_y^3\left(\frac{2x^2}{3}-\frac{4x}{3}+\frac{2}{3}\right)+\pi^2\left(\frac{9x^2}{4}-x$$
$$-\frac{5}{24}\Big)+\mathrm{L}_y\left(-8x^2+\frac{13x}{2}+\pi^2\left(2x^2-3x+\frac{3}{2}\right)-\frac{5}{2}\right)+\left(\frac{37x^2}{3}-\frac{28x}{3}+\frac{14}{3}\right)\zeta_3-\frac{67}{18}\Big]$$
$$+\mathrm{L}_s\Big[\left(2x^2+6x-3\right)\mathrm{L}_x^3+\left(-\frac{23x}{3}+\mathrm{L}_y\left(-6x^2+x-\frac{1}{2}\right)+\frac{5}{6}\right)\mathrm{L}_x^2+\mathrm{Li}_2(x)\left(-12x^2+6x-3\right)\mathrm{L}_x$$
$$+\left((2x-1)\mathrm{L}_y^2+2\mathrm{L}_y+\frac{100x^2}{3}-\frac{181x}{6}+\pi^2\left(\frac{62x^2}{3}-\frac{41x}{3}+\frac{41}{6}\right)+\frac{68}{3}\right)\mathrm{L}_x-\frac{1957x^2}{54}+\mathrm{L}_y^2\left(\frac{4}{3}\right.$$
$$-\frac{2x}{3}\Big)+\frac{1957x}{54}+\mathrm{L}_y^3\left(-\frac{4x^2}{3}+\frac{14x}{3}-\frac{7}{3}\right)+\pi^2\left(\frac{67x^2}{36}-\frac{229x}{36}+\frac{421}{72}\right)+\mathrm{S}_{1,2}(x)\left(8x^2-12x\right.$$
$$+6)+\mathrm{Li}_3(x)\left(12x^2-6x+3\right)+\mathrm{L}_y\left(-\frac{172x^2}{3}+57x+\pi^2\left(-\frac{16x^2}{3}+\frac{22x}{3}-\frac{11}{3}\right)-\frac{97}{3}\right)$$
$$+\left(-\frac{91x^2}{3}+\frac{73x}{3}-\frac{73}{6}\right)\zeta_3-\frac{553}{27}\Big]+\left(3x^2+\frac{x}{8}-\frac{1}{16}\right)\mathrm{L}_x^4+\left(-\frac{x^2}{18}-\frac{35x}{12}+\mathrm{L}_y\left(-\frac{32x^2}{3}\right.\right.$$
$$+\frac{16x}{3}-\frac{8}{3}\Big)+\frac{7}{4}\Big)\mathrm{L}_x^3+\left(\left(\frac{7x^2}{2}+\frac{5x}{4}-\frac{17}{8}\right)\mathrm{L}_y^2+\left(\frac{x^2}{6}-\frac{29x}{12}-\frac{13}{6}\right)\mathrm{L}_y+\frac{365x}{72}+\pi^2\left(\frac{95x^2}{6}\right.\right.$$
$$-\frac{103x}{12}+\frac{103}{24}\Big)-\frac{7}{1-x}+\frac{1027}{144}\Big)\mathrm{L}_x^2+\left(\left(-\frac{2x^2}{3}+\frac{4x}{3}-\frac{2}{3}\right)\mathrm{L}_y^3+\left(\frac{x}{2}-\frac{1}{4}\right)\mathrm{L}_y^2+\left(\pi^2\left(-\frac{43x^2}{3}\right.\right.\right.$$
$$+\frac{15x}{2}-\frac{11}{4}\Big)+2\Big)\mathrm{L}_y-\frac{304x^2}{9}+\frac{153x}{4}+\pi^2\left(-\frac{125x^2}{18}+\frac{149x}{36}+\frac{155}{36}\right)+\left(-54x^2+45x\right.$$
$$-\frac{45}{2}\Big)\zeta_3-\frac{1181}{72}\Big)\mathrm{L}_x+\frac{38959x^2}{648}-\frac{38959x}{648}+\mathrm{S}_{2,2}(x)\left(-10x^2+25x-\frac{1}{2}\right)+\mathrm{Li}_4(x)\left(-6x^2\right.$$
$$-x+\frac{1}{2}\Big)+\mathrm{L}_y^4\left(-x^2+\frac{25x}{12}-\frac{25}{24}\right)+\mathrm{L}_y^3\left(-\frac{7x^2}{9}+\frac{2x}{9}-\frac{13}{36}\right)+\pi^4\left(\frac{77x^2}{45}-\frac{29x}{24}+\frac{101}{240}\right)$$
$$+\mathrm{S}_{1,3}(x)\left(4x^2+2x-1\right)+\mathrm{L}_y^2\left(-\frac{281x}{36}+\pi^2\left(-4x^2+\frac{19x}{3}-\frac{19}{6}\right)-\frac{163}{72}+\frac{5}{x}\right)$$
$$+\mathrm{S}_{1,2}(x)\left(\frac{14x^2}{3}-\frac{47x}{3}+\mathrm{L}_y\left(8x^2-8x+4\right)+\mathrm{L}_x\left(10x^2-9x-\frac{3}{2}\right)+\frac{46}{3}\right)+\mathrm{Li}_3(x)\left(-\frac{x^2}{3}\right.$$
$$+\frac{23x}{6}+\mathrm{L}_y\left(-10x^2-3x+\frac{15}{2}\right)+\mathrm{L}_x\left(24x^2-10x+5\right)+\frac{29}{6}\Big)+\mathrm{Li}_2(x)\left(\left(-21x^2+\frac{21x}{2}\right.\right.$$
$$-\frac{21}{4}\Big)\mathrm{L}_x^2+\left(\frac{x^2}{3}-\frac{23x}{6}+\mathrm{L}_y\left(10x^2+3x-\frac{15}{2}\right)-\frac{29}{6}\right)\mathrm{L}_x+\pi^2\left(-5x^2-\frac{x}{6}+\frac{25}{12}\right)\Big)$$
$$+\left(\frac{43x^2}{18}+\frac{28x}{9}-\frac{169}{18}\right)\zeta_3+\mathrm{L}_y\left(\frac{520x^2}{9}-\frac{829x}{18}+\pi^2\left(-\frac{50x^2}{9}+\frac{28x}{3}-\frac{25}{3}\right)+\left(\frac{98x^2}{3}\right.\right.$$
$$-\frac{107x}{3}+\frac{71}{6}\Big)\zeta_3+\frac{605}{36}\Big)+\pi^2\left(-\frac{3797x^2}{216}+\frac{1585x}{108}+\left(2x^2-2x+1\right)\log(2)-\frac{2353}{216}\right)$$
$$+\frac{41473}{1296},\tag{19}$$

$$
\begin{aligned}
C \;=\;& \frac{1}{\varepsilon^4}\left\{\frac{x^2}{2}-\frac{x}{2}+\frac{1}{4}\right\}+\frac{1}{\varepsilon^3}\left\{\mathrm{L}_m\left[x^2-x+\frac{1}{2}\right]+\mathrm{L}_s\left[-x^2+x-\frac{1}{2}\right]+\frac{5x^2}{2}-\frac{5x}{2}+\mathrm{L}_x\left(-4x^2+4x\right.\right.\\
&\left.-2\right)+\mathrm{L}_y\left(4x^2-4x+2\right)+1\Big\}+\frac{1}{\varepsilon^2}\left\{\mathrm{L}_m\mathrm{L}_s\left[-2x^2+2x-1\right]+\mathrm{L}_s^2\left[x^2-x+\frac{1}{2}\right]+\mathrm{L}_m\left[3x^2-3x\right.\right.\\
&+\mathrm{L}_x\left(-4x^2+4x-2\right)+\mathrm{L}_y\left(4x^2-4x+2\right)+1\Big]+\mathrm{L}_s\left[-5x^2+5x+\mathrm{L}_y\left(-8x^2+8x-4\right)+\mathrm{L}_x\left(8x^2\right.\right.\\
&\left.-8x+4\right)-2\Big]+\left(6x^2-7x+\frac{7}{2}\right)\mathrm{L}_x^2+\left(-10x^2+11x+\mathrm{L}_y\left(-12x^2+12x-6\right)-4\right)\mathrm{L}_x+\frac{73x^2}{8}\\
&-\frac{73x}{8}+\pi^2\left(-\frac{13x^2}{4}+\frac{13x}{4}-\frac{13}{8}\right)+\mathrm{L}_y^2\left(6x^2-7x+\frac{7}{2}\right)+\mathrm{L}_y\left(10x^2-9x+3\right)+\frac{53}{16}\Big\}\\
&+\frac{1}{\varepsilon}\left\{\mathrm{L}_m^3\left[-\frac{x^2}{3}+\frac{x}{3}-\frac{1}{6}\right]+\mathrm{L}_m\mathrm{L}_s^2\left[2x^2-2x+1\right]+\mathrm{L}_s^3\left[-\frac{2x^2}{3}+\frac{2x}{3}-\frac{1}{3}\right]+\mathrm{L}_m^2\left[-x^2+x\right.\right.\\
&\left.+\mathrm{L}_y\left(-2x^2+2x-1\right)+\mathrm{L}_x\left(2x^2-2x+1\right)-\frac{1}{2}\right]+\mathrm{L}_m\mathrm{L}_s\left[-6x^2+6x+\mathrm{L}_y\left(-8x^2+8x-4\right)\right.\\
&+\mathrm{L}_x\left(8x^2-8x+4\right)-2\Big]+\mathrm{L}_s^2\left[5x^2-5x+\mathrm{L}_x\left(-8x^2+8x-4\right)+\mathrm{L}_y\left(8x^2-8x+4\right)+2\right]\\
&+\mathrm{L}_m\left[\left(\frac{1}{2}-x\right)\mathrm{L}_x^2+\left(3x-2x^2\right)\mathrm{L}_x+\frac{37x^2}{4}+\mathrm{L}_y^2\left(\frac{1}{2}-x\right)-\frac{37x}{4}+\pi^2\left(-\frac{7x^2}{6}+\frac{7x}{6}-\frac{7}{12}\right)\right.\\
&+\mathrm{L}_y\left(2x^2-x-1\right)+\frac{25}{8}\Big]+\mathrm{L}_s\left[\left(-12x^2+14x-7\right)\mathrm{L}_x^2+\left(20x^2-22x+\mathrm{L}_y\left(24x^2-24x+12\right)\right.\right.\\
&\left.+8\right)\mathrm{L}_x-\frac{73x^2}{4}+\frac{73x}{4}+\mathrm{L}_y\left(-20x^2+18x-6\right)+\mathrm{L}_y^2\left(-12x^2+14x-7\right)+\pi^2\left(\frac{13x^2}{2}-\frac{13x}{2}\right.\\
&\left.+\frac{13}{4}\right)-\frac{53}{8}\Big]+\left(\frac{2x^2}{3}+3x-\frac{3}{2}\right)\mathrm{L}_x^3+\left(-6x+\mathrm{L}_y\left(-2x^2-2x+1\right)+\frac{3}{2}\right)\mathrm{L}_x^2+\mathrm{Li}_2(x)\left(-4x^2\right.\\
&\left.+2x-1\right)\mathrm{L}_x+\left(\left(3x-\frac{3}{2}\right)\mathrm{L}_y^2+3\mathrm{L}_y-24x^2+\frac{55x}{2}+\pi^2\left(\frac{46x^2}{3}-\frac{31x}{3}+\frac{31}{6}\right)-10\right)\mathrm{L}_x\\
&-\frac{3\mathrm{L}_y^2}{2}+\frac{429x^2}{16}-\frac{429x}{16}+\pi^2\left(-\frac{32x^2}{3}+\frac{17x}{3}-\frac{29}{24}\right)+\mathrm{L}_y^3\left(-\frac{2x^2}{3}-\frac{5x}{3}+\frac{5}{6}\right)+\mathrm{S}_{1,2}(x)\left(4x^2\right.\\
&\left.-6x+3\right)+\mathrm{Li}_3(x)\left(4x^2-2x+1\right)+\mathrm{L}_y\left(24x^2-\frac{41x}{2}+\pi^2\left(-\frac{46x^2}{3}+\frac{61x}{3}-\frac{61}{6}\right)+\frac{13}{2}\right)\\
&+\left(-\frac{34x^2}{3}+\frac{28x}{3}-\frac{14}{3}\right)\zeta_3+\frac{283}{32}\Big\}+\mathrm{L}_m^4\left[\frac{x^2}{4}-\frac{x}{4}+\frac{1}{8}\right]+\mathrm{L}_m^3\mathrm{L}_s\left[\frac{2x^2}{3}-\frac{2x}{3}+\frac{1}{3}\right]\\
&+\mathrm{L}_m\mathrm{L}_s^3\left[-\frac{4x^2}{3}+\frac{4x}{3}-\frac{2}{3}\right]+\mathrm{L}_s^4\left[\frac{x^2}{3}-\frac{x}{3}+\frac{1}{6}\right]+\mathrm{L}_m^3\left[\mathrm{L}_x\left(-\frac{2x^2}{3}+\frac{2x}{3}-\frac{1}{3}\right)+\mathrm{L}_y\left(\frac{2x^2}{3}-\frac{2x}{3}\right.\right.\\
&\left.+\frac{1}{3}\right)+\frac{1}{6}\Big]+\mathrm{L}_m^2\mathrm{L}_s\left[2x^2-2x+\mathrm{L}_x\left(-4x^2+4x-2\right)+\mathrm{L}_y\left(4x^2-4x+2\right)+1\right]+\mathrm{L}_m\mathrm{L}_s^2\left[6x^2\right.\\
&\left.-6x+\mathrm{L}_x\left(-8x^2+8x-4\right)+\mathrm{L}_y\left(8x^2-8x+4\right)+2\right]+\mathrm{L}_s^3\left[-\frac{10x^2}{3}+\frac{10x}{3}+\mathrm{L}_y\left(-\frac{16x^2}{3}+\frac{16x}{3}\right.\right.\\
&\left.-\frac{8}{3}\right)+\mathrm{L}_x\left(\frac{16x^2}{3}-\frac{16x}{3}+\frac{8}{3}\right)-\frac{4}{3}\Big]+\mathrm{L}_m^2\left[\left(\frac{x}{2}-\frac{1}{4}\right)\mathrm{L}_x^2+\left(x^2-\frac{3x}{2}\right)\mathrm{L}_x-\frac{5x^2}{2}+\mathrm{L}_y^2\left(\frac{x}{2}\right.\right.\\
&\left.-\frac{1}{4}\right)+\frac{5x}{2}+\mathrm{L}_y\left(-x^2+\frac{x}{2}+\frac{1}{2}\right)+\pi^2\left(\frac{2x^2}{3}-\frac{2x}{3}+\frac{1}{3}\right)-\frac{3}{4}\Big]+\mathrm{L}_m\mathrm{L}_s\left[(2x-1)\mathrm{L}_x^2+\left(4x^2\right.\right.
\end{aligned}
$$

$$-6x)\,\mathrm{L}_x - \frac{37x^2}{2} + \frac{37x}{2} + \mathrm{L}_y^2(2x-1) + \mathrm{L}_y\left(-4x^2+2x+2\right) + \pi^2\left(\frac{7x^2}{3} - \frac{7x}{3} + \frac{7}{6}\right) - \frac{25}{4}\Bigg]$$
$$+\mathrm{L}_s^2\Bigg[\left(12x^2-14x+7\right)\mathrm{L}_x^2 + \left(-20x^2+22x+\mathrm{L}_y\left(-24x^2+24x-12\right)-8\right)\mathrm{L}_x + \frac{73x^2}{4} - \frac{73x}{4}$$
$$+\pi^2\left(-\frac{13x^2}{2} + \frac{13x}{2} - \frac{13}{4}\right) + \mathrm{L}_y^2\left(12x^2-14x+7\right) + \mathrm{L}_y\left(20x^2-18x+6\right) + \frac{53}{8}\Bigg] + \mathrm{L}_m\Bigg[\frac{2x^2\mathrm{L}_x^3}{3}$$
$$+\left(-x+\mathrm{L}_y\left(-2x^2+x-\frac{1}{2}\right)+\frac{1}{2}\right)\mathrm{L}_x^2 + \mathrm{Li}_2(x)\left(-4x^2+2x-1\right)\mathrm{L}_x + \left(-8x^2 + \frac{19x}{2} + \pi^2\left(2x^2\right.\right.$$
$$\left.\left.-x+\frac{1}{2}\right)-4\right)\mathrm{L}_x + \frac{189x^2}{8} + \mathrm{L}_y^2\left(\frac{1}{2}-x\right) - \frac{189x}{8} + \pi^2\left(-\frac{5x^2}{3} + \frac{2x}{3} + \frac{1}{4}\right) + \mathrm{L}_y^3\left(-\frac{2x^2}{3} + \frac{4x}{3}\right.$$
$$\left.-\frac{2}{3}\right) + \mathrm{S}_{1,2}(x)\left(4x^2-6x+3\right) + \mathrm{Li}_3(x)\left(4x^2-2x+1\right) + \mathrm{L}_y\left(8x^2 - \frac{13x}{2} + \pi^2\left(-2x^2+3x\right.\right.$$
$$\left.\left.-\frac{3}{2}\right)+\frac{5}{2}\right) + \left(-\frac{44x^2}{3} + \frac{38x}{3} - \frac{19}{3}\right)\zeta_3 + \frac{115}{16}\Bigg] + \mathrm{L}_s\Bigg[\left(-\frac{4x^2}{3} - 6x + 3\right)\mathrm{L}_x^3 + \left(12x + \mathrm{L}_y\left(4x^2\right.\right.$$
$$\left.\left.+4x-2\right)-3\right)\mathrm{L}_x^2 + \mathrm{Li}_2(x)\left(8x^2-4x+2\right)\mathrm{L}_x + \left((3-6x)\mathrm{L}_y^2 - 6\mathrm{L}_y + 48x^2 - 55x + \pi^2\left(-\frac{92x^2}{3}\right.\right.$$
$$\left.\left.+\frac{62x}{3} - \frac{31}{3}\right) + 20\right)\mathrm{L}_x + 3\mathrm{L}_y^2 - \frac{429x^2}{8} + \frac{429x}{8} + \mathrm{Li}_3(x)\left(-8x^2+4x-2\right) + \mathrm{S}_{1,2}(x)\left(-8x^2\right.$$
$$\left.+12x-6\right) + \mathrm{L}_y^3\left(\frac{4x^2}{3} + \frac{10x}{3} - \frac{5}{3}\right) + \pi^2\left(\frac{64x^2}{3} - \frac{34x}{3} + \frac{29}{12}\right) + \mathrm{L}_y\left(-48x^2+41x+\pi^2\left(\frac{92x^2}{3}\right.\right.$$
$$\left.\left.-\frac{122x}{3} + \frac{61}{3}\right) - 13\right) + \left(\frac{68x^2}{3} - \frac{56x}{3} + \frac{28}{3}\right)\zeta_3 - \frac{283}{16}\Bigg] + \left(-3x^2 - \frac{x}{12} + \frac{1}{24}\right)\mathrm{L}_x^4 + \left(\frac{5x^2}{3}\right.$$
$$\left.+\frac{7x}{3} + \mathrm{L}_y\left(12x^2 - \frac{13x}{3} + \frac{13}{6}\right) - \frac{17}{12}\right)\mathrm{L}_x^3 + \left(\left(-7x^2 + \frac{7x}{2} - \frac{7}{4}\right)\mathrm{L}_y^2 + \left(-5x^2 - \frac{1}{4}\right)\mathrm{L}_y\right.$$
$$\left.-\frac{83x}{4} + \pi^2\left(-\frac{58x^2}{3} + \frac{34x}{3} - \frac{17}{3}\right) + \frac{3}{1-x} + \frac{35}{8}\right)\mathrm{L}_x^2 + \left(\left(2x^2-4x+2\right)\mathrm{L}_y^3 + \left(\frac{3}{4} - \frac{3x}{2}\right)\mathrm{L}_y^2\right.$$
$$\left.+\left(\pi^2\left(\frac{116x^2}{3} - \frac{89x}{3} + \frac{89}{6}\right) - 6\right)\mathrm{L}_y - 48x^2 + \frac{125x}{2} + \pi^2\left(25x^2 - 13x + \frac{7}{6}\right) + \left(\frac{128x^2}{3}\right.\right.$$
$$\left.\left.-\frac{164x}{3} + \frac{82}{3}\right)\zeta_3 - \frac{97}{4}\right)\mathrm{L}_x + \frac{2479x^2}{32} - \frac{2479x}{32} + \mathrm{S}_{1,3}(x)\left(-36x^2+62x-31\right) + \mathrm{L}_y^4\left(-3x^2\right.$$
$$\left.+\frac{71x}{12} - \frac{71}{24}\right) + \mathrm{L}_y^3\left(-\frac{5x^2}{3} + \frac{2x}{3} + \frac{11}{12}\right) + \pi^4\left(-\frac{829x^2}{720} - \frac{191x}{720} + \frac{191}{1440}\right) + \mathrm{S}_{2,2}(x)\left(8x^2\right.$$
$$\left.-56x+28\right) + \mathrm{Li}_4(x)\left(12x^2-14x+7\right) + \mathrm{L}_y^2\left(\frac{37x}{4} + \pi^2\left(-\frac{20x^2}{3} + \frac{29x}{3} - \frac{29}{6}\right) - \frac{37}{8} + \frac{3}{x}\right)$$
$$+\mathrm{S}_{1,2}(x)\left(10x^2 - 11x + \mathrm{L}_x\left(-16x^2+56x-28\right) + \mathrm{L}_y\left(8x^2-8x+4\right)\right) + \mathrm{Li}_3(x)\left(10x^2+3x\right.$$
$$\left.+\mathrm{L}_x\left(-28x^2+18x-9\right) + \mathrm{L}_y\left(16x^2-20x+10\right)-1\right) + \mathrm{Li}_2(x)\left(\left(22x^2 - 11x + \frac{11}{2}\right)\mathrm{L}_x^2\right.$$
$$\left.+\left(-10x^2 - 3x + \mathrm{L}_y\left(-16x^2+20x-10\right)+1\right)\mathrm{L}_x + \pi^2\left(\frac{76x^2}{3} - \frac{52x}{3} + \frac{26}{3}\right)\right) + \left(-\frac{265x^2}{6}\right.$$
$$\left.+\frac{259x}{6} - \frac{293}{12}\right)\zeta_3 + \mathrm{L}_y\left(48x^2 - \frac{67x}{2} + \pi^2\left(-25x^2 + 30x - \frac{61}{6}\right) + \left(-\frac{128x^2}{3} + \frac{140x}{3}\right.\right.$$

$$-\frac{70}{3}\Big)\zeta_3+\frac{39}{4}\Big)+\pi^2\left(-\frac{1631x^2}{48}+\frac{285x}{16}+\left(-2x^2+2x-1\right)\log(2)-\frac{247}{96}\right)+\frac{1621}{64}, \tag{20}$$

$$\begin{aligned}
D_l &= \frac{1}{\varepsilon^3}\left\{-\frac{x^2}{2}+\frac{x}{2}-\frac{1}{4}\right\}+\frac{1}{\varepsilon^2}\left\{\mathrm{L}_m\left[-\frac{x^2}{3}+\frac{x}{3}-\frac{1}{6}\right]+\mathrm{L}_s\left[-\frac{x^2}{3}+\frac{x}{3}-\frac{1}{6}\right]+\frac{5x^2}{9}-\frac{5x}{9}+\mathrm{L}_x\left(\frac{2x^2}{3}\right.\right.\\
&\left.-\frac{2x}{3}+\frac{1}{3}\right)+\frac{19}{36}\Big\}+\frac{1}{\varepsilon}\left\{\mathrm{L}_m\mathrm{L}_s\left[-\frac{2x^2}{3}+\frac{2x}{3}-\frac{1}{3}\right]+\mathrm{L}_s^2\left[x^2-x+\frac{1}{2}\right]+\mathrm{L}_m\left[\frac{5x^2}{3}-\frac{5x}{3}+1\right]\right.\\
&+\mathrm{L}_s\left[-\frac{40x^2}{9}+\frac{40x}{9}+\mathrm{L}_x\left(\frac{4x^2}{3}-\frac{4x}{3}+\frac{2}{3}\right)-\frac{37}{18}\right]+\frac{649x^2}{108}-\frac{649x}{108}+\mathrm{L}_x\left(-\frac{10x^2}{3}+\frac{10x}{3}-2\right)\\
&\left.+\pi^2\left(-\frac{11x^2}{36}+\frac{11x}{36}-\frac{11}{72}\right)+\frac{589}{216}\right\}+\mathrm{L}_m^3\left[\frac{x^2}{9}-\frac{x}{9}+\frac{1}{18}\right]+\mathrm{L}_m^2\mathrm{L}_s\left[\frac{2x^2}{3}-\frac{2x}{3}+\frac{1}{3}\right]\\
&+\mathrm{L}_m\mathrm{L}_s^2\left[\frac{4x^2}{3}-\frac{4x}{3}+\frac{2}{3}\right]+\mathrm{L}_s^3\left[-\frac{8x^2}{9}+\frac{8x}{9}-\frac{4}{9}\right]+\mathrm{L}_m^2\left[-\frac{23x^2}{18}+\frac{23x}{18}-\frac{23}{36}\right]+\mathrm{L}_m\mathrm{L}_s\left[-4x^2\right.\\
&\left.+4x-\frac{5}{3}\right]+\mathrm{L}_s^2\left[\frac{11x^2}{9}-\frac{11x}{9}+\mathrm{L}_x\left(-\frac{8x^2}{3}+\frac{8x}{3}-\frac{4}{3}\right)+\frac{1}{9}\right]+\mathrm{L}_m\left[\frac{73x^2}{18}-\frac{73x}{18}+\pi^2\left(-\frac{x^2}{6}\right.\right.\\
&\left.+\frac{x}{6}-\frac{1}{12}\right)+\frac{43}{36}\Big]+\mathrm{L}_s\left[\left(\frac{2x}{3}-\frac{1}{3}\right)\mathrm{L}_x^2+\left(\frac{20x^2}{3}-\frac{22x}{3}+\frac{10}{3}\right)\mathrm{L}_x+\frac{91x^2}{18}-\frac{91x}{18}+\pi^2\left(\frac{17x^2}{18}\right.\right.\\
&\left.-\frac{17x}{18}+\frac{17}{36}\right)+\frac{19}{4}\Big]+\frac{2x^2\mathrm{L}_x^3}{9}+\left(-\frac{13x}{9}+\mathrm{L}_y\left(-\frac{2x^2}{3}+\frac{2x}{3}-\frac{1}{3}\right)+\frac{5}{9}\right)\mathrm{L}_x^2+\mathrm{Li}_2(x)\left(-\frac{4x^2}{3}\right.\\
&\left.+\frac{4x}{3}-\frac{2}{3}\right)\mathrm{L}_x+\left(-\frac{56x^2}{9}+8x+\pi^2\left(\frac{25x^2}{9}-\frac{17x}{9}+\frac{17}{18}\right)-\frac{29}{9}\right)\mathrm{L}_x-\frac{2639x^2}{216}+\frac{2639x}{216}\\
&\left.+\pi^2\left(\frac{2x^2}{27}-\frac{14x}{27}+\frac{137}{216}\right)+\mathrm{Li}_3(x)\left(\frac{4x^2}{3}-\frac{4x}{3}+\frac{2}{3}\right)+\left(-\frac{73x^2}{9}+\frac{73x}{9}-\frac{73}{18}\right)\zeta_3-\frac{3937}{432},\right.
\end{aligned} \tag{21}$$

$$\begin{aligned}
D_h &= \frac{1}{\varepsilon^2}\left\{\mathrm{L}_m\left[-\frac{2x^2}{3}+\frac{2x}{3}-\frac{1}{3}\right]+\mathrm{L}_s\left[-\frac{4x^2}{3}+\frac{4x}{3}-\frac{2}{3}\right]+\frac{10x^2}{9}-\frac{10x}{9}+\frac{5}{9}\right\}+\frac{1}{\varepsilon}\left\{\mathrm{L}_m^2\left[-\frac{x^2}{3}+\frac{x}{3}\right.\right.\\
&\left.-\frac{1}{6}\right]+\mathrm{L}_s^2\left[2x^2-2x+1\right]+\mathrm{L}_m\left[-\frac{5x^2}{9}+\frac{5x}{9}+\mathrm{L}_x\left(\frac{4x^2}{3}-\frac{4x}{3}+\frac{2}{3}\right)+\frac{1}{18}\right]+\mathrm{L}_s\left[-\frac{50x^2}{9}\right.\\
&\left.+\frac{50x}{9}+\mathrm{L}_x\left(\frac{8x^2}{3}-\frac{8x}{3}+\frac{4}{3}\right)-\frac{19}{9}\right]+\frac{131x^2}{27}-\frac{131x}{27}+\mathrm{L}_x\left(-\frac{20x^2}{9}+\frac{20x}{9}-\frac{10}{9}\right)\\
&\left.+\pi^2\left(-\frac{x^2}{3}+\frac{x}{3}-\frac{1}{6}\right)+\frac{101}{54}\right\}+\mathrm{L}_m^3\left[\frac{7x^2}{9}-\frac{7x}{9}+\frac{7}{18}\right]+\mathrm{L}_m^2\mathrm{L}_s\left[\frac{4x^2}{3}-\frac{4x}{3}+\frac{2}{3}\right]+\mathrm{L}_m\mathrm{L}_s^2\left[\frac{2x^2}{3}\right.\\
&\left.-\frac{2x}{3}+\frac{1}{3}\right]+\mathrm{L}_s^3\left[-\frac{14x^2}{9}+\frac{14x}{9}-\frac{7}{9}\right]+\mathrm{L}_m^2\left[\frac{37x^2}{18}-\frac{37x}{18}+\mathrm{L}_x\left(-\frac{4x^2}{3}+\frac{4x}{3}-\frac{2}{3}\right)+\frac{43}{36}\right]\\
&+\mathrm{L}_m\mathrm{L}_s\left[\frac{4x^2}{9}-\frac{4x}{9}+\mathrm{L}_x\left(-\frac{8x^2}{3}+\frac{8x}{3}-\frac{4}{3}\right)+\frac{2}{9}\right]+\mathrm{L}_s^2\left[\frac{7x^2}{3}-\frac{7x}{3}+\mathrm{L}_x\left(-4x^2+4x-2\right)+\frac{1}{6}\right]\\
&+\mathrm{L}_m\left[\frac{76x^2}{9}-\frac{76x}{9}+\mathrm{L}_x\left(-\frac{20x^2}{9}+\frac{20x}{9}-\frac{16}{9}\right)+\frac{9}{2}\right]+\mathrm{L}_s\left[\left(\frac{2x}{3}-\frac{1}{3}\right)\mathrm{L}_x^2+\left(\frac{40x^2}{9}-\frac{46x}{9}\right.\right.\\
&\left.+\frac{14}{9}\right)\mathrm{L}_x+\frac{199x^2}{27}-\frac{199x}{27}+\pi^2\left(x^2-x+\frac{1}{2}\right)+\frac{349}{54}\Big]+\frac{2x^2\mathrm{L}_x^3}{9}+\left(-\frac{13x}{9}+\mathrm{L}_y\left(-\frac{2x^2}{3}+\frac{2x}{3}\right.\right.
\end{aligned}$$

$$-\frac{1}{3}\Big)+\frac{5}{9}\Big)\,\mathrm{L}_x^2+\mathrm{Li}_2(x)\left(-\frac{4x^2}{3}+\frac{4x}{3}-\frac{2}{3}\right)\mathrm{L}_x+\left(-\frac{224x^2}{27}+\frac{272x}{27}+\pi^2\left(\frac{8x^2}{3}-\frac{16x}{9}+\frac{8}{9}\right)\right.$$
$$\left.-\frac{130}{27}\right)\mathrm{L}_x+\frac{113x^2}{54}-\frac{113x}{54}+\pi^2\left(\frac{17x^2}{54}-\frac{41x}{54}+\frac{83}{108}\right)+\mathrm{Li}_3(x)\left(\frac{4x^2}{3}-\frac{4x}{3}+\frac{2}{3}\right)$$
$$+\left(-\frac{76x^2}{9}+\frac{76x}{9}-\frac{38}{9}\right)\zeta_3-\frac{149}{108}, \tag{22}$$

$$\begin{aligned}
E_l \;=\;& \frac{1}{\varepsilon^3}\left\{\frac{x^2}{2}-\frac{x}{2}+\frac{1}{4}\right\}+\frac{1}{\varepsilon^2}\left\{\mathrm{L}_m\left[\frac{x^2}{3}-\frac{x}{3}+\frac{1}{6}\right]+\mathrm{L}_s\left[\frac{x^2}{3}-\frac{x}{3}+\frac{1}{6}\right]-\frac{5x^2}{9}+\frac{5x}{9}+\mathrm{L}_x\left(-\frac{4x^2}{3}+\frac{4x}{3}\right.\right.\\
&\left.-\frac{2}{3}\right)+\mathrm{L}_y\left(\frac{4x^2}{3}-\frac{4x}{3}+\frac{2}{3}\right)-\frac{19}{36}\Big\}+\frac{1}{\varepsilon}\left\{\mathrm{L}_m\mathrm{L}_s\left[\frac{2x^2}{3}-\frac{2x}{3}+\frac{1}{3}\right]+\mathrm{L}_s^2\left[-x^2+x-\frac{1}{2}\right]\right.\\
&+\mathrm{L}_m\left[-\frac{5x^2}{3}+\frac{5x}{3}-1\right]+\mathrm{L}_s\left[\frac{40x^2}{9}-\frac{40x}{9}+\mathrm{L}_x\left(-\frac{8x^2}{3}+\frac{8x}{3}-\frac{4}{3}\right)+\mathrm{L}_y\left(\frac{8x^2}{3}-\frac{8x}{3}+\frac{4}{3}\right)\right.\\
&\left.+\frac{37}{18}\right]-\frac{649x^2}{108}+\frac{649x}{108}+\mathrm{L}_y\left(-\frac{20x^2}{3}+\frac{20x}{3}-4\right)+\pi^2\left(\frac{59x^2}{36}-\frac{59x}{36}+\frac{59}{72}\right)+\mathrm{L}_x\left(\frac{20x^2}{3}\right.\\
&\left.-\frac{20x}{3}+4\right)-\frac{589}{216}\Big\}+\mathrm{L}_m^3\left[-\frac{x^2}{9}+\frac{x}{9}-\frac{1}{18}\right]+\mathrm{L}_m^2\mathrm{L}_s\left[-\frac{2x^2}{3}+\frac{2x}{3}-\frac{1}{3}\right]+\mathrm{L}_m\mathrm{L}_s^2\left[-\frac{4x^2}{3}+\frac{4x}{3}\right.\\
&\left.-\frac{2}{3}\right]+\mathrm{L}_s^3\left[\frac{8x^2}{9}-\frac{8x}{9}+\frac{4}{9}\right]+\mathrm{L}_m^2\left[\frac{23x^2}{18}-\frac{23x}{18}+\frac{23}{36}\right]+\mathrm{L}_m\mathrm{L}_s\left[4x^2-4x+\frac{5}{3}\right]+\mathrm{L}_s^2\left[-\frac{55x^2}{9}\right.\\
&\left.+\frac{55x}{9}+\mathrm{L}_y\left(-\frac{16x^2}{3}+\frac{16x}{3}-\frac{8}{3}\right)+\mathrm{L}_x\left(\frac{16x^2}{3}-\frac{16x}{3}+\frac{8}{3}\right)-\frac{23}{9}\right]+\mathrm{L}_m\left[-\frac{73x^2}{18}+\frac{73x}{18}\right.\\
&\left.+\pi^2\left(\frac{x^2}{6}-\frac{x}{6}+\frac{1}{12}\right)-\frac{43}{36}\right]+\mathrm{L}_s\left[\left(\frac{2}{3}-\frac{4x}{3}\right)\mathrm{L}_x^2+\left(-\frac{40x^2}{3}+\frac{44x}{3}-\frac{20}{3}\right)\mathrm{L}_x+\frac{1027x^2}{54}\right.\\
&+\mathrm{L}_y^2\left(\frac{2}{3}-\frac{4x}{3}\right)-\frac{1027x}{54}+\pi^2\left(-\frac{89x^2}{18}+\frac{89x}{18}-\frac{89}{36}\right)+\mathrm{L}_y\left(\frac{40x^2}{3}-12x+\frac{16}{3}\right)+\frac{787}{108}\Big]\\
&-\frac{4}{9}x^2\mathrm{L}_x^3+\left(\frac{26x}{9}+\mathrm{L}_y\left(\frac{4x^2}{3}-\frac{4x}{3}+\frac{2}{3}\right)-\frac{10}{9}\right)\mathrm{L}_x^2+\mathrm{Li}_2(x)\left(\frac{8x^2}{3}-\frac{8x}{3}+\frac{4}{3}\right)\mathrm{L}_x+\left(\frac{112x^2}{9}\right.\\
&\left.-16x+\pi^2\left(-\frac{50x^2}{9}+\frac{34x}{9}-\frac{17}{9}\right)+\frac{58}{9}\right)\mathrm{L}_x-\frac{13963x^2}{648}+\frac{13963x}{648}+\mathrm{L}_y^2\left(\frac{26x}{9}-\frac{16}{9}\right)\\
&+\mathrm{Li}_3(x)\left(-\frac{8x^2}{3}+\frac{8x}{3}-\frac{4}{3}\right)+\mathrm{S}_{1,2}(x)\left(-\frac{8x^2}{3}+\frac{8x}{3}-\frac{4}{3}\right)+\mathrm{L}_y^3\left(\frac{4x^2}{9}-\frac{8x}{9}+\frac{4}{9}\right)\\
&+\pi^2\left(\frac{250x^2}{27}-\frac{202x}{27}+\frac{631}{216}\right)+\mathrm{L}_y\left(-\frac{112x^2}{9}+\frac{80x}{9}+\pi^2\left(\frac{50x^2}{9}-\frac{22x}{3}+\frac{11}{3}\right)-\frac{26}{9}\right)\\
&+\left(\frac{13x^2}{9}-\frac{13x}{9}+\frac{13}{18}\right)\zeta_3-\frac{10069}{1296},
\end{aligned} \tag{23}$$

$$\begin{aligned}
E_h \;=\;& \frac{1}{\varepsilon^2}\left\{\mathrm{L}_m\left[\frac{2x^2}{3}-\frac{2x}{3}+\frac{1}{3}\right]+\mathrm{L}_s\left[\frac{4x^2}{3}-\frac{4x}{3}+\frac{2}{3}\right]+-\frac{10x^2}{9}+\frac{10x}{9}-\frac{5}{9}\right\}+\frac{1}{\varepsilon}\left\{\mathrm{L}_m^2\left[\frac{x^2}{3}-\frac{x}{3}+\frac{1}{6}\right]\right.\\
&\left.+\mathrm{L}_s^2\left[-2x^2+2x-1\right]+\mathrm{L}_m\left[\frac{5x^2}{9}-\frac{5x}{9}+\mathrm{L}_x\left(-\frac{8x^2}{3}+\frac{8x}{3}-\frac{4}{3}\right)+\mathrm{L}_y\left(\frac{8x^2}{3}-\frac{8x}{3}+\frac{4}{3}\right)-\frac{1}{18}\right]\right.\\
&\left.+\mathrm{L}_s\left[\frac{50x^2}{9}-\frac{50x}{9}+\mathrm{L}_x\left(-\frac{16x^2}{3}+\frac{16x}{3}-\frac{8}{3}\right)+\mathrm{L}_y\left(\frac{16x^2}{3}-\frac{16x}{3}+\frac{8}{3}\right)+\frac{19}{9}\right]-\frac{131x^2}{27}\right.
\end{aligned}$$

$$+\frac{131x}{27}+\mathrm{L}_y\left(-\frac{40x^2}{9}+\frac{40x}{9}-\frac{20}{9}\right)+\pi^2\left(\frac{5x^2}{3}-\frac{5x}{3}+\frac{5}{6}\right)+\mathrm{L}_x\left(\frac{40x^2}{9}-\frac{40x}{9}+\frac{20}{9}\right)$$
$$-\frac{101}{54}\Bigg\}+\mathrm{L}_m^3\left[-\frac{7x^2}{9}+\frac{7x}{9}-\frac{7}{18}\right]+\mathrm{L}_m^2\mathrm{L}_s\left[-\frac{4x^2}{3}+\frac{4x}{3}-\frac{2}{3}\right]+\mathrm{L}_m\mathrm{L}_s^2\left[-\frac{2x^2}{3}+\frac{2x}{3}-\frac{1}{3}\right]$$
$$+\mathrm{L}_s^3\left[\frac{14x^2}{9}-\frac{14x}{9}+\frac{7}{9}\right]+\mathrm{L}_m^2\left[-\frac{37x^2}{18}+\frac{37x}{18}+\mathrm{L}_y\left(-\frac{8x^2}{3}+\frac{8x}{3}-\frac{4}{3}\right)+\mathrm{L}_x\left(\frac{8x^2}{3}-\frac{8x}{3}+\frac{4}{3}\right)\right.$$
$$\left.-\frac{43}{36}\right]+\mathrm{L}_m\mathrm{L}_s\left[-\frac{4x^2}{9}+\frac{4x}{9}+\mathrm{L}_y\left(-\frac{16x^2}{3}+\frac{16x}{3}-\frac{8}{3}\right)+\mathrm{L}_x\left(\frac{16x^2}{3}-\frac{16x}{3}+\frac{8}{3}\right)-\frac{2}{9}\right]$$
$$+\mathrm{L}_s^2\left[-\frac{65x^2}{9}+\frac{65x}{9}+\mathrm{L}_y\left(-8x^2+8x-4\right)+\mathrm{L}_x\left(8x^2-8x+4\right)-\frac{47}{18}\right]+\mathrm{L}_m\left[-\frac{76x^2}{9}+\frac{76x}{9}\right.$$
$$\left.+\mathrm{L}_y\left(-\frac{40x^2}{9}+\frac{40x}{9}-\frac{32}{9}\right)+\mathrm{L}_x\left(\frac{40x^2}{9}-\frac{40x}{9}+\frac{32}{9}\right)-\frac{9}{2}\right]+\mathrm{L}_s\left[\left(\frac{2}{3}-\frac{4x}{3}\right)\mathrm{L}_x^2+\left(-\frac{80x^2}{9}\right.\right.$$
$$\left.+\frac{92x}{9}-\frac{28}{9}\right)\mathrm{L}_x+\frac{451x^2}{27}+\mathrm{L}_y^2\left(\frac{2}{3}-\frac{4x}{3}\right)-\frac{451x}{27}+\pi^2\left(-5x^2+5x-\frac{5}{2}\right)+\mathrm{L}_y\left(\frac{80x^2}{9}\right.$$
$$\left.\left.-\frac{68x}{9}+\frac{16}{9}\right)+\frac{301}{54}\right]+-\frac{4}{9}x^2\mathrm{L}_x^3+\left(\frac{26x}{9}+\mathrm{L}_y\left(\frac{4x^2}{3}-\frac{4x}{3}+\frac{2}{3}\right)-\frac{10}{9}\right)\mathrm{L}_x^2+\mathrm{Li}_2(x)\left(\frac{8x^2}{3}\right.$$
$$\left.-\frac{8x}{3}+\frac{4}{3}\right)\mathrm{L}_x+\left(\frac{448x^2}{27}-\frac{544x}{27}+\pi^2\left(-\frac{16x^2}{3}+\frac{32x}{9}-\frac{16}{9}\right)+\frac{260}{27}\right)\mathrm{L}_x-\frac{5809x^2}{162}$$
$$+\frac{5809x}{162}+\mathrm{L}_y^2\left(\frac{26x}{9}-\frac{16}{9}\right)+\mathrm{Li}_3(x)\left(-\frac{8x^2}{3}+\frac{8x}{3}-\frac{4}{3}\right)+\mathrm{S}_{1,2}(x)\left(-\frac{8x^2}{3}+\frac{8x}{3}-\frac{4}{3}\right)$$
$$+\mathrm{L}_y^3\left(\frac{4x^2}{9}-\frac{8x}{9}+\frac{4}{9}\right)+\pi^2\left(\frac{487x^2}{54}-\frac{391x}{54}+\frac{301}{108}\right)+\mathrm{L}_y\left(-\frac{448x^2}{27}+\frac{352x}{27}+\pi^2\left(\frac{16x^2}{3}\right.\right.$$
$$\left.\left.-\frac{64x}{9}+\frac{32}{9}\right)-\frac{164}{27}\right)+\left(\frac{16x^2}{9}-\frac{16x}{9}+\frac{8}{9}\right)\zeta_3-\frac{5023}{324}, \tag{24}$$

$$F = \mathrm{L}_s^2\left[\frac{4x^2}{9}-\frac{4x}{9}+\frac{2}{9}\right]+\mathrm{L}_s\left[-\frac{40x^2}{27}+\frac{40x}{27}-\frac{20}{27}\right]+\frac{100x^2}{81}-\frac{100x}{81}+\pi^2\left(-\frac{4x^2}{9}+\frac{4x}{9}-\frac{2}{9}\right)+\frac{50}{81}. \tag{25}$$

Notice that our results are expressed not only in terms of the classic polylogarithms up to weight four, but also in terms of Nielsen polylogarithms

$$\mathrm{S}_{n,p}(x)=\frac{(-1)^{n+p-1}}{(n-1)!p!}\int_0^1 dy\frac{\log^{n-1}(y)\log^p(1-xy)}{y}. \tag{26}$$

Conclusions

In this letter, we have presented the two-loop virtual QCD corrections to the production of heavy-quarks in the light quark annihilation channel in the ultra-relativistic limit. Our results form a crucial part of the NNLO predictions for heavy-quark production in hadron-hadron collisions. However, depending on the mass of the heavy quark (bottom or top) and the kinematics of the process under consideration power corrections in the heavy-quark mass may have to be considered

as well. Our results have been derived by combining two completely different methods which have substantial overlap. This provides direct and highly non-trivial checks of the QCD factorization approach of Ref. [13], of the direct calculation of massive Feynman diagrams and, last but not least, also on the available masseless results of Ref. [14].

In order to yield physical cross sections, our result for $\langle \mathcal{M}^{(0)} | \mathcal{M}^{(2)} \rangle$ still has to be combined with the tree-level $2 \to 4$, the one-loop $2 \to 3$ as well as the square of the one-loop $2 \to 2$ processes. While some of the matrix elements (including the full mass dependence) can be easily generated, others became available in the literature only rather recently, see e.g. Refs. [22,38]. The combination of all these contributions enables the analytic cancellation of the remaining infrared divergences as well as the isolation of the initial state singularities which need to be absorbed into parton distribution functions. This is a necessary prerequisite e.g. to the construction of numerical programs which provide NNLO QCD estimates of observable scattering cross sections.

Finally, while we have chosen to work in this letter with squared matrix elements, it should, of course, be clear that with the help of Refs. [39,40] analogous results can be derived for the massive two-loop amplitude $|\mathcal{M}^{(2)}\rangle$ itself.

A MATHEMATICA file with our results can be obtained by downloading the source from the preprint server `http://arXiv.org`. The results are also available from the authors upon request.

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