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Abstract

In this article we derive longitudinal impedance and wake function for an undulator setup with arbitrary undulator parameter, taking into account a finite transverse size of the electron bunch. Earlier studies considered a line density-distribution of electrons instead. We focus our attention on the long-wavelength asymptote (compared with resonance wavelength), at large distance of the electron bunch from the undulator entrance compared to the overtaking length, and for large vacuum-chamber size compared to the typical transverse size of the field. These restrictions define a parameter region of interest for practical applications. We calculate a closed expression for impedance and wake function that may be evaluated numerically in the most general case. Such expression allows us to derive an analytical solution for a Gaussian transverse and longitudinal bunch shape. Finally, we study the feasibility of current-enhanced SASE schemes (ESASE) recently proposed for LCLS, that fall well-within our approximations. Numerical estimations presented in this paper indicate that impedance-induced energy spread is sufficient to seriously degrade the FEL performance. Our conclusion is in contrast with results in literature, where wake calculations for the LCLS case are given in free-space, as if the presence of the undulator were negligible.

Key words:
longitudinal impedance, longitudinal wake-function, X-Ray Free-Electron Laser (XFEL), Enhanched SASE schemes
PACS: 41.60.Ap, 41.60.-m, 41.20.-q

1 Introduction

Self-Amplified Spontaneous Emission Free Electron Lasers (SASE FELs) [1, 2, 3] are nowadays considered as a unique tool for production of intense, polarized, short-pulse radiation tunable throughout the VUV and X-ray wavelength range, with peak and average brilliance exceeding both modern Synchrotron Radiation and Laser Plasma sources by many order of magnitudes [4, 5, 6, 7, 8, 9]. Successful operation of SASE FELs requires high quality (low emittance and low energy spread), intense electron beams. One of the trends for SASE FELs is production of ultra-short radiation pulses. These can be obtained by exploiting electron bunches with an ultra-short charge concentration (spike). At the FLASH [1] facility at DESY, Hamburg [4, 5, 6, 7], electron bunches with sharp spikes have been produced in the framework of a nonlinear bunch-compression scheme. Experimental [4, 5, 6, 7] and theoretical [10, 11] studies of FLASH operation have shown that properties of ultra-short pulses are significantly influenced by collective effects, the most important of them being space-charge effects. Space-charge plays an important role in the beam-formation system, in the drift space and also in a long undulator. Collective effects might be crucially important for X-ray SASE FELs (XFELs) as well.

This article presents a description of longitudinal wake fields in XFELs. In particular, our study is of importance in connection with novel schemes of radiation production, like Enhanced SASE schemes (ESASE) [12, 13, 14]. ESASE proposals rely on two steps. First, the electron beam is modulated in energy by interacting with a GW-level optical laser in a modulator wiggler placed in the accelerator section. Second, a dispersive section transforms the energy modulation into density modulation, eventually leading to a subfemtosecond-long spike in the beam current before the entrance in the FEL undulator. The peak current of this spike can reach tens of kA without emittance worsening, because only a small charge is concentrated in the high-current region. As the electron beam undergoes the SASE process, the enhanced current part should saturate faster than the rest of the bunch. Alternatively, the x-ray wavelength may be reduced, for a fixed undulator length. Moreover ultra-short pulses (in the attosecond range) are produced as a result of the presence of the short lasing spike. Faster saturation of emission from the enhanced-current spike also suggests that ESASE schemes may be used to obtain saturation even in situations when beam parameters deteriorate with respect to design values.

A detailed study of longitudinal wake fields arising after the dispersive section, in particular dominant space-charge wake fields is due in order to

[1] Free-Electron Laser in Hamburg.

assess the magnitude of detrimental effects on the FEL process. It is important to note that the undulator parameter K for XFEL setups obeys $K^2 \gg 1$. As a result, the average longitudinal Lorentz factor $\bar{\gamma}_z = \gamma/\sqrt{1+K^2/2}$ is such that $\bar{\gamma}_z^2 \ll \gamma^2$, γ being the Lorentz factor of the beam. Based on $\bar{\gamma}_z^2 \ll \gamma^2$, we will demonstrate that the presence of the undulator strongly influences the space-charge wake. In contrast to this, in [13, 14], wake calculations for the LCLS case are given in free-space, as if the presence of the undulator were negligible. Authors of references [13, 14] incorrectly conclude that the FEL process is basically unaffected by space-charge wakes.

This paper is devoted to the calculation of impedance and longitudinal wake field in XFELs, with particular attention to the LCLS case, for which ESASE schemes have been first proposed. This means that we will restrict our attention to a very specific region of parameters, discussed in the next Section 2. First, the longitudinal size of the beam is much larger than the FEL wavelength. Second, electrons are assumed to have travelled into the undulator for a distance longer than the overtaking length. Third, effects of metallic surroundings can be neglected. When the electron-beam size is larger than the radiation diffraction size calculated from a single undulator period [2], major simplifications arise. In fact, radiation from the undulator is drastically suppressed and calculations of impedance and wake function can be performed considering a non-radiating beam, and thus accounting for space-charge interactions only. Then, space-charge impedance and wake function is found to reproduce the free-space case. Only, the Lorentz factor γ must be consistently substituted with the average longitudinal Lorentz factor $\bar{\gamma}_z$. In Section 3 we derive the electric field that will be used to calculate impedance and wake. In Section 4 we introduce concepts of impedance and wake field. Fields are calculated in Section 3, while impedances and wakes are respectively dealt with in Section 5 and Section 6. Then, in Section 7, we apply our theory to the ESASE setup referring to the LCLS facility. We calculate the energy chirp associated with wakes inside the undulator and between dispersive section and undulator. Subsequently, the magnitude of their effect is estimated by calculating the linear energy chirp parameter [15, 16]. We find that the gain of the FEL process is sensibly reduced, and that longitudinal wake fields constitute a reason of concern regarding the practical realization of ESASE schemes. Conclusions end our treatment in Section 8.

[2] Of order $\sqrt{\lambda\lambda_w}$, λ being the reduced wavelength of coherent radiation and λ_w the undulator period.

2 Parameter-space of the problem

As has been said in the Introduction, results of this paper can be applied to calculate effects of wake fields in planar [3] undulators under a specific choice of parameters corresponding to an XFEL system. Quantities of interest are defined once the bunch and the undulator system are specified. The bunch is characterized by an rms length σ_z, a transverse rms dimension $\sigma_\perp$ and Lorentz factor γ. Moreover, we define the undulator period λ_w, the vacuum chamber transverse dimension a and the undulator parameter K, where $K = \lambda_w e H_w/(2\pi m_e c^2)$, $(-e)$ being the negative electron charge, H_w the peak undulator magnetic field on-axis, and m_e the rest mass of the electron. Finally, L_s is the saturation length of the FEL process.

The bunch length σ_z corresponds to the reduced wavelength of the coherent field generated by the bunch, $\sigma_z \simeq \lambda\!\!\!^{-} \equiv \lambda/(2\pi)$. This (reduced) wavelength is much longer than the reduced resonant wavelength $\lambda\!\!\!^{-}_r \simeq \lambda\!\!\!^{-}_w/(2\bar{\gamma}_z^2)$, where $\lambda\!\!\!^{-}_w = \lambda_w/(2\pi)$, and $\bar{\gamma}_z = \gamma/\sqrt{1+K^2/2}$ is the already defined average longitudinal Lorentz factor. This means $\lambda\!\!\!^{-} \gg \lambda\!\!\!^{-}_r$.

The overtaking length is defined by the quantity $2\bar{\gamma}_z^2\lambda\!\!\!^{-}$. When the bunch has travelled inside the undulator for more than $2\bar{\gamma}_z^2\lambda\!\!\!^{-}$ a steady state is reached, and asymptotic expressions for the wake fields can be given. In the present study we will work with such asymptotic expressions only. This means that the saturation length of the FEL process, L_s, must be much longer than the overtaking length, i.e. $L_s \gg 2\bar{\gamma}_z^2\lambda\!\!\!^{-}$.

Also, in this paper we will neglect the presence of the vacuum chamber. This is possible when the vacuum chamber dimension is much larger than $\bar{\gamma}_z\lambda\!\!\!^{-}$, i.e. $a \gg \bar{\gamma}_z\lambda\!\!\!^{-}$, a typical transverse dimension associated with the coherent field, that is verified for ESASE XFEL setups.

Summing up, we will work under the following constraints:

$$\begin{aligned} \lambda\!\!\!^{-} &\gg \lambda\!\!\!^{-}_r \quad , \\ L_s &\gg 2\bar{\gamma}_z^2\lambda\!\!\!^{-} \, , \\ a &\gg \bar{\gamma}_z\lambda\!\!\!^{-} \quad . \end{aligned} \tag{1}$$

Based on conditions in (1), we will develop a theory of wake fields from undulators in XFELs. In particular, the first assumption greatly simplifies

[3] Although we presented final expressions of our theory in the case of a planar undulator, there are no specific effects related with the choice of a planar undulator. Our work may be straightforwardly extended to the case of a helical undulator as well.

our consideration allowing for a long-wavelength asymptotic treatment. Under the second and the third assumption we will be able to present an expression for the impedance in terms of a double convolution involving the charge density distribution and Bessel functions. Similarly, an analytical expression for the wake function could be given. However, it will not be necessary to explicitly calculate this expression. In fact, when discussing practical applications, we will work in the asymptotic case

$$\sigma_\perp^2 \gg \lambda\lambda_w \,. \tag{2}$$

Extra-condition (2) greatly simplifies the treatment of wake fields. Our results can be directly applied to realistic situations as the ESASE scheme analyzed in Section 7, where we will refer, explicitly, to the LCLS case.

3 Field calculation

Calculation of longitudinal wake field and impedance from an FEL undulator is subject to the characterization of the electric field generated at a given position (that is the position of a test electron) by the entire bunch.

We perform an analysis in terms of harmonics, i.e. we consider sinusoidal dependence of the electric field of the kind $\vec{E} = \vec{\tilde{E}}(\vec{r},\omega)\exp[-i\omega t] + C.C.$, the symbol "C.C." indicating complex conjugation [4]. Here t is the time, $\omega = 2\pi c/\lambda$ is the frequency, with c the speed of light in vacuum. The complex amplitude $\vec{\tilde{E}}(\vec{r},\omega)$ can actually be considered as the representation of the electric field in the space-frequency domain, and it will be referred to as "the field".

We assume that particles proceed along an undulator, under the constraints discussed in Section 2.

3.1 Transverse field

The transverse field $\vec{\tilde{E}}_\perp$ can be treated in terms of Paraxial Maxwell's equations in the space-frequency domain (see e.g. [17, 18]). From the paraxial

[4] In the following, for simplicity, we will consider $\omega > 0$. Expressions for the field at negative values of ω can be obtained based on the property $\vec{\tilde{E}}(-\omega) = \vec{\tilde{E}}^*(\omega)$ starting from explicit expressions for $\vec{\tilde{E}}$ at $\omega > 0$.

approximation follows that the electric field envelope $\vec{\vec{E}}_\perp = \vec{\vec{E}}_\perp \exp[-i\omega z/c]$ does not vary much along z on the scale of the reduced wavelength $\lambda\!\!\!^{-} = \lambda/(2\pi)$. As a result, the following field equation holds:

$$\mathcal{D}\left[\vec{\vec{E}}_\perp(z, \vec{r}_\perp, \omega)\right] = \vec{g}(z, \vec{r}_\perp, \omega) \,, \tag{3}$$

where the differential operator $\mathcal{D}$ is defined by

$$\mathcal{D} \equiv \left({\nabla_\perp}^2 + \frac{2i\omega}{c}\frac{\partial}{\partial z}\right) , \tag{4}$$

${\nabla_\perp}^2$ being the Laplacian operator over transverse cartesian coordinates. Eq. (3) is Maxwell's equation in paraxial approximation. The source-term vector $\vec{g}(z, \vec{r}_\perp)$ is specified by the trajectory of the source electrons, and can be written in terms of the Fourier transform of the transverse current density, $\vec{\vec{j}}_\perp(z, \vec{r}_\perp, \omega)$, and of the charge density, $\bar{\rho}(z, \vec{r}_\perp, \omega)$, as

$$\vec{g} = -4\pi \exp\left[-\frac{i\omega z}{c}\right]\left(\frac{i\omega}{c^2}\vec{\vec{j}}_\perp - \vec{\nabla}_\perp \bar{\rho}\right) . \tag{5}$$

$\vec{\vec{j}}_\perp$ and $\bar{\rho}$ are regarded as given data. In this paper we will treat $\vec{\vec{j}}_\perp$ and $\bar{\rho}$ as macroscopic quantities, without investigating individual electron contributions. We consider transverse and longitudinal distribution densities of the current constant through the undulator. In the time domain, we may write the charge density $\rho(\vec{r}, t)$ and the current density $\vec{j}(\vec{r}, t)$ as

$$\rho(\vec{r}, t) = \frac{1}{v_{oz}(z)}\rho_\perp(\vec{r}_\perp - \vec{r'}_{o\perp}(z)) f\left(t - \frac{s_o(z)}{v_o}\right) \tag{6}$$

and

$$\vec{j}(\vec{r}, t) = \frac{1}{v_{oz}(z)}\vec{v}(z)\rho_\perp(\vec{r}_\perp - \vec{r'}_{o\perp}(z)) f\left(t - \frac{s_o(z)}{v_o}\right) . \tag{7}$$

The quantity $\rho_\perp$ has the meaning of transverse electron beam distribution, while f is the longitudinal charge density distribution. $\vec{r'}_{o\perp}(z)$, $s_o(z)$ and v_o pertain a reference electron with Lorentz factor γ that is injected on axis with no deflection and is guided by the undulator field only. Such electron follows a trajectory specified by $\vec{r'}_{o\perp}(z) = r'_{ox}\vec{e}_x + r'_{oy}\vec{e}_y$, $\vec{e}_x$ and $\vec{e}_y$ being the unit vectors in the horizontal and vertical directions respectively, with

$$r'_{ox}(z) = \frac{K}{\gamma k_w}\cos(k_w z) = r_w \cos(k_w z) \ , \quad r'_{oy}(z) = 0 \ , \tag{8}$$

where we defined the transverse amplitude of oscillations $r_w = K/(\gamma k_w)$. The corresponding velocity is indicated with $\vec{v}_{o\perp}(z) = v_{ox}\vec{e}_x + v_{oy}\vec{e}_y$:

$$v_{ox}(z) = -\frac{Kc}{\gamma}\sin(k_w z) \ , \quad v_{oy}(z) = 0 \ . \tag{9}$$

Finally, $s_o(z)$ is the curvilinear abscissa measured along the trajectory of the reference particle.

Note that, according to Eq. (6) and Eq. (7), $\vec{j} = \vec{v}_o\rho$. In fact, for each particle in the beam $\delta\gamma/\gamma \ll 1$. Therefore we can neglect differences between the average transverse velocity of electrons $\langle\vec{v}\rangle$ and $\vec{v}_o$.

In the space-frequency domain, Eq. (6) and Eq. (7) transform to:

$$\bar{\rho}(\vec{r}_\perp, z, \omega) = \rho_o\left(\vec{r}_\perp - \vec{r'}_{o\perp}(z)\right)\bar{f}(\omega)\exp\left[i\omega s_o(z)/v_o\right] \tag{10}$$

and

$$\vec{\bar{j}}(\vec{r}_\perp, z, \omega) = \vec{v}_o(z)\rho_o\left(\vec{r}_\perp - \vec{r'}_{o\perp}(z)\right)\bar{f}(\omega)\exp\left[i\omega s_o(z)/v_o\right] \ , \tag{11}$$

where, for simplicity, we introduced the symbol

$$\rho_o(\vec{r}_\perp) = \frac{1}{v_{oz}(z)}\rho_\perp(\vec{r}_\perp) \ . \tag{12}$$

It should be remarked that $\bar{\rho}$ and $\vec{\bar{j}} = \bar{\rho}\vec{v}_o$ satisfy the continuity equation. In other words, one can find $\vec{\nabla}\cdot\vec{\bar{j}} = i\omega\bar{\rho}$.

We note that for a generic motion one has

$$\omega\left(\frac{s_o(z_2) - s_o(z_1)}{v} - \frac{z_2 - z_1}{c}\right) = \int_{z_1}^{z_2} d\bar{z}\frac{\omega}{2\gamma_z^2(\bar{z})c} \ , \tag{13}$$

Also,

$$\int_0^z \frac{\omega}{2c\gamma_z^2(\bar{z})}d\bar{z} = \frac{\omega}{2c\bar{\gamma}_z^2}z - \frac{\omega K^2}{8\gamma^2 k_w c}\sin(2k_w z) \simeq \frac{\omega}{2c\bar{\gamma}_z^2}z \ , \tag{14}$$

where the average longitudinal Lorentz factor $\bar{\gamma}_z$ is defined as

$$\bar{\gamma}_z = \frac{\gamma}{\sqrt{1 + K^2/2}} \,. \tag{15}$$

The approximate equality in Eq. (14) follows from the fact that we are interested in wavelengths $\omega \ll \omega_r$, where the fundamental $\omega_r = 2k_w c\bar{\gamma}_z^2$ is fixed imposing resonance condition between electric field and reference particle. The term in $\sin(2k_w z')$ is of order ω/ω_r, that is our accuracy, and can be neglected everywhere.

With the help of Eq. (10) and Eq. (11), Eq. (5) can be presented as (see also [19]):

$$\vec{g} = -4\pi \exp\left[i \int_0^z d\bar{z} \frac{\omega}{2\bar{\gamma}_z^2 c} \right] \left[\frac{i\omega}{c^2} \vec{v}_{o\perp}(z) - \vec{\nabla}_\perp \right] \rho_o \left(\vec{r}_\perp - \vec{r'}_{o\perp}(z) \right) \bar{f}(\omega) \,. \tag{16}$$

We find an exact solution of Eq. (4) without any other assumption about the parameters of the problem. A Green's function for Eq. (4), namely the solution corresponding to the unit point source can be written as (see [17]):

$$G(z - z'; \vec{r_\perp} - \vec{r'_\perp}) = -\frac{1}{4\pi(z - z')} \exp\left\{ i\omega \frac{|\, \vec{r_\perp} - \vec{r'_\perp}\,|^2}{2c(z - z')} \right\} \,, \tag{17}$$

assuming $z - z' > 0$. When $z - z' < 0$ the paraxial approximation does not hold, and the paraxial wave equation Eq. (3) should be substituted, in the space-frequency domain, by a more general Helmholtz equation. Yet, the radiation formation length for $z - z' < 0$ is very short with respect to the case $z - z' > 0$, i.e. we can neglect contributions from sources located at $z - z' < 0$.

Thus, after integration by parts, we obtain the solution

$$\begin{aligned} \vec{\bar{E}}_\perp(z, \vec{r}_\perp) = \frac{i\omega}{c} \int_0^z dz' \int d\vec{r'}_\perp \exp\left\{ i\omega \left[\frac{|\, \vec{r}_\perp - \vec{r'}_\perp\,|^2}{2c(z - z')} \right] + i \int_0^{z'} d\bar{z} \frac{\omega}{2c\gamma_z^2(\bar{z})} \right\} \\ \times \frac{1}{z - z'} \rho_o \left(\vec{r'}_\perp - \vec{r'}_{\perp o}(z') \right) \bar{f}(\omega) \left(\frac{\vec{v}_\perp(z')}{c} - \frac{\vec{r}_\perp - \vec{r'}_\perp}{z - z'} \right) \,. \end{aligned} \tag{18}$$

Eq. (18) describes the field at any position z. Note that ρ_o depends on the

difference $\vec{r'}_\perp - \vec{r'}_{\perp o}(z')$. This dependence is important concerning the effect studied in this paper, as it will be seen later on.

Eq. (18) consists of two terms: one in $\vec{v}_\perp$ and the other in $\vec{r}_\perp - \vec{r'}_\perp$. We will sometimes name the first term the "current term" $\vec{\widetilde{E}}_{\perp c}$, while the second will be indicated as the gradient term $\vec{\widetilde{E}}_{\perp g}$. With the help of Eq. (8), Eq. (9) and Eq. (14) we can re-write Eq. (18) as

$$\begin{aligned}\vec{\widetilde{E}}_\perp(z,\vec{r}_\perp) = -\frac{i\omega}{c}\int_0^z dz' \frac{1}{z-z'}\int d\vec{r'}_\perp \left(\frac{K\vec{e}_x}{\gamma}\sin(k_w z') + \frac{\vec{r}_\perp - \vec{r'}_\perp}{z-z'}\right) \\ \times\rho_o\left(\vec{r'}_\perp - r_w\cos(k_w z')\vec{e}_x\right)\bar{f}(\omega)\exp\left\{i\omega\left[\frac{|\,\vec{r}_\perp - \vec{r'}_\perp\,|^2}{2c(z-z')}\right] + \frac{i\omega z'}{2c\bar{\gamma}_z^2}\right\}\,.\end{aligned} \tag{19}$$

3.2 Longitudinal field

A similar expression can be found for the longitudinal field. Since $v_z(z) \simeq c$, we can write the longitudinal equivalent of Eq. (5) as

$$g_z = -4\pi\exp\left[-\frac{i\omega z}{c}\right]\left(\frac{i\omega}{c^2}\bar{j}_z - \partial_z\bar{\rho}\right)\,, \tag{20}$$

that is

$$g_z = -4\pi\exp\left[i\int_0^z d\bar{z}\frac{\omega}{2\bar{\gamma}_z^2 c}\right]\left[-\frac{i\omega}{\bar{\gamma}_z^2 c} - \frac{\partial}{\partial z}\right]\rho_o\left(\vec{r}_\perp - \vec{r'}_{o\perp}(z)\right)\bar{f}(\omega)\,, \tag{21}$$

having used the fact that $v_{oz}(z) \simeq c$.

It follows that the longitudinal component of the field, can be written analogously to Eq. (18) as

$$\begin{aligned}\widetilde{E}_z(z,\vec{r}_\perp) = \int_0^z dz'\int d\vec{r'}_\perp \exp\left\{i\omega\left[\frac{|\,\vec{r}_\perp - \vec{r'}_\perp\,|^2}{2c(z-z')}\right] + i\int_0^{z'} d\bar{z}\frac{\omega}{2c\gamma_z^2(\bar{z})}\right\} \\ \times\frac{1}{z-z'}\left[-\frac{i\omega}{\bar{\gamma}_z^2 c} - \frac{\partial}{\partial z'}\right]\rho_o\left(\vec{r'}_\perp - \vec{r'}_{\perp o}(z')\right)\bar{f}(\omega)\,.\end{aligned}$$

(22)

Note that the integral in Eq. (22) is performed for z' ranging from 0 up to z exactly as the integral in Eq. (19), for the same reasons.

Use of Eq. (8), Eq. (9) and Eq. (14) allow to re-write Eq. (22) as

$$\widetilde{E}_z(z,\vec{r}_\perp) = \int_0^z dz' \frac{1}{z-z'} \int d\vec{r'}_\perp \exp\left\{ i\omega \left[\frac{|\,\vec{r}_\perp - \vec{r'}_\perp\,|^2}{2c(z-z')} \right] + \frac{i\omega z'}{2c\bar{\gamma}_z^2} \right\}$$
$$\times \left[-\frac{i\omega}{\bar{\gamma}_z^2 c} - \frac{K}{\gamma} \sin(k_w z') \frac{\partial}{\partial[x' - r'_{ox}(z')]} \right] \rho_o\left(\vec{r'}_\perp - r_w \cos(k_w z')\vec{e}_x\right) \bar{f}(\omega)\ . \tag{23}$$

Finally, integration by parts of the term in $\partial/\{\partial[x' - r'_{ox}(z')]\}$ gives

$$\widetilde{E}_z(z,\vec{r}_\perp) = -\frac{i\omega}{c} \int_0^z dz' \frac{1}{z-z'} \int d\vec{r'}_\perp \left[\frac{1}{\bar{\gamma}_z^2} + \frac{K}{\gamma} \sin(k_w z') \frac{x-x'}{z-z'} \right]$$
$$\times \rho_o\left(\vec{r'}_\perp - r_w \cos(k_w z')\vec{e}_x\right) \bar{f}(\omega) \exp\left\{ i\omega \left[\frac{|\,\vec{r}_\perp - \vec{r'}_\perp\,|^2}{2c(z-z')} \right] + \frac{i\omega z'}{2c\bar{\gamma}_z^2} \right\}\ . \tag{24}$$

Note that, in contrast with Eq. (19), we cannot clearly distinguish between "current" and "gradient" terms in Eq. (24): the term in $\sin(k_w z')$ can be traced back to the gradient of the charge density, but the one in $1/\bar{\gamma}_z^2$ is a combination between current and gradient term.

3.3 Perturbation theory

It is possible to analyze Eq. (19) and Eq. (24) in the framework of a perturbation theory, based on expansion in the small parameter $\lambda\!\!\!^{-}_r/\lambda\!\!\!^{-} \ll 1$ according to the first of conditions (1). This allows simplified treatment of impedance and wakes.

The first step towards this direction is a presentation of of $\vec{\widetilde{E}}_\perp$ and $\widetilde{E}_z$ with the help of the following expansions in plane waves:

$$\frac{1}{(z-z')} \exp\left\{ i\omega \left[\frac{|\,\vec{r}_\perp - \vec{r'}_\perp\,|^2}{2c(z-z')} \right] \right\} =$$
$$\frac{ic}{2\pi\omega} \int d\vec{k}_\perp \exp\left[-i\vec{k}_\perp \cdot \left(\vec{r}_\perp - \vec{r'}_\perp\right) \right] \exp\left[\frac{ik_\perp^2 c}{2\omega}(z'-z) \right] \tag{25}$$

and

$$\frac{\vec{r}_\perp - \vec{r'}_\perp}{(z-z')^2}\exp\left\{i\omega\left[\frac{|\,\vec{r}_\perp - \vec{r'}_\perp\,|^2}{2c(z-z')}\right]\right\} = -\frac{ic^2}{2\pi\omega^2}$$
$$\int d\vec{k}_\perp \vec{k}_\perp \exp\left[-i\vec{k}_\perp\cdot\left(\vec{r}_\perp - \vec{r'}_\perp\right)\right]\exp\left[\frac{ik_\perp^2 c}{2\omega}(z'-z)\right]. \quad (26)$$

We obtain

$$\vec{\widetilde{E}}_\perp(z,\vec{r}_\perp) = \frac{\bar{f}(\omega)}{2\pi}\int d\vec{k}_\perp \int_0^z dz' \int d\vec{r'}_\perp \left[\frac{K\vec{e}_x}{\gamma}\sin(k_w z') - \frac{c\vec{k}_\perp}{\omega}\right]\exp\left[\frac{i\omega z'}{2c\bar{\gamma}_z^2}\right]$$
$$\times\rho_o\left(\vec{r'}_\perp - r_w\cos(k_w z')\vec{e}_x\right)\exp\left[-i\vec{k}_\perp\cdot\left(\vec{r}_\perp - \vec{r'}_\perp\right)\right]\exp\left[\frac{ik_\perp^2 c}{2\omega}(z'-z)\right] \quad (27)$$

and

$$\widetilde{E}_z(z,\vec{r}_\perp) = \frac{\bar{f}(\omega)}{2\pi}\int d\vec{k}_\perp \int_0^z dz' \int d\vec{r'}_\perp \left[\frac{1}{\bar{\gamma}_z^2} - \frac{Kck_x}{\gamma\omega}\sin(k_w z')\right]\exp\left[\frac{i\omega z'}{2c\bar{\gamma}_z^2}\right]$$
$$\times\rho_o\left(\vec{r'}_\perp - r_w\cos(k_w z')\vec{e}_x\right)\exp\left[-i\vec{k}_\perp\cdot\left(\vec{r}_\perp - \vec{r'}_\perp\right)\right]\exp\left[\frac{ik_\perp^2 c}{2\omega}(z'-z)\right]. \quad (28)$$

Performing a change of variables $\vec{r'}_\perp \longrightarrow \vec{r'}_\perp - r_w\cos(k_w z')\vec{e}_x$ and introducing notation $\vec{\theta} = \vec{k}_\perp c/\omega$ we re-write Eq. (27) and Eq. (28) as

$$\vec{\widetilde{E}}_\perp(z,\vec{r}_\perp) = \frac{\omega^2\bar{f}(\omega)}{2\pi c^2}\int d\vec{\theta}\int_0^z dz' \int d\vec{r'}_\perp \left[\frac{K\vec{e}_x}{\gamma}\sin(k_w z') - \vec{\theta}\right]\exp\left[\frac{i\omega z'}{2c\bar{\gamma}_z^2}\right]$$
$$\times\rho_o\left(\vec{r'}_\perp\right)\exp\left[-\frac{i\omega}{c}\vec{\theta}\cdot\left(\vec{r}_\perp - \vec{r'}_\perp - r_w\cos(k_w z')\vec{e}_x\right)\right]\exp\left[\frac{i\omega\theta^2}{2c}(z'-z)\right] \quad (29)$$

and

$$\widetilde{E}_z(z,\vec{r}_\perp) = \frac{\omega^2\bar{f}(\omega)}{2\pi c^2}\int d\vec{\theta}\int_0^z dz' \int d\vec{r'}_\perp \left[\frac{1}{\bar{\gamma}_z^2} - \frac{K\theta_x}{\gamma}\sin(k_w z')\right]\exp\left[\frac{i\omega z'}{2c\bar{\gamma}_z^2}\right]$$
$$\times\rho_o\left(\vec{r'}_\perp\right)\exp\left[-\frac{i\omega}{c}\vec{\theta}\cdot\left(\vec{r}_\perp - \vec{r'}_\perp - r_w\cos(k_w z')\vec{e}_x\right)\right]\exp\left[\frac{i\omega\theta^2}{2c}(z'-z)\right]. \quad (30)$$

Note that the maximal range of angles $\theta_{x,y}$ is limited by the last exponential function in Eq. (29) and Eq. (30) and by the fact that $z - z' \gtrsim \lambda\!\!\!^{-}_w$. It

follows that $\theta_{x,y}$ cannot be larger than about $\sqrt{\lambda\!\!\!^{-}/\lambda\!\!\!^{-}_w}$. Then, the trigonometric terms inside the exponential functions in both Eq. (29) and Eq. (30) are of magnitude $\omega\theta_x r_w/c \lesssim \sqrt{\lambda\!\!\!^{-}_r/\lambda\!\!\!^{-}} \ll 1$. It follows that we may expand $\exp\{i\omega r_w\theta_x \cos[k_w z']/c\} \simeq 1 + i\omega r_w\theta_x \cos[k_w z']/c$. Using exponential representation for all trigonometric functions we obtain:

$$\vec{\widetilde{E}}_\perp(z,\vec{r}_\perp) = \frac{\omega^2 \bar{f}(\omega)}{2\pi c^2}\int d\vec{\theta}\exp\left[-\frac{i\omega\theta^2 z}{2c}\right]\int_0^z dz'\int d\vec{r'}_\perp$$
$$\times\rho_o\left(\vec{r'}_\perp\right)\exp\left[-\frac{i\omega}{c}\vec{\theta}\cdot\left(\vec{r}_\perp-\vec{r'}_\perp\right)\right]\left\{\frac{K\vec{e}_x}{2i\gamma}\left[\exp(ik_w z')-\exp(-ik_w z')\right]-\vec{\theta}\right\}$$
$$\times\left\{1+\frac{i\omega\theta_x r_w}{2c}\left[\exp(ik_w z')+\exp(-ik_w z')\right]\right\}\exp\left[\frac{i\omega\theta^2 z'}{2c}\right]\exp\left[\frac{i\omega z'}{2c\bar{\gamma}_z^2}\right]. \quad (31)$$

and

$$\widetilde{E}_z(z,\vec{r}_\perp) = \frac{\omega^2 \bar{f}(\omega)}{2\pi c^2}\int d\vec{\theta}\exp\left[-\frac{i\omega\theta^2 z}{2c}\right]\int_0^z dz'\int d\vec{r'}_\perp$$
$$\times\rho_o\left(\vec{r'}_\perp\right)\exp\left[-\frac{i\omega}{c}\vec{\theta}\cdot\left(\vec{r}_\perp-\vec{r'}_\perp\right)\right]\left\{\frac{1}{\bar{\gamma}_z^2}-\frac{K\theta_x}{2i\gamma}\left[\exp(ik_w z')-\exp(-ik_w z')\right]\right\}$$
$$\times\left\{1+\frac{i\omega\theta_x r_w}{2c}\left[\exp(ik_w z')+\exp(-ik_w z')\right]\right\}\exp\left[\frac{i\omega\theta^2 z'}{2c}\right]\exp\left[\frac{i\omega z'}{2c\bar{\gamma}_z^2}\right]. \quad (32)$$

Eq. (31) and Eq. (32) have been found exploiting the small parameter $\lambda\!\!\!^{-}_r/\lambda\!\!\!^{-}$. In both Eq. (31) and Eq. (32) products of factors within {...} brackets are of the form $\exp[\pm ipk_w z']$, with $p = 0, 1, 2...$, terms for $p > 1$ being obtainable considering higher orders in $\sqrt{\lambda\!\!\!^{-}/\lambda\!\!\!^{-}_w}$ in the previous expansion of the exponential of trigonometric function. Note that when $p = 0$, the magnitude of $\theta_{x,y}$ can be estimated from the last two exponential functions in Eq. (31) and Eq. (32), giving characteristic a scale $\theta_{x,y} \sim 1/\bar{\gamma}_z$. For other values of p, instead, we have a characteristic scale $\theta_{x,y} \sim \sqrt{\lambda\!\!\!^{-}/\lambda\!\!\!^{-}_w}$.

Consider first Eq. (31). Magnitudes of factors within {...} brackets are $K/(2\gamma)$ and θ for the first bracket, 1 and $\omega\theta_x r_w/(2c)$ for the second bracket. Terms of the form $\exp[\pm ipk_w z']$ with $p = 0$ can have magnitudes $\theta \sim 1/\bar{\gamma}_z$ or $[K/(2\gamma)]\cdot[\omega\theta_x r_w/(2c)] \sim K^2\lambda\!\!\!^{-}_r\bar{\gamma}_z/(\gamma^2\lambda\!\!\!^{-})$, this last kind being negligible. When $p = 1$ terms have magnitudes $K/(2\gamma)$ or $\theta\omega\theta_x r_w/(2c) \sim K/\gamma$, and both kinds have to be kept. Similarly, it can be shown that all other values of p give negligible terms.

Consider now Eq. (32). Magnitudes of factors within {...} brackets are $1/\bar{\gamma}_z^2$ and $K\theta_x/(2\gamma)$ for the first bracket, 1 and $\omega\theta_x r_w/(2c)$ for the second bracket.

Terms of the form $\exp[\pm ipk_w z']$ with $p = 0$ can have magnitudes $1/\bar{\gamma}_z^2$ or $[K\theta_x/(2\gamma)] \cdot [\omega\theta_x r_w/(2c)] \sim K^2 \lambda\!\!\!^{-}_r/(\gamma^2 \lambda\!\!\!^{-})$, this last kind being negligible. When $p = 1$ terms have magnitudes $K\theta_x/2\gamma \sim K/\gamma \cdot \sqrt{\lambda\!\!\!^{-}/\lambda\!\!\!^{-}_w}$ or $\omega\theta_x r_w/(2c\bar{\gamma}_z^2) \sim \sqrt{\lambda\!\!\!^{-}_r/\lambda\!\!\!^{-}} \cdot 1/(\bar{\gamma}_z\gamma)$, and this last kind can be neglected. Similarly, it can be shown that all other values of p give negligible terms.

Altogether, we obtain the following expressions for transverse and longitudinal field:

$$\begin{aligned}
\vec{\widetilde{E}}_\perp(z,\vec{r}_\perp) &= \frac{\omega^2 \bar{f}(\omega)}{2\pi c^2} \int d\vec{\theta} \int_0^z dz' \int d\vec{r'}_\perp \\
&\times \rho_o\left(\vec{r'}_\perp\right) \exp\left[-\frac{i\omega}{c}\vec{\theta}\cdot\left(\vec{r}_\perp - \vec{r'}_\perp\right)\right] \exp\left[\frac{i\omega\theta^2(z'-z)}{2c}\right] \exp\left[\frac{i\omega z'}{2c\bar{\gamma}_z^2}\right] \\
&\times \left\{-\vec{\theta} + \left[\frac{K\vec{e}_x}{2i\gamma} - \frac{i\omega\theta_x r_w \vec{\theta}}{2c}\right]\exp(ik_w z') - \left[\frac{K\vec{e}_x}{2i\gamma} + \frac{i\omega\theta_x r_w \vec{\theta}}{2c}\right]\exp(-ik_w z')\right\}
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
\widetilde{E}_z(z,\vec{r}_\perp) &= \frac{\omega^2 \bar{f}(\omega)}{2\pi c^2} \int d\vec{\theta} \int_0^z dz' \int d\vec{r'}_\perp \rho_o\left(\vec{r'}_\perp\right) \exp\left[-\frac{i\omega}{c}\vec{\theta}\cdot\left(\vec{r}_\perp - \vec{r'}_\perp\right)\right] \\
&\times \exp\left[\frac{i\omega\theta^2(z'-z)}{2c}\right] \exp\left[\frac{i\omega z'}{2c\bar{\gamma}_z^2}\right] \left\{\frac{1}{\bar{\gamma}_z^2} - \frac{K\theta_x}{2\gamma i}\left[\exp(ik_w z') - \exp(-ik_w z')\right]\right\} .
\end{aligned} \tag{34}$$

Eq. (33) and Eq. (34) are the first order result of our perturbation theory, where the small parameter $\lambda\!\!\!^{-}_r/\lambda\!\!\!^{-}$ has been exploited through the expansion of exponential functions in Eq. (29) and Eq. (30), and non-negligible terms are kept.

We now go back to the space-frequency domain performing the integral in $d\vec{\theta}$ with the help of Eq. (25), Eq. (26) and using also

$$\begin{aligned}
&\int d\vec{\theta} \exp\left[-\frac{i\omega}{c}\vec{\theta}\cdot\left(\vec{r}_\perp - \vec{r'}_\perp\right)\right] \exp\left[\frac{i\omega\theta^2(z'-z)}{2c}\right] \theta_x^2 \\
&= \frac{2c\pi\left[c(z-z') + i\omega(x-x')^2\right]}{(z'-z)^3} \exp\left[\frac{i\omega\left|\vec{r}_\perp - \vec{r'}_\perp\right|^2}{2c(z-z')}\right] .
\end{aligned} \tag{35}$$

Finally, we obtain:

$$
\begin{aligned}
\vec{\widetilde{E}}_{\perp}(z,\vec{r}_{\perp}) &= -\frac{i\omega\bar{f}(\omega)}{c}\int d\vec{r'}_{\perp}\rho_o\left(\vec{r'}_{\perp}\right)\exp\left[\frac{i\omega z}{2c\bar{\gamma}_z^2}\right] \\
&\times\Bigg\{\exp[+ik_w z]\int_0^z \frac{dz'}{z-z'}\exp\left[i\omega\frac{|\,\vec{r}_{\perp}-\vec{r'}_{\perp}\,|^2}{2c(z-z')}\right]\left[+\frac{K\vec{e}_x}{2i\gamma}\exp[ik_w(z'-z)]\right] \\
&\quad+\exp[-ik_w z]\int_0^z \frac{dz'}{z-z'}\exp\left[i\omega\frac{|\,\vec{r}_{\perp}-\vec{r'}_{\perp}\,|^2}{2c(z-z')}\right]\left[-\frac{K\vec{e}_x}{2i\gamma}\exp[ik_w(z-z')]\right] \\
&\quad+\exp[+ik_w z]\int_0^z \frac{dz'}{z-z'}\exp\left[i\omega\frac{|\,\vec{r}_{\perp}-\vec{r'}_{\perp}\,|^2}{2c(z-z')}\right] \\
&\qquad\times\left[-\frac{r_w\vec{e}_x}{2(z-z')}-\frac{i\omega r_w(x-x')(\vec{r}_{\perp}-\vec{r'}_{\perp})}{2c(z-z')^2}\right]\exp[ik_w(z'-z)] \\
&\quad+\exp[-ik_w z]\int_0^z \frac{dz'}{z-z'}\exp\left[i\omega\frac{|\,\vec{r}_{\perp}-\vec{r'}_{\perp}\,|^2}{2c(z-z')}\right] \\
&\qquad\times\left[-\frac{r_w\vec{e}_x}{2(z-z')}-\frac{i\omega r_w(x-x')(\vec{r}_{\perp}-\vec{r'}_{\perp})}{2c(z-z')^2}\right]\exp[ik_w(z-z')] \\
&+\int_0^z \frac{dz'}{z-z'}\exp\left[i\omega\frac{|\,\vec{r}_{\perp}-\vec{r'}_{\perp}\,|^2}{2c(z-z')}\right]\exp\left[\frac{i\omega(z'-z)}{2c\bar{\gamma}_z^2}\right]\left[\frac{\vec{r}_{\perp}-\vec{r'}_{\perp}}{z-z'}\right]\Bigg\}
\end{aligned}
\tag{36}
$$

for the transverse field, and

$$
\begin{aligned}
\widetilde{E}_z(z,\vec{r}_{\perp}) &= -\frac{i\omega\bar{f}(\omega)}{c}\int d\vec{r'}_{\perp}\rho_o\left(\vec{r'}_{\perp}\right)\exp\left[\frac{i\omega z}{2c\bar{\gamma}_z^2}\right] \\
&\times\Bigg\{\exp[+ik_w z]\int_0^z \frac{dz'}{z-z'}\exp\left[i\omega\frac{|\,\vec{r}_{\perp}-\vec{r'}_{\perp}\,|^2}{2c(z-z')}\right]\left[+\frac{K}{2i\gamma}\frac{x-x'}{z-z'}\exp[ik_w(z'-z)]\right] \\
&+\exp[-ik_w z]\int_0^z \frac{dz'}{z-z'}\exp\left[i\omega\frac{|\,\vec{r}_{\perp}-\vec{r'}_{\perp}\,|^2}{2c(z-z')}\right]\left[-\frac{K}{2i\gamma}\frac{x-x'}{z-z'}\exp[-ik_w(z'-z)]\right] \\
&+\int_0^z \frac{dz'}{z-z'}\exp\left[i\omega\frac{|\,\vec{r}_{\perp}-\vec{r'}_{\perp}\,|^2}{2c(z-z')}\right]\exp\left[\frac{i\omega(z'-z)}{2c\bar{\gamma}_z^2}\right]\frac{1}{\bar{\gamma}_z^2}\Bigg\}
\end{aligned}
\tag{37}
$$

for the longitudinal field. Here we neglected factors $\exp[i\omega(z'-z)/(2c\bar{\gamma}_z^2)]$ in integral terms in dz' including $\exp[\pm k_w z]$, because $\omega\lambda_w/(2c\bar{\gamma}_z^2) \ll 1$.

Note that there exists a mathematical shortcut to obtain Eq. (36) and Eq. (37) from Eq. (19) and Eq. (24). In fact, if we perform a change of variables $\vec{r'}_{\perp} \longrightarrow \vec{r'}_{\perp} - r_w\cos(k_w z')\vec{e}_x$, we formally expand the Green's function exponential

$\exp\{i\omega[|\vec{r}_\perp - \vec{r'}_\perp - r_w \cos(k_w z')\vec{e}_x|^2/2c(z-z')]\}$ to the first order in r_w and keep non-negligible first-harmonic terms in $\exp[\pm ik_w z']$, we obtain Eq. (36) and Eq. (37). We will regard it as a mnemonic rule, that will be useful later on.

Fields are calculated under conditions (1). In the limit for $z \gg \bar{\gamma}_z^2 \lambda\!\!\!^{-}$, as we will see in the next Section 3.4, integrals in dz' depend on z only through phase factors, i.e. a steady state solution is reached.

Analysis of Eq. (36) and Eq. (37) presents an interesting picture of the fields generated by the electron beam. Eq. (36) and Eq. (37) consist of the sum of integrals in dz'. Some include exponential factors $\exp[\pm k_w z]$, other not.

Terms not including $\exp[\pm k_w z]$ (the last integrals in dz' in both Eq. (36) and Eq. (37)) oscillate, as a function of z, on a scale $\lambda\!\!\!^{-}\bar{\gamma}_z^2$. The field $\vec{\tilde{E}}$ is given by $\vec{\tilde{E}} \exp[i\omega z/c]$. It follows that the phase velocity of terms not including $\exp[\pm k_w z]$ is the same as that of the electron beam harmonic $\bar{\rho}$. We can interpret this fact by saying that this part of the field is entangled with the electron beam. It is natural to identify these terms as space-charge terms. The formation length of the space-charge field is determined by the factor $\exp[i\omega(z'-z)/(2c\bar{\gamma}_z^2)]$ under integral sign, and amounts to $2\lambda\!\!\!^{-}\bar{\gamma}_z^2$. Similarly, the diffraction size of the space-charge field is given by $\bar{\gamma}_z \lambda\!\!\!^{-}$.

Terms including $\exp[\pm ik_w z]$ are indicative of fields $\vec{\tilde{E}}$ performing a cycle of oscillation on the scale of an undulator period with respect to the electron-beam harmonic $\bar{\rho}$. Phase velocity of terms including $\exp[+ik_w z]$ is slower than that of the beam harmonic. These field terms have a phase velocity slower than the speed of light. Phase velocity of terms including $\exp[-ik_w z]$ is faster than that of the beam harmonic. These field terms have a phase velocity faster than the speed of light. We can interpret these facts by saying that these parts of the field are not entangled with the electron beam. It is natural to identify these terms as radiation terms. The formation length of radiation field terms is determined by the factor $\exp[\pm ik_w(z'-z)]$ under integral sign, and amounts to $\lambda\!\!\!^{-}_w$. Similarly, the diffraction size of the space-charge field is given by $\sqrt{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w}$.

It is interesting to trace each term in Eq. (36) and Eq. (37) back to the source terms that originated them, distinguishing between gradient and current terms. Considering Eq. (36) it can be seen that the first and the second integral are (radiative) current density terms. The third and the fourth term are (radiative) gradient terms, while the last term is a (space charge) gradient term. However, one can see from Eq. (37) that the third integral is a (space-charge) term originated from a mixture of gradient and current sources. Thus, although the first and the second integral are (radiative) gradient terms, it does not make sense to separately talk about gradient and current term for the longitudinal component of the field.

An interesting picture emerges, where radiation field and space-charge field are treated on equal foot, through paraxial Maxwell's equation. On the one hand, as we have seen, these fields have different formation lengths, and different diffraction sizes. On the other hand, our theory allows for generic transverse sizes of the electron beam $\sigma_\perp$, that makes it possible to compare $\sigma_\perp$ with both diffraction sizes, thus obtaining different regimes. As we will see, when $\sigma_\perp \gg \sqrt{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w}$, impedance and wakes are essentially dominated by the longitudinal space-charge term. It is important to remark, for future use in the next Sections, that $\bar{\gamma}_z$ enters the expression of the space-charge field, and not γ.

3.4 Explicit expressions for the field

We can consider Eq. (36) and Eq. (37) as starting point for our investigations, and calculate explicit expressions for the field to be used later on in the calculation of the impedance. First, we make a change in the integration variable from z' to $\xi \equiv z - z'$. In the limit for $z \longrightarrow \infty$, corresponding to the second of conditions (1), i.e. $z \gg \bar{\gamma}_z^2 \lambda\!\!\!^{-}$, we can write

$$
\begin{aligned}
\vec{\tilde{E}}_\perp(z,\vec{r}_\perp) = -\frac{i\omega \bar{f}(\omega)}{c} \int d\vec{r'}_\perp \rho_o\left(\vec{r'}_\perp\right) \exp\left[\frac{i\omega z}{2c\bar{\gamma}_z^2}\right] \\
\times\Bigg\{ + \frac{K\vec{e}_x}{2i\gamma} \exp[+ik_w z] \int_0^\infty \frac{d\xi}{\xi} \exp\left[+i\omega \frac{|\,\vec{r}_\perp - \vec{r'}_\perp\,|^2}{2c\xi} - ik_w\xi\right] \\
- \frac{K\vec{e}_x}{2i\gamma} \exp[-ik_w z] \int_0^\infty \frac{d\xi}{\xi} \exp\left[+i\omega \frac{|\,\vec{r}_\perp - \vec{r'}_\perp\,|^2}{2c\xi} + ik_w\xi\right] \\
+ \exp[+ik_w z]\Bigg[+ \frac{icr_w\vec{e}_x}{2\omega\,|\,\vec{r}_\perp - \vec{r'}_\perp\,|} \cdot \frac{d}{d\left[|\,\vec{r}_\perp - \vec{r'}_\perp\,|\right]} \\
+ \frac{2icr_w}{\omega}(x-x')(\vec{r}_\perp - \vec{r'}_\perp) \cdot \frac{d^2}{d\left[|\,\vec{r}_\perp - \vec{r'}_\perp\,|^2\right]^2}\Bigg] \\
\times \int_0^\infty \frac{d\xi}{\xi} \exp\left[+i\omega \frac{|\,\vec{r}_\perp - \vec{r'}_\perp\,|^2}{2c\xi} - ik_w\xi\right] \\
+ \exp[-ik_w z]\Bigg[+ \frac{icr_w\vec{e}_x}{2\omega\,|\,\vec{r}_\perp - \vec{r'}_\perp\,|} \cdot \frac{d}{d\left[|\,\vec{r}_\perp - \vec{r'}_\perp\,|\right]} \\
+ \frac{2icr_w}{\omega}(x-x')(\vec{r}_\perp - \vec{r'}_\perp) \cdot \frac{d^2}{d\left[|\,\vec{r}_\perp - \vec{r'}_\perp\,|^2\right]^2}\Bigg]
\end{aligned}
$$

$$\times \int_0^\infty \frac{d\xi}{\xi} \exp\left[+i\omega \frac{|\vec{r}_\perp - \vec{r'}_\perp|^2}{2c\xi} + ik_w\xi\right]$$
$$+ \left[\frac{ic}{\omega} \frac{\vec{r}_\perp - \vec{r'}_\perp}{|\vec{r}_\perp - \vec{r'}_\perp|} \cdot \frac{d}{d\left[|\vec{r}_\perp - \vec{r'}_\perp|\right]}\right]$$
$$\times \int_0^\infty \frac{d\xi}{\xi} \exp\left[+i\omega \frac{|\vec{r}_\perp - \vec{r'}_\perp|^2}{2c\xi} - \frac{i\omega\xi}{2c\bar{\gamma}_z^2}\right]\Bigg\} \quad (38)$$

for the transverse field and

$$\widetilde{E}_z(z, \vec{r}_\perp) = -\frac{i\omega \bar{f}(\omega)}{c} \int d\vec{r'}_\perp \rho_o\left(\vec{r'}_\perp\right) \exp\left[\frac{i\omega z}{2c\bar{\gamma}_z^2}\right]$$
$$\times\Bigg\{ -\exp[ik_w z]\left[\frac{cK}{2\omega\gamma} \frac{x - x'}{|\vec{r}_\perp - \vec{r'}_\perp|} \cdot \frac{d}{d\left[|\vec{r}_\perp - \vec{r'}_\perp|\right]}\right]$$
$$\times \int_0^\infty \frac{d\xi}{\xi} \exp\left[i\omega \frac{|\vec{r}_\perp - \vec{r'}_\perp|^2}{2c\xi} - ik_w\xi\right]$$
$$-\exp[-ik_w z]\left[\frac{cK}{2\omega\gamma} \frac{x - x'}{|\vec{r}_\perp - \vec{r'}_\perp|} \cdot \frac{d}{d\left[|\vec{r}_\perp - \vec{r'}_\perp|\right]}\right]$$
$$\times \int_0^\infty \frac{d\xi}{\xi} \exp\left[+i\omega \frac{|\vec{r}_\perp - \vec{r'}_\perp|^2}{2c\xi} + ik_w\xi\right]$$
$$+ \frac{1}{\bar{\gamma}_z^2} \int_0^\infty \frac{d\xi}{\xi} \exp\left[+i\omega \frac{|\vec{r}_\perp - \vec{r'}_\perp|^2}{2c\xi} - \frac{i\omega\xi}{2c\bar{\gamma}_z^2}\right]\Bigg\} \quad (39)$$

for the longitudinal field.

We now use the fact that, for any real number $\alpha > 0$:

$$\begin{cases} \int_0^\infty d\xi \exp\left[i\left(-\xi + \alpha/\xi\right)\right]/\xi = 2K_0\left(2\sqrt{\alpha}\right) \\ \int_0^\infty d\xi \exp\left[i\left(+\xi + \alpha/\xi\right)\right]/\xi = 2K_0\left(-2i\sqrt{\alpha}\right) = \pi\left[iJ_0(2\sqrt{\alpha}) - Y_0(2\sqrt{\alpha})\right], \end{cases} \quad (40)$$

where K_n is the n-th order modified Bessel function of the second kind, Y_n is n-th order Bessel function of the second kind and J_n is the n-th order Bessel function of the first kind. Using Eq. (40) and the fact that $k_w \gg \omega/(2c\bar{\gamma}_z^2)$, and remembering that

$$\frac{d^2}{d\left[\mid \vec{r}_\perp - \vec{r'}_\perp \mid^2\right]^2} =$$
$$\frac{1}{4 \mid \vec{r}_\perp - \vec{r'}_\perp \mid^2} \frac{d^2}{d\left[\mid \vec{r}_\perp - \vec{r'}_\perp \mid\right]^2} - \frac{1}{4 \mid \vec{r}_\perp - \vec{r'}_\perp \mid^3} \frac{d}{d\left[\mid \vec{r}_\perp - \vec{r'}_\perp \mid\right]} \tag{41}$$

we can write Eq. (38) and Eq. (39) as

$$\begin{aligned}
\vec{\tilde{E}}_\perp(z,\vec{r}_\perp) = & -\frac{i\omega \bar{f}(\omega)}{c} \int d\vec{r'}_\perp \rho_o\left(\vec{r'}_\perp\right) \exp\left[\frac{i\omega z}{2c\bar{\gamma}_z^2}\right] \\
\times & \Bigg\{ + \exp[+ik_w z] \frac{K\vec{e}_x}{i\gamma} K_0\left(\sqrt{2}\frac{\mid \vec{r}_\perp - \vec{r'}_\perp \mid}{\sqrt{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w}}\right) \\
& - \exp[-ik_w z] \frac{K\vec{e}_x}{i\gamma} K_0\left(-\sqrt{2}i\frac{\mid \vec{r}_\perp - \vec{r'}_\perp \mid}{\sqrt{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w}}\right) \\
& + \exp[+ik_w z] \frac{icr_w}{\omega}\Bigg[- \frac{\sqrt{2}\vec{e}_x}{\sqrt{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w} \mid \vec{r}_\perp - \vec{r'}_\perp \mid} K_1\left(\sqrt{2}\frac{\mid \vec{r}_\perp - \vec{r'}_\perp \mid}{\sqrt{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w}}\right) \\
& \qquad + \frac{2(x-x')(\vec{r}_\perp - \vec{r'}_\perp)}{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w \mid \vec{r}_\perp - \vec{r'}_\perp \mid^2} K_2\left(\sqrt{2}\frac{\mid \vec{r}_\perp - \vec{r'}_\perp \mid}{\sqrt{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w}}\right)\Bigg] \\
& - \exp[-ik_w z] \frac{icr_w}{\omega}\Bigg[- \frac{\sqrt{2}i\vec{e}_x}{\sqrt{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w} \mid \vec{r}_\perp - \vec{r'}_\perp \mid} K_1\left(-\sqrt{2}i\frac{\mid \vec{r}_\perp - \vec{r'}_\perp \mid}{\sqrt{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w}}\right) \\
& \qquad + \frac{2(x-x')(\vec{r}_\perp - \vec{r'}_\perp)}{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w \mid \vec{r}_\perp - \vec{r'}_\perp \mid^2} K_2\left(-\sqrt{2}i\frac{\mid \vec{r}_\perp - \vec{r'}_\perp \mid}{\sqrt{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w}}\right)\Bigg] \\
& - \left[\frac{ic}{\omega}\frac{\vec{r}_\perp - \vec{r'}_\perp}{\mid \vec{r}_\perp - \vec{r'}_\perp \mid}\right] \frac{2}{\bar{\gamma}_z \lambda\!\!\!^{-}} K_1\left(\frac{\mid \vec{r}_\perp - \vec{r'}_\perp \mid}{\bar{\gamma}_z \lambda\!\!\!^{-}}\right)\Bigg\}
\end{aligned} \tag{42}$$

and

$$\begin{aligned}
\widetilde{E}_z(z,\vec{r}_\perp) = & -\frac{i\omega \bar{f}(\omega)}{c} \int d\vec{r'}_\perp \rho_o\left(\vec{r'}_\perp\right) \exp\left[\frac{i\omega z}{2c\bar{\gamma}_z^2}\right] \\
& \times \Bigg\{ + \frac{\sqrt{2}}{\sqrt{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w}} \exp[+ik_w z]\left[\frac{cK}{\omega\gamma}\frac{x-x'}{\mid \vec{r}_\perp - \vec{r'}_\perp \mid}\right] K_1\left(\sqrt{2}\,\frac{\mid \vec{r}_\perp - \vec{r'}_\perp \mid}{\sqrt{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w}}\right) \\
& \quad + \frac{\sqrt{2}i}{\sqrt{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w}} \exp[-ik_w z]\left[\frac{cK}{\omega\gamma}\frac{x-x'}{\mid \vec{r}_\perp - \vec{r'}_\perp \mid}\right] K_1\left(\sqrt{2}i\frac{\mid \vec{r}_\perp - \vec{r'}_\perp \mid}{\sqrt{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w}}\right) \\
& \quad + \frac{2}{\bar{\gamma}_z^2} K_0\left(\frac{\mid \vec{r}_\perp - \vec{r'}_\perp \mid}{\bar{\gamma}_z \lambda\!\!\!^{-}}\right)\Bigg\}.
\end{aligned} \tag{43}$$

Note that, similarly as in Eq. (36) Eq. (37), it is possible to recognize in Eq. (42) and Eq. (43) radiative and space-charge terms, as well as gradient and current terms (for the transverse field components).

3.5 Cross-check with Gauss law

It is possible to cross-check our expressions for the field with the help of Gauss law:

$$\vec{\nabla} \cdot \vec{\bar{E}} = 4\pi\bar{\rho} \, . \tag{44}$$

This cross-check will constitute a general cross-check of the correctness of our calculation and allow a better understanding of the interplay of different field contributions in the complicated machinery of Maxwell's equation.

As said before, in the present study we work in the steady state, when the bunch has travelled inside the undulator for more than $2\bar{\gamma}_z^2\lambda\!\!\!^{-}$. In this case, an explicit expression for transverse and longitudinal fields are given in Eq. (42) and Eq. (43).

We will demonstrate that the field $\vec{\bar{E}} = \vec{\bar{E}}_{\rm rad} + \vec{\bar{E}}_{\rm sc}$ obeys Eq. (44) by separately showing that radiation field $\vec{\bar{E}}_{\rm rad}$ and space-charge field $\vec{\bar{E}}_{\rm sc}$ verify:

$$\begin{aligned} &\vec{\nabla} \cdot \vec{\bar{E}}_{\rm sc} = 4\pi\bar{\rho} \\ &\vec{\nabla} \cdot \vec{\bar{E}}_{\rm rad} = 0 \, . \end{aligned} \tag{45}$$

Relations (45) can be interpreted saying that the radiation field is not entangled with sources, while the space-charge field is. Hence the different right hand sides.

Let us begin with the space-charge field. First we can write:

$$\frac{\partial \bar{E}_{z\ {\rm sc}}}{\partial z} = \frac{\partial}{\partial z}\left\{\widetilde{E}_{z\ {\rm sc}} \exp[i\omega z/c]\right\} = \frac{i\omega}{c}\left(1 + \frac{1}{2\bar{\gamma}_z}^2\right)\bar{E}_{z{\rm sc}} \simeq \frac{i\omega}{c}\bar{E}_{z{\rm sc}} \, , \tag{46}$$

because, in the steady state, $\widetilde{E}_{z\ {\rm sc}}$ depends on z only through $\exp[i\omega z/(2c\bar{\gamma}_z^2]$.

Thus, in order to verify the first of Eq. (45) we should prove that

$$\vec{\nabla}_\perp \cdot \vec{\bar{E}}_{\perp{\rm sc}} = 4\pi\bar{\rho} - \frac{i\omega}{c}\bar{E}_{z{\rm sc}} \, . \tag{47}$$

From Eq. (42) we have

$$\vec{\vec{E}}_{\perp \mathrm{sc}} = -\frac{2\bar{f}(\omega)}{\bar{\gamma}_z \lambda\!\!\!^{-}} \exp\left[\frac{i\omega z}{2c\bar{\gamma}_z^2}\right] \int d\vec{r'}_\perp \rho_o\left(\vec{r'}_\perp\right) \frac{\vec{r}_\perp - \vec{r'}_\perp}{|\,\vec{r}_\perp - \vec{r'}_\perp\,|} K_1\left(\frac{|\,\vec{r}_\perp - \vec{r'}_\perp\,|}{\bar{\gamma}_z \lambda\!\!\!^{-}}\right) . \tag{48}$$

In order to calculate the left hand side of Eq. (47) we can use the divergence theorem in two dimensions, to find:

$$\vec{\nabla}_\perp \cdot \left\{ \frac{\vec{R}_\perp}{\left|\vec{R}_\perp\right|} K_1\left(\frac{\omega\left|\vec{R}_\perp\right|}{c\bar{\gamma}_z}\right)\right\} = -\frac{\omega}{c\bar{\gamma}_z} K_o\left(\frac{\omega\left|\vec{R}_\perp\right|}{c\bar{\gamma}_z}\right) + \frac{2\pi c\bar{\gamma}_z}{\omega}\delta\left(\vec{R}_\perp\right) , \tag{49}$$

where δ indicates the Dirac-delta function and derivation is understood in weak sense, we set $\vec{R}_\perp = \vec{r}_\perp - \vec{r'}_\perp$. Remembering $\vec{\vec{E}}_\perp = \vec{\bar{E}}_\perp \exp[-i\omega z/c]$ we obtain

$$\begin{aligned}\vec{\nabla}_\perp \cdot \vec{\bar{E}}_{\perp \mathrm{sc}} = &\left[-\frac{2\omega^2}{c^2\bar{\gamma}_z^2}\int d\vec{r'}_\perp \rho_o(\vec{r'}_\perp) K_o\left(\frac{\left|\vec{r}_\perp - \vec{r'}_\perp\right|}{\lambda\!\!\!^{-}\bar{\gamma}_z}\right) + 4\pi\rho_o(\vec{r}_\perp)\right] \\ &\times \bar{f}(\omega)\exp\left[\frac{i\omega z}{v_z}\right] .\end{aligned} \tag{50}$$

Now, from Eq. (43) we have:

$$\bar{E}_{z\ \mathrm{sc}} = -\frac{2i\omega}{c\bar{\gamma}_z^2}\int d\vec{r'}_\perp \rho_o\left(\vec{r'}_\perp\right)\exp\left[\frac{i\omega z}{2c\bar{\gamma}_z^2}\right] K_0\left(\frac{|\,\vec{r}_\perp - \vec{r'}_\perp\,|}{\lambda\!\!\!^{-}\bar{\gamma}_z}\right)\bar{f}(\omega)\exp\left[\frac{i\omega z}{v_z}\right] . \tag{51}$$

Substitution in the right hand side of Eq. (47) yields Eq. (50), thus verifying Eq. (47).

Let us now consider the radiative fields, and show that the second of Eq. (45) is also verified. Presentations in Eq. (42) and Eq. (43) include now many terms, and it is convenient to start with alternative presentations for the transverse and longitudinal field, namely Eq. (33) and Eq. (34). In the limit for $z \gg \bar{\gamma}_z^2 \lambda\!\!\!^{-}$ The radiative part of the field is given by:

$$\begin{aligned}\vec{\bar{E}}_{\perp \mathrm{rad}}(z,\vec{r}_\perp) = &\exp\left[\frac{i\omega z}{c}\right]\frac{\omega^2\bar{f}(\omega)}{2\pi c^2}\int d\vec{\theta}\int_0^\infty dz' \int d\vec{r'}_\perp \rho_o\left(\vec{r'}_\perp\right) \\ &\times\exp\left[-\frac{i\omega}{c}\vec{\theta}\cdot\left(\vec{r}_\perp - \vec{r'}_\perp\right)\right]\exp\left[\frac{i\omega\theta^2(z'-z)}{2c}\right]\end{aligned}$$

$$\times \left\{ \frac{K\vec{e}_x}{\gamma} \sin[k_w z'] - \frac{i\omega\theta_x r_w \vec{\theta}}{c} \cos[k_w z'] \right\} \tag{52}$$

and

$$\bar{E}_{z\ \mathrm{rad}}(z,\vec{r}_\perp) = -\frac{\omega^2 \bar{f}(\omega) K}{2\pi c^2 \gamma} \exp\left[\frac{i\omega z}{c}\right] \int d\vec{\theta} \int_0^\infty dz' \int d\vec{r'}_\perp \rho_o\left(\vec{r'}_\perp\right)$$
$$\times \exp\left[-\frac{i\omega}{c}\vec{\theta}\cdot\left(\vec{r}_\perp - \vec{r'}_\perp\right)\right] \exp\left[\frac{i\omega\theta^2(z'-z)}{2c}\right] \theta_x \sin[k_w z'] \,. \tag{53}$$

Here we neglected factors $\exp[i\omega z'/(2c\bar{\gamma}_z^2)]$ because $\omega\lambda_w/(2c\bar{\gamma}_z^2) \ll 1$. We may now calculate directly $\vec{\nabla}_\perp \cdot \bar{E}_{\perp\mathrm{rad}}$ and $\partial_z \bar{E}_{z\ \mathrm{rad}}$. We obtain

$$\vec{\nabla}_\perp \cdot \vec{\bar{E}}_{\perp\mathrm{rad}}(z,\vec{r}_\perp) = -\frac{i\omega}{c}\frac{K\omega^2 \bar{f}(\omega)}{2\pi c^2\gamma} \exp\left[\frac{i\omega z}{c}\right] \int d\vec{\theta} \int_0^\infty dz' \int d\vec{r'}_\perp \rho_o\left(\vec{r'}_\perp\right)$$
$$\times \exp\left[-\frac{i\omega}{c}\vec{\theta}\cdot\left(\vec{r}_\perp - \vec{r'}_\perp\right)\right] \left\{ \theta_x \sin[k_w z'] \exp\left[\frac{i\omega\theta^2(z'-z)}{2c}\right] \right.$$
$$\left. -2\theta_x \lambda\!\!\!^{-}_w \cos[k_w z'] \frac{\partial}{\partial z'} \exp\left[\frac{i\omega\theta^2(z'-z)}{2c}\right] \right\} \tag{54}$$

and

$$\partial_z \bar{E}_{z\ \mathrm{rad}}(z,\vec{r}_\perp) = -\frac{i\omega}{c}\frac{K\omega^2 \bar{f}(\omega)}{2\pi c^2\gamma} \exp\left[\frac{i\omega z}{c}\right] \int d\vec{\theta} \int_0^\infty dz' \int d\vec{r'}_\perp \rho_o\left(\vec{r'}_\perp\right)$$
$$\times \exp\left[-\frac{i\omega}{c}\vec{\theta}\cdot\left(\vec{r}_\perp - \vec{r'}_\perp\right)\right] \exp\left[\frac{i\omega\theta^2(z'-z)}{2c}\right] \cdot \{\theta_x \sin[k_w z']\} \ , \tag{55}$$

In Eq. (55) we neglected an extra-term $-\theta^2\theta_x/2 \sin[k_w z']$ in parenthesis $\{...\}$, because $\lambda\!\!\!^{-}/\lambda\!\!\!^{-}_w \ll 1$. We now integrate by parts the term in $\cos[k_w z']$ in Eq. (54) to obtain

$$\vec{\nabla}_\perp \cdot \vec{\bar{E}}_{\perp\mathrm{rad}}(z,\vec{r}_\perp) = -\frac{i\omega}{c}\frac{K\omega^2 \bar{f}(\omega)}{2\pi c^2\gamma} \exp\left[\frac{i\omega z}{c}\right] \int d\vec{\theta} \int_0^\infty dz' \int d\vec{r'}_\perp \rho_o\left(\vec{r'}_\perp\right)$$
$$\times \exp\left[-\frac{i\omega}{c}\vec{\theta}\cdot\left(\vec{r}_\perp - \vec{r'}_\perp\right)\right] \exp\left[\frac{i\omega\theta^2(z'-z)}{2c}\right] \exp\left[\frac{i\omega z'}{2c\bar{\gamma}_z^2}\right] \cdot \{-\theta_x \sin[k_w z']\} \ . \tag{56}$$

Obviously $\vec{\nabla}_\perp \cdot \vec{\bar{E}}_{\perp\mathrm{rad}} + \partial_z \bar{E}_{z\ \mathrm{rad}} = 0$, and also the second of Eq. (45) is satisfied.

4 Wake function and impedance

The knowledge of the electric field at the position of any test particle inside the beam, that has been derived in Section (3), allows to derive wake function and impedance related with the system under study. Let us briefly review here concepts of longitudinal impedance and wake function, that will be used later on. The longitudinal impedance of a system, $Z_o(\omega)$, can be given as the Fourier transform of the wake function $G_o(\Delta s)$:

$$Z_o = \int_{-\infty}^{\infty} \frac{d(\Delta s)}{\beta c}\, G_o(\Delta s) \exp\left[i\omega \frac{\Delta s}{\beta c}\right] ,$$

$$G_o = \frac{1}{(-e)} \int_{-\infty}^{\infty} d\vec{r} \cdot \vec{E}^o(\Delta s, \vec{r}'(t), t)|_{t=z'/(\beta_z c)} . \tag{57}$$

Here the integral in the expression for G_o is a line integral calculated along the trajectory of a test particle. In fact, $\vec{E}^o(\Delta s, \vec{r}'(t), t)$ indicates the time-domain electric field generated by a source particle acting on the test particle at longitudinal distance Δs from the source. In the calculation of $\vec{E}^o(\Delta s, \vec{r}'(t), t)$ we assume that effects from the vacuum chamber are negligible, i.e. $a \gg \bar{\gamma}_z \lambda\!\!\!^{-}$, that is the third of conditions (1). $\vec{E}^o(\Delta s, \vec{r}'(t), t)$ is integrated along the test particle trajectory, and divided by the electron charge $(-e)$, so that $e^2 G_o(\Delta s)$ is the energy (gained, or lost) by the test particle due to the action of the source. In agreement with [20] we take the test particle behind the source for positive values of Δs.

According to the given definition of wake function, one should integrate the field over the entire trajectory. However, there is no principle difficulty in considering only part of the trajectory, let us say, up to longitudinal position z. Mathematically, this means that the line integral for G_o should be performed up to the trajectory point of the test electron corresponding to longitudinal position z. In this way, $G = G(\Delta s, z)$.

Note that Eq. (57) is automatically dependent on a particular source electron, and a particular test electron. In order to formulate this statement in a mathematical way, we may introduce test and source particle initial transverse position $\vec{r}_{\perp T}$ and $\vec{r}_{\perp S}$ and write $G_o = G_o(\Delta s, z, \vec{r}_{\perp S}, \vec{r}_{\perp T})$, where we neglect differences in energy between the two particles. Following our previous work [21], we will slightly modify the concepts of wake and impedance by substituting test and source particles with disks of total charge (-e), longitudinally separated by a distance Δs. This amounts to an integration over the transverse particle distribution in $d\vec{r}_{\perp T}$ and $d\vec{r}_{\perp S}$, that makes our definitions

independent of $\vec{r}_{\perp T}$ and $\vec{r}_{\perp S}$. We thus obtain

$$G(\Delta s, z) = c^2 \int d\vec{r'}_{\perp} \int d\vec{r''}_{\perp} \rho_o(\vec{r'}_{\perp}) \rho_o(\vec{r''}_{\perp}) G_o(\Delta s, z, \vec{r'}_{\perp}, \vec{r''}_{\perp}) \, . \tag{58}$$

In Eq. (58) we used the fact that ρ_o is independent of the longitudinal position. With the redefinition in Eq. (58) we can further consider the impedance $Z(\omega, z)$ proceeding as in (57) for the definition of $Z_o(\omega)$, but Fourier substituting $G_o(\Delta s)$ with $G(\Delta s, z)$. Note that by definition of $Z(\omega, z)$ we have

$$Z(\omega, z) = \frac{1}{|\bar{f}(\omega)|^2} \int_V \vec{\vec{j}}^{\,*} \cdot \vec{\vec{E}} \, dV = \frac{1}{|\bar{f}(\omega)|^2} \int_0^z dz' \int_A d\vec{r'}_{\perp} \vec{\vec{j}}^{\,*} \cdot \vec{\vec{E}} \, , \tag{59}$$

where $|\bar{f}(\omega)|^{-2}$ accounts for the fact that test and source disks have total charge (-e), while $\vec{\vec{j}}^{\,*} \cdot \vec{\vec{E}} \propto |\bar{f}(\omega)|^2$, $\bar{f}(\omega)$ being the already defined Fourier transform of the longitudinal bunch-profile. Here the volume V is a cylinder of base A including the undulator up to position $z' = z$. The integration in z' is performed from 0 to z, because we will be interested in impedance and wakes generated inside the undulator, and we will assume that the undulator begins at position $z = 0$.

5 Impedance calculations

According to Eq. (59), the expressions for the longitudinal impedance associated with $\vec{\vec{E}}_{\perp}$ and $\vec{\vec{E}}_z$, $Z_{\perp}$ and Z_z are given by

$$\begin{aligned}
Z_{\perp} &= \frac{1}{|\bar{f}(\omega)|^2} \int_0^z dz' \exp\left[-\frac{i s_o(z')}{v_o}\right] \\
&\quad \times \int d\vec{r'}_{\perp} \bar{f}^*(\omega) \rho_o^*[\vec{r'}_{\perp} - r_w \cos(k_w z)] \vec{v}_{o\perp}(z') \cdot \vec{\vec{E}}_{\perp}(z', \omega, \vec{r'}_{\perp}) \\
Z_z &= \frac{1}{|\bar{f}(\omega)|^2} \int_0^z dz' \exp\left[-\frac{i s_o(z')}{v_o}\right] \\
&\quad \times \int d\vec{r'}_{\perp} \bar{f}^*(\omega) \rho_o^*[\vec{r'}_{\perp} - r_w \cos(k_w z) \vec{e}_x] c \bar{E}_z(z', \omega, \vec{r'}_{\perp}) \, .
\end{aligned} \tag{60}$$

When we calculated the field, we saw that the dependence of the source transverse charge density ρ_o on the electron motion $\vec{r'}_{\perp o}(z)$ had to be accounted for. We will show that, in the calculation of the impedance, the dependence of the test transverse charge density ρ_o on $\vec{r'}_{\perp} - r_w \cos(k_w z)\vec{e}_x$ must also be accounted for. With the help of a change of variables we rewrite Eq. (60) as a sum $Z = Z_\perp + Z_z$:

$$
\begin{aligned}
Z = \frac{1}{|\bar{f}(\omega)|^2} \int_0^z dz' \exp\left[-\frac{i s_o(z')}{v_o}\right] \int d\vec{r'}_{\perp} \bar{f}^*(\omega) \rho_o^*(\vec{r'}_{\perp}) \\
\times \Bigg\{ -\frac{Kc}{2i\gamma} \left[\exp(ik_w z') - \exp(-ik_w z')\right] \bar{E}_x(z', \omega, \vec{r'}_{\perp} + r_w \cos(k_w z)\vec{e}_x) \\
+ c\bar{E}_z(z', \omega, \vec{r'}_{\perp} + r_w \cos(k_w z)\vec{e}_x) \Bigg\} .
\end{aligned}
\tag{61}
$$

Calculations can be drastically simplified, because transverse radiative gradient terms in $Z_\perp$ cancel with longitudinal radiative terms in Z_z. It is easier to show this facts with the help of Eq. (36) and Eq. (37), rather than using explicit expressions Eq. (42) and Eq. (43).

First, with the help of Eq. (36) and Eq. (37), we write down the part of the impedance from the radiative transverse field, $Z_{\perp \mathrm{r}}$ and from the radiative longitudinal field, $Z_{z\,\mathrm{r}}$. In principle, in order to dispose of the oscillating terms in $\vec{\bar{E}}_{\mathrm{r}}(z', \omega, \vec{r'}_{\perp} + r_w \cos(k_w z)\vec{e}_x)$, we may use the same mathematical shortcut that can be exploited to obtain Eq. (36) and Eq. (37) from Eq. (19) and Eq. (24). In fact, we may formally expand the Green's function exponential $\exp\{i\omega[|\, \vec{r'}_{\perp} - \vec{r''}_{\perp} + r_w \cos(k_w z')\vec{e}_x\,|^2/2c(z' - z'')]\}$ to the first order in r_w and keep non-negligible first-harmonic terms in $\exp[\pm ik_w z']$. However, $\vec{\bar{E}}_{\perp \mathrm{r}}$ is multiplied by $\vec{v}_{o\perp}$ in the expression for the impedance $Z_\perp$. Since $\vec{v}_{o\perp}$ oscillates with period $\lambda\!\!\!^{-}_w$, we can neglect the oscillatory contributions in $\cos(k_w z)\vec{e}_x$ in the expression of $\vec{\bar{E}}_{\perp \mathrm{r}}$, because they would give oscillatory contributions that average to zero after integration in dz'. We therefore obtain:

$$
\begin{aligned}
Z_{\perp \mathrm{r}} = \frac{K\omega}{2\gamma} \int d\vec{r'}_{\perp} \int d\vec{r''}_{\perp} \rho_o^*(\vec{r'}_{\perp}) \rho_o\left(\vec{r''}_{\perp}\right) \int_0^z dz' \\
\times \Bigg\{ -\int_0^{z'} \frac{dz''}{z' - z''} \exp\left[i\omega \frac{|\, \vec{r'}_{\perp} - \vec{r''}_{\perp}\,|^2}{2c(z' - z'')}\right] \Bigg[+ \frac{K\vec{e}_x}{2i\gamma} \exp[ik_w(z'' - z')] \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^{z'} \frac{dz''}{z'-z''} \exp\left[i\omega \frac{|\,\vec{r'}_\perp - \vec{r''}_\perp\,|^2}{2c(z'-z'')}\right]\left[-\frac{K\vec{e}_x}{2i\gamma}\exp[ik_w(z'-z'')]\right] \\
&+\int_0^{z'} \frac{dz''}{z'-z''} \exp\left[i\omega \frac{|\,\vec{r'}_\perp - \vec{r''}_\perp\,|^2}{2c(z'-z'')}\right] \\
&\qquad\times\left[+\frac{r_w}{2(z'-z'')} + \frac{i\omega r_w (x'-x'')^2}{2c(z'-z'')^2}\right]\exp[ik_w(z''-z')] \\
&+\int_0^{z'} \frac{dz''}{z'-z''} \exp\left[i\omega \frac{|\,\vec{r'}_\perp - \vec{r''}_\perp\,|^2}{2c(z'-z'')}\right] \\
&\qquad\times\left[-\frac{r_w}{2(z'-z'')} - \frac{i\omega r_w (x'-x'')^2}{2c(z'-z'')^2}\right]\exp[ik_w(z'-z'')]\,.
\end{aligned}
\tag{62}
$$

for the transverse field. On the contrary, the longitudinal velocity $\vec{v}_{o\ z}$ is a sum of a constant term, whose magnitude is about c and a negligible oscillates with period $2\lambda\!\!\!^{-}_w$. As a result, the oscillatory contributions in $\cos(k_w z)\vec{e}_x$ in the expression of $\bar{E}_{z\ \mathrm{r}}$ must be kept, and an expansion of the exponential in the Green's function must be performed, leading to

$$
\begin{aligned}
Z_{z\ \mathrm{r}} = &-\frac{K\omega}{2\gamma}\int d\vec{r'}_\perp \int d\vec{r''}_\perp \rho_o^*(\vec{r'}_\perp)\rho_o\left(\vec{r''}_\perp\right)\int_0^z dz' \\
&\times\Bigg\{\int_0^{z'} \frac{dz''}{z'-z''} \exp\left[i\omega \frac{|\,\vec{r'}_\perp - \vec{r''}_\perp\,|^2}{2c(z'-z'')}\right] \\
&\qquad\times\left[+\frac{i\omega r_w (x'-x'')^2}{2c(z'-z'')^2} + \frac{r_w}{2(z'-z'')}\right]\exp[ik_w(z''-z')] \\
&+\int_0^{z'} \frac{dz''}{z'-z''} \exp\left[i\omega \frac{|\,\vec{r'}_\perp - \vec{r''}_\perp\,|^2}{2c(z'-z'')}\right] \\
&\qquad\times\left[-\frac{i\omega r_w (x'-x'')^2}{2c(z'-z'')^2} - \frac{r_w}{2(z'-z'')}\right]\exp[ik_w(z'-z'')]\Bigg\}
\end{aligned}
\tag{63}
$$

Partial cancellation can be exploited between the last two terms of Eq. (62) and Eq. (63). Note that the longitudinal radiative impedance is completely cancelled, and one obtains

$$
Z_{\mathrm{r}} = Z_{\perp r} + Z_{z\ \mathrm{r}} =
$$

$$\frac{K\omega}{2\gamma}\int d\vec{r'}_\perp \int d\vec{r''}_\perp \rho_o^*(\vec{r'}_\perp)\rho_o\left(\vec{r''}_\perp\right)\int_0^z dz'$$
$$\times\Bigg\{-\int_0^{z'}\frac{dz''}{z'-z''}\exp\left[i\omega\frac{|\,\vec{r'}_\perp-\vec{r''}_\perp\,|^2}{2c(z'-z'')}\right]\left[+\frac{K}{2i\gamma}\exp[ik_w(z''-z')]\right]$$
$$+\int_0^{z'}\frac{dz''}{z'-z''}\exp\left[i\omega\frac{|\,\vec{r'}_\perp-\vec{r''}_\perp\,|^2}{2c(z'-z'')}\right]\left[-\frac{K}{2i\gamma}\exp[ik_w(z'-z'')]\right]\Bigg\}. \qquad (64)$$

Performing the integrals in dz'' with the help of Eq. (40) or, equivalently, calculating $Z_{\rm r}$ with the help of the transverse radiative current terms in Eq. (42) we finally obtain

$$Z_{\rm r}=Z_{\perp r}+Z_{z\ {\rm r}}=i\frac{K^2\omega z}{2\gamma^2}\int d\vec{r'}_\perp\int d\vec{r''}_\perp\rho_o^*(\vec{r'}_\perp)\rho_o\left(\vec{r''}_\perp\right)$$
$$\times\Bigg\{K_o\left[\frac{\sqrt{2}\,|\,\vec{r'}_\perp-\vec{r''}_\perp\,|}{\sqrt{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w}}\right]+K_o\left[\frac{-\sqrt{2}i\,|\,\vec{r'}_\perp-\vec{r''}_\perp\,|}{\sqrt{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w}}\right]\Bigg\}. \qquad (65)$$

Let us now consider the space-charge part of the impedance. One finds that the space-charge term in $Z_\perp$, i.e. $Z_{\perp{\rm sc}}$, averages to zero, as can directly be seen by inspecting the last term in $K_1[|\vec{r}_\perp-\vec{r'}_\perp|/(\bar{\gamma}_z\lambda\!\!\!^{-})]$ of Eq. (42). In fact, such term is independent of z. Now, according to Eq. (61), in order to obtain the correspondent impedance contribution, this term must be multiplied by $\vec{v}_\perp$ (i.e. by $\exp[\pm ik_w z']$) and integrated in dz' for a saturation length. It follows that, during the integration process in dz', one integrates a fast varying function of z on the scale of $\lambda\!\!\!^{-}_w$. As a result, we obtain a negligible effective impedance contribution over many undulator periods, and we can neglect the transverse space-charge contribution in the calculation of wake and impedance. The total space-charge part of the impedance coincides with the longitudinal space-charge impedance. It can be shown that oscillatory contributions in $\cos(k_w z')\vec{e}_x$ in the expression of $\vec{\vec{E}}_{\rm zsc}$ are of higher order in ω/ω_r and they can thus be neglected. As a result we obtain

$$Z_{\rm sc}=Z_{z\ sc}=-i\omega z\frac{2+K^2}{\gamma^2}\int d\vec{r'}_\perp\int d\vec{r''}_\perp\rho_o^*(\vec{r'}_\perp)\rho_o\left(\vec{r''}_\perp\right)K_0\left(\frac{|\,\vec{r'}_\perp-\vec{r''}_\perp\,|}{\bar{\gamma}_z\lambda\!\!\!^{-}}\right). \qquad (66)$$

We thus reach the conclusion that only longitudinal space-charge terms and transverse radiative terms enter the expression for the impedance that can now be calculated in integral from for any transverse beam-distribution under conditions (1).

Finally, we obtain the total impedance $Z = Z_{\rm r} + Z_{\rm sc}$. The real part Z_R is given by

$$Z_R = -\frac{K^2 \pi \omega z}{4\gamma^2} \int d\vec{r'}_\perp \int d\vec{r''}_\perp \rho_o^*(\vec{r'}_\perp) \rho_o \left(\vec{r''}_\perp\right) J_0 \left(\frac{\sqrt{2}\left|\vec{r'}_\perp - \vec{r''}_\perp\right|}{\sqrt{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w}} \right) . \tag{67}$$

The imaginary part Z_I, instead, amounts to

$$\begin{aligned} Z_I = -\frac{K^2 \omega z}{2\gamma^2} \int d\vec{r'}_\perp \int d\vec{r''}_\perp \rho_o^*(\vec{r'}_\perp) \rho_o \left(\vec{r''}_\perp\right) \left\{ \frac{\pi}{2} Y_0 \left(\frac{\sqrt{2}\left|\vec{r'}_\perp - \vec{r''}_\perp\right|}{\sqrt{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w}} \right) \right. \\ \left. - K_0 \left(\frac{\sqrt{2}\left|\vec{r'}_\perp - \vec{r''}_\perp\right|}{\sqrt{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w}} \right) + \frac{4 + 2K^2}{K^2} K_0 \left(\frac{\left|\vec{r'}_\perp - \vec{r''}_\perp\right|}{\bar{\gamma}_z \lambda\!\!\!^{-}} \right) \right\} , \end{aligned} \tag{68}$$

having used the fact that $K_0(-ix) = (\pi/2)[iJ_0(x) - Y_0(x)]$.

5.1 *Asymptotic case for $\sigma_\perp^2 \ll \lambda\!\!\!^{-}\lambda\!\!\!^{-}_w$*

Before proceeding with the analysis of the wake, it is interesting to derive asymptotic limits of Eq. (67) and Eq. (68) in the case for $\sigma_\perp^2 \ll \lambda\!\!\!^{-}\lambda\!\!\!^{-}_w$. Bessel functions in Eq. (67) and Eq. (68) can be expanded for small argument values. In particular, using $J_0(x) \simeq 1$ for $x \ll 1$, and recalling that ρ_o is normalized to $1/c$, the real part of the impedance becomes

$$Z_R = -\frac{K^2 \omega \pi z}{4\gamma^2} \int d\vec{r'}_\perp \int d\vec{r''}_\perp \rho_o^*(\vec{r'}_\perp) \rho_o \left(\vec{r''}_\perp\right) = -\frac{K^2 \pi z}{4c\lambda\!\!\!^{-}\gamma^2} , \tag{69}$$

independently of the choice of ρ_o. Subsequently, we use $K_0(x) \simeq -\gamma_E - \ln(x/2)$ and $Y_0 \simeq 2/\pi[\gamma_E + \ln(x/2)]$, $\gamma_E \simeq 0.577216$ being the Euler Gamma constant in the imaginary part of the impedance, Eq. (68). We obtain

$$\begin{aligned} Z_I &= -\frac{K^2 \omega z}{\gamma^2} \int d\vec{r'}_\perp \int d\vec{r''}_\perp \rho_o^*(\vec{r'}_\perp) \rho_o \left(\vec{r''}_\perp\right) \\ &\quad \times \left\{ \ln\left(\sqrt{\frac{\lambda\!\!\!^{-}}{\lambda\!\!\!^{-}_r}} \right) - \frac{2}{K^2} \ln\left(\sqrt{1 + \frac{K^2}{2}} \right) - \frac{2\gamma_E}{K^2} - \frac{2}{K^2} \ln\left(\frac{\left|\vec{r'}_\perp - \vec{r''}_\perp\right|}{2\lambda\!\!\!^{-}\gamma} \right) \right\} \\ &= -\frac{K^2 z}{c\lambda\!\!\!^{-}\gamma^2} \ln\left(\sqrt{\frac{\lambda\!\!\!^{-}}{\lambda\!\!\!^{-}_r}} \right) + \frac{2z}{c\lambda\!\!\!^{-}\gamma^2} \ln\left(\sqrt{1 + \frac{K^2}{2}} \right) + Z_{I\ \rm free} , \end{aligned} \tag{70}$$

where

$$Z_{I\,\mathrm{free}} = \frac{2z\gamma_E}{c\lambda\gamma^2} + \frac{2\omega z}{\gamma^2}\int d\vec{r'}_\perp \int d\vec{r''}_\perp \rho_o^*(\vec{r'}_\perp)\rho_o\left(\vec{r''}_\perp\right)\ln\left(\frac{\left|\vec{r'}_\perp - \vec{r''}_\perp\right|}{2\lambda\gamma}\right). \tag{71}$$

$Z_{I\,\mathrm{free}}$ is the only model-part of the impedance. In particular, assuming a Gaussian transverse profile:

$$\rho_o(r_\perp) = \frac{1}{2\pi\sigma_\perp^2 c}\exp\left[-\frac{r_\perp^2}{2\sigma_\perp^2}\right] \tag{72}$$

we obtain

$$\begin{aligned} Z_{I\,\mathrm{free}} &= \frac{2z\gamma_E}{c\lambda\gamma^2} + \frac{z}{2\pi^2\sigma_\perp^4 c\lambda\gamma^2} \\ &\quad\times\int d\vec{r'}_\perp \int d\vec{r''}_\perp \exp\left[-\frac{r_\perp'^2}{2\sigma_\perp^2}\right]\exp\left[-\frac{r_\perp''^2}{2\sigma_\perp^2}\right]\ln\left(\frac{\left|\vec{r'}_\perp - \vec{r''}_\perp\right|}{2\lambda\gamma}\right) \\ &= \frac{2z\gamma_E}{c\lambda\gamma^2} + \frac{2z}{c\lambda\gamma^2}\ln\left(\frac{\sigma_\perp}{\gamma\lambda}\right). \end{aligned} \tag{73}$$

Also, $Z_{I\,\mathrm{free}}$, is logarithmically divergent on $\sigma_\perp$. This is, in fact, the free-space impedance. The renormalized impedance[5], i.e. the difference of Eq. (70) with the free-space impedance is independent of $\sigma_\perp$ and constitute a result valid for any value of K.

Here we underline the fact that in the limit for $\sigma_\perp \longrightarrow 0$ the difference between the impedance of an electron beam moving through a magnetic system and the free-space impedance is independent of the electron beam model. It is finite, and can thus be applied in a one-dimensional approximation whereby the electron bunch is modelled by a line density. This one-dimensional approach was first proposed in the time-domain to study Coherent Synchrotron Radiation (CSR) in [22] and is currently used in CSR codes. Also in this case, the renormalized wake is obtained by subtracting the free-space wake (as is natural, because the wake is the Fourier transform

[5] Note that, without slow-wave radiative contributions to the field (proportional to $\exp[+ik_w z]$), it would be impossible to recover Eq. (70) and Eq. (80). In other words, the renormalization process would fail. This underlines the fact that, although slow-wave radiative contributions have no realization in the far zone, they are of fundamental importance in the calculation of the impedance.

of the impedance, i.e. its time-domain counterpart). The renormalization [6] procedure used here and introduced in [22] has to be seen as a mathematical algorithm to deal with calculation of self-forces of a moving charged line. Such calculation is problematic due to incompleteness of electromagnetic theory, yielding to divergence. Such divergence is cancelled by subtracting the longitudinal force that would be present in a straight-line motion from the force calculated on a curved trajectory. The finite difference can be entirely ascribed to curvature.

5.2 Existing studies of the asymptote $\sigma_\perp^2 \ll \lambda\lambda_w$

Analytical and numerical studies can be found in literature, treating longitudinal impedance and wake from an undulator setup in the case of a line density distribution of electrons (see e.g. [24], [25] and [26]).

Reference [24] deals with the one-dimensional renormalized wake of an electron beam with a Gaussian longitudinal profile

$$f(s) = \frac{(-e)N}{\sqrt{2\pi}\sigma_z} \exp\left[-\frac{s^2}{2\sigma_z^2}\right] , \tag{74}$$

N being the number of electrons in the beam. In particular, in reference [24], the following expression for the energy gained or lost by a particle at position z down the beamline and position s within the bunch [7] was obtained:

$$\Delta\mathcal{E} = \frac{e^2 N K^2 z}{\sqrt{2\pi}\sigma_z \gamma^2} \bar{G}(p, K, x) , \tag{75}$$

where $x = -s/\sigma_z$, $p \gg 1$ is the bunch length parameter

$$p = \frac{\gamma^2 k_w \sigma_z}{1 + K^2/2} \tag{76}$$

and $\bar{G}$ is given by

[6] It may be worth to note here, that in the renormalization procedure used in [22] only retarded fields are used whereas in previous works (see e.g. [23]) devoted to renormalization in classical electrodynamics a radiation field is used, that is half the difference of retarded and advanced fields.

[7] It should be noted that the definition of s in this paper differs for a sign with respect to that in [24].

$$\bar{G}(p,K,x) = \frac{x}{2}\exp\left[-\frac{x^2}{2}\right][\ln(p) + g(K)] + F(x)\,, \tag{77}$$

with

$$\begin{aligned} F(x) = {} & \frac{1}{4}[\gamma_E + 3\ln(2) - 2]x\exp\left[-\frac{x^2}{2}\right] \\ & - \sqrt{\frac{\pi}{8}}\left\{1 + \mathrm{erf}\left[\frac{x}{\sqrt{2}}\right] - x\exp\left[-\frac{x^2}{2}\right]\right. \\ & \left.\times \int_0^x dx' \exp\left[\frac{(x')^2}{2}\right]\left[1 + \mathrm{erf}\left[\frac{x'}{\sqrt{2}}\right]\right]\right\}. \end{aligned} \tag{78}$$

Moreover, in the limit for $K^2 \ll 1$, $g(K) \longrightarrow 0$, while in the limit for $K^2 \gg 1$, $g(K) \longrightarrow 1$. For arbitrary values of K, $g(K)$ was presented in [24] as a plot, using numerical integration techniques. Now $g(K)$ can be expressed fully analytically:

$$g(K) = 1 - \frac{\ln\left[1 + K^2/2\right]}{K^2/2}\,. \tag{79}$$

Eq. (75) was already been independently cross-checked, with the help of the code TraFiC^4, in [25]. In this paper we underline the correctness of Eq. (75).

Reference [26] deals with the renormalized impedance in the case of a line density distribution. Such impedance is presented in Eq. (26) of [26] in the asymptotic case for $K \gg 1$. When $\sigma_\perp \ll \sqrt{\lambda\lambda_w}$ and $K^2 \gg 1$ only the first term of Eq. (70) survives, and the total renormalized impedance Z_{ren} reads:

$$Z_{\mathrm{ren}} = Z - iZ_{I\ \mathrm{free}} = -\frac{K^2\pi z}{4c\lambda\gamma^2} - i\frac{K^2 z}{2c\lambda\gamma^2}\ln\left(\frac{\lambda}{\lambda_r}\right)\,. \tag{80}$$

Eq. (80) is in agreement with reference [26], where the impedance per unit length is given [8] .

5.3 Energy conservation law for $\sigma_\perp^2 \ll \lambda\lambda_w$

In general, the real part of the impedance can always be cross-checked with the energy conservation law, that requires:

[8] An extra factor $-1/c$ in our expression is the result of different definition of impedance.

$$\frac{dW}{d\omega} = -\frac{1}{\pi}\left|\bar{f}(\omega)\right|^2 Z_R(\omega)\,, \tag{81}$$

where the energy spectrum of the radiation, $dW/d\omega$, is defined as the integral over all angles of the total energy emitted per unit frequency per unit solid angle $d\Omega = \theta d\theta d\phi$:

$$\frac{dW}{d\omega} = \int_0^{2\pi} d\phi \int_0^{\infty} d\theta \frac{dW}{d\omega d\Omega} = \frac{cz_o^2}{4\pi^2}\int_0^{2\pi} d\phi \int_0^{\infty} d\theta\ \theta \left|\vec{\vec{E}}\right|^2 \ . \tag{82}$$

It is easy to verify Eq. (81) in the case $\sigma_\perp^2 \ll \lambda\!\!\!^{-}\lambda\!\!\!^{-}_w$. In this case, the electron beam transverse size is much smaller than the radiation diffraction size, and a filament-beam model can be used.

Our theory has been developed for any value of the undulator parameter K, and in the long-wavelength asymptote, i.e. for $\lambda\!\!\!^{-}_r/\lambda\!\!\!^{-} \ll 1$. In this case it is possible to give a simple mathematical description of the radiation energy spectrum. Based on this expression we will then verify Eq. (81).

In order to calculate the energy spectrum according to Eq. (82) we must first calculate $\vec{\vec{E}}_\perp$ in the far zone. We can specify "how near" ω is to the resonant frequency $\omega_r = 2k_w c\bar{\gamma}_z^2$ by introducing a detuning parameter C, defined as $C = \omega/(2\bar{\gamma}_z^2 c) - k_w = (\Delta\omega/\omega_r)k_w$, where $\omega = \omega_r + \Delta\omega$. Then, the field generated by a filament-beam is well-known and is given, in paraxial approximation (see e.g. Eq. (13) of [18]), by:

$$\vec{\vec{E}}_\perp = \exp\left[i\frac{\omega\theta^2 z}{2c}\right]\frac{i\omega|\bar{f}(\omega)|}{c^2 z}\int_{-L_w/2}^{L_w/2} dz'\left\{\frac{K}{2i\gamma}\left[\exp\left(2ik_w z'\right) - 1\right]\vec{e}_x + \vec{\theta}\exp\left(ik_w z'\right)\right\}$$
$$\times \exp\left[i\left(C + \frac{\omega\theta^2}{2c}\right)z' - \frac{K\theta_x}{\gamma}\frac{\omega}{k_w c}\cos(k_w z') - \frac{K^2}{8\gamma^2}\frac{\omega}{k_w c}\sin(2k_w z')\right]\,. \tag{83}$$

As first proposed in [27] one may use the Anger-Jacobi expansion $\exp\left[ia\sin(\psi)\right] = \sum_{p=-\infty}^{\infty} J_p(a)\exp\left[ip\psi\right]$, where $J_p(\cdot)$ indicates the Bessel function of the first kind of order p, to write the integral in Eq. (83) in a different way:

$$\vec{\vec{E}}_\perp = \exp\left[i\frac{\omega\theta^2 z}{2c}\right]\frac{i\omega|\bar{f}(\omega)|}{c^2 z}\sum_{m,n=-\infty}^{\infty} J_m(u)J_n(v)\exp\left[\frac{i\pi n}{2}\right]$$
$$\times\int_{-L_w/2}^{L_w/2} dz'\exp\left[i\left(C + \frac{\omega\theta^2}{2c}\right)z'\right]\left\{\frac{K}{2i\gamma}\left[\exp\left(2ik_w z'\right) - 1\right]\vec{e}_x + \vec{\theta}\exp\left(ik_w z'\right)\right\}$$

$$\times \exp\left[i(n+2m)k_w z'\right] \ , \tag{84}$$

where $u = -K^2\omega/(8\gamma^2 k_w c)$ and $v = -K\theta_x\omega/(\gamma k_w c)$. There is no simple result valid at the same time for arbitrary K values and arbitrary detuning. However, there are asymptotes for $\lambda\!\!\!^{-}_r/\lambda\!\!\!^{-} \ll 1$ and arbitrary K, or $K^2 \ll 1$ and arbitrary detuning. We will first assume $K^2 \ll 1$ and arbitrary values for $\lambda\!\!\!^{-}/\lambda\!\!\!^{-}_r$ [9]. Consider the case $|\Delta\omega|/\omega_r \gg 1/N_w$, N_w being the number of undulator periods, $C < 0$ and $K^2 \ll 1$. Because of these assumptions, both $u \ll 1$ and $v \ll 1$ (here $\theta^2 \sim 2c|C|/\omega$). This means that asymptotic expansions of Bessel functions can be used. Non-negligible contributions are for $n = m = 0$ or for $n = -1$ and $m = 0$. It follows that

$$\begin{aligned}
\vec{\tilde{E}}_\perp &= -\frac{\omega|\bar{f}(\omega)|K}{2c^2 z\gamma}\exp\left[i\frac{\omega\theta^2 z}{2c}\right]\left\{\vec{e}_x - \vec{\theta}\frac{\theta_x\omega}{k_w c}\right\} \\
&\quad\times\int_{-L_w/2}^{L_w/2} dz' \exp\left[i\left(C+\frac{\omega\theta^2}{2c}\right)z'\right] \\
&= -\frac{\omega|\bar{f}(\omega)|KL_w}{2c^2 z\gamma}\exp\left[i\frac{\omega\theta^2 z}{2c}\right]\left\{\left[1-\frac{\theta_x^2\omega}{k_w c}\right]\vec{e}_x + \left[\frac{\theta_x\theta_y\omega}{k_w c}\right]\vec{e}_y\right\} \\
&\quad\times\mathrm{sinc}\left[\frac{L_w}{4}\left(C+\frac{\omega\theta^2}{2c}\right)\right] .
\end{aligned} \tag{85}$$

Note that since both $u \ll 1$ and $v \ll 1$, the Anger-Jacobi expansion is not really necessary here, and we might have derived Eq. (85), based on $K^2 \ll 1$ and $|\Delta\omega|/\omega_r \gg 1/N_w$, directly from Eq. (83) by directly expanding in Taylor series the exponential function of trigonometric arguments.

The total energy emitted per unit frequency per unit solid angle $K^2 \ll 1$ and $|\Delta\omega|/\omega_r \gg 1/N_w$ is

$$\frac{dW}{d\omega d\Omega} = \frac{\omega^2|\bar{f}(\omega)|^2K^2L_w^2}{16\pi^2c^3\gamma^2}\left\{\left[1-\frac{\theta_x^2\omega}{k_w c}\right]^2 + \left[\frac{\theta_x\theta_y\omega}{k_w c}\right]^2\right\}\mathrm{sinc}^2\left[\frac{L_w}{4}\left(C+\frac{\omega\theta^2}{2c}\right)\right] \tag{86}$$

in agreement with [28]. Substituting Eq. (86) in Eq. (82) and using the fact that $N_w \gg 1$ and $\mathrm{sinc}^2[x/a]/(\pi a) \longrightarrow \delta(x)$ for $a \longrightarrow 0$ we obtain

[9] Note that in the long wavelength asymptote, $\lambda\!\!\!^{-} \gg \lambda\!\!\!^{-}_r$, i.e. $\omega/\omega_r \ll 1$, we always have $|\Delta\omega|/\omega_r \gg 1/N_w$, but not viceversa. Thus, here we are considering $|\Delta\omega|/\omega_r \gg 1/N_w$, but arbitrary value for $\lambda\!\!\!^{-}/\lambda\!\!\!^{-}_r$.

$$\frac{dW}{d\omega} = \frac{\omega|\bar{f}(\omega)|^2 K^2 L_w}{8c^2\gamma^2}\left[1+\left(\frac{\omega}{ck_w\bar{\gamma}_z^2}-1\right)^2\right], \tag{87}$$

Also Eq. (87) is valid for $K \ll 1$ and $|\Delta\omega|/\omega_r \gg 1/N_w$, and is in agreement with [27] (where the energy spectrum was first calculated) and, [28] [10] and [24]. In the limit for $\lambda\!\!\!^{-} \gg \lambda\!\!\!^{-}_r$ we write

$$\frac{dW}{d\omega} = \frac{\omega|\bar{f}(\omega)|^2 K^2 L_w}{4c^2\gamma^2}, \tag{88}$$

Eq. (88) is at the left hand side of Eq. (81) and has been calculated for $K \ll 1$ and $\lambda\!\!\!^{-} \gg \lambda\!\!\!^{-}_r$.

The right hand side can be written using the real part of Eq. (69), that is valid for arbitrary K values and $\lambda\!\!\!^{-} \gg \lambda\!\!\!^{-}_r$, and calculating the impedance along an undulator of length L_w. We obtain:

$$-\frac{1}{\pi}\left|\bar{f}(\omega)\right|^2 L_w \frac{dZ_R}{dz} = \frac{\omega|\bar{f}(\omega)|K^2 L_w}{4c^2\gamma^2}, \tag{89}$$

thus verifying Eq. (81) and energy conservation. Agreement between the asymptote of Eq. (87) for $\lambda\!\!\!^{-} \gg \lambda\!\!\!^{-}_r$ and Eq. (69) is due to the fact that Eq. (69) is valid for arbitrary values of K. However, Eq. (88) can also be calculated from Eq. (83) under the only assumption $\lambda\!\!\!^{-} \gg \lambda\!\!\!^{-}_r$, i.e. Eq. (88) is not only valid for $K^2 \ll 1$, but for any value of K. More generally, we can say that in the long wavelength asymptote ($\lambda\!\!\!^{-} \gg \lambda\!\!\!^{-}_r$) it is sufficient to account for the first harmonic only, independently of the value of K.

5.4 Asymptotic case for $\sigma_\perp^2 \gg \lambda\!\!\!^{-}\lambda\!\!\!^{-}_w$

As we already discussed, radiation field and space-charge field exhibit different formation lengths and different transverse scales, namely $\sqrt{\lambda\!\!\!^{-}\lambda\!\!\!^{-}_w}$ and $\lambda\!\!\!^{-}\bar{\gamma}_z$. The same transverse scales are also present in Eq. (67) and Eq. (68).

The first two terms in Y_0 and K_0 in Eq. (68), as well as the entire real part of the impedance, are linked to the presence of transverse current density and to radiation field. The last term in Eq. (68) instead, is due to the presence of longitudinal space-charge field, a combination of current and gradient terms. The corresponding Bessel functions yield different characteristic transverse scales. Bessel functions related with the radiation field are linked with a

[10] A typing error is present in Eq. (2.11) of [28].

transverse size $\sqrt{\lambda\lambda_w} \sim \sqrt{\lambda\lambda_r\bar{\gamma}_z^2}$. Those related with the longitudinal space-charge field are linked with a transverse size $\lambda\bar{\gamma}_z \sim \sqrt{\lambda\lambda\bar{\gamma}_z^2}$. Since $\lambda \gg \lambda_r$, it follows that the characteristic transverse size related with the radiation field contribution is much smaller than that related with the space-charge field contribution, $\sqrt{\lambda\lambda_w/2} \ll \lambda\bar{\gamma}_z$. By inspection of Eq. (67) and Eq. (68) one can see that the value of $\left|\vec{r'}_\perp - \vec{r''}_\perp\right|$ is limited by $\sigma_\perp$, because of the presence of the exponential functions under the integration sign. Therefore, assuming constant total charge of the beam, when the electron beam transverse size $\sigma_\perp$ increases beyond $\sqrt{\lambda\lambda_w}$ the radiation contribution is suppressed with respect to the space-charge one.

Summing up, when condition (2) is valid together with (1), i.e. $\sigma_\perp^2 \gg \lambda\lambda_w$, we may neglect the real part of the impedance Z_R and approximate the total impedance with

$$Z = -i\frac{2\omega z}{\bar{\gamma}_z^2}\int d\vec{r'}_\perp \int d\vec{r''}_\perp \rho_o^*(\vec{r'}_\perp)\rho_o\left(\vec{r''}_\perp\right) K_0\left(\frac{\left|\vec{r'}_\perp - \vec{r''}_\perp\right|}{\lambda\bar{\gamma}_z}\right). \tag{90}$$

This means that, in the limit $\sigma_\perp^2 \gg \lambda\lambda_w$, the only field to be accounted for when calculating impedance (and wake), is the effective longitudinal space-charge field.

5.5 Discussion

Results obtained in Section 5.4, namely Eq. (90), mean that in the limit $\sigma_\perp^2 \gg \lambda\lambda_w$ radiation is suppressed, so that the beam can be considered as non-radiating, and only space-charge impedance is present. Such impedance amounts to the free-space impedance, where γ is consistently substituted with $\bar{\gamma}_z$. Eq. (90) gives the correct impedance at position z inside the undulator, as an asymptotic limit for $\sigma_\perp^2 \gg \lambda\lambda_w$ of our general theory.

Our results are in contrast with [13]. Authors of [13] first noted, correctly, that the presence of a finite transverse dimension of the beam $\sigma_\perp$ suppresses radiation in the far zone. Thus, the real part of the renormalized impedance in Eq. (80) (valid in the one-dimensional limit of a pencil beam) can be generalized to the case when a finite transverse dimension of the beam $\sigma_\perp$ is present, by multiplying it by an exponentially suppressing factor. However, they extended such understanding to the imaginary part as well, that is incorrect. This led them to obtain the following expression for the renormalized undulator impedance accounting for a finite $\sigma_\perp$ (see Eq. (2) of reference [13]):

$$Z_{\sigma_\perp} = Z_{\rm ren} \exp\left[-\frac{2\omega}{c} k_w \sigma_\perp^2\right] = \left[-\frac{K^2 \pi z}{4c\lambda\gamma^2} - i\frac{K^2 z}{2c\lambda\gamma^2} \ln\left(\frac{\lambda}{\lambda_r}\right)\right] \exp\left[-\frac{2\omega}{c} k_w \sigma_\perp^2\right], \tag{91}$$

the explicit expression for $Z_{\rm ren}$ being given by Eq. (80).

Eq. (91) is not derived, in [13], as the asymptotic result from a complete theory. It is the outcome of an analogy with the fact that radiation in the far zone is suppressed when a finite transverse electron beam size $\sigma_\perp$ is considered. However, the imaginary part of $Z_{\rm ren}$ results as a combination (a difference, actually) of two logarithmic contributions, that can be respectively ascribed to space-charge and radiative field (see Section 5.1) and present different characteristic transverse scales ($\bar{\gamma}_z\lambda$ and $\sqrt{\lambda\lambda_w}$) as a consequence of different field formation-lengths. $Z_{\rm ren}$ is calculated in the limit for $\sigma_\perp^2 \ll \lambda\lambda_w$. Only in this limit the dependence on $\sigma_\perp$ in the imaginary part of $Z_{\rm ren}$ is cancelled as a result of the combination of the before-mentioned logarithmic contributions. For finite transverse size $\sigma_\perp$ such compensation does not take place at all. For example, for the ESASE scheme at LCLS (see Section 7), one has $\sigma_\perp = 30\ \mu$m, $\sqrt{\lambda\lambda_w} \simeq 6\ \mu$m and $\bar{\gamma}_z\lambda = 500\mu$m. As a result, $\sigma_\perp$ is small with respect to $\bar{\gamma}_z\lambda$, but large with respect to $\sqrt{\lambda\lambda_w}$. The incorrectness of Eq. (91) (i.e. Eq. (2) of [13]) follows from these observations.

Authors of reference [13] conclude that the impedance in Eq. (91), that is related with curved trajectory, is suppressed. Thus, the space-charge induced, free-space impedance (Eq. (5) in [13]) is finally considered when calculating the energy spread inside the undulator. This result is counterintuitive. According to it, when the electron beam does not radiate ($\sigma_\perp^2 \gg \lambda\lambda_w$), the presence of the undulator does not influence the impedance, independently of the value of the undulator parameter K. However, its incorrectness was not trivial to prove, because it relies on an apparently correct analogy between real and imaginary part of the impedance. Only developing a complete theory the presence of two separate logarithmic dependencies can be spotted. Thus, in Eq. (90) we saw, as an asymptotic case of our theory, that only space-charge impedance is relevant for $\sigma_\perp^2 \gg \lambda\lambda_w$, but we additionally demonstrated that γ must be consistently substituted with $\bar{\gamma}_z$.

Our conclusion is that when $K^2 \gtrsim 1$ the presence of the undulator strongly influences the longitudinal impedance, whether the beam radiates or not (i.e. independently of the transverse size $\sigma_\perp$).

6 Analytical expression for the wake function in the steady state case for $\sigma_\perp^2 \gg \lambda\lambda_w$

As discussed in the previous Sections, our derivations hold under conditions (1) and drastically simplify under condition (2), i.e. for $\sigma_\perp^2 \gg \lambda\lambda_w$. In fact, as we have seen before, when condition (2) holds transverse contributions to impedance and wake are negligible. In the following we will consider the case when both (1) and (2) are satisfied, that is the case for ESASE schemes at LCLS, as we will see in Section 7.

We will consider a transverse longitudinal profile, as specified in Eq. (72), and a longitudinal bunch profile, specified by Eq. (74). Note that the *rms* bunch length σ_z is connected to the *rms* bunch duration σ_t by $\sigma_z = \beta c \sigma_t$, so that, in terms of time and frequency we have

$$f(t) = \frac{(-e)N}{\sqrt{2\pi}\sigma_t} \exp\left[-\frac{t^2}{2\sigma_t^2}\right] \quad \longleftrightarrow \quad \bar{f}(\omega) = (-e)N \exp\left[-\frac{\omega^2\sigma_t^2}{2}\right], \tag{92}$$

When $\sigma_\perp^2 \gg \lambda\lambda_w$, an expression for the wake can be found by Fourier-transforming the impedance given in Eq. (90) [11] . As already noted, Eq. (90) is mathematically identical to the free-space expression where only γ has been substituted by $\bar{\gamma}_z$. Here we will present only the final result for the wake function. For mathematical details regarding wake calculations, we refer the interested reader to a previous work of us [21]. That paper dealt with a different subject, namely wake fields and impedances for electron beams accelerated through ultra-high field gradients. However, in [21], we also analyzed the steady-state ($z \gg 2\bar{\gamma}_z^2\sigma_z$), free-space case of Gaussian transverse and longitudinal distribution for the beam [12] . We find [13] that the antisymmetric part of the wake G_A is given by $G_A(\Delta\xi) = \bar{\gamma}_z\eta\hat{z}/\sigma_\perp \cdot H_A(\Delta\xi)$, where

$$H_A(\Delta\xi) = -\frac{1}{2\sqrt{\pi}}(\Delta\xi)\left\{2\frac{\sqrt{\pi}}{|\Delta\xi|} - \pi\exp\left[\frac{(\Delta\xi)^2}{4}\right]\mathrm{erfc}\left[\frac{|\Delta\xi|}{2}\right]\right\}. \tag{93}$$

[11] In the more general case when only conditions (1) hold, we should Fourier-transform both Eq. (67) and Eq. (68).

[12] In the limit for $z \gg 2\bar{\gamma}_z^2\sigma_z$, the antisymmetric part of the longitudinal wake function G_A (always defined, as in Eq. (58), from a source disk to a test disk separated of Δs) is dominant with respect to the symmetric part. In this paper we will only analyze this part, that will be used later on to discuss the feasibility of ESASE schemes.

[13] See Eq. (21) of reference [21], where γ has been substituted with $\bar{\gamma}_z$.

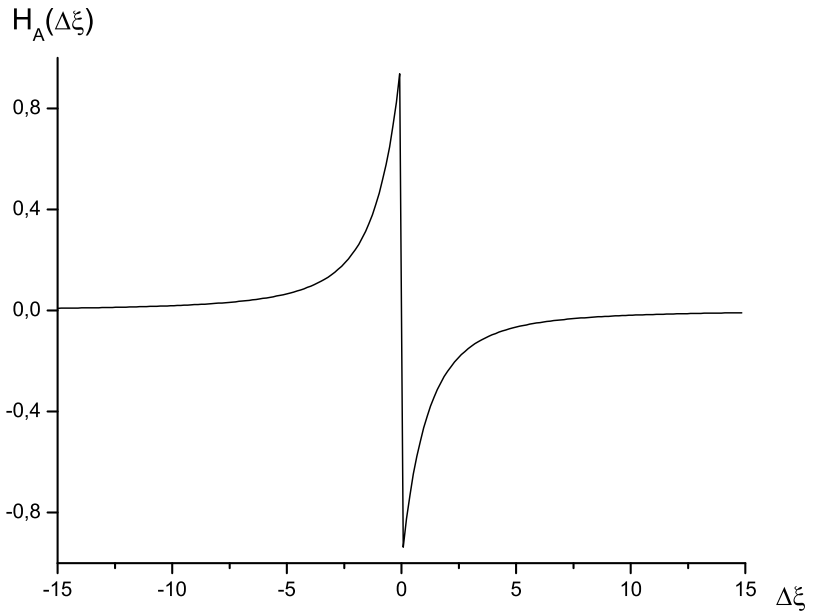

Fig. 1. Plot of the universal function H_A as a function of $\Delta\xi$.

Here we defined $\Delta\xi = \bar{\gamma}_z(\Delta s)/\sigma_\perp$, $\eta = \bar{\gamma}_z\sigma_z/\sigma_\perp$ and $\hat{z} = z/(2\bar{\gamma}_z^2\sigma_z)$. A plot of the universal function H_A, that is the symmetric part of the wake in units of $\bar{\gamma}_z\hat{z}/\sigma_\perp$, as a function of $\Delta\xi$ is given in Fig. 1.

The energy change of a single particle at position s within the bunch due to the reactive part of the wake, averaged over transverse coordinates is given by the convolution $\Delta\mathcal{E}_A(s) = (-e)\int_{-\infty}^{\infty} G_A(\Delta s) f(s-\Delta s) d(\Delta s)$. An explicit expression for $\Delta\mathcal{E}_A/\mathcal{E}_o$, with $\mathcal{E}_o = \gamma m_e c^2$, as a function of $\xi = \bar{\gamma}_z s/\sigma_\perp$:

$$\frac{\Delta\mathcal{E}_A}{\mathcal{E}_o}(\xi) = \frac{I_{\max}}{\gamma I_A}\,\eta\hat{z}\int_{-\infty}^{\infty} d(\Delta\xi) H_A(\xi-\Delta\xi)\exp\left[-\frac{(\Delta\xi)^2}{2\eta^2}\right]. \tag{94}$$

Note that Eq. (94) is a function of ξ but also depends parametrically on η, and may be presented as

$$\frac{\Delta\mathcal{E}_A}{\mathcal{E}_o}\left(\frac{s}{\sigma_z};\eta\right) = \frac{I_{\max}\hat{z}}{\gamma I_A} F\left(\frac{s}{\sigma_z};\eta\right). \tag{95}$$

where we indicated the parametric dependence of η after the semicolon and

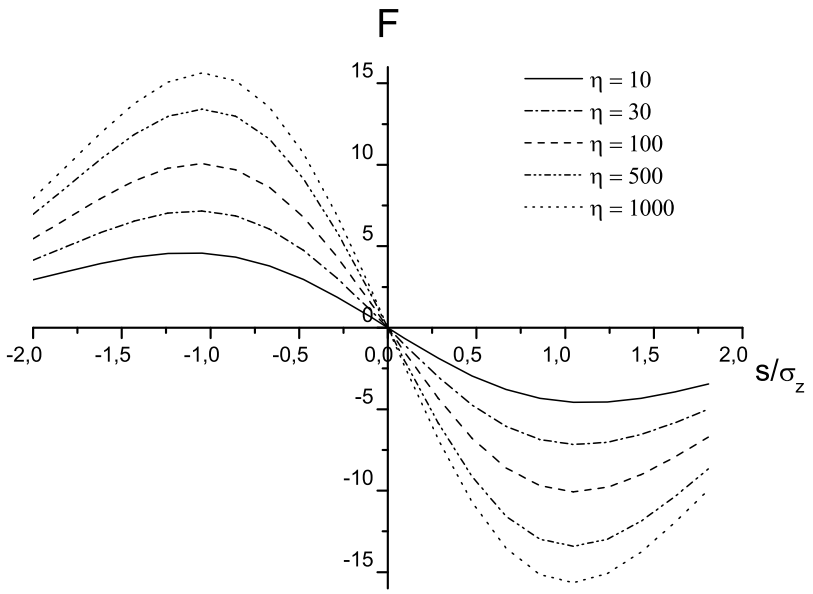

Fig. 2. Plot of F in Eq. (96) as a function of s/σ_z for different values of η.

$$F\left(\frac{s}{\sigma_z};\eta\right)=\int_{-\infty}^{\infty} d(\Delta\xi)\,\eta\,H_A\left(\eta\frac{s}{\sigma_z}-\Delta\xi\right)\exp\left[-\frac{(\Delta\xi)^2}{2\eta^2}\right]. \tag{96}$$

A plot of F is given as a function of s/σ_z in Fig. 2 for different values of η.

7 Application to ESASE schemes

We can now give a practical example of application of our work. Namely, we calculate the impact of longitudinal wake fields in ESASE schemes [12, 13, 14]. Here we propose an analysis on a set of parameters referring to the LCLS [8] setup considered in [14]. Similar calculations may be performed on other parameter sets like those for the European XFEL [9].

We consider a beam with normalized emittance after the dispersive section $\epsilon_n \simeq 1.2$ mm mrad (like in Fig. 3 of [13]). We take the average betatron function in the focusing lattice $\beta_f = 18$ m, and $\gamma = 2.8 \cdot 10^4$. This gives a transverse beam size $\sigma_\perp = (\epsilon_n\beta_f/\gamma)^{1/2} \simeq 30$ μm. The longitudinal size of the bunch is $\sigma_z = 50$ nm. The maximal current is about the Alfven current $I_A \simeq 17$ kA; in fact, $I_{\rm peak} \simeq 18$kA. Finally, the undulator has a period $\lambda_w = 0.03$ m, $K = 3.7$, and the vacuum chamber dimension is $a = 2.5$ mm.

We consider a wavelength $\lambda\!\!\!^{-} \simeq \sigma_z = 50$ nm. We can neglect the vacuum

chamber influence (see Section 2), because $\bar{\gamma}_z \lambda\!\!\!^{-} = 500\ \mu$m, as $\bar{\gamma}_z \simeq 10^4$, and $\bar{\gamma}_z \lambda\!\!\!^{-} \ll a = 2.5$ mm. The overtaking length is $2\lambda\!\!\!^{-}\bar{\gamma}_z^2 \simeq 10$ m. The saturation length is about $L_s = 50$ m. Thus $\hat{z} = 5$, according to the definition in Section 6 and we can use our asymptotic expression. Moreover $\eta = \bar{\gamma}_z \sigma_z / \sigma_\perp \simeq 16.7$. From fig. 2 (or from direct calculations) one can see that the maximal value assumed by $F(s/\sigma_z, \eta)$ for $\eta = 16.7$ is about $F_{\max} \simeq 6$. It follows that the energy-chirp peak-to-peak is given by (see Eq. (94) or Eq. (95)) :

$$\Delta\mathcal{E}_{A,\text{peak}} = 2 m_e c^2 \frac{I_{\max}}{I_A} \hat{z} F_{\max} \simeq 30 \text{ MeV} . \tag{97}$$

In contrast to this, estimations in [14] indicate "a swing in energy of 2.4 MeV". The reason for this large discrepancy is due to the fact that in reference [14], where it is correctly recognized that the "most significant cause for concern is the longitudinal space charge forces", the Lorentz factor γ is incorrectly used in place of $\bar{\gamma}_z$. In fact, as it is clearly stated in that reference : "While this expression [14] has been derived for beam lines containing only drift sections and focusing elements, we apply it *without modification* to the present case where the electron beam is passing through the undulator and oscillates almost rigidly with a deviation of less than 1μm".

In addition to this, it should be noted that energy chirp is also accumulated in the free-space between the dispersive section and the undulator, worsening the situation even more. In the LCLS case [13], the dispersive section is a dogleg located about 200 m from the undulator. One should account for the energy chirp accumulated in this region too, and sum it to that in Eq. (97). To this extent, reference [21] can be used. The overtaking length is now $2\lambda\!\!\!^{-}\gamma^2 \simeq 80$ m, so that $\hat{z} = 2.5$ and our asymptotic expression for the wake are still valid with some accuracy. Using the same procedure as for the wake inside the undulator (but considering γ instead of $\bar{\gamma}_z$), we obtain an extra energy chirp of about $\Delta\mathcal{E}_{A,\text{peak}} \simeq 20$ MeV.

The sum of contributions from the straight section after the dogleg and from the undulator amounts to about 50 MeV. Although the energy chirp is non-linear, in order to estimate the magnitude of the effect we can use the linear energy chirp parameter $\hat{\alpha}$ defined in [15, 16]. The effect of linear energy chirp starts to play a significant role on the FEL gain when $\hat{\alpha} \gtrsim 1$. Intuitively, this means that the relative energy change becomes comparable with the FEL parameter on the scale of the coherence length. The chirp parameter is defined as $\hat{\alpha} = -(\gamma\omega\rho_{1D}^2)^{-1} \cdot d\gamma/dt$, ρ_{1D} being the one-dimensional ρ-parameter in FEL theory defined as (see [29]):

[14] Eq. (3) of reference [14].

$$\rho_{1\mathrm{D}} = \frac{\lambda_w}{4\pi}\left[\frac{2\pi^2 j_o K^2 A_{JJ}^2}{I_A \lambda_w \gamma^3}\right]^{1/3}, \tag{98}$$

where j_o is the beam current density and the coupling factor A_{JJ}, for a planar undulator, is given by $A_{JJ} = J_0(Q) - J_1(Q)$, where $Q = K^2/(2+K^2)$. For ESASE schemes at LCLS $I_{\mathrm{peak}} = 18$ kA, and we have $\rho_{1D} \simeq 10^{-3}$. Using an estimated peak-to-peak chirp of 50 MeV we obtain $\hat{\alpha} \simeq 1$. Thus, the saturation length is significantly modified [16]. This is a reason of concern, because ESASE schemes are based on the assumption that the nominal saturation length of about 80 m is shortened to about 50 m, that is only 37.5% less. The effect described here is fundamental, in the sense that it cannot be avoided by fine tuning of the setup parameters.

Finally, it should be noted that in this paper we did not account for the symmetric part of the wake, related with Transition Undulator Radiation. This part constitutes only a correction to our calculations, as the space-charge wake accumulates along the longitudinal axis, being proportional to $\hat{z}$. We did not include it in this article, because the space-charge wake alone is enough to raise concern. Numerical estimations presented in this paper indicate that effects of energy chirp induces by space-charge longitudinal wake pose a serious threat to the operation of ESASE schemes at LCLS.

8 Conclusions

In this paper we presented a theory of wake fields in an XFEL system, with particular emphasis to ESASE schemes [12, 13, 14].

We worked with specific constraints on parameters, that are fulfilled in XFEL setups (see Section 2). Namely, we neglected the influence of the vacuum chamber and we assumed that the saturation length is long with respect to the overtaking length. Our results are valid for arbitrary values of the undulator parameter K and in the long wavelength asymptotic, i.e. $\lambda\!\!\!^{-} \gg \lambda\!\!\!^{-}_r$, $\lambda\!\!\!^{-}_r$ being the reduced wavelength of the fundamental harmonic. Note that, for any FEL setup, the lasing part of the bunch is always much longer than $\lambda\!\!\!^{-}_r$ so that condition $\lambda\!\!\!^{-} \gg \lambda\!\!\!^{-}_r$ is very natural. It follows that our results are of practical importance not only in relation with ESASE schemes, but for any FEL setup. We derived expressions for the steady state impedance, that is composed of a radiative and a space-charge part. Radiation field and space-charge field are characterized by different formation lengths: the undulator period $\lambda\!\!\!^{-}_w$ and the overtaking length $2\lambda\!\!\!^{-}\bar{\gamma}_z^2$, respectively. As a result, the steady state radiative part of the impedance can be applied for any undulator system (with $N_w \gg 1$), whereas the steady state space-charge part of the impedance can be used only assuming that the saturation length is

long with respect to the overtaking length, which limits its practical region of applicability. Non-steady state results for the space-charge part of the impedance can be obtained applying methods presented in [21] and [30].

After having dealt with a generic expression for the steady-state impedance, we specialized our theory to the case when the transverse beam size $\sigma_\perp^2 \gg \lambda\lambda_w$. Major simplifications arise in this case: in particular, space-charge contributions to impedance and wake dominate with respect to radiative contributions. In this particular condition, that is practically fulfilled for ESASE XFEL setups, we showed that the (antisymmetric) wake can be given in terms of an asymptotic expression for the wake generated by a beam in uniform motion along the longitudinal axis (see [21]), provided that the Lorentz factor γ is consistently substituted with the average longitudinal Lorentz factor $\bar{\gamma}_z$. Final expressions are presented in the case of a planar undulator. However, there are no specific effects related with such choice, and our work may be straightforwardly extended to the case of a helical undulator as well.

We applied our theory to calculate the effects of longitudinal wake fields on ESASE schemes. Our conclusion is that longitudinal wake fields pose a threat to the practical realization of ESASE schemes. This finding is in contrast with estimations in literature, where no important detrimental effect is foreseen. The reason for this contrast is an incorrect application, in literature, of expressions that are valid for beam lines containing drift sections and focusing elements to describe the case of XFEL undulators, where the longitudinal Lorentz factor is sensibly different.

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