A non-perturbative study of non-commutative U(1) gauge theory *)

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We study U(1) gauge theory on a 4d non-commutative torus, where two directions are non-commutative. Monte Carlo simulations are performed after mapping the regularized theory onto a U(N) lattice gauge theory in $d = 2$. At intermediate coupling strength, we find a phase in which open Wilson lines acquire non-zero vacuum expectation values, which implies the spontaneous breakdown of translational invariance. In this phase, various physical quantities obey clear scaling behaviors in the continuum limit with a fixed non-commutativity parameter θ, which provides evidence for a possible continuum theory. In the weak coupling symmetric phase, the dispersion relation involves a negative IR-singular term, which is responsible for the observed phase transition.

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§1. Introduction

Non-commutative (NC) geometry has been studied extensively as a modification of our notion of space-time at short distances. It has recently attracted much attention since gauge theories on a NC geometry have been shown to appear as a low energy limit of string theories with a background tensor field.[1)] NC geometry also appears naturally in matrix model formulations of string theory.[2),3)] Introducing non-commutativity to the space-time coordinates may change the infrared dynamics drastically at the quantum level due to the so-called UV/IR mixing.[4)] The same effect also poses a severe problem in the renormalization procedure within perturbation theory since a new type of IR divergences appears in non-planar diagrams.

The finite lattice formulation[5)] (extending an earlier work[3)]) regularizes such divergences as well as the ordinary UV divergences. It therefore provides a non-

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perturbative framework to establish the existence of a consistent field theory on a NC geometry. In ref. 6) a simple field theory — 2d U(1) gauge theory — on a NC plane has been studied by Monte Carlo simulation, and the existence of a finite continuum limit has been confirmed. The ultraviolet dynamics is described by the commutative 2d U(∞) gauge theory. On the other hand, Wilson loops of large area (with a variety of shapes) pick up a complex phase linear in the loop area, in the spirit of the Aharonov-Bohm effect[6] (but their expectation values are shape-dependent, in contrast to the commutative plane[7]).

As an interesting physical consequence of the UV/IR mixing in the case of scalar field theory, ref. 8) predicted the existence of a "striped phase" for the NC $\lambda\phi^4$ model in dimensions $d > 2$. In this phase, non-zero Fourier modes of the scalar field acquire a vacuum expectation value and break the translational invariance spontaneously. The existence of this new phase was fully established by Monte Carlo simulations.[9] Such a phase has also been observed in the 2d lattice model,[10),9] but in that case the continuum limit is still under investigation. An analogous phase was observed in simulations involving a fuzzy sphere instead of a NC plane.[11]

In the case of 4d U(1) NC gauge theories, the one-loop calculation of the effective action[12] suggests that the perturbative vacuum is unstable against the condensation of the Wilson line. This causes the spontaneous symmetry breaking (SSB) of the translational invariance, since the open Wilson lines carry non-zero momenta.[13] Whether a stable *non-perturbative* vacuum exists or not is an interesting question, which we have addressed in ref. 14) from first principles using the lattice formulation.[5]

§2. Lattice gauge theory on a NC geometry

NC geometry is characterized by the commutation relation $[\hat{x}_\mu, \hat{x}_\nu] = i\,\Theta_{\mu\nu}$ among the space-time coordinates $\hat{x}_\mu$, where $\Theta_{\mu\nu}$ is the non-commutativity tensor. Here we consider the 4d Euclidean space-time $\hat{x}_\mu$ ($\mu = 1, \cdots, 4$) with the non-commutativity introduced only in the $\mu = 1, 2$ directions, i.e.

$$\Theta_{12} = -\Theta_{21} = \theta \ , \tag{2·1}$$

and $\Theta_{\mu\nu} = 0$ otherwise. Since we have two commutative directions, we may regard one of them as the Euclidean time. This allows us to alleviate the well-known problems concerning causality and unitarity.

The lattice regularized version of gauge theory on NC geometry can be defined by an analog of Wilson's plaquette action[5]

$$S = -\beta \sum_x \sum_{\mu<\nu} U_\mu(x) \star U_\nu(x + a\hat{\mu}) \star U_\mu(x + a\hat{\nu})^* \star U_\nu(x)^* + \text{c.c.} \ , \tag{2·2}$$

where the symbol $\hat{\mu}$ represents a unit vector in the μ-direction and we have introduced the lattice spacing a. Here the star product (denoted by $\star$) encodes the non-commutativity of the space-time.

In order to study the lattice NC theory (2·2) by Monte Carlo simulations, it is crucial to reformulate it in terms of matrices.[5] In the present setup (2·1), with

two NC directions and two commutative ones, the transcription applies only to the NC plane whereas the commutative plane remains untouched. Let us decompose the four-dimensional coordinate as $x \equiv (y, z)$, where $y \equiv (x_1, x_2)$ and $z \equiv (x_3, x_4)$ represent two-dimensional coordinates in the NC and in the commutative plane, respectively. We use a one-to-one map between a field $\varphi(x)$ on the four-dimensional $N \times N \times L \times L$ lattice and a $N \times N$ matrix field $\hat{\varphi}(z)$ on a two-dimensional $L \times L$ lattice. This map yields the following correspondence

$$\varphi_1(y,z) \star \varphi_2(y,z) \quad \Longleftrightarrow \quad \hat{\varphi}_1(z)\,\hat{\varphi}_2(z)\ , \tag{2·3}$$

$$\varphi(y + a\hat{\mu}, z) \quad \Longleftrightarrow \quad \Gamma_\mu\,\hat{\varphi}(z)\,\Gamma_\mu^\dagger\ , \tag{2·4}$$

$$\frac{1}{N^2}\sum_y \varphi(y,z) \quad \Longleftrightarrow \quad \frac{1}{N}\,\mathrm{tr}\,\hat{\varphi}(z)\ . \tag{2·5}$$

The SU(N) matrices Γ_μ ($\mu = 1, 2$), which represent a shift in a NC direction, satisfy the 't Hooft-Weyl algebra with the matrix size N being odd. For this construction,[6),9)] the non-commutativity parameter θ in (2·1) is given by

$$\theta = \frac{1}{\pi} N a^2\ . \tag{2·6}$$

Note that the extent in the NC directions Na goes to infinity in the continuum limit $a \to 0$ at fixed θ.

Using this map, the link variables $U_\mu(x)$ are mapped to a $N \times N$ unitary matrix field $\hat{U}_\mu(z)$ on the two-dimensional $L \times L$ lattice. The action (2·2) can be rewritten in terms of $\hat{U}_\mu(z)$. By performing a field redefinition $V_\mu(z) \equiv \hat{U}_\mu(z)\Gamma_\mu$ for $\mu = 1, 2$ and $V_\mu(z) \equiv \hat{U}_\mu(z)$ otherwise, we arrive at

$$\begin{aligned}
S &= S_{\mathrm{NC}} + S_{\mathrm{com}} + S_{\mathrm{mixed}}\ , \\
S_{\mathrm{NC}} &= -N\beta\mathcal{Z}_{12}\sum_z \mathrm{tr}\,\Big(V_1(z)\,V_2(z)\,V_1(z)^\dagger\,V_2(z)^\dagger\Big) + \mathrm{c.c.}\ , \\
S_{\mathrm{com}} &= -N\beta\sum_z \mathrm{tr}\,\Big(V_3(z)\,V_4(z+a\hat{3})\,V_3(z+a\hat{4})^\dagger\,V_4(z)^\dagger\Big) + \mathrm{c.c.}\ , \\
S_{\mathrm{mixed}} &= -N\beta\sum_z\sum_{\mu=1}^{2}\sum_{\nu=3}^{4} \mathrm{tr}\,\Big(V_\mu(z)\,V_\nu(z)\,V_\mu(z+a\hat{\nu})^\dagger\,V_\nu(z)^\dagger\Big) + \mathrm{c.c.}\ .
\end{aligned} \tag{2·7}$$

This action possesses the U(1)^2 symmetry $V_\mu(z) \mapsto \mathrm{e}^{i\alpha_\mu}V_\mu(z)$ for $\mu = 1, 2$, which is related to the translational symmetry in the NC directions of the action (2·2).

§3. Phase diagram

Let us first investigate the phase structure of the lattice model (2·7). As an order parameter for the spontaneous breaking of the U(1)^2 symmetry, we define a gauge invariant operator

$$P_\mu(n) = \frac{1}{NL^2}\sum_z \mathrm{tr}\,\Big(V_\mu(z)^n\Big)\ , \tag{3·1}$$

for $\mu = 1, 2$. This operator corresponds to the open Wilson line carrying a momentum with the absolute value[5),6)]

$$p = \frac{2\pi k}{Na} , \quad k = \begin{cases} \frac{n}{2} & \text{for even } n , \\ \frac{n+N}{2} & \text{for odd } n . \end{cases} \tag{3·2}$$

Since the operator $P_\mu(n)$ with odd n carries a momentum on the cutoff scale, it does not couple to excitations that survive in the continuum limit. Therefore we will focus mainly on the even n case in what follows.

In fig. 1 we plot $\langle |P_1(n)| \rangle$ against β for $n = 2$. We observe that there is a phase, in which the order parameter becomes non-zero. On the other hand, the quantity $\langle |P_1(n)| \rangle$ for *odd* n takes tiny values throughout the whole region of β. This implies that the $\mathrm{U}(1)^2$ symmetry is broken down to $(\mathrm{Z}_2)^2$ in this phase, which we refer to as the "broken phase". The critical point between the broken phase and the weak coupling phase, denoted as β_{c}, increases as $\beta_{\mathrm{c}} \sim N^2$.

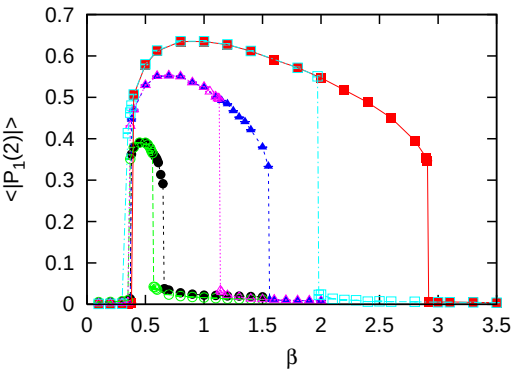

Fig. 1. The order parameter $\langle |P_1(n)| \rangle$ is plotted against β for $n = 2$. The system size is $N = 15$ (circles), $N = 25$ (triangles) and $N = 35$ (squares). The closed (open) symbols represent results obtained with increasing (decreasing) β, which show a clear hysteresis behavior.

§4. Continuum limit

In this section we investigate whether it is possible to fine-tune β as a function of N in such a way that physical quantities scale. For that purpose, let us next consider closed Wilson loops, which play an important role in commutative gauge theories as a criterion for confinement. In the present case, since we introduce non-commutativity only in two directions, there are three kinds of square-shaped Wilson loops depending on their orientations. For instance, the Wilson loops in the NC plane and the commutative plane are defined respectively as

$$W_{12}(n) = (\mathcal{Z}_{12})^{n^2} \frac{1}{NL^2} \sum_z \mathrm{tr} \left(V_1(z)^n \, V_2(z)^n \, V_1(z)^{\dagger n} \, V_2(z)^{\dagger n} \right) ,$$

$$W_{34}(n) = \frac{1}{NL^2} \sum_z \mathrm{tr} \left(\mathcal{V}_3(z,n) \, \mathcal{V}_4(z + na\hat{3}, n) \mathcal{V}_3(z + na\hat{4}, n)^\dagger \, \mathcal{V}_4(z,n)^\dagger \right) , \tag{4·1}$$

using the parallel transporter in the commutative directions

$$\mathcal{V}_\nu(z,n) \equiv V_\nu(z) V_\nu(z + a\hat{\nu}) \cdots V_\nu(z + (n-1)a\hat{\nu}) . \tag{4·2}$$

In the following we set $a = 1$ for $N = 45$ as a convention, and the lattice spacing a for other N is determined through (2·6) with $\theta = \frac{45}{\pi} \simeq 14.3$. As a practical strategy

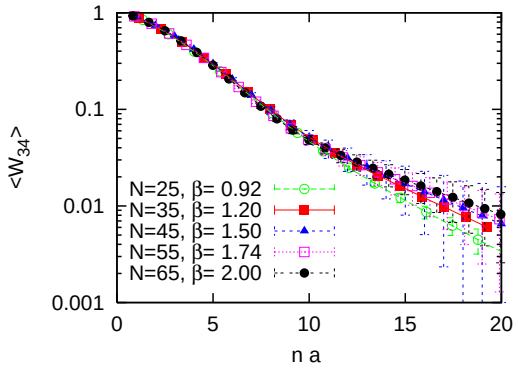

Fig. 2. The expectation value of the Wilson loop in the commutative plane.

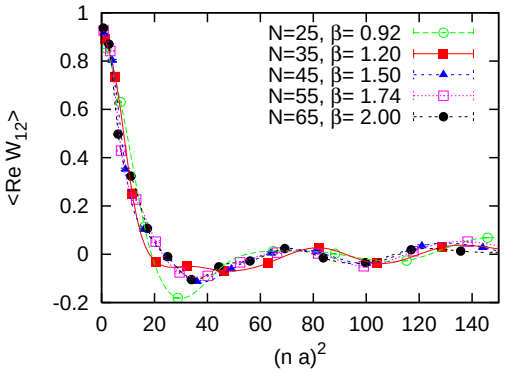

Fig. 3. The expectation value of the Wilson loop in the NC plane.

to fine-tune β, we first optimize the scaling behavior for the expectation value of the square Wilson loop $W_{34}(n)$ in the commutative directions. Fig. 2 shows the result. The horizontal axis represents the physical side length na of the loop. We observe a clear scaling behavior, and the scaling region extends as N increases. In fact the optimal β increases with N much slower than the lower critical point β_c between the broken phase and the weak coupling phase, which grows as N^2. This implies that we remain in the broken phase in the continuum limit. In fig. 3 we plot the expectation value of the Wilson loop in the NC plane with the *same* sets of parameters as the ones used to obtain fig. 2. We do observe a compelling scaling behavior. We have also confirmed the scaling of other quantities.[14)]

§5. Dispersion relation

In this section we discuss the dispersion relation in the symmetric phase. As we mentioned before, in the present setup we regard one of the commutative coordinates (say, x_4) as the Euclidean time. From the exponential decay of the two-point correlation function of open Wilson line operators separated in the time direction, we can extract the energy of a state that couples to this operator. Similar studies have been done also in the case of NC scalar field theory.[9)]

Let us define the open Wilson line operator at a fixed time x_4 as

$$P_\mu(x_4, n) \equiv \frac{1}{NL} \sum_{x_3} \mathrm{tr} \left(V_\mu(x_3, x_4)^n \right)$$

for $\mu = 1, 2$, which has a zero momentum component in the x_3 direction and a non-zero momentum component (3·2) in a NC direction depending on n. Then we define the two-point correlation function of the open Wilson lines

$$C_n(\tau) \equiv \frac{1}{2} \sum_{\mu=1}^{2} \sum_{x_4} \left\langle P_\mu(x_4, n)^* \cdot P_\mu(x_4 + \tau, n) \right\rangle \qquad (5\cdot1)$$

with a separation τ in the temporal direction.

Fig. 4 shows the dispersion relation obtained from the two-point correlation function (5·1) for even n. The parameters (N, β) are chosen as in the broken phase. Namely, a is still determined through (2·6) with $\theta = \frac{45}{\pi} \simeq 14.3$, and we fine-tune β at each N in such a way that the scaling behavior of the Wilson loops in the commutative plane is optimized. It turns out that the data points (E, p) for different N lie to a good approximation on a single curve

$$E^2 = p^2 - \frac{c}{(\theta p)^2} \qquad (5\cdot2)$$

with $c \simeq 0.1285$. Due to the negative sign of the second term in (5·2), the usual Lorentz invariant (massless) dispersion relation is bent down. The IR singularity is regularized on the finite lattice since the smallest non-zero momentum (which corresponds to $n = 2$) is given by $\frac{2\pi}{Na} \propto \frac{1}{\sqrt{N}}$. However, if one increases N at fixed θ, the energy at the smallest non-zero momentum vanishes at some N, and one enters the broken phase. Therefore we cannot take the continuum limit in the symmetric phase. We have also studied the dispersion relation in the *broken* phase, which reveals the existence of a Nambu-Goldstone mode associated with the SSB of the $U(1)^2$ symmetry.[14)]

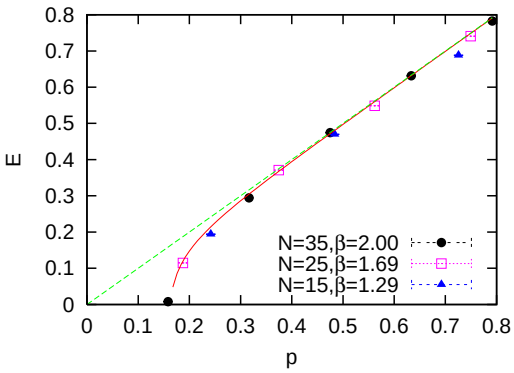

Fig. 4. The dispersion relation in the symmetric phase. The energy E obtained from the two-point correlation function (5·1) is plotted against the momentum p.

§6. Summary and discussion

To summarize, we studied four-dimensional gauge theory in NC geometry from first principles based on its lattice formulation. In particular we clarified the fate of the tachyonic instability encountered in perturbative calculations. The lattice formulation is suited for such a study since the IR singularity responsible for the instability is regularized in a gauge invariant manner, and we can trace the behavior of the system as the regularization is removed. This revealed the existence of a first order phase transition associated with the spontaneous breakdown of the $U(1)^2$ symmetry, which corresponds to the translational symmetry in the NC directions.

The dynamical extent in the NC directions — defined through the eigenvalue distribution of matrices — turns out to be finite in the continuum limit.[14)] An analogous first order phase transition is found in gauge theories on fuzzy manifolds,[15)] where the fuzzy manifolds collapse at sufficiently large couplings. The instability in those cases is due to the uniform condensation of a scalar field on the fuzzy manifold. The phenomenon that the space-time itself becomes a dynamical object is characteristic to gauge theories on NC geometry. This may be related to the dynamical compactification of extra dimensions in string theory.[16)] See also ref. 17), which uses fuzzy spheres for compactified dimensions.

On the other hand, if we wish to obtain a phenomenologically viable *4d model*,

we may stay in the symmetric phase by keeping the UV cutoff finite and view the NC gauge theory as an effective theory of a more fundamental theory. A θ-deformed dispersion relation for the photon such as the one displayed in fig. 4 should then have implications on observational data from blazars (highly active galactic nuclei), which are assumed to emit bursts of photons simultaneously, covering a broad range of energy. In particular, a relative delay of these photons depending on the frequency could hint at the existence of a NC geometry.[18)]

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