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Heavy Flavour Production in Deep–Inelastic Scattering - Two–Loop Massive Operator Matrix Elements and Beyond*

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We calculate the O(ε)–term of the two–loop massive operator matrix elements for twist 2–operators, which contribute to the heavy flavour Wilson coefficients in unpolarized deep–inelastic scattering in the asymptotic limit $Q^2 \gg m^2$. Our calculation was performed in Mellin space using Mellin–Barnes integrals and generalized hypergeometric functions. The O(ε)–term contributes in the renormalization at 3–loop order.

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1. Introduction

In the range of small values of the Bjørken variable x, the contribution of heavy flavour corrections to deep–inelastic structure functions is of the order of 20–40% and hence has to be known in the QCD analyses of the structure functions for high precision extractions of the parton densities and the QCD scale $\Lambda_{\rm QCD}$ [1]. In the full kinematic range, a semi–analytic result for the heavy flavour Wilson coefficients up to next–to–leading order exists [2], with a fast implementation to Mellin–space given in [3], whereas a fully analytic result to O(α_s^2) could be achieved in the limit $Q^2 \gg m^2$ in [4], Q^2 denoting the virtuality of the exchanged photon and m^2 the mass of the heavy quark. The corresponding Wilson coefficient for F_L at $O(\alpha_s^3)$ was calculated in [5]. In this limit, the heavy–flavour contributions can be expressed as a convolution of light–flavour Wilson coefficients and massive operator matrix elements (OMEs) between light partonic states. The results in [4] have been obtained using integration–by–parts techniques. We performed a first recalculation of these OMEs in Mellin–space [6, 7], using both Mellin–Barnes integrals and generalized hypergeometric functions. This shifts the

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problem of solving complicated integrals of Nielsen–type in [4], to the calculation of sums over products of harmonic sums [8, 9] depending on the Mellin–parameter N, weighted binomials and Euler Beta–functions. The expressions in our result are even on the diagrammatic level considerably smaller than the ones obtained in [4], and are seemingly more suitable to the problem. In this paper, we show a first step towards the $O(\alpha_s^3)$–term of the heavy–flavour Wilson coefficients, by calculating in dimensional regularization the $O(\varepsilon)$–term of the two–loop OMEs, with $\varepsilon = D - 4$.

2. Method

Our calculation is performed in the asymptotic limit $Q^2 \gg m^2$, applying the light–cone expansion, where, as the massless renormalization group equation (RGE) gives a splitting of the deep–inelastic structure functions $F_{2/L}$ into a convolution of perturbatively calculable Wilson coefficients and non–perturbative parton distribution functions, the massive RGE allows to write the heavy flavour contribution to the twist–2 Wilson coefficients as a convolution of light–flavour Wilson coefficients and massive operator matrix elements [4]:

$$H^{\rm S,NS}_{(2,L),i}\left(\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = \underbrace{A^{\rm S,NS}_{k,i}\left(\frac{m^2}{\mu^2}\right)}_{\text{massive OMEs}} \otimes \underbrace{C^{\rm S,NS}_{(2,L),k}\left(\frac{Q^2}{\mu^2}\right)}_{\text{light Wilson coefficients}}.$$

These OMEs are universal objects, calculable via the corresponding flavour singlet, pure–singlet and non–singlet operators between partonic states, determining the non–power contributions in m^2/Q^2. The process dependence is then solely given by the massless light Wilson coefficients [10].
The OMEs contain ultraviolet and collinear divergences, the former being removed through renormalization, the latter absorbed into the parton distribution functions. To two–loop order, the renormalized gluonic OME reads:

$$\begin{aligned} A^{(2)}_{Qg} &= \frac{1}{8}\left\{\widehat{P}^{(0)}_{qg}\otimes\left[P^{(0)}_{qq}-P^{(0)}_{gg}+2\beta_0\right]\right\}\ln^2\left(\frac{m^2}{\mu^2}\right) - \frac{1}{2}\widehat{P}^{(1)}_{qg}\ln\left(\frac{m^2}{\mu^2}\right) \\ &\quad + \overline{a}^{(1)}_{Qg}\otimes\left[P^{(0)}_{qq}-P^{(0)}_{gg}+2\beta_0\right] + a^{(2)}_{Qg}\,, \end{aligned}$$

with similar expressions for the quarkonic contributions. Here, $P^{(k-1)}_{ij}$ are the kth–loop splitting functions, β_0 is the lowest order expansion coefficient of the β–function, and μ^2 the renormalization and factorization scale. $a^{(k)}_{ij}$ and $\bar{a}^{(k)}_{ij}$ are the $O(\varepsilon^0)$ resp. $O(\varepsilon)$-terms in the expansion of the OME. As

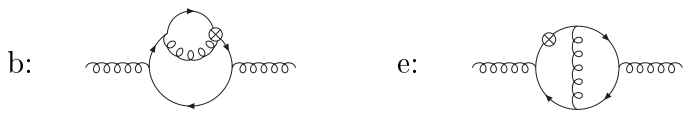

Fig. 1. Two example diagrams contributing to the gluonic OME, with all fermion lines massive and external momentum $p^2 = 0$.

a first step towards a $O(\alpha_s^3)$ calculation, one needs each of these quantities to one additional order in ε, since they then enter the constant term of the OME by multiplying the corresponding splitting functions. The $O(\varepsilon)$–term of the OME $A_{ij}^{(2)}$, $\bar{a}_{ij}^{(2)}$, is a new result presented here and the main topic of this calculation.

3. Calculation

The diagrams can be grouped into two sets: one–loop in one–loop insertions and generic two–loop diagrams. They are calculated using `FORM` [11] and `MAPLE` programs. Figure 1 shows some diagrams contributing to the gluonic OME.

The rules for operator insertion are, e.g., given in [12]. The calculation is done on the one hand by the use of Mellin–Barnes integrals to produce numeric results. These results serve as a check for the analytic results, obtained by expressing the diagrams as generalized hypergeometric functions which are first expanded in ε and then summed up to the desired order.
The application of Mellin–Barnes integrals for scalar diagrams in our framework has already been explained in some detail before, cf. e.g. [6] (see also [13]). The idea is to express in a loop–by–loop manner the sub–diagram of a full diagram into a Mellin–Barnes representation and to combine this with the remaining part. For full diagrams with a numerator structure, one can make heavily use of the fact that the light–like vector Δ occurring in the numerators obeys $\Delta^2 = 0$. This reduces the integrals to be calculated to a smaller set. After finding a suitable Mellin–Barnes integral representation, we use the mathematica package `MB` [14] to numerically calculate the

Table 1. Mellin moments N= 2 and N= 6 for diagram e.

Diagram	N	$1/\varepsilon^2$	$1/\varepsilon$	1	ε	ε^2
e	2	8.88889	−11.2593	9.82824	−12.8921	2.39145
	6	2.93878	−4.24257	3.39094	−4.3892	0.826978

Mellin–Barnes integrals for fixed values of Mellin–N, up to a given order in ε. As an example, Table 1 shows up to some generic multiplicative factors the results for the full diagram e of Figure 1.

By closing the contour and applying the theorem of residues, it is in principle possible to even obtain analytic results from the Mellin–Barnes representation [6]. However, the way of hypergeometric functions turned out to be more appropriate for the calculation of analytic results. In this case, one first introduces Feynman parameters and does the two momentum integrations. As an example, the scalar version of diagram e of Figure 1 with all propagators to the power one, evaluates to:

$$I_e := \frac{(\Delta p)^{N-1}\Gamma(1-\varepsilon)}{N(N+1)(4\pi)^{4+\varepsilon}(m^2)^{1-\varepsilon}} \int_0^1 dz\, dw\, \frac{w^{-1-\varepsilon/2}(1-z)^{\varepsilon/2}z^{-\varepsilon/2}}{(z+w-wz)^{1-\varepsilon}} \left[1 - w^{N+1} - (1-w)^{N+1}\right] .$$

On rewriting this Feynman parameter integral into a ${}_PF_Q$–function, one obtains a product of ${}_3F_2$–functions and the Euler Beta–function $B(a,b) := \Gamma(a)\Gamma(b)/\Gamma(a+b)$:

$$\begin{aligned} I_e = \frac{c}{N(N+1)} \exp\left\{\sum_{i=2}^{\infty} \zeta_i \frac{\varepsilon^i}{i}\right\} \Bigg\{ & \\ & B(\varepsilon/2+1, 1-\varepsilon/2) B(1, -\varepsilon/2)\ {}_3F_2\left[\begin{array}{c} 1-\varepsilon, 1, 1+\varepsilon/2 \\ 2, 1-\varepsilon/2 \end{array}; 1\right] \\ & - B(\varepsilon/2+1, 1-\varepsilon/2) B(1, N+1-\varepsilon/2)\ {}_3F_2\left[\begin{array}{c} 1-\varepsilon, 1, 1+\varepsilon/2 \\ 2, N+2-\varepsilon/2 \end{array}; 1\right] \\ & - B(\varepsilon/2+1, 1-\varepsilon/2) B(N+2, -\varepsilon/2)\ {}_3F_2\left[\begin{array}{c} 1-\varepsilon, N+2, 1+\varepsilon/2 \\ 2, N+2-\varepsilon/2 \end{array}; 1\right] \Bigg\} \\ c := \frac{S_\varepsilon^2}{(4\pi)^4 (m^2)^{1-\varepsilon}} (\Delta p)^{N-1} & \end{aligned}$$

One then expands this expression up to the desired order in ε, in this obtaining finite and infinite sums over harmonic sums and Beta–functions:

$$I_e = \frac{c}{N(N+1)} \sum_{s=1}^{\infty} \left\{ \frac{1}{s^2} - \frac{S_1(s)}{s} + \frac{S_1(N+s)}{s} - \frac{B(N+1,s)}{s} \right\} + O(\varepsilon)$$

It is the next step of summing up the various sums over harmonic sums and more complicated expressions, which constitutes the most difficult part

of the calculation. These sums could be solved among other things using their integral representations, where a certain amount of more complicated sums could be calculated using the mathematica package `SIGMA` [15, 16]. For integral e, one obtains up to $O(\varepsilon)$:

$$I_e = \frac{c}{N(N+1)} \left\{ \frac{S_1^2(N) + 3S_2(N)}{2} + \frac{S_1^3(N) + 3S_1(N)S_2(N) + 8S_3(N)}{12} \varepsilon \right\}$$

In a similar manner, it was possible to calculate all diagrams contributing to the calculation of the $O(\varepsilon)$–term of the two–loop unpolarized OMEs. Here algebraic relations between harmonic sums were used [17].

4. Results

The $O(\varepsilon)$ contributions to the mass–renormalized unpolarized OMEs for the singlet, pure–singlet, and non–singlet cases read:

$$\begin{aligned}
\overline{a}_{Qg}^{(2)} &= T_R C_F \left\{ \frac{2}{3} \frac{(N^2+N+2)(3N^2+3N+2)}{N^2(N+1)^2(N+2)} \zeta_3 + \frac{P_1}{N^3(N+1)^3(N+2)} S_2 \right. \\
&+ \frac{N^4-5N^3-32N^2-18N-4}{N^2(N+1)^2(N+2)} S_1^2 + \frac{N^2+N+2}{N(N+1)(N+2)} \Big(16 S_{2,1,1} - 8 S_{3,1} \\
&- 8 S_{2,1} S_1 + 3 S_4 - \frac{4}{3} S_3 S_1 - \frac{1}{2} S_2^2 - S_2 S_1^2 - \frac{1}{6} S_1^4 + 2 \zeta_2 S_2 - 2 \zeta_2 S_1^2 - \frac{8}{3} \zeta_3 S_1 \Big) \\
&- 8 \frac{N^2-3N-2}{N^2(N+1)(N+2)} S_{2,1} + \frac{2}{3} \frac{3N+2}{N^2(N+2)} S_1^3 \\
&+ \frac{2}{3} \frac{3N^4+48N^3+43N^2-22N-8}{N^2(N+1)^2(N+2)} S_3 + 2 \frac{3N+2}{N^2(N+2)} S_2 S_1 + 4 \frac{S_1}{N^2} \zeta_2 \\
&+ \frac{N^5+N^4-8N^3-5N^2-3N-2}{N^3(N+1)^3} \zeta_2 \\
&\left. - 2 \frac{2N^5-2N^4-11N^3-19N^2-44N-12}{N^2(N+1)^3(N+2)} S_1 + \frac{P_2}{N^5(N+1)^5(N+2)} \right\} \\
&+ T_R C_A \left\{ \frac{N^2+N+2}{N(N+1)(N+2)} \Big(16 S_{-2,1,1} - 4 S_{2,1,1} - 8 S_{-3,1} - 8 S_{-2,2} - 4 S_{3,1} - \frac{2}{3} \beta''' \right. \\
&+ 9 S_4 - 16 S_{-2,1} S_1 + \frac{40}{3} S_1 S_3 + 4 \beta'' S_1 - 8 \beta' S_2 + \frac{1}{2} S_2^2 - 8 \beta' S_1^2 + 5 S_1^2 S_2 + \frac{1}{6} S_1^4 \\
&\left. - \frac{10}{3} S_1 \zeta_3 - 2 S_2 \zeta_2 - 2 S_1^2 \zeta_2 - 4 \beta' \zeta_2 - \frac{17}{5} \zeta_2^2 \Big) \right. \\
&+ \frac{4(N^2-N-4)}{(N+1)^2(N+2)^2} \Big(-4 S_{-2,1} + \beta'' - 4 \beta' S_1 \Big) - \frac{2}{3} \frac{N^3+8N^2+11N+2}{N(N+1)^2(N+2)^2} S_1^3 \\
&+ 8 \frac{N^4+2N^3+7N^2+22N+20}{(N+1)^3(N+2)^3} \beta' + 2 \frac{3N^3-12N^2-27N-2}{N(N+1)^2(N+2)^2} S_2 S_1
\end{aligned}$$

$$
\begin{aligned}
&-\frac{16}{3}\frac{N^5+10N^4+9N^3+3N^2+7N+6}{(N-1)N^2(N+1)^2(N+2)^2}S_3 - 8\frac{N^2+N-1}{(N+1)^2(N+2)^2}\zeta_2 S_1 \\
&-\frac{2}{3}\frac{9N^5-10N^4-11N^3+68N^2+24N+16}{(N-1)N^2(N+1)^2(N+2)^2}\zeta_3 - \frac{P_3}{(N-1)N^3(N+1)^3(N+2)^3}S_2 \\
&-\frac{2P_4}{(N-1)N^3(N+1)^3(N+2)^2}\zeta_2 - \frac{P_5}{N(N+1)^3(N+2)^3}S_1^2 \\
&+\frac{2P_6}{N(N+1)^4(N+2)^4}S_1 - \frac{2P_7}{(N-1)N^5(N+1)^5(N+2)^5}\Bigg\} .
\end{aligned}
$$

$$
\begin{aligned}
\overline{a}_{qq,Q}^{\rm NS,(2)} &= T_F C_F \Bigg\{ \frac{4}{3}S_4 + \frac{4}{3}S_2\zeta_2 - \frac{8}{9}S_1\zeta_3 - \frac{20}{9}S_3 - \frac{20}{9}S_1\zeta_2 + 2\frac{3N^2+3N+2}{9N(N+1)}\zeta_3 \\
&\quad + \frac{112}{27}S_2 + \frac{3N^4+6N^3+47N^2+20N-12}{18N^2(N+1)^2}\zeta_2 - \frac{656}{81}S_1 + \frac{P_8}{648N^4(N+1)^4}\Bigg\} \\
\overline{a}_{Qq}^{\rm PS,(2)} &= T_F C_F \Bigg\{ -2\frac{(5N^3+7N^2+4N+4)(N^2+5N+2)}{(N-1)N^3(N+1)^3(N+2)^2}\Big(2S_2+\zeta_2\Big) \\
&\quad - \frac{4}{3}\frac{(N^2+N+2)^2\Big(3S_3+\zeta_3\Big)}{(N-1)N^2(N+1)^2(N+2)} + 2\frac{P_9}{(N-1)N^5(N+1)^5(N+2)^4}\Bigg\}.
\end{aligned}
$$

with the polynomials P_i given by

$$
\begin{aligned}
P_1 =& 3N^6+30N^5+15N^4-64N^3-56N^2-20N-8 ,\\
P_2 =& 8N^{10}+24N^9-11N^8-128N^7-195N^6-119N^5-23N^4-27N^3-45N^2 \\
&-24N-4 ,\\
P_3 =& N^9+21N^8+85N^7+105N^6+42N^5+290N^4+600N^3+456N^2+256N+64 \\
P_4 =& (N^3+3N^2+12N+4)(N^5-N^4+5N^2+N+2) ,\\
P_5 =& N^6+6N^5+7N^4+4N^3+18N^2+16N-8 ,\\
P_6 =& 2N^8+22N^7+117N^6+386N^5+759N^4+810N^3+396N^2+72N+32 ,\\
P_7 =& 4N^{15}+50N^{14}+267N^{13}+765N^{12}+1183N^{11}+682N^{10}-826N^9-1858N^8 \\
&-1116N^7+457N^6+1500N^5+2268N^4+2400N^3+1392N^2+448N+64 \\
P_8 =& 1551N^8+6204N^7+15338N^6+17868N^5+8319N^4+944N^3+528N^2 \\
&-144N-432 \\
P_9 =& 5N^{11}+62N^{10}+252N^9+374N^8-400N^6+38N^7-473N^5 \\
&-682N^4-904N^3-592N^2-208N-32 .
\end{aligned}
$$

Here, $S_{i,\ldots,j} \equiv S_{i,\ldots,j}(N)$, $\beta \equiv \beta(N+1)$, and $\zeta_i \equiv \zeta(i)$ is Riemann's Zeta-function. In all results, there is an overall factor $S_\varepsilon^2 a_s^2 (m^2/\mu^2)^\varepsilon$ and an overall factor $(1+(-1)^N)/2$ in all singlet and pure–singlet cases. The above Mellin-space expressions can be converted to x-space using analytic

continuation in N and Mellin inversion [18].

5. Comparison

The results in [4], which are up to constant order in ε, involved 48 basic functions, cf. [7]. Our results for the constant part of the OMEs gave raise to only six harmonic sums, where five of them can be obtained from $S_1(N)$ through differentiation after analytic continuation and using algebraic relations. This leads to an amount of only two basic functions [19]. To order $O(\varepsilon)$, we encounter the following 14 harmonic sums:

$$\{S_1,\ S_2,\ S_3,\ S_4,\ S_{-2}, S_{-3}, S_{-4}\},\quad S_{2,1},\quad S_{-2,1},\quad S_{-3,1},\quad S_{2,1,1},\quad S_{-2,1,1},$$

$$S_{-2,2},\quad S_{3,1}.$$

We can again group the first seven functions into the same class. Additionally, one finds that the function $S_{-2,2}$ depends on $S_{-2,1}$ and $S_{-3,1}$, and the harmonic sum $S_{2,1}$ depends on $S_{3,1}$, which leaves us to order ε with only six basic harmonic sums, as also observed for a large variety of other two–loop processes, cf. [20].

6. Conclusion

We have calculated the $O(\varepsilon)$–term of the unpolarized massive two–loop OMEs, contributing to the heavy–flavour Wilson coefficients in the asymptotic limit $Q^2 \gg m^2$, as a first step towards the $O(\alpha_s^3)$–term of these Wilson coefficients. The calculation was done in Mellin space, where numeric results were obtained by the use of Mellin–Barnes integrals, whereas analytic results were calculated using generalized hypergeometric functions. After applying algebraic relations, the analytic result for the $O(\varepsilon)$–term is expressible in only six basic harmonic sums.

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