Revised Version

PoS-(ACAT-)081-(2007-)
DESY 07-104
SFB/CPP-07-39
HEPTOOLS 07-017

DESY 07–104

SFB/CPP–07–39

HEPTOOLS 07-017

arXiv:0707.3567v2 [hep-ph] 3 Jul 2008

Automatizing the application of Mellin-Barnes representations for Feynman integrals

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Feynman diagrams may be evaluated by Mellin-Barnes representations of their Feynman parameter integrals in $d = 4 - 2\varepsilon$ dimensions. Recently, the Mathematica toolkit AMBRE has been developed for the automatic derivation of such representations with a loop-by-loop approach. We describe the package and exemplify its use with the ε-expansion of the massive one-loop QED vertex function.

XI International Workshop on Advanced Computing and Analysis Techniques in Physics Research, ACAT, Amsterdam, The Netherlands, April 23-27, 2007

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1. Introduction

For many of the precision predictions of observables for LHC and ILC we need the automated evaluation of multi-leg and/or multi-loop Feynman diagrams; a summary of the needs and the present status may be found in [1].

A promising approach is the representation of the Feynman diagrams by Feynman parameter integrals and the subsequent use of Mellin-Barnes (MB) representations for their further evaluation. The method was invented in 1975 for the finite massive scalar three-point function in [2]. Massive one-loop self-energies and vertices were investigated in d dimensions in [3]. An important step was the explicit evaluation, in terms of polylogarithms, of the planar on-shell double box in [4–7] and of the non-planar case in [8, 7]. Slightly different algorithms are used for the derivation of proper MB-representations for divergent diagrams as expansions in powers of $\varepsilon = (4-d)/2$. The algorithm of Tausk was automated in Mathematica and Maple [9] and in Mathematica [10]; the latter package, MB, is publicly available. As input it needs some MB-integral representation and performs the ε expansion. Quite recently, the Mathematica package AMBRE was published which prepares for a large class of Feynman diagrams the MB-integral representation [11].

In this contribution, we describe the package AMBRE and give a simple example of its use [1].

2. Construction of Mellin-Barnes representations

We will use the MB-representation

$$\frac{1}{(A+B)^\nu} = \frac{B^{-\nu}}{2\pi i \Gamma(\nu)} \int\limits_{-i\infty+R}^{i\infty+R} d\sigma A^\sigma\, B^{-\sigma}\, \Gamma(-\sigma)\Gamma(\nu+\sigma), \tag{2.1}$$

where the integration contour separates the poles of the Γ-functions.

We evaluate L-loop Feynman integrals[2] in $d = 4-2\varepsilon$ dimensions with N internal lines with momenta q_i and masses m_i, and E external legs with momenta p_e:

$$G_L[T(k)] = \frac{1}{(i\pi^{d/2})^L} \int \frac{d^dk_1 \ldots d^dk_L\; T(k)}{(q_1^2-m_1^2)^{\nu_1}\ldots(q_i^2-m_i^2)^{\nu_i}\ldots(q_N^2-m_N^2)^{\nu_N}}. \tag{2.2}$$

The numerator $T(k)$ is a tensor in the integration variables:

$$T(k) \;=\; 1, k_l^\mu, k_l^\mu k_n^\nu, \ldots \tag{2.3}$$

The momenta of the denominator functions d_i are:

$$d_i \;=\; q_i^2 - m_i^2 \;=\; \;=\; \left(\sum_{l=1}^{L} \alpha_{il} k_l - \sum_{e=1}^{E} \beta_{ie} p_e\right)^2 - m_i^2. \tag{2.4}$$

The momentum integrals are replaced by standard Feynman parameter integrals:

$$G_L[T(k)] \;=\; \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod_{i=1}^N \Gamma(\nu_i)} \int_0^1 \prod_{j=1}^N dx_j\, x_j^{\nu_j-1} \delta(1-\sum_{i=1}^N x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}\, P_L(T),$$

[1] Examples of different complexity may be found at the webpages [12, 13].

[2] Often one uses the additional normalization $e^{\varepsilon\gamma_E L}$; we leave this to the later evaluation with the package MB [10].

(2.5)

with

$$N_\nu \;=\; \sum_{i=1}^{N} \nu_i. \tag{2.6}$$

The two functions U and F may be derived from

$$\mathscr{N} \;=\; \sum_{i=1}^{N} x_i (q_i^2 - m_i^2) \;\equiv\; kMk - 2kQ + J, \tag{2.7}$$

where $M_{ll'} = \sum_{i=1}^{N} \alpha_{il'} \alpha_{il} x_i$, and $Q_l = \sum_{i=1}^{N} \alpha_{il} P_i x_i$, and $J = \sum_{i=1}^{N} (P_i^2 - m_i^2) x_i$; namely:

$$U(x) \;=\; \det(M), \tag{2.8}$$

$$F(x) \;=\; -\det(M)\, J + Q\tilde{M}Q. \tag{2.9}$$

Where we may assume $M^+ = M$. The U and F as well as $\tilde{M} = \det(M)\, M^{-1}$ are polynomials in x. We will evaluate L-loop integrals with the loop-by-loop iteration procedure, because the formulae simplify for one-loop integrals:

$$U \;=\; M \;=\; \tilde{M} \;=\; \det(M) \;=\; \sum_{i}^{N} x_i \;=\; 1, \tag{2.10}$$

$$F \;=\; -UJ + Q^2 \;=\; \sum_{i,j}^{N} [P_i P_j - P_i^2 + m_i^2] x_i x_j \;\equiv\; \sum_{i \le j}^{N} f_{ij} x_i x_j. \tag{2.11}$$

The simplest tensor factors $P_1(T)$ in (2.5) become:

$$P_1(1) \;=\; 1, \tag{2.12}$$

$$P_1(k^\alpha) \;=\; \sum_{i=1}^{N} x_i P_i^\alpha, \tag{2.13}$$

$$P_1(k^\alpha k^\beta) \;=\; \sum_{i=1}^{N} x_i P_i^\alpha \sum_{j=1}^{N} x_j P_j^\beta - \frac{\Gamma\left(N_\nu - \frac{d}{2} - 1\right)}{\Gamma\left(N_\nu - \frac{d}{2}\right)}\, F\, \frac{g^{\alpha\beta}}{2}. \tag{2.14}$$

The P_i^α are the so-called chords introduced in (2.4). The general case is:

$$\begin{aligned} G_1(T_m) \;&\equiv\; G_1(k^{\mu_1} \cdots k^{\mu_m}) \\ &=\; \frac{(-1)^{N_\nu}}{\prod_{i=1}^{N} \Gamma(\nu_i)} \int \prod_{i=1}^{N} dx_i x_i^{\nu_i - 1} \delta(1 - \sum_{j=1}^{N} x_j) \sum_{r=0}^{m} \frac{\Gamma\left(N_\nu - \frac{d+r}{2}\right)}{(-2)^{\frac{r}{2}} F^{N_\nu - \frac{d+r}{2}}} \left\{ \mathscr{A}_r P^{m-r} \right\}^{(\mu_1, \ldots, \mu_m)}, \end{aligned} \tag{2.15}$$

with the abbreviations $F \equiv F(x)$ and $P \equiv P_1(k^\mu) = \sum_i x_i P_i = \sum_{i,e} x_i \beta_{ie} p_e^\mu$. The r starts from zero (with $\mathscr{A}_0 = 1$), and it is $\mathscr{A}_r = 0$ for r odd, and $\mathscr{A}_r = g^{\mu_{i_1}\mu_{i_2}} \cdots g^{\mu_{i_{r-1}}\mu_{i_r}}$ for r even. The convention $(\mu_{i_1} \ldots)$ means the totally symmetric combination of the arguments.

In AMBRE, the tensorial numerators are assumed to be contracted with a tensor $P(m)$ composed of external momenta p_e, so that the following quantity is evaluated:

$$P(m)\, G_1(T_m) \;\equiv\; \left(p_{e_1}^{\mu_1} \cdots p_{e_m}^{\mu_m}\right)\; G_1(k^{\mu_1} \cdots k^{\mu_m}). \tag{2.16}$$

One now has to perform the x-integrations. We do this by the following simple formula:

$$\int_0^1 \prod_{i=1}^{N} dx_i\, x_i^{q_i-1}\, \delta\left(1-\sum_j x_j\right) = \frac{\Gamma(q_1)\cdots\Gamma(q_N)}{\Gamma(q_1+\cdots+q_N)}. \tag{2.17}$$

From the above text it is evident that the integrand of (2.5) contains besides simple sums of monomials $\prod_i x_i^{n_i}$ (like e.g. Q and $\tilde{M}$) also sums of monomials with non-integer exponents. This is due to the appearance of the factors $U(x)$ and $F(x)$. One may rewrite $F(x)$ and $U(x)$ so that (2.17) becomes applicable. For the one-loop case this has to be done only for $F(x)$. In (2.11), $F(x)$ is written as a sum of $N_F \leq \frac{1}{2}N(N+1)$ non-vanishing, bilinear terms in x_i[3]:

$$\begin{aligned} F(x)^{-(N_\nu-dL/2)} &= \left[\sum_{n=1}^{N_F} f_n(i,j)x_ix_j\right]^{-(N_\nu-dL/2)} \\ &= \frac{1}{\Gamma(N_\nu-dL/2)}\frac{1}{(2\pi i)^{N_F}}\prod_{i=1}^{N_F}\int_{-i\infty+u_i}^{i\infty+u_i} dz_i \prod_{n=2}^{N_F}[f_n(i,j)x_ix_j]^{z_n} \\ &\quad [f_1(i,j)x_ix_j]^{-(N_\nu-dL/2)-\Sigma_{j=2}^{N_F}z_j}\,\Gamma\left(N_\nu-\frac{dL}{2}+\sum_{j=2}^{N_F}z_j\right)\prod_{j=2}^{N_F}\Gamma(-z_j). \end{aligned} \tag{2.18}$$

Here, $f_n(i,j) = f_{ij}$ if $f_{ij} \neq 0$. Inserting (2.18) and the tensor function $P(T)$ into (2.2) allows to apply (2.17) for an evaluation of the x-integrations.

As a result, L-loop scalar Feynman integrals may be represented by a single multi-dimensional MB-integral and tensor Feynman integrals by finite sums of MB-integrals.

3. The Mathematica package AMBRE

In this section we describe the use of the package AMBRE [11]. AMBRE stands for **A**utomatic **M**ellin-**B**arnes **Re**presentation. It is a (semi-)automatic procedure written for multi-loop calculations. The package works with Mathematica 5.0 and later versions of it. One has to perform the following tasks:

(i) define kinematical invariants which depend on the external momenta;

(ii) make a decision about the order in which L one-loop subloops $(L \geq 1)$ will be worked out sequentially;

(iii) construct a Feynman integral for the chosen subloop and perform manipulations on the corresponding F-polynomial to make it optimal for later use of the MB representations;

(iv) use equation (2.18);

(v) perform the integrations over Feynman parameters with equation (2.17);

[3] If useful, one may also consider $F(x)$ with linear and bilinear terms in the x_i.

(vi) if needed, go back to step (iii) and repeat the steps for the next subloop until F in the last, L^{th} subloop will be changed into an MB-integral.

The steps (ii) and (iii) must be analyzed carefully, because there exists some freedom of choice on the order of loop integrations in step (ii) and also on the order of MB integrations in step (iii). Different choices may lead to different forms of MB-representations.

The basic functions of AMBRE are:

- **Fullintegral[{numerator},{propagators},{internal momenta}]** – is the basic function for input Feynman integrals
- **invariants** – is a list of invariants, e.g. **invariants = {p1*p1 → s}**
- **IntPart[iteration]** – prepares a subintegral for a given internal momentum by collecting the related numerator, propagators, integration momentum
- **Subloop[integral]** – determines for the selected subintegral the U and F polynomials and an MB-representation
- **ARint[result,i_]** – displays the MB-representation number i for Feynman integrals with numerators
- **Fauto[0]** – allows user specified modifications of the F polynomial **fupc**
- **BarnesLemma[repr,1,Shifts->True]** – function tries to apply Barnes' first lemma to a given MB-representation; when **Shifts->True** is set, AMBRE will try a simplifying shift of variables; the default is **Shifts->False**
  BarnesLemma[repr,2,Shifts->True] – function tries to apply Barnes' second lemma

4. V3l2m: the one-loop massive QED vertex

As an example, we evaluate the one-loop massive QED vertex function V3l2m (vertex with 3 internal lines, 2 of them being massive, $m^2 = 1$):

$$\texttt{V3l2m} = e^{\varepsilon\gamma_E} \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{(k^2)^{n_0}[(k+p_1)^2-1]^{n_1}[(k-p_2)^2-1]^{n_2}}. \tag{4.1}$$

The corresponding definition in AMBRE:

» V3l2m = Fullintegral[{1},{PR[k, 0, n0] PR[k + p1, 1, n1] PR[k - p2, 1, n2]},{k}]

In a next step we define the invariants:

» invariants = $\{p1^2 -> 1, p2^2 -> 1, p1p2 -> (s-2)/2\}$

With **IntPart[1]** and **SubLoop[integral]**, we determine the F-polynomial for the diagram,

$$F = (x_1+x_2)^2 + [-s]x_1x_2, \tag{4.2}$$

As usual, we have to replace here a positive s by $s+i\varepsilon$. After setting **{n1 -> 1, n2 -> 1, n0 -> 1}** and applying Barnes' first lemma,

» MBV3l2m = BarnesLemma[V3l2m, 1]

an MB-representation for the Feynman integral is obtained:

$$\texttt{V3l2m[y]} \;=\; -\frac{e^{\varepsilon\gamma_E}\Gamma(-2\varepsilon)}{\Gamma(1-2\varepsilon)2\pi i}\int dz(-s)^{-\varepsilon-1-z}\frac{\Gamma^2(-\varepsilon-z)\Gamma(-z)\Gamma(1+\varepsilon+z)}{\Gamma(-2\varepsilon-2z)}. \tag{4.3}$$

For the subsequent step, the derivation of the ε-expansion, the package MB may be used:

$$\texttt{V3l2m[y]} \;=\; \frac{\texttt{V3l2m[}-1,\texttt{y]}}{\varepsilon} + \texttt{V3l2m[0,y]} + \varepsilon\ \texttt{V3l2m[1,y]} + \cdots, \tag{4.4}$$

where we already introduced the conformal variable

$$y = \frac{\sqrt{-s+4}-\sqrt{-s}}{\sqrt{-s+4}+\sqrt{-s}}. \tag{4.5}$$

Representations for the first terms of the ε-expansion are easily obtained with the following MB commands

» rules = MBoptimizedRules[MBV3l2m, eps -> 0, {}, {eps}]

» integrals = MBcontinue[MBV3l2m, eps -> 0, rules]

» expV3l2m = MBexpand[integrals, Exp[eps*EulerGamma], {eps, 0, n}]

We reproduce a few of them here:

$$\texttt{V3l2m[}-1,\texttt{y]} \;=\; \frac{1}{2}\frac{1}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dz(-s)^{-1-z}\,\frac{\Gamma^3[-z]\Gamma[1+z]}{\Gamma[-2z]}, \tag{4.6}$$

$$\begin{aligned}\texttt{V3l2m[0,y]} \;=\; & \frac{1}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dz(-s)^{-1-z}\,\frac{\Gamma^3[-z]\Gamma[1+z]}{\Gamma[-2z]} \\ & \frac{1}{2}\left[\gamma_E - \ln(-s) + 2\Psi[-2z] - 2\Psi[-z] + \Psi[1+z]\right],\end{aligned} \tag{4.7}$$

$$\begin{aligned}\texttt{V3l2m[1,y]} \;=\; & \frac{1/4}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dz(-s)^{-1-z}\,\frac{\Gamma^3[-z]\Gamma[1+z]}{\Gamma[-2z]} \\ & \Big[\gamma_E^2 + Log[-s]^2 + Log[-s](-2\gamma_E - 4\Psi[-2z] + 4\Psi[-z] - 2\Psi[1+z]) \\ & +\gamma_E(4\Psi[-2z] - 4\Psi[-z] + 2\Psi[1+z]) \\ & -4\Psi[1,-2z] + 2\Psi[1,-z] + \Psi[1,1+z] \\ & +4(\Psi[-2z]^2 - 2\Psi[-2z]\Psi[-z] + \Psi[-z]^2 + \Psi[-2z]\Psi[1+z] \\ & -\Psi[-z]\Psi[1+z]) + \Psi[1+z]^2\Big],\end{aligned} \tag{4.8}$$

$$\begin{aligned}\texttt{V3l2m[2,y]} \;=\; & \frac{1/12}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dz(-s)^{-1-z}\,\frac{\Gamma^3[-z]\Gamma[1+z]}{\Gamma[-2z]} \\ & \Big[a(z) + b(z)\Psi(0,1+z) + \Psi(2,1+z) + 2\Psi(0,1+z)^2 + \Psi(0,1+z)^3 \\ & + 3\Psi(0,1+z)\Psi(1,1+z) + c(z)[\Psi(1,1+z) + 2\Psi(0,1+z)^2],\Big]\end{aligned} \tag{4.9}$$

with some coefficients $a(z), b(z), c(z)$, which depend on $Log[-s], \gamma_E, \Psi(k,-z), \Psi(k,-2z)$ ($k = 0,1,2$). Here, $\Psi[z]$ and $\Psi[k,z]$ are polygamma functions:

$$\text{PolyGamma [z]} \;=\; \text{PolyGamma [0,z]} \;=\; \Psi(z) \;=\; \frac{\Gamma'(z)}{\Gamma(z)} \tag{4.10}$$

$$\text{PolyGamma [0,n+1]} = S_1(n) - \gamma_E = \sum_{i=1}^{n} \frac{1}{i} - \gamma_E,$$
$$\text{PolyGamma [n,z]} = \Psi[n,z] = \frac{d^n \Psi(z)}{dz^n} = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}}. \tag{4.11}$$

The integration path includes the straight line ranging from $(-i\infty + 1/2)$ to $(+i\infty + 1/2)$. Closing the path to the left, allows to express the integral as an infinite series of residua arising from the poles of

$$\Gamma[1+z], \Psi[k, 1+z], \tag{4.12}$$

at $z = -n, n = 1, 2, \cdots$ with different weight functions $G(z)$, e.g. for `V3l2m[-1,y]` with

$$G(z) = (-s)^{-1-z} \frac{\Gamma^3[-z]}{\Gamma[-2z]}. \tag{4.13}$$

The resulting respresentations are inverse binomial sums:

$$\texttt{V3l2m}[-1, \texttt{y}] = \frac{1}{2} \sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)}, \tag{4.14}$$
$$\texttt{V3l2m}[0, \texttt{y}] = \frac{1}{2} \sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} S_1(n), \tag{4.15}$$
$$\texttt{V3l2m}[1, \texttt{y}] = \frac{1}{4} \sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} \left[S_1(n)^2 + \zeta_2 - S_2(n)\right], \tag{4.16}$$
$$\texttt{V3l2m}[2, \texttt{y}] = \sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} \left[\frac{1}{12} S_1[n]^3 - \frac{1}{4} S_1[n] S_2[n] + \frac{1}{4} \zeta_2 S_1[n] + \frac{1}{6} S_3[n] - \frac{1}{6} \zeta_3\right], \tag{4.17}$$

etc., and the harmonic numbers $S_k(n)$ are

$$\text{HarmonicNumber[n,k]} = S_k(n) = \sum_{i=1}^{n} \frac{1}{i^k}. \tag{4.18}$$

The simplest of the sums may be done with Mathematica:

$$\texttt{V3l2m}[-1, \texttt{y}] = \frac{1}{2} \frac{4 \arcsin(\sqrt{s}/2)}{\sqrt{4-s}\sqrt{s}} = \frac{1}{2} \frac{-2y}{1-y^2} \ln y. \tag{4.19}$$

The other sums appearing above may be obtained from sums listed in Table 1 of Appendix D in [14]:

$$\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} = \frac{y}{y^2-1} 2 \ln(y), \tag{4.20}$$
$$\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} S_1(n) = \frac{y}{y^2-1} \left[-4\mathrm{Li}_2(-y) - 4\ln(y)\ln(1+y) + \ln^2(y) - 2\zeta_2\right], \tag{4.21}$$
$$\sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}(2n+1)} S_1(n)^2 = \frac{y}{y^2-1} \Big[16 S_{1,2}(-y) - 8\mathrm{Li}_3(-y) + 16\mathrm{Li}_2(-y)\ln(1+y)$$

$$
\begin{aligned}
&+8\ln^2(1+y)\ln(y) - 4\ln(1+y)\ln^2(y) + \frac{1}{3}\ln^3(y) + 8\zeta_2\ln(1+y) \\
&-4\zeta_2\ln(y) - 8\zeta_3\Big], \qquad (4.22)
\end{aligned}
$$

$$
\sum_{n=0}^{\infty}\frac{s^n}{\binom{2n}{n}(2n+1)}S_2(n) = -\frac{y}{3(y^2-1)}\ln^3(y), \qquad (4.23)
$$

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{s^n}{\binom{2n}{n}(2n+1)}S_1(n)^3 &= \frac{y}{y^2-1}\Big[-96S_{1,2}(-y)\ln(1+y) - 96S_{1,3}(-y) + 48S_{2,2}(-y) \\
&-24\zeta_2\ln^2(1+y) - 48\ln^2(1+y)\mathrm{Li}_2(-y) + 48\zeta_3\ln(1+y) \\
&+48\ln(1+y)\mathrm{Li}_3(-y) - 16\ln(y)\ln^3(1+y) + 24\zeta_2\ln(y)\ln(1+y) \\
&+12\ln^2(y)\ln^2(1+y) - 2\ln^3(y)\ln(1+y) + \frac{1}{12}\ln^4(y) - 3\zeta_2\ln^2(y) \\
&+6\ln^2(y)\mathrm{Li}_2(-y) + 2\ln^2(y)\mathrm{Li}_2(y) - 10\zeta_3\ln(y) - 24\ln(y)\mathrm{Li}_3(-y) \\
&-8\ln(y)\mathrm{Li}_3(y) + 3\zeta_4 + 24\mathrm{Li}_4(-y) + 12\mathrm{Li}_4(y)\Big], \qquad (4.24)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{s^n}{\binom{2n}{n}(2n+1)}S_3(n) &= \frac{y}{y^2-1}\Big[\frac{1}{12}\ln^4(y) + 12\mathrm{Li}_4(y) + 2\ln^2(y)\mathrm{Li}_2(y) \\
&-4\zeta_3\ln(y) - 8\ln(y)\mathrm{Li}_3(y) - 12\zeta_4\Big], \qquad (4.25)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{s^n}{\binom{2n}{n}(2n+1)}S_1(n)S_2(n) &= \frac{y}{y^2-1}\Big[\frac{2}{3}\ln^3(y)\ln(1+y) - \frac{1}{12}\ln^4(y) + \zeta_2\ln^2(y) \\
&+2\ln^2(y)\mathrm{Li}_2(-y) + 2\ln^2(y)\mathrm{Li}_2(y) + 2\zeta_3\ln(y) - 8\ln(y)\mathrm{Li}_3(-y) \\
&-8\ln(y)\mathrm{Li}_3(y) + 2\zeta_4 + 16\mathrm{Li}_4(-y) + 12\mathrm{Li}_4(y)\Big]. \qquad (4.26)
\end{aligned}
$$

The one-loop QED vertex function is a relatively simple one-scale problem, and in fact one may derive the general term of the ε-expansion. So far, we applied the Mathematica packages without an interference by the user. The packages have been created for the automatization of this type of calculations. But here we have an instructive example of the limitations of that. Let us go back to Equation (4.3) and shift the integration variable z according to $z = z' - \varepsilon$, with a related shift of the real part of the integration path, $u' = u + \varepsilon$. The resulting MB-integral is:

$$
\texttt{V3l2m[y]} = -\frac{e^{\varepsilon\gamma_E}\Gamma(-2\varepsilon)}{\Gamma(1-2\varepsilon)2\pi i}\int dz'(-s)^{-1-z'}\frac{\Gamma^2(-z')\Gamma(-z'+\varepsilon)\Gamma(1+z')}{\Gamma(-2z')} \qquad (4.27)
$$

After taking residua as before, but now without using `MB`, one has to evaluate:

$$
\texttt{V3l2m[y]} = \frac{e^{\varepsilon\gamma_E}}{2\varepsilon}\sum_{n=0}^{\infty}\frac{s^n}{\binom{2n}{n}(2n+1)}\frac{\Gamma(n+1+\varepsilon)}{\Gamma(n+1)}. \qquad (4.28)
$$

We may apply here the following well-known relation [15][4]:

$$
\frac{\Gamma(n+1+\varepsilon)}{\Gamma(n+1)} = \Gamma(1+\varepsilon)\exp\left[-\sum_{k=1}^{\infty}\frac{(-\varepsilon)^k}{k}S_k(n)\right], \qquad (4.29)
$$

[4]Express $\Gamma(-n+\varepsilon)$ by $\Gamma(\varepsilon)$. Thereby factors $1/(-n+\varepsilon+k)$ are collected, inverse them for small ε.

and obtain the general term of the inverse binomial sum for the n^{th} term in the ε-expansion of V3l2m[y]. The first terms agree with (4.14)–(4.17). The evaluation of the sums may be performed for the general case by introducing integral representations for the Γ functions and the harmonic numbers (for the latter see e.g. [16]), and then performing the two-dimensional integral.

Alternatively to the approach described here, one may determine the vertex function in terms of Harmonic Polylogarithms (HPLs) [17] by the differential equation method, which allows easily to derive the ε-expansion of the Feynman integral SE2l2m. The vertex may be related to SE2l2m and to the tadpole T1l1m by integration by parts; see e.g. [18]:

$$\mathtt{V3l2m} = \frac{(d-2)\mathtt{T1l1m} - 2(d-3)\mathtt{SE2l2m}}{(d-4)(s-4)}. \tag{4.30}$$

The master integrals T1l1m and SE2l2m are tabulated e.g. in the file mastersHPL.m [19, 20]. From the corresponding formulae it may be seen that only HPLs with arguments 0 and -1 appear. An approach to describe the vertex in d dimensions with hypergeometric functions and the subsequent derivation of the ε-expansion may be found in [21].

5. Summary

To summarize, for many applications of present phenomenological or more theoretical interest the package AMBRE solves an important part of the calculational problem: the semi-automatic derivation of MB-representations for a large class of Feynman integrals.

Acknowledgments

We would like to thank J. Blümlein, S.-O. Moch and B. Tausk for useful discussions and suggestions. The present work is supported in part by the European Community's Marie-Curie Research Training Networks MRTN-CT-2006-035505 'HEPTOOLS' and MRTN-CT-2006-035482 'FLAVIAnet', and by Sonderforschungsbereich/Transregio 9–03 of Deutsche Forschungsgemeinschaft 'Computergestützte Theoretische Teilchenphysik'.

References

[1] S.Weinzierl, *Automated calculations for multi-leg processes*, talk given at this conference.

[2] N. Usyukina, *On a representation for three point function*, *Teor. Mat. Fiz.* **22** (1975) 300–306 (in Russian).

[3] E. Boos and A. Davydychev, *A method of evaluating massive Feynman integrals*, *Theor. Math. Phys.* **89** (1991) 1052–1063.

[4] V. Smirnov, *Analytical result for dimensionally regularized massless on-shell double box, Phys. Lett.* **B460** (1999) 397–404, [hep-ph/9905323].

[5] V. Smirnov and O. Veretin, *Analytical results for dimensionally regularized massless on-shell double boxes with arbitrary indices and numerators, Nucl. Phys.* **B566** (2000) 469–485, [hep-ph/9907385].

[6] V. Smirnov, *Analytical result for dimensionally regularized massive on- shell planar double box*, *Phys. Lett.* **B524** (2002) 129–136, [http://arXiv.org/abs/hep-ph/0111160].

[7] G. Heinrich and V. Smirnov, *Analytical evaluation of dimensionally regularized massive on-shell double boxes*, *Phys. Lett.* **B598** (2004) 55–66, [hep-ph/0406053].

[8] B. Tausk, *Non-planar massless two-loop Feynman diagrams with four on- shell legs*, *Phys. Lett.* **B469** (1999) 225–234, [hep-ph/9909506].

[9] C. Anastasiou and A. Daleo, *Numerical evaluation of loop integrals*, *JHEP* **10** (2006) 031, [hep-ph/0511176].

[10] M. Czakon, *Automatized analytic continuation of Mellin-Barnes integrals*, *Comput. Phys. Commun.* **175** (2006) 559–571, [hep-ph/0511200].

[11] J. Gluza, K. Kajda, and T. Riemann, *AMBRE - a Mathematica package for the construction of Mellin-Barnes representations for Feynman integrals*, *to appear in Comput. Phys. Commun.* (2007) [arXiv:0704.2423 [hep-ph]].

[12] Silesian Univ., Katowice, webpage http://prac.us.edu.pl/∼gluza/ambre.

[13] DESY, webpage http://www-zeuthen.desy.de/theory/research/CAS.html.

[14] A. Davydychev and M. Kalmykov, *Massive Feynman diagrams and inverse binomial sums*, *Nucl. Phys.* **B699** (2004) 3–64, [hep-th/0303162].

[15] S. Weinzierl, *Expansion around half-integer values, binomial sums and inverse binomial sums*, *J. Math. Phys.* **45** (2004) 2656–2673, [hep-ph/0402131].

[16] J. Blümlein and S. Kurth, *Harmonic sums and Mellin transforms up to two-loop order*, *Phys. Rev.* **D60** (1999) 014018, [hep-ph/9810241].

[17] E. Remiddi and J. Vermaseren, *Harmonic polylogarithms*, *Int. J. Mod. Phys.* **A15** (2000) 725–754, [hep-ph/9905237].

[18] J. Fleischer, J. Gluza, A. Lorca, and T. Riemann, *First order radiative corrections to Bhabha scattering in d dimensions*, *Eur. J. Phys.* **48** (2006) 35–52, [hep-ph/0606210].

[19] M. Czakon, J. Gluza, and T. Riemann, *Master integrals for massive two-loop Bhabha scattering in QED*, *Phys. Rev.* **D71** (2005) 073009, [hep-ph/0412164].

[20] DESY, webpage http://www-zeuthen.desy.de/theory/research/bhabha/bhabha.html.

[21] A. Davydychev and M. Kalmykov, *New results for the epsilon-expansion of certain one-, two- and three-loop Feynman diagrams*, *Nucl. Phys.* **B605** (2001) 266–318, [http://arXiv.org/abs/hep-th/0012189].