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Towards mirror symmetry *à la* SYZ for generalized Calabi–Yau manifolds

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Abstract

Fibrations of flux backgrounds by supersymmetric cycles are investigated. For an internal six-manifold M with static $SU(2)$ structure and mirror $\hat{M}$, it is argued that the product $M \times \hat{M}$ is doubly fibered by supersymmetric three-tori, with both sets of fibers transverse to M and $\hat{M}$. The mirror map is then realized by T-dualizing the fibers. Mirror-symmetric properties of the fluxes, both geometric and non-geometric, are shown to agree with previous conjectures based on the requirement of mirror symmetry for Killing prepotentials. The fibers are conjectured to be destabilized by fluxes on generic $SU(3) \times SU(3)$ backgrounds, though they may survive at type-jumping points. T-dualizing the surviving fibers ensures the exchange of pure spinors under mirror symmetry.

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1 Introduction

The study of flux compactifications is strongly motivated by the necessity to fix the moduli of the compact space. It leads to the consideration of flux backgrounds which lack certain geometric features of Calabi–Yau manifolds: typically the closure of the two- and three-forms of Calabi–Yau manifolds are spoiled by intrinsic torsion. Moreover, the duality symmetries of string theory lead to backgrounds that are non-geometric in the sense that the closed-string metric is not globally defined. This concept appeared first in various incarnations in [1, 2, 3, 4, 5] and a unifying picture connecting these various points of view was proposed using generalized geometry in [6].

However, there is still some structure surviving in flux backgrounds preserving eight supercharges in four dimensions: such backgrounds have to possess $SU(3) \times SU(3)$ structure [7, 8, 9]. This implies the existence of a pair of pure spinors of different parity Φ_+ and Φ_-, one being closed and inducing a generalized complex structure, so that the internal space is a generalized Calabi–Yau manifold [10, 11]. The other one is not closed in the presence of Ramond-Ramond fluxes, but its imaginary part is, and gives rise to calibrations [12, 13, 14, 15, 16, 17].

$SU(3)$-structures form a geometric subclass of $SU(3) \times SU(3)$ structure manifolds, where the pure spinors are denoted by Ω and e^{iJ}, and there is no type-jumping. These manifolds were established to be the mirrors of Calabi-Yau with so-called geometric or electric H–flux in [18], which in the case of torus-bundles reduces to the statement that T-duality exchanges the Chern-class of the bundle with the integral of the H-flux along the T-dualized direction [1, 19, 20].

Special cases of non-geometric backgrounds have been identified as physical realizations of the type-jumping phenomenon previously studied in generalized complex geometry [11, 6, 21]. Furthermore, the effective actions of string theory on backgrounds admitting $SU(3) \times SU(3)$ structure exhibit symmetry properties under the exchange of Φ_+ and Φ_- [9]. This exchange extends the action of mirror symmetry beyond the realm of Calabi–Yau manifolds, in which the pure spinors are $\Phi_+ = e^{iJ}$ and $\Phi_- = \Omega$, where J and Ω denote the Kähler form and holomorphic three-form, respectively. The generalized calibrations are exchanged in the same way as the ones governing stability of D-branes of type A and B on Calabi–Yau manifolds [22, 15]. Fortunately generalized complex submanifolds share a lot of properties with Abelian D-branes [23, 24, 25].

Flux backgrounds, while fixing moduli, have therefore violently shaken the geometric framework of Calabi–Yau compactifications, but still happen to possess good mirror-symmetric properties. This begs for an explanation in terms of the action of T-duality on the internal space in the presence of fluxes. In other words, we would like to know what remains of the Strominger–Yau–Zaslow (SYZ) picture of mirror symmetry [26, 27], in the case of $SU(3) \times SU(3)$ structure backgrounds.

The purpose of this paper is therefore to investigate the moduli space of calibrated cycles in backgrounds with $SU(3) \times SU(3)$ structure, and to formulate the exchange between Φ_+ and Φ_- in terms of T-duality along such cycles, thus extending mirror symmetry to cases where much of the structure available in Calabi–Yau manifolds is missing.[1]

This can be done in several steps. After recalling the connection between the pure spinors and the supercharges, we specialize to the case of internal manifolds with a so-called static $SU(2)$ structure. The type of the pure spinors are constant on such manifolds, but never maximal, since they are equal to one and two, respectively. We shall see that supersymmetric tori transverse to the product of the internal space M and its mirror $\hat{M}$ have the entire $M \times \hat{M}$ as moduli space. In particular M and $\hat{M}$ are still fibered by three-tori, but the fibers are not supersymmetric by themselves. We illustrate this generalized SYZ proposal for static $SU(2)$ structure manifolds in various special cases and show that it is compatible with the mirror map advocated in [9].

Then we address the case of generic $SU(3) \times SU(3)$ structures, that exhibit type-jumping phenomena, and correspondingly open-string moduli fixing. We shall see that zeroes or critical points of the coefficients relating the supercharges to each other dictate the position moduli of supersymmetric cycles. Finally, we may perform T-duality along the existing supersymmetric cycles, and obtain the type-jumping phenomena from the naturality properties of Fourier–Mukai transform with respect to the so-called B- and β-transforms of generalized complex geometry. This will be related to the covariance properties of the differential operators on flux backgrounds, and confirm the mirror-symmetric form of the superpotentials for $SU(3) \times SU(3)$

[1] For generalized Kähler manifolds an argument of mirror symmetry via T-duality for the topological sigma-models was put forward in [28].

backgrounds.

2 Review and notations

2.1 Supersymmetry, pure spinors and structures

Generalized complex geometry contains both complex geometry and symplectic geometry. An almost generalized complex structure on a manifold M is defined as an almost complex structure on the sum of the tangent and cotangent bundles. It is a generalized complex (GC) structure if its $+i$-eigenbundle is stable under the action of the Courant bracket [10, 11, 29]. We will give a more detailed review of the concepts in generalized geometry, including GC submanifolds, in the next sub-section. Here we review the definition of pure spinors in terms of supercharges. There is a one-to-one correspondence between GC structures and pure spinors. A pure spinor is a sum of differential forms and may locally be written in a unique way as the wedge product of k complex one-forms and the exponential of a two-form:

$$\theta_1 \wedge \cdots \wedge \theta_k \wedge e^{B+i\omega}. \tag{2.1}$$

The integer k is called the *type* of the pure spinor. From now on we only consider six-dimensional manifolds. The special case $k = 0$ corresponds to a symplectic structure on the manifold, and the special case $k = 3$ to a complex structure. Not only can the type assume other values, but it can also vary on the manifold. This is called the *type-jumping* phenomenon [11]. We will mostly work with the pure spinors as the objects encoding the GC structure.

Consider Type II compactifications on six-manifolds with $SU(3) \times SU(3)$ structure [10, 11, 7, 8, 9] (for more references see [30]). These are characterized by a pair of no-where vanishing $SU(3)$-invariant spinors $\eta^{1,2}$, which arise in the decomposition of the two $SO(9,1)$ spinors $\epsilon^{1,2}$ of Type II under $SO(3,1) \times SO(6)$.

Let M and $\hat{M}$ be a (real) six-dimensional manifold and its mirror, both assumed to have $SU(3) \times SU(3)$ structure. As such they respectively possess pure spinors Φ_-, Φ_+ and $\hat{\Phi}_-, \hat{\Phi}_+$, where the signs denote the parity of the type. The pure spinors on M are constructed as bilinears of spinors:

$$\begin{aligned} \Phi_+ &= \eta^1_+ \otimes \eta^{2\dagger}_+ \\ \Phi_- &= \eta^1_+ \otimes \eta^{2\dagger}_- \,, \end{aligned} \tag{2.2}$$

where η^1 and η^2 are related to each other by the equation

$$\eta^2_+ = c\eta^1_+ + (v + iw)_m \gamma^m \eta^1_- \,, \tag{2.3}$$

defining the complex one-form $v + iw$ and complex number c, which have to satifsy the normalization condition

$$|c|^2 + |v + iw|^2 = 1 \,. \tag{2.4}$$

There are analogous objects on $\hat{M}$ and we shall occasionally refer to them just by putting hats on the symbols we explicitly define on M.

At points where $|c| = 1$ (zeroes of $v + iw$), the two $SU(3)$ spinors η^1 and η^2 become proportional to each other, and the two $SU(3)$ structures defined by bilinears of η^1 and η^2 agree. At such points the pure spinors Φ_- and Φ_+ have type three and type zero respectively,

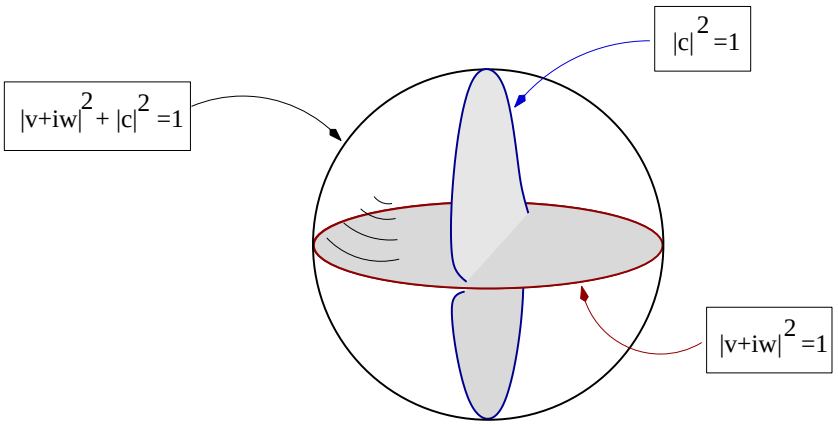


Figure 1: Moduli space of spinors for $SU(3) \times SU(3)$ structure manifolds (depicted one dimension lower as an S^2). The blue and red circles depict the $SU(3)$ and the static $SU(2)$ structures, respectively, which do not intersect.

just as they do in the case of manifolds of $SU(3)$ structures. Up to a B-transform they read at such points

$$\begin{aligned} \Phi_-|_{|c|=1} &= \Omega \\ \Phi_+|_{|c|=1} &= e^{iJ} \,. \end{aligned} \tag{2.5}$$

If $|c| = 1$ on the whole of M, then M has an $SU(3)$ structure. Calabi–Yau manifolds form the subclass of those manifolds for which both of Ω and e^{iJ} are closed.

At generic points though, the spinors η^1 and η^2 are linearly independent, the two $SU(3)$ structures constructed from them do not agree, and their fundamental two-form and complex three-form may be written as

$$\begin{aligned} J^1 &= j + v \wedge w \,, & J^2 &= j - v \wedge w \\ \Omega^1 &= \omega \wedge (v + iw) \,, & \Omega^2 &= \omega \wedge (v - iw) \,. \end{aligned} \tag{2.6}$$

The pure spinors in turn are expressed [7] in a way that allows to read-off their types, as

$$\begin{aligned} \Phi_- &= -\frac{1}{8} (v + iw) \wedge e^{i(j + c\omega)} \\ \Phi_+ &= -\frac{1}{8} \bar{c}\, e^{-i(j + v\wedge w + \frac{1}{c}\omega)} \,. \end{aligned} \tag{2.7}$$

It can be observed that type-jumping (from one to three) occurs for Φ_- at points where $|c| = 1$ (zeroes of $v + iw$). The limit where $v + iw$ goes to zero is ill-defined in those expressions, and the pure spinors at such points are expressed as in formulas (2.5).

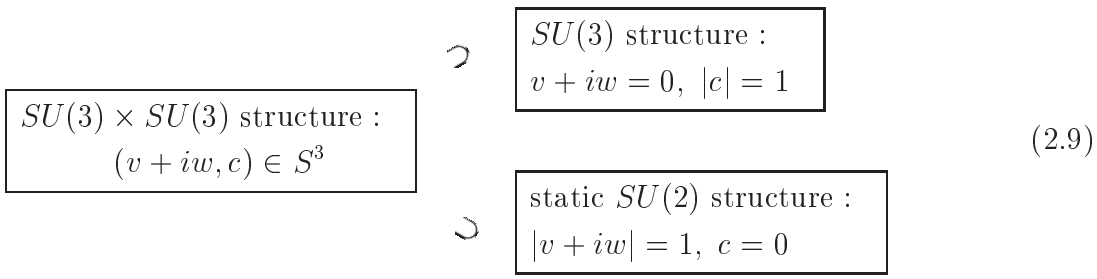
Another type-jumping phenomenon occurs at zeroes of c. At these points the two spinors η^1 and η^2 become orthogonal, and there is a local $SU(2)$ structure. The pure spinors then read:

$$\begin{aligned} \Phi_- &= (v + iw) \wedge e^{ij} \\ \Phi_+ &= \omega \wedge e^{iv\wedge w} \,. \end{aligned} \tag{2.8}$$

We notice that due to the normalization constraint relating c to $v + iw$, type-jumping occurs at critical points of $|c|$ and $|v + iw|$. The situation is depicted in Figure 1. Manifolds

with $v + iw = 0$ everywhere form the particular class of manifolds with $SU(3)$ structure. Those with $c = 0$ everywhere form another particular class, the one of manifolds with *static* $SU(2)$ structure. On such manifolds the pure spinors Φ_- and Φ_+ have type one and type two everywhere. The Euler characteristic of any manifold with static $SU(2)$ structure is zero, because otherwise the vector field corresponding to $v + iw$ would have zeroes.

In summary the set of manifolds with $SU(3) \times SU(3)$ structure has two important subclasses:



(2.9)

2.2 Generalized geometry, generalized submanifolds and D-branes

For the sake of completeness, let us recall a few definitions from generalized complex (GC) geometry [11]. Given an n-dimensional manifold M, with even n, a generalized almost complex structure on M is defined as an almost complex structure on the sum of tangent and cotangent bundles $TM \oplus T^*M$. For example, such a structure can be induced by an ordinary complex structure J on M

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix} , \tag{2.10}$$

in which case it will sometimes be termed a diagonal GC structure, or by a symplectic form ω on M

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} , \tag{2.11}$$

where the matrices are written in a basis adapted to the direct sum $TM \oplus T^*M$. Hybrid examples, other than these two extreme ones, are classified by a generalized Darboux theorem [11], saying that any GC space is locally the sum of a complex space and a symplectic space. Hybrid GC structures with no underlying complex or symplectic structure do appear in $\mathcal{N} = 1$ supersymmetric compactifications of string theory [31, 32].

Around every point p, the sum $T_pM \oplus T_p^*M$ is naturally endowed with an inner product of signature (n, n),

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\iota_X \eta + \iota_Y \xi) . \tag{2.12}$$

It also acts naturally on polyforms on M:

$$(X + \xi).\phi = \iota_X \phi + \xi \wedge \phi. \tag{2.13}$$

Acting twice on ϕ yields a Clifford algebra, and the $+i$ eigenbundle of a GC structure is an n-dimensional subspace, hence the one-to-one correspondence between GC structures and pure spinors (polyforms with an n-dimensional annihilator). On a Calabi–Yau manifold, the pure

spinor associated to the diagonal GC structure induced by the ordinary complex structure is the holomorphic n-form, while the pure spinor associated to the off-diagonal GC structure induced by the symplectic structure is $e^{i\omega}$, where ω denotes the Kähler form.

The inner product is conserved by an action of the group $O(n,n)$, whose generic element contains off-diagonal blocks that can be exponentiated into the so-called B- and β-transforms

$$\begin{aligned} \exp B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} : \qquad & B : X + \xi \mapsto X + \xi + \iota_X B \\ \exp \beta = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} : \qquad & \beta : X + \xi \mapsto X + \iota_\xi \beta + \xi \,, \end{aligned} \tag{2.14}$$

where B and β are antisymmetric blocks identified with a two-form $B_{\mu\nu}$ and a bivector $\beta^{\mu\nu}$. The correponding transforms act by conjugation on the matrices of the GC structures, and by left-multiplication by e^B or e^β on the corresponding pure spinors. These actions will occur in section 6.

Let H be a closed three-form. A generalized submanifold is defined in [11] as a submanifold N endowed with a two-form B such that $H|_N = dB$. The generalized tangent bundle τ_N^B of this generalized submanifold is defined as the B-transform of the sum of the tangent bundle TN and conormal bundle (or annihilator) $\operatorname{Ann} TN$, namely:

$$\tau_N^B = \{X + \xi \in TN \oplus T^*M|_N,\ \xi|_N = \iota_X B\} \ , \tag{2.15}$$

so that $\tau_N^0 = TN \oplus \operatorname{Ann} TN$. A generalized tangent bundle is a maximally isotropic subspace (*i.e.*, it is isotropic with respect to natural pairing and it has the maximal possible dimension for an isotropic space in ambient signature (n,n), namely n.) Moreover, all the maximally isotropic subspaces are of this form, for some submanifold N and two-form B.

Given a GC structure $\mathcal{J}$, a generalized complex brane is defined in [11] as a generalized submanifold whose generalized tangent bundle is stable under the action of $\mathcal{J}$. In the case of a diagonal GC structure, the compatibility condition gives rise to the B-branes, as expected due to the localization properties of the B-model on complex parameters [33]. The submanifold N namely has to be a complex submanifold, and F has to be of type $(1,1)$ with respect to J

$$J(TN) \subset TN \,, \qquad J^*(\iota_X F) + \iota_{JX} F = 0 \,. \tag{2.16}$$

In the other extreme case of a symplectic structure, the definition yields all possible types of A-branes, including the non-Lagrangian ones [34, 35]. These are two tests of the idea that D-branes in generalized geometries are generalized submanifolds. This idea has passed further tests: calibrating forms and pure spinors encoding stability conditions for topological branes [36] are correctly exchanged by mirror symmetry [37, 22, 23, 14].

3 The SYZ argument for Calabi–Yau manifolds

Let us sketch the SYZ argument [26], assuming for a moment that M is an ordinary Calabi–Yau manifold with a Calabi–Yau mirror $\hat{M}$. We break the argument up into steps, which we shall then extend to generalized Calabi–Yau manifolds.

Step 1: Consider the D0-branes of the B-model on M.
As there is an ordinary complex structure on M, one can always put stable D0-branes on it. In other words, the moduli space of a D0-brane consists of the entire manifold M.

Step 2: Consider the A-model on the mirror manifold $\hat{M}$.
As mirror symmetry does not change moduli spaces, there must be a stable D-brane L on $\hat{M}$ (a special Lagrangian submanifold (SLag) of $\hat{M}$) that has the same moduli space, namely M. It is safe to disregard the coisotropic D-branes of the A-model in this context [34, 38, 39], because they are five-dimensional and one eventually considers D-branes that can be obtained from D0-branes by three T-dualities, which rules out dimension five.

Step 3: Project out the gauge-bundle moduli.
Moreover, this moduli space has a fibered structure: it is fibered over the set of geometric moduli called $\mathcal{M}_{\mathrm{SLag}}(L)$, with fiber given by the gauge-bundle moduli (the projection map π is given by "forgetting the bundle data"):

$$M \xrightarrow{\pi} \mathcal{M}_{\mathrm{SLag}}(L). \tag{3.1}$$

M is therefore fibered by the gauge bundle data, with fiber given by the set of Wilson lines $T^{b_1(L)}$.

Step 4: Describe the local tangent space to the moduli space of supersymmetric three-cycles.
The tangent space at L to the moduli space of SLags [40] with flat connections is given by

$$H^1(L, \mathbf{C}) \simeq H^1(L) \oplus H^1(L), \tag{3.2}$$

with the first term corresponding to geometric moduli and the second one to gauge-bundle moduli (the Lagrangian and special condition are preserved by exactly those deformations that are induced by harmonic one-forms, and the flat gauge connections are described by the set of b_1 monodromies around the non-trivial homology cycles in L).

Step 5: Use the result of step 1 to compute the dimension of the fibers.
The moduli space of SLags with flat connections on $\hat{M}$ (continuously connected to L) therefore has real dimension $2b_1(L)$, half of which comes from the moduli of flat connections. But the fiber in the fibration (3.1) is a torus $T^{b_1(L)}$. As this moduli space is M itself, we learn that $2b_1 = 6$, and that M is fibered by three-tori.

Step 6: T-dualize along the three-cycles.
Consider a D3-brane with flat connection wrapping a T^3 fiber on M. T-dualizing along the three $U(1)$ directions produces a D0-brane on a T-dual manifold called M', whose moduli space is the whole of M'. Consider a D0-brane on M. Its moduli space is the whole of M. It sits at some point in a T^3 fiber. T-dualizing along the three $U(1)$ directions of this fiber produces a D3-brane with flat connection wrapping a three-cycle on M'. This describes a fibration of M' by three-tori, whose moduli space is M. This is the same situation as with the couple of branes on M and $\hat{M}$ described above. Therefore $\hat{M} = M'$ and T-duality along the torus fibers is mirror symmetry.

4 Fibrations *à la* SYZ for static $SU(2)$ structure manifolds

Manifolds with static $SU(2)$ structure form an interesting but still tractable subclass of backgrounds because they substantially differ from Calabi–Yau manifolds (in that they admit no closed type-three pure spinor), and because they do not exhibit type-jumping phenomena. They are relatively tractable, for the price of considering cycles that are transverse to M and its mirror $\hat{M}$. Having type-one and type-two closed pure spinors, we find it natural to form their wedge product, which induces a GC structure on the product $M \times \hat{M}$, because the wedge product starts with a complex three-form and allows for some parallel treatment of the SYZ argument.

4.1 Supersymmetric cycles on $M \times \hat{M}$

Consider a generalized Calabi–Yau manifold M and its mirror $\hat{M}$, both with static $SU(2)$-structure.[2] There is a nowhere-vanishing complex one-form field $v + iw$, inducing on every local four-dimensional transverse space a real two-form j and a complex two-form ω. The corresponding two pure spinors are

$$\begin{aligned} \Phi_- &= (v + iw) \wedge e^{ij} \\ \Phi_+ &= \omega \wedge e^{iv \wedge w} \,. \end{aligned} \tag{4.1}$$

They are exchanged under mirror symmetry with analogous objects on the mirror $\hat{M}$ built from a nowhere-vanishing complex one-form field $\hat{v} + i\hat{w}$, inducing on every local four-dimensional transverse space a real two-form $\hat{j}$ and a complex two-form $\hat{\omega}$:

$$\begin{aligned} \hat{\Phi}_- &= (\hat{v} + i\hat{w}) \wedge e^{i\hat{j}} \\ \hat{\Phi}_+ &= \hat{\omega} \wedge e^{i\hat{v} \wedge \hat{w}} \,. \end{aligned} \tag{4.2}$$

This is a case of the generalized Darboux theorem with types one and two, and we can choose local coordinates that are adapted to it:

$$\begin{aligned} j &=: dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \,, & \hat{j} &=: d\hat{x}^3 \wedge d\hat{x}^4 + d\hat{x}^5 \wedge d\hat{x}^6 , \\ \omega &=: d(x^3 + ix^4) \wedge d(x^5 + ix^6) \,, & \hat{\omega} &=: d(\hat{x}^3 + i\hat{x}^4) \wedge d(\hat{x}^5 + i\hat{x}^6) \,. \end{aligned} \tag{4.3}$$

The pure spinors Φ_-, Φ_+, $\hat{\Phi}_-$ and $\hat{\Phi}_+$ induce almost GC structures on M and $\hat{M}$ denoted by $\mathcal{J}_-$, $\mathcal{J}_+$, $\hat{\mathcal{J}}_-$ and $\hat{\mathcal{J}}_+$.

Step 1. Where can we place points? In order to parallel the first step of the SYZ argument for Calabi–Yau manifolds, we need to be able to move points on a six-dimensional space. This cannot be M or $\hat{M}$, because the GC structure induced by Φ_- always maps some tangent vectors to some normal vectors. This prevents the generalized tangent bundle to a point from being stable under the action of the GC structure.

Consider instead supersymmetric cycles on $M \times \hat{M}$. There are several possible choices for structures and calibrations, and we will be interested in the following combinations:

[2] For recent developments based on the physics of $SU(2)$ structure manifolds as gravity duals of deformations of super Yang–Mills theories, see for instance [41].

- GC branes w.r.t. the GC structure $\mathcal{J}_- \oplus \hat{\mathcal{J}}_+$, calibrated by $\Phi_+ \wedge \hat{\Phi}_-$, which we call Σ
- GC branes w.r.t. the GC structure $\mathcal{J}_+ \oplus \hat{\mathcal{J}}_-$, calibrated by $\Phi_- \wedge \hat{\Phi}_+$, which we call $\hat{\Sigma}$.

In a basis of the local tangent space to $M \times \hat{M}$ adapted to the local splitting into $2+4$ dimensions, we have the following matrix representation for the GC structures, where the symbols J_ω and $J_{\hat{\omega}}$ denote the almost complex structures corresponding to ω and $\hat{\omega}$ in the local four-dimensional subspaces, so that we obtain

$$\begin{aligned}
\mathcal{J}_- \oplus \hat{\mathcal{J}}_+ &= \begin{pmatrix} J_{v+iw} & 0 & 0 & 0 \\ 0 & -J^*_{v+iw} & 0 & 0 \\ 0 & 0 & 0 & -j^{-1} \\ 0 & 0 & j & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -\hat{v} \wedge \hat{w}^{-1} & 0 & 0 \\ \hat{v} \wedge \hat{w} & 0 & 0 & 0 \\ 0 & 0 & J_{\hat{\omega}} & 0 \\ 0 & 0 & 0 & -J^*_{\hat{\omega}} \end{pmatrix}, \\
\mathcal{J}_+ \oplus \hat{\mathcal{J}}_- &= \begin{pmatrix} 0 & -v \wedge w^{-1} & 0 & 0 \\ v \wedge w & 0 & 0 & 0 \\ 0 & 0 & J_\omega & 0 \\ 0 & 0 & 0 & -J^*_\omega \end{pmatrix} \oplus \begin{pmatrix} J_{\hat{v}+i\hat{w}} & 0 & 0 & 0 \\ 0 & -J^*_{\hat{v}+i\hat{w}} & 0 & 0 \\ 0 & 0 & 0 & -\hat{j}^{-1} \\ 0 & 0 & \hat{j} & 0 \end{pmatrix}.
\end{aligned} \tag{4.4}$$

Let us describe the generalized tangent bundle τ^0_Σ (with zero field strength), of the GC submanifold Σ of $M \times \hat{M}$. As the two GC structures we consider on $M \times \hat{M}$ are block-diagonal with blocks of the same size , the projections of the generalized tangent bundle onto the sums of blocks and dual blocks are separately generalized complex and calibrated w.r.t. the corresponding blocks.

We may choose Σ to have zero-dimensional projections onto $\mathrm{Vect}(v,w)$ and $\mathrm{Vect}(\hat{v},\hat{w})^\perp$. chosen to be trivial, the projections of τ^0_Σ onto $\mathrm{Vect}(\hat{v},\hat{w})$ and $\mathrm{Vect}(v,w)^\perp$ have to be Lagrangian w.r.t. $\hat{v} \wedge \hat{w}$ and j respectively, and calibrated by $\hat{v} + i\hat{w}$ and ω. This gives one-dimensional and two-dimensional projections on $\mathrm{Vect}(\hat{v},\hat{w})$ and $\mathrm{Vect}(v,w)^\perp$ respectively for the world-volume of Σ

$$j + \hat{v} \wedge \hat{w}|_\Sigma = 0 \tag{4.5}$$
$$\mathrm{Im}\left(\omega \wedge (\hat{v} + i\hat{w})\right)|_\Sigma = 0\,. \tag{4.6}$$

They look like the Lagrangian and special conditions, but live on a six-dimensional subspace of $M \times \hat{M}$, transverse to both M and $\hat{M}$. To sum up, a possible local generalized tangent bundle is given in the coordinates chosen above as:

$$\tau^0_\Sigma = \left\langle v, w, \frac{\partial}{\partial x^3} + i dx^4, \frac{\partial}{\partial x^5} + i dx^6, \hat{v}_* + i\hat{w}_*, d\hat{x}^3, d\hat{x}^4, d\hat{x}^5, d\hat{x}^6 \right\rangle. \tag{4.7}$$

The supersymmetric cycle Σ is therefore three-dimensional, but neither of its projections on M or $\hat{M}$ is (they are two- and one-dimensional respectively). The situation is depicted in fig. 2.

The same linear-algebraic exercise can be repeated with hats exchanged to yield the local generalized tangent bundle of the cycle called $\hat{\Sigma}$ (with a somewhat misleading notation because $\hat{\Sigma}$ is not mirror to Σ; both are their own mirror):

$$v \wedge w + \hat{j}|_{\hat{\Sigma}} = 0 \tag{4.8}$$
$$\mathrm{Im}\left((v + iw) \wedge \hat{\omega}\right)|_{\hat{\Sigma}} = 0\,, \tag{4.9}$$

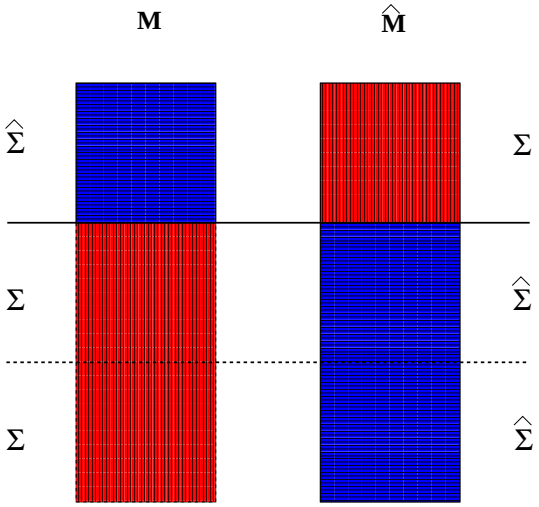


Figure 2: Supersymmetric cycles Σ and $\hat{\Sigma}$ and their location within $M \times \hat{M}$.

with the generalized tangent bundle given by

$$\tau^0_{\hat{\Sigma}} = \left\langle v_* + iw_*, dx^3, dx^4, dx^5, dx^6, \hat{v}, \hat{w}, \frac{\partial}{\partial \hat{x}^3} + id\hat{x}^4, \frac{\partial}{\partial \hat{x}^5} + id\hat{x}^6 \right\rangle . \tag{4.10}$$

We thus obtain the situation in Figure 2. The supersymmetric cycles we have just described are sketched as submanifolds of $M \times \hat{M}$ that are transverse to both M and $\hat{M}$, whereas the tree-dimensional supersymmetric cycles on a mirror pair of Calabi–Yau manifolds are longitudinal either to M or to $\hat{M}$. If one thinks of a supersymmetric three-cycle as a leg, then the SYZ picture of mirror pairs correspond to standing on $M \times \hat{M}$ with one leg on M and one leg on $\hat{M}$. What we have just argued is that standing on $M \times \hat{M}$ with M and $\hat{M}$ generalized Calabi–Yau manifolds with static $SU(2)$ structure can be achieved, but only with the legs crossed.

4.2 The fibration of $M \times \hat{M}$

We are ready to turn to *Step 3* and *Step 4*. So far we have exhibited two three-dimensional supersymmetric cycles on $M \times \hat{M}$, called Σ and $\hat{\Sigma}$, each of which possesses six position moduli given by the projections onto the subspaces of $M \times \hat{M}$ that are complex w.r.t. $(v + iw) \wedge \hat{\omega}$ and $\omega \wedge (\hat{v} + i\hat{w})$. We want to show that they both have the topology of a three-torus, and that the moduli space of $\Sigma \times \hat{\Sigma}$ on $M \times \hat{M}$ is the whole of $M \times \hat{M} \times M \times \hat{M}$.

We have worked out the generalized tangent spaces of both cycles Σ and $\hat{\Sigma}$. This gives only local informations, essentially counting dimensions. For some point p in $M \times \hat{M}$, on the local tangent space $T_p(M \times \hat{M})$ in which Φ_-, Φ_+, $\hat{\Phi}_-$ and $\hat{\Phi}_+$ have the generalized Darboux expressions we wrote above, the projections of τ^0_Σ onto the subspace $\mathrm{Vect}(v, w)$ and $\mathrm{Vect}(\hat{v}, \hat{w})^\perp$ have dimension zero. So have the projections of $\tau^0_{\hat{\Sigma}}$ on $\mathrm{Vect}(v, w)^\perp$ and $\mathrm{Vect}(\hat{v}, \hat{w})$. We have just described a projector

$$\tau^0_\Sigma \times \tau^0_{\hat{\Sigma}} \to T_p(M \times \hat{M}) . \tag{4.11}$$

This is the tangent application to the projection

$$\Sigma \times \hat{\Sigma} \mapsto p \in M \times \hat{M} . \tag{4.12}$$

There are therefore position moduli for $\Sigma \times \hat{\Sigma}$ in all the twelve directions, which correspond to moving $\Sigma \times \hat{\Sigma}$ around p in the coordinate patch.

Let us call $\mathcal{M}$ the moduli space of supersymmetric cycles of $M \times \hat{M}$ that are continuously connected to $\Sigma \times \hat{\Sigma}$. We have just argued that there is a twelve-dimensional subspace consisting of translation moduli, so there must exist other moduli, which make up some subspace $\mathcal{M}'$ consisting of deformations that leave the projection of $\Sigma \times \hat{\Sigma}$ onto the local complex subspaces of $M \times \hat{M}$ fixed:

$$T_{\Sigma\times\hat{\Sigma}}\mathcal{M} = T_p(M \times \hat{M}) \times T_{\Sigma\times\hat{\Sigma}}\mathcal{M}' \,. \tag{4.13}$$

Consider now the projection π that "forgets the gauge bundle" along the two cycles Σ and $\hat{\Sigma}$. It induces a fibration of $\mathcal{M}'$ over some base $\mathcal{B}$ consisting of the Lagrangian deformations of $\Sigma \times \hat{\Sigma}$ w.r.t. $v \wedge w \wedge j \wedge \hat{v} \wedge \hat{w} \wedge \hat{j}$:

$$\mathcal{M}' \xrightarrow{\pi} \mathcal{B} \tag{4.14}$$

The space $\mathcal{M}'$ has the following topological meaning, as the fiberwise projection of the generalized tangent bundle is isomorphic to the complexified dual of the ordinary tangent space to $\Sigma \times \hat{\Sigma}$ as a bundle and as a Lie algebroid (cf. section 7.2 of [17]) :

$$\mathcal{M}' = H^1\left(L(\Sigma \times \hat{\Sigma}, 0)|_p\right) = H^1_{\mathrm{dR}}(\Sigma \times \hat{\Sigma}, \mathbf{C}). \tag{4.15}$$

The dimension of the moduli space $\mathcal{M}'$ is therefore twice the Betti number of the six-dimensional cycle $\Sigma \times \hat{\Sigma}$.

In order to compute this dimension, we are going to perform T-duality along $\Sigma \times \hat{\Sigma}$, with the point p still fixed. This implies that gauge connections will be fixed on the image of $\Sigma \times \hat{\Sigma}$, which will only have translation moduli. The moduli space is not changed by T-duality, but it is now described as follows. The one-dimensional projection of Σ onto T_pM, with moduli from flat connection and normal deformations (all in $\mathcal{M}'$), is mapped to a point-like projection onto $T_p\hat{M}$ with two translation moduli. The fixed zero-dimensional projection of Σ onto $T_p\hat{M}$, is mapped to a one-dimensional projection onto T_pM sitting at some fixed $w_* = 0$, extended along v_* and with fixed flat connection. The two-dimensional projection of Σ onto $T_p\hat{M}$ is mapped to a point of M with four translation moduli. So Σ is mapped to itself, but the fiber moduli are traded for translational ones, living in the subspace $\mathrm{Vect}(\hat{v_*}, \hat{w_*}) \oplus \mathrm{Vect}(v_*, w_*)^\perp$. By T-dualizing $\hat{\Sigma}$ along $\Sigma \times \hat{\Sigma}$, one trades in the same way the deformation moduli for translational ones, living in the subspace $\mathrm{Vect}(v_*, w_*) \oplus \mathrm{Vect}(\hat{v_*}, \hat{w_*})^\perp$, so that the tangent space at $\Sigma \times \hat{\Sigma}$ to the moduli space $\mathcal{M}'$ is isomorphic (by dimension counting) to

$$\mathrm{Vect}(\hat{v_*}, \hat{w_*}) \oplus \mathrm{Vect}(v_*, w_*)^\perp \oplus \mathrm{Vect}(v_*, w_*) \oplus \mathrm{Vect}(\hat{v_*}, \hat{w_*})^\perp = T_p(M \times \hat{M}) \,. \tag{4.16}$$

The moduli (sub)-space $\mathcal{M}'$ therefore has dimension twelve.

Let us move to *Step 5*. We have just computed the dimension of $\mathcal{M}'$, which is accessible to our local computations, but its T-dual interpretation in homology promotes the result to a Betti number, a global quantity. From this T-duality argument we learn that

$$\mathrm{dim} T_{\Sigma\times\hat{\Sigma}}\mathcal{M}' = 2b_1(\Sigma \times \hat{\Sigma}) = \mathrm{dim}(M \times \hat{M}) = 12 \,. \tag{4.17}$$

Hence $\Sigma \times \hat{\Sigma}$ is a six-torus, the product of two supersymmetric three-tori, and its moduli space is $M \times \hat{M} \times M \times \hat{M}$.

Note that Σ or $\hat{\Sigma}$ by itself does not have $M \times \hat{M}$ as its moduli space, nor $M \times M$ nor $\hat{M} \times \hat{M}$, as it is only the case for $SU(3)$ structures.

5 Illustrations in flux compactifications

So far we have drawn the conclusions of there being transverse three-dimensional supersymmetric cycles on a mirror pair of manifolds with static $SU(2)$ structures. This begs for a few checks. We shall first T-dualize the three-tori and check that the pure spinors are exchanged by this transformation. We shall then turn to the example of $K3 \times T^2$, which was of course available in the Calabi–Yau case, but can also be endowed with a static $SU(2)$ structure. Finally, in order to make contact with open problems in flux compactifications (where the nature of non-geometric fluxes is still under investigation), we shall take the analog of *Step 6* by turning on all the possible fluxes on a six-torus with static $SU(2)$-structure, thus putting our T-duality proposal to the test.

5.1 Mirror images of the pure spinors

Let us perform a Fourier–Mukai transform (F.T.) on the pure spinors, by weighting them with the Poincaré connection on $\Sigma \times \hat{\Sigma}$ we worked out. As we have established that the three-dimensional intersection of $\Sigma \times \hat{\Sigma}$ and M are the directions which are T-dualized, the Fourier–Mukai transform of the pure spinors reads

$$\text{F.T.}\,(\Phi_-) = \int_{(\Sigma\times\hat{\Sigma})\cap M} (v+iw)\wedge e^{ij}\wedge e^{v\wedge\hat{v}+dx^3\wedge d\hat{x}^3+dx^5\wedge d\hat{x}^5} = e^{i\hat{v}\wedge w}\wedge\omega = \hat{\Phi}_+, \tag{5.1}$$

$$\text{F.T.}\,(\Phi_+) = \int_{(\Sigma\times\hat{\Sigma})\cap M} e^{iv\wedge w}\wedge\omega\wedge e^{v\wedge\hat{v}+dx^3\wedge d\hat{x}^3+dx^5\wedge d\hat{x}^5} = (\hat{v}+iw)\wedge e^{i\hat{j}} = \hat{\Phi}_-, \tag{5.2}$$

with the value of the base coordinates unchanged, namely provided $w = \hat{w}$, which makes sense, because the local coordinates w or $\hat{w}$ are not T-dualized. The mapping of pure spinors under Fourier–Mukai transform coincides with what is expected from mirror symmetry.

5.2 The $K3 \times T^2$ example

As the Euler characteristic is multiplicative, the manifold $K3 \times T^2$ has Euler characteristic zero. There may therefore be a nowhere-vanishing vector field on it. Real and imaginary part of the complex coordinate of T^2 as an elliptic curve indeed serve as v and w vector fields.[3]

In the present case, ω and j are a complex and a Kähler form on $K3$, while $\hat{\omega}$ and $\hat{j}$ are the same objects on the mirror $K3$. Of course in this case we have a global picture of the cycles: Σ is a point in T^2 times a special Lagrangian torus with respect to j, times a Lagrangian circle in the mirror torus times a point in the mirror $K3$, while $\hat{\Sigma}$ is the mirror circle on the first torus times a point in the first $K3$ times a point in the second T^2 times the dual torus in the second $K3$. The projection is just given by associating the points to $\Sigma \times \hat{\Sigma}$. This is just the ordinary SYZ case but with the complex structures of the two-tori exchanged. It is a straightforward consequence of the Calabi–Yau case because crossing the legs amounts to permuting the two two-tori.

[3]For a thorough treatment of the reduction of IIA supergravity on $K3 \times T^2$ endowed with an $SU(2)$ structure, see [42].

5.3 Static $SU(2)$ structure with non-geometric fluxes

Let us apply this analysis to the case of a six-torus endowed with a static $SU(2)$ structure. This seems of course to be an over-simplification, as many torus fibrations can be explicitly found in such a geometry. However, T-duality leads from geometric to non-geometric fluxes, which in the terminology [5] are called Q- and R-fluxes according to the number of T-dualized directions supporting a B-field. With each double arrow symbolizing one T-duality, these notations are summarized in the following way:

$$H_{abc} \leftrightarrow f^a_{bc} \leftrightarrow Q^{ab}_c \leftrightarrow R^{abc}. \tag{5.3}$$

The embedding of three-tori into $M \times \hat{M}$ along which T-duality is performed is key to the map between geometric and non-geometric fluxes. Finding the mirror of a generic flux configuration is therefore a non-trivial check of our proposal[4]. We are going to complete the study of fluxes on the $SU(2)$ structure background of [9], first including all the non-geometric fluxes (which indeed fill all the entries of the charge matrix), and then to obtain the mirror configuration by T-duality along the transverse supersymmetric fibers.

5.3.1 Charge matrix

We consider a six-torus endowed with a static $SU(2)$ structure. The holomorphic vector $e^3 = v + iw$ is completed to a basis by (e^1, e^2, e^3), and likewise for the mirror the basis is denoted by $(\hat{e}^1, \hat{e}^2, \hat{e}^3)$. The GC submanifolds Σ and $\hat{\Sigma}$ solving the structure and stability equations (4.5)-(4.9) are chosen as

$$\begin{array}{c|c} \Sigma & \hat{\Sigma} \\ \hline \Re(e^1) & \Re(\hat{e}^1) \\ \hline \Re(e^2) & \Re(\hat{e}^2) \\ \hline \Re(\hat{e}^3) & \Re(e^3) \end{array}, \tag{5.4}$$

which have trivial projection onto the base spanned by $(\Im(e^1), \Im(e^2), \Im(e^3))$ and Σ projects trivially upon $e^3 = v + iw$ etc. as required.

The generic $SU(3) \times SU(3)$ structure is described by a symplectic basis with forms that are not necessarily closed. Denote the two bases by

$$\Sigma^- = \begin{pmatrix} \alpha_I \\ \beta^I \end{pmatrix}, \qquad \Sigma^+ = \begin{pmatrix} \omega_A \\ \tilde{\omega}^B \end{pmatrix}, \tag{5.5}$$

where the entries of $\Sigma^\pm$ are odd/even formal sums of forms. In particular $d\Sigma^\pm \neq 0$ and can therefore be expanded in $\Sigma^\mp$, *i.e.*

$$d\Sigma^- = \mathcal{Q}\Sigma^+ . \tag{5.6}$$

The matrix $\mathcal{Q}$ is called the charge matrix. In the present case it is a four-by-four matrix.

[4]Choosing a static $SU(2)$ structure protects us against type-jumping phenomena; those will of course be crucial in the generic $SU(3) \times SU(3)$ case, which will be elaborated on in the next section, in a much less thorough way though.

Furthermore define the generalized symplectic basis $\Sigma^{\pm}$ in terms of the basis e^i as follows

$$\Sigma^- = \begin{pmatrix} 2\Re(e^3) \\ -2\Im(e^3) + \Re(e^3)j \\ -\Im(e^3)j^2 \\ \frac{1}{3}\Re(e^3)j^2 + \frac{4}{3}\Im(e^3)j \end{pmatrix}, \tag{5.7}$$

and

$$\Sigma^+ = \begin{pmatrix} 4\Re(e^1)\wedge\Re(e^2) \\ 8\left(\Im(e^1)\wedge\Re(e^2) + \Re(e^1)\wedge\Im(e^2)\right) - 16\Re(e^1)\wedge\Re(e^2)\wedge\Re(e^3)\wedge\Im(e^3) \\ 16\Im(e^1)\wedge\Im(e^2)\wedge\Re(e^3)\wedge\Im(e^3) \\ \frac{4}{3}\Im(e^1)\wedge\Im(e^2) + \frac{4}{3}(\Im(e^1)\wedge\Re(e^2) + \Re(e_1)\wedge\Im(e^2))\wedge\Re(e^3)\wedge\Im(e^3) \end{pmatrix}, \tag{5.8}$$

where we defined

$$j = 2i(e^1\wedge e^{\bar{1}} + e^2\wedge e^{\bar{2}}) = 4\left(\Re(e^1)\wedge\Im(e^1) + \Re(e^2)\wedge\Im(e^2)\right). \tag{5.9}$$

As discussed earlier, the standard relation between the two symplectic basis vectors is (5.6). Turning on fluxes – both geometric H-flux and non-geometric Q- and R-fluxes – has the effect of twisting the the differential operator d

$$(d + H\wedge + Q\cdot + R\cdot)\Sigma^- \sim \mathcal{Q}\Sigma^+ . \tag{5.10}$$

Here we denote by $\sim$ equality up to terms that are perpendicular to all elements in the symplectic basis with respect to the symplectic pairing

$$\int_M \langle\sigma,\rho\rangle = \int_M \left(\sum_p (-1)^{[\frac{p+1}{2}]}\sigma_p\wedge\rho_{6-p}\right), \tag{5.11}$$

where $\sigma = \sum_p \sigma_p$ is a polyform (the sum runs over the degrees) and $\langle,\rangle$ denotes the Mukai pairing. In particular the symplectic basis obeys

$$\int_M \langle\alpha^I,\beta_J\rangle = \delta^I_J , \qquad \int_M \langle\omega_A,\tilde{\omega}^B\rangle = \delta^B_A . \tag{5.12}$$

Note that the action on cohomologies is as follows

$$\begin{aligned} d: &\quad H^p \to H^{p+1} \\ H: &\quad H^p \to H^{p+3} \\ Q\cdot: &\quad H^p \to H^{p-1} \\ R\cdot: &\quad H^p \to H^{p-3} , \end{aligned} \tag{5.13}$$

in agreement with Q having two vector and one form index and R being a tri-vector. Note that d acts on the one-forms as $de^i = f^i_{jk}e^j\wedge e^k$. The mapping of the various degrees under the fluxes (5.13) can be depicted as in Fig. 3. Here $[p]$, with $p = 1, 2, \cdots$ denotes the degree of the forms.

The various flux components then follow by noting that

$$\Sigma^- = \begin{pmatrix} [1] \\ [1]+[3] \\ [5] \\ [3]+[5] \end{pmatrix}, \qquad \Sigma^+ = \begin{pmatrix} [2] \\ [2]+[4] \\ [4] \\ [2]+[4] \end{pmatrix}, \tag{5.14}$$

and further allowing additional terms compatible with the equivalence relation $\sim$.

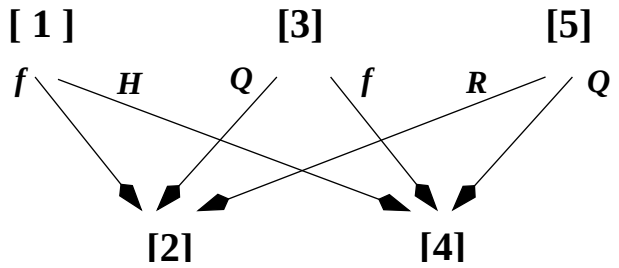


Figure 3: Mapping of cohomology degrees under the fluxes f, H, Q and R.

5.3.2 Geometric fluxes

The effect of the geometric fluxes (both H and f) was already discussed in [9]. There it was found that with the geometric flux parameters one can switch on the following entries in the charge matrix

$$(d + H\wedge)\Sigma^- \sim \left(\begin{array}{c|c|c|c} \mathbb{F}_{11} & \mathbb{F}_{12}+\mathbb{H}_{12} & \mathbb{H}_{13} & \mathbb{F}_{14}+\mathbb{H}_{14} \\ \hline \mathbb{F}_{21} & \mathbb{F}_{22}+\mathbb{H}_{22} & \mathbb{F}_{23}+\mathbb{H}_{23} & \mathbb{F}_{24}+\mathbb{H}_{24} \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & \mathbb{F}_{42} & \mathbb{F}_{43} & \mathbb{F}_{44} \end{array}\right) \Sigma^+ \,. \tag{5.15}$$

The geometric flux charges (a.k.a. torsion charges) $\mathbb{F}_i$ follow from the relation

$$d\mathfrak{e}^I = f^I_{JK}\mathfrak{e}^J \wedge \mathfrak{e}^K \,, \tag{5.16}$$

where $\mathfrak{e}^I = \Re(e^i)$ for $I = i$ and $\mathfrak{e}^I = \Im(e^i)$ for $I = \bar{i}$.

To sum up, the f-flux we have to turn on in order to generate the above charge entries are

$$\begin{aligned} f = &+ 2\mathbb{F}_{11}\Re(\partial_3)\wedge\Re(e^1)\wedge\Re(e^2) \\ &+ 8\mathbb{F}_{12}\left(\Re(\partial_3)\wedge\Im(e^1)\wedge\Re(e^2)+\Re(\partial_3)\wedge\Re(e^1)\wedge\Im(e^2)\right) \\ &+ \frac{4}{3}\mathbb{F}_{14}\Re(\partial_3)\wedge\Im(e^1)\wedge\Im(e^2) \\ &- 2\mathbb{F}_{21}\Im(\partial_3)\wedge\Re(e^1)\wedge\Re(e^2) \\ &- 4\mathbb{F}_{22}(\Im(\partial_3)\wedge\Im(e^1)\wedge\Re(e^2)+\Im(\partial_3)\wedge\Re(e^1)\wedge\Im(e^2)) \\ &- 2\mathbb{F}_{22}(\Im(\partial_1)\wedge\Re(e^2)\wedge\Im(e^3)+\Im(\partial_2)\wedge\Im(e^3)\wedge\Re(e^1)) \\ &+ 2\mathbb{F}_{23}(\Re(\partial_1)\wedge\Im(e^3)\wedge\Im(e^2)+\Re(\partial_2)\wedge\Im(e^1)\wedge\Im(e^3)) \\ &- \frac{1}{3}\mathbb{F}_{24}\left(2\Im(\partial_3)\wedge\Im(e^1)\wedge\Im(e^2)+\Re(\partial_1)\wedge\Re(e^2)\wedge\Im(e^3)+\Re(\partial_2)\wedge\Im(e^3)\wedge\Re(e^1)\right) \\ &+ 3\mathbb{F}_{42}(-\Re(e^2)\wedge\Im(\partial_1)+\Re(e^1)\wedge\Im(\partial_2))\wedge\Re(e^3) \\ &+ \frac{3}{2}\mathbb{F}_{43}(\Re(\partial_1)\wedge\Im(e^2)\wedge\Re(e^3)+\Re(\partial_2)\wedge\Re(e^3)\wedge\Im(e^1)) \\ &+ \frac{1}{2}\mathbb{F}_{44}(\Im(e^2)\wedge\Im(\partial_1)+\Im(\partial_2)\wedge\Im(e^1))\wedge\Re(e^3)\,. \end{aligned} \tag{5.17}$$

We should perhaps add a word of explanation. Recall that the relations between the two symplectic basis is only up to the equivalence w.r.t. $\sim$. This in particular allows one to switch

on f-flux to generate the $\mathcal{Q}_{12}$ charge entry, without turning on H-flux simultaneously. To be more explicit

$$f = 8\mathbb{F}_{12}\left(\Re(\partial_3)\wedge\Im(e^1)\wedge\Re(e^2)+\Re(\partial_3)\wedge\Re(e^1)\wedge\Im(e^2)\right) \tag{5.18}$$

acting upon $\Sigma_1^- = 2\Re(e^3)$ will only generate the two-form part of Σ_2^+, denoted by $\Sigma_2^+\big|_{[2]}$

$$f\Sigma_1^- = 2\mathbb{F}_{12}\ \Sigma_2^+\big|_{[2]}\ . \tag{5.19}$$

However, this can be written as

$$f\Sigma_1^- = \mathbb{F}_{12}\Sigma_2^+ + \Omega\,, \tag{5.20}$$

where

$$\Omega = \mathbb{F}_{12}\left(\Sigma_2^+\big|_{[2]} - \Sigma_2^+\big|_{[4]}\right)\ , \tag{5.21}$$

which is perpendicular to all other basis elements

$$\langle\Omega,\Sigma_i^\pm\rangle = 0\,, \tag{5.22}$$

and thus

$$f\Sigma_1^- \sim \mathbb{F}_{12}\Sigma_2^+\ . \tag{5.23}$$

Likewise the H-flux can be determined as

$$\begin{aligned} H =& +16\mathbb{H}_{12}\Re(e^1)\wedge\Re(e^2)\wedge\Im(e^3)\\ &-8\mathbb{H}_{13}\Im(e^1)\wedge\Im(e^2)\wedge\Im(e^3)\\ &+\frac{4}{3}\mathbb{H}_{14}\left(\Re(e^2)\wedge\Im(e^1)-\Re(e^1)\wedge\Im(e^2)\right)\wedge\Im(e^3)\\ &+16\mathbb{H}_{22}\Re(e^1)\wedge\Re(e^2)\wedge\Re(e^3)\\ &-8\mathbb{H}_{23}\Im(e^1)\wedge\Im(e^2)\wedge\Re(e^3)\\ &-\frac{4}{3}\mathbb{H}_{24}\left(\Im(e^1)\wedge\Re(e^2)\wedge\Re(e^3)+\Re(e^1)\wedge\Im(e^2)\wedge\Re(e^3)\right)\ . \end{aligned} \tag{5.24}$$

The resulting charge matrix entries are as we indicated in (5.15).

5.3.3 Non-geometric fluxes

Here we wish to study the effect of the Q- and R-fluxes, which can be done by linear superposition with the results from [9]. We find by simple dimensional analysis that the effect of these non-geometric fluxes on the charge matrix can be only of the following type:

$$(Q+R)\Sigma^- \sim \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0\\ \hline \mathbb{Q}_{21} & \mathbb{Q}_{22} & 0 & \mathbb{Q}_{24}\\ \hline \mathbb{R}_{31} & \mathbb{Q}_{32}+\mathbb{R}_{32} & \mathbb{Q}_{33} & \mathbb{Q}_{34}+\mathbb{R}_{34}\\ \hline \mathbb{R}_{41} & \mathbb{Q}_{42}+\mathbb{R}_{42} & \mathbb{Q}_{43} & \mathbb{Q}_{44}+\mathbb{R}_{44} \end{array}\right)\Sigma^+\,. \tag{5.25}$$

We can determine the corresponding non-geometric fluxes which will turn on these charge entries by analyzing the structure of the linear equations and keeping in mind the liberty to

add terms perpendicular to all basis elements in the symplectic basis. We find the following Q-fluxes (are one-forms and bi-vectors)

$$\begin{aligned}
Q = &+ \frac{1}{2}\mathbb{Q}_{21}\left(\Re(e^2)\wedge\Im(\partial_1) - \Re(e^1)\wedge\Im(\partial_2)\right)\wedge\Re(\partial_3)\\
&+ \mathbb{Q}_{22}\left(-\Re(e^2)\wedge\Re(\partial_1) + \Im(e^2)\wedge\Im(\partial_1) - \Im(e^1)\wedge\Im(\partial_2) + \Re(e^1)\wedge\Re(\partial_2)\right)\wedge\Re(\partial_3)\\
&+ \frac{1}{3}\mathbb{Q}_{24}\left(\Im(e^1)\wedge\Re(\partial_2) - \Im(e^2)\wedge\Re(\partial_1)\right)\wedge\Re(\partial_3)\\
&+ \mathbb{Q}_{32}\,\Im(\partial_1)\wedge\Im(\partial_2)\wedge\Re(e^3)\\
&- \frac{1}{2}\mathbb{Q}_{33}\Re(\partial_1)\wedge\Re(\partial_2)\wedge\Re(e^3)\\
&- \frac{1}{12}\mathbb{Q}_{34}(\Im(\partial_2)\wedge\Re(\partial_1) + \Re(\partial_2)\wedge\Im(\partial_1))\wedge\Re(e^3)\\
&+ 3\mathbb{Q}_{42}\,\Im(\partial_1)\wedge\Im(\partial_2)\wedge\Im(e^3)\\
&- \frac{3}{2}\mathbb{Q}_{43}\Re(\partial_1)\wedge\Re(\partial_2)\wedge\Im(e^3)\\
&- \frac{1}{4}\mathbb{Q}_{44}(\Im(\partial_2)\wedge\Re(\partial_1) + \Re(\partial_2)\wedge\Im(\partial_1))\wedge\Im(e^3)\,,
\end{aligned} \tag{5.26}$$

as well as R-fluxes of the type

$$\begin{aligned}
R = &- \frac{1}{8}\mathbb{R}_{31}\,\Im(\partial_1)\wedge\Im(\partial_2)\wedge\Im(\partial_3)\\
&- \frac{1}{2}\mathbb{R}_{32}(\Re(\partial_1)\wedge\Im(\partial_2) - \Re(\partial_2)\wedge\Im(\partial_1))\wedge\Im(\partial_3)\\
&- \frac{1}{12}\mathbb{R}_{34}\Re(\partial_1)\wedge\Re(\partial_2)\wedge\Im(\partial_3)\\
&+ \frac{3}{8}\mathbb{R}_{41}\,\Im(\partial_1)\wedge\Im(\partial_2)\wedge\Re(\partial_3)\\
&- \frac{3}{2}\mathbb{R}_{42}(\Re(\partial_1)\wedge\Im(\partial_2) - \Re(\partial_2)\wedge\Im(\partial_1))\wedge\Re(\partial_3)\\
&+ \frac{1}{4}\mathbb{R}_{44}\Re(\partial_1)\wedge\Re(\partial_2)\wedge\Re(\partial_3)\,.
\end{aligned} \tag{5.27}$$

In summary we have shown that the full charge matrix can be constructed by switching on geometric as well as non-geometric fluxes:

$$\mathcal{Q} = \left(\begin{array}{c|c|c|c} \mathbb{F}_{11} & \mathbb{F}_{12}+\mathbb{H}_{12} & \mathbb{H}_{13} & \mathbb{F}_{14}+\mathbb{H}_{14} \\ \hline \mathbb{F}_{21}+\mathbb{Q}_{21} & \mathbb{F}_{22}+\mathbb{H}_{22}+\mathbb{Q}_{22} & \mathbb{F}_{23}+\mathbb{H}_{23} & \mathbb{F}_{24}+\mathbb{H}_{24}+\mathbb{Q}_{24} \\ \hline \mathbb{R}_{31} & \mathbb{Q}_{32}+\mathbb{R}_{32} & \mathbb{Q}_{33} & \mathbb{Q}_{34}+\mathbb{R}_{34} \\ \hline \mathbb{R}_{41} & \mathbb{F}_{42}+\mathbb{Q}_{42}+\mathbb{R}_{42} & \mathbb{F}_{43}+\mathbb{Q}_{43} & \mathbb{F}_{44}+\mathbb{Q}_{44}+\mathbb{R}_{44} \end{array}\right)\,. \tag{5.28}$$

5.3.4 Mirror symmetry

We now wish to test out generalized SYZ proposal in this setup. This should in particular be compatible with the proposed mirror map of [9]. The mirror fluxes are obtained by first recalling that we dualize along $\Re(e^1)$, $\Re(e^2)$ and $\Re(e^3)$ and that thereby the mirror map is realized as

$$\Re(e^i) \quad \longleftrightarrow \quad \Re(\partial_i)\,. \tag{5.29}$$

The mirror fluxes are determined straight-forwardly from our expressions for the fluxes. The mirrors of the geometric fluxes are

$$\begin{aligned}
\widehat{f} = &+ 2\mathbb{F}_{11}\Re(e^3)\wedge\Re(\partial_1)\wedge\Re(\partial_2)\\
&+ 8\mathbb{F}_{12}\left(\Re(e^3)\wedge\Im(e^1)\wedge\Re(\partial_2)+\Re(e^3)\wedge\Re(\partial_1)\wedge\Im(e^2)\right)\\
&+ \frac{4}{3}\mathbb{F}_{14}\Re(e^3)\wedge\Im(e^1)\wedge\Im(e^2)\\
&- 2\mathbb{F}_{21}\Im(\partial_3)\wedge\Re(\partial_1)\wedge\Re(\partial_2)\\
&- 4\mathbb{F}_{22}(\Im(\partial_3)\wedge\Im(e^1)\wedge\Re(\partial_2)+\Im(\partial_3)\wedge\Re(\partial_1)\wedge\Im(e^2))\\
&- 2\mathbb{F}_{22}(\Im(\partial_1)\wedge\Re(\partial_2)\wedge\Im(e^3)+\Im(\partial_2)\wedge\Im(e^3)\wedge\Re(\partial_1))\\
&+ 2\mathbb{F}_{23}(\Re(e^1)\wedge\Im(e^3)\wedge\Im(e^2)+\Re(e^2)\wedge\Im(e^1)\wedge\Im(e^3))\\
&- \frac{1}{3}\mathbb{F}_{24}\left(2\Im(\partial_3)\wedge\Im(e^1)\wedge\Im(e^2)+\Re(e^1)\wedge\Re(\partial_2)\wedge\Im(e^3)+\Re(e^2)\wedge\Im(e^3)\wedge\Re(\partial_1)\right)\\
&+ 3\mathbb{F}_{42}(-\Re(\partial_2)\wedge\Im(\partial_1)+\Re(\partial_1)\wedge\Im(\partial_2))\wedge\Re(\partial_3)\\
&+ \frac{3}{2}\mathbb{F}_{43}(\Re(e^1)\wedge\Im(e^2)\wedge\Re(\partial_3)+\Re(e^2)\wedge\Re(\partial_3)\wedge\Im(e^1))\\
&+ \frac{1}{2}\mathbb{F}_{44}(\Im(e^2)\wedge\Im(\partial_1)+\Im(\partial_2)\wedge\Im(e^1))\wedge\Re(\partial_3)\,,
\end{aligned} \tag{5.30}$$

and

$$\begin{aligned}
\widehat{H} = &+ 16\mathbb{H}_{12}\Re(\partial_1)\wedge\Re(\partial_2)\wedge\Im(e^3)\\
&- 8\mathbb{H}_{13}\Im(e^1)\wedge\Im(e^2)\wedge\Im(e^3)\\
&+ \frac{4}{3}\mathbb{H}_{14}\left(\Re(\partial_2)\wedge\Im(e^1)-\Re(\partial_1)\wedge\Im(e^2)\right)\wedge\Im(e^3)\\
&+ 16\mathbb{H}_{22}\Re(\partial_1)\wedge\Re(\partial_2)\wedge\Re(\partial_3)\\
&- 8\mathbb{H}_{23}\Im(e^1)\wedge\Im(e^2)\wedge\Re(\partial_3)\\
&- \frac{4}{3}\mathbb{H}_{24}\left(\Im(e^1)\wedge\Re(\partial_2)\wedge\Re(\partial_3)+\Re(\partial_1)\wedge\Im(e^2)\wedge\Re(\partial_3)\right)\,.
\end{aligned} \tag{5.31}$$

These include of course both geometric and non-geometric fluxes.

Likewise the non-geometric mirrors are

$$\begin{aligned}
\widehat{Q} = &+ \frac{1}{2}\mathbb{Q}_{21}\left(\Re(\partial_2)\wedge\Im(\partial_1)-\Re(\partial_1)\wedge\Im(\partial_2)\right)\wedge\Re(e^3)\\
&+ \mathbb{Q}_{22}\left(-\Re(\partial_2)\wedge\Re(e^1)+\Im(e^2)\wedge\Im(\partial_1)-\Im(e^1)\wedge\Im(\partial_2)+\Re(\partial_1)\wedge\Re(e^2)\right)\wedge\Re(e^3)\\
&+ \frac{1}{3}\mathbb{Q}_{24}\left(\Im(e^1)\wedge\Re(e^2)-\Im(e^2)\wedge\Re(e^1)\right)\wedge\Re(e^3)\\
&+ \mathbb{Q}_{32}\,\Im(\partial_1)\wedge\Im(\partial_2)\wedge\Re(\partial_3)\\
&- \frac{1}{2}\mathbb{Q}_{33}\,\Re(e^1)\wedge\Re(e^2)\wedge\Re(\partial_3)\\
&- \frac{1}{12}\mathbb{Q}_{34}(\Im(\partial_2)\wedge\Re(e^1)+\Re(e^2)\wedge\Im(\partial_1))\wedge\Re(\partial_3)\\
&+ 3\mathbb{Q}_{42}\,\Im(\partial_1)\wedge\Im(\partial_2)\wedge\Im(e^3)\\
&- \frac{3}{2}\mathbb{Q}_{43}\,\Re(e^1)\wedge\Re(e^2)\wedge\Im(e^3)\\
&- \frac{1}{4}\mathbb{Q}_{44}(\Im(\partial_2)\wedge\Re(e^1)+\Re(e^2)\wedge\Im(\partial_1))\wedge\Im(e^3)\,,
\end{aligned} \tag{5.32}$$

and

$$\begin{aligned}
\widehat{R} = &- \frac{1}{8}\mathbb{R}_{31}\,\Im(\partial_1)\wedge\Im(\partial_2)\wedge\Im(\partial_3)\\
&- \frac{1}{2}\mathbb{R}_{32}(\Re(e^1)\wedge\Im(\partial_2)-\Re(e^2)\wedge\Im(\partial_1))\wedge\Im(\partial_3)\\
&- \frac{1}{12}\mathbb{R}_{34}\,\Re(e^1)\wedge\Re(e^2)\wedge\Im(\partial_3)\\
&+ \frac{3}{8}\mathbb{R}_{41}\,\Im(\partial_1)\wedge\Im(\partial_2)\wedge\Re(e^3)\\
&- \frac{3}{2}\mathbb{R}_{42}(\Re(e^1)\wedge\Im(\partial_2)-\Re(e^2)\wedge\Im(\partial_1))\wedge\Re(e^3)\\
&+ \frac{1}{4}\mathbb{R}_{44}\,\Re(e^1)\wedge\Re(e^2)\wedge\Re(e^3)\,.
\end{aligned} \tag{5.33}$$

Acting with the mirror fluxes on the basis yields the mirror charge matrix $\widehat{\mathcal{Q}}$ to be

$$\widehat{\mathcal{Q}} = \left(\begin{array}{c|c|c|c}
-\frac{1}{4}\mathbb{Q}_{33} & \frac{3}{16}\mathbb{F}_{43}-\frac{3}{32}\mathbb{Q}_{43} & \mathbb{H}_{13} & -6\mathbb{F}_{23}-6\mathbb{H}_{23}\\
\hline
-\frac{1}{6}\mathbb{Q}_{34}+\frac{1}{24}\mathbb{R}_{34} & \frac{1}{8}\mathbb{F}_{44}+\frac{1}{8}\mathbb{Q}_{44}+\frac{1}{16}\mathbb{R}_{44} & -\frac{1}{6}\mathbb{F}_{14}+\frac{2}{3}\mathbb{H}_{14} & -\mathbb{F}_{24}-2\mathbb{H}_{24}+\frac{1}{4}\mathbb{Q}_{24}\\
\hline
\mathbb{R}_{31} & \frac{3}{8}\mathbb{R}_{41} & -4\mathbb{F}_{11} & -24\mathbb{F}_{21}-6\mathbb{Q}_{21}\\
\hline
\frac{8}{3}\mathbb{Q}_{32}+\frac{2}{3}\mathbb{R}_{32} & -2\mathbb{F}_{42}+\mathbb{Q}_{42}+\frac{1}{2}\mathbb{R}_{42} & \frac{16}{3}\mathbb{F}_{12}-\frac{32}{3}\mathbb{H}_{12} & 16\mathbb{F}_{22}-64\mathbb{H}_{22}+2\mathbb{Q}_{22}
\end{array}\right)\,. \tag{5.34}$$

Note this is nicely confirming the conjectured mirror map on the charge matrix as of [9] where it was conjectured that the charge entries appear as

$$\mathcal{Q} = \begin{pmatrix} p_I{}^A & e_{IB} \\ q^{IA} & m^I{}_B \end{pmatrix} \quad \to \quad \widehat{\mathcal{Q}} = \begin{pmatrix} m^I{}_A & e_{BI} \\ q^{AI} & -p_I{}^B \end{pmatrix}\,. \tag{5.35}$$

Recall that this was derived by comparing the Killing prepotentials, and thus does not fix the mapping of the charges up to linear transformations that leave the blocks invariant. We confirmed the mapping of the charges and explicitly worked out the charge entries of $\widehat{\mathcal{Q}}$.

We should note that in addition to the linear conditions that arise from the action of the fluxes on the basis, there are also quadratic constraints, which arise from the condition that the differential has to be nilpotent, upon the entries of the charge matrix. These will have to be taken into account, in order to discuss physical flux configurations. The factors in the above matrix could then be taken care of by allowing only fluxes that solve the quadratic constraints.

6 Supersymmetric cycles on generic $SU(3) \times SU(3)$ structure backgrounds

In this section we want to investigate the generic case of $SU(3) \times SU(3)$ structure backgrounds, where the underlying manifold (in some duality frame) has non-zero Euler number. Relaxing the topological condition $\chi(M) = 0$ implies that there is no static $SU(2)$ structure at all. Not only do we have to face the loss of ordinary complex structure on M, but we are going to encounter type-jumping phenomena. The following two closed subsets are indeed going to be of special interest:

$$\{c^{-1}(1)\} \ : \qquad \text{type-three and type-zero pure spinors}\,, \tag{6.1}$$

as in the case of $SU(3)$ structures (this set was empty in the previous part of our analysis), and

$$\{c^{-1}(0)\} \ : \qquad \text{type-one and type-two pure spinors}\,, \tag{6.2}$$

as in the case of $SU(2)$ structures. They correspond to the two big circles we have depicted on figure (1). So far we have been confined to only one of them, because of the topological assumption we have made.

Some three-tori will be supersymmetric on $M \times \hat{M}$, either in transverse or longitudinal position, but they will always be situated above points of these two special subsets. Away from those subsets, types of pure spinors are too low to allow for stable D0-branes. This is T-dual to the disappearance of most of the supersymmetric three-torus fibers. We shall describe this in terms of mass generation for moduli through fluxes.

Motivated by this observation concerning D0-branes, we want to address the existence, stability and moduli space of three-dimensional supersymmetric cycles. We shall see that for $SU(3) \times SU(3)$ structures that are not static $SU(2)$ structures, such cycles still exist at type-jumping points. As T-duality does not change moduli spaces, we expect some moduli of those cycles to be fixed. In particular, three-dimensional supersymmetric cycles are not likely to give rise to a fibered structure of a whole manifold with generic $SU(3) \times SU(3)$ structure. But they can still allow for the exchange of pure spinors Φ_- and Φ_+ by mirror symmetry as a T-duality along a three-dimensional supersymmetric cycle.

6.1 D3-branes and D0-branes through maximum-type points

Consider a mirror pair of manifolds M and $\hat{M}$ with $SU(3) \times SU(3)$ structures, that do not fall into the class of static $SU(2)$ structures (as they have opposite Euler characteristics, assuming that one has non-zero Euler characteristic is sufficient to ensure the condition). We assume both sides of the mirror correspondence to have a geometric description in the sense of a sigma

model. Consider some point p on M at which the complex one-form $v + iw$ vanishes. At that point the pure spinors assume the same forms as in the Calabi–Yau case. We may write for some complex coordinates X, Y, Z

$$\begin{aligned} \Phi_-|_p &= \Omega|_p = dX \wedge dY \wedge dZ \\ \Phi_+|_p &= e^{iJ}|_p = e^{\frac{i}{2}(dX\wedge d\bar{X} + dY\wedge d\bar{Y} + dZ\wedge d\bar{Z})} \,, \end{aligned} \tag{6.3}$$

and one may put a D0-brane of the B-model, that is generalized complex w.r.t. to Ω, or a D3-brane of the A-model, *i.e.* a Lagrangian D-brane which will be denoted by L.

Let us T-dualize along L, which we assume to have the topology of a torus, corresponding to the three isometries we need to perform T-duality[5]. Let there be local coordinates x',y' and z' on L (that are imaginary parts of complex coordinates $X = x + ix'$, $Y = y + iy'$, $Z = z + iz'$ defined on the locus with equation $v + iw = 0$), so that Fourier–Mukai transform yields

$$\begin{aligned} \text{F.T.}\left(\Phi_-|_{\text{F.T.}(L)}\right) &= \int_L e^{i(dx'\wedge d\hat{x}' + dy'\wedge d\hat{y}' + dz'\wedge d\hat{z}')} dX \wedge dY \wedge dZ \\ &= \exp\Big(i(d\hat{x}' \wedge dx + d\hat{y}' \wedge dy + d\hat{z}' \wedge dz)\Big) \\ &= \hat{\Phi}_+|_{\hat{p}} =: e^{i\hat{J}}|_{\hat{p}} \\ \text{F.T.}\left(\Phi_+|_{\text{F.T.}(L)}\right) &= \hat{\Phi}_-|_{\hat{p}} =: \hat{\Omega}|_{\hat{p}} \,, \end{aligned} \tag{6.4}$$

which are the expressions of the pure spinors on the T-dual point $\hat{p}$ on which a supersymmetric D0-brane sits. Of course $\hat{p}$ has to be in the set of zeroes of $\hat{v} + i\hat{w}$, or $\hat{c}^{-1}(1)$, which is not empty since the mirror manifold $\hat{M}$ also has non-zero Euler number.

6.2 Away from maximum-type points through fluxes

It has long been appreciated that the behaviour of pure spinors under mirror symmetry is transparent to B-transforms by a two-form whose components are extended in directions transverse to the T-dualized directions, while non-geometry occurs when the two-form has components that are longitudinal. In terms of the previous local complex coordinates, B-transforms by two-forms of type $(1,1)$ are still B-transforms on the mirror, while those of type $(0,2)$ or $(2,0)$ are β-transforms on the mirror. This can be seen in local charts by wedging together pairs of the following naturality properties derived in lemma 6.2 of the second reference in [23], where v and ϕ denote a longitudinal vector and one-form, and w and ψ denote a transverse vector and one-form:

$$\begin{aligned} i) \quad & \text{F.T.}(\iota_v \wedge \Phi) &&= \hat{v} \wedge (\text{F.T.}(\Phi)) \\ ii) \quad & \text{F.T.}(\iota_w \wedge \Phi) &&= \iota_w(\text{F.T.}(\Phi)) \\ iii) \quad & \text{F.T.}(\phi \wedge \Phi) &&= \iota_{\hat{\phi}}(\text{F.T.}(\Phi)) \\ iv) \quad & \text{F.T.}(\psi \wedge \Phi) &&= \psi \wedge \text{F.T.}(\Phi) \,. \end{aligned}$$

In other words covariant and contravariant tensors stay so under T-duality if their components are transverse to the dualized directions, while they are flipped if they are longitudinal.

[5]The assumption is reasonable because we have two $SU(3)$ structures, each of which gives rise to a fibration by three-tori, and at points the two fibers are the same, the fiber is supersymmetric; but such points are exactly the zeroes of $v + iw$.

So far we have seen how T-duality maps pure spinors Φ_+ and Φ_- to each other along the maximum-type locus of equation $v + iw = 0$. It looked formally the same as in the Calabi–Yau or $SU(3)$ structure case. Suppose an H-flux is turned on on both sides of the mirror correspondence. Choosing a gauge for the local B-field from which the flux derives induces various B- and β-transforms on M and $\hat{M}$, according to the way the support of the B-field intersects with the T-dualized directions. Generically, going away from the maximum-type locus should induce a β-transform that will lower the type of Φ_- to one, which is the most generic type for an odd pure spinor (*i.e.* the lowest type allowed by parity).

Thanks to property *iv*), B-fields of type $(1,1)$ in the complex structure described above pull back to zero on the three-cycle L. They act as B-transforms on both sides of the mirror correspondence and do not lower the type of the pure spinors

$$e^B \wedge \Phi_\pm \longleftrightarrow e^B \wedge \hat{\Phi}_\mp \,. \tag{6.5}$$

We have to take into account possible B-transforms by longitudinal B-fields, that give rise to non-geometric fluxes on the mirror (we will restrict to the case of Q-fluxes on the mirror, with two indices of the B-field along T-dualized directions). Consider an H-flux on the space M, with one unit of flux along a three-cycle $\mathcal{C}$:

$$\int_{\mathcal{C}} H = 1 \,. \tag{6.6}$$

In order for L to be a supersymmetric three-cycle, the local B-field, which gives rise to the flux, has to pull-back to zero on L.

We are interested in the application of T-duality in two directions carrying indices of non-zero components of the H-flux. These directions, denoted by y and z, are two $U(1)$ isometries, spanning a two-torus. Consider the one-form valued integral of H along this torus. It is closed because the three-form H is:

$$d \int_{T^2} H = 0. \tag{6.7}$$

One can locally integrate the one-form, so that there exists (locally) a scalar function X such that

$$\int_{T^2} H = dX \,, \tag{6.8}$$

which amounts to a gauge choice, because the B-field

$$B := X \mathrm{vol}_{T^2} \,, \tag{6.9}$$

where vol_{T^2} is the volume form of T^2, is compatible with the quantization of H. Upon T-duality along the two $U(1)$ isometry directions y and z, this B-field is mapped to a bivector living on the T-dual manifold

$$\mathrm{F.T.}(e^{i(J+X dy\wedge dz)}) = X \iota_{\partial_{\hat{y}}} \iota_{\partial_{\hat{z}}} \left(\mathrm{F.T.}(e^{iJ})\right) + \mathrm{F.T.}(e^{iJ}) \,. \tag{6.10}$$

This way the lowest component of the odd pure spinor we read-off from the RHS is the one-form $X \iota_{\partial_{\hat{y}}} \iota_{\partial_{\hat{z}}} \Omega$, that appears to be weighted by the local coordinate X. We may thus identify the first term in the expansion of the polyform on the RHS with the mirror of the fiberwise components of the H-field. It should also be equal to the one-form $v + iw$. Thus, we

have seen that the $(0,2)$ part of the argument of the exponential in the expression of Φ_+ is mirror to the complex vector $v+iw$. We can rewrite this mapping in a coordinate-independent way as

$$\Phi_+ = e^{B^{(0,2)}} \wedge e^{iJ} \longrightarrow e^{\beta}\hat{\Omega} = \Phi_-, \tag{6.11}$$

where the RHS has now type one and contains an overall factor of $\hat{v} + i\hat{w}$.

Likewise we can start with Φ_-. Again H can be locally written as $H = d(Xdy \wedge dz)$ and thus

$$\text{F.T.}(e^{Xdy\wedge dz} \wedge \Omega) = e^{X\partial_y\wedge\partial_z}\text{F.T.}(\Omega)\,. \tag{6.12}$$

Furthermore $\text{F.T.}(\Omega) = e^{iJ}$ with $J = J^{1,2} = j \pm v \wedge w$ being one of the two-forms of the two $SU(3)$ structures (2.6). We have thus established that

$$e^{B^{(0,2)}} \wedge \Omega \longrightarrow e^{\beta}e^{iJ}\,, \tag{6.13}$$

with $\beta = X\partial_y \wedge \partial_z$. Apart from this we know that the contraction between β and j vanishes, just as $\omega \wedge j$ vanishes for $SU(2)$ structures. Hence contractions between the bivector β and the polyform e^{iJ} only involve j and is therefore unambiguous. So the contraction between β and the higher powers of J just selects the square of j, and gives rise to a $(2,0)$ form called B', which squares to zero. Using the expansion (6.3) we find:

$$e^{\beta}(e^{iJ}) = 1 + iJ + \beta\left(-\frac{J^2}{2}\right) - \frac{J^2}{2} + \beta\left(-i\frac{J^3}{6}\right) - i\frac{J^3}{6}\,. \tag{6.14}$$

Defining

$$B' := \beta\left(-\frac{J^2}{2}\right)\,, \tag{6.15}$$

this can be rewritten in the following way

$$e^{\beta}\left(e^{iJ}\right) = 1 + iJ + B' + B' \wedge iJ - \frac{J^2}{2} - i\frac{J^3}{6} = e^{B'} \wedge e^{iJ}\,. \tag{6.16}$$

Thus, one may say that the β-transform of the type-zero pure spinors assumes the same form as a B-transform for accidental dimensional reasons. We therefore write the β-transform of the type-zero spinor as a B-transform by B', which of course is still of the most generic type zero:

$$e^{B^{(0,2)}} \wedge \Omega \longrightarrow e^{B'+iv\wedge w+ij}\,. \tag{6.17}$$

This formula was already derived assuming a T^3-fibration in [32] as a clue that $SU(3) \times SU(3)$ structures could account for non-geometric situations involving T-dualizing with a B-field extended along two fiberwise directions. Here we see that it actually holds for a mere topological reason on spaces with non-zero Euler number and $SU(3) \times SU(3)$ structure. On such spaces $v + iw$ have zeroes on which odd pure spinors have type three, thus giving rise to supersymmetric three-cycles; the mirror formula between Φ_- and Φ_+ follows from the naturality properties of B- and β-transforms w.r.t. T-duality along the three-cycles, even if the SYZ argument is spoiled away from the zeroes of $v + iw$ due to the absence of supersymmetric D0-branes. Moreover, T-dualizing along L and exploiting properties of the Fourier–Mukai transform allowed us to go the other way around, which lowers the type of Φ_-. To sum up,

putting all the possible B- and β-transforms on both sides, we have argued that the following T-duality holds in an open neighborhood of type-jumping point:

$$\begin{aligned} \Phi_+ = e^{\tilde{\beta}+B'^{(0,2)}+B^{(1,1)}}\exp(iJ) \quad &\longleftrightarrow \quad e^{\beta'+\tilde{B}^{(0,2)}+B^{(1,1)}}\hat{\Omega} = \hat{\Phi}_- \\ \tilde{\beta} \quad &\longleftrightarrow \quad \tilde{B}^{(0,2)} \\ B'^{(0,2)} \quad &\longleftrightarrow \quad \beta' \\ B^{(1,1)} \quad &\longleftrightarrow \quad B^{(1,1)}\,, \end{aligned} \tag{6.18}$$

where the odd pure spinor has type one as β' (or equivalently $B'^{(0,2)}$) is non-zero.

6.3 Moduli spaces

As D0-branes can only be stable at points where the odd pure spinors has type three, their moduli space must be evaluated by looking at massless infinitesimal deformations at such exceptional points. Going away from such a point involves a β-transform. If one goes along the subset $c^{-1}(1)$ the β-transform is trivial and we have found a translation modulus; otherwise the direction along which we are going is a fixed modulus. On the other hand, the first cohomology of the Lie algebroid of a D0-brane was evaluated in [17] as the set of vectors X^l such that the β-transform that acts on the pure spinor satisfies

$$\partial_l \beta^{\mu\nu} X^l \partial_\mu \wedge \partial_\nu = 0\,. \tag{6.19}$$

This makes for a five-dimensional moduli space, as a gauge may be chosen in which β only depends on one coordinate, the one along the direction v. Consider the three-dimenional D-brane going through such a point. It also has a five-dimensional moduli space, since the normal deformation in the direction v is not allowed anymore, and it is exactly the modulus that has disappeared for D0-branes.

In the more generic cases we want to investigate here, we have to compute the mass matrix of the deformations of our three-dimensional supersymmetric tori. Moduli that are fixed by the flux should get a mass.

One can make an observation in local coordinates around a point where the pure spinors have type zero and type three. The fundamental two-form J takes the expression

$$J = \frac{i}{2}\left(dX \wedge d\bar{X} + dY \wedge d\bar{Y} + dZ \wedge d\bar{Z}\right)\,, \tag{6.20}$$

and imagine we start with a supersymmetric three-torus extended along the directions x, y and Z and the T-dual of an H-flux deriving from the coordinate x is a β-transform with $\beta = X\partial_y \wedge \partial_z$, so that it is easy to repeat the argument of the previous subsection for the computation of B'. In a neighborhood of the point we considered, Φ_+ assumes the form:

$$e^{\beta}e^{iJ} = e^{Xd\bar{Y}d\bar{Z}}e^{iJ}\,, \tag{6.21}$$

so that a four-chain $\mathcal{B}$ that is bounded by the supersymmetric three-cycle and some generalized cycle (Σ, F) at the other end will go (along the X direction) through cycles carrying non-zero field strength

$$F = P_\Sigma(Xd\bar{Y}d\bar{Z}), \tag{6.22}$$

where P_Σ denotes the pull-back to Σ. Hence the three-cycle (Σ, F) cannot be generalized complex if it goes into the X direction. This loss of structure fixes the position moduli X for the three-cycle, which fact is mirror to X acquiring a mass as a translation modulus for a D0-brane.

7 Conclusions and outlook

The SYZ argument has been shown to extend to a class of generalized Calabi-Yau spaces, namely so-called static $SU(2)$ structure manifolds. We have shown that there are no supersymmetric three-tori on M or its mirror $\hat{M}$, but the product $M \times \hat{M}$ is doubly fibered by three-tori, both families of fibers are transverse to M and $\hat{M}$, and the resulting six-tori are calibrated generalized submanifolds of $M \times \hat{M}$. Moreover mirror symmetry is performed by T-dualizing the three-dimensional intersection of such generalized submanifold with M. This transversality property is reminiscent of the (much more general) conjectures formulated by Gualtieri in the final chapter of [11].

It is somewhat surprising that this argument is applicable also when including non-geometric fluxes, in particular R-fluxes. These non-geometric fluxes are expected to spoil the geometric description of the background even locally. In the R-flux case, the geometry is expected to be replaced by some non-associative algebra [43]. However we did not encounter such a necessity. We suspect that the case of static $SU(2)$ structure, which prevents the type of the pure spinors from jumping, guards us against the destabilizing effects of non-geometric fluxes on D-branes.

The large-volume limit which was assumed in the SYZ argument for Calabi–Yau manifolds is also highly questionable in generic flux backgrounds. Again the topological condition of a static $SU(2)$ with a non-vanishing vector field allows for more globally well-defined quantities than the ordinary complex torus studied in [44, 6]. This is consistent with the observation made in [9] that more charges can be turned on geometrically on $SU(2)$ structure backgrounds than on generic ones.

The case of generic $SU(3) \times SU(3)$ structures is much less transparent.[6] We have identified a set of three-cycles, T-dual to type-jumping points on the mirror. They cannot fiber the manifold or even its product with its mirror. This fact is mirror to the mass that fluxes give to the translation moduli of D0-branes, spoiling the very first step of the SYZ argument. T-dualizing the surviving three-tori and asking for functorial properties w.r.t. B- and β-transforms of generalized geometry gives however a correct mirror exchange between type-zero and type-one pure spinors. Our argument was limited to the use of classical geometry.

In order to formulate an SYZ argument for the generic case, it seems natural to consider non-commutative fibrations. It has been observed that T-dualizing directions that support more than one index of a non-zero component of a B-field leads to non-commutative fibrations through an *uncertainty principle* for D-branes [46, 47]. Of course allowing non-commutative fibers, with non-commutativity scale proportional to the quanta of fluxes and to the discrepancy between the pair of $SU(3) \times SU(3)$ structures, would be a way of fibering generalized backgrounds by (further) generalized submanifolds. The only fibers we are able to see in the

[6] We have disregarded Ramond–Ramond fluxes, in the presence of which a one of the two pure spinors cannot be closed [13]. Bianchi identity in the presence of Ramond–Ramond fluxes requires an orientifold projection, see [45].

present approach are the ones along which the two structures agree, which results in type-jumping and in a commutative fiber. It might be that non-commutative fibrations on more general bases than a torus will be equivalent to fibrations by T-folds [4], and that going away from type jumping points will require acting on the fibers with transition functions involving T-dualities.

We hope to gain more insight into these issues by studying the proper reduction on generic $SU(3) \times SU(3)$ structure manifolds. Initial results have appeared in [9] and the case of $SU(3)$ structures was discussed in [48]. We trust that the analog of harmonic forms will be generalized or twisted harmonic forms, *i.e.* forms that are harmonic w.r.t. the Laplacian twisted by all fluxes (geometric and non-geometric). This should in particular allow one to determine the mass terms that we discussed in at the end of this paper, and thus the disappearance of geometric moduli will become more transparent. We shall come back to these points in due time.

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