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Path Integral Representations on the Complex Sphere

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Abstract

In this paper we discuss the path integral representations for the coordinate systems on the complex sphere S_{3C}. The Schrödinger equation, respectively the path integral, separates in exactly 21 orthogonal coordinate systems. We enumerate these coordinate systems and we are able to present the path integral representations explicitly in the majority of the cases. In each solution the expansion into the wave-functions is stated. Also, the kernel and the corresponding Green function can be stated in closed form in terms of the invariant distance on the sphere, respectively on the hyperboloid.

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1 Introduction

It is a well-known fact that the number of coordinate systems which separate the Schrödinger equation on the three-dimensional sphere is six: cylindrical, spherical, conical, prolate spheroidal, oblate spheroidal, and elliptic [10, 12]. On the two-dimensional sphere there are just two, spherical and elliptical. On the other hand, the corresponding number of coordinate systems on the two- and three-dimensional hyperboloid are nine, respectively 34 [37]. Furthermore, on the two- and three-dimensional Euclidean and pseudo-Euclidean spaces the number of systems is 10 and 54, and 4 and 11, respectively. All the spaces listed above can have an Euclidean metric signature, or a Minkowskian metric signature, and they are all real. The situation changes, if we start to consider the corresponding complex spaces. On two-dimensional and three-dimensional complex Euclidean space there are 6 and 18, respectively, and on the two- and three-dimensional complex sphere there are 5 and 21 coordinate systems which separate the Schrödinger equation

Table 1: Coordinate Systems on S_{3C}

No. of System	Complexifcation of Real Space	Two-Dimensional Subsystem	Three-Dimensional Systems
No 1	$S^{(3)}, \Lambda^{(3)}$, O(2,2)	$S(2,\mathbb{R})$ Polar	Cylindrical
No 2	$\Lambda^{(3)}$, O(2,2)	$E(2,\mathbb{R})$ Cartesian	Horicyclic
No 3	$S^{(3)}, \Lambda^{(3)}$, O(2,2)	$S(2,\mathbb{R})$ Polar	Spherical
No 4	$\Lambda^{(3)}$, O(2,2)	$S(2,\mathbb{C})$ Horospherical	Horospherical
No 5	$\Lambda^{(3)}$, O(2,2)	$E(2,\mathbb{R})$ Polar	Horicyclic-polar
No 6	$S^{(3)}, \Lambda^{(3)}$, O(2,2)	$S(2,\mathbb{R})$ Conical	Sphero-elliptic
No 7	$\Lambda^{(3)}$, O(2,2)	$S(2,\mathbb{C})$ Degenerate elliptic I	Spherical-degenerate elliptic I
No 8	$\Lambda^{(3)}$, O(2,2)	$S(2,\mathbb{C})$ Degenerate elliptic II	Spherical-degenerate elliptic II
No 9	$\Lambda^{(3)}$, O(2,2)	$E(2,\mathbb{R})$ Elliptic	Horicyclic-elliptic
No 10	O(2,2)	$E(2,\mathbb{C})$ Hyperbolic	Horicyclic-hyperbolic
No 11	$\Lambda^{(3)}$, O(2,2)	$E(2,\mathbb{R})$ Parabolic	Horicyclic-parabolic I
No 12	O(2,2)	$E(2,\mathbb{C})$ Semi-parabolic	Horicyclic-parabolic II
No 13	$S^{(3)}, \Lambda^{(3)}$, O(2,2)	-	Elliptic-Cylindrical
No 14	$\Lambda^{(3)}$, O(2,2)	-	Elliptic-Parabolic
No 15	$\Lambda^{(3)}$, O(2,2)	-	Elliptic-Hyperbolic
No 16	O(2,2)	-	Parabolic
No 17	$S^{(3)}, \Lambda^{(3)}$, O(2,2)	-	Ellipsoidal
No 18	$\Lambda^{(3)}$, O(2,2)	-	*System 18*
No 19	O(2,2)	-	*System 19*
No 20	$\Lambda^{(3)}$, O(2,2)	-	*System 20*
No 21	O(2,2)	-	*System 21*

[3]. In comparison to the real two- and three-dimensional sphere there is a richer structure. This is not surprising because the two-dimensional flat space and the hyperboloids are contained as subgroup cases in the complex sphere.

In Table 1 I have listed some properties of the coordinate systems on the three-dimensional complex sphere. The coordinate systems, which contain two-dimensional flat systems, i.e. the Euclidean plane (real and complex) are, (2), (5), (9)–(11) [36]. The coordinates systems which have the two-dimensional sphere (real and complex) as a subsystem are (3), (4), (6)–(8) [21, 23, 27, 36]. The coordinate systems which exist also on the real three-dimensional sphere are (1), (3), (6), (13) and (17) [28, 37].

According to [25], the complexification of the two elliptic cylindrical coordinate systems (i.e. spheroidal systems) on the $S(3,\mathbb{R})$-sphere and on the three-dimensional hyperboloid give just one coordinate system on $S(3,\mathbb{C})$, i.e. (13). In particular in [34, 27] coordinate systems on the two-dimensional complex sphere and corresponding superintegrable potentials, and in [27] coordinate systems on the two-dimensional (complex) plane and corresponding superintegrable potentials were discussed, including the corresponding interbases expansions. The goal of [27] was to extend the notion of superintegrable potentials of real spaces to the corresponding complexified spaces. The findings were such that there are in addition to the four coordinate systems on the real two-dimensional Euclidean plane three more coordinate systems and also three more superintegrable potentials. Similarly, in addition to the two coordinate systems on the real two-dimensional sphere there are three more coordinate systems on the complex sphere

and also four more superintegrable potentials. This is not surprising because the complex plane contains not only the Euclidean plane but also the pseudo-Euclidean plane (10 coordinate systems [12, 21, 20]) and the complex sphere contains not only the real sphere but also the two-dimensional hyperboloid (9 coordinate systems [12, 21, 22, 37]).

On the other hand, it is also possible to complexify the 34 coordinate systems on the three-dimensional hyperboloid [24, 37]. This gives all 21 systems except (19, (12), (16), (19) and (21). These remaining system can be found be complexifying the coordinate systems on the real hyperboloid SO(2, 2) [25].

In Table 1 I have indicated which coordinate systems emerges from which real space by complexification; these spaces are indicated by $S^{(3)}$ (real sphere), $\Lambda^{(3)}$ (real hyperboloid), and the systems emerging from O(2,2) (real hyperboloid), respectively.

In this paper I apply the path integral method [7, 29, 19, 40] to the complex sphere S_{3C}. We are able to find in the majority of the coordinate representations a path integral representation. Many known solutions from other path integral problems can be applied in a straightforward way to find the corresponding complex-sphere representation.

In Table 1 those coordinate systems where a new path integral representation can be stated are underlined. The most important "new" solution consist in the path integral formulation of the complex periodic Liouville potential. It has a real spectrum with eigenvalues $\propto J(J+2), J \in \mathbb{N}$, in accordance with the spectrum of the Hamiltonian of the three-dimensional sphere. The fact that a complex periodic potential has a real spectrum is at first sight surprising, but has actually attracted in recent years a lot of attention, e.g. the workshop series [41]. This property of "pseudo-hermitian" Hamiltonians has also become known as "$\mathcal{PT}$-symmetry" of Hamiltonians.

In Table 1 those ellipsoidal systems where no path integral evaluation is possible are *emphasized*. There is hardly a solution known, because the *only* ellipsoidal systems where a solution of the free Schrödinger is know is the one on the real sphere [1]. There is nothing known about the solutions of the Schrödinger equations on the hyperboloids in paraboloidal or the remaining "ellipsoidal" coordinates, let alone a path integral representation.

The paper has a review-like character and is organized as follows: In the next Section we present the relevant path integral representations for those coordinate systems which have a subgroup structure, i.e. where a lower dimensional case is contained. In many cases we can rely on already solved path integral problem to find the specific one in question. Section III contains the remaining cases which are not of the subspace type. For some evaluations I also will make use of the technique of space-time transformation, where I refer for more details to the literature [6, 18, 19, 29]. Therefore our principal emphasis is to bring together both already known and new results in order to complete matters.

Obviously, Sections II and III take on the form of an enumeration. We will rely heavily on already known solutions in the sequel, and these solution will not be re-derived again.

We start with the definition of the coordinates, calculate the relevant metric terms, the momentum operator and the quantum Hamiltonian in terms of he momentum operators. In our formulation of the path integral we always use a lattice definition which we have called "product form" and consists mainly of the fact that all metric terms in the energy term in the Lagrangian in the its lattice form are given by geometric means [19]. In our first path integral representation we will state this lattice formulation explicitly.

Let us note that the complexification requires also the following consideration: The eigenvalues of the Hamiltonian on the three-dimensional complex sphere are denoted by $\propto -\sigma(\sigma+2)$. On the real sphere this yields with $\sigma = J$ the eigenvalues $\propto J(J+2)$ whereas on the real three-

dimensional hyperboloid one has $\sigma = 1+\mathrm{i}p$ and therefore the eigenvalues are $\propto (p^2+1)$. We must therefore look carefully which manifold we consider if we specify the coordinates including their ranges. Usually, an analytic continuation may be required, which is not performed, however.

We will denote in the following the quantum number of the Eigenvalues of the Hamiltonian by J, irrespective whether there is a discrete or a continuous spectrum. On the real sphere the spectrum is always discrete and on the hyperboloid (two-sheeted) continuous. The kernel in its wave-functions expansion and spectrum on the complex sphere will be displayed in the most cases in a discrete formulation. The corresponding restriction to the sphere or hyperboloid then will decide whether one can keep the discrete formulation as it is or one must analytically continue to the continuous spectrum. However, on the single-sheeted hyperboloid and on the O(2,2)-hyperboloid one has in fact both: a discrete and a continuous part. The latter is not discussed in the sequel and is postponed to a future study. In the enumeration of the coordinate system we keep the convention of the corresponding systems on the sphere and hyperboloid from previous publications [12, 37]. In some cases we note explicitly the correspondences to the sphere and hyperboloid cases to illustrate the examples.

The third section contains a summary and discussion of our results.

2 The Path Integral Representations: Part I

2.1 System 1: Cylindrical

The cylindrical coordinate system on the complex sphere S_{3C} has the form:

$$\left.\begin{array}{ll} z_1 = \sin\vartheta\cos\varphi_1 & z_2 = \sin\vartheta\sin\varphi_1 \\ z_3 = \cos\vartheta\cos\varphi_2 & z_4 = \cos\vartheta\sin\varphi_2 \end{array}\right\} \qquad (\vartheta \in (0, \tfrac{\pi}{2}), \varphi_1, \varphi_2 \in [0, 2\pi)) \ . \tag{2.1}$$

The set of commuting operators characterizing the cylindrical coordinate system are

$$\mathcal{L}_1 = I_{23}^2, \qquad \mathcal{L}_2 = I_{14}^2 \ . \tag{2.2}$$

The metric terms have the form

$$\left.\begin{array}{l} \mathrm{d}s^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi_1^2 + \cos^2\vartheta\,\mathrm{d}\varphi_2^2 \ , \\ \sqrt{g} = \sin\vartheta\cos\vartheta \ , \\ \Gamma_\vartheta = \cot\vartheta - \tan\vartheta, \quad \Gamma_{\varphi_1} = 0, \quad \Gamma_{\varphi_2} = 0 \ . \end{array}\right\} \tag{2.3}$$

Therefore we have for the momentum operators:

$$p_\vartheta = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\vartheta} + \frac{1}{2}\cot\vartheta - \frac{1}{2}\tan\vartheta\right), \qquad p_{\varphi_1} = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\varphi_1}, \qquad p_{\varphi_2} = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\varphi_2}, \tag{2.4}$$

and the Hamiltonian is given by

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial\vartheta^2} + (\cot\vartheta - \tan\vartheta)\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi_1^2} + \frac{1}{\cos^2\vartheta}\frac{\partial^2}{\partial\varphi_2^2}\right] \\ &= \frac{1}{2m}\left(p_\vartheta^2 + \frac{1}{\sin^2\vartheta}\,p_{\varphi_1}^2 + \frac{1}{\cos^2\vartheta}\,p_{\varphi_2}^2\right) - \frac{\hbar^2}{8m}\left(4 + \frac{1}{\cos^2\vartheta} + \frac{1}{\sin^2\vartheta}\right) \ . \end{aligned} \tag{2.5}$$

In the second line in (2.5) a quantum potential $\propto \hbar^2$ appears which is due to the ordering of the position and momentum operators in H. The path integral solution on the complex sphere S_{3C} is identical to the the usual sphere $S_{3\mathbb{R}}$ and is well-known from the literature and I just state the result.
In the canonical formulation we have [11, 12, 18, 19] (compare also [2, 4, 39])

$$
\begin{aligned}
&K^{(S_{3C})}(\vartheta'', \vartheta', \varphi_1'', \varphi_1', \varphi_2'', \varphi_2'; T) \\
&= \lim_{N\to\infty} \left(\frac{m}{2\pi \mathrm{i}\epsilon\hbar}\right)^{3N/2} \prod_{j=1}^{N-1} \int \mathrm{d}\theta_j \sin\vartheta_j \cos\vartheta_j \, \mathrm{d}\varphi_{1,j} d\varphi_{2,j} \\
&\quad \times \exp\left\{ \frac{\mathrm{i}}{\hbar} \sum_{j=1}^{N} \left[\frac{m}{2}\Big(\Delta^2\vartheta_j + \widehat{\cos^2\vartheta}_j \Delta^2\varphi_{1,j} + \widehat{\sin^2\vartheta}_j \Delta^2\varphi_{2,j} \Big) + \frac{\hbar^2}{8m}\left(4 + \frac{1}{\cos^2\vartheta_j} + \frac{1}{\sin^2\vartheta_j}\right)\right] dt \right\} \\
&= \int_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t) \sin\vartheta\cos\vartheta \int_{\varphi_1(t')=\varphi_1'}^{\varphi_1(t'')=\varphi_1''} \mathcal{D}\varphi_1(t) \int_{\varphi_2(t')=\varphi_2'}^{\varphi_2(t'')=\varphi_2''} \mathcal{D}\varphi_2(t) \\
&\quad \times \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2}\Big(\dot\vartheta^2 + \cos^2\vartheta\dot\varphi_1^2 + \sin^2\vartheta\dot\varphi_2^2 \Big) + \frac{\hbar^2}{8m}\left(4 + \frac{1}{\cos^2\vartheta} + \frac{1}{\sin^2\vartheta}\right)\right] dt \right\} \\
&= \sum_{J=0}^{\infty} \sum_{k_1,k_2\in\mathbb{Z}} \frac{\mathrm{e}^{\mathrm{i}[k_1(\varphi_1''-\varphi_1')+k_2(\varphi_2''-\varphi_2')]}}{4\pi^2} \frac{2(J+1)\left(\frac{J-|m_1|-|m_2|}{2}\right)!\left(\frac{J+|m_1|+|m_2|}{2}\right)!}{\left(\frac{J-|m_1|+|m_2|}{2}\right)!\left(\frac{J+|m_1|-|m_2|}{2}\right)!} \\
&\quad \times (\sin\vartheta'\sin\vartheta'')^{|m_1|} (\cos\vartheta'\cos\vartheta'')^{|m_2|} \\
&\quad \times P^{(|m_1|,|m_2|)}_{(J-|m_1|-|m_2|)/2}(\cos 2\vartheta') P^{(|m_1|,|m_2|)}_{(J-|m_1|-|m_2|)/2}(\cos 2\vartheta'') \ \mathrm{e}^{-\mathrm{i}\hbar T J(J+2)/2m}
\end{aligned}
\tag{2.6}
$$

($u_j = u(t_j)$, $\Delta u_j = u_j - u_{j-1}$, $u_j = u(t' + t_j)$, for $u = \vartheta, \varphi_1, \varphi_2$, $\epsilon = T/N$, $\widehat{\sin^2\vartheta}_j = \sin^2\vartheta_j \sin^2\vartheta_{j-1}$, and for $\cos^2\vartheta$ similarly). Here we have stated the explicit lattice definition of the path integral. Thus we have for the wave-functions on S_{3C} and the energy-spectrum

$$
\begin{aligned}
\Psi_{J,m_1,m_2}(\vartheta,\varphi_1,\varphi_2) &= \sqrt{\frac{2(J+1)\left(\frac{J-|m_1|-|m_2|}{2}\right)!\left(\frac{J+|m_1|+|m_2|}{2}\right)!}{\left(\frac{J-|m_1|+|m_2|}{2}\right)!\left(\frac{J+|m_1|-|m_2|}{2}\right)!}} \\
&\quad \times \frac{\mathrm{e}^{\mathrm{i}(k_1\varphi_1+k_2\varphi_2)}}{2\pi} (\sin\vartheta)^{|m_1|}(\cos\vartheta)^{|m_2|} P^{(|m_1|,|m_2|)}_{(J-|m_1|-|m_2|)/2}(\cos 2\vartheta) \ ,
\end{aligned}
\tag{2.7}
$$

$$
E_J = \frac{\hbar^2}{2m} J(J+2) \ . \tag{2.8}
$$

The kernel $K^{(S^{(3)})}(T)$ can be cast into the form of a Θ-function [9]

$$
K^{(S^{(3)})}(\psi_{S^{(3)}}, T) = \frac{\mathrm{e}^{\mathrm{i}\hbar T/2m}}{4\pi^2} \frac{\mathrm{d}}{\mathrm{d}\cos\psi_{S^{(3)}}} \Theta_3\left(\frac{\psi_{S^{(3)}}}{2}\middle| -\frac{\hbar T}{2\pi m}\right) \ . \tag{2.9}
$$

The corresponding Green function (resolvent kernel) has the form

$$
G^{(S^{(3)})}(\psi_{S^{(3)}}, E) = \frac{m}{2\pi\hbar^2} \frac{\sin[(\pi - \psi_{S^{(3)}})(\gamma + 1/2)]}{\sin[\pi(\gamma+1/2)]\sin\psi_{S^{(3)}}} \ , \tag{2.10}
$$

where the quantity $\cos\psi_{S^{(3)}}$ (invariant distance) in spherical coordinates is given by

$$\cos\psi_{S^{(3)}}(\mathbf{z}'',\mathbf{z}') = \cos\vartheta_1'\cos\vartheta_1'' + \sin\vartheta_1'\sin\vartheta_1''\Big(\cos\vartheta_2'\cos\vartheta_2'' + \sin\vartheta_2'\sin\vartheta_2''\cos(\varphi''-\varphi')\Big)\,, \quad (2.11)$$

and $\gamma = -1/2 + \sqrt{2mE/\hbar^2+1}$. The expressions (2.9) and (2.10) are independent of the particular coordinate representation.

Note that the corresponding representation of the Green function of the three-dimensional hyperboloid has the form ($K_{\Lambda^{(3)}}(T)$ cannot be stated in closed form [12])

$$G_{\Lambda^{(3)}}(d_{\Lambda^{(3)}}(\mathrm{u}'',\mathrm{u}'),E) = \frac{-m}{\pi^2\hbar^2\sinh d_{\Lambda^{(3)}}(\mathrm{u}'',\mathrm{u}')}\mathcal{Q}^{1/2}_{-1/2-\mathrm{i}\sqrt{2mE/\hbar^2-1}}\Big(\cosh d_{\Lambda^{(3)}}(\mathrm{u}'',\mathrm{u}')\Big)\,, \quad (2.12)$$

$\cosh d$ is the invariant distance on $\Lambda^{(3)}$ given by

$$\cosh d_{\Lambda^{(3)}}(\mathrm{u}'',\mathrm{u}') = \cosh\tau'\cosh\tau'' - \sinh\tau'\sinh\tau''\Big(\cos\vartheta'\cos\vartheta'' + \sin\vartheta'\sin\vartheta''\cos(\varphi''-\varphi')\Big)\,, \quad (2.13)$$

with τ,ϑ,φ spherical coordinates on $\Lambda^{(3)}$. $\mathcal{Q}$ denotes a Legendre function of the second kind. The cylindrical system exists on the three-dimensional sphere (System I.) and on the three-dimensional hyperboloid (System I., in the following we use the notations and enumerations of [12, 37]). Note the difference in the Legendre-function (first or second kind) depending whether we consider the real sphere or the real hyperboloid, respectively.

2.2 System 2: Horicyclic

The horicyclic coordinate system on the complex sphere S_{3C} has the form:

$$\left.\begin{array}{ll} z_1 = \frac{1}{2}\Big[\mathrm{e}^{-\mathrm{i}x} + (1+y^2+z^2)\,\mathrm{e}^{\mathrm{i}x}\Big] & z_2 = \mathrm{i}y\,\mathrm{e}^{\mathrm{i}x} \\ z_3 = \mathrm{i}z\,\mathrm{e}^{\mathrm{i}x} & z_4 = \frac{\mathrm{i}}{2}\Big[\mathrm{e}^{-\mathrm{i}x} + (1+y^2+z^2)\,\mathrm{e}^{\mathrm{i}x}\Big] \end{array}\right\} \quad (2.14)$$

($x,y,z\in\mathbb{R}$). The set of commuting operators characterizing the horicyclic coordinate system are

$$\mathcal{L}_1 = (I_{42}+\mathrm{i}I_{21})^2, \qquad \mathcal{L}_2 = (I_{34}+\mathrm{i}I_{13})^2\ . \quad (2.15)$$

The metric terms have the form

$$\left.\begin{array}{l} \mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{e}^{2\mathrm{i}x}(\,\mathrm{d}y^2 + \mathrm{d}z^2)\ , \\ \sqrt{g} = \mathrm{e}^{2\mathrm{i}x}\ , \\ \Gamma_x = 2\mathrm{i}, \quad \Gamma_y = 0, \quad \Gamma_z = 0\ . \end{array}\right\} \quad (2.16)$$

This coordinate system corresponds to a subgroup algebra of $\mathfrak{so}(4,\mathbb{C})$, namely $\mathcal{E}(2,\mathbb{C})$. According to Ref.[36] there exist six such systems (which make up the six coordinate systems on $\mathcal{E}(2,\mathbb{C})$), and together with the horicyclic systems they are systems (5), (9)–(12). The momentum operators are:

$$p_x = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial x} + \mathrm{i}\right), \qquad p_y = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial y}, \qquad p_z = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial z}. \quad (2.17)$$

The Hamiltonian is given by

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2} + 2\mathrm{i}\frac{\partial}{\partial x} + \mathrm{e}^{-2\mathrm{i}x}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right] \\ &= \frac{1}{2m}\left[p_x^2 + \mathrm{e}^{-2\mathrm{i}x}\left(p_y^2 + p_z^2\right)\right] - \frac{\hbar^2}{2m} \ . \end{aligned} \tag{2.18}$$

For the path integral we find

$$\begin{aligned} & K^{(S_{3C})}(x'', x', y'', y', z'', z' : T) \\ &= \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \int\limits_{y(t')=y'}^{y(t'')=y''} \mathcal{D}y(t) \int\limits_{z(t')=z'}^{z(t'')=z''} \mathcal{D}z(t)\, \mathrm{e}^{2\mathrm{i}x} \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T \left[\frac{m}{2}\dot{x}^2 + \mathrm{e}^{2\mathrm{i}x}(\dot{y}^2 + \dot{z}^2) + \frac{\hbar^2}{2m}\right] \mathrm{d}t\right\} \\ &= \mathrm{e}^{-\mathrm{i}(x''-x')} \int_{\mathbb{R}} \mathrm{d}k_y \int_{\mathbb{R}} \mathrm{d}k_z \frac{\mathrm{e}^{\mathrm{i}k_y(y''-y')+\mathrm{i}k_z(z''-z')}}{(2\pi)^2} \\ &\quad \times \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T \left[\frac{m}{2}\dot{x}^2 - \frac{\hbar^2}{2m}(k_y^2 + k_z^2)\,\mathrm{e}^{-2\mathrm{i}x}\right] \mathrm{d}t\right\}. \end{aligned} \tag{2.19}$$

The path integration in the variables y and z are just plane waves. The remaining path integral in the variable x we can solve this path integral by an analytic continuation of the Liouville path integral solution [19]. In [5] it was shown that the proper continuation are Hankel functions $2^{-1/2}H^{(1)}_{n+1/2}(k\ \mathrm{e}^{-\mathrm{i}x})$ [9], hence we obtain:

$$\begin{aligned} & K^{(S_{3C})}(x'', x', y'', y', z'', z' : T) \\ &= \mathrm{e}^{-\mathrm{i}(x''-x')} \int_{\mathbb{R}} \mathrm{d}k_y \int_{\mathbb{R}} \mathrm{d}k_z \frac{\mathrm{e}^{\mathrm{i}k_y(y''-y')+\mathrm{i}k_z(k_z(z''-z')}}{(2\pi)^2} \\ &\quad \times \sum_{J\in\mathbb{N}_0} \tfrac{1}{2} H^{(1)}_{J+1/2}\left(\sqrt{k_y^2+k_z^2}\ \mathrm{e}^{-\mathrm{i}x''}\right) H^{(1)}_{J+1/2}\left(\sqrt{k_y^2+k_z^2}\ \mathrm{e}^{-\mathrm{i}x'}\right) \exp\left[-\frac{\mathrm{i}}{\hbar}\frac{\hbar^2 J(J+2)}{2m}T\right]. \end{aligned} \tag{2.20}$$

The normalization follows from the consideration $(1/\sqrt{2})H^{(1)}_n(k\,\mathrm{e}^{\mathrm{i}x}) \to \mathrm{e}^{\mathrm{i}kx}/\sqrt{2\pi}\ (x \to \infty)$. Note the relation $H^{(1)}_\nu(\mathrm{i}z) = (2/\mathrm{i}\pi)\,\mathrm{e}^{\mathrm{i}\pi\nu/2}K_\nu(z)$. The wave-functions on S_{3C} and the energy-spectrum are

$$\Psi_{Jk_yk_z}(x,y,z) = \frac{\mathrm{e}^{\mathrm{i}(k_y y + k_z z)}}{2\pi} \cdot \mathrm{e}^{-\mathrm{i}x}\, \frac{1}{\sqrt{2}} H^{(1)}_{J+1/2}\left(\sqrt{k_y^2+k_z^2}\ \mathrm{e}^{-\mathrm{i}x}\right) \ , \tag{2.21}$$

$$E_J = \frac{\hbar^2}{2m} J(J+2) \ . \tag{2.22}$$

The horicyclic system exists only on the three-dimensional hyperboloid (System II.).

2.3 System 3: Spherical

The spherical coordinate system is the best-known coordinate system and has been discussed extensively in the literature, including its generalization to D dimensions. We have:

$$\left.\begin{array}{ll} z_1 = \sin\chi\cos\vartheta & z_2 = \sin\chi\sin\vartheta\cos\varphi \\ z_3 = \sin\chi\sin\vartheta\sin\varphi & z_4 = \cos\chi \end{array}\right\} \qquad (\chi\in[0,\pi), \vartheta\in[0,\pi), \varphi\in[0,2\pi)) \ . \tag{2.23}$$

The set of commuting operators is given by

$$\mathcal{L}_1 = I_{12}^2 + I_{13}^2 + I_{23}^2, \qquad \mathcal{L}_2 = I_{23}^2 \ . \tag{2.24}$$

The metric terms are given by

$$\left.\begin{array}{l} \mathrm{d}s^2 = \mathrm{d}\chi^2 + \sin^2\chi(\,\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2) \\ \sqrt{g} = \sin\vartheta\sin^2\chi \ , \\ \Gamma_\vartheta = \cot\vartheta, \quad \Gamma_\chi = 2\cot\chi, \quad \Gamma_\varphi = 0 \ . \end{array}\right\} \tag{2.25}$$

The spherical system corresponds to the $\mathfrak{so}(3,\mathbb{C})$ subalgebra of $\mathfrak{so}(4,\mathbb{C})$ such that the subsystem-coordinates are coordinates of the two-dimensional complex sphere S_{2C}: $z_1^2+z_2^2+z_3^2=1$. According to Refs.[21, 23, 27] there are exactly five such systems, and together with the spherical system these are the systems (4), (6), (7), and (8). The momentum operators are:

$$p_\chi = \frac{\hbar}{\mathrm{i}}\Big(\frac{\partial}{\partial\chi} + \cot\chi\Big), \quad p_\vartheta = \frac{\hbar}{\mathrm{i}}\Big(\frac{\partial}{\partial\vartheta} + \frac{1}{2}\cot\vartheta\Big), \quad p_\varphi = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\varphi} \ , \tag{2.26}$$

and the Hamiltonian is given by

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial\chi^2} + 2\cot\chi\frac{\partial}{\partial\chi} + \frac{1}{\sin^2\chi}\Big(\frac{\partial^2}{\partial\vartheta^2} + \cot\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\Big)\right] \\ &= \frac{1}{2m}\left[p_\chi^2 + \frac{1}{\sin^2\chi}\Big(p_\vartheta^2 + \frac{1}{\sin^2\vartheta}p_\varphi^2\Big)\right] - \frac{\hbar^2}{8m}\left(4 + \frac{1}{\sin^2\chi}\Big(1 + \frac{1}{\sin^2\vartheta}\Big)\right) \ . \end{aligned} \tag{2.27}$$

I do not go into the details of the path integral solution of the spherical system, because this has been done extensively in the literature, e.g. [2, 4, 18, 19, 11, 12, 39]. We state

$$\begin{aligned} &K^{(S_{3C})}(\chi'',\chi',\vartheta'',\vartheta',\varphi'',\varphi';T) \\ &= \int_{\chi(t')=\chi'}^{\chi(t'')=\chi''} \mathcal{D}\chi(t)\sin^2\chi \int_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t)\sin\vartheta \int_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t) \\ &\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\Big(\dot\chi^2 + \sin^2\chi(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2)\Big) + \frac{\hbar^2}{8m}\left(4+\frac{1}{\sin^2\chi}\Big(1+\frac{1}{\sin^2\vartheta}\Big)\right)\right]dt\right\} \\ &= \frac{1}{2\pi^2}\sum_{J=0}^{\infty}(J+1)C_J^1(\cos\psi_{S^{(3)}})\exp\Big[-\frac{\mathrm{i}\hbar T}{2m}J(J+2)\Big] \qquad (2.28) \\ &= \sum_{J=0}^{\infty}\sum_{m_1,m_2}\Psi_{J,m_1,m_2}(\chi'',\vartheta'',\varphi'')\Psi^*_{J,m_1,m_2}(\chi',\vartheta',\varphi')\exp\Big[-\frac{\mathrm{i}\hbar T}{2m}J(J+2)\Big] \ , \qquad (2.29) \end{aligned}$$

$$\Psi_{J,m_1,m_2}(\chi,\vartheta,\varphi) = N^{-1/2}\,\mathrm{e}^{\mathrm{i}m_1\varphi}(\sin\chi)^{m_1}C_{J-m_1}^{m_1+2}(\cos\chi)(\sin\vartheta)^{m_2}C_{m_1-m_2}^{m_2+3/2}(\cos\vartheta)\ , \tag{2.30}$$

$$N = \frac{2\pi^3 2^{-(1+2m_1+2m_2)}}{(J+1)(m_1+3/2)(J-m_1)!(m_1-m_2)!)}\frac{\Gamma(J+m_1+2)\Gamma(m_1+m_2+1)}{\Gamma^2(m_1+1)\Gamma^2(m_2+3/2)}\ , \tag{2.31}$$

and the energy-spectrum (2.8). The spherical system exists on the three-dimensional sphere (System III.) and on the three-dimensional hyperboloid (System X.).

2.4 System 4: Horospherical

This coordinate system is defined as

$$\left.\begin{aligned} z_1 &= \frac{1}{2}\Big[\mathrm{e}^{-\mathrm{i}x} + (1-y^2)\,\mathrm{e}^{\mathrm{i}x}\Big]\sin\chi \qquad & z_2 &= y\,\mathrm{e}^{\mathrm{i}x}\sin\chi \\ z_3 &= -\frac{\mathrm{i}}{2}\Big[\mathrm{e}^{-\mathrm{i}x} - (1+y^2)\,\mathrm{e}^{\mathrm{i}x}\Big]\sin\chi \qquad & z_4 &= \cos\chi \end{aligned}\right\} \tag{2.32}$$

($\chi \in (0,\pi), x, y \in \mathbb{R}$). The set of commuting operators is given by

$$\mathcal{L}_1 = I_{12}^2 + I_{13}^2 + I_{23}^2, \qquad \mathcal{L}_2 = (I_{32} + \mathrm{i}I_{21})^2\ , \tag{2.33}$$

and the metric terms have the form

$$\left.\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\chi^2 + \sin^2\chi(\,\mathrm{d}x^2 + \mathrm{e}^{2\mathrm{i}x}\,\mathrm{d}y^2)\ , \\ \sqrt{g} &= \mathrm{e}^{\mathrm{i}x}\sin^2\chi\ , \\ \Gamma_\chi &= 2\cot\chi, \quad \Gamma_x = \mathrm{i}, \quad \Gamma_y = 0\ . \end{aligned}\right\} \tag{2.34}$$

The momentum operators are

$$p_\chi = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\chi} + \cot\chi\right), \quad p_x = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial x} + \frac{\mathrm{i}}{2}\right), \quad p_y = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial y}\ , \tag{2.35}$$

and the Hamiltonian is given by

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial\chi^2} + 2\cot\chi\frac{\partial}{\partial\chi} + \frac{1}{\sin^2\chi}\left(\frac{\partial^2}{\partial x^2} + 2\mathrm{i}x\frac{\partial}{\partial x} + \mathrm{e}^{-2\mathrm{i}x}\frac{\partial^2}{\partial y^2}\right)\right] \\ &= \frac{1}{2m}\left[p_\chi^2 + \frac{1}{\sin^2\chi}\left(p_x^2 + \mathrm{e}^{-2\mathrm{i}x}p_y^2\right)\right] - \frac{\hbar^2}{8m}\left(4 + \frac{1}{\sin^2\chi}\right)\ . \end{aligned} \tag{2.36}$$

For the path integral we find by separating off the y-path integration (plane waves):

$$\begin{aligned} &K^{(S_{3C})}(\chi'',\chi',x'',x',y'',y';T) \\ &= \int_{\chi(t')=\chi'}^{\chi(t'')=\chi''}\mathcal{D}\chi(t)\int_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\int_{y(t')=y'}^{y(t'')=y''}\mathcal{D}y(t)\,\mathrm{e}^{\mathrm{i}x}\sin^2\chi \\ &\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T\left[\frac{m}{2}\dot\chi^2 + \sin^2\chi(\dot x^2 + \mathrm{e}^{2\mathrm{i}x}\dot y^2) + \frac{\hbar^2}{8m}\left(4 + \frac{1}{\sin^2\chi}\right)\right]\mathrm{d}t\right\} \end{aligned}$$

$$= (\sin\chi'\sin\chi'')^{-1/2}\,\mathrm{e}^{-\mathrm{i}(x''-x')/2}\,\mathrm{e}^{\mathrm{i}\hbar T/2m}\int_{\mathbb{R}}\frac{\mathrm{e}^{\mathrm{i}k_y(y''-y')}}{2\pi}$$
$$\times\int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''}\mathcal{D}\chi(t)\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\sin\chi$$
$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T\left[\frac{m}{2}\dot\chi^2+\sin^2\chi\left(\dot x^2-\frac{\hbar^2k_y^2}{2m}\,\mathrm{e}^{2\mathrm{i}x}\right)+\frac{\hbar^2}{8m\sin^2\chi}\right]\mathrm{d}t\right\}\ . \tag{2.37}$$

In the x-path integration we can use the result of the horicyclic system (2) with energy spectrum $E_{n_x}=(n_x+\frac{1}{2})^2\hbar^2/2m$, $n_x\in\mathbb{N}_0$, yielding

$$K^{(S_{3C})}(\chi'',\chi',x'',x',y'',y';T)=(\sin\chi'\sin\chi'')^{-1}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}(x''-x')}\,\mathrm{e}^{\mathrm{i}\hbar T/2m}$$
$$\times\int_{\mathbb{R}}\frac{\mathrm{e}^{\mathrm{i}k_y(y''-y')}}{2\pi}\sum_{n_x\in\mathbb{N}_0}\tfrac{1}{2}H^{(1)}_{n_x+1/2}(|k_y|\,\mathrm{e}^{-\mathrm{i}x''})H^{(1)}_{n_x+1/2}(|k_y|\,\mathrm{e}^{\mathrm{i}x'})$$
$$\times\int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''}\mathcal{D}\chi(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^T\left(\frac{m}{2}\dot\chi^2-\frac{\hbar^2}{2m}\frac{(n_x+\frac{1}{2})^2-\frac{1}{4}}{\sin^2\chi}\right)\mathrm{d}t\right]$$
$$=(\sin\chi'\sin\chi'')^{-1/2}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}(x''-x')}\,\mathrm{e}^{\mathrm{i}\hbar T/2m}\int_{\mathbb{R}}\frac{\mathrm{e}^{\mathrm{i}k_y(y''-y')}}{2\pi}$$
$$\times\sum_{n_x\in\mathbb{N}_0}\tfrac{1}{2}H^{(1)}_{n_x+1/2}(|k_y|\,\mathrm{e}^{-\mathrm{i}x''})H^{(1)}_{n_x+1/2}(|k_y|\,\mathrm{e}^{\mathrm{i}x'})$$
$$\times\sum_{l\in\mathbb{N}_0}\left[(l+n_x+1)\frac{\Gamma(l+2n_x+2))}{l!}\right]P^{-n_x-1/2}_{l+n_x+1/2}(\cos\chi')P^{-n_x-1/2}_{l+n_x+1/2}(\cos\chi'')$$
$$\times\exp\left[-\frac{\mathrm{i}}{\hbar}\frac{\hbar^2(l+n_x)(l+n_x+2)}{2m}T\right]\ . \tag{2.38}$$

The path integral in the variable χ is just one for the symmetric Pöschl–Teller potential [8, 19, 30]. This yields for the wave-functions on S_{3C}

$$\Psi_{l,n_x,k_y}(\chi,x,y) \;=\; \frac{\mathrm{e}^{\mathrm{i}k_yy}}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{2}}\,\mathrm{e}^{-\mathrm{i}x/2}H^{(1)}_{n_x+1/2}(k_y\,\mathrm{e}^{-\mathrm{i}x})$$
$$\times\left[(l+n_x+1)\frac{\Gamma(l+2n_x+2))}{l!\sin\chi}\right]^{1/2}P^{-n_x-1/2}_{l+n_x+1/2}(\cos\chi)\ , \tag{2.39}$$

and the energy-spectrum (2.8), by observing that we can re-define $J=l+n_x$.

2.5 System 5: Horicyclic-Polar

This coordinate system is defined as

$$\left.\begin{array}{ll} z_1=\frac{1}{2}\left[\,\mathrm{e}^{-\mathrm{i}x}+(1+\varrho^2)\,\mathrm{e}^{\mathrm{i}x}\right] & z_2=\mathrm{i}\rho\,\mathrm{e}^{\mathrm{i}x}\cos\varphi \\ z_3=\mathrm{i}\varrho\,\mathrm{e}^{\mathrm{i}x}\sin\varphi & z_4=\frac{\mathrm{i}}{2}\left[\,\mathrm{e}^{-\mathrm{i}x}-(1-\varrho^2)\,\mathrm{e}^{\mathrm{i}x}\right] \end{array}\right\} \tag{2.40}$$

($\varphi \in [0, 2\pi), x \in \mathbb{R}\varrho > 0$). and the set of commuting operators is given by

$$\mathcal{L}_1 = (I_{42}^2 + \mathrm{i}I_{21})^2 + (I_{34}^2 + \mathrm{i}I_{13})^2, \qquad \mathcal{L}_2 = I_{23}^2 \ . \tag{2.41}$$

For the metric terms we have

$$\left.\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{e}^{2\mathrm{i}x}(\,\mathrm{d}\varrho^2 + \varrho^2\,\mathrm{d}\varphi^2)\ , \\ \sqrt{g} &= \varrho\,\mathrm{e}^{2\mathrm{i}x}\ , \\ \Gamma_\varrho &= \frac{1}{\varrho}, \quad \Gamma_x = 2\mathrm{i}, \quad \Gamma_\varphi = 0\ , \end{aligned}\right\} \tag{2.42}$$

and the momentum operators are

$$p_x = \frac{\hbar}{\mathrm{i}}\Big(\frac{\partial}{\partial x} + \mathrm{i}\Big), \quad p_\varrho = \frac{\hbar}{\mathrm{i}}\Big(\frac{\partial}{\partial \varrho} + \frac{1}{2\varrho}\Big), \quad p_\varphi = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial \varphi}\ . \tag{2.43}$$

Thus, the Hamiltonian is given by

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2} + 2\mathrm{i}\frac{\partial}{\partial x} + \mathrm{e}^{-2\mathrm{i}x}\Big(\frac{\partial^2}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial}{\partial \varrho} + \frac{1}{\varrho^2}\frac{\partial^2}{\partial \varphi^2}\Big)\right] \\ &= \frac{1}{2m}\left[p_x^2 + \mathrm{e}^{-2\mathrm{i}x}\Big(p_\varrho^2 + \frac{1}{\varrho^2}p_\varphi^2\Big)\right] - \mathrm{e}^{-2\mathrm{i}x}\frac{\hbar^2}{8m\varrho^2}\ . \end{aligned} \tag{2.44}$$

In the following path integral, the variable φ can be separated off in terms of circular wave, the variable ϱ in terms of a radial path integral (free motion), and for the remaining path integration in x we find similarly as in the horicyclic system (2)

$$\begin{aligned} &K^{(S_{3C})}(x'', x', \varrho'', \varrho', \varphi'', \varphi'; T) \\ &= \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \int_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''} \mathcal{D}\varrho(t) \int_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)\varrho\,\mathrm{e}^{2\mathrm{i}x} \\ &\quad \times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T \left[\frac{m}{2}\dot{x}^2 + \mathrm{e}^{2\mathrm{i}x}(\dot{\varrho}^2 + \varrho^2\dot{\varphi}^2) + \mathrm{e}^{-2\mathrm{i}x}\frac{\hbar^2}{8m\varrho^2}\right]\mathrm{d}t\right\} \\ &= e^{-\mathrm{i}(x''-x')}\sum_{\nu\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}\nu(\varphi''-\varphi')}}{2\pi}\int_0^\infty \mathrm{d}k_\varrho k_\varrho J_\nu(k_\varrho\varrho')J_\nu(k_\varrho\varrho'') \\ &\quad \times \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^T\left(\frac{m}{2}\dot{x}^2 - \mathrm{e}^{-2\mathrm{i}x}\frac{\hbar^2k_\varrho^2}{2m}\right)\mathrm{d}t\right] \\ &= e^{-\mathrm{i}(x''-x')}\sum_{\nu\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}\nu(\varphi''-\varphi')}}{2\pi}\int_0^\infty \mathrm{d}k_\varrho k_\varrho J_\nu(k_\varrho\varrho')J_\nu(k_\varrho\varrho'') \\ &\quad \times \sum_{J\in\mathbb{N}_0}\tfrac{1}{2}H_{J+1/2}^{(1)}(|k_\varrho|\,\mathrm{e}^{-\mathrm{i}x''})H_{J+1/2}^{(1)}(|k_\varrho|\,\mathrm{e}^{\mathrm{i}x'})\exp\left[-\frac{\mathrm{i}}{\hbar}\frac{\hbar^2J(J+2)}{2m}T\right]\ . \end{aligned} \tag{2.45}$$

The wave-functions on S_{3C} are given by ($k_\varrho > 0$)

$$\Psi_{Jk_\varrho\nu}(x, \varrho, \varphi) = \frac{\mathrm{e}^{\mathrm{i}\nu\varphi}}{\sqrt{2\pi}}\sqrt{k_\varrho}\,J_\nu(k_\varrho\varrho')\,\mathrm{e}^{-\mathrm{i}x}\cdot\frac{1}{\sqrt{2}}H_{J+1/2}^{(1)}(k_\varrho\,\mathrm{e}^{-\mathrm{i}x})\ , \tag{2.46}$$

with the energy-spectrum (2.8).

2.6 System 6: Sphero-Elliptic

This coordinate system is defined as

$$\left.\begin{aligned} z_1 &= \sin\chi\,\mathrm{sn}(\alpha,k)\,\mathrm{dn}(\beta,k') & z_2 &= \sin\chi\,\mathrm{cn}(\alpha,k)\,\mathrm{cn}(\beta,k') \\ z_3 &= \sin\chi\,\mathrm{dn}(\alpha,k)\,\mathrm{sn}(\beta,k') & z_4 &= \cos\chi. \end{aligned}\right\} \tag{2.47}$$

($\chi \in (0,\pi), \alpha \in [-K,K], \beta \in [-2K',2K']$). The set of commuting operators is given by

$$\mathcal{L}_1 = I_{12}^2 + I_{13}^2 + I_{23}^2, \qquad \mathcal{L}_2 = I_{23}^2 + k^2 I_{13}^2 \ . \tag{2.48}$$

The elliptic coordinate system reads in algebraic form as follows ($a_1 \le \varrho_1 \le a_2 \le \varrho_2 \le a_3$)

$$s_1^2 = R^2\frac{(\varrho_1-a_1)(\varrho_2-a_1)}{(a_2-a_1)(a_3-a_1)},\ s_2^2 = R^2\frac{(\varrho_1-a_2)(\varrho_2-a_2)}{(a_3-a_2)(a_1-a_2)},\ s_3^2 = R^2\frac{(\varrho_1-a_3)(\varrho_2-a_3)}{(a_1-a_3)(a_2-a_3)} \ . \tag{2.49}$$

If we put $\varrho_1 = a_1 + (a_2-a_1)\,\mathrm{sn}^2(\alpha,k)$ and $\varrho_2 = a_2 + (a_3-a_2)\,\mathrm{cn}^2(\beta,k')$, where $\mathrm{sn}(\alpha,k)$, $\mathrm{cn}(\alpha,k)$ and $\mathrm{dn}(\alpha,k)$ are the Jacobi elliptic functions with modulus k, we obtain for the coordinates $\mathbf{s}$ on the sphere (also called conical coordinates)

$$\begin{aligned} s_1 &= R\,\mathrm{sn}(\alpha,k)\,\mathrm{dn}(\beta,k') & (-K \le \alpha \le K)\ , \\ s_2 &= R\,\mathrm{cn}(\alpha,k)\,\mathrm{cn}(\beta,k') & (-2K' \le \beta \le 2K')\ , \\ s_3 &= R\,\mathrm{dn}(\alpha,k)\,\mathrm{sn}(\beta,k')\ , \end{aligned} \tag{2.50}$$

where

$$k^2 = \frac{a_2-a_1}{a_3-a_1} = \sin^2 f\ , \qquad k'^2 = \frac{a_3-a_2}{a_3-a_1} = \cos^2 f\ , \qquad k^2+k'^2 = 1\ . \tag{2.51}$$

$K = K(k) = \frac{\pi}{2}\,{}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)$ and $K' = K(k')$ are complete elliptic integrals, and $2f$ is the interfocus distance on the upper semi-sphere of the ellipses on the sphere. Note the relations $\mathrm{cn}^2\alpha + \mathrm{sn}^2\alpha = 1$ and $\mathrm{dn}^2\alpha = 1 - k^2\,\mathrm{sn}^2\alpha$. In the following we omit the moduli k and k' of the Jacobi elliptic functions if it is obvious that the variable α goes with k and β goes with k'. The metric terms are now

$$\left.\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\chi^2 + \sin^2\chi(k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta)(\,\mathrm{d}\alpha^2 + \,\mathrm{d}\beta^2)\ , \\ \sqrt{g} &= (k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta)\sin^2\chi\ , \\ \Gamma_\chi &= 2\cot\chi\ , \quad \Gamma_\alpha = -2\frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\alpha\,\mathrm{dn}\alpha}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\alpha}, \quad \Gamma_\beta = -2\frac{k'^2\,\mathrm{sn}\beta\,\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\,\mathrm{cn}^2\beta + k'^2\,\mathrm{cn}^2\beta}\ . \end{aligned}\right\} \tag{2.52}$$

The momentum operators are (p_χ as in (2.35))

$$p_\alpha = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\alpha} - \frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\alpha\,\mathrm{dn}\alpha}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\alpha}\right)\ , \quad p_\beta = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\beta} - \frac{k'^2\,\mathrm{sn}\beta\,\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\,\mathrm{cn}^2\beta + k'^2\,\mathrm{cn}^2\beta}\right)\ , \tag{2.53}$$

and the Hamiltonian has the form

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial\chi^2} + 2\cot\chi\frac{\partial}{\partial\chi} + \frac{1}{\sin^2\chi}\frac{1}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\alpha}\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right)\right] \\ &= \frac{1}{2m}\left[p_\chi^2 + \frac{1}{\sin^2\chi}\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta}}(p_\alpha^2+p_\beta^2)\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta}}\right] - \frac{\hbar^2}{2m}\ . \end{aligned} \tag{2.54}$$

The corresponding path integral representation has been discussed in [12, 17] and we just state the result:

$$
\begin{aligned}
&K^{(S_{3C})}(\chi'',\chi',\alpha'',\alpha',\beta'',\beta';T) \\
&= \int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''} \mathcal{D}\chi(t)\sin^2\chi \int\limits_{\alpha(t')=\alpha'}^{\alpha(t'')=\alpha''} \mathcal{D}\alpha(t) \int\limits_{\beta(t')=\beta'}^{\beta(t'')=\beta''} \mathcal{D}\beta(t)(k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta) \\
&\qquad \times \exp\left\{\frac{\mathrm{i}m}{2\hbar}\int_{t'}^{t''}\left[\dot\chi^2 + \sin^2\chi(k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta)(\dot\alpha^2+\dot\beta^2)\right]dt + \frac{\mathrm{i}\hbar T}{2m}\right\} \\
&= \sum_{J=0}^{\infty}\sum_{l=-J}^{J}\sum_{\lambda}\sum_{p,q=\pm}\Lambda^{p}_{l,h}(\alpha'')\Lambda^{p\,*}_{l,h}(\alpha')\Lambda^{q}_{l,\tilde h}(\beta'')\Lambda^{q\,*}_{l,\tilde h}(\beta') \\
&\qquad \times (J+1)\frac{(l+J+1)!}{|J-l|!}\,\mathrm{e}^{-\mathrm{i}\hbar T J(J+2)/2m}P^{-l-1/2}_{J+1/2}(\sin\chi'')P^{-l-1/2}_{J+1/2}(\sin\chi')\ .
\end{aligned}
\tag{2.55}
$$

The solution in the variable χ is again of the symmetric Pöschl–Teller type. For the *periodic Lamé polynomials* $\Lambda^p_{ln}(z)$ we have adopted the notation of [38]. In [38] it is shown that the wave-functions of the spherical basis $|lm>$ can be expanded into the wave-functions of the elliptical basis $|l\lambda>$ and vice versa. The are Lamé polynomials Λ^p_{lh} are satisfying ($k^2 = 1-k'^2$)

$$
\left.
\begin{aligned}
&\frac{\mathrm{d}^2\Lambda^p_{lh}}{\mathrm{d}\alpha^2} + [h - l(l+1)k^2\,\mathrm{sn}^2(\alpha,k)]\Lambda^p_{lh} = 0\ , \\
&\Lambda^p_{lh}(-\alpha) = p\Lambda_{lh}(\alpha)\ , \qquad h = -\frac{\lambda}{4} + l(l+1)\ .
\end{aligned}
\right\}
\tag{2.56}
$$

The functions $\Lambda^q_{lh'}(\beta)$ satisfies the same equation with $\alpha\to\beta, k\to k', h\to\tilde h = \lambda/4$ and $p\to q$. These functions are also called *ellipsoidal harmonics* which satisfy the orthonormality relation

$$
\int_{-K}^{K}d\alpha\int_{-2K'}^{2K'}d\beta(k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta)\Lambda^{p'}_{l',\tilde h}(\alpha)\Lambda^{p\,*}_{l,h}(\alpha)\Lambda^{q'}_{l',\tilde h'}(\beta)\Lambda^{q\,*}_{l,h'}(\beta) = \delta_{ll'}\delta_{qq'}\delta_{pp'}\delta_{h\tilde h}\ . \tag{2.57}
$$

Here λ is the eigenvalue of the operator $E = \mathcal{L}_2$ [38] which commutes with the Hamiltonian. The wave-functions on S_{3C} have the form

$$
\Psi_{J,l,\lambda,p} = \Lambda^p_{l,h}(\alpha)\Lambda^q_{l,\tilde h}(\beta)\sqrt{(J+1)\frac{(l+J+1)!}{|J-l|!}}\,P^{-l-1/2}_{J+1/2}(\sin\chi)\ , \tag{2.58}
$$

and the energy-spectrum (2.8). The sphero-elliptic system exists on the three-dimensional sphere (System II.) and on the three-dimensional hyperboloid (System III.).

2.7 System 7: Spherical-degenerate elliptic I

This coordinate system is defined as

$$
\left.
\begin{aligned}
z_1 &= \frac{1}{2}\sin\chi\left(\frac{\cosh\tau_2}{\cosh\tau_1} + \frac{\cosh\tau_1}{\cosh\tau_2}\right) & z_2 &= \sin\chi\tanh\tau_1\tanh\tau_2 \\
z_3 &= \sin\chi\left[\frac{1}{\cosh\tau_1\cosh\tau_2} - \frac{1}{2}\left(\frac{\cosh\tau_2}{\cosh\tau_1} + \frac{\cosh\tau_1}{\cosh\tau_2}\right)\right] & z_4 &= \cos\chi
\end{aligned}
\right\}
\tag{2.59}
$$

($\chi \in [0,\pi), \tau_1, \tau_2 \in \mathbb{R}$). The set of commuting operators is given by

$$\mathcal{L}_1 = I_{12}^2 + I_{13}^2 + I_{23}^2, \qquad \mathcal{L}_2 = -I_{12}^2 + I_{23}^2 + \mathrm{i}\{I_{31}, I_{32}\} \ . \tag{2.60}$$

$\{I,J\} = IJ + JI$ denotes the anti-commutator of the operators I and J. The metric terms are

$$\left.\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\chi^2 + \sin^2\chi\Big(\frac{1}{\cosh^2\tau_1} - \frac{1}{\cosh^2\tau_2}\Big)(\,\mathrm{d}\tau_1^2 - \,\mathrm{d}\tau_2^2) \ , \\ \sqrt{g} &= \sin^2\chi\Big(\frac{1}{\cosh^2\tau_1} - \frac{1}{\cosh^2\tau_2}\Big) \ , \\ \Gamma_\chi &= 2\cot\chi, \quad \Gamma_{\tau_{1,2}} = \mp 2\frac{\sinh\tau_{1,2}}{\cosh^3\tau_{1,2}} \cdot \frac{1}{\cosh^{-2}\tau_1 - \cosh^{-2}\tau_2} \ . \end{aligned}\right\} \tag{2.61}$$

The momentum operators have the form

$$p_\chi = \frac{\hbar}{\mathrm{i}}\Big(\frac{\partial}{\partial\chi} + \cot\chi\Big), \quad p_{\tau_1} = \frac{\hbar}{\mathrm{i}}\Big(\frac{\partial}{\partial\tau_1} + \tfrac{1}{2}\Gamma_{\tau_1}\Big), \quad p_{\tau_2} = \frac{\hbar}{\mathrm{i}}\Big(\frac{\partial}{\partial\tau_2} + \tfrac{1}{2}\Gamma_{\tau_2}\Big) \ , \tag{2.62}$$

and the Hamiltonian reads

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}\Bigg[\frac{\partial^2}{\partial\chi^2} + 2\cot\chi\frac{\partial}{\partial\chi} \\ &\qquad + \frac{1}{\sin^2\chi}\Big(\frac{1}{\cosh^2\tau_1} - \frac{1}{\cosh^2\tau_2}\Big)^{-1}\Big(\frac{\partial^2}{\partial\tau_1^2} + \tfrac{1}{2}\Gamma_{\tau_1}\frac{\partial}{\partial\tau_1} - \frac{\partial^2}{\partial\tau_2^2} - \tfrac{1}{2}\Gamma_{\tau_2}\frac{\partial}{\partial\tau_2}\Big)\Bigg] \\ &= \frac{1}{2m}\Bigg[p_\chi^2 + \frac{1}{\sin^2\chi}\Big(\frac{1}{\cosh^2\tau_1} - \frac{1}{\cosh^2\tau_2}\Big)^{-1/2}(p_{\tau_1}^2 - p_{\tau_2}^2)\Big(\frac{1}{\cosh^2\tau_1} - \frac{1}{\cosh^2\tau_2}\Big)^{-1/2}\Bigg] \\ &\qquad - \frac{\hbar^2}{8m}\Big(4 + \frac{1}{\sin^2\chi}\Big) \ . \end{aligned} \tag{2.63}$$

In the calculation I use now for the (τ_1,τ_2)-subpath integration a path integral solution on the two-dimensional hyperboloid [12, p.97], i.e. the kernel in terms of elliptic parabolic coordinates. The χ-path integration is the usual symmetric Pöschl–Teller case [8, 19, 30], therefore we obtain

$$\begin{aligned} &K^{(S_{3C})}(\chi'',\chi',\tau_1'',\tau_1',\tau_2'',\tau_2';T) \\ &= \int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''} \mathcal{D}\chi(t) \int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \mathcal{D}\tau_1(t) \int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''} \mathcal{D}\tau_2(t)\sin^2\chi\Big(\frac{1}{\cosh^2\tau_1} - \frac{1}{\cosh^2\tau_2}\Big) \\ &\quad \times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T\left[\frac{m}{2}\dot\chi^2 + \sin^2\chi\Big(\frac{1}{\cosh^2\tau_1} - \frac{1}{\cosh^2\tau_2}\Big)(\dot\tau_1^2 - \dot\tau_2^2) + \frac{\hbar^2}{8m}\Big(4 + \frac{1}{\sin^2\chi}\Big)\right]\mathrm{d}t\right\} \\ &= (\sin\chi'\sin\chi'')^{-1}\,\mathrm{e}^{\mathrm{i}\hbar T/2m}\int_0^\infty dp\,p\sinh\pi p\int_0^\infty \frac{dk\,k\sinh\pi k}{(\cosh^2\pi k + \sinh^2\pi p)^2} \\ &\quad \times \sum_{\epsilon,\epsilon'=\pm1} P^{\mathrm{i}k}_{\mathrm{i}p-1/2}(\epsilon\tanh\tau_1'')P^{-\mathrm{i}k}_{\mathrm{i}p-1/2}(\epsilon\tanh\tau_1')P^{\mathrm{i}p}_{\mathrm{i}k-1/2}(\epsilon'\tanh\tau_2'')P^{-\mathrm{i}p}_{\mathrm{i}k-1/2}(\epsilon'\tanh\tau_2') \\ &\quad \times \int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''}\mathcal{D}\chi(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^T\left(\frac{m}{2}\dot\chi^2 - \frac{\hbar^2}{2m}\frac{p^2-\frac{1}{4}}{\sin^2\chi}\right)\mathrm{d}t\right] \end{aligned}$$

$$
\begin{aligned}
&= (\sin\chi'\sin\chi'')^{-1}\,\mathrm{e}^{\mathrm{i}\hbar T/2m}\int_0^\infty dp\,p\sinh\pi p\int_0^\infty \frac{dk\,k\sinh\pi k}{(\cosh^2\pi k+\sinh^2\pi p)^2}\\
&\quad\times\sum_{\epsilon,\epsilon'=\pm1} P^{\mathrm{i}k}_{\mathrm{i}p-1/2}(\epsilon\tanh\tau_1'')P^{-\mathrm{i}k}_{\mathrm{i}p-1/2}(\epsilon\tanh\tau_1')P^{\mathrm{i}p}_{\mathrm{i}k-1/2}(\epsilon'\tanh\tau_2'')P^{-\mathrm{i}p}_{\mathrm{i}k-1/2}(\epsilon'\tanh\tau_2')\\
&\quad\times\sum_{J\in\mathbb{N}_0}\left[(J+p+\tfrac12)\frac{\Gamma(p+2J+1)}{J!}\right]P^{-J-1/2}_{J+p+1/2}(\cos\chi')P^{-J-1/2}_{J+p+1/2}(\cos\chi'')\\
&\quad\times\exp\left[-\frac{\mathrm{i}}{\hbar}\frac{\hbar^2J(J+2)}{2m}T\right]\ .
\end{aligned}
\tag{2.64}
$$

The wave-functions on S_{3C} have the form

$$
\begin{aligned}
\Psi_{J,p,k,\pm}(\chi,\tau_1,\tau_2) &= \frac{\sqrt{p\sinh\pi pk\sinh\pi k}}{\cosh^2\pi k+\sinh^2\pi p}P^{\mathrm{i}k}_{\mathrm{i}p-1/2}(\epsilon\tanh\tau_1)P^{\mathrm{i}p}_{\mathrm{i}k-1/2}(\epsilon'\tanh\tau_2)\\
&\quad\times(\sin\chi)^{-1}\left[(J+p+\tfrac12)\frac{\Gamma(p+2J+1)}{J!}\right]^{1/2}P^{-J-1/2}_{J+p+1/2}(\cos\chi')\ ,
\end{aligned}
\tag{2.65}
$$

and the energy-spectrum (2.8).

2.8 System 8: Spherical-degenerate elliptic II

This coordinate system is defined as

$$
\left.
\begin{aligned}
z_1 &= -\frac{\mathrm{i}\sin\chi}{8\xi\eta}\big[(\xi^2-\eta^2)^2+4\big] & z_2 &= \frac{\sin\chi}{2\xi\eta}(\xi^2+\eta^2)^2\\
z_3 &= \frac{\sin\chi}{8\xi\eta}\big[-(\xi^2-\eta^2)^2+4\big] & z_4 &= \cos\chi-\{I_{12},I_{13}\}+\mathrm{i}\{I_{12},I_{23}\}
\end{aligned}
\right\}
\tag{2.66}
$$

($\chi\in(0,\pi),\xi,\eta>0$). The set of commuting operators is given by

$$
\mathcal{L}_1 = I_{12}^2+I_{13}^2+I_{23}^2,\qquad \mathcal{L}_2 = -I_{12}^2+I_{13}^2+\mathrm{i}\{I_{12},I_{23}\}\ ,
\tag{2.67}
$$

and the metric terms are

$$
\left.
\begin{aligned}
&\mathrm{d}s^2 = \mathrm{d}\chi^2+\sin^2\chi\left(\frac{1}{\eta^2}-\frac{1}{\xi^2}\right)(\mathrm{d}\xi^2-\mathrm{d}\eta^2)\ ,\\
&\sqrt{g} = \sin^2\chi\left(\frac{1}{\eta^2}-\frac{1}{\xi^2}\right)\ ,\\
&\Gamma_\chi = 2\cot\chi,\quad \Gamma_\xi = -\frac{2}{\xi^3}\cdot\frac{1}{1/\eta^2-1/\xi^2},\quad \Gamma_\eta = \frac{2}{\eta^3}\cdot\frac{1}{1/\eta^2-1/\xi^2},\qquad .
\end{aligned}
\right\}
\tag{2.68}
$$

The momentum operators have the form

$$
p_\chi = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\chi}+\cot\chi\right),\quad p_\xi = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\xi}+\tfrac12\Gamma_\xi\right),\quad p_\eta = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\eta}+\tfrac12\Gamma_\eta\right)\ ,
\tag{2.69}
$$

and the Hamiltonian reads

$$
\begin{aligned}
H &= -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial\chi^2}+2\cot\chi\frac{\partial}{\partial\chi}+\frac{1}{\sin^2\chi}\left(\frac{1}{\eta^2}-\frac{1}{\xi^2}\right)^{-1}\left(\frac{\partial^2}{\partial\xi^2}+\tfrac12\Gamma_\xi\frac{\partial}{\partial\xi}-\frac{\partial^2}{\partial\eta^2}+\tfrac12\Gamma_\eta\frac{\partial}{\partial\eta}\right)\right]\\
&= \frac{1}{2m}\left[p_\chi^2+\frac{1}{\sin^2\chi}\left(\frac{1}{\eta^2}-\frac{1}{\xi^2}\right)^{-1/2}(p_\xi^2-p_\eta^2)\left(\frac{1}{\eta^2}-\frac{1}{\xi^2}\right)^{-1/2}\right]-\frac{\hbar^2}{8m}\left(4+\frac{1}{\sin^2\chi}\right)\ .
\end{aligned}
\tag{2.70}
$$

For the path integral we obtain

$$K^{(S_{3C})}(\chi'',\chi',\xi'',\xi',\eta'',\eta';T) = \int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''} \mathcal{D}\chi(t) \int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t) \int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t) \sin^2\chi\left(\frac{1}{\eta^2}-\frac{1}{\xi^2}\right)$$
$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T \left[\frac{m}{2}\dot{\chi}^2+\sin^2\chi\left(\frac{1}{\eta^2}-\frac{1}{\xi^2}\right)(\dot{\xi}^2-\dot{\eta}^2)+\frac{\hbar^2}{8m}\left(4+\frac{1}{\sin^2\chi}\right)\right]\mathrm{d}t\right\} \quad . \tag{2.71}$$

For the evaluation of this path integral we first consider the (ξ,η)-subpath integration, denoted by $\hat{K}(\xi'',\xi',\eta'',\eta';T)$. By the usual technique of space-time transformation we obtain for the corresponding transformed path integral

$$\hat{K}(\xi'',\xi',\eta'',\eta';s'') = \int\limits_{\xi(0)=\xi'}^{\xi(s'')=\xi''} \mathcal{D}\xi(s) \int\limits_{\eta(0)=\eta'}^{\eta(s'')=\eta''} \mathcal{D}\eta(s) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}(\dot{\xi}^2-\dot{\eta}^2)-\frac{\mathcal{E}}{\xi^2}+\frac{\mathcal{E}}{\eta^2}\right]\mathrm{d}s\right\}. \tag{2.72}$$

We therefore obtain two path integrations for the inverse-square potential. Problems like these have been discussed in [12, 13]. The corresponding Green function we can write as follows:

$$\begin{aligned}
&\hat{G}(\xi'',\xi',\eta'',\eta';\mathcal{E}) = \int_0^\infty \mathrm{d}s''\hat{K}(\xi'',\xi',\eta'',\eta';s'') \\
&= \frac{4m^2}{\hbar^3}\sqrt{\xi'\xi''\eta'\eta''}\int\frac{\mathrm{d}\mathcal{E}}{2\pi\mathrm{i}} I_\lambda\left(\sqrt{2m\mathcal{E}}\,\frac{\xi_>}{\hbar}\right)K_\lambda\left(\sqrt{2m\mathcal{E}}\,\frac{\xi_>}{\hbar}\right)I_{\tilde{\lambda}}\left(\sqrt{-2m\mathcal{E}}\,\frac{\eta_<}{\hbar}\right)K_{\tilde{\lambda}}\left(\sqrt{-2m\mathcal{E}}\,\frac{\eta_>}{\hbar}\right) \\
&= \frac{m^2}{\hbar^3}\sqrt{\xi'\xi''\eta'\eta''}\int_0^\infty\frac{\mathrm{d}s''}{s''}\int\frac{\mathrm{d}\mathcal{E}}{2\pi\mathrm{i}}\,\mathrm{e}^{\mathrm{i}\mathcal{E}s''/\hbar} \\
&\quad\times\exp\left[\frac{m}{2\mathrm{i}\hbar s''}(\xi'^2+\xi''^2)\right]I_\lambda\left(\frac{\mathrm{i}m\xi'\xi''}{\hbar s''}\right)I_{\tilde{\lambda}}\left(\sqrt{-2m\mathcal{E}}\,\frac{\eta_<}{\hbar}\right)K_{\tilde{\lambda}}\left(\sqrt{-2m\mathcal{E}}\,\frac{\eta_>}{\hbar}\right)
\end{aligned} \tag{2.73}$$

($\mathcal{E}=(\lambda^2-\frac{1}{4})/2m$). $\xi_{<,>}$ denotes the smaller/larger of ξ',ξ'', and similarly for η. By means of a similar analysis as in [12, 13] the Green function $\hat{G}(\mathcal{E})$ is found to read

$$\begin{aligned}
\hat{G}(\xi'',\xi',\eta'',\eta';\mathcal{E}) = \frac{1}{4\pi^2}\sqrt{\mu'\mu''\nu'\nu''}\int\mathrm{d}\kappa\int_0^\infty\frac{\mathrm{d}k\,k\sinh^2\pi k}{\hbar^2k^2/2m-\mathcal{E}} \\
\times H^{(1)}_{-\mathrm{i}\tilde{p}}(\sqrt{\kappa}\,\xi')H^{(1)*}_{-\mathrm{i}\tilde{p}}(\sqrt{\kappa}\,\xi'')H^{(1)}_{-\mathrm{i}\tilde{p}}(\sqrt{\kappa}\,\eta')H^{(1)*}_{-\mathrm{i}\tilde{p}}(\sqrt{\kappa}\,\eta'') \quad .
\end{aligned} \tag{2.74}$$

We insert this result in (2.71) and get together with the remaining path integral in χ (symmetric Pöschl–Teller case):

$$\begin{aligned}
&K^{(S_{3C})}(\chi'',\chi',\xi'',\xi',\eta'',\eta';T) \\
&= \frac{1}{4\pi^2}\sqrt{\mu'\mu''\nu'\nu''}\int\mathrm{d}k\int_0^\infty\mathrm{d}p\,p\sinh^2\pi p H^{(1)}_p(\sqrt{k}\,\xi')H^{(1)*}_p(\sqrt{k}\,\xi'')H^{(1)}_p(\sqrt{k}\,\eta')H^{(1)*}_p(\sqrt{k}\,\eta'') \\
&\quad\times\sum_{J\in\mathbb{N}_0}\left[(J+p+\tfrac{1}{2})\frac{\Gamma(p+2J+1)}{J!}\right]P^{-J-1/2}_{J+p+1/2}(\cos\chi')P^{-J-1/2}_{J+p+1/2}(\cos\chi'') \\
&\quad\times\exp\left[-\frac{\mathrm{i}}{\hbar}\frac{\hbar^2J(J+2)}{2m}T\right] \quad .
\end{aligned} \tag{2.75}$$

The wave-functions on S_{3C} have the form

$$\Psi_{J,k,\kappa}(\chi,\xi,\eta) = \frac{\sqrt{p\mu\nu}}{2\pi}\sinh\pi p H_p^{(1)}(\sqrt{k}\,\xi)H_p^{(1)}(\sqrt{k}\,\eta)$$
$$\times(\sin\chi)^{-1}\left[(J+p+\tfrac{1}{2})\frac{\Gamma(p+2J+1)}{J!}\right]^{1/2} P^{-J-1/2}_{J+p+1/2}(\cos\chi') \ , \tag{2.76}$$

and the energy-spectrum (2.8).

2.9 System 9: Horicyclic-Elliptic

This coordinate system is defined as

$$\left.\begin{array}{ll} z_1 = \frac{1}{2}\Big[\mathrm{e}^{-\mathrm{i}x} + (1+\cosh^2\tau_1+\sinh^2\tau_2)\,\mathrm{e}^{\mathrm{i}x}\Big] & z_2 = \mathrm{i}\cosh\tau_1\cosh\tau_2\,\mathrm{e}^{\mathrm{i}x} \\ z_3 = \sinh\tau_1\sinh\tau_2\,\mathrm{e}^{\mathrm{i}x} & z_4 = \frac{\mathrm{i}}{2}\Big[\mathrm{e}^{-\mathrm{i}x} + (-1+\cosh^2\tau_1+\sinh^2\tau_2)\,\mathrm{e}^{\mathrm{i}x}\Big] \end{array}\right\} \tag{2.77}$$

($x,\tau_1,\tau_2\in\mathbb{R}$), the set of commuting operators read

$$\mathcal{L}_1 = (I_{42}+\mathrm{i}I_{21})^2 + (I_{34}+\mathrm{i}I_{13})^2, \qquad \mathcal{L}_2 = I_{23}^2 + (I_{34}+\mathrm{i}I_{13})^2 \ , \tag{2.78}$$

the metric terms are given by

$$\left.\begin{array}{l} \mathrm{d}s^2 = \mathrm{d}x + \mathrm{e}^{2\mathrm{i}x}(\cosh^2\tau_1-\cosh^2\tau_2)(\,\mathrm{d}\tau_1^2 - \,\mathrm{d}\tau_2^2) \ , \\ \sqrt{g} = \mathrm{e}^{2\mathrm{i}x}(\cosh^2\tau_1-\cosh^2\tau_2) \ , \\ \Gamma_x = 2\mathrm{i}, \quad \Gamma_{\tau_1} = \dfrac{2\sinh\tau_2\cosh\tau_2}{\cosh^2\tau_2-\cosh^2\tau_2}, \quad \Gamma_{\tau_1} = \dfrac{-2\sinh\tau_2\cosh\tau_2}{\cosh^2\tau_2-\cosh^2\tau_2} \ , \end{array}\right\} \tag{2.79}$$

the momentum operators have the form

$$p_x = \frac{\hbar}{\mathrm{i}}\Big(\frac{\partial}{\partial x}+\mathrm{i}\Big), \quad p_{\tau_1} = \frac{\hbar}{\mathrm{i}}\Big(\frac{\partial}{\partial\tau_1}+\tfrac{1}{2}\Gamma_{\tau_1}\Big), \quad p_{\tau_2} = \frac{\hbar}{\mathrm{i}}\Big(\frac{\partial}{\partial\tau_2}+\tfrac{1}{2}\Gamma_{\tau_2}\Big) \ , \tag{2.80}$$

and the Hamiltonian reads

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2}+2\mathrm{i}\frac{\partial}{\partial\chi}+\frac{\mathrm{e}^{-2\mathrm{i}x}}{\cosh^2\tau_2-\cosh^2\tau_2}\left(\frac{\partial^2}{\partial\tau_1^2}+\Gamma_{\tau_1}\frac{\partial}{\partial\tau_1}-\frac{\partial^2}{\partial\tau_2^2}-\Gamma_{\tau_2}\frac{\partial}{\partial\tau_2}\right)\right] \\ &= \frac{1}{2m}\left[p_x^2+\frac{\mathrm{e}^{-2\mathrm{i}x}}{\sqrt{\cosh^2\tau_2-\cosh^2\tau_2}}(p_{\tau_1}^2-p_{\tau_2}^2)\frac{1}{\sqrt{\cosh^2\tau_2-\cosh^2\tau_2}}\right]-\frac{\hbar^2}{2m} \ . \end{aligned} \tag{2.81}$$

For the path integral representation we obtain

$$\begin{aligned} &K^{(S_{3C})}(x'',x',\tau_1'',\tau_1',\tau_2'',\tau_2';T) \\ &= \int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\mathcal{D}\tau_2(t)\,\mathrm{e}^{2\mathrm{i}x}(\cosh^2\tau_1-\cosh^2\tau_2) \\ &\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T\left[\frac{m}{2}\dot{x}^2+\mathrm{e}^{2\mathrm{i}x}(\cosh^2\tau_1-\cosh^2\tau_2)(\dot{\tau}_1^2-\dot{\tau}_2^2)+\frac{\hbar^2}{2m}\right]\mathrm{d}t\right\} \ . \end{aligned} \tag{2.82}$$

We first consider the path integral in (τ_1, τ_2). It has exactly the form of the path integral representation of an elliptic coordinate system on the two-dimensional pseudo-Euclidean plane [12, 20, 26]. We can immediately use the corresponding result, and together with the x-path integration of the horicyclic-polar system this yields the final result

$$\begin{aligned} & K^{(S_{3C})}(x'', x', \tau_1'', \tau_1', \tau_2'', \tau_2'; T) \\ & = \frac{1}{8\pi} \int_0^\infty p dp \int_{\mathbb{R}} dk \ \mathrm{e}^{-\pi k} \mathrm{Me}_{\mathrm{i}k}(\tau_2''; \tfrac{p^2}{4}) \mathrm{Me}^*_{\mathrm{i}k}(\tau_2'; \tfrac{p^2}{4}) M^{(3)}_{\mathrm{i}k}(\tau_1''; \tfrac{p}{2}) M^{(3)\,*}_{\mathrm{i}k}(\tau_1'; \tfrac{p}{2}) \\ & \quad \times \sum_{J \in \mathbb{N}_0} \tfrac{1}{2} H^{(1)}_{J+1/2}(|k_\varrho| \ \mathrm{e}^{-\mathrm{i}x''}) H^{(1)}_{J+1/2}(|k_\varrho| \ \mathrm{e}^{\mathrm{i}x'}) \exp\left[-\frac{\mathrm{i}}{\hbar} \frac{\hbar^2 J(J+2)}{2m} T \right] . \end{aligned} \tag{2.83}$$

The wave-functions on S_{3C} are given by

$$\Psi_{Jpk}(x, \tau_1, \tau_2) = \sqrt{\frac{p}{16\pi}} \, \mathrm{e}^{-\pi k/2} \mathrm{Me}_{\mathrm{i}k}(\tau_2; \tfrac{p^2}{4}) M^{(3)}_{\mathrm{i}k}(\tau_1; \tfrac{p}{2}) \, \mathrm{e}^{-\mathrm{i}x} H^{(1)}_{J+1/2}(p \ \mathrm{e}^{-\mathrm{i}x}) \ , \tag{2.84}$$

and the energy-spectrum (2.8). $\mathrm{Me}_\nu(z)$ and $\mathrm{Me}^{(3)}_\nu(z)$ are Mathieu-functions [35], which are typical for the quantum motion in elliptic coordinates in two dimensions [11].

2.10 System 10: Horicyclic-Hyperbolic

This coordinate system is defined as

$$\left.\begin{aligned} z_1 &= \frac{1}{2}\Big[\mathrm{e}^{-\mathrm{i}x} + (1 + \mathrm{e}^{2y} - \mathrm{e}^{2z}) \, \mathrm{e}^{\mathrm{i}x} \Big] & z_2 &= \frac{\mathrm{i}}{\sqrt{2}} \Big[\sinh(y - z) + \mathrm{e}^{y+z} \Big] \mathrm{e}^{\mathrm{i}x} \\ z_3 &= \frac{1}{\sqrt{2}} \Big[\sinh(y - z) - \mathrm{e}^{y+z} \Big] \mathrm{e}^{\mathrm{i}x} & z_4 &= \frac{\mathrm{i}}{2} \Big[\mathrm{e}^{-\mathrm{i}x} + (-1 + \mathrm{e}^{2y} - \mathrm{e}^{2z}) \, \mathrm{e}^{\mathrm{i}x} \Big] \mathrm{e}^{\mathrm{i}x} \end{aligned} \right\} \tag{2.85}$$

$(x, y, z \in \mathbb{R})$. The set of commuting operators is given by

$$\mathcal{L}_1 = (I_{42} + \mathrm{i}I_{21})^2 + (I_{34} + \mathrm{i}I_{13})^2, \qquad \mathcal{L}_2 = I_{23}^2 - (I_{42} + I_{31} + \mathrm{i}I_{12} + \mathrm{i}I_{34})^2 \ . \tag{2.86}$$

The metric terms are

$$\left.\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{e}^{2\mathrm{i}x}(\mathrm{e}^{2y} + \mathrm{e}^{2z})(\mathrm{d}y^2 - \mathrm{d}z^2) \ , \\ \sqrt{g} &= \mathrm{e}^{2\mathrm{i}x}(\mathrm{e}^{2y} + \mathrm{e}^{2z}) \ , \\ \Gamma_x &= 2\mathrm{i}, \quad \Gamma_y = \frac{2\,\mathrm{e}^{2y}}{\mathrm{e}^{2y} + \mathrm{e}^{2z}}, \quad \Gamma_z = \frac{-2\,\mathrm{e}^{2z}}{\mathrm{e}^{2y} + \mathrm{e}^{2z}} \ . \end{aligned} \right\} \tag{2.87}$$

The momentum operators have the form

$$p_x = \frac{\hbar}{\mathrm{i}} \Big(\frac{\partial}{\partial x} + \mathrm{i} \Big), \quad p_y = \frac{\hbar}{\mathrm{i}} \Big(\frac{\partial}{\partial y} + \tfrac{1}{2}\Gamma_y \Big), \quad p_z = \frac{\hbar}{\mathrm{i}} \Big(\frac{\partial}{\partial z} + \tfrac{1}{2}\Gamma_z \Big) \ , \tag{2.88}$$

with the Hamiltonian given by

$$\begin{aligned} H &= -\frac{\hbar^2}{2m} \left[\frac{\partial^2}{\partial x^2} + 2\mathrm{i}\frac{\partial}{\partial \chi} + \frac{\mathrm{e}^{-2\mathrm{i}x}}{\mathrm{e}^{2y} + \mathrm{e}^{2z}} \left(\frac{\partial^2}{\partial y^2} + \Gamma_y \frac{\partial}{\partial y} - \frac{\partial^2}{\partial z^2} + \Gamma_z \frac{\partial}{\partial z} \right) \right] \\ &= \frac{1}{2m} \left[p_x^2 + \frac{\mathrm{e}^{-2\mathrm{i}x}}{\sqrt{\mathrm{e}^{2y} + \mathrm{e}^{2z}}} (p_y^2 - p_z^2) \frac{1}{\sqrt{\mathrm{e}^{2y} - \mathrm{e}^{2z}}} \right] - \frac{\hbar^2}{2m} \ . \end{aligned} \tag{2.89}$$

For the path integral we obtain

$$
\begin{aligned}
&K^{(S_{3C})}(x'',x',y'',y',z'',z';T)\\
&=\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\int\limits_{y(t')=y'}^{y(t'')=y''}\mathcal{D}y(t)\int\limits_{z(t')=z'}^{z(t'')=z''}\mathcal{D}z(t)\,\mathrm{e}^{2\mathrm{i}x}(\,\mathrm{e}^{2y}+\,\mathrm{e}^{2z})\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T\left[\frac{m}{2}\dot{x}^2+\,\mathrm{e}^{2\mathrm{i}x}(\,\mathrm{e}^{2y}+\,\mathrm{e}^{2z})(\dot{y}^2-\dot{z}^2)+\frac{\hbar^2}{2m}\right]\mathrm{d}t\right\}\ .
\end{aligned}
\tag{2.90}
$$

We start by considering the (y,z)-subpath integration. This two-dimensional sub-system corresponds to the second of the hyperbolic systems on the two-dimensional pseudo-Euclidean plane $E(1,1)$ [12, 20, 26], in particular [11]. The result is given by

$$
\begin{aligned}
\hat{K}^{(E(1,1))}(y'',y',z'',z';T)=\frac{2}{\pi^4}\int_0^\infty dk\,k\sinh\pi k\int_0^\infty dp\,p\\
\times K_{\mathrm{i}k}(\,\mathrm{e}^{y_1''}p)K_{\mathrm{i}k}(\,\mathrm{e}^{y_1'}p)K_{\mathrm{i}k}(-\mathrm{i}\,\mathrm{e}^{y_2'}p)K_{\mathrm{i}k}(\mathrm{i}\,\mathrm{e}^{y_2''}p)\,\,\mathrm{e}^{-\mathrm{i}\hbar p^2T/2m}.
\end{aligned}
\tag{2.91}
$$

Using the result of the x-path integration of the horicyclic-polar system we get finally

$$
\begin{aligned}
&K^{(S_{3C})}(x'',x',y'',y',z'',z';T)\\
&=\frac{2}{\pi^4}\int_0^\infty dk\,k\sinh\pi k\int_0^\infty dp\,pK_{\mathrm{i}k}(\,\mathrm{e}^{y''}p)K_{\mathrm{i}k}(\,\mathrm{e}^{y'}p)K_{\mathrm{i}k}(-\mathrm{i}\,\mathrm{e}^{z'}p)K_{\mathrm{i}k}(\mathrm{i}\,\mathrm{e}^{z''}p)\\
&\quad\times\sum_{J\in\mathbb{N}_0}\tfrac{1}{2}H^{(1)}_{J+1/2}(px\,\mathrm{e}^{-\mathrm{i}x''})H^{(1)}_{J+1/2}(p\,\mathrm{e}^{\mathrm{i}x'})\exp\left[-\frac{\mathrm{i}}{\hbar}\frac{\hbar^2J(J+2)}{2m}T\right]\ .
\end{aligned}
\tag{2.92}
$$

The wave-functions on S_{3C} are given by

$$
\Psi_{Jpk}(x,y,z)=\frac{\sqrt{pk\sinh\pi k}}{\pi^2}K_{\mathrm{i}k}(p\,\mathrm{e}^{y})K_{\mathrm{i}k}(-\mathrm{i}p\,\mathrm{e}^{z})\,\mathrm{e}^{-\mathrm{i}x}H^{(1)}_{J+1/2}(p\,\,\mathrm{e}^{-\mathrm{i}x})\ ,
\tag{2.93}
$$

and the energy-spectrum (2.8).

2.11 System 11: Horicyclic-Parabolic I

This coordinate system is defined as

$$
\left.
\begin{aligned}
z_1&=\tfrac{1}{2}\Big[\,\mathrm{e}^{-\mathrm{i}x}+(1+\tfrac{1}{4}(\xi^2+\eta^2)^2)\,\mathrm{e}^{\mathrm{i}x}\Big] & z_2&=\tfrac{\mathrm{i}}{2}(\xi^2-\eta^2)\,\mathrm{e}^{\mathrm{i}x}\\
z_3&=\mathrm{i}\xi\eta\,\mathrm{e}^{\mathrm{i}x} & z_4&=\tfrac{\mathrm{i}}{2}\Big[\,\mathrm{e}^{-\mathrm{i}x}+(-1+\tfrac{1}{4}(\xi^2+\eta^2)^2)\,\mathrm{e}^{\mathrm{i}x}\Big]
\end{aligned}
\right\}
\tag{2.94}
$$

$(x,\xi\in\mathbb{R},\eta>0)$, with the set of commuting operators given by

$$
\mathcal{L}_1=(I_{42}+\mathrm{i}I_{21})^2+(I_{34}+\mathrm{i}I_{13})^2,\qquad \mathcal{L}_2=\{I_{23},I_{42}+\mathrm{i}I_{21}\}\ .
\tag{2.95}
$$

The metric terms are

$$
\left.
\begin{aligned}
\mathrm{d}s^2&=\mathrm{d}x^2+\,\mathrm{e}^{2\mathrm{i}x}(\xi^2+\eta^2)(\,\mathrm{d}\xi^2+\,\mathrm{d}\eta^2)\ ,\\
\sqrt{g}&=\,\mathrm{e}^{2\mathrm{i}x}(\xi^2+\eta^2)\ ,\\
\Gamma_x&=2\mathrm{i},\quad \Gamma_\xi=\frac{2\xi}{\xi^2+\eta^2},\quad \Gamma_\eta=\frac{2\eta}{\xi^2+\eta^2}\ .
\end{aligned}
\right\}
\tag{2.96}
$$

The momentum operators read

$$p_x = \frac{\hbar}{\mathrm{i}}\Big(\frac{\partial}{\partial x} + \mathrm{i}\Big), \quad p_\xi = \frac{\hbar}{\mathrm{i}}\Big(\frac{\partial}{\partial \xi} + \frac{\xi}{\xi^2+\eta^2}\Big), \quad p_\eta = \frac{\hbar}{\mathrm{i}}\Big(\frac{\partial}{\partial \eta} + \frac{\eta}{\xi^2+\eta^2}\Big) \ , \tag{2.97}$$

and the Hamiltonian has the form

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2} + 2\mathrm{i}\frac{\partial}{\partial \chi} + \frac{\mathrm{e}^{-2\mathrm{i}x}}{\xi^2+\eta^2}\left(\frac{\partial^2}{\partial \xi^2} + \frac{2\xi}{\xi^2+\eta^2}\frac{\partial}{\partial \xi} + \frac{\partial^2}{\partial z^2} + \frac{2\eta}{\xi^2+\eta^2}\frac{\partial}{\partial \eta}\right)\right] \\ &= \frac{1}{2m}\left[p_x^2 + \frac{\mathrm{e}^{-2\mathrm{i}x}}{\sqrt{\xi^2+\eta^2}}(p_\xi^2+p_\eta^2)\frac{1}{\sqrt{\xi^2+\eta^2}}\right] - \frac{\hbar^2}{2m} \ . \end{aligned} \tag{2.98}$$

For the path integral representation we obtain

$$\begin{aligned} &K(x'',x',\xi'',\xi',\eta'',\eta';T) \\ &= \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t) \int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t)\, \mathrm{e}^{2\mathrm{i}x}(\xi^2+\eta^2) \\ &\quad \times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T \left[\frac{m}{2}\Big(\mathrm{d}x^2 + \mathrm{e}^{2\mathrm{i}x}(\xi^2+\eta^2)(\mathrm{d}\xi^2+\mathrm{d}\eta^2)\Big) + \frac{\hbar^2}{2m}\right]\mathrm{d}t\right\} \\ &= \int_{\mathbb{R}} d\zeta \int_{\mathbb{R}} \frac{dp}{32\pi^4} \\ &\quad \times \left\{ \begin{array}{l} |\Gamma(\frac{1}{4}+\frac{\mathrm{i}\zeta}{2p})|^2 E^{(0)}_{-1/2+\mathrm{i}\zeta/p}(\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2p}\,\xi'') E^{(0)}_{-1/2-\mathrm{i}\zeta/p}(\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2p}\,\eta'') \\ |\Gamma(\frac{3}{4}+\frac{\mathrm{i}\zeta}{2p})|^2 E^{(1)}_{-1/2+\mathrm{i}\zeta/p}(\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{2p}\,\xi'') E^{(1)}_{-1/2-\mathrm{i}\zeta/p}(\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{2p}\,\eta'') \end{array} \right\} \\ &\quad \times \left\{ \begin{array}{l} |\Gamma(\frac{1}{4}+\frac{\mathrm{i}\zeta}{2p})|^2 E^{(0)}_{-1/2-\mathrm{i}\zeta/p}(\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{2p}\,\xi') E^{(0)}_{-1/2+\mathrm{i}\zeta/p}(\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{2p}\,\eta') \\ |\Gamma(\frac{3}{4}+\frac{\mathrm{i}\zeta}{2p})|^2 E^{(1)}_{-1/2-\mathrm{i}\zeta/p}(\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2p}\,\xi') E^{(1)}_{-1/2+\mathrm{i}\zeta/p}(\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2p}\,\eta') \end{array} \right\} \\ &\quad \times \sum_{J\in\mathbb{N}_0} \tfrac{1}{2} H^{(1)}_{J+1/2}(p\,\mathrm{e}^{-\mathrm{i}x''}) H^{(1)}_{J+1/2}(p\,\mathrm{e}^{\mathrm{i}x'}) \exp\left[-\frac{\mathrm{i}}{\hbar}\frac{\hbar^2 J(J+2)}{2m}T\right] \ . \end{aligned} \tag{2.99}$$

In this solution we have exploited the path integral representation on the two-dimensional Euclidean plane in parabolic coordinates [11, 12], in particular [11] for details. The wave-functions on S_{3C} $E^{(0)}_{-1/2-\mathrm{i}\zeta/p}(z)$ are of even parity, whereas the wave-functions $E^{(1)}_{-1/2-\mathrm{i}\zeta/p}(z)$ are of odd parity. The wave-functions on S_{3C} for even and odd parity, respectively, are given by

$$\begin{aligned} &\Psi_{Jp\zeta}(x,\xi,\eta) = \frac{1}{8\pi^2} \\ &\quad \times \left\{ \begin{array}{l} |\Gamma(\frac{1}{4}+\frac{\mathrm{i}\zeta}{2p})|^2 E^{(0)}_{-1/2+\mathrm{i}\zeta/p}(\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2p}\,\xi) E^{(0)}_{-1/2-\mathrm{i}\zeta/p}(\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2p}\,\eta) \\ |\Gamma(\frac{3}{4}+\frac{\mathrm{i}\zeta}{2p})|^2 E^{(1)}_{-1/2+\mathrm{i}\zeta/p}(\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{2p}\,\xi) E^{(1)}_{-1/2-\mathrm{i}\zeta/p}(\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{2p}\,\eta) \end{array} \right\} \\ &\quad \times \mathrm{e}^{-\mathrm{i}x} H^{(1)}_{J+1/2}(p\ \mathrm{e}^{-ix}) \ , \end{aligned} \tag{2.100}$$

and the energy-spectrum (2.8).

2.12 System 12: Horicyclic-Parabolic II

This coordinate system is defined as

$$\left.\begin{aligned} z_1 &= \tfrac{1}{2}\Big\{ \mathrm{e}^{-\mathrm{i}x} + [1+2(\xi-\eta)^2(\xi+\eta)]\,\mathrm{e}^{\mathrm{i}x}\Big\} & z_2 &= \mathrm{i}[\tfrac{1}{2}(\xi-\eta)^2 + (\xi+\eta)]\,\mathrm{e}^{\mathrm{i}x} \\ z_3 &= [\tfrac{1}{2}(\xi-\eta)^2 - (\xi+\eta)]\,\mathrm{e}^{\mathrm{i}x} & z_4 &= \tfrac{1}{2}\Big\{ \mathrm{e}^{-\mathrm{i}x} + [-1+2(\xi-\eta)^2(\xi+\eta)]\,\mathrm{e}^{\mathrm{i}x}\Big\}. \end{aligned}\right\} \tag{2.101}$$

The set of commuting operators is given by

$$\mathcal{L}_1 = (I_{12}+\mathrm{i}I_{21})^2+(I_{34}+\mathrm{i}I_{13})^2, \qquad \mathcal{L}_2 = \{I_{23}, I_{42}+I_{31}+\mathrm{i}I_{21}+\mathrm{i}I_{34}\}-\mathrm{i}(I_{42}-I_{31}+\mathrm{i}I_{21}-\mathrm{i}I_{34})^2 \ . \tag{2.102}$$

The metric terms are

$$\left.\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}x^2 + 4\,\mathrm{e}^{2\mathrm{i}x}(\xi-\eta)(\,\mathrm{d}\xi^2 - \mathrm{d}\eta^2)\ , \\ \sqrt{g} &= 4\,\mathrm{e}^{2\mathrm{i}x}(\xi-\eta)\ , \\ \Gamma_x &= 2\mathrm{i}, \quad \Gamma_\xi = \frac{1}{\xi-\eta}, \quad \Gamma_\eta = \frac{-1}{\xi-\eta}\ . \end{aligned}\right\} \tag{2.103}$$

The momentum operators have the form

$$p_x = \frac{\hbar}{\mathrm{i}}\Big(\frac{\partial}{\partial x}+\mathrm{i}\Big), \quad p_\xi = \frac{\hbar}{\mathrm{i}}\Big(\frac{\partial}{\partial \xi}+\frac{1}{2(\xi-\eta)}\Big), \quad p_\eta = \frac{\hbar}{\mathrm{i}}\Big(\frac{\partial}{\partial \eta}-\frac{1}{2(\xi-\eta)}\Big)\ , \tag{2.104}$$

and the Hamiltonian reads

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2}+2\mathrm{i}\frac{\partial}{\partial \chi}+\frac{\mathrm{e}^{-2\mathrm{i}x}}{\xi-\eta}\left(\frac{\partial^2}{\partial \xi^2}+\frac{1}{\xi-\eta}\frac{\partial}{\partial \xi}-\frac{\partial^2}{\partial \eta^2}+\frac{1}{\xi-\eta}\frac{\partial}{\partial \eta}\right)\right] \\ &= \frac{1}{2m}\left[p_x^2+\frac{\mathrm{e}^{-2\mathrm{i}x}}{\sqrt{\xi-\eta}}(p_\xi^2-p_\eta^2)\frac{1}{\sqrt{\xi-\eta}}\right]-\frac{\hbar^2}{2m}\ . \end{aligned} \tag{2.105}$$

For the path integral we find

$$\begin{aligned} &K^{(S_{3C})}(x'',x',\xi'',\xi',\eta'',\eta';T) \\ &= \int_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\int_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathcal{D}\xi(t)\int_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)4\,\mathrm{e}^{2\mathrm{i}x}(\xi-\eta) \\ &\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T\left[\frac{m}{2}\dot{x}^2+4\,\mathrm{e}^{2\mathrm{i}x}(\xi-\eta)(\dot{\xi}^2-\dot{\eta}^2)+\frac{\hbar^2}{2m}\right]\mathrm{d}t\right\}\ . \end{aligned} \tag{2.106}$$

The (ξ,η)-subpath integration corresponds to the path integration of the third parabolic system on the two-dimensional pseudo-Euclidean plane [12, 20, 26]. Using the result of [12] and the horicyclic system we get

$$\begin{aligned} &K^{(S_{3C})}(x'',x',\xi'',\xi',\eta'',\eta';T) \\ &= 16\int_0^\infty\frac{dp}{p^{1/3}}\int_{\mathbb{R}}d\zeta\,\mathrm{Ai}\left[-\left(\xi'+\sqrt{2m}\frac{\zeta}{p^2}\right)p^{2/3}\right]\mathrm{Ai}\left[-\left(\xi''+\sqrt{2m}\frac{\zeta}{p^2}\right)p^{2/3}\right] \end{aligned}$$

$$\times \mathrm{Ai}\Big[-\Big(\eta' + \sqrt{2m}\frac{\zeta}{p^2}\Big)p^{2/3}\Big]\mathrm{Ai}\Big[-\Big(\eta'' + \sqrt{2m}\frac{\zeta}{p^2}\Big)p^{2/3}\Big]$$
$$\times \sum_{J\in\mathbb{N}_0} \tfrac{1}{2}H^{(1)}_{J+1/2}(p\,\mathrm{e}^{-\mathrm{i}x''})H^{(1)}_{J+1/2}(px\,\mathrm{e}^{\mathrm{i}x'})\exp\Big[-\frac{\mathrm{i}}{\hbar}\frac{\hbar^2 J(J+2)}{2m}T\Big] \ . \qquad (2.107)$$

Here, the Ai are Airy-functions [9]. The wave-functions on S_{3C} are given by

$$\Psi_{Jp\zeta}(x,\xi,\eta) = \frac{4}{p^{1/6}}\mathrm{Ai}\Big[-\Big(\xi + \sqrt{2m}\frac{\zeta}{p^2}\Big)p^{2/3}\Big]\mathrm{Ai}\Big[-\Big(\eta + \sqrt{2m}\frac{\zeta}{p^2}\Big)p^{2/3}\Big]$$
$$\times\frac{1}{\sqrt{2}}\ \mathrm{e}^{-\mathrm{i}x}H^{(1)}_{J+1/2}(p\ \mathrm{e}^{-\mathrm{i}x}) \ , \qquad (2.108)$$

and the energy-spectrum (2.8).

3 The Path Integral Representations: Part II

3.1 System 13: Elliptic-Cylindrical

We now come the those coordinate systems which do not have a subgroup structure. There are nine of them, and we can find for six of these cases a path integral representation.

This coordinate system of the elliptic-cylindrical type is defined as

$$\left.\begin{array}{ll} z_1 = k\,\mathrm{sn}(\alpha,k)\,\mathrm{sn}(\beta,k) & z_2 = -\mathrm{i}\dfrac{k}{k'}\,\mathrm{cn}(\alpha,k)\,\mathrm{cn}(\beta,k)\cos\varphi \\ z_3 = -\mathrm{i}\dfrac{k}{k'}\,\mathrm{cn}(\alpha,k)\,\mathrm{cn}(\beta,k)\sin\varphi & z_4 = \dfrac{1}{k'}\,\mathrm{dn}(\alpha,k)\,\mathrm{dn}(\beta,k) \end{array}\right\} \ , \qquad (3.1)$$

The set of commuting operators is given by

$$\mathcal{L}_1 = I_{23}^2, \qquad \mathcal{L}_2 = I_{12}^2 + I_{13}^2 + kI_{14}^2 \ . \qquad (3.2)$$

and the metric terms are

$$\left.\begin{array}{l} \mathrm{d}s^2 = -k^2(\,\mathrm{sn}^2\alpha - \,\mathrm{sn}^2\beta)(\,\mathrm{d}\alpha^2 + \,\mathrm{d}\beta^2) + \dfrac{k^2}{k'^2}\,\mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta\,\mathrm{d}\varphi^2 \ , \\ \sqrt{g} = k^2(\,\mathrm{sn}^2\alpha - \,\mathrm{sn}^2\beta)\dfrac{k}{k'}\ \mathrm{cn}\alpha\,\mathrm{cn}\beta \ , \\ \Gamma_\alpha = -\dfrac{2k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\alpha\,\mathrm{dn}\alpha}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta} + \dfrac{\mathrm{cn}\alpha\,\mathrm{dn}\alpha}{\mathrm{sn}\alpha}, \quad \Gamma_\beta = -\dfrac{2k'^2\,\mathrm{sn}\beta\,\mathrm{cn}\beta\,\mathrm{dn}\beta}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta} - k'^2\dfrac{\mathrm{sn}\beta\,\mathrm{cn}\beta}{\mathrm{dn}\beta} \ , \end{array}\right\} \qquad (3.3)$$

and $\Gamma_\varphi = 0$. In [12, 17] we have constructed a kernel for the prolate elliptic coordinate system on the sphere $S^{(3)}$. We used the definition for the coordinates ($a_1 \le \rho_1 \le a_2 \le \rho_2 \le a_3$, algebraic from)

$$\left.\begin{array}{l} s_1^2 = R^2\dfrac{(\rho_1-a_2)(\rho_2-a_2)}{(a_3-a_2)(a_1-a_2)}\cos^2\varphi \ , \\ s_2^2 = R^2\dfrac{(\rho_1-a_2)(\rho_2-a_2)}{(a_3-a_2)(a_1-a_2)}\sin^2\varphi \ , \\ s_3^2 = R^2\dfrac{(\rho_1-a_1)(\rho_2-a_1)}{(a_2-a_1)(a_3-a_1)} \ , \\ s_4^2 = R^2\dfrac{(\rho_1-a_3)(\rho_2-a_3)}{(a_2-a_3)(a_1-a_3)} \ . \end{array}\right\} \qquad (3.4)$$

In terms of the Jacobi elliptic functions we have ($-K \le \alpha \le K, -2K' \le \beta \le 2K', 0 \le \varphi < 2\pi$)

$$\left.\begin{aligned} s_1 &= R\,\mathrm{cn}(\alpha,k)\,\mathrm{cn}(\beta,k')\cos\varphi \ , \\ s_2 &= R\,\mathrm{cn}(\alpha,k)\,\mathrm{cn}(\beta,k')\sin\varphi \ , \\ s_3 &= R\,\mathrm{sn}(\alpha,k)\,\mathrm{dn}(\beta,k') \ , \\ s_4 &= R\,\mathrm{dn}(\alpha,k)\,\mathrm{sn}(\beta,k') \ . \end{aligned}\right\} \tag{3.5}$$

The momentum operators are given by

$$p_\alpha = \frac{\hbar}{\mathrm{i}}\Big(\frac{\partial}{\partial\alpha} + \tfrac{1}{2}\Gamma_\alpha\Big), \quad p_\beta = \frac{\hbar}{\mathrm{i}}\Big(\frac{\partial}{\partial\beta} + \tfrac{1}{2}\Gamma_\alpha\Big), \quad p_\varphi = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\varphi} \ . \tag{3.6}$$

Therefore we have for the Hamiltonian

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}\Bigg[\frac{1}{(k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta)}\Big(\frac{\partial^2}{\partial\alpha^2} + \frac{\mathrm{cn}\alpha\,\mathrm{dn}\alpha}{\mathrm{sn}\alpha}\frac{\partial}{\partial\alpha} \\ &\qquad + \frac{\partial^2}{\partial\beta^2} - k'^2\frac{\mathrm{sn}\beta\,\mathrm{cn}\beta}{\mathrm{dn}\beta}\frac{\partial}{\partial\beta}\Big) + \frac{1}{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\beta}\frac{\partial^2}{\partial\varphi^2}\Bigg] \qquad (3.7) \\ &= \frac{\hbar^2}{2m}\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta}}(p_\alpha^2 + p_\beta^2)\frac{1}{\sqrt{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta}} \\ &\qquad -\frac{\hbar^2}{8m}\Bigg[4 + \frac{1}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta}\Big(\frac{\mathrm{cn}^2\alpha\,\mathrm{dn}^2\alpha}{\mathrm{sn}^2\alpha} + k'^4\frac{\mathrm{sn}^2\beta\,\mathrm{cn}^2\beta}{\mathrm{dn}^2\beta}\Big)\Bigg] \ . \qquad (3.8) \end{aligned}$$

We found the following representation [12] ($\alpha \in [-K,K], \beta \in [-K',K'], \varphi \in [0,2\pi), a \in [-1,0]$):

$$\begin{aligned} &K^{(S_{3C})}(\alpha'',\alpha',\beta'',\beta',\varphi'',\varphi';T) \\ &= \int_{\alpha(t')=\alpha'}^{\alpha(t'')=\alpha''}\mathcal{D}\alpha(t)\int_{\beta(t')=\beta'}^{\beta(t'')=\beta''}\mathcal{D}\beta(t)(k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta)\,\mathrm{cn}\alpha\,\mathrm{cn}\beta\int_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t) \\ &\quad\times\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\Bigg[\frac{m}{2}\Big((k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta)(\dot\alpha^2 + \dot\beta^2) + \mathrm{cn}^2\alpha\,\mathrm{cn}^2\beta\dot\varphi^2\Big) \\ &\qquad + \frac{\hbar^2}{8m}\frac{1}{k^2\,\mathrm{cn}^2\alpha + k'^2\,\mathrm{cn}^2\beta}\Big(\frac{\mathrm{sn}^2\beta\,\mathrm{dn}^2\beta}{\mathrm{cn}^2\beta} + k^4\frac{\mathrm{sn}^2\alpha\,\mathrm{dn}^2\alpha}{\mathrm{cn}^2\alpha}\Big)\Bigg]dt + \frac{\mathrm{i}\hbar T}{2m}\Bigg\} \\ &= \frac{1}{2\pi}\sum_{J=0}^{\infty}\sum_{r,p=\pm1}\sum_{qk_2}\mathrm{e}^{\mathrm{i}k_2(\varphi''-\varphi')}\,\mathrm{e}^{-2\mathrm{i}\hbar TJ(J+2)/2m} \\ &\quad\times\psi^{(r,p)}_{1,Jqk_2}(\alpha'';a)\psi^{(r,p)\,*}_{1,Jqk_2}(\alpha';a)\psi^{(r,p)}_{2,Jqk_2}(\beta'';a)\psi^{(r,p)\,*}_{2,Jqk_2}(\beta';a) \ . \qquad (3.9) \end{aligned}$$

This representation can be derived by a group path integration together with an interbasis expansion from the cylindrical or spherical basis to the ellipso-cylindrical ones. For instance, one has [16] ($r,p = \pm1$)

$$\Psi^{(r,p)}_{J,k_1,k_2}(\vartheta,\varphi_1,\varphi_2) = \sum_q T^{(r,p)\,*}_{Jqk_2}\Psi^{(r,p)}_{Jqk_2}(\alpha,\beta,\varphi;a) \ , \tag{3.10}$$

where the $a \geq 0$ corresponds to the oblate elliptic case (which we do not discuss here), and $a \in [-1, 0]$ to the prolate elliptic system, respectively. We have the factorization $\Psi(\alpha, \beta, \varphi; a) = \psi_1(\alpha; a)\psi_2(\beta; a)\, \mathrm{e}^{\mathrm{i}k_2\varphi}/\sqrt{2\pi}$. For the associated Lamé polynomials ψ_{i,Jqk_2}, $i = 1, 2$, we adopt the notations of [16, 28]. The relevant quantum numbers have the following meaning: The functions $\psi^{(r,p)}_{i,Jqk_2}(z)$ are called *associated Lamé polynomials* and satisfy the associated Lamé equation. We take them for normalized. For the principal quantum number we have $l \in \mathbb{N}_0$. $(r, p) = \pm 1$ denotes one of the four parity classes of solutions of dimension $(J + 1)^2$, i.e., the multiplicity of the degeneracy of the level J, one for each class of the corresponding recurrence relations as given in [16, 28] and the parity classes from the periodic Lamé functions Λ^p_{lh} from the spherical harmonics on the sphere can be applied. These expansions have been considered in [16, 28] together with three-term recurrence relations for the interbasis coefficients. They can be determined by taking into account that a basis in O(4) is related in a unique way to the cylindrical and spherical bases on $S^{(3)}$ by using the properties of the elliptic operator Λ on $S^{(3)}$ with eigenvalue q. Details can be found in [16, 28]. Due to the unitarity of these coefficients the path integration is then performed by inserting in each short-time kernel in the cylindrical system first the expansions (3.10), second, exploiting the unitarity, and thus yielding the result (3.9). The elliptic-cylindrical system exists on the three-dimensional sphere (oblate and prolate spheroidal (System IV and V.), on the three-dimensional hyperboloid (System XVII. and XVIII – prolate and oblate elliptic).

3.2 System 14: Elliptic-Parabolic

This coordinate system is defined as

$$\left.\begin{aligned} z_1 &= \frac{1}{2}\left(\frac{\cosh\tau_1}{\cosh\tau_2} + \frac{\cosh\tau_2}{\cosh\tau_1}\right) \qquad z_2 = \tanh\tau_1 \tanh\tau_2 \cosh\tau_3 \\ z_3 &= -\mathrm{i}\tanh\tau_1 \tanh\tau_2 \sinh\tau_3 \quad z_4 = \frac{-\mathrm{i}}{\cosh\tau_1 \cosh\tau_2} + \frac{\mathrm{i}}{2}\left(\frac{\cosh\tau_1}{\cosh\tau_2} + \frac{\cosh\tau_2}{\cosh\tau_1}\right) \end{aligned}\right\} , \tag{3.11}$$

$(\tau_1, \tau_2 > 0, \tau_3 \in \mathbb{R})$. The set of commuting operators is given by

$$\mathcal{L}_1 = I^2_{23}, \qquad \mathcal{L}_2 = I^2_{24} + I^2_{34} - I^2_{12} + I^2_{13} - I^2_{14} - \mathrm{i}\{I_{12}, I_{42}\} - \mathrm{i}\{I_{13}, I_{43}\} \ . \tag{3.12}$$

The metric terms are

$$\left.\begin{aligned} \mathrm{d}s^2 &= (\tanh^2\tau_1 - \tanh^2\tau_2)(\,\mathrm{d}\tau_1^2 - \,\mathrm{d}\tau_2^2) + \tanh^2\tau_1 \tanh^2\tau_2\, \mathrm{d}\tau_3^2 \ , \\ \sqrt{g} &= (\tanh^2\tau_1 - \tanh^2\tau_2)\tanh\tau_1 \tanh\tau_2 \ , \\ \Gamma_{\tau_{1,2}} &= \pm\frac{2\sinh\tau_{1,2}}{\cosh^3\tau_{1,2}} \frac{1}{\tanh^2\tau_1 - \tanh^2\tau_2} + \frac{1}{\cosh^2\tau_{1,2}}, \quad \Gamma_{\tau_3} = 0 \ . \end{aligned}\right\} \tag{3.13}$$

The momentum operators have the form

$$p_{\tau_1} = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\tau_1} + \tfrac{1}{2}\Gamma_{\tau_1}\right), \quad p_{\tau_2} = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\tau_2} + \tfrac{1}{2}\Gamma_{\tau_2}\right), \quad p_{\tau_3} = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\tau_3} \ , \tag{3.14}$$

with the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\left[\frac{1}{\tanh^2\tau_1 - \tanh^2\tau_2}\left(\frac{\partial^2}{\partial\tau_1^2} + \Gamma_{\tau_1}\frac{\partial}{\partial\tau_1} - \frac{\partial^2}{\partial\tau_2^2} + \Gamma_{\tau_2}\frac{\partial}{\partial\tau_2}\right) + \frac{1}{\tanh^2\tau_1 \tanh^2\tau_2}\frac{\partial^2}{\partial\tau_3^2}\right]$$

$$= \frac{1}{2m}\left[\frac{1}{\sqrt{\tanh^2\tau_1-\tanh^2\tau_2}}(p_{\tau_1}^2-p_{\tau_2}^2)\frac{1}{\sqrt{\tanh^2\tau_1-\tanh^2\tau_2}}+\frac{p_{\tau_3}^2}{\tanh^2\tau_1\tanh^2\tau_2}\right]+\frac{3\hbar^2}{8m}. \tag{3.15}$$

The solution cannot be directly derived from one of the lower-dimensional subcases. For the path integral we find by separating off the τ_3-path integration which corresponds to plane waves

$$\begin{aligned}
&K^{(S_{3C})}(\tau_1'',\tau_1',\tau_2'',\tau_2',\tau_3'',\tau_3';T)\\
&=\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\mathcal{D}\tau_2(t)\int\limits_{\tau_3(t')=\tau_3'}^{\tau_3(t'')=\tau_3''}\mathcal{D}\tau_3(t)(\tanh^2\tau_1-\tanh^2\tau_2)\tanh\tau_1\tanh\tau_2\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T\left[\frac{m}{2}\left((\tanh^2\tau_1-\tanh^2\tau_2)(\dot\tau_1^2-\dot\tau_2^2)+\tanh^2\tau_1\tanh^2\tau_2\dot\tau_3^2\right)-\frac{3\hbar^2}{8m}\right]\mathrm{d}t\right\}\\
&=(\tanh\tau_1'\tanh\tau_1''\tanh\tau_2'\tanh\tau_2'')^{-1/2}\int_{\mathbb{R}}\mathrm{d}k_{\tau_3}\frac{\mathrm{e}^{\mathrm{i}k_{\tau_3}(\tau_3''-\tau_3')}}{2\pi}\\
&\quad\times\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\int\limits_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\mathcal{D}\tau_2(t)(\tanh^2\tau_1-\tanh^2\tau_2)\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T\left[\frac{m}{2}(\tanh^2\tau_1-\tanh^2\tau_2)(\dot\tau_1^2-\dot\tau_2^2)-\frac{\hbar^2k_{\tau_3}^2}{2m}\coth^2\tau_1\coth^2\tau_2-\frac{3\hbar^2}{8m}\right]\mathrm{d}t\right\}\\
&=(\tanh\tau_1'\tanh\tau_1''\tanh\tau_2'\tanh\tau_2'')^{-1/2}\\
&\quad\times\int_{\mathbb{R}}\mathrm{d}k_{\tau_3}\frac{\mathrm{e}^{\mathrm{i}k_{\tau_3}(\tau_3''-\tau_3')}}{2\pi}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int_0^\infty\mathrm{d}s''K_k(\tau_1'',\tau_1',\tau_2'',\tau_2';s'')\ ,
\end{aligned}\tag{3.16}$$

with the space-time transformed path integral $K_k(s'')$ given by:

$$\begin{aligned}
K_k(\tau_1'',\tau_1',\tau_2'',\tau_2';s'')=&\int\limits_{\tau_1(0)=\tau_1'}^{\tau_1(s'')=\tau_1''}\mathcal{D}\tau_1(s)\int\limits_{\tau_2(0)=\tau_2'}^{\tau_2(s'')=\tau_2''}\mathcal{D}\tau_2(s)\\
&\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}(\dot\tau_1^2-\dot\tau_2^2)+\left(E+\frac{3\hbar^2}{8m}\right)\left(\frac{1}{\cosh^2\tau_1}-\frac{1}{\cosh^2\tau_2}\right)\right.\right.\\
&\left.\left.+\frac{\hbar^2k_{\tau_3}^2}{2m}\left(\frac{1}{\sinh^2\tau_1}-\frac{1}{\sinh^2\tau_2}\right)\right]\mathrm{d}s\right\}.
\end{aligned}\tag{3.17}$$

For this path integral we now use the path integral representation of the modified Pöschl–Teller potential

$$V^{(mPT)}(\tau)=\frac{\hbar^2}{2m}\left(\frac{\eta^2-\frac{1}{4}}{\sinh^2\tau}-\frac{\nu^2-\frac{1}{4}}{\cosh^2\tau}\right)\ . \tag{3.18}$$

We therefore obtain with $\eta=\mathrm{i}\sqrt{k_{\tau_3}^2-\frac{1}{4}}$ and $\nu=2mE/\hbar^2+1\equiv p^2+1$

$$K^{(S_{3C})}(\tau_1'',\tau_1',\tau_2'',\tau_2',\tau_3'',\tau_3';T)=(\tanh\tau_1'\tanh\tau_1''\tanh\tau_2'\tanh\tau_2'')^{-1/2}$$

$$\times \int_{\mathbb{R}} \mathrm{d}k \frac{\mathrm{e}^{\mathrm{i}k_{\tau_3}(\tau_3''-\tau_3')}}{2\pi} \int_0^\infty dk \int_0^\infty dp \, \exp\left[-\frac{\mathrm{i}\hbar T}{2m}(p^2+1)\right]$$
$$\times \Psi_k^{(\eta,\nu)}(\tau_2'')\Psi_k^{(\eta,\nu)\,*}(\tau_2')\Psi_k^{(\eta,\nu)}(\tau_1'')\Psi_k^{(\eta,\nu)\,*}(\tau_1') \ . \tag{3.19}$$

In this solution we have explicitly $J(J+2) \equiv (p^2+1)$. The $\Psi_p^{(\mu,\nu)}(\omega)$ are the modified Pöschl–Teller functions, which are given by

$$\begin{aligned}
\Psi_n^{(\eta,\nu)}(r) &= N_n^{(\eta,\nu)}(\sinh r)^{2k_2-\frac{1}{2}}(\cosh r)^{-2k_1+\frac{3}{2}} \\
&\quad \times {}_2F_1(-k_1+k_2+\kappa, -k_1+k_2-\kappa+1; 2k_2; -\sinh^2 r) \qquad (3.20)\\
N_n^{(\eta,\nu)} &= \frac{1}{\Gamma(2k_2)}\left[\frac{2(2\kappa-1)\Gamma(k_1+k_2-\kappa)\Gamma(k_1+k_2+\kappa-1)}{\Gamma(k_1-k_2+\kappa)\Gamma(k_1-k_2-\kappa+1)}\right]^{1/2} . \qquad (3.21)
\end{aligned}$$

The scattering states are given by:

$$\begin{aligned}
V(r) &= \frac{\hbar^2}{2m}\left(\frac{\eta^2-\frac{1}{4}}{\sinh^2 r} - \frac{\nu^2-\frac{1}{4}}{\cosh^2 r}\right) \\
\Psi_p^{(\eta,\nu)}(r) &= N_p^{(\eta,\nu)}(\cosh r)^{2k_1-\frac{1}{2}}(\sinh r)^{2k_2-\frac{1}{2}} \\
&\quad \times {}_2F_1(k_1+k_2-\kappa, k_1+k_2+\kappa-1; 2k_2; -\sinh^2 r) \qquad (3.22)\\
N_p^{(\eta,\nu)} &= \frac{1}{\Gamma(2k_2)}\sqrt{\frac{p\sinh\pi p}{2\pi^2}}\Big[\Gamma(k_1+k_2-\kappa)\Gamma(-k_1+k_2+\kappa) \\
&\quad \times \Gamma(k_1+k_2+\kappa-1)\Gamma(-k_1+k_2-\kappa+1)\Big]^{1/2} , \qquad (3.23)
\end{aligned}$$

k_1, k_2 defined by: $k_1 = \frac{1}{2}(1 \pm \nu)$, $k_2 = \frac{1}{2}(1 \pm \eta)$, where the correct sign depends on the boundary-conditions for $r \to 0$ and $r \to \infty$, respectively. The number N_M denotes the maximal number of states with $0, 1, \ldots, N_M < k_1 - k_2 - \frac{1}{2}$. $\kappa = k_1 - k_2 - n$ for the bound states and $\kappa = \frac{1}{2}(1 + \mathrm{i}p)$ for the scattering states. ${}_2F_1(a, b; c; z)$ is the hypergeometric function [9, p.1057]. The bound states are needed for the bound states on the O(2,2) hyperboloid.

3.3 System 15: Elliptic-Hyperbolic

This coordinate system is defined as

$$\left.\begin{aligned}
z_1 &= -\frac{1}{2}\left(\frac{\cosh\tau_2}{\cosh\tau_1} + \frac{\cosh\tau_1}{\cosh\tau_2}\right) - \frac{\tau_3^2}{2\cosh\tau_1\cosh\tau_2} \\
z_2 &= \frac{\mathrm{i}\tau_3}{\cosh\tau_1\cosh\tau_2} \\
z_3 &= \tanh\tau_1\tanh\tau_2 \\
z_4 &= \mathrm{i}\left[\frac{2-\tau_3^2}{2\cosh\tau_1\cosh\tau_2} - \frac{1}{2}\left(\frac{\cosh\tau_1}{\cosh\tau_2} + \frac{\cosh\tau_2}{\cosh\tau_1}\right)\right]
\end{aligned}\right\} , \tag{3.24}$$

$(\tau_1, \tau_2, \tau_3 \in \mathbb{R})$. The set of commuting operators is given by

$$\mathcal{L}_1 = (I_{42} + \mathrm{i}I_{21})^2, \qquad \mathcal{L}_2 = 2I_{12}^2 + I_{13}^2 + +I_{14}^2 - I_{34}^2 + \mathrm{i}(\{I_{12}, I_{42}\} + \{I_{13}, I_{43}\}) \ . \tag{3.25}$$

The metric terms are given by

$$\left.\begin{aligned}
\mathrm{d}s^2 &= \left(\frac{1}{\cosh^2\tau_1}-\frac{1}{\cosh^2\tau_2}\right)(\,\mathrm{d}\tau_1^2-\,\mathrm{d}\tau_2^2)+\frac{1}{\cosh^2\tau_1\cosh^2\tau_2}\,\mathrm{d}\tau_3^2\ ,\\
\sqrt{g} &= \left(\frac{1}{\cosh^2\tau_1}-\frac{1}{\cosh^2\tau_2}\right)\frac{1}{\cosh\tau_1\cosh\tau_2}\ ,\\
\Gamma_{\tau_1} &= \frac{-2\sinh\tau_1}{\cosh^3\tau_1}\frac{1}{1/\cosh^2\tau_1-1/\cosh^2\tau_2}-\tanh\tau_1\ ,\\
\Gamma_{\tau_2} &= \frac{2\sinh\tau_2}{\cosh^3\tau_2}\frac{1}{1/\cosh^2\tau_1-1/\cosh^2\tau_2}-\tanh\tau_2\ ,
\end{aligned}\right\} \tag{3.26}$$

and $\Gamma_{\tau_3}=0$. We have for the momentum operators

$$p_{\tau_1}=\frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\tau_1}+\tfrac{1}{2}\Gamma_{\tau_1}\right),\qquad p_{\tau_2}=\frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\tau_2}+\tfrac{1}{2}\Gamma_{\tau_2}\right),\qquad p_{\tau_3}=\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\tau_3}\ . \tag{3.27}$$

This gives for the Hamiltonian

$$\begin{aligned}
H &= -\frac{\hbar^2}{2m}\left[\frac{\cosh^2\tau_1\cosh^2\tau_2}{\cosh^2\tau_2-\cosh^2\tau_1}\left(\frac{\partial^2}{\partial\tau_1^2}-\tanh\tau_1\frac{\partial}{\partial\tau_1}-\frac{\partial^2}{\partial\tau_2^2}+\tanh\tau_2\frac{\partial}{\partial\tau_2}\right)\right.\\
&\qquad\left.+\cosh^2\tau_1\cosh^2\tau_2\frac{\partial^2}{\partial\tau_3^2}\right]\\
&= \frac{1}{2m}\left[\sqrt{\frac{\cosh^2\tau_1\cosh^2\tau_2}{\cosh^2\tau_2-\cosh^2\tau_1}}(p_{\tau_1}^2-p_{\tau_2}^2)\sqrt{\frac{\cosh^2\tau_1\cosh^2\tau_2}{\cosh^2\tau_2-\cosh^2\tau_1}}+\cosh^2\tau_1\cosh^2\tau_2 p_{\tau_3}^2\right]+\frac{3\hbar^2}{8m}.
\end{aligned} \tag{3.28}$$

For the path integral we find

$$\begin{aligned}
&K^{(S_{3C})}(\tau_1'',\tau_1',\tau_2'',\tau_2',\tau_3'',\tau_3';T)\\
&=\int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\int_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''}\mathcal{D}\tau_2(t)\int_{\tau_3(t')=\tau_3'}^{\tau_3(t'')=\tau_3''}\mathcal{D}\tau_3(t)\left(\frac{1}{\cosh^2\tau_1}-\frac{1}{\cosh^2\tau_2}\right)\frac{1}{\cosh\tau_1\cosh\tau_2}\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T\left[\frac{m}{2}\left(\frac{\cosh^2\tau_2-\cosh^2\tau_1}{\cosh^2\tau_1\cosh^2\tau_2}(\dot\tau_1^2-\dot\tau_2^2)+\frac{\dot\tau_3^2}{\cosh^2\tau_1\cosh^2\tau_2}\right)-\frac{3\hbar^2}{8m}\right]\mathrm{d}t\right\}\ .
\end{aligned} \tag{3.29}$$

The path integral for this coordinate system corresponds to the path integral for the first elliptic parabolic system on the three-dimensional hyperboloid in its complexified form. We have from [12] the path integral representation ($a>0$, $|\vartheta|<\pi/2$, $\varrho\in\mathbb{R}$):

$$\begin{aligned}
&\int_{a(t')=a'}^{a(t'')=a''}\mathcal{D}a(t)\int_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\frac{\cosh^2 a-\cos^2\vartheta}{\cosh^3 a\cos^3\vartheta}\int_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''}\mathcal{D}\varrho(t)\\
&\qquad\times\exp\left[\frac{\mathrm{i}m}{2\hbar}\int_{t'}^{t''}\frac{(\cosh^2 a-\cos^2\vartheta)(\dot a^2+\dot\vartheta^2)+\dot\varrho^2}{\cosh^2 a\cos^2\vartheta}dt-\frac{3\mathrm{i}\hbar T}{8m}\right]\\
&=\sqrt{\cosh a'\cosh a''}\,\cos\vartheta'\cos\vartheta''\int_{\mathbb{R}}\frac{dk_\varrho}{2\pi}\,\mathrm{e}^{\mathrm{i}k_\varrho(\varrho''-\varrho')}
\end{aligned}$$

$$\times \int_0^\infty dp\, \sinh \pi p \int_0^\infty \frac{dk\, k \sinh \pi k}{(\cosh^2 \pi k + \sinh^2 \pi p)^2} \exp\left[-\frac{\mathrm{i}\hbar T}{2m}(p^2+1)\right]$$
$$\times \sum_{\epsilon,\epsilon'=\pm 1} S^{\mathrm{i}k\,(1)}_{\mathrm{i}p-1/2}(\epsilon \tanh a''; \mathrm{i}k_\varrho) S^{\mathrm{i}k\,(1)\,*}_{\mathrm{i}p-1/2}(\epsilon \tanh a'; \mathrm{i}k_\varrho)$$
$$\times \mathrm{ps}^{\mathrm{i}p}_{\mathrm{i}k-1/2}(\epsilon' \sin \vartheta''; -k_\varrho^2)\, \mathrm{ps}^{\mathrm{i}p\,*}_{\mathrm{i}k-1/2}(\epsilon' \sin \vartheta'; -k_\varrho^2) \ . \tag{3.30}$$

The ps^ν_μ and $S^{\nu\,(1)}_\mu$ are spheroidal wave-functions. We can achieve the connection by the coordinate substitution $\vartheta = \mathrm{i}\tau_2$ and $\varrho = \tau_3$. Equation (3.30) is actually the solution to the original problem, with continuous spectrum as in the previous case, because its formulation comes from the complexification of the corresponding coordinate system on the three-dimensional hyperboloid, and we are done.

3.4 System 16: Parabolic

This coordinate system is defined as

$$\left.\begin{aligned} z_1 &= \frac{(\xi^2+\eta^2)^2+4}{8\xi\eta} + \frac{\tau^2}{2\xi\eta} & z_2 &= -\mathrm{i}\frac{\tau}{\xi\eta} \\ z_3 &= -\frac{\mathrm{i}}{2}\left(\frac{\xi}{\eta} - \frac{\eta}{\xi}\right) & z_4 &= \mathrm{i}\left(\frac{(\xi^2+\eta^2)^2-4}{8\xi\eta} + \frac{\tau^2}{2\xi\eta}\right) \end{aligned}\right\} \tag{3.31}$$

$(\xi, \eta > 0, \tau \in \mathbb{R})$. The set of commuting operators is given by

$$\mathcal{L}_1 = (I_{42} + \mathrm{i}I_{21})^2, \qquad \mathcal{L}_2 = \{I_{32}, I_{42} + \mathrm{i}I_{21}\} - \{I_{41}, \mathrm{i}I_{34} - I_{31}\} \ . \tag{3.32}$$

The metric terms are

$$\left.\begin{aligned} \mathrm{d}s^2 &= \left(\frac{1}{\xi^2} + \frac{1}{\eta^2}\right)(\mathrm{d}\xi^2 + \mathrm{d}\eta^2) + \frac{1}{\xi^2\eta^2}\,\mathrm{d}\tau^2 \ , \\ \sqrt{g} &= \left(\frac{1}{\xi^2} + \frac{1}{\eta^2}\right)\frac{1}{\xi\eta} \ , \\ \Gamma_\xi &= -\frac{2/\xi^3}{1\xi^2 + 1/\eta^2} - \frac{1}{\xi}, \quad \Gamma_\eta = -\frac{2/\eta^3}{1/\xi^2 + 1/\eta^2} - \frac{1}{\eta}, \quad \Gamma_\tau = 0 \ . \end{aligned}\right\} \tag{3.33}$$

We have for the momentum operators

$$p_\xi = = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\xi} + \frac{\xi}{\xi^2+\eta^2} - \frac{3}{2\xi}\right) \ , \tag{3.34}$$

$$p_\eta = = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\eta} + \frac{\eta}{\xi^2+\eta^2} - \frac{3}{2\eta}\right) \ , \tag{3.35}$$

$$p_\tau = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\tau} \ , \tag{3.36}$$

and the Hamiltonian is given by

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}\left[\frac{\xi^2\eta^2}{\xi^2+\eta^2}\left(\frac{\partial^2}{\partial\xi^2} - \frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial\eta^2} - \frac{1}{\eta}\frac{\partial}{\partial\eta}\right) + \xi^2\eta^2\frac{\partial^2}{\partial\tau^2}\right] \\ &= \frac{1}{2m}\left[\sqrt{\frac{\xi^2\eta^2}{\xi^2+\eta^2}}\,(p_\xi^2 + p_\eta^2)\sqrt{\frac{\xi^2\eta^2}{\xi^2+\eta^2}} + \xi^2\eta^2\,p_\tau^2\right] + \frac{3\hbar^2}{8m} \ . \end{aligned} \tag{3.37}$$

For the path integral we find by separating off the τ-path integration (plane waves)

$$
\begin{aligned}
&K^{(S_{3C})}(\xi'',\xi',\eta'',\eta',\tau'',\tau';T)\\
&=\int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathcal{D}\xi(t)\int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\frac{\xi^2+\eta^2}{\xi^3\eta^3}\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T\left[\frac{m}{2}\left(\frac{\xi^2+\eta^2}{\xi^2\eta^2}(\dot\xi^2+\dot\eta^2)+\frac{\dot\tau^2}{\xi^2\eta^2}\right)-\frac{3\hbar^2}{8m}\right]\mathrm{d}t\right\}\\
&=(\xi'\xi''\eta'\eta'')^{1/2}\int_{\mathbb{R}}\mathrm{d}k\frac{\mathrm{e}^{\mathrm{i}k(\tau''-\tau')}}{2\pi}\int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathcal{D}\xi(t)\int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)\frac{\xi^2+\eta^2}{\xi^2\eta^2}\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T\left[\frac{m}{2}\frac{\xi^2+\eta^2}{\xi^2\eta^2}(\dot\xi^2+\dot\eta^2)-\frac{\hbar^2k^2}{2m}\xi^2\eta^2-\frac{3\hbar^2}{8m}\right]\mathrm{d}t\right\}\\
&=(\xi'\xi''\eta'\eta'')^{1/2}\int_{\mathbb{R}}\mathrm{d}k\frac{\mathrm{e}^{\mathrm{i}k(\tau''-\tau')}}{2\pi}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int_0^\infty\mathrm{d}s''K_k(\xi'',\xi',\eta'',\eta';s'')\ ,
\end{aligned}
\tag{3.38}
$$

with the transformed path integral $K_k(s'')$ given by:

$$
\begin{aligned}
K_k(\xi'',\xi',\eta'',\eta';s'')&=\int\limits_{\xi(0)=\xi'}^{\xi(s'')=\xi''}\mathcal{D}\xi(s)\int\limits_{\eta(0)=\eta'}^{\eta(s'')=\eta''}\mathcal{D}\eta(s)\\
&\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\Big((\dot\xi^2+\dot\eta^2)-\omega^2(\xi^2+\eta^2)\Big)-\hbar^2\frac{\lambda^2-\frac14}{2m}\left(\frac{1}{\xi^2}+\frac{1}{\eta^2}\right)\right]\mathrm{d}s\right\}\ .
\end{aligned}
\tag{3.39}
$$

I have set $\omega=\hbar|k|/m$ and $\lambda=\pm\sqrt{1+2mE/\hbar^2}$. Each of the two path integrals in ξ and η are path integrals for the radial harmonic oscillator. We insert the expansions into the discrete wave-functions and obtain

$$
\begin{aligned}
K_k(\xi'',\xi',\eta'',\eta';s'')&=\frac{4m^2\omega^2}{\hbar^2}\sqrt{\xi'\xi''\eta'\eta''}\\
&\times\sum_{n_\xi,n_\eta}\frac{n_\xi!}{\Gamma(n_\xi+\lambda+1)}\frac{n_\eta!}{\Gamma(n_\eta+\lambda+1)}\left(\frac{m^2\omega^2}{\hbar^2}\xi'\xi''\eta'\eta''\right)^\lambda\mathrm{e}^{-2\mathrm{i}\omega(n_\xi+n_\eta+\lambda+1)s''}\\
&\times L_{n_\xi}^{(\lambda)}\left(\frac{m\omega}{\hbar}\xi'^2\right)L_{n_\xi}^{(\lambda)}\left(\frac{m\omega}{\hbar}\xi''^2\right)L_{n_\eta}^{(\lambda)}\left(\frac{m\omega}{\hbar}\eta'^2\right)L_{n_\eta}^{(\lambda)}\left(\frac{m\omega}{\hbar}\eta''^2\right)\\
&\times\exp\left[-\frac{m\omega}{2\hbar}(\xi'^2+\xi''^2+\eta'^2+\eta''^2)\right]\ .
\end{aligned}
\tag{3.40}
$$

Since $G(E)=\frac{\mathrm{i}}{\hbar}\int_0^\infty K(s'')\,\mathrm{d}s''$ we obtain by taking the minus sign in the square-root of λ by performing the s''-integration the energy-levels

$$
E_{n_\xi n_\eta}=\frac{\hbar^2}{2m}(n_\xi+n_\eta)(n_\xi+n_\eta+2)\equiv\frac{\hbar^2}{2m}J(J+2)\ ,
\tag{3.41}
$$

with $J=n_\xi+n_\eta$. Evaluating the residua and ordering factors therefore yields

$$
K^{(S_{3C})}(\xi'',\xi',\eta'',\eta',\tau'',\tau';T)=\int_{\mathbb{R}}\sum_{n_\xi,n_\eta}\Psi_{kn_\xi n_\eta}(\xi',\eta',x')\,\Psi^*_{kn_\xi n_\eta}(\xi'',\eta'',x'')\,\mathrm{e}^{-\mathrm{i}E_jT/\hbar}
\tag{3.42}
$$

with the wave-functions on S_{3C} given by

$$\begin{aligned}\Psi_{kn_\xi n_\eta}(\xi,\eta,\tau) &= \frac{\mathrm{e}^{\mathrm{i}k\tau}}{\sqrt{2\pi}}\sqrt{\hbar|k|\frac{\xi n_\xi!}{\Gamma(n_\xi+\lambda+1)}\frac{\eta n_\eta!}{\Gamma(n_\eta+\lambda+1)}}\left(\frac{m^2\omega^2}{\hbar^2}\xi\eta\right)^{J+1}\\ &\quad\times L_{n_\xi}^{(J+1)}\left(\frac{m\omega}{\hbar}\xi^2\right)L_{n_\eta}^{(J+1)}\left(\frac{m\omega}{\hbar}\eta^2\right)\exp\left[-\frac{m\omega}{2\hbar}(\xi^2+\eta^2)\right]\ ,\end{aligned}\tag{3.43}$$

and the energy-spectrum (2.8). This concludes the discussion.

3.5 System 17: Ellipsoidal

This coordinate system is defined as

$$\left.\begin{aligned} z_1^2 &= -\frac{\varrho_1\varrho_2\varrho_3}{ab} & z_2^2 &= \frac{(\varrho_1-1)(\varrho_2-1)(\varrho_3-1)}{(a-1)(b-1)}\\ z_3^2 &= -\frac{(\varrho_1-b)(\varrho_2-b)(\varrho_3-b}{(a-b)(b-1)b} & z_4^2 &= \frac{(\varrho_1-a)(\varrho_2-a)(\varrho_3-a)}{(a-b)(a-1)a}\ . \end{aligned}\right\}\tag{3.44}$$

The set of commuting operators is given by

$$\left.\begin{aligned} \mathcal{L}_1 &= abI_{12}^2+aI_{13}^2+bI_{14}^2\ ,\\ \mathcal{L}_2 &= (a+b)I_{12}^2+(a+1)I_{13}^2+(b+1)I_{14}^2+aI_{32}^2+bI_{42}^2+I_{43}^2\ . \end{aligned}\right\}\tag{3.45}$$

The metric terms are given by

$$\mathrm{d}s^2=\frac{(\varrho_1-\varrho_2)(\varrho_1-\varrho_3)}{f(\varrho_1)}\,\mathrm{d}\varrho_1^2+\frac{(\varrho_2-\varrho_3)(\varrho_2-\varrho_1)}{f(\varrho_2)}\,\mathrm{d}\varrho_2^2+\frac{(\varrho_3-\varrho_1)(\varrho_3-\varrho_2)}{f(\varrho_3)}\,\mathrm{d}\varrho_3^2\ ,\tag{3.46}$$

with $f(\varrho)=-4(\varrho-a)(\varrho-b)(\varrho-1)\varrho$. For the path integral we find

$$\begin{aligned}K^{(S_{3C})}(\varrho_1'',\varrho_1',\varrho_2'',\varrho_2',\varrho_3'',\varrho_3';T) &= \int_{\varrho_1(t')=\varrho_1'}^{\varrho_1(t'')=\varrho_1''}\mathcal{D}\varrho_1(t)\int_{\varrho_2(t')=\varrho_2'}^{\varrho_2(t'')=\varrho_2''}\mathcal{D}\varrho_2(t)\int_{\varrho_3(t')=\varrho_3'}^{\varrho_3(t'')=\varrho_3''}\mathcal{D}\varrho_3(t)\\ &\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T\left[\frac{m}{2}\frac{(\varrho_1-\varrho_2)(\varrho_1-\varrho_3)}{f(\varrho_1)}\dot\varrho_1^2+\frac{(\varrho_2-\varrho_3)(\varrho_2-\varrho_1)}{f(\varrho_2)}\dot\varrho_2^2\right.\right.\\ &\qquad\left.\left.+\frac{(\varrho_3-\varrho_1)(\varrho_3-\varrho_2)}{f(\varrho_3)}\dot\varrho_3^2-\Delta V(\varrho_1,\varrho_2,\varrho_3)\right]\mathrm{d}t\right\}\ .\end{aligned}\tag{3.47}$$

It is obvious that such a path integral in ellipsoidal coordinates is not tractable. We let the result as it stands, and the same statement is valid for the remaining "ellipsoidal"-related coordinate systems. Let us, however, note that we can state the propagator in a formal way by a construction it from the wave-functions according to [12]. We have found the following representation on the real sphere $S^{(3)}$:

$$\int_{\varrho_1(t')=\varrho_1'}^{\varrho_1(t'')=\varrho_1''}\mathcal{D}\varrho_1(t)\int_{\varrho_2(t')=\varrho_2'}^{\varrho_2(t'')=\varrho_2''}\mathcal{D}\varrho_2(t)\int_{\varrho_3(t')=\varrho_3'}^{\varrho_3(t'')=\varrho_3''}\mathcal{D}\varrho_3(t)\frac{(\varrho_2-\varrho_1)(\varrho_3-\varrho_2)(\varrho_3-\varrho_1)}{8\sqrt{P(\varrho_1)P(\varrho_2)P(\varrho_3)}}$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\sum_{i=1}^{3} g_{\varrho_i\varrho_i}\dot{\varrho}_i^2 - \Delta V_{PF}(\boldsymbol{\varrho})\right]dt\right\}$$

$$= \sum_{2s=0}^{\infty}\sum_{\lambda,\mu} \mathrm{e}^{-2\mathrm{i}\hbar T s(s+1)/m}\Psi^*_{s,\lambda,\mu}(\varrho_1',\varrho_2',\varrho_3')\Psi_{s,\lambda,\mu}(\varrho_1'',\varrho_2'',\varrho_3'') \tag{3.48}$$

with ellipsoidal coordinates defined on $S^{(3)}$ $(d<\varrho_3<c<\varrho_2<b<\varrho_1<a)$:

$$\left.\begin{aligned} s_1^2 &= \frac{(\varrho_1-d)(\varrho_2-d)(\varrho_3-d)}{(a-d)(b-d)(c-d)} \\ s_2^2 &= \frac{(\varrho_1-c)(\varrho_2-c)(\varrho_3-c)}{(a-c)(b-c)(d-c)} \\ s_3^2 &= \frac{(\varrho_1-a)(\varrho_2-a)(\varrho_3-a)}{(d-a)(c-a)(b-a)} \\ s_4^2 &= \frac{(\varrho_1-b)(\varrho_2-b)(\varrho_3-b)}{(d-b)(c-b)(a-b)}\ . \end{aligned}\right\} \tag{3.49}$$

The metric tensor in ellipsoidal coordinates then has the form

$$(g_{ab}) = -\frac{1}{4}\mathrm{diag}\left(\frac{(\varrho_1-\varrho_2)(\varrho_1-\varrho_3)}{P(\varrho_1)}, \frac{(\varrho_2-\varrho_3)(\varrho_2-\varrho_1)}{P(\varrho_2)}, \frac{(\varrho_3-\varrho_1)(\varrho_3-\varrho_2)}{P(\varrho_3)}\right)\ , \tag{3.50}$$

and $P(\varrho)=(\varrho-a)(\varrho-b)(\varrho-c)(\varrho-d)$. Here, we have adopted the notation of Karayan et al. [1, 16]. The quantum numbers are the eigenvalues of the operators which characterize the ellipsoidal system on the sphere, thus giving a complete set of observables corresponding to the ellipsoidal coordinates on the sphere. However, this representation remains on a formal level and the corresponding wave-functions are explicitly know only on the real three-dimensional sphere. For details see [1, 16, 32, 28, 33, 31]. The ellipsoidal system exists on the three-dimensional sphere (System VI.) on the three-dimensional hyperboloid (System XVXIII.).

3.6 System 18

For the remaining coordinate system we state only their definition. The corresponding quantum theory setup is formally the same as for the Ellipsoidal coordinates, however with a different function $f(\varrho)$.

This coordinate system is defined as

$$\left.\begin{aligned} (\mathrm{i}z_1+z_2)^2 &= \frac{\varrho_1\varrho_2\varrho_3}{a} & z_1^2+z_2^2 &= \frac{1}{a^2}\big[(a+1)\varrho_1\varrho_2\varrho_3 - a(\varrho_1\varrho_2+\varrho_1\varrho_3+\varrho_2\varrho_3)\big] \\ z_3^2 &= \frac{(\varrho_1-1)(\varrho_2-1)(\varrho_3-1)}{1-a} & z_4^2 &= \frac{(\varrho_1-a)(\varrho_2-a)(\varrho_3-a)}{a^2(a-1)} \end{aligned}\right\} \tag{3.51}$$

The set of commuting operators is given by

$$\left.\begin{aligned} \mathcal{L}_1 &= (I_{42}-\mathrm{i}I_{14})^2 - a(I_{32}+\mathrm{i}I_{13})^2 - aI_{12}^2\ , \\ \mathcal{L}_2 &= (a+1)I_{12}^2 + I_{14}^2 + I_{42}^2 - a(I_{13}^2+I_{32}^2) + (I_{42}+\mathrm{i}I_{14})^2 + (I_{32}+\mathrm{i}I_{13})^2\ . \end{aligned}\right\} \tag{3.52}$$

The metric terms are given by

$$\mathrm{d}s^2 = \frac{(\varrho_1-\varrho_2)(\varrho_1-\varrho_3)}{f(\varrho_1)}\,\mathrm{d}\varrho_1^2 + \frac{(\varrho_2-\varrho_3)(\varrho_2-\varrho_1)}{f(\varrho_2)}\,\mathrm{d}\varrho_2^2 + \frac{(\varrho_3-\varrho_1)(\varrho_3-\varrho_2)}{f(\varrho_3)}\,\mathrm{d}\varrho_3^2\ , \tag{3.53}$$

with $f(\varrho) = -4(\varrho-2)(\varrho-1)\varrho^2$. Systems 18 corresponds to Systems 31–33 on the three-dimensional hyperboloid.

3.7 System 19

This coordinate system is defined as

$$\left.\begin{array}{ll} (z_1+\mathrm{i}z_2)^2 = -(\varrho_1-1)(\varrho_2-1)(\varrho_3-1) & z_1^2+z_2^2 = 2\varrho_1\varrho_2\varrho_3 - (\varrho_1\varrho_3+\varrho_2\varrho_3+\varrho_1\varrho_2)+1 \\ (z_3+\mathrm{i}z_4)^2 = -\varrho_1\varrho_2\varrho_3 & z_3^2+z_4^2 = \varrho_1\varrho_3+\varrho_2\varrho_3+\varrho_1\varrho_2-2\varrho_1\varrho_2\varrho_3 \ . \end{array}\right\} \tag{3.54}$$

The set of commuting operators is given by

$$\left.\begin{array}{l} \mathcal{L}_1 = 2(I_{31}+\mathrm{i}I_{32})^2 + \{I_{31}+\mathrm{i}I_{32}, I_{24}+\mathrm{i}I_{41}\} + I_{12}^2 \ , \\ \mathcal{L}_2 = 2(I_{31}+\mathrm{i}I_{32})^2 + \{I_{31}+\mathrm{i}I_{32}, I_{24}+I_{41}\} - I_{34}^2 \ . \end{array}\right\} \tag{3.55}$$

The metric terms are given by

$$\mathrm{d}s^2 = \frac{(\varrho_1-\varrho_2)(\varrho_1-\varrho_3)}{f(\varrho_1)}\,\mathrm{d}\varrho_1^2 + \frac{(\varrho_2-\varrho_3)(\varrho_2-\varrho_1)}{f(\varrho_2)}\,\mathrm{d}\varrho_2^2 + \frac{(\varrho_3-\varrho_1)(\varrho_3-\varrho_2)}{f(\varrho_3)}\,\mathrm{d}\varrho_3^2 \ , \tag{3.56}$$

with $f(\varrho) = -4(\varrho-1)^2\varrho^2$.

3.8 System 20

This coordinate system is defined as

$$\left.\begin{array}{l} (z_2-\mathrm{i}z_1)^2 z_1^2 + z_2^2 + z_3^2 = \varrho_1\varrho_2\varrho_3 \\ -2z_3(z_2-\mathrm{i}z_1) = \varrho_1\varrho_2+\varrho_1\varrho_3+\varrho_2\varrho_3-\varrho_1\varrho_2\varrho_3 \\ z_1^2+z_2^2+z_3^2 = \varrho_1\varrho_2\varrho_3-\varrho_1\varrho_2-\varrho_1\varrho_3-\varrho_2\varrho_3+\varrho_1+\varrho_2+\varrho_3 \\ z_4^2 = -(\varrho_1-1)(\varrho_2-1)(\varrho_3-1) \ . \end{array}\right\} \tag{3.57}$$

The set of commuting operators is given by

$$\left.\begin{array}{l} \mathcal{L}_1 = (I_{41}+\mathrm{i}I_{42})^2 + \{I_{32}-\mathrm{i}I_{13}, I_{12}\} \ , \\ \mathcal{L}_2 = I_{41}^2+I_{42}^2-I_{34}^2-(I_{41}+\mathrm{i}I_{42})^2 + \{I_{41}-\mathrm{i}I_{42}, I_{34}\} \ . \end{array}\right\} \tag{3.58}$$

The metric terms are given by

$$\mathrm{d}s^2 = \frac{(\varrho_1-\varrho_2)(\varrho_1-\varrho_3)}{f(\varrho_1)}\,\mathrm{d}\varrho_1^2 + \frac{(\varrho_2-\varrho_3)(\varrho_2-\varrho_1)}{f(\varrho_2)}\,\mathrm{d}\varrho_2^2 + \frac{(\varrho_3-\varrho_1)(\varrho_3-\varrho_2)}{f(\varrho_3)}\,\mathrm{d}\varrho_3^2 \ , \tag{3.59}$$

with $f(\varrho) = -4(\varrho-1)\varrho^3$. Systems 20 corresponds to System 34 on the three-dimensional hyperboloid.

3.9 System 21

This coordinate system is defined as

$$\left.\begin{aligned}&(z_1+\mathrm{i}z_2)^2=2\varrho_1\varrho_2\varrho_3\\&(z_1+\mathrm{i}z_2)((z_3+\mathrm{i}z_4)=-(\varrho_1\varrho_2+\varrho_2\varrho_3+\varrho_1\varrho_3)\\&-(z_1+\mathrm{i}z_2)(z_3-\mathrm{i}z_4)+\tfrac{1}{2}(z_3+\mathrm{i}z_4)^2=\varrho_1+\varrho_2+\varrho_3\ .\end{aligned}\right\}\tag{3.60}$$

The set of commuting operators is given by

$$\left.\begin{aligned}\mathcal{L}_1&=\tfrac{1}{2}\{I_{21},I_{41}+I_{23}+\mathrm{i}(I_{31}+I_{24})\}-\tfrac{1}{4}[I_{13}+I_{24}+\mathrm{i}(I_{23}+I_{41})]^2\ ,\\\mathcal{L}_2&=\tfrac{1}{2}\{I_{21}+I_{43},I_{32}+I_{14}+\mathrm{i}(I_{13}+I_{24})\}\\&\quad+\tfrac{1}{2}\{I_{41}+I_{23}+\mathrm{i}(I_{31}+I_{24}),I_{43}\}+\tfrac{1}{2}(I_{42}+\mathrm{i}I_{23})^2-\tfrac{1}{2}(I_{13}+\mathrm{i}I_{14})^2\ .\end{aligned}\right\}\tag{3.61}$$

The metric terms are given by

$$\mathrm{d}s^2=\frac{(\varrho_1-\varrho_2)(\varrho_1-\varrho_3)}{f(\varrho_1)}\,\mathrm{d}\varrho_1^2+\frac{(\varrho_2-\varrho_3)(\varrho_2-\varrho_1)}{f(\varrho_2)}\,\mathrm{d}\varrho_2^2+\frac{(\varrho_3-\varrho_1)(\varrho_3-\varrho_2)}{f(\varrho_3)}\,\mathrm{d}\varrho_3^2\ ,\tag{3.62}$$

with $f(\varrho)=-4\varrho^4$.

4 Summary and Discussion

The archived results are very satisfactory. We could find many path integral representations on the complex sphere, several of them included the extension and application of already know results, several others of them are completely new. The most important result consists of the incorporation of the complex Liouville potential into the path integral formalism. However, we must always keep in mind that the complex sphere is an abstract space, which means that the various path integral representations require an interpretation depending whether one considers a compact or non-compact variable range. In the compact case, the abstract complex space allows the interpretation of the real three-dimensional sphere with its discrete spectrum. Here, we can identify the six coordinate systems as indicated in Table 1: They are systems (1), (3), (6), (13), and (17), where No.(13) is counted twice to include the prolate as well as the oblate spheroidal cases. In the non-compact case allows the interpretation of the three-dimensional $\Lambda^{(3)}$ and O(2,2)-hyperboloid, respectively, with a continuous spectrum. Therefore the eigenvalues of the complex sphere

$$\frac{\hbar^2}{2m}\sigma(\sigma+2)\longleftarrow\left\{\begin{array}{lll}\dfrac{\hbar^2}{2m}l(l+2) & l=0,1,2,\ldots & \text{sphere}\\[2ex]\dfrac{\hbar^2}{2m}(p^2+1) & p>0 & \text{hyperboloid.}\end{array}\right.\tag{4.63}$$

This includes the replacement of the summation of the discrete principal quantum number l, say, by the principal continuous quantum number p, i.e., $\sum_l\to\int_0^\infty\mathrm{d}p$. Furthermore, the wave-functions have to analytically continued, say the discrete wave-functions for the spherical coordinate system (3) on the real sphere:

$$\Psi_{J,m_1,m_2}(\chi,\vartheta,\varphi)=N^{-1/2}\,\mathrm{e}^{\mathrm{i}m_1\varphi}(\sin\chi)^{m_1}C_{J-m_1}^{m_1+2}(\cos\chi)(\sin\vartheta)^{m_2}C_{m_1-m_2}^{m_2+3/2}(\cos\vartheta)\ ,\tag{4.64}$$

must be replaced by the continuous wave-functions for the spherical system on the hyperboloid $\Lambda^{(3)}$ ($Y_l^m(\vartheta,\varphi)$ are the usual spherical harmonics on the two-dimensional sphere, c.f. Section 2.3) [12]:

$$\Psi_{p,l,m}(\tau,\vartheta,\varphi) = Y_l^m(\vartheta,\varphi)(\sinh\tau)^{-1/2}\sqrt{\frac{p\sinh\pi p}{\pi}}\Gamma(\mathrm{i}p+l+1)\mathcal{P}^{-\frac{1}{2}-l}_{\mathrm{i}p-1/2}(\cosh\tau)\ . \tag{4.65}$$

Also, the invariant distance (under rotations) on the real sphere must be replaced by the invariant distance on the hyperboloid, c.f. Eqs. (2.11,2.13) with the corresponding Green functions (2.10,2.12), respectively.

In order to find the corresponding solutions on the O(2,2) hyperboloid matters are more difficult, because in addition to a continuous spectrum also a discrete spectrum is present. These issues will be discussed in future publication.

In Section II, I have displayed the path integral solutions (1) to (12). They are characterized by the property that they have a subgroup, respectively a subspace structure. This has also been emphasized in Table 1, where the explicit subspace with its corresponding coordinate representation has been displayed. Remarkably, for all coordinate systems a path integral representation could be found. The principal new representation was the one for the horicyclic system, respectively the complex Liouville potential, i.e.

$$\begin{aligned}&\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^T\left(\frac{m}{2}\dot{x}^2-\frac{\hbar^2}{2m}k^2\,\mathrm{e}^{-2\mathrm{i}x}\right)\,\mathrm{d}t\right]\\ &\quad=\sum_{J\in\mathbb{N}_0}\tfrac{1}{2}H^{(1)}_{J+1/2}\left(k\,\mathrm{e}^{-\mathrm{i}x''}\right)H^{(1)}_{J+1/2}\left(k\,\mathrm{e}^{-\mathrm{i}x'}\right)\exp\left[-\frac{\mathrm{i}}{\hbar}\frac{\hbar^2J(J+2)}{2m}T\right]\ .\end{aligned} \tag{4.66}$$

This path integral representation was very useful in evaluating several path integral representations involving other horicyclic coordinate possibilities.

In several other coordinate systems we could exploit already known results, for instance from the two-dimensional sphere (e.g. sphero-elliptic (6)), from the two-dimensional hyperboloid (e.g. spherical-degenerate elliptic I (7)), from the two-dimensional pseudo-Euclidean plane (e.g. horicyclic-elliptic (9)), the Euclidean plane (e.g. horicyclic-parabolic I (11)), and others as noted in the text.

In Section III, I have discussed the path integral representations of the non-subspace cases which are much more involved. Starting with the elliptic-cylindrical system we can only state the formal solution as known from the three-dimensional sphere, which can be represented by Lamé polynomials. Four more representations could be explicitly stated, the last for the ellipsoidal one also only as a formal solutions in terms of *ellipsoidal wave-functions* as known from the three-dimensional sphere. For the remaining "ellipsoidal" systems nothing is known which is not surprising due to their very complicated structure.

There are several three-dimensional other spaces where a path integral treatment for various coordinate systems is possible: these are the single-sheeted hyperboloid, the O(2,2)-hyperboloid [12], Darboux spaces in three dimensions [14], and Koenigs spaces in two and three dimensions [15]. The latter two open the possibility to discuss quantum motion on spaces of non-constant curvature, whereas the former two have the property that in addition to the continuous spectrum also an infinite discrete spectrum exists. In particular, the quantum motion on the O(2,2)

hyperboloid is of special interest due to the fact that on the O(2,2) hyperboloid a discrete *and* a continuous spectrum exists. Therefore the complete spectrum consists of both contributions of (4.63), and almost all path integral representations with their wave-function expansions must be analytically continued to a discrete *and* a continuous contribution. These issues will be discussed in a future publication.

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