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Light quark masses and pseudoscalar decay constants from $N_f = 2$ Lattice QCD with twisted mass fermions

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Abstract

We present the results of a lattice QCD calculation of the average up-down and strange quark masses and of the light meson pseudoscalar decay constants with $N_f = 2$ dynamical fermions. The simulation is carried out at a single value of the lattice spacing with the twisted mass fermionic action at maximal twist, which guarantees automatic $\mathcal{O}(a)$-improvement of the physical quantities. Quark masses are renormalized by implementing the non perturbative RI-MOM renormalization procedure. Our results for the light quark masses are $m_{ud}^{\overline{\rm MS}}(2\ {\rm GeV}) = 3.85 \pm 0.12 \pm 0.40$ MeV, $m_s^{\overline{\rm MS}}(2\ {\rm GeV}) = 105 \pm 3 \pm 9$ MeV and $m_s/m_{ud} = 27.3 \pm 0.3 \pm 1.2$. We also obtain $f_K = 161.7 \pm 1.2 \pm 3.1$ MeV and the ratio $f_K/f_\pi = 1.227 \pm 0.009 \pm 0.024$. From this ratio, by using the experimental determination of $\Gamma(K \to \mu\bar{\nu}_\mu(\gamma))/\Gamma(\pi \to \mu\bar{\nu}_\mu(\gamma))$ and the average value of $|V_{ud}|$ from nuclear beta decays, we obtain $|V_{us}| = 0.2192(5)(45)$, in agreement with the determination from K_{l3} decays and the unitarity constraint.

1 Introduction

In this paper we extend to the kaon sector our previous lattice study of the pion mass and decay constant [1]. We present a determination of the light quark masses, strange quark mass m_s and the average up-down quark mass m_{ud}, of the kaon pseudoscalar decay constant f_K, and of the ratio f_K/f_π. We have simulated the theory with $N_f = 2$ dynamical quarks, taken to be degenerate in mass, and two valence quarks. In order to investigate the properties of the K meson, we consider in the present analysis a partially quenched setup, namely we take the valence quark masses μ_1 and μ_2 different in value between each other and different from the sea quark mass μ_S.

The strategy of the calculation is the following. We first compute the pseudoscalar meson masses and decay constants for different values of the sea and valence quark masses, and study their mass dependence. We then use the experimental values of the ratios M_π/f_π and M_K/M_π to determine the average up-down and the strange quark mass respectively. The lattice spacing is fixed from f_π. The results obtained for the quark masses are finally used to evaluate f_K and the ratio f_K/f_π.

The calculation is based on a set of gauge field configurations generated with the tree-level improved Symanzik gauge action at $\beta = 3.9$, corresponding to $a = 0.087(1)$ fm ($a^{-1} \simeq 2.3$ GeV) [1], and the twisted mass fermionic action at maximal twist. We have simulated 5 values of the bare sea quark mass,

$$a\mu_S = \{0.0040,\ 0.0064,\ 0.0085,\ 0.0100,\ 0.0150\}\,, \tag{1}$$

and computed quark propagators for 8 values of the valence quark mass,

$$a\mu_{1,2} = \{0.0040,\ 0.0064,\ 0.0085,\ 0.0100,\ 0.0150,\ 0.0220,\ 0.0270,\ 0.0320\}\ . \tag{2}$$

The first five masses are equal to the sea quark masses, and lie in the range $1/6\, m_s \lesssim \mu_{1,2} \lesssim 2/3\, m_s$, where m_s is the physical strange quark mass, while the heaviest three are around the strange quark mass.

We implement non-degenerate valence quarks in the twisted mass formulation of lattice QCD as discussed for instance in refs. [2, 3]. We introduce two twisted doublets of degenerate valence quarks, $(u_1,\, d_1)$ and $(u_2,\, d_2)$, with masses μ_1 and μ_2 respectively, and simulate charged mesons $\bar{u}_1 d_2$ and $\bar{d}_1 u_2$. Within each doublet, the two valence quarks are regularized in the physical basis with Wilson parameters of opposite values ($r_u = -r_d = 1$).

At each value of the sea quark mass we have computed the two-point correlation functions of charged pseudoscalar mesons, with both degenerate and non degenerate valence quarks, on a set of 240 independent gauge field configurations, separated by 20 HMC trajectories one from the other (each trajectory being of length 1/2). To improve the statistical accuracy, we have evaluated the meson correlators using a stochastic method to include all spatial sources. The method involves a real stochastic source ($Z(2)$-noise) for all colour and spatial indices at one Euclidean time slice randomly moved when passing from one gauge configuration to another. This "one-end" method is similar to that pioneered in ref. [4] and implemented in ref. [5]. Statistical errors on the meson masses and decay constants are evaluated using the jackknife procedure, by decimating 10 configurations out of 240 in each jackknife bin. Statistical errors on the fit results, which are based on data obtained at different sea quark masses, are evaluated using a bootstrap procedure. Further details on the numerical simulation can be found in [6].

The use of twisted mass fermions in the present calculation turns out to be beneficial in several aspects [7, 2]: i) the pseudoscalar meson masses and decay constants, which represent the basic ingredients of the calculation, are automatically improved at $\mathcal{O}(a)$; [1] ii) once maximal twist is realized, the physical quark mass is directly related to the twisted mass parameter of the action, and it is subject only to multiplicative renormalization; iii) the determination of the pseudoscalar decay constant does not require the introduction of any renormalization constant, and it is based on the relation

$$f_{PS} = (\mu_1 + \mu_2)\, \frac{|\langle 0|P^1(0)|P\rangle|}{M_{PS}^2}\,. \tag{3}$$

Concerning the size of discretization effects, it is worth noting that, since the two valence quarks are regularized in the physical basis with Wilson parameters of opposite values, the meson mass M_{PS}^2 differs from its continuum counterpart only by terms of $\mathcal{O}(a^2\mu)$ and $\mathcal{O}(a^4)$, whereas f_{PS} differs from its continuum limit by terms of $\mathcal{O}(a^2)$ [8, 9]. Therefore, at $\mathcal{O}(a^2)$ the cutoff effects on M_{PS}^2 and f_{PS} are as in a chiral invariant lattice formulation.

The meson mass M_{PS} and the matrix element $|\langle 0|P^1(0)|P\rangle|$ have been extracted from a fit of the two-point pseudoscalar correlation function in the time interval $t/a \in [10, 21]$. In order to illustrate the quality of the data, we show in fig. 1 the effective masses of pseudoscalar mesons, as a function of the time, in the degenerate cases $\mu_S = \mu_1 = \mu_2$.

[1] Strictly speaking, automatic O(a) improvement was proved in [7, 2] to hold in a unitary as well as in a mixed action framework. Actually the same proof goes through also in the present partially quenched setup. The reason is that all the symmetries entering the discussion of the renormalizability and O(a) improvement are valid for generic values of the masses of the various valence and sea quarks.

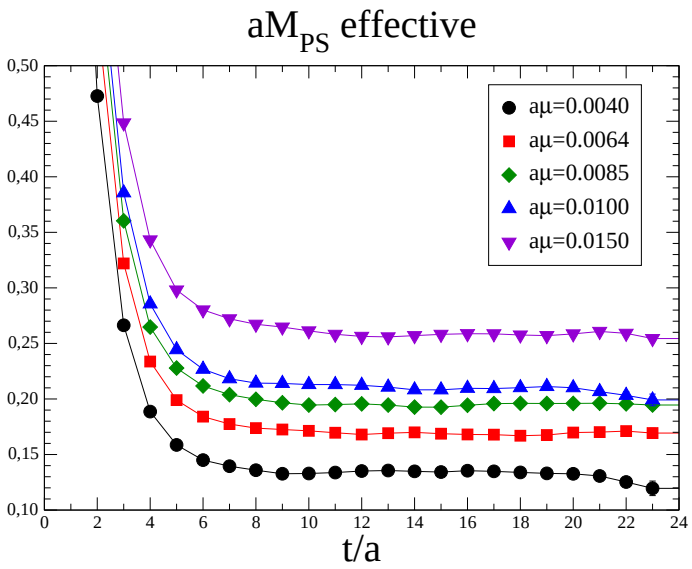


Figure 1: *Effective masses of pseudoscalar mesons, as a function of the time, in the degenerate cases* $\mu_S = \mu_1 = \mu_2$*. Error bars are smaller than the symbol sizes.*

2 Quark mass dependence of pseudoscalar meson masses and decay constants

The determination of the physical properties of K mesons requires a study of the quark mass dependence of the corresponding observables over a large range of masses, extending from the physical strange quark down to the light up and down quarks. In this work, we study the quark mass dependence of the pseudoscalar meson masses and decay constants by investigating two different functional forms. The first one is the dependence predicted by continuum partially quenched chiral perturbation theory (PQChPT), whereas in the second case we consider a simple polynomial dependence. For a recent precision study of the quark mass dependence of meson masses and decay constants in the partially quenched theory with $N_f = 2$ dynamical fermions see also ref. [10].

2.1 PQChPT fits

Within PQChPT we consider the full next-to-leading order (NLO) expressions with the addition of the local NNLO contributions, i.e. terms quadratic in the quark masses, which turn out to be needed for a good description of the meson masses and decay constants up to the region of quark masses around the strange quark mass. The PQChPT predictions have been derived in ref. [11] and can be written in the form

$$M^2_{PS}(\mu_S,\mu_1,\mu_2) \;=\; B_0\,(\mu_1+\mu_2)\cdot\left[1+\frac{\xi_1\,(\xi_S-\xi_1)\,\ln 2\xi_1}{(\xi_2-\xi_1)}-\frac{\xi_2\,(\xi_S-\xi_2)\,\ln 2\xi_2}{(\xi_2-\xi_1)}+\right.$$
$$\left.+a_V\,\xi_{12}+a_S\,\xi_S+a_{VV}\,\xi^2_{12}+a_{SS}\,\xi^2_S+a_{VS}\,\xi_{12}\,\xi_S+a_{VD}\,\xi^2_{D12}\right]\,, \qquad (4)$$

$$f_{PS}(\mu_S,\mu_1,\mu_2) \;=\; f\cdot\Bigg[1-\xi_{1S}\,\ln 2\xi_{1S}-\xi_{2S}\,\ln 2\xi_{2S}+\frac{\xi_1\,\xi_2-\xi_S\,\xi_{12}}{2\,(\xi_2-\xi_1)}\ln\left(\frac{\xi_1}{\xi_2}\right)+$$
$$+(b_V+1/2)\,\xi_{12}+(b_S-1/2)\,\xi_S+b_{VV}\,\xi_{12}^2+b_{SS}\,\xi_S^2+b_{VS}\,\xi_{12}\,\xi_S+b_{VD}\,\xi_{D12}^2\Bigg]\;,$$

where $\xi_i = 2B_0\mu_i/(4\pi f)^2$, $\xi_{ij} = B_0(\mu_i+\mu_j)/(4\pi f)^2$ and $\xi_{Dij} = B_0(\mu_i-\mu_j)/(4\pi f)^2$. The parameters B_0 and f are the low energy constants (LECs) entering the chiral Lagrangian at the LO [2], whereas a_V, a_S, b_V and b_S are related to the NLO LECs [11] by

$$a_V = 4\alpha_8 - 2\alpha_5 \quad , \quad a_S = 8\alpha_6 - 4\alpha_4 \quad , \quad b_V = \alpha_5 \quad , \quad b_S = 2\alpha_4\,. \tag{5}$$

The quadratic terms in the quark masses in eq. (4) represent the local NNLO contributions. The corresponding chiral logarithms at two loops in the partially quenched theory are also known [12]. They involve, however, a larger number of NLO LECs whose values, in the $N_f = 2$ theory, cannot be fixed from phenomenology. Introducing their contribution in the fit would increase significantly the number of free parameters, thus limiting, at the same time, the predictive power of the calculation.

In the limit of degenerate valence quark masses, $\mu_1 = \mu_2 \equiv \mu_V$, eq. (4) is finite and reduces to

$$M_{PS}^2(\mu_S,\mu_V,\mu_V) = 2\,B_0\,\mu_V\,\cdot[1+(2\,\xi_V-\xi_S)\,\ln 2\xi_V+(a_V+1)\,\xi_V+(a_S-1)\,\xi_S+$$
$$+a_{VV}\,\xi_V^2+a_{SS}\,\xi_S^2+a_{VS}\,\xi_V\,\xi_S\Big]\;, \tag{6}$$
$$f_{PS}(\mu_S,\mu_V,\mu_V) = f\cdot\Big[1-2\,\xi_{VS}\,\ln 2\xi_{VS}+b_V\,\xi_V+b_S\,\xi_S+b_{VV}\,\xi_V^2+b_{SS}\,\xi_S^2+b_{VS}\,\xi_V\,\xi_S\Big]\;.$$

2.1.1 Finite volume corrections

In a lattice QCD calculation aiming at a percent precision on the physical predictions, the impact of finite size corrections cannot be neglected. The lattice in our simulation has spatial extension $L = 24a \simeq 2.1$ fm, and the pseudoscalar meson mass at the lightest value of the quark mass is such that $M_{PS}L \simeq 3.2$. Since we have not performed yet a systematic study of non-degenerate meson masses and decay constants on different lattice volumes, we will estimate the finite size effects by including in the fits the corrections predicted by one-loop chiral perturbation theory, which, in the partially quenched case, are expressed by [13] [3]

$$M_{PS}^2(\mu_S,\mu_1,\mu_2;L) = M_{PS}^2(\mu_S,\mu_1,\mu_2)\cdot\left[1+\frac{\xi_1\,(\xi_S-\xi_1)\,\tilde g_1(L,\xi_1)}{(\xi_2-\xi_1)}-\frac{\xi_2\,(\xi_S-\xi_2)\,\tilde g_1(L,\xi_2)}{(\xi_2-\xi_1)}\right]\;,$$
$$f_{PS}(\mu_S,\mu_1,\mu_2;L) = f_{PS}(\mu_S,\mu_1,\mu_2)\cdot$$
$$\Bigg[1-\xi_{1S}\,\tilde g_1(L,\xi_{1S})-\xi_{2S}\,\tilde g_1(L,\xi_{2S})+\frac{\xi_{12}-\xi_S}{2\,(\xi_2-\xi_1)}\,(\xi_1\,\tilde g_1(L,\xi_1)-\xi_2\,\tilde g_1(L,\xi_2))+$$
$$+\frac{1}{4}\,(\xi_S-\xi_1)\,\tilde g_2(L,\xi_1)+\frac{1}{4}\,(\xi_S-\xi_2)\,\tilde g_2(L,\xi_2)\Bigg]\;. \tag{7}$$

[2] The pseudoscalar decay constant f is normalised such that $f_\pi = 130.7$ MeV at the physical pion mass.

[3] We thank D.Becirevic and G.Villadoro for having provided us with the expression of finite volume corrections to $M_{PS}^2(\mu_S,\mu_1,\mu_2)$ which is not given in ref. [13].

The functions $\tilde{g}_s$ $(s = 1, 2)$ in eq. (7) are defined as

$$\tilde{g}_s(L, M^2) = \frac{(4\pi)^{3/2}}{(M^2)^{2-s}}\,\Gamma(s - 1/2)\,\xi_{s-1/2}(L, M^2)\,, \tag{8}$$

where M is the pseudoscalar meson mass at the LO, $M^2 = 2B_0\mu = (4\pi f)^2\xi$,

$$\xi_s(L, M^2) = \frac{1}{(4\pi)^{3/2}\Gamma(s)}\int_0^\infty d\tau\ \tau^{s-5/2}e^{-\tau M^2}\left[\vartheta^3\left(\frac{L^2}{4\tau}\right) - 1\right]\,, \tag{9}$$

and $\vartheta(\tau)$ is the elliptic theta function

$$\vartheta(\tau) \equiv \sum_{n=-\infty}^{\infty} e^{-\tau\, n^2}\,. \tag{10}$$

The limits of eq. (7) in the case of degenerate valence quark masses, $\mu_1 = \mu_2 \equiv \mu_V$, can be obtained by using the identity

$$M^2\,\frac{d}{dM^2}\,\tilde{g}_s(L, M^2) = -(2 - s)\,\tilde{g}_s(L, M^2) - \tilde{g}_{s+1}(L, M^2) \tag{11}$$

and are given by

$$\begin{aligned} M^2_{PS}(\mu_S, \mu_V, \mu_V; L) &= M^2_{PS}(\mu_S, \mu_V, \mu_V)\cdot\left[1 + \xi_V\,\tilde{g}_1(L, \xi_V) - (\xi_V - \xi_S)\,\tilde{g}_2(L, \xi_V)\right]\,, \\ f_{PS}(\mu_S, \mu_V, \mu_V; L) &= f_{PS}(\mu_S, \mu_V, \mu_V)\cdot\left[1 - 2\,\xi_{VS}\,\tilde{g}_1(L, \xi_{VS})\right]\,. \end{aligned} \tag{12}$$

2.2 Polynomial fits

The inclusion of the local NNLO contributions in the PQChPT predictions expressed by eq. (4) is required by the observation that the pure NLO predictions are not accurate enough to describe the quark mass dependence of pseudoscalar meson masses and decay constants up to the region of the strange quark. However, not having considered the full NNLO chiral predictions, we regard eq. (4) mostly as an effective description of the quark mass dependence of these observables. In order to evaluate the associated systematic uncertainty, we also consider in the analysis an alternative description based on a simple polynomial dependence on the quark masses, for both the pseudoscalar meson masses and decay constants:

$$\begin{aligned} M^2_{PS}(\mu_S, \mu_1, \mu_2) \;&=\; B_0\,(\mu_1 + \mu_2)\cdot \\ &\quad\cdot\left[1 + a_V\,\xi_{12} + a_S\,\xi_S + a_{VV}\,\xi^2_{12} + a_{SS}\,\xi^2_S + a_{VS}\,\xi_{12}\,\xi_S + a_{VD}\,\xi^2_{D12}\right]\,, \\ f_{PS}(\mu_S, \mu_1, \mu_2) \;&=\; f\cdot\left[1 + (b_V + 1/2)\,\xi_{12} + (b_S - 1/2)\,\xi_S + b_{VV}\,\xi^2_{12} + b_{SS}\,\xi^2_S + \right. \\ &\quad\left. + b_{VS}\,\xi_{12}\,\xi_S + b_{VD}\,\xi^2_{D12}\right]\,. \end{aligned} \tag{13}$$

Note that, though we are adopting in eq. (13) the same notation for the coefficients of the chiral expansions as in eq. (4), the physical meaning of these coefficients, i.e. their relation to the derivatives of M^2_{PS} and f_{PS} with respect to the quark masses, is actually different.

It also worth observing that, in the case of the polynomial fits (13), a change in the values of the LECs f and B_0 only amounts to a redefinition of the fit parameters of M_{PS}^2 and f_{PS} respectively. Therefore, in this case, the two fits are independent one from the other. The differences between the results obtained by performing either chiral or polynomial fits will be included in the final estimates of the systematic errors.

3 Chiral extrapolations

The input data in the present analysis are the lattice results for the pseudoscalar meson masses and decay constants obtained at each value of the sea quark mass, with both degenerate and non degenerate valence quarks. We exclude from the fits the heaviest mesons having both the valence quark masses in the strange mass region, namely with $a\mu_{1,2} = \{0.0220, 0.0270, 0.0320\}$. Overall, we have considered therefore 150 combinations of quark masses for both the meson masses and the decay constants. The full sets of results are collected in tables 4 and 5 of the appendix. The number of free parameters in the combined fit of M_{PS}^2 and f_{PS} is 14, but a first analysis shows that some of them, in the various cases, are compatible with zero within one standard deviation, and are kept fixed to zero in the final estimates of the fit parameters (see table 1).

In order to extrapolate the pseudoscalar meson masses and decay constants to the points corresponding to the physical pion and kaon, we have considered three different fits:

- **Polynomial fit:** a polynomial dependence on the quark masses is assumed for the pseudoscalar meson masses and decay constants, according to eq. (13).

- **PQChPT fit:** the pseudoscalar meson masses and decay constants are fitted according to the predictions of PQChPT expressed by eq. (4) to which we add the finite volume corrections of eq. (7).

- **Constrained PQChPT fit:** this fit, denoted as C-PQChPT in the following, deserves a more detailed explanation. The main uncertainty in using eqs. (4) and (13) to effectively describe the quark mass dependence of M_{PS}^2 and f_{PS} is related to the extrapolation toward the physical up and down quark masses. On the other hand, we have shown in ref. [1] that pure NLO ChPT, with the inclusion of finite volume corrections, is sufficiently accurate in describing the lattice results for both the pseudoscalar meson masses and decay constants when the analysis is restricted to our lightest four quark masses in the unitary setup (i.e. $\mu_1 = \mu_2 = \mu_S$). In order to take advantage of this information, when performing the C-PQChPT fit we first determine the LO parameters B_0 and f and the NLO combinations $a_V + a_S$ and $b_V + b_S$ from a fit based on pure NLO ChPT performed on the lightest four unitary points. In other words, we repeat here as a preliminary step the same analysis done in ref. [1], but on the smaller statistical sample of data used for the present study.[4] In this way we

[4]Note that in the limit $\mu_1 = \mu_2 = \mu_S$, and when all the coefficients of the quadratic terms are sent to zero, the PQChPT expressions (4), as well as the finite volume corrections expressed by eq. (7), reduce to the pure NLO ChPT predictions used in the chiral fit of ref. [1].

determine

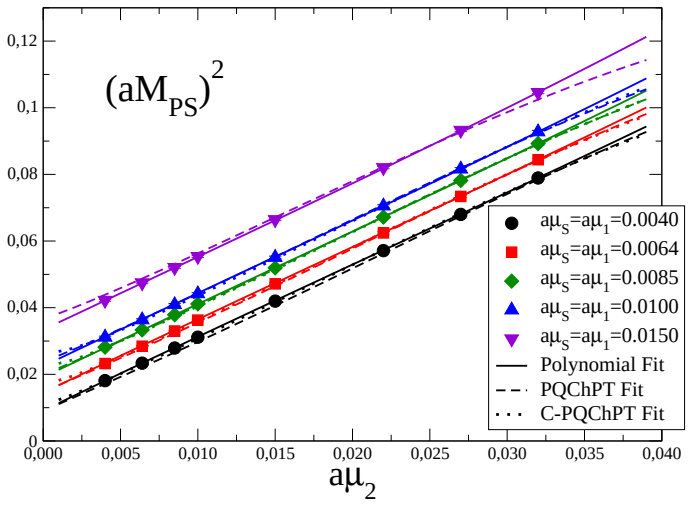
$$2aB_0 = 4.82(10) \ , \quad af = 0.0552(12) \ ,$$
$$a_V + a_S = 0.80(23) \ , \quad b_V + b_S = 0.62(24) \ . \tag{14}$$

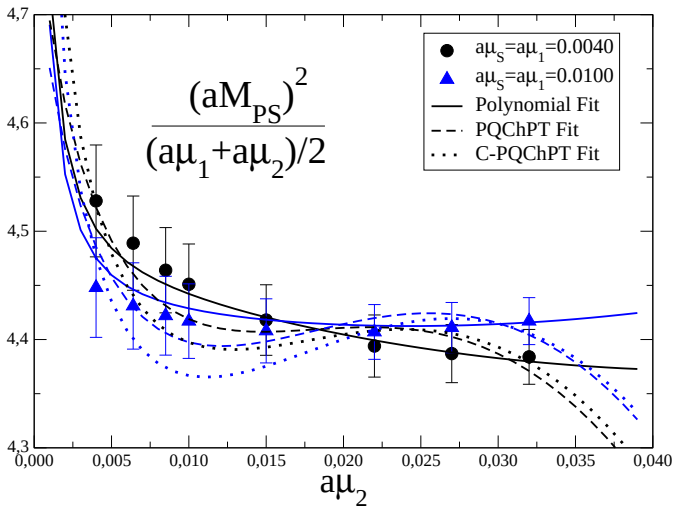
These results, are perfectly consistent, at the level of $\sim 1.5\ \sigma$, with those obtained in ref. [1]. By using the constraints of eq. (14), the other parameters entering the chiral expansions of M_{PS}^2 and f_{PS} are then obtained from a fit to eq. (4) over the non unitary points. For consistency with the previous unitary fit, we exclude also in this case from the analysis the data at the highest value of sea quark mass, $a\mu_S = 0.0150$.

In table 1 ("All data") we collect the results obtained for the fit parameters in the three cases: polynomial, PQChPT and C-PQChPT fits. In the last line we also quote the corresponding values of the χ^2 per degree of freedom. From these values we see that, though the quality of the fit is better in the polynomial case, all three analyses provide a good description of the lattice data, in the whole region of masses explored in the simulation. This is only true, however, if the terms quadratic in the quark masses are taken into account.

A potential problem in the partially quenched theory is the divergence of the chiral logarithms in the limit in which the light valence quark mass goes to zero at fixed sea quark mass (see eq. (4)). This divergence does not affect the extrapolation of the lattice results to the physical point, since the sea and the light valence quark masses are degenerate in this case. However, in order to verify that this unphysical behaviour of the partially quenched chiral logarithms does not modify the result of the extrapolation, we have repeated the analysis by restricting both the polynomial and the chiral fits to the 30 quark mass combinations (26 in the case of the C-PQChPT fit) that, satisfying the constraint $\mu_2 \geq \mu_1 = \mu_S$, are not affected by the dangerous chiral logarithms. The results obtained for the free parameters of these fits are also shown in table 1 (last three columns). By comparing these results with those obtained by using the full set of data, we find some differences in the estimates of the coefficients of the quadratic terms, particularly those involving the sea quark mass (a_{VS}, a_{SS}, ...). These differences reflect the relative importance in the fit of the various quadratic terms in the different quark mass regions. For instance, in the case of the highest sea quark mass, $a\mu_S = 0.0150$, only 4 out of 30 combinations of masses are included in the fit restricted by the condition $\mu_2 \geq \mu_1 = \mu_S$. On the other hand, when we compare the results for the extrapolated physical quantities (am_{ud}, am_s, af_π, ...) obtained from the two fits, we find that they are almost indistinguishable (see table 2). This is reassuring, as it shows that the effects of potentially divergent chiral logarithms are well under control in our analysis.

The mass dependence of the pseudoscalar meson masses and decay constants is illustrated in fig. 2, where we also compare the lattice data with the results of the polynomial, PQChPT and C-PQChPT fits. We have shown in the plots the cases in which one of the valence quark mass (μ_1) is equal to the sea quark mass, and the results are presented as a function of the second valence quark mass (μ_2). The points corresponding to the physical pion and kaon are thus obtained by extrapolating/interpolating the results shown in fig. 2 to the limits $\mu_1 \to m_{ud}$ and $\mu_2 \to m_s$.





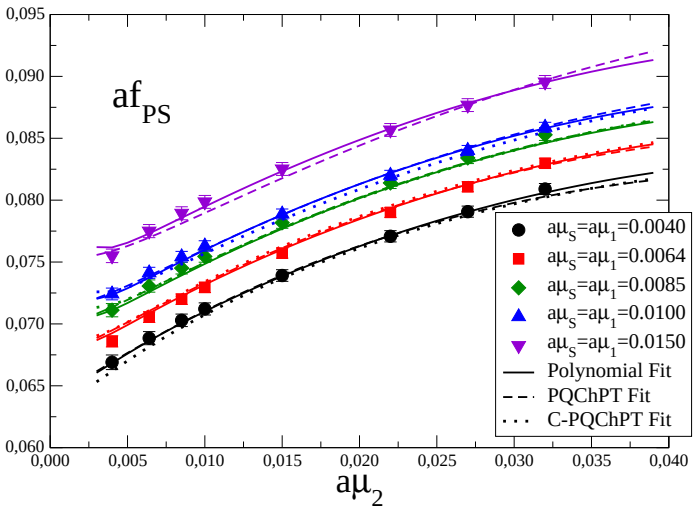


Figure 2: *Lattice results for $a^2 M_{PS}^2$ (top), $a^2 M_{PS}^2/\frac{1}{2}(a\mu_1+a\mu_2)$ (center) and af_{PS} (bottom) as a function of the valence quark mass $a\mu_2$, with $a\mu_1 = a\mu_S$. The solid, dashed and dotted curves represent the results of the polynomial, PQChPT and C-PQChPT fits respectively. For better clarity, results at only two values of the see quark mass have been shown in the center plot.*

	All data			Only $\mu_2 \geq \mu_1 = \mu_S$		
Fit	Polynomial	PQChPT	C-PQChPT	Polynomial	PQChPT	C-PQChPT
$2aB_0$	4.59(3)	4.79(6)	4.82(10)	4.55(6)	4.86(12)	4.82(10)
af	0.0607(6)	0.0577(6)	0.0552(12)	0.0606(9)	0.0574(14)	0.0552(12)
a_V	-0.63(7)	2.37(10)	2.15(18)	-0.52(16)	1.91(15)	2.15(18)
a_S	0.0	-1.44(10)	-1.35(12)	0.0	-1.04(37)	-1.35(12)
b_V	2.66(4)	0.68(5)	0.86(8)	2.56(13)	0.49(12)	0.75(8)
b_S	0.86(13)	-1.22(15)	-0.25(23)	1.03(15)	-0.94(34)	-0.13(24)
a_{VV}	2.6(2)	-9.3(3)	-8.3(6)	2.3(5)	-7.8(18)	-5.8(7)
a_{VS}	0.0	7.6(4)	6.9(3)	0.0	6.0(38)	0.0
a_{SS}	0.0	0.0	0.0	0.0	0.0	5.9(7)
a_{VD}	-0.6(1)	-3.8(2)	-3.2(3)	-0.9(6)	-2.6(21)	-5.1(4)
b_{VV}	-4.0(2)	1.2(2)	0.9(1)	-4.1(8)	0.0	2.3(5)
b_{VS}	0.0	6.0(6)	3.7(12)	0.0	7.1(21)	0.0
b_{SS}	0.0	0.0	-5.3(14)	0.0	0.0	-2.0(6)
b_{VD}	-3.7(2)	-3.8(2)	-3.0(3)	-2.6(6)	0.0	-3.1(6)
χ^2/d.o.f.	0.38	1.34	1.11	0.28	0.40	0.78

Table 1: *Values of the fit parameters as obtained from the polynomial, PQChPT and C-PQChPT fits (see text for details), by analysing all combinations of quark masses or only the combinations satisfying the constraint $\mu_2 \geq \mu_1 = \mu_S$. In the last line, the corresponding χ^2 per degree of freedom are also given.*

In order to illustrate the impact of finite volume corrections in the PQChPT fits, we compare in fig. 3 the best fit curves for the pseudoscalar meson masses and decay constants as obtained with or without including these corrections. In the plots the differences between the two curves are barely visible. Obviously, a different question is whether the theoretical formulae based on ChPT can accurately describe at the NLO the dependence of M_{PS}^2 and f_{PS} on the lattice volume. We postpone this issue to a future investigation, in which we plan to better quantify the systematic error due to finite size effects by extending the calculation of light pseudoscalar meson masses and decay constants on lattices with different spatial sizes.

By having determined the fit parameters, we are now ready to extrapolate eqs. (4) and (13) to the physical pion and kaon. We follow the procedure outlined in sect. 1: we use the experimental values of the ratios M_π/f_π and M_K/M_π to determine the average up-down and the strange quark mass respectively. Once these masses have been determined, we use again eqs. (4) and (13) to compute the values of the pion and kaon decay constants as well as their ratio f_K/f_π.[5]

[5]In order to account for the electromagnetic isospin breaking effects which are not introduced in the

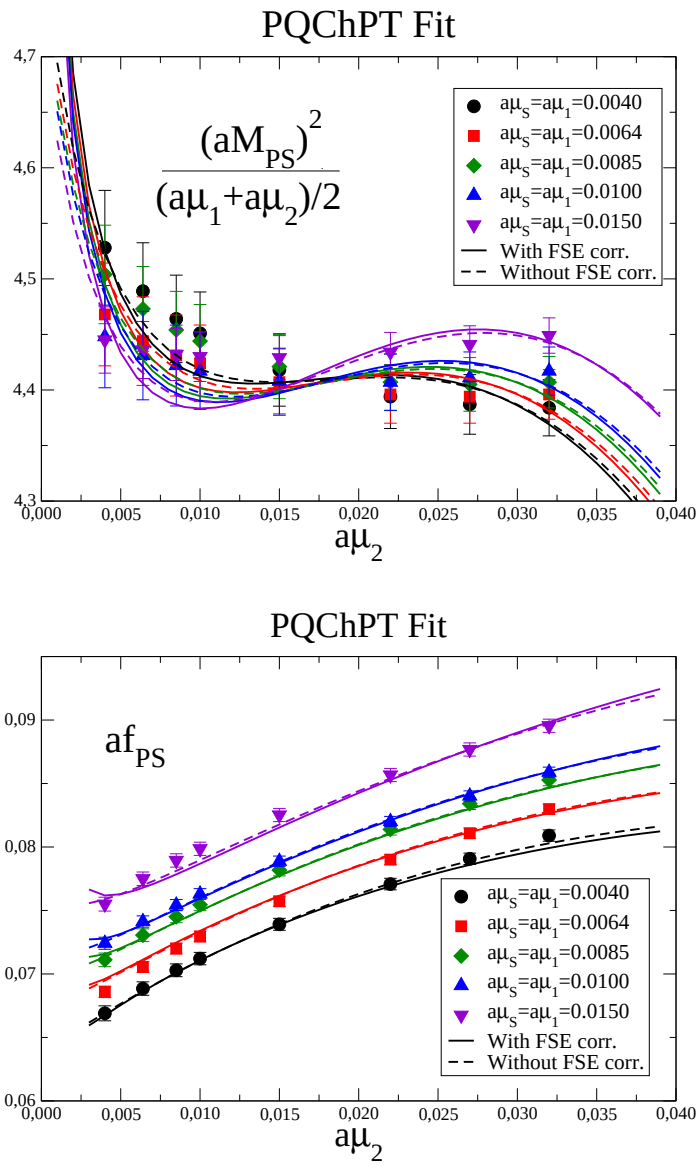


Figure 3: *PQChPT fits of the pseudoscalar meson masses and decay constants performed with (solid lines) and without (dashed lines) including the finite volume corrections of eq. (7). The results are shown as a function of the valence quark mass $a\mu_2$, with $a\mu_1 = a\mu_S$.*

In table 2 we collect the values of the quark masses, meson masses and decay constants, in lattice units, as obtained from the three fits by analysing all combinations of quark masses or only the combinations that satisfy the constraint $\mu_2 \geq \mu_1 = \mu_S$. Note that the

lattice simulation, we use as "experimental" values of the pion and kaon mass the combinations [14]

$$(M_\pi^2)_{QCD} = M_{\pi^0}^2 \ , \quad (M_K^2)_{QCD} = \frac{1}{2}\left[M_{K^0}^2 + M_{K^+}^2 - (1+\Delta_E)(M_{\pi^+}^2 - M_{\pi^0}^2)\right]$$

with $\Delta_E = 1$.

	All data			Only $\mu_2 \geq \mu_1 = \mu_S$		
Fit	Polynomial	PQChPT	C-PQChPT	Polynomial	PQChPT	C-PQChPT
$am_{ud} \cdot 10^3$	0.90(2)	0.86(2)	0.79(4)	0.91(3)	0.84(5)	0.79(4)
am_s	0.0243(5)	0.0235(5)	0.0218(10)	0.0243(7)	0.0234(12)	0.0217(10)
m_s/m_{ud}	26.9(1)	27.4(2)	27.5(3)	26.7(2)	27.9(2)	27.4(3)
aM_π	0.0642(6)	0.0632(6)	0.0610(12)	0.0642(9)	0.0629(14)	0.0610(12)
aM_K	0.235(2)	0.232(2)	0.224(4)	0.235(3)	0.231(5)	0.224(4)
af_π	0.0622(6)	0.0612(6)	0.0591(11)	0.0622(8)	0.0609(13)	0.0591(11)
af_K	0.0756(7)	0.0744(7)	0.0730(11)	0.0755(8)	0.0747(11)	0.0731(12)
f_K/f_π	1.216(3)	1.215(4)	1.236(8)	1.214(8)	1.225(11)	1.238(7)

Table 2: *Values of the quark masses, meson masses and decay constants in lattice units as obtained from the polynomial, PQChPT and C-PQChPT fits by analysing all combinations of quark masses or only the combinations satisfying the constraint* $\mu_2 \geq \mu_1 = \mu_S$.

values of the quark mass ratio m_s/m_{ud} and of the ratio of decay constants f_K/f_π, being dimensionless and well normalised quantities, are obtained at this step without need of fixing the scale nor of introducing the quark mass renormalization constant. For these quantities, therefore, the results presented in table 2 already represent physical predictions of the calculation.

As a further investigation, we have studied how the results for the quark masses and decay constants change when the analysis is performed only on mesons with degenerate valence quarks. In this case, we find values of quark mass in good agreement with those given in table (2), whereas for f_K and f_K/f_π we obtain results that are larger by about 5% than those quoted in the table. This reflects the fact that the mass difference between valence quarks represents only a small effect in meson masses, while it turns out to be relevant in decay constants, at the present level of accuracy, as shown by the contribution of the a_{VD} and b_{VD} terms respectively in the simple polynomial fits (see table 1).

4 Physical results

In order to convert into physical units the results obtained for the strange quark mass and the kaon decay constants we fix the scale within each analysis (polynomial, PQChPT and C-PQChPT fits) by using f_π as physical input. This choice deserves some discussion. By looking at table 2, we see that the value of the pion decay constant in lattice units as obtained from the C-PQChPT fit is in agreement, at the level of 1.4 σ, with the result of our previous study, $af_\pi = 0.0576(7)$ [1]. Indeed, from the present analysis we obtain the estimate $a = 0.089(2)$ fm, to be compared with the determination $a = 0.087(1)$ fm of ref. [1]. We also find that the estimate of the lattice spacing obtained from the C-PQChPT

analysis coincides with the one derived from the pure NLO ChPT analysis performed over the lightest four unitary points. This is expected, since as explained before the NLO unitary fit over the four lightest quark masses is used as a constraint in the C-PQChPT analysis, and the effect of the quadratic terms which are left out in the first fit is negligible in the evaluation of f_π. We then conclude that the difference between the determination $a = 0.089(2)$ fm and the one given in ref. [1] is a purely statistical effect and, as such, is properly accounted for by the quoted statistical errors. In the analyses based on the PQChPT and polynomial fits, instead, we obtain the estimates $a = 0.092(2)$ fm and $a = 0.094(1)$ fm respectively. In this case, the differences with respect to the C-PQChPT determination, which are at the level of 3% and 6% respectively, have a systematic origin related to the uncertainty in the chiral extrapolation.

As mentioned before, rather than choosing a common estimate of the scale for the polynomial, PQChPT and C-PQChPT analyses, we prefer to fix the scale by relying on the determination of af_π as obtained within each separate fit. This choice has the important advantage that also the pion and kaon masses are fixed in this way to their physical values within each fit, since the experimental results for the ratios M_π/f_π and M_K/M_π have been used to determine the light and strange quark masses. Therefore, the absolute normalization of the fit functions describing the quark mass dependence of both the meson masses and the decay constants is always correct, independently of the assumptions done on the chiral behaviour. As a result, we find that the systematic differences among the various determinations of am_s and af_K given in table 2, which are at the level of 6% and 2% respectively, reduce by approximately a factor of two when the results are converted in physical units. Nevertheless, in the case of the polynomial and PQChPT fits we conservatively add in the calculation of the dimensionful quantities a 6% and 3% of systematic error coming from the different estimates of the scale.

The determination of the physical strange and up-down quark masses also requires implementing a renormalization procedure. The relation between the bare twisted mass at maximal twist, μ_q, and the renormalized quark mass, m_q, is given by

$$m_q(\mu_R) = Z_m(g^2, a\mu_R)\, \mu_q(a)\,, \tag{15}$$

where μ_R is the renormalization scale, conventionally fixed to 2 GeV for the light quarks. Z_m is the inverse of the flavour non-singlet pseudoscalar density renormalization constant, $Z_m = Z_P^{-1}$. We have used the $\mathcal{O}(a)$-improved non-perturbative RI-MOM determination of Z_P, which gives $Z_P^{\rm RI-MOM}(1/a) = 0.39(1)(2)$ at $\beta = 3.9$ [15], and converted the result to the $\overline{\rm MS}$ scheme at the scale $\mu_R = 2$ GeV by using renormalization group improved continuum perturbation theory at the N^{3}LO [16].

In table 3 we collect the results for the light quark masses and pseudoscalar decay constants, in physical units, as obtained from the polynomial, PQChPT and C-PQChPT fits. For completeness, we also show in the table the results for the ratios m_s/m_{ud} and f_K/f_π already given in table 2. To be conservative, we only consider from now on the results obtained from the analysis of the quark mass combinations satisfying the constraint $\mu_2 \geq \mu_1 = \mu_S$ which, though being affected by larger statistical errors, are safe from the effects of the potentially divergent chiral logarithms. In table 3 we quote as a systematic error within each fit the uncertainty associated with the determination of the lattice spacing

Fit	Polynomial	PQChPT	C-PQChPT
$m_{ud}^{\overline{\rm MS}}$ (MeV)	4.07(9)(33)	3.82(15)(25)	3.74(13)(21)
$m_{s}^{\overline{\rm MS}}$ (MeV)	109(2)(9)	107(3)(7)	102(3)(6)
m_s/m_{ud}	26.7(2)(0)	27.9(2)(0)	27.4(3)(0)
f_K (MeV)	158.7(11)(89)	160.2(15)(54)	161.8(10)(0)
f_K/f_π	1.214(8)(0)	1.225(11)(0)	1.238(7)(0)

Table 3: *Results for the light quark masses and pseudoscalar decay constants, in physical units, as obtained from the polynomial, PQChPT and C-PQChPT fits respectively, by analysing only the combinations of quark masses satisfying the constraint* $\mu_2 \geq \mu_1 = \mu_S$. *The quoted errors are statistical (first) and systematic (second), the latter coming from the uncertainties in the determination of the lattice scale and of the quark mass renormalization constant.*

and of the quark mass renormalization constant.

In order to derive our final estimates for the quark masses and decay constants, we perform a weighted average of the results of the three analyses presented in table 3 and conservatively add the whole spread among these results to the systematic uncertainty. In this way, we obtain as our final estimates of the light quark masses the results

$$m_{ud}^{\overline{\rm MS}}(2\ {\rm GeV}) = 3.85 \pm 0.12 \pm 0.40\ {\rm MeV} \quad , \quad m_{s}^{\overline{\rm MS}}(2\ {\rm GeV}) = 105 \pm 3 \pm 9\ {\rm MeV}\,, \tag{16}$$

and the ratio

$$m_s/m_{ud} = 27.3 \pm 0.3 \pm 1.2\,, \tag{17}$$

where the first error is statistical and the second systematic.

For the kaon decay constant and the ratio f_K/f_π we obtain the accurate determinations

$$f_K = 161.7 \pm 1.2 \pm 3.1\ {\rm MeV} \quad , \quad f_K/f_\pi = 1.227 \pm 0.009 \pm 0.024\,. \tag{18}$$

It is interesting to compare our result for the strange quark mass with other lattice QCD determinations of the same quantity. This comparison is illustrated in fig. 4.

The HPQCD-MILC-UKQCD Collaboration, using the MILC extensive simulations of lattice QCD performed with $N_f = 2+1$ dynamical improved staggered fermions, initially quoted the result $m_s^{\overline{\rm MS}}(2\ {\rm GeV}) = 76(3)(7)$ MeV [14], significantly lower than our prediction in eq. (16). In [14], the quark mass renormalization constant was determined using one-loop perturbation theory. The two-loop calculation has then led to a significant increase of the quark mass estimate [24], and the most recent determination presented by MILC now reads $m_s^{\overline{\rm MS}}(2\ {\rm GeV}) = 90(5)(4)$ MeV [25]. Recently, a similar result has been also obtained by the CP-PACS and JLQCD Collaborations, using $\mathcal{O}(a)$-improved Wilson fermions with $N_f = 2+1$ and implementing the quark mass renormalization at one loop: $m_s^{\overline{\rm MS}}(2\ {\rm GeV}) = 91.1(^{+14.6}_{-6.2})$ [26]. It is worth noting that this result is perfectly consistent with the previous $N_f = 2$ determinations of the same quantity obtained by the two collaborations [17, 18].

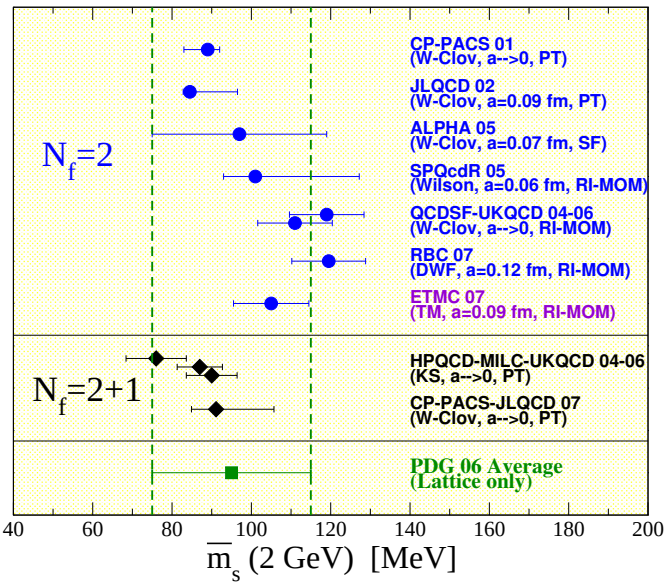


Figure 4: *Lattice QCD determinations of the strange quark mass obtained from simulations with $N_f = 2$ [17]-[23] and $N_f = 2 + 1$ [14, 24, 25, 26] dynamical fermions. The PDG average (from lattice only) [27] is also shown for comparison.*

In the present analysis, we find that the use of non-perturbative renormalization plays a crucial role in the determination of the quark masses. The estimate $Z_P^{\rm RI-MOM}(1/a) = 0.39(1)(2)$ obtained with the RI-MOM method is in fact significantly smaller than the prediction $Z_P^{\rm BPT}(1/a) \simeq 0.57(5)$ given by one-loop boosted perturbation theory (in the same RI-MOM renormalization scheme) [15]. Had we used the perturbative estimate of Z_P we would have obtained $m_{ud}^{\overline{\rm MS}}(2\ {\rm GeV}) = 2.63 \pm 0.08 \pm 0.36$ MeV and $m_s^{\overline{\rm MS}}(2\ {\rm GeV}) = 72 \pm 2 \pm 9$ MeV. As shown in fig. 4, our prediction for the strange quark mass in eq. (16) is in good agreement with other determinations based on a non-perturbative evaluation of the mass renormalization constant. These include the results obtained by ALPHA, $m_s^{\overline{\rm MS}}(2\ {\rm GeV}) = 97(22)$ MeV [19], by SPQ$_{\rm CD}$R, $m_s^{\overline{\rm MS}}(2\ {\rm GeV}) = 101(8)(^{+25}_{-0})$ MeV [20], by QCDSF-UKQCD, $m_s^{\overline{\rm MS}}(2\ {\rm GeV}) = 119(5)(8)$ MeV from the vector Ward identity [21] and $m_s^{\overline{\rm MS}}(2\ {\rm GeV}) = 111(6)(4)(6)$ MeV from the axial one [22], and by RBC, $m_s^{\overline{\rm MS}}(2\ {\rm GeV}) = 119.5(56)(74)$ MeV [23]. It is often found that, in lattice determinations of quark masses, implementing a non-perturbative renormalization method has an impact that can be larger even than the quenching effect. We believe that this observation should be always kept in mind, particularly when the lattice results for quark masses are considered for producing final averages.

Our result for the ratio f_K/f_π is compared in fig. 5 with other recent lattice determinations based on simulations with $N_f = 2$ and $N_f = 2 + 1$ dynamical fermions.

Our calculation and those based on the MILC improved staggered gauge configurations are the only ones in which light quark masses significantly lower than $m_s/3$ have been simulated ($m_q \simeq m_s/6$ for our lightest quark mass). Therefore, it is interesting to compare our determination of f_K/f_π with the more recent results quoted by MILC, $f_K/f_\pi = 1.208(2)(^{+7}_{-14})$ [25], and by HPQCD-UKQCD, $f_K/f_\pi = 1.189(7)$ [33]. Despite the

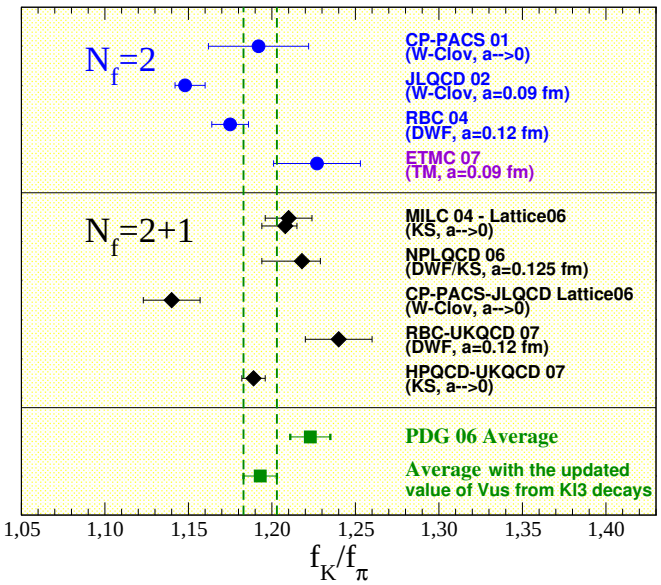


Figure 5: *Lattice QCD determinations of the ratio f_K/f_π obtained from simulations with $N_f = 2$ [17, 18, 28] and $N_f = 2+1$ [25], [29]-[33] dynamical fermions. The results are also compared with the PDG 2006 average [27] and with the average based on the updated determination of V_{us} from $K_{\ell 3}$ decays [34].*

strange quark is still quenched in our simulation, and our results are still obtained at a single value of the lattice spacing, we find the agreement between these determinations quite satisfactory. In order to better quantify the size of discretization effects, which are of $\mathcal{O}(a^2)$ in the present calculation, we plan to extend the simulation to other two values of the lattice spacing (corresponding to $\beta = 3.8$ and $\beta = 4.05$). This should also allow us to eventually perform the extrapolation to the continuum limit.

Our result for the ratio f_K/f_π can be combined with the experimental measurement of $\Gamma(K \to \mu\bar{\nu}_\mu(\gamma))/\Gamma(\pi \to \mu\bar{\nu}_\mu(\gamma))$ [27] to get a determination of the ratio $|V_{us}|/|V_{ud}|$ [35]. We obtain

$$|V_{us}|/|V_{ud}| = 0.2251(5)(47)\,, \tag{19}$$

where the first error is the experimental one and the second is the theory error coming from the uncertainty on f_K/f_π. Eq (19), combined with the determination $|V_{ud}| = 0.97377(27)$ [36] from nuclear beta decays, yields the estimate

$$|V_{us}| = 0.2192(5)(45)\,, \tag{20}$$

in agreement with the value extracted from $K_{\ell 3}$ decays, $|V_{us}| = 0.2255(19)$ [34], and leads to the constraint due to the unitarity of the CKM matrix

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 - 1 = (-3.7 \pm 2.0) \cdot 10^{-3}\,. \tag{21}$$

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Appendix

In this appendix we collect in tables 4 and 5 the values of the pseudoscalar meson masses and decay constants obtained at the various combinations of simulated sea and valence quark masses.

μ_1	μ_2	$\mu_S = 0.0040$	$\mu_S = 0.0064$	$\mu_S = 0.0085$	$\mu_S = 0.0100$	$\mu_S = 0.0150$
0.0040	0.0040	0.1346(8)	0.1342(8)	0.1357(10)	0.1349(11)	0.1364(9)
0.0040	0.0064	0.1528(7)	0.1524(8)	0.1537(9)	0.1529(10)	0.1542(8)
0.0040	0.0085	0.1670(7)	0.1667(8)	0.1678(8)	0.1670(9)	0.1682(7)
0.0040	0.0100	0.1765(7)	0.1762(8)	0.1771(8)	0.1765(9)	0.1775(7)
0.0040	0.0150	0.2049(8)	0.2046(8)	0.2052(8)	0.2048(9)	0.2055(7)
0.0040	0.0220	0.2390(8)	0.2388(8)	0.2391(8)	0.2390(9)	0.2393(7)
0.0040	0.0270	0.2608(8)	0.2606(8)	0.2608(8)	0.2608(9)	0.2609(7)
0.0040	0.0320	0.2809(8)	0.2808(8)	0.2809(8)	0.2811(9)	0.2809(7)
0.0064	0.0064	0.1690(7)	0.1687(8)	0.1697(8)	0.1690(9)	0.1701(7)
0.0064	0.0085	0.1820(7)	0.1817(7)	0.1826(8)	0.1819(9)	0.1829(7)
0.0064	0.0100	0.1908(7)	0.1905(7)	0.1912(8)	0.1906(9)	0.1915(7)
0.0064	0.0150	0.2174(7)	0.2171(7)	0.2176(7)	0.2172(8)	0.2179(6)
0.0064	0.0220	0.2501(7)	0.2499(7)	0.2500(8)	0.2499(8)	0.2503(6)
0.0064	0.0270	0.2711(7)	0.2709(7)	0.2710(8)	0.2710(8)	0.2712(6)
0.0064	0.0320	0.2907(7)	0.2905(7)	0.2905(8)	0.2907(8)	0.2908(6)
0.0085	0.0085	0.1942(7)	0.1939(7)	0.1946(8)	0.1940(8)	0.1949(6)
0.0085	0.0100	0.2025(7)	0.2022(7)	0.2027(8)	0.2022(8)	0.2030(6)
0.0085	0.0150	0.2279(7)	0.2276(7)	0.2279(7)	0.2276(8)	0.2282(6)
0.0085	0.0220	0.2594(7)	0.2592(7)	0.2592(8)	0.2591(8)	0.2595(6)
0.0085	0.0270	0.2799(7)	0.2796(7)	0.2796(8)	0.2796(8)	0.2799(6)
0.0085	0.0320	0.2990(6)	0.2988(7)	0.2987(8)	0.2989(8)	0.2991(6)
0.0100	0.0100	0.2104(7)	0.2101(7)	0.2106(7)	0.2102(8)	0.2109(6)
0.0100	0.0150	0.2351(6)	0.2348(7)	0.2350(7)	0.2347(8)	0.2353(6)
0.0100	0.0220	0.2659(6)	0.2656(7)	0.2656(8)	0.2655(8)	0.2660(6)
0.0100	0.0270	0.2859(6)	0.2857(7)	0.2856(8)	0.2857(8)	0.2860(6)
0.0100	0.0320	0.3048(6)	0.3046(7)	0.3045(8)	0.3046(7)	0.3048(6)
0.0150	0.0150	0.2576(6)	0.2573(7)	0.2574(7)	0.2572(8)	0.2577(6)
0.0150	0.0220	0.2863(6)	0.2861(7)	0.2859(7)	0.2859(7)	0.2864(6)
0.0150	0.0270	0.3053(6)	0.3050(7)	0.3049(8)	0.3049(7)	0.3054(6)
0.0150	0.0320	0.3233(6)	0.3230(7)	0.3228(8)	0.3230(7)	0.3234(6)

Table 4: *Values of the pseudoscalar meson masses $aM_{PS}(\mu_S, \mu_1, \mu_2)$ for the various combinations of simulated sea and valence quark masses.*

μ_1	μ_2	$\mu_S = 0.0040$	$\mu_S = 0.0064$	$\mu_S = 0.0085$	$\mu_S = 0.0100$	$\mu_S = 0.0150$
0.0040	0.0040	0.0669(6)	0.0666(5)	0.0674(6)	0.0681(6)	0.0676(7)
0.0040	0.0064	0.0689(5)	0.0686(5)	0.0696(6)	0.0701(5)	0.0700(6)
0.0040	0.0085	0.0703(5)	0.0701(4)	0.0711(5)	0.0715(5)	0.0716(6)
0.0040	0.0100	0.0712(5)	0.0710(4)	0.0721(5)	0.0724(5)	0.0726(6)
0.0040	0.0150	0.0739(5)	0.0738(4)	0.0749(5)	0.0751(4)	0.0755(5)
0.0040	0.0220	0.0771(5)	0.0772(4)	0.0782(5)	0.0783(4)	0.0787(5)
0.0040	0.0270	0.0791(4)	0.0792(4)	0.0802(5)	0.0804(4)	0.0807(5)
0.0040	0.0320	0.0809(4)	0.0811(4)	0.0821(5)	0.0822(4)	0.0826(6)
0.0064	0.0064	0.0707(5)	0.0706(4)	0.0716(5)	0.0719(5)	0.0722(6)
0.0064	0.0085	0.0721(5)	0.0720(4)	0.0731(5)	0.0732(5)	0.0737(5)
0.0064	0.0100	0.0730(5)	0.0729(4)	0.0740(5)	0.0741(4)	0.0747(5)
0.0064	0.0150	0.0757(4)	0.0757(4)	0.0768(5)	0.0768(4)	0.0775(5)
0.0064	0.0220	0.0789(4)	0.0790(4)	0.0800(5)	0.0799(4)	0.0807(5)
0.0064	0.0270	0.0809(4)	0.0811(4)	0.0820(5)	0.0819(4)	0.0827(5)
0.0064	0.0320	0.0827(4)	0.0830(4)	0.0839(5)	0.0838(4)	0.0845(5)
0.0085	0.0085	0.0735(5)	0.0734(4)	0.0745(5)	0.0746(4)	0.0752(5)
0.0085	0.0100	0.0744(5)	0.0744(4)	0.0754(5)	0.0754(4)	0.0762(5)
0.0085	0.0150	0.0771(4)	0.0771(4)	0.0782(5)	0.0780(4)	0.0789(5)
0.0085	0.0220	0.0802(4)	0.0804(4)	0.0814(5)	0.0812(4)	0.0821(5)
0.0085	0.0270	0.0822(4)	0.0825(4)	0.0834(5)	0.0832(4)	0.0841(5)
0.0085	0.0320	0.0841(4)	0.0844(4)	0.0853(5)	0.0850(4)	0.0859(5)
0.0100	0.0100	0.0753(4)	0.0753(4)	0.0764(5)	0.0763(4)	0.0771(5)
0.0100	0.0150	0.0779(4)	0.0780(4)	0.0791(4)	0.0789(4)	0.0799(5)
0.0100	0.0220	0.0811(4)	0.0813(4)	0.0823(4)	0.0820(4)	0.0830(5)
0.0100	0.0270	0.0831(4)	0.0834(4)	0.0843(4)	0.0840(4)	0.0850(5)
0.0100	0.0320	0.0850(4)	0.0853(4)	0.0862(5)	0.0859(4)	0.0868(5)
0.0150	0.0150	0.0806(4)	0.0808(4)	0.0817(4)	0.0814(4)	0.0825(5)
0.0150	0.0220	0.0838(4)	0.0841(4)	0.0849(4)	0.0846(4)	0.0857(5)
0.0150	0.0270	0.0858(4)	0.0862(4)	0.0870(4)	0.0866(4)	0.0877(5)
0.0150	0.0320	0.0877(4)	0.0881(4)	0.0889(4)	0.0885(4)	0.0896(5)

Table 5: *Values of the pseudoscalar decay constants $af_{PS}(\mu_S, \mu_1, \mu_2)$ for the various combinations of simulated sea and valence quark masses.*

References

[1] Ph. Boucaud *et al.* [ETM Collaboration], Phys. Lett. B **650** (2007) 304 [arXiv:hep-lat/0701012].

[2] R. Frezzotti and G. C. Rossi, JHEP **0410** (2004) 070 [arXiv:hep-lat/0407002].

[3] A. M. Abdel-Rehim, R. Lewis, R. M. Woloshyn and J. M. S. Wu, Phys. Rev. D **74** (2006) 014507 [arXiv:hep-lat/0601036].

[4] M. Foster and C. Michael [UKQCD Collaboration], Phys. Rev. D **59** (1999) 074503 [arXiv:hep-lat/9810021].

[5] C. McNeile and C. Michael [UKQCD Collaboration], Phys. Rev. D **73** (2006) 074506 [arXiv:hep-lat/0603007].

[6] ETM Collaboration, in preparation. See also C. Urbach, plenary talk given at LATTICE 2007, `http://www.physik.uni-regensburg.de/lat07`.

[7] R. Frezzotti and G. C. Rossi, JHEP **0408** (2004) 007 [arXiv:hep-lat/0306014].

[8] S. R. Sharpe and J. M. S. Wu, Phys. Rev. D **71** (2005) 074501 [arXiv:hep-lat/0411021].

[9] R. Frezzotti, G. Martinelli, M. Papinutto and G. C. Rossi, JHEP **0604** (2006) 038 [arXiv:hep-lat/0503034].

[10] L. Del Debbio, L. Giusti, M. Luscher, R. Petronzio and N. Tantalo, JHEP **0702** (2007) 082 [arXiv:hep-lat/0701009].

[11] S. R. Sharpe, Phys. Rev. D **56**, 7052 (1997) [Erratum-ibid. D **62**, 099901 (2000)] [arXiv:hep-lat/9707018].

[12] J. Bijnens and T. A. Lahde, Phys. Rev. D **72** (2005) 074502 [arXiv:hep-lat/0506004].

[13] D. Becirevic and G. Villadoro, Phys. Rev. D **69** (2004) 054010 [arXiv:hep-lat/0311028].

[14] C. Aubin *et al.* [HPQCD Collaboration], Phys. Rev. D **70** (2004) 031504 [arXiv:hep-lat/0405022].

[15] ETM Collaboration, in preparation. See also P. Dimopoulos, talk given at LATTICE 2007, `http://www.physik.uni-regensburg.de/lat07`.

[16] K. G. Chetyrkin and A. Retey, Nucl. Phys. B **583** (2000) 3 [arXiv:hep-ph/9910332].

[17] A. Ali Khan *et al.* [CP-PACS Collaboration], Phys. Rev. D **65** (2002) 054505 [Erratum-ibid. D **67** (2003) 059901] [arXiv:hep-lat/0105015].

[18] S. Aoki *et al.* [JLQCD Collaboration], Phys. Rev. D **68** (2003) 054502 [arXiv:hep-lat/0212039].

[19] M. Della Morte, R. Hoffmann, F. Knechtli, J. Rolf, R. Sommer, I. Wetzorke and U. Wolff [ALPHA Collaboration], Nucl. Phys. B **729** (2005) 117 [arXiv:hep-lat/0507035].

[20] D. Becirevic *et al.*, Nucl. Phys. B **734** (2006) 138 [arXiv:hep-lat/0510014].

[21] M. Gockeler, R. Horsley, A. C. Irving, D. Pleiter, P. E. L. Rakow, G. Schierholz and H. Stuben [QCDSF Collaboration], Phys. Lett. B **639** (2006) 307 [arXiv:hep-ph/0409312].

[22] M. Gockeler *et al.*, Phys. Rev. D **73** (2006) 054508 [arXiv:hep-lat/0601004].

[23] T. Blum, T. Doi, M. Hayakawa, T. Izubuchi and N. Yamada, arXiv:0708.0484 [hep-lat].

[24] Q. Mason, H. D. Trottier, R. Horgan, C. T. H. Davies and G. P. Lepage [HPQCD Collaboration], Phys. Rev. D **73** (2006) 114501 [arXiv:hep-ph/0511160].

[25] C. Bernard *et al.* [MILC Collaboration], PoS **LAT2006** (2006) 163 [arXiv:hep-lat/0609053].

[26] T. Ishikawa *et al.* [JLQCD Collaboration], arXiv:0704.1937 [hep-lat].

[27] W. M. Yao *et al.* [Particle Data Group], J. Phys. G **33** (2006) 1.

[28] Y. Aoki *et al.*, Phys. Rev. D **72** (2005) 114505 [arXiv:hep-lat/0411006].

[29] C. Aubin *et al.* [MILC Collaboration], Phys. Rev. D **70** (2004) 114501 [arXiv:hep-lat/0407028].

[30] S. R. Beane, P. F. Bedaque, K. Orginos and M. J. Savage, Phys. Rev. D **75** (2007) 094501 [arXiv:hep-lat/0606023].

[31] T. Ishikawa *et al.*, PoS **LAT2006** (2006) 181 [arXiv:hep-lat/0610050].

[32] C. Allton *et al.* [RBC and UKQCD Collaborations], Phys. Rev. D **76** (2007) 014504 [arXiv:hep-lat/0701013].

[33] E. Follana, C. T. H. Davies, G. P. Lepage and J. Shigemitsu [HPQCD Collaboration], arXiv:0706.1726 [hep-lat].

[34] G. Isidori, conference summary talk at KAON'07, `http://www.lnf.infn.it/conference/kaon07`; see also G. Isidori, arXiv:0709.2438 [hep-ph].

[35] W. J. Marciano, Phys. Rev. Lett. **93** (2004) 231803 [arXiv:hep-ph/0402299].

[36] W. J. Marciano and A. Sirlin, Phys. Rev. Lett. **96** (2006) 032002 [arXiv:hep-ph/0510099].