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Cosmological horizons and reconstruction of quantum field theories.

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Dedicated to Professor Klaus Fredenhagen on the occasion of his 60th birthday.

Abstract. As a starting point, we state some relevant geometrical properties enjoyed by the cosmological horizon of a certain class of Friedmann-Robertson-Walker backgrounds. Those properties are generalised to a larger class of expanding spacetimes M admitting a geodesically complete cosmological horizon $\Im^-$ common to all co-moving observers. This structure is later exploited in order to recast, in a cosmological background, some recent results for a linear scalar quantum field theory in spacetimes asymptotically flat at null infinity. Under suitable hypotheses on M, encompassing both the cosmological de Sitter background and a large class of other FRW spacetimes, the algebra of observables for a Klein-Gordon field is mapped into a subalgebra of the algebra of observables $\mathcal{W}(\Im^-)$ constructed on the cosmological horizon. There is exactly one pure quasifree state λ on $\mathcal{W}(\Im^-)$ which fulfils a suitable energy-positivity condition with respect to a generator related with the cosmological time displacements. Furthermore λ induces a preferred physically meaningful quantum state λ_M for the quantum theory in the bulk. If M admits a timelike Killing generator preserving $\Im^-$, then the associated self-adjoint generator in the GNS representation of λ_M has positive spectrum (i.e. energy). Moreover λ_M turns out to be invariant under every symmetry of the bulk metric which preserves the cosmological horizon. In the case of an expanding de Sitter spacetime, λ_M coincides with the Euclidean (Bunch-Davies) vacuum state, hence being Hadamard in this case. Remarks on the validity of the Hadamard property for λ_M in more general spacetimes are presented.

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1 Introduction

In the framework of quantum field theory over curved backgrounds we witnessed, in the past few year, an increased display of new and important formal results. In many cases we can track their origin in the existence of a non trivial interplay between some field theories living on a Lorentzian background - say M - and a suitable counterpart constructed over a co-dimension one submanifold of M, often chosen as the conformal boundary of the spacetime. Usually thought of as a realization of the so-called holographic principle, this research line provided its most remarkable results in the framework of (asymptotically) AdS backgrounds. As a matter of fact, concepts such as Maldacena's conjecture [AGM00] - in a string framework - or Rehren's duality (see [DR02] and references therein) - in the algebraic quantum field theory setting - are appearing nowadays almost ubiquitously in the theoretical high-energy physics literature. More recently a similar philosophy has been also adopted to deal with a rather different scenario, namely asymptotically flat spacetimes, where it is future null infinity – $\Im^+ \sim \mathbb{R}\times\mathbb{S}^2$, *i.e.* the conformal boundary – which plays the role of the above-mentioned co-dimension one submanifold [DMP06, Mo06, Mo07, Da07].

Although one could safely claim that all these mentioned results are compelling, one should also actively seek connections to those theoretical models which are nowdays testable and, within this respect, one can safely claim that cosmology is a rather natural playground. In this realm, one of the most widely known theories is inflation where, as in other models, the pivotal role is played by a single scalar field living on an (almost) de Sitter background. Although, within this framework, most of the results are mainly, though not only, at a classical level, it is to a certain extent mandatory to look for a deep-rooted analysis of the full-fledged underlying quantum field theory in order to achieve a more firm understanding of the model under analysis.

To this avail, the first, but to a certain extent, not appealing chance is to perform a case-by-case analysis of the quantum structure of all the possible models nowadays available. In our opinion a more attractive possibility is to look for some mean allowing us to draw some general conclusions or to point out some universal feature, independently from the chosen model or from the chosen background. Taking into account this philosophy, a natural "first step" to undertake would be to try to implement the previously discussed bulk-to-boundary correspondence which appears to encode, almost per construction, all the criteria of universality we are seeking for, in the case of a large class of cosmological models.

As a starting point point let us assume the *Cosmological Principle* which leads the underlying background to be endowed with the widely-used Friedmann-Robertson-Walker (FRW) metrics. A direct

inspection of the geometric properties of these spacetimes points out that, in most of the relevant physical cases, such as de Sitter to quote just one example, it exists a natural submanifold which, at first glance, appears to be a good candidate as the preferred co-dimension 1 hypersurface: the *cosmological (future or past) horizon* as defined by Rindler [Ri06]. More precisely, in this paper we shall consider the cosmological past horizon $\Im^-$, in common with all the co-moving observers, in order to deal with expanding universes. The first of the main aims of this manuscript is indeed to discuss some non trivial geometric features of the cosmological horizon $\Im^-$. Particularly, under some technical restrictions on the analytic form of the expanding factor in the FRW metric with flat spatial section, the horizon has a universal structure and, hence, it represents the natural setting where to stage a bulk-to-boundary correspondence. An expanding universe admits a preferred future-oriented timelike vector field X defining the worldlines of co-moving observers, whose common expanding rest-frames are the 3-surfaces orthogonal to X. In FRW metrics X is a conformal Killing field which becomes tangent to the cosmological horizon and, in the class of FRW metrics we consider, it individuates complete null geodesics on $\Im^-$.

This extent will be generalised to expanding spacetimes M equipped with a geodesically complete cosmological horizon $\Im^-$ and an asymptotical conformal Killing field X, generally different from FRW spacetimes. The leading role of X in such a construction is strengthened by its intertwining relation with the conformal factor which is a primary condition to take into account if one wants to study the structure of the *symmetry group of the horizon* (actually a subgroup of the huge full isometry group of the horizon viewed as a semi-Riemannian manifold). We also address such an issue and we discover that such a group is actually an infinite dimensional group $SG_{\Im^-}$ which has the structure of an iterated semidirect product *i.e.* it is $SO(3) \ltimes \left(C^\infty(\mathbb{S}^2) \ltimes C^\infty(\mathbb{S}^2)\right)$ where $SO(3)$ is the special orthogonal group with a three dimensional algebra, whereas $C^\infty(\mathbb{S}^2)$ stands for the set of smooth functions over $\mathbb{S}^2$ thought as an Abelian group under addition. The geometric interpretation of $SG_{\Im^-}$ is intertwined to the following result. The subgroup of isometries of the spacetime which preserves the cosmological horizon structure is injectively mapped to a subgroup of $SG_{\Im^-}$ which, hence, encodes some of the possible symmetries of the spacetime. However it must be remarked that $SG_{\Im^-}$ is universal in the sense that it does not depend on the particular spacetime M in the class under consideration.

As a result we find that, under suitable hypotheses on M – valid, in particular, for certain FRW spacetimes which are de Sitter asymptotically – the algebra of observables $\mathcal{W}(M)$ of a Klein-Gordon field in M is one-to-one (isometrically) mapped to a subalgebra of the algebra of observables $\mathcal{W}(\Im^-)$ naturally constructed on the cosmological horizon. In this sense information of quantum theory in the bulk M is encoded in the quantum theory defined on the boundary $\Im^-$. It turns out that there is exactly one pure quasifree state λ on $\mathcal{W}(\Im^-)$ which fulfils a certain energy-positivity condition with respect to some generators of $SG_{\Im^-}$. The relevant generators are here those which can be interpreted as limit values on $\Im^-$ of timelike Killing vectors of M, whenever one fixes a spacetime M admitting $\Im^-$ as the cosmological horizon. However, exactly as the geometric structure of $\Im^-$, λ is universal in the sense that it does not depend on the particular spacetime M in the class under consideration. The GNS-Fock representation of λ individuates a unitary irreducible representation of $SG_{\Im^-}$. Fixing an expanding spacetime M with complete cosmological horizon, λ induces a preferred quantum state λ_M for the quantum theory in M and it enjoys remarkable properties. It turns out to be invariant under all those isometries of M (if any) that preserve the cosmological horizon structure. If M admits a timelike Killing generator preserving $\Im^-$, the associated self-adjoint generator in the GNS representation of λ_M has positive spectrum, *i.e.*, energy. Eventually, if M is the expanding de Sitter spacetime, λ_M coincides to the Euclidean (Bunch-Davies) vacuum state, so that it is Hadamard in that case at least. Actually, Hadamard property seems to be valid in general, but that issue will be investigated elsewhere.

As a final technical remark we would like to report that in the derivation of many results reported here we have been guided by similar analyses previously performed in the case of asymptotically flat

spacetime, using the null infinity as co-dimension one submanifold. However, to follow the subsequent discussion there is no need of being familiar with the tricky notion of asymptotically flat spacetime.

1.1. Notation, mathematical conventions. Throughout $\mathbb{R}^+ := [0, +\infty)$, $\mathbb{N} := \{0, 1, 2, \ldots\}$. For smooth manifolds M, N, $C^\infty(M; N)$ (omitting N whenever $N = \mathbb{R}$) is the space of smooth functions $f : M \to N$. $C_0^\infty(M; N) \subset C^\infty(M; N)$ is the subspace of compactly-supported functions. If $\chi : M \to N$ is a diffeomorphism, χ^* is the natural extension to tensor bundles (counter-, co-variant and mixed) from M to N (Appendix C in [Wa84]). A spacetime (M, g) is a Hausdorff, second-countable, smooth, four-dimensional connected manifold M, whose smooth metric has signature $-+++$. We shall also assume that a spacetime is oriented and time oriented. We adopt definitions of causal structures of Chap. 8 in [Wa84]. If $S \subset M \cap \widehat{M}$, (M, g) and $(\widehat{M}, \widehat{g})$ being spacetimes, $J^\pm(S; M)$ ($I^\pm(S; M)$) and $J^\pm(S; \widehat{M})$ ($I^\pm(S; \widehat{M})$) indicate the causal (chronological) sets associated to S and respectively referred to the spacetime M or $\widehat{M}$.

1.2. Outline of the paper. In section 2 we introduce and discuss the geometric set-up of the backgrounds we are going to take into account throughout this paper. Particularly we find under which analytic conditions on the expanding factor, a Friedmann-Robertson-Walker (FRW) spacetime can be smoothly extended to a larger spacetime that encompasses the cosmological horizon. In section 3 we provide a generalisation of the results of section 2 and we study their implications. Furthermore we introduce and discuss the structure of the horizon symmetry group showing its interplay with the possible isometries of the bulk metric. In section 4 we study the structure of bulk scalar QFT and of the associated Weyl algebra and its the horizon counterpart. Furthermore we discuss the existence of a preferred algebraic state invariant under the full symmetry group, which enjoys some uniqueness/energy-positivity properties. Subsections 4.3 and 4.4 are devoted to the development of the interplay between the bulk and the boundary theory; a particular emphasis is given to the selection of a natural preferred bulk states and on the analysis of its properties. Since all these conclusions are based upon some a priori assumptions on the behaviour of the solutions in the bulk of the Klein-Gordon equation with a generic coupling to curvature, we shall devote section 4.5 to test these requirements. Eventually, in section 5, we draw some conclusions and we provide some hints on future research perspectives.

2 Cosmological horizons and asymptotically flatness

2.1. *Friedmann-Robertson-Walker spacetime and cosmological horizons.* A homogeneous and isotropic universe can be locally described by a smooth spacetime, in the following indicated by (M, g_{FRW}), where M is a smooth Lorentzian manifold equipped with the following **Friedmann-Robertson-Walker** (FRW) **metric**

$$g_{FRW} = -dt \otimes dt + a(t)^2 \left[\frac{1}{1 - \kappa r^2} dr \otimes dr + r^2 d\mathbb{S}^2(\theta, \varphi) \right]. \tag{1}$$

Above, $d\mathbb{S}^2(\theta, \varphi) = d\theta \otimes d\theta + \sin^2\theta \, d\phi \otimes d\phi$ is the standard metric on the unit 2-sphere and, up to normalisation, κ can take the values $-1, 0, 1$ corresponding respectively to an hyperbolic, flat and closed spaces. The coordinate t ranges in some open interval I. Here $a(t)$ is a smooth function of t with constant sign (since g is nondegenerate). Henceforth we shall assume that $a(t) > 0$ when $t \in I$. We also suppose that the field ∂_t individuates the time orientation of the spacetime.
Physically speaking and in the universe observed nowadays, the sections of M at fixed t are the isotropic

and homogeneous 3-spaces containing the matter of the universe, the world lines describing the histories of those particles of matter being integral curves of ∂_t. In this picture, the **cosmic time** t is the *proper-time* measured at rest with each of these particles, whereas the scale $a(t)$ measures the size of the observed cosmic expansion in function of t.
The metric (1) may enjoy two physically important features. Consider a co-moving observer pictured by a integral line $\gamma = \gamma(t)$, $t \in I$, of the field ∂_t and focus on $J^-(\gamma)$. If $J^-(\gamma)$ does not cover the whole spacetime M, the observer γ cannot receive physical information from some events of M during his/her history: causal future-directed signals starting from $M \setminus J^-(\gamma)$ cannot achieve any point on γ. In other words, and adopting the terminology of [Ri06], a **cosmological horizon** takes place for γ and it is the null 3-hypersurface $\partial J^-(\gamma)$. Conversely, whenever $J^+(\gamma)$ does not cover the whole spacetime M, physical information sent by the observer γ during his/her story is prevented from getting to some events of M: Causal future-directed signals starting from γ do not reach any point in $M \setminus J^+(\gamma)$. In this case, exploiting again the terminology of [Ri06], a **cosmological past horizon** exists for γ. It is the null 3-hypersurface $\partial J^+(\gamma)$.
As it is well-known, a sufficient condition for the appearance of cosmological horizons can be obtained from the following analysis. One re-arranges the metric (1) into the form

$$g_{FRW} = a^2(\tau)\left[-d\tau \otimes d\tau + \frac{1}{1-\kappa r^2} dr \otimes dr + r^2 d\mathbb{S}^2(\theta,\varphi)\right] \doteq a^2(\tau) g(\tau, r, \theta, \varphi), \tag{2}$$

where

$$\tau(t) = d + \int a^{-1}(t) dt \tag{3}$$

is the **conformal cosmological time**, $d \in \mathbb{R}$ being any fixed constant. By construction $\tau = \tau(t)$ is a diffeomorphism from I to some open, possibly infinite, interval $(\alpha, \beta) \ni \tau$. Notice both that ∂_τ is a conformal Killing vector field whose integral lines coincide, up to the parametrisation, to the integral lines of ∂_t and that (M, g_{FRW}) is globally hyperbolic.

As causal structures are preserved under conformal rescaling of the metric, a straightforward analysis based on the shape of g in (2) establishes that $J^-(\gamma)$ does not cover he whole spacetime M whenever $\beta < +\infty$. In that case a cosmological event horizon takes place for γ. Similarly $J^+(\gamma)$ does not cover the whole spacetime M whenever $\alpha > -\infty$. In that case a cosmological past horizon takes place for γ. In both cases the horizons $\partial J^-(\gamma)$ and $\partial J^+(\gamma)$ are null 3-hypersurfaces diffeomorphic to $\mathbb{R} \times \mathbb{S}^2$, made of null geodesics of g_{FRW}. One may think of these surfaces as the limit light-cones emanating from $\gamma(t)$, respectively towards the past or towards the future, as t tends to $\sup I$ or $\inf I$ respectively. The tips of the cones generally get lost in the limit procedure: In realistic models α and β correspond, when they are finite, to a *big bang* or a *big crunch* respectively. As a general comment, we stress that *the cosmological horizons introduced above generally depend on the fixed observer* γ.

Remark 2.1. The requirement on the finiteness of the bounds α and β for the range of the conformal cosmological time τ are *sufficient* conditions for the existence of the cosmological horizons, but they are by no means necessary. Indeed it may happen that – and this is the case of de Sitter spacetime – there is, indeed a cosmological horizon *arbitrarily close to* M*, but outside* M. This happens when the spacetime M and its metric can be extended beyond its original region M to a larger spacetime $(\widehat{M}, \widehat{g})$ so that it happens that $\Im^+ = \partial J^-(M; \widehat{M}) = \partial M$ and $\Im^- = \partial J^+(M; \widehat{M}) = \partial M$. Hence the cosmological horizon $\Im^+$ or $\Im^-$ coincides with the boundary ∂M and, by construction, it does not depend on the considered observer γ (an integral curve of the field ∂_t) evolving in M. Referring in particular to a conformally static region M (equipped with the metric (1) for $\kappa = 0$) embedded in the complete de Sitter spacetime $\widehat{M}$,

∂M turns out to be a null surface with the topology of $\mathbb{R}\times\mathbb{S}^2$. In the following we shall focus on this type of cosmological horizons.

2.2. *FRW metrics with $\kappa=0$ and associated geometric structure.* Here, we would like to pinpoint some geometrical properties enjoyed by a subclass if the FRW spacetimes that will be used later in order to get the main results presented in this paper. To this end we consider here the spacetime (M, g_{FRW}), where $M\simeq(\alpha,\beta)\times\mathbb{R}^3$ and the metric g_{FRW} is like in (2), but with $\kappa=0$.
Furthermore we shall restrict our attention to the case where the factor $a(\tau)$ in (2) has the following form

$$a(\tau)=\frac{\gamma}{\tau}+O\left(\frac{1}{\tau^2}\right)\ ,\quad \frac{da(\tau)}{d\tau}=-\frac{\gamma}{\tau^2}+O\left(\frac{1}{\tau^3}\right) \tag{4}$$

for either $(\alpha,\beta):=(-\infty,0)$ and $\gamma<0$, or $(\alpha,\beta):=(0,+\infty)$ and $\gamma>0$. The above asymptotic values are meant to be taken as $\tau\to-\infty$ or $\tau\to+\infty$ respectively. The first issue we are going to discuss is the extension of the spacetime (M, g_{FRW}) to a larger spacetime $(\widehat{M},\widehat{g})$ that encompasses $\Im^+$ and/or $\Im^-$. To this end, if we introduce the new coordinates $U=\tan^{-1}(\tau+r)$ and $V=\tan^{-1}(\tau-r)$ ranging in subsets of $\mathbb{R}$ individuated by $\tau\in(\alpha,\beta)$ and $r\in(0,+\infty)$, (2) can be written as:

$$g_{FRW}=\frac{a^2(\tau(U,V))}{\cos^2U\cos^2V}\left[-\frac{1}{2}dU\otimes dV-\frac{1}{2}dV\otimes dU+\frac{\sin^2(U-V)}{4}d\mathbb{S}^2(\theta,\varphi)\right]. \tag{5}$$

The metric, obtained cancelling the overall factor $a^2(\tau(U,V))/(\cos^2U\cos^2V)$, is well-behaved and smooth for $U,V\in\mathbb{R}$ removing the axis $U=V$. This is nothing but the apparent singularity appearing for $r=0$ in the original metric (2). Consider $\mathbb{R}^2$ equipped with null coordinates U,V with respect to the standard Minkowskian metric on $\mathbb{R}^2$ and assume that every point is a 2-sphere with radius $|\sin(U-V)|/2$ (hence the spheres for $U=V$ are degenerate). Then, let us focus on the segments in $\mathbb{R}^2$

$$\begin{array}{l} a,\ V=U \text{ with } U\in(-\pi/2,\pi/2),\\ b,\ U=\pi/2 \text{ with } V\in(-\pi/2,\pi/2),\\ c, V=-\pi/2 \text{ with } U\in(-\pi/2,\pi/2).\end{array}$$

The original spacetime M is realized as a suitable subset of the union of the segment a, *i.e.* $r=0$, and the *interior* of the triangle abc, *i.e.* $r>0$, as in the figure 1. In this picture it is natural to assume that the null *endless* segments b and c representing null 3-hypersurfaces diffeomorphic to $\mathbb{R}\times\mathbb{S}^2$, individuate respectively $\Im^+$ and $\Im^-$ provided that $\beta=+\infty$ in the first case and/or $\alpha=-\infty$ in the second case where (α,β) is the domain of τ. Otherwise the points of M cannot get closer and closer to all the points of those segments. Therefore we are committed to assume $\alpha=-\infty$ and/or $\beta=+\infty$ and we stick with this assumption in the following discussion.
Summarising, we wish to extend g_{FRW} smoothly to a region larger than the open triangle abc joined with a, and including one of the endless segments b and c at least. In the case $a(\tau)$ is of the form (4), the function $a^2(\tau(U,V))/(\cos^2U\cos^2V)$ is smooth in neighbourhoods of the open segments b and c only if $\gamma\neq0$, and in particular it does not vanish on b and c, making nondegenerate $\widehat{g}$ thereon. However, a bad singularity appears as soon as $U=-V$, that is $\tau=0$. Therefore either:

$(\alpha,\beta)=(0,+\infty)$ – and in this case M $(r\geq0,\ \tau\in(0,+\infty))$ coincides with the upper half of the triangle abc, and it may be extended to a larger spacetime $(\widehat{M},\widehat{g})$ by adding a neighbourhood of the endless segment b viewed as $\Im^+$ – or

$(\alpha,\beta)=(-\infty,0)$ – and in this case M $(r\geq0,\ \tau\in(-\infty,0))$ coincides to the lower half of the triangle abc, and it may be extended to a larger spacetime $(\widehat{M},\widehat{g})$ by adding a neighbourhood the endless segment c viewed as $\Im^-$.

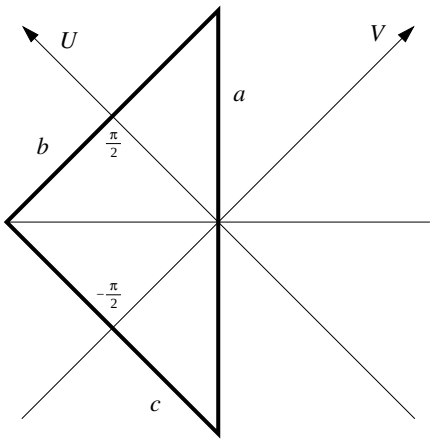

Figure 1: The interior of the triangle represents the original FRW background seen as an open subset of Einstein's static universe. Each point in the (U, V)-plane represents a 2-sphere and, furthermore, the segments b and c are respectively $\Im^+$ and $\Im^-$.

In both cases the line $U = -V$ does not belong to M and to its extension, and the metric $\widehat{g}$ coincides with the right-hand side of (5).

The function $a(\tau)$ and its interplay with the vector field ∂_τ when approaching the cosmological horizon will play a distinguished role in our construction for this reason let's enumerate below some of its properties that we are going to generalise in the next section. To this end, notice that $a(\tau)$ is smooth in $\widehat{M}$ and vanishes exactly either on $\Im^+ = \partial J^-(M; \widehat{M})$ or on $\Im^- = \partial J^+(M; \widehat{M})$, depending on the considered values for the interval (α, β) and for γ as discussed below formula (4). On the other hand, by direct inspection

$$da \restriction_{\Im^+} = -2\gamma dU\,, \quad da \restriction_{\Im^-} = -2\gamma dV,\,. \tag{6}$$

and hence da does not vanish either on $\Im^+$ or on $\Im^-$, provided $\gamma \neq 0$. By direct inspection one finds that, restricting either to $\Im^+$ or $\Im^-$, the metric $\widehat{g}$ takes the following distinguished form called **Bondi form**:

$$\widehat{g} \restriction_{\Im^\pm} = \gamma^2 \left(-d\ell \otimes da - da \otimes d\ell + d\mathbb{S}^2(\theta, \varphi)\right)\,,$$

where, with $\Im^\pm$, it is implicitly assumed that one must choose either $\Im^+$ or $\Im^-$ and where, for arbitrarily fixed constants k_+, k_-

$$\ell(U) = -\gamma^{-1} \tan U + k_- \quad \text{on } \Im^-\,, \qquad \ell(V) = -\gamma^{-1} \tan V + k_+ \quad \text{on } \Im^+\,,$$

hence $\ell \in \mathbb{R}$ turns out to be the parameter of the integral lines of $n \doteq \nabla a$.
Consider then the vector field ∂_τ, it is an easy task to check that it is a conformal Killing vector for $\widehat{g}$ in M with conformal Killing equation

$$\mathscr{L}_{\partial_\tau} \widehat{g} = -2\partial_\tau(\ln a)\, \widehat{g}\,.$$

where the right-hand side vanishes approaching either $\Im^+$ or $\Im^-$. Furthermore, ∂_τ tends to become tangent to either $\Im^+$ or $\Im^-$ approaching it and it coincides to $-\gamma \widehat{\nabla}^b a$ thereon, as can be directly seen from the form of ℓ.

3 Expanding universes with cosmological horizon and its group.

3.1. *Expanding universes with cosmological horizon* $\Im^-$. The previous discussion remarked that in an expanding FRW spacetimes the scale factor a and its interplay with the conformal Killing field ∂_τ play a distinguished role when approaching the cosmological horizon. A reader interested in asymptotically flat spacetime could have noticed that many of the above mentioned geometrical properties are shared by the structure of null infinity. In that realm, in [DMP06, Mo06, Mo07], it was shown that, when dealing with quantum field theory issues, a key role is played by a certain symmetry group of diffeomorphisms defined on $\Im^+$, the so called BMS group, which has the most notable property to embody the isometries of the bulk spacetime [Ge77, AX78] through a suitable geometric correspondence of generators. In the following we first generalise the result presented in the section 2.2 and then we shall construct the counterpart of the BMS group for the found class of spacetimes and the particular form of cosmological horizons.

Definition 3.1. *A globally hyperbolic spacetime* (M, g) *equipped with a positive smooth function* $\Omega : M \to \mathbb{R}^+$, *a future-oriented timelike vector* X *defined on* M, *and a constant* $\gamma \neq 0$, *will be called an* **expanding universe with (geodesically complete) cosmological (past) horizon** *when the following facts hold:*

1. **Existence and causal properties of horizon**. (M, g) *can be isometrically embedded as the interior of a sub manifold-with-boundary of a larger spacetime* $(\widehat{M}, \widehat{g})$, *the boundary* $\Im^- := \partial M$ *verifying* $\Im^- \cap J^+(M; \widehat{M}) = \emptyset$.

2. **Data interplay 1)**. Ω *extends to a smooth function on* $\widehat{M}$ *such that (i)* $\Omega\restriction_{\Im^-} = 0$ *and (ii)* $d\Omega \neq 0$ *everywhere on* $\Im^-$.

3. **Data interplay 2)**. X *is a conformal Killing vector for* $\widehat{g}$ *in a neighbourhood of* $\Im^-$ *in* M, *with*

$$\mathscr{L}_X(\widehat{g}) = -2X(\ln\Omega)\,\widehat{g}\,, \tag{7}$$

   where (i) $X(\ln\Omega) \to 0$ *approaching* $\Im^-$ *and (ii)* X *does not tend everywhere to the zero vector approaching* $\Im^-$.

4. **Global Bondi-form of the metric on** $\Im^-$ **and geodesic completeness**. *(i)* $\Im^-$ *is diffeomorphic to* $\mathbb{R} \times \mathbb{S}^2$, *(ii) the metric* $\widehat{g}\restriction_{\Im^-}$ *takes the* **Bondi form** *globally up to the constant factor* $\gamma^2 > 0$*:*

$$\widehat{g}\restriction_{\Im^-} = \gamma^2\left(-d\ell \otimes d\Omega - d\Omega \otimes d\ell + d\mathbb{S}^2(\theta,\phi)\right)\,, \quad \ell \in \mathbb{R}\,,\ (\theta,\phi) \in \mathbb{S}^2\,,\ \Omega = 0 \tag{8}$$

   $d\mathbb{S}^2$ *being the standard metric on the unit* 2*-sphere. Hence* $\Im^-$ *is a null* 3*-submanifold, and (iii) the curves* $\mathbb{R} \ni \ell \mapsto (\ell, \theta, \phi)$ *are complete null* $\widehat{g}$*-geodesics.*

The manifold $\Im^-$ *is called the* **cosmological (past) horizon of** M. *The integral parameter of* X *is called the* **conformal cosmological time**. *There is a completely analogous definition of* **contracting universe** *referring to the existence of* $\Im^+$ *in the future instead of* $\Im^-$.

Remark 3.1.
(1) In view of condition 3, the vector X is a Killing vector of the metric $g_0 := \Omega^{-2}g$ in a neighbourhood of $\Im^-$ in M. In such a neighbourhood, one can think of Ω^2 as an expansion scale evolving with rate $X(\Omega^2)$ referred to the conformal cosmological time.
(2) $\Im^- \cap J^+(M; \widehat{M}) = \emptyset$ entails $M = I^+(M; \widehat{M})$ and $\Im^- = \partial M = \partial I^+(M; \widehat{M}) = \partial J^+(M; \widehat{M})$, so that

$\Im^-$ has the proper interpretation as a past cosmological horizon in common for all the observers in (M, g) evolving along the integral lines of X.
(3) It is worth stressing that the spacetimes considered in the given definition are neither homogeneous nor isotropic in general; hence we can deal with a larger class of manifolds than simply the FRW spacetimes.

Similarly to the particular case examined previously, also in the general case pictured by Definition 3.1, the conformal Killing vector field X becomes tangent to $\Im^-$ and it coincides with ∂_ℓ up to a nonnegative factor, which now may depend on angular variables, as we go to establish. The proof of the following proposition is in the Appendix.

Proposition 3.1. *If $(M, g, \Omega, X, \gamma)$ is an expanding universe with cosmological horizon, the following holds.*

(a) *X extends smoothly to a unique smooth vector field $\widetilde{X}$ on $\Im^-$, which may vanish on a closed subset of $\Im^-$ with empty interior at most. Then X fulfils the $\widehat{g}$-Killing equation on $\Im^-$.*

(b) *$\widetilde{X}$ has the form $f\partial_\ell$, where, referring to the representation $\Im^- \equiv \mathbb{R} \times \mathbb{S}^2$, f depends only on the variables $\mathbb{S}^2$ and, furthermore, it is smooth and nonnegative.*

Since, for the FRW spacetimes, the function $f = f(\theta, \phi)$ appearing in $\widetilde{X} = f\partial_\ell$ takes the constant value 1, the presence of a nontrivial function f is related to the failure of isotropy for the more general spacetimes considered in Definition 3.1.

3.2. *The horizon symmetry group $SG_{\Im^-}$.* In the forthcoming discussion we shall make use several times of the following technical fact. In the representation $\Im^- \equiv \mathbb{R} \times \mathbb{S}^2 \ni (\ell, s)$, the null $\widehat{g}$-geodesic segments imbedded in $\Im^-$ are all of the curves

$$J \ni \ell \mapsto (\alpha\ell + \beta, s)\,, \quad \text{for constants } \alpha \neq 0, \beta \in \mathbb{R},\ s \in \mathbb{S}^2, \text{ and some interval } J \subset \mathbb{R}. \tag{9}$$

In this section, in the hypotheses of definition 3.1, we select a subgroup $SG_{\Im^-}$ of physically relevant isometries of $\Im^-$. We shall see in Proposition 3.3 that, as matter of fact, $SG_{\Im^-}$ contains the isometries generated by Killing vectors obtained as a limit towards $\Im^-$ of (all possible) Killing vectors of (M, g), when these vectors tend to become tangent to $\Im^-$. As a preliminary proposition, it holds:

Proposition 3.2. *If $(M, g, \Omega, X, \gamma)$ is an expanding universe with cosmological horizon and Y is a Killing vector field of (M, g), Y can be extended to a smooth vector field $\widehat{Y}$ defined on $\widehat{M}$ and*

(a) *$\mathscr{L}_{\widehat{Y}}\widehat{g} = 0$ on $M \cup \Im^-$;*

(b) *$\widetilde{Y} := \widehat{Y}|_{\Im^-}$ is uniquely determined by Y, and it is tangent to $\Im^-$ if and only if $g(Y, X)$ vanishes approaching $\Im^-$ from M. Restricting to the linear space of the Killing fields Y on (M, g) such that $g(Y, X) \to 0$ approaching $\Im^-$, the following further facts hold.*

(c) *If $\widetilde{Y}$ vanishes in some $A \subset \Im^-$ and $A \neq \emptyset$ is open with respect to the topology of $\Im^-$, then $Y = 0$ everywhere in M as well as $\widehat{Y}$ in $M \cup \Im^-$.*

(d) *The linear map $Y \mapsto \widetilde{Y}$ is injective, i.e. Killing vectors of (M, g) are represented on $\Im^-$* faithfully

The proof of the proposition above is given in the Appendix.
The statements (a) and (b) of Proposition 3.2 establish that the Killing vectors Y in M with $g(Y, X) \to 0$ approaching $\Im^-$ extend to *Killing vectors* of $(\Im^-, h)$, h being the degenerate metric on $\Im^-$ induced by $\widehat{g}$.

Since the vector fields $\widehat{Y}$ tangent to $\Im^-$ admit $\Im^-$ as invariant manifold, we can define

Definition 3.2. *If $(M, g, \Omega, X, \gamma)$ is an expanding universe with cosmological horizon, a Killing vector field of (M, g), Y, is said to* **to preserve** *$\Im^-$ if $g(Y, X) \to 0$ approaching $\Im^-$. Similarly, the Killing isometries of the (local) one-parameter group generated by Y are said* **to preserve** *$\Im^-$.*

In the rest of this part we shall consider the one-parameter group of isometries of $(\Im^-, h)$ generated by such Killing vectors $\widehat{Y}\restriction_{\Im^-}$. These isometries amount to a little part of the huge group of isometries of $(\Im^-, h)$. For instance, referring to the representation $(\ell, s) \in \mathbb{R} \times \mathbb{S}^2 \equiv \Im^-$, for *every* smooth diffeomorphism $f : \mathbb{R} \to \mathbb{R}$, the transformation $\ell \to f(\ell), \quad s \to s$ is an isometry of $(\Im^-, h)$. However only diffeomorphisms of the form $f(\ell) = a\ell + b$ with $a \neq 0$ can be isometries generated by the restriction $\widehat{Y}\restriction_{\Im^-}$ to $\Im^-$ of extensions of Killing fields Y of (M, g) as in the proposition 3.2. This is because those isometries are restrictions of isometries of the manifolds-with-boundary $(M \cup \{\Im^-\}, \widehat{g}\restriction_{M\cup\{\Im^-\}})$, and thus they *preserve the null $\widehat{g}$-geodesics in $\Im^-$*. These geodesics have the form (9). The requirement that, for every constants $a, b \in \mathbb{R}$, $a \neq 0$, there must be constants $a', b' \in \mathbb{R}$, $a' \neq 0$ such that $f(a\ell + b) = a'\ell + b'$ for all ℓ varying in a fixed nonempty interval J, is fulfilled only if f is an affine transformation as said above. We relax now the constraints on the above transformations allowing them also to be dependant on the angular coordinates. Hence we aim to study the class $G_{\Im^-}$ of diffeomorphisms $F : \Im^- \to \Im^-$

$$\ell \to \ell' := f(\ell, s)\,, \quad s \to s' := g(\ell, s) \quad \text{with } \ell \in \mathbb{R} \text{ and } s \in \mathbb{S}^2, \tag{10}$$

such that: (i) they are isometries of the degenerate metric h induced by $\widehat{g}\restriction_{\Im^-}$ (8) and (ii) they may be restrictions to $\Im^-$ of isometries of $\widehat{g}$ in $M \cup \Im^-$.
Assume that $F \in G_{\Im^-}$. The curve $\gamma : \mathbb{R} \ni \ell \to \gamma_s(\ell) \equiv (\ell, s)$ (with $s \in \mathbb{S}^2$ arbitrarily fixed) is a null geodesic forming $\Im^-$, therefore $\mathbb{R} \ni \ell \to F(\gamma_s(\ell))$ has to be, first of all, a null curve. In other words

$$\widehat{g}\restriction_{\Im^-}\left(\frac{\partial f}{\partial \ell}\frac{\partial}{\partial \ell} + \frac{\partial g}{\partial \ell}\frac{\partial}{\partial \theta} + \frac{\partial g}{\partial \ell}\frac{\partial}{\partial \phi}, \frac{\partial f}{\partial \ell}\frac{\partial}{\partial \ell} + \frac{\partial g}{\partial \ell}\frac{\partial}{\partial \theta} + \frac{\partial g}{\partial \ell}\frac{\partial}{\partial \phi}\right) = 0\,.$$

Using (8) and arbitrariness of $s \equiv (\theta, \phi)$, it implies that g does not depend on ℓ since the standard metric on the unital sphere is strictly positive definite. The map g has to be an isometry of $\mathbb{S}^2$ equipped with its standard metric. In other words $g \in O(3)$. Moreover, $\mathbb{R} \ni \ell \to F(\gamma_s(\ell)) = (f(\ell, s), g(s))$ has to be a null geodesic which belongs to $\Im^-$. As a consequence of (2) in remark 3.1, $(f(\ell, s), g(s)) = (c(s)\ell + b(s), g(s))$ for some fixed numbers $c(s), b(s) \in \mathbb{R}$ with $c(s) > 0$, and for every $\ell \in \mathbb{R}$. Summarising, it must be $g(\ell, s) = R(s)$ for all ℓ, s and $f(\ell, s) = c(s)\ell + b(s)$, for all ℓ, s, for some $R \in O(3)$, $c, b \in C^\infty(\mathbb{S}^2)$ with $c(s) \neq 0$. It is obvious that, conversely, every such a diffeomorphism fulfils (i) and (ii).

Remark 3.2. **(1)** By direct inspection one sees that the class $G_{\Im^-}$ of all diffeomorphisms F as above is a group with respect to the composition of diffeomorphisms.
(2) Only transformations $F \in G_{\Im^-}$, associated with R lying in the component connect to the identity of $O(3)$, *i.e.*, $SO(3)$ belong to a one-parameter group of isometries induced by Killing vectors in M.
From now on we shall restrict ourselves to the subgroup of $G_{\Im^-}$ whose elements are constructed using elements of $SO(3)$ and each element of the one-parameter group of diffeomorphisms generated by a vector field Z will be denoted by $\exp\{tZ\}$ being $t \in \mathbb{R}$.

Definition 3.3. *The* **horizon symmetry group** *$SG_{\Im^-}$ is the group (with respect to the composition of functions) of all diffeomorphisms of $\mathbb{R} \times \mathbb{S}^2$,*

$$F_{(a,b,R)} : \mathbb{R} \times \mathbb{S}^2 \ni (\ell, s) \mapsto \Big(e^{a(s)}\ell + b(s), R(s)\Big) \in \mathbb{R} \times \mathbb{S}^2 \quad \textit{with } \ell \in \mathbb{R} \textit{ and } s \in \mathbb{S}^2, \tag{11}$$

where $a, b \in C^\infty(\mathbb{S}^2)$ are arbitrary smooth functions and $R \in SO(3)$.
The **Horizon Lie algebra** $\mathfrak{g}_{\Im^-}$ *is the infinite-dimensional Lie algebra of smooth vector fields on $\mathbb{R} \times \mathbb{S}^2$ generated by the fields*

$$S_1\,, S_2\,, S_3\,,\ \beta\partial_\ell\,,\ \ell\alpha\partial_\ell\,, \quad \text{for all } \alpha, \beta \in C^\infty(\mathbb{S}^2).$$

S_1, S_2, S_3 indicate the three smooth vector fields on the unit sphere $\mathbb{S}^2$ generating rotations about the orthogonal axes, respectively, x, y and z.

It is worth noticing that $SG_{\Im^-}$ depends on the geometric structure of $\Im^-$ but not on the attached spacetime (M, g), which, in principle, *could not even admit any Killing vector preserving* $\Im^-$. In this sense $SG_{\Im^-}$ is a *universal* object for the whole class of expanding spacetimes with cosmological horizon. $SG_{\Im^-}$ may be seen as an abstract group defined on the set $SO(3) \times C^\infty(\mathbb{S}^2) \times C^\infty(\mathbb{S}^2)$, without reference to any expanding spacetime with cosmological horizon (M, g). Adopting this point of view, if we indicate $F_{a,b,R}$ by the abstract triple (R, a, b), the composition between elements in $SG_{\Im^-}$ reads

$$(R, a, b)(R', a', b') = \left(RR',\ \ a' + a \circ R',\ \ e^{a \circ R'} b' + b \circ R'\right), \tag{12}$$

for any $(R, a, b), (R', a', b') \in SO(3) \times C^\infty(\mathbb{S}^2) \times C^\infty(\mathbb{S}^2)$ and where $\circ$ denotes the usual composition of functions.
The relationship between $SG_{\Im^-}$ and $\mathfrak{g}_{\Im^-}$ is clarified in the following proposition.

Proposition 3.3. *Referring to the definition 3.3, the following facts hold:*
(a) *Each vector field $Z \in \mathfrak{g}_{\Im^-}$ is complete and the generated global one-parameter group of diffeomorphisms of $\mathbb{R} \times \mathbb{S}^2$, $\{\exp\{tZ\}\}_{t\in\mathbb{R}}$, is a subgroup of $SG_{\Im^-}$.*
(b) *For every $F \in SG_{\Im^-}$ there are $Z_1, Z_2 \in \mathfrak{g}_{\Im^-}$ – with, possibly, $Z_1 = Z_2$ – such that $F = \exp\{t_1 Z_1\} \exp\{t_2 Z_2\}$ for some real numbers t_1, t_2.*

The proof of this proposition is in the Appendix.
Furthermore, we have the following important result which finally makes explicit the interplay between Killing vectors Y in M preserving $\Im^-$, the group $SG_{\Im^-}$ and the Lie algebra $\mathfrak{g}_{\Im^-}$.

Theorem 3.1. *Let $(M, g, \Omega, X, \gamma)$ be an expanding universe with cosmological horizon and Y a Killing vector field of (M, g) preserving $\Im^-$. The following holds.*
(a) *The restriction of the unique smooth extension $\widetilde{Y}$ of Y to $\Im^-$ (see Prop. 3.2) belongs to $\mathfrak{g}_{\Im^-}$.*
(b) *$\{\exp\{t\widetilde{Y}\}\}_{t\in\mathbb{R}}$ is a subgroup of $SG_{\Im^-}$.*

The proof of this theorem is in the Appendix.
As an example consider the expanding universe M with cosmological horizon associated with the metric g_{FRW} (2) with $\kappa = 1$ and a as in (4). In this case $X := \partial_\tau$ and there is a lot of Killing vectors Y of (M, g_{FRW}) satisfying $g_{FRW}(Y, X) \to 0$ approaching $\Im^-$. The most trivial ones are all of the Killing vectors of the surfaces at $\tau =$constant with respect to the induced metric. We have here a Lie algebra generated by 6 independent Killing vectors Y associated, respectively, space translations and space rotations. In this case $g_{FRW}(Y, X) = 0$ so that the associated Killing vectors $\widehat{Y}\restriction_{\Im^-}$ belongs to $\mathfrak{g}_{\Im^-}$. This is not the whole story in the sharp case $a(\tau) = \gamma/\tau$ with $\gamma < 0$ which corresponds to the expanding de Sitter spacetime. Indeed, in this case, there is another Killing vector B of g_{FRW} fulfilling $g_{FRW}(B, X) \to 0$ approaching $\Im^-$. It is $B := \tau\partial_\tau + r\partial_r$. B, extended to $M \cup \Im^-$, gives rise to the structure of a *bifurcate Killing horizon* [KW91].
A last technical result, proved in the Appendix and useful in the forthcoming discussion, is

Proposition 3.4. *Let $(M, g, \Omega, X, \gamma)$ be an expanding universe with cosmological horizon and Y a smooth vector field of (M, g) which tends to the smooth field $\widetilde{Y} \in \mathfrak{g}_{\Im^-}$ pointwisely.*
If there is an open set $A \subset \widehat{M}$ with $A \supset \Im^-$ where $Y\restriction_{A\cap M}$ is timelike and future directed, then, everywhere on $\Im^-$,

$$\widetilde{Y}(\ell, s) = f(s)\partial_\ell \,, \quad \text{for some } f \in C^\infty(\mathbb{S}^2), \text{ with } f(s) \geq 0 \text{ on } \mathbb{S}^2. \tag{13}$$

4 Preferred states induced by the cosmological horizon.

In this section $(M, g, \Omega, X, \gamma)$ is an expanding universe with cosmological horizon. Since (M, g) is globally hyperbolic per definition, one can study properties of quantum fields propagating therein, following the algebraic approach in the form presented in [KW91, Wa94].

4.1. *QFT in the bulk.* Consider real linear bosonic QFT in (M, g) based on the symplectic space $(\mathcal{S}(M), \sigma_M)$, where $\mathcal{S}(M)$ is the space of real smooth, compactly supported on Cauchy surfaces, solutions φ of

$$P\varphi = 0 \,, \quad \text{where } P \text{ is the Klein-Gordon operator } P = \Box + \xi R + m^2 \,. \tag{14}$$

with $\Box = -\nabla_a \nabla^a$, $m > 0$ and $\xi \in \mathbb{R}$ constants. The nondegenerate, Cauchy-surface independent, symplectic form σ_M is:

$$\sigma_M(\varphi_1, \varphi_2) := \int_S (\varphi_2 \nabla_N \varphi_1 - \varphi_1 \nabla_N \varphi_2)\, d\mu_g^{(S)} \quad \forall \varphi_1, \varphi_2 \in \mathsf{S}(M) \,, \tag{15}$$

S being any Cauchy surface of M with normal unit future-directed vector N and 3-volume measure $d\mu_g^{(S)}$ induced by g. As is well known [BR021, BR022], it is possible to associate canonically any symplectic space, for instance $(\mathsf{S}(M), \sigma_M)$, a **Weyl C^*-algebra**, $\mathcal{W}(M)$ in this case. This is, up to (isometric) $*$-isomorphisms, unique and its generators $W_M(\varphi) \neq 0$, $\varphi \in \mathsf{S}(M)$, satisfy **Weyl commutation relations** (from now on we employ conventions as in [Wa94])

$$W_M(-\varphi) = W_M(\varphi)^* \,, \qquad W_M(\varphi) W_M(\varphi') = e^{i\sigma_M(\varphi,\varphi')/2} W(\varphi + \varphi') \,. \tag{16}$$

$\mathcal{W}(M)$ represents the basic set of quantum observable associated with the bosonic field ϕ propagating in the bulk spacetime (M, g).

The main goal of this section is to prove that the geometric structures on $(M, g, \Omega, X, \gamma)$ pick out a very remarkable algebraic state ω on $\mathcal{W}(M)$, which, among other properties turns out to be invariant under the natural action of every Killing isometry of (M, g) which preserves $\Im^-$. This happens provided a certain algebraic interplay between QFT in M and QFT on $\Im^-$ exists.

4.2. *Bosonic QFT on $\Im^-$ and $SG_{\Im^-}$-invariant states.* Referring to $\Im^- \equiv \mathbb{R} \times \mathbb{S}^2$, consider

$$\mathsf{S}(\Im^+) := \left\{ \psi \in C^\infty(\mathbb{R} \times \mathbb{S}^2) \;\middle|\; \psi \,, \partial_\ell \psi \in L^2(\mathbb{R} \times \mathbb{S}^2, d\ell \wedge \epsilon_{\mathbb{S}^2}(\theta, \phi)) \right\} \,, \tag{17}$$

$\epsilon_{\mathbb{S}^2}$ being the standard volume form of the unit 2-sphere, and the nondegenerate symplectic form σ

$$\sigma(\psi_1, \psi_2) := \int_{\mathbb{R}\times\mathbb{S}^2} \left(\psi_2 \frac{\partial \psi_1}{\partial \ell} - \psi_1 \frac{\partial \psi_2}{\partial \ell} \right) d\ell \wedge \epsilon_{\mathbb{S}^2}(\theta, \phi) \quad \forall \psi_1, \psi_2 \in \mathsf{S}(\Im^+) \,. \tag{18}$$

As in the previous section, we associate to $(\mathsf{S}(\Im^-), \sigma)$ the C^*-algebra $\mathcal{W}(\Im^-)$ whose generators $W(\psi) \neq 0$ satisfy the Weyl commutation relations (16)

Remark 4.1. Exploiting the given definitions, it is straightforwardly proved that $(\mathsf{S}(\Im^+), \sigma)$ is *invariant under the pull-back action of* $SG_{\Im^-}$. In other words (i) $\psi \circ g \in \mathsf{S}(\Im^-)$ if $\psi \in \mathsf{S}(\Im^-)$ and also (ii) $\sigma(\psi_1 \circ g, \psi_2 \circ g) = \sigma(\psi_1, \psi_2)$ for all $g \in SG_{\Im^-}$ and $\psi_1, \psi_2 \in \mathsf{S}(\Im^-)$. As a well known consequence [BR022, BGP96], $SG_{\Im^-}$ induces a $*$-automorphism $G_{\Im^-}$-representation $\alpha : \mathcal{W}(\Im^-) \to \mathcal{W}(\Im^-)$, uniquely individuated by linearity and continuity by the requirement

$$\alpha_g(W(\psi)) := W(\psi \circ g^{-1}) , \quad \psi \in \mathsf{S}(\Im^-) \text{ and } g \in G_{\Im^-} . \tag{19}$$

Since we are interested in physical properties which are $SG_{\Im^-}$-invariant, we face the issue about the existence of α_g-invariant algebraic states on $\mathcal{W}(\Im^-)$ with $g \in SG_{\Im^-}$.

We adopt here the definition of **quasifree state** given in [KW91], and also adopted in [DMP06, Mo06, Mo07]. Consider the quasifree state λ defined on $\mathcal{W}(\mathsf{S}(\Im^-))$ unambiguously defined as follows: if $\psi, \psi' \in \mathsf{S}(\Im^-)$, then

$$\lambda(W(\psi)) = e^{-\mu(\psi,\psi)/2} , \quad \mu(\psi, \psi') := Re \int_{\mathbb{R}\times\mathbb{S}^2} 2k\Theta(k)\overline{\widehat{\psi}(k,\theta,\phi)}\widehat{\psi}'(k,\theta,\phi)dk \wedge \epsilon_{\mathbb{S}^2}(\theta,\phi) , \tag{20}$$

the bar denoting the complex conjugation, $\Theta(k) := 0$ for $k < 0$ and $\Theta(k) := 1$ for $k \geq 0$; here we have used the ℓ-Fourier-Plancherel transform $\widehat{\psi}$ of ψ:

$$\widehat{\psi}(k,\theta,\phi) := \int_{\mathbb{R}} \frac{e^{ik\ell}}{\sqrt{2\pi}}\psi(\ell,\theta,\phi)d\ell , \quad (k,\theta,\phi) \in \mathbb{R}\times\mathbb{S}^2 . \tag{21}$$

The constraint

$$|\sigma(\psi,\psi')|^2 \leq 4\,\mu(\psi,\psi)\mu(\psi',\psi') , \qquad \text{for every } \psi, \psi' \in \mathsf{S} , \tag{22}$$

which must hold for every quasifree state (see Appendix A in [Mo06]), is fulfilled by the scalar product μ, as the reader can verify by inspection exploiting (21) and the definition of σ Consider the GNS representation of λ, $(\mathfrak{H}, \Pi, \Upsilon)$. Since λ is quasifree, $\mathfrak{H}$ is a bosonic Fock space $\mathcal{F}_+(\mathcal{H})$ with cyclic vector Υ given by the Fock vacuum and 1-particle Hilbert $\mathcal{H}$ space obtained as the Hilbert completion of the complex space generated by the "positive-frequency parts" $\Theta\widehat{\psi} =: K_\mu\psi$, of every wavefunction $\psi \in \mathsf{S}(\Im^-)$, with the scalar product $\langle\cdot,\cdot\rangle$ individuated by μ, as stated in (ii) of Lemma A1 in the Appendix A of [Mo06]. In our case

$$\langle K_\mu\psi, K_\mu\psi'\rangle = \int_{\mathbb{R}\times\mathbb{S}^2} 2k\Theta(k)\overline{\widehat{\psi}(k,\theta,\phi)}\widehat{\psi}'(k,\theta,\phi)dk \wedge \epsilon_{\mathbb{S}^2}(\theta,\phi). \tag{23}$$

The map $K_\mu : \mathsf{S}(\Im^-) \to \mathcal{H}$ is $\mathbb{R}$-linear and has a dense complexified range. A state similar to λ, and denoted by the same symbol, has been defined on $\Im^+ \simeq \mathbb{R}\times\mathbb{S}^2$ in [DMP06, Mo06, Mo07][1] and, barring minor adaption, it enjoys exactly the form (20). Therefore, we can make use of Theorem 2.12 in [DMP06] we know that λ *is pure*. Furthermore the one-particle space $\mathcal{H}$ of its GNS representation is isomorphic to the separable Hilbert space $L^2(\mathbb{R}^+ \times \mathbb{S}^2; 2kdk \wedge \epsilon_{\mathbb{S}^2})$.

[1] In [DMP06, Mo06] a different, but unitarily-equivalent, Hilbert space representation was used referring to the measure dk instead of $2kdk$. Features of Fourier-Plancherel theory on $\mathbb{R}\times\mathbb{S}^2$ were discussed in the Appendix C of [Mo07].

The state λ enjoys further remarkable properties in reference to the group $SG_{\Im^-}$. Particularly, since $(\mathfrak{H}, \Pi, \Upsilon)$ is its GNS triple, λ *turns out to be invariant under the* $*$*-automorphisms representation* (19) *for all* $g \in SG_{\Im^-}$. In other words $\lambda(\alpha_g(A))$ turns out to be equal to $\lambda(a)$ for all $A \in \mathcal{W}(\Im^-)$ and for all $g \in SG_{\Im^-}$ as it can be realized out of the straightforward extension to the whole algebra of the the following unitary action V of $SG_{\Im^-}$ on the one-particle Hilbert space $\mathcal{H}$:

$$\left(V_{(R,a,b)}\varphi\right)(k,s) := e^{a(R^{-1}(s))}e^{-ikb(R^{-1}(s))}\varphi\left(e^{a(R^{-1}(s))}k, R^{-1}(s)\right) \quad \text{for all } \varphi \in \mathcal{H}\,, \tag{24}$$

being $g = (R, a, b) \in SG_{\Im^-}$ and $s = (\theta, \phi)$. Furthermore, by standard manipulation, one can realize that the unique unitary representation $U : SG_{\Im^-} \ni g \mapsto U_g$ that implements α in $\mathfrak{H}$ while leaving Υ invariant, preserves $\mathcal{H}$ and it is unambiguously determined by $U\restriction_{\mathcal{H}}$. U has the following tensorialised form

$$U = I \oplus U\restriction_{\mathcal{H}} \oplus (U\restriction_{\mathcal{H}} \otimes U\restriction_{\mathcal{H}}) \oplus (U\restriction_{\mathcal{H}} \otimes U\restriction_{\mathcal{H}} \otimes U\restriction_{\mathcal{H}}) \oplus \cdots \tag{25}$$

Finally the restriction of U on the one-particle Hilbert space $\mathcal{H}$ is an irreducible representation.

A second important result concerns the positive-energy/uniqueness properties of λ. In Minkowski QFT positivity of energy, is a stability requirement and in general spacetimes the notion of energy is associated to that of a Killing time. This interpretation can be extended to this case too, namely to the theory on $\Im^-$. The positive-energy requirement is fulfilled for the "asymptotic" notion of time associated to the limit values $\widetilde{Y}$ towards $\Im^-$ of a timelike future-directed vector field Y in M, when $\widetilde{Y} \in \mathfrak{g}_{\Im^-}$. Notice that Y may not be a Killing vector outside $\Im^-$; it is enough that $Y \to \widetilde{Y} \in \mathfrak{g}_{\Im^-}$. This includes the case $Y = X$ in particular, due to Proposition 3.1.
In the following, $\{\exp\{tZ\}\}_{t\in\mathbb{R}}$ is the one-parameter subgroup of $G_{\Im^-}$ generated by any $Z \in \mathfrak{g}_{\Im^-}$ and $\{\alpha_t^{(Z)}\}_{t\in\mathbb{R}}$ is the associated one-parameter group of $*$-automorphisms of $\mathcal{W}(\Im^-)$ (19).

Proposition 4.1. *Consider an expanding universe with cosmological horizon* $(M, g, X, \Omega, \gamma)$*, the quasifree, pure,* $SG_{\Im^-}$*-invariant state* λ *on* $\mathcal{W}(\Im^-)$ *defined in (20) and a timelike future-directed vector field* Y *in* M *such that* $Y \to \widetilde{Y} \in \mathfrak{g}_{\Im^-}$ *pointwisely approaching* $\Im^-$ *(*$Y = X$ *in particular, in view of Proposition 3.1). The following holds.*
(a) *The unitary group* $\{U_t^{(\widetilde{Y})}\}_{t\in\mathbb{R}}$ *which implements* $\alpha^{(\widetilde{Y})}$ *leaving fixed the cyclic GNS vector in the GNS representation of* λ *is strongly continuous with nonnegative self-adjoint generator* $H^{(\widetilde{Y})} = -i\frac{d}{dt_s}U_t^{(\widetilde{Y})}|_{t=0}$.
(b) *The restriction of* $H^{(\widetilde{Y})}$ *to the one-particle space has no zero modes if and only if* $\widetilde{Y}$ *vanishes on a zero-measure subset of* $\Im^-$.

Proof. From Proposition 3.4 one has that $\widetilde{Y}(\ell, s) = f(s)\partial_\ell$ for some non negative smooth function $f : \mathbb{S}^2 \to \mathbb{R}$. Therefore $\exp\{t\widehat{Y}\}$ amounts to the displacement $(\ell, s) \to (\ell + f(s)t, s)$. As a consequence of the previous discussion, the one parameter group $\alpha^{(\widetilde{Y})}$ is unitarily represented by $\{U_t^{(\widetilde{Y})}\}_{t\in\mathbb{R}}$. $U_t^{(\widetilde{Y})}$ is the tensorialisation (as in (25)) of the (representation of the) unitary group in the one-particle space $V_t : \mathcal{H} \to \mathcal{H}$, with

$$(V_t\phi)(k,s) = e^{itkf(s)}\psi(k,s) = \left(e^{ith^{(\widetilde{Y})}}\psi\right)(k,s)\,, \quad \text{for all } \phi \in \mathcal{H}.$$

From standard theorems of operator theory one obtains that $\mathbb{R} \ni t \mapsto V_t$ is strongly continuous with self-adjoint generator $h^{(\widetilde{Y})}$, in the one-particle space $\mathcal{H} = L^2(\mathbb{R}^+ \times \mathbb{S}^2; 2kdk \wedge \epsilon_{\mathbb{S}^2})$, given by $(h^{(\widetilde{Y})}\phi)(k,s) = kf(s)\phi(k,s)$, defined in the dense domains $\mathcal{D}(h^{(\widetilde{Y})})$ made of the elements of the Hilbert space $L^2(\mathbb{R}^+ \times$

$\mathbb{S}^2; 2kdk \wedge \epsilon_{\mathbb{S}^2})$ such that the right-hand side belongs to $L^2(\mathbb{R}^+ \times \mathbb{S}^2; 2kdk \wedge \epsilon_{\mathbb{S}^2})$. It is so evident that, since $f \geq 0$, for every $\psi \in \mathcal{D}(H)$

$$\langle \phi, h^{(\widetilde{Y})} \phi \rangle = \int_0^{+\infty} 2kdk \int_{\mathbb{S}^2} \epsilon_{\mathbb{S}^2}(s) |\phi(k,s)|^2 k f(s) \geq 0 \,, \tag{26}$$

and thus $\sigma(h^{(\widetilde{Y})}) \subset [0, +\infty)$. Passing to the whole Fock space by (25) the result remains unchanged for the whole generator $H^{(\widetilde{Y})} = 0 + h^{(\widetilde{Y})} \oplus I \otimes h^{(\widetilde{Y})} \oplus h^{(\widetilde{Y})} \otimes I \oplus \cdots$ using standard properties of generators. The last statement is a trivial consequence of (26) using $\widetilde{Y} = f\partial_\ell$. □

The result applies in particular for $\widetilde{Y} = \partial_\ell$, since it is always possible to view ∂_ℓ as the limit value of some timelike vector field of M. For expanding universes with cosmological horizon as described in section 2.2, if $X := -\gamma \partial_\tau$, then $X \to \partial_\ell$ while approaching $\Im^-$. In this above case the energy-positivity property applies for X and there are no zero modes.
This is not the whole story, since the positive-energy property for ∂_ℓ, determines completely λ.

Theorem 4.1. *Consider the state λ defined in (20) and its GNS representation. The following holds.*
(a) *The state λ is the unique pure quasifree state on $\mathcal{W}(\Im^-)$ satisfying both:*
(i) it is invariant under $\alpha^{(\partial_\ell)}$,
(ii) the unitary group which implements $\alpha^{(\partial_\ell)}$ leaving fixed the cyclic GNS vector is strongly continuous with nonnegative self-adjoint generator (energy positivity condition).
(b) *Each folium of states on $\mathcal{W}(\Im^-)$ contains at most one pure $\alpha^{(\partial_\ell)}$-invariant state.*

Proof. The proofs of (a) and (b), though rather technical, are identical to those of the corresponding statements in Theorem 3.1 of [Mo06], where, in the cited proof, $\mathcal{F}$ refers to a Bondi frame. This holds since the self-adjoint generator of the unitary group $t \mapsto U_t$, implementing $\{\alpha_t^{(\partial_\ell)}\}_{t \in \mathbb{R}}$ and leaving Υ invariant, is the tensorialisation of the positive self-adjoint generator H acting in the one-particle space $L^2(\mathbb{R}^+ \times \mathbb{S}^2; 2kdk \wedge \epsilon_{\mathbb{S}^2})$ as $(H\widehat{\psi})(k,\theta,\phi) = k\widehat{\psi}(k,\theta,\phi)$. Note that H is defined in the dense domains of the elements of the Hilbert space $L^2(\mathbb{R}^+ \times \mathbb{S}^2; 2kdk \wedge \epsilon_{\mathbb{S}^2})$ such that the right-hand side is still in $L^2(\mathbb{R}^+ \times \mathbb{S}^2; 2kdk \wedge \epsilon_{\mathbb{S}^2})$. Hence $\sigma(H) = \sigma_c(H) = [0, +\infty)$.
The action of the one-parameter subgroup $\mathbb{R} \ni t \mapsto g^{(\partial_\ell)}(t)$ of $G_{\Im^-}$ on fields defined on $\Im^-$ coincides exactly with the one-parameter subgroup of the BMS group on fields defined on $\Im^-$. Furthermore also the unitary representations of $SG_{\Im^-}$ and of the BMS group are identical when restricted to those subgroups. □

4.3. *Interplay of QFT in M and QFT on $\Im^-$.* While in the previous section we have shown that it exists a preferred quasifree pure state λ invariant under the action of $SG_{\Im^-}$ and enjoying some uniqueness properties, we wonder now if it is possible to induce a state λ_M on the algebra of field observables in the bulk starting from λ. If this is the case, we would expect λ_M to fulfil some invariance properties with respect to the possible isometries individuated by Killing vectors which preserve $\Im^-$. To this avail, we concentrate beforehand on algebraic properties, establishing the existence of a nice interplay between $\mathcal{W}(\Im^-)$ and $\mathcal{W}(M)$ under suitable hypotheses on the considered symplectic forms. That interplay will be used to define λ_M in the next subsection.
The symplectic form σ_M on $\mathcal{S}(M)$ defined in (15) can be equivalently rewritten as the integral of a 3-form,

$$\sigma_M(\varphi_1, \varphi_2) := \int_S \chi(\varphi_1, \varphi_2) = \int_S \frac{1}{6} \left(\varphi_1 \nabla^\mu \varphi_2 - \varphi_2 \nabla^\mu \varphi_1\right) \sqrt{-\widehat{g}}\, \epsilon_{\mu\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma \,, \tag{27}$$

where $\epsilon_{\mu\alpha\beta\gamma}$ is the totally antisymmetric Levi Civita symbol, S is a future oriented Cauchy surface and the second equality holds in any local coordinate patch.

Notice that, even though S is moved back in the past and it seems to tend to coincide with $\Im^-$, this is not necessarily the case, since $\Im^-$ and Cauchy surfaces in M may have different topologies. In particular, information could get lost through the time-like past infinity i^-, the tip of the cone representing $\Im^-$. That point does not belong to $\widehat{M}$ in our hypotheses. However one may expect that, in certain cases at least, assuming that each φ_i extends to $\Gamma\varphi_i \in \mathsf{S}(\Im^-)$ smoothly, it holds

$$\sigma_M(\varphi_1, \varphi_2) = \int_{\Im^-} \chi(\Gamma\varphi_1, \Gamma\varphi_2)\,. \tag{28}$$

Now, by direct inspection one verifies that, for $\psi_1, \psi_2 \in \mathsf{S}(\Im^-)$,

$$\int_{\Im^-} \chi(\psi_1, \psi_2) = \gamma^2 \int_{\mathbb{R}\times\mathbb{S}^2} \left(\psi_2 \frac{\partial \psi_1}{\partial \ell} - \psi_1 \frac{\partial \psi_2}{\partial \ell}\right) d\ell \wedge \epsilon_{\mathbb{S}^2}(\theta, \phi)\,, \tag{29}$$

where γ is the last constant in $(M, g, \Omega, X, \gamma)$. Following this way one is led to expect that

$$\sigma_M(\varphi_1, \varphi_2) = \sigma(\gamma\Gamma\varphi_1, \gamma\Gamma\varphi_2)\,. \tag{30}$$

Notice that this result is by no means trivial and it might not hold, since it strictly depends on the behaviour of the solutions of Klein-Gordon equations across $\Im^-$.

Here we investigate the consequences of (30) under the hypothesis that such an identity holds true. The existence of $\Gamma : \mathcal{S}(M) \to \mathsf{S}(\Im^-)$ fulfilling (30) implies the existence of a isometric $*$-homomorphism $\imath : \mathcal{W}(M) \to \mathcal{W}(\Im^-)$. In this way the field observables of the bulk are mapped into observables of the theory on $\Im^-$. Moreover, the state λ on $\Im^-$ induces a preferred state λ_M on $\mathcal{W}(M)$ via pull-back. This state enjoys interesting invariance properties with respect to the symmetries of (M, g) which preserve $\Im^-$, as well as a positivity property with respect to timelike Killing vectors of M which preserve $\Im^-$.

Theorem 4.2. *Consider an expanding universe with cosmological horizon $(M, g, X, \Omega, \gamma)$ and suppose that every $\varphi \in \mathcal{S}(M)$ extends smoothly to some $\Gamma\phi \in \mathsf{S}(\Im^-)$ in order that (30) holds true:*

$$\sigma_M(\varphi_1, \varphi_2) = \sigma(\gamma\Gamma\varphi_1, \gamma\Gamma\varphi_2)\,, \quad \textit{for every } \varphi_1, \varphi_2 \in \mathcal{S}(M).$$

In these hypotheses, there is an (isometric) $*$-homomorphism $\imath : \mathcal{W}(M) \to \mathcal{W}(\Im^-)$ that identifies the Weyl C^*-algebra of the bulk M with a sub C^*-algebra of the boundary $\Im^-$; it is completely determined by the requirement:

$$\imath\left(W_M(\varphi)\right) := W(\gamma\Gamma\varphi)\,, \quad \textit{for all } \varphi \in \mathcal{W}(M). \tag{31}$$

Proof. Notice that the linear map $\gamma\Gamma : \mathcal{S}(M) \to \mathsf{S}(\Im^-)$ has to be injective due to nondegenerateness of σ and (30). Consider the sub Weyl-C^*-algebra A_M of $\mathcal{W}(\Im^-)$ generated by the elements $W(\gamma\Gamma\varphi)$ with $\varphi \in \mathcal{S}(M)$. Since Weyl C^*-algebras are determined up to (isometric) $*$-algebra isomorphisms, A_M is nothing but the Weyl C^*-algebra associated with the symplectic space $(\gamma\Gamma(\mathcal{S}(M)), \sigma)$ and the map $\gamma\Gamma : \mathcal{S}(M) \to \Gamma(\mathcal{S}(M))$ is an isomorphism of symplectic spaces. Under these hypotheses [BR022], there is a unique (isometric) $*$-isomorphism $\imath : \mathcal{W}(M) \to A_M \subset \mathcal{W}(\Im^-)$ completely individuated by (31). □

4.4. *The preferred invariant state λ_M.* We proceed to show that, in the hypotheses of Theorem 4.2, a preferred state λ_M on $\mathcal{W}(M)$ is induced by λ. That state enjoys very remarkable physical properties. From now on, if Y is a complete Killing vector of (M, g), the associated one-parameter group of g-isometries, $\{\exp\{tY\}\}_{t\in\mathbb{R}}$, preserves under pull-back action σ_M. Hence [BR022, BGP96] there is a unique isometric $*$-isomorphism $\beta_t^{(Y)} : \mathcal{W}(M) \to \mathcal{W}(M)$ induced by

$$\beta_t^{(Y)}\left(W_M(\varphi)\right) := W_M(\varphi \circ \exp\{-tY\})\,, \quad \text{for every } \varphi \in \mathcal{S}(M).$$

In the following we shall call $\beta^{(Y)} := \{\beta_t^{(Y)}\}_{t\in\mathbb{R}}$ the **natural $*$-isomorphism action of** $\{\exp\{tY\}\}_{t\in\mathbb{R}}$ **on** $\mathcal{W}(M)$. Similarly, every $Z \in \mathfrak{g}_{\Im^-}$ has a **natural action** $\alpha^{(Z)}$ **on** $\mathcal{W}(\Im^-)$ in terms of isometric $*$-isomorphism, obtained by requiring,

$$\alpha_t^{(Z)}(W(\psi)) := W(\psi \circ \exp\{-tZ\}) , \quad \text{for every } \psi \in \mathsf{S}(\Im^-),$$

since the pull-back action of $\{\exp\{tZ\}\}_{t\in\mathbb{R}}$, generated by Z on fields of $\mathsf{S}(\Im^-)$ preserves σ.
To stress a further important point, let us consider an expanding universe with cosmological horizon $(M, g, X, \Omega, \gamma)$ and let us suppose that every $\varphi \in \mathcal{E}(M)$ extends smoothly to some $\Gamma\varphi \in \mathsf{S}(\Im^-)$ in order that (30) holds true. In this case there is a uniquely defined smooth function $\widehat{\varphi}$ defined on $M \cup \Im^-$, that reduces to φ in M and to $\Gamma\varphi$ on $\Im^-$. If Y is a complete Killing vector of (M, g) preserving $\Im^-$, the one parameter group generated by its unique extension $\widehat{Y}$ to $M \cup \Im^-$ (Proposition 3.2 and Theorem 3.1) acts on $\widehat{\varphi}$ globally. Taking the relevant restrictions of scalar fields and Killing vector fields we obtain:

$$(\Gamma\varphi) \circ \exp\{t\widetilde{Y}\} = \Gamma\left(\varphi \circ \exp\{tY\}\right) , \tag{32}$$

where, as usual, $\widetilde{Y} := \widehat{Y}\restriction_{\Im^-}$. As a straightforward consequence it holds

$$\imath\left(\beta_t^{(Y)}(a)\right) = \alpha_t^{(\widetilde{Y})}(\imath(a)) , \quad \text{for all } a \in \mathcal{W}(M) \text{ and } t \in \mathbb{R} . \tag{33}$$

Theorem 4.3. *Consider an expanding universe with cosmological horizon $(M, g, X, \Omega, \gamma)$ fulfilling the hypotheses of Theorem 4.2. Let $\lambda_M : \mathcal{W}(M) \to \mathbb{C}$ be the state induced by λ defined in (20) through the isometric $*$-homomorphism $\imath$ (31):*

$$\lambda_M(a) := \lambda(\imath(a)) , \quad \textit{for all } a \in \mathcal{W}(M). \tag{34}$$

λ_M enjoys the following properties:
(a) *Whenever (M, g) admits some complete Killing vector field Y preserving $\Im^-$, then letting $\beta^{(Y)}$ be the natural action on $\mathcal{W}(M)$, λ_M is invariant under $\beta^{(Y)}$ and the unitary one-parameter group $\{U_t^{(Y)}\}_{t\in\mathbb{R}}$, which implements $\beta^{(Y)}$ in the GNS representation of λ_M leaving fixed the cyclic vector, is strongly continuous.*
(b) *If Y above is everywhere timelike and future-directed in M, then (i) the one-parameter group $\{U_t^{(Y)}\}_{t\in\mathbb{R}}$ has positive self-adjoint generator, (ii) that generator has no zero-modes in the one-particle subspace, if $\widetilde{Y} = 0$ on a zero-measure subset of $\Im^-$.*

Remark 4.2. As noticed before Proposition 4.1, positivity of energy is a stability requirement. The statement (b) of the theorem assures that, in the presence of a timelike Killing vector out of which defining the notion of energy, if it preserves $\Im^-$, the condition of energy positivity holds true. If such a timelike Killing vector is absent, then Proposition 4.1 assures nonetheless the validity of a positivity-energy condition, particularly with respect to the conformal Killing vector X.

4.5. *Testing the construction for the de Sitter case and for other FRW metrics.* We proceed to show that the hypotheses of Theorem 4.2 are valid when $(M, g, X, \Omega, \gamma)$ is in the class of the FRW metrics considered in section 2.2, so that the preferred state λ_M exists for those spacetimes. That class includes the expanding region of de Sitter spacetime (see [BMG94, BM96] for a related analysis in the framework of Wightman's axioms). We shall verify, in this last case, that the preferred state λ_M is nothing but the well-known *de Sitter Euclidean vacuum* or *Bunch-Davies state*, ω_E [SS76, BD78, Al85]. Let us start with de Sitter scenario. The expanding de Sitter region is

$$M \simeq (-\infty, 0) \times \mathbb{R}^3 , \quad g = a^2(\tau)\left[-d\tau \otimes d\tau + dr \otimes dr + r^2 d\mathbb{S}^2(\theta, \varphi)\right] , \tag{35}$$

where $\tau \in (-\infty, 0)$ and where r, θ, ϕ are standard spherical coordinates on $\mathbb{R}^3$, whereas $a(\tau) = \gamma/\tau$ for some constant $\gamma < 0$, so that and $R = 12/\gamma^2$. A class of, generally complex, solutions $\Phi_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{R}^3$ of (14) is[2]

$$\Phi_{\mathbf{k}}(\tau, \mathbf{x}) := \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \frac{\chi_{\mathbf{k}}(\tau)}{a(\tau)}\,, \tag{36}$$

where, according to [SS76], it holds

$$\chi_{\mathbf{k}}(\tau) := \frac{1}{2}\sqrt{-\pi\tau}\, e^{i\pi\nu/2}\overline{H^{(2)}_{\nu}(-k\tau)}\,, \quad \text{where} \quad \nu := \sqrt{\frac{9}{4} - 12(m^2R^{-1} + \xi)} \tag{37}$$

being $k := |\mathbf{k}|$ and $H^{(2)}_{\nu}$ is the second-type Hankel function. The sign in front of the square root in the definition of ν (which may be imaginary) does not affect the right-hand side of (37) and it could be fixed arbitrarily (either for ν real or imaginary). With these choices one finds the time-independent normalisation

$$\frac{d\overline{\chi_{\mathbf{k}}(\tau)}}{d\tau}\chi_{\mathbf{k}}(\tau) - \overline{\chi_{\mathbf{k}}(\tau)}\frac{d\chi_{\mathbf{k}}(\tau)}{d\tau} = i\,, \quad \text{for all } \tau \in (-\infty, 0). \tag{38}$$

Let us now show how ω_E is defined. To this end, take any $\varphi \in \mathcal{S}(M)$ and a Cauchy surface Σ_τ in (M, g) at fixed τ. Define

$$\widetilde{\varphi}(\mathbf{k}) := -i\int_{\mathbb{R}^3}\left[\frac{\partial\overline{\Phi_{\mathbf{k}}(\tau,\mathbf{x})}}{\partial\tau}\varphi(\tau,\mathbf{x}) - \overline{\Phi_{\mathbf{k}}(\tau,\mathbf{x})}\frac{\partial\varphi(\tau,\mathbf{x})}{\partial\tau}\right]a(\tau)^2 d\mathbf{x},\,. \tag{39}$$

where, per direct inspection, the right-hand side of (39) does not depend on the choice of τ.

Furthermore, $H^{(2)}_{\nu}(z)$ decays as $z^{-1/2}$ as $|z| \to \infty$, $\widetilde{\varphi} \in C^\infty(\mathbb{R}^3 \setminus \{0\})$ and it vanishes for $|\mathbf{k}| \to \infty$ faster than every power $|\mathbf{k}|^{-n}$, $n \in \mathbb{N}$. From the known behaviour of the functions $H^{(2)}_{\nu}(z)$ in a neighbourhood of $z = 0$ [GR95], one sees both that the leading divergence as $\mathbf{k} \to 0$ due to the functions $\chi_{\mathbf{k}}$ is of order $|\mathbf{k}|^{-|Re\nu|}$ and that $|\widetilde{\varphi}|^2$, as well as $|\widetilde{\varphi}|$, is integrable with respect to $d\mathbf{k}$ whenever $|Re\nu| < 3/2$ or, equivalently, $m^2 + \xi R > 0$. Once one constructs $\widetilde{\varphi}$ out of (39), then φ is

$$\varphi(\tau, \mathbf{x}) = \int_{\mathbb{R}^3}\left[\Phi_{\mathbf{k}}(\tau,\mathbf{x})\widetilde{\varphi}(\mathbf{k}) + \overline{\Phi_{\mathbf{k}}(\tau,\mathbf{x})\widetilde{\varphi}(\mathbf{k})}\right]\, d\mathbf{k}\,. \tag{40}$$

This holds out of (36), (38), (39), and of the properties of Fourier transform for functions in $C^\infty_0(\mathbb{R}^3)$. Since when $m^2 + \xi R > 0$ and $\varphi \in \mathcal{S}(M)$, $\widetilde{\varphi} \in L^2(\mathbb{R}^3; d\mathbf{k}) \cap L^1(\mathbb{R}^3; d\mathbf{k})$ then

$$-2Im\left\{\int_{\mathbb{R}^3}\overline{\widetilde{\varphi}_1}(\mathbf{k})\widetilde{\varphi}_2(\mathbf{k})d\mathbf{k}\right\} = \int_{\mathbb{R}^3}(\varphi_2\partial_\tau\varphi_1 - \varphi_1\partial_\tau\varphi_2)\, a^2(\tau)d\mathbf{x} =: \sigma_M(\varphi_1,\varphi_2) \quad \forall\varphi_1,\varphi_2 \in \mathcal{S}(M)\,. \tag{41}$$

The (restriction to M of the) **Euclidean vacuum** in de Sitter space is nothing but the quasifree state ω_E on $\mathcal{W}(M)$ completely identified by

$$\omega_E(W_M(\varphi)) = e^{-\frac{1}{2}\int_{\mathbb{R}^3}\overline{\widetilde{\varphi}(\mathbf{k})}\widetilde{\varphi}(\mathbf{k})\, d\mathbf{k}}\,, \quad \text{for every } \varphi \in \mathcal{S}(M). \tag{42}$$

[2] The form of the modes as presented in [BD78, BD82] is different both since in [SS76, BD78] the contracting region of de Sitter spacetime was considered and due to the absence of the overall exponential $\exp -i\pi\nu/2$, which would affect the final results and the normalisation (38) for ν imaginary, but not the final form of the two-point function.

Notice that the constraint (22) is automatically fulfilled in view of (41).

Remark 4.3. The maximally extended de Sitter spacetime can be realized by glueing together two isometric spacetimes – one expanding and the other contracting, when moving towards the future – on the common cosmological horizon. The obtained spacetime is maximally symmetric and admits $SO(1,5)$ as group of isometries. The state ω_E extends to a globally defined state on the whole de Sitter spacetime [Al85] and such a state is $O(1,5)$-invariant, hence it is invariant also under symmetries which do not preserve the horizon.

Theorem 4.4. *Consider the expanding universe (M,g,X,Ω,γ) given by (35) with $a(\tau)=\gamma/\tau$. Consider a quantum scalar Klein-Gordon field propagating in (M,g) with $m^2+\xi R>0$. Then,*
(a) *If $m^2+\xi R>\frac{5}{48}R$ (see also Remark 4.4), every $\varphi\in\mathcal{S}(M)$ extends smoothly to some $\Gamma\phi\in\mathsf{S}(\Im^-)$, (30) holds true and*
(b) *λ_M on $\mathcal{S}(M)$ coincides with the restriction to M of ω_E.*
The proof will be given in the appendix.

Remark 4.4. The requirement $m^2+\xi R>\frac{5}{48}R$, *i.e.* $|Re\nu|<1$ is used to assure that $\Gamma\varphi\in\mathsf{S}(\Im^-)$ if $\varphi\in\mathcal{S}(M)$. Actually the requirement can be dropped preserving only $m^2+\xi R>0$ if we change definition (15) of $\mathsf{S}(\Im^-)$, namely

$$\mathsf{S}(\Im^-):=\left\{\psi\in C^\infty(\mathbb{R}\times\mathbb{S}^2)\ \left|\ \int_{\mathbb{R}\times\mathbb{S}^2}|\widehat{\psi}(k,\theta,\phi)|^2|k|\,dk\wedge\epsilon_{\mathbb{S}^2}(\theta,\phi)<+\infty\right.\right\}$$

where $\widehat{\psi}$ indicates the Fourier-Plancherel transform of the *Schwartz distribution* ψ (as discussed in the Appendix C of [Mo07]). Then the symplectic form on $\Im^-$ could be defined Fourier transforming along the $\mathbb{R}$-direction (18). In this way, the identity (18) would hold true in a weaker limit sense, employing a suitable regularisation of ψ_1 and or ψ_2 by means of sequences of smooth compactly supported functions. Then the construction of λ on $\mathcal{W}(\Im^-)$ and of its GNS triple as well as the uniqueness/positive energy theorems would closely resemble to our previous analysis.
To conclude we have the last promised theorem proved in the Appendix: The hypotheses of Theorem 4.2 are fulfilled, and thus λ_M is defined, for FRW metrics as described in section 2.2 with $a(\tau)$ as in (4), provided the mass m of the Klein-Gordon field and/or the constant ξ are large enough.

Theorem 4.5. *Consider a quantum scalar Klein-Gordon field φ, satisfying (14) and propagating in an expanding universe (M,g,X,Ω,γ). Consider $a(\tau)$ as in (4) and with $\ddot{a}(\tau)=2\gamma/\tau^3+O(1/\tau^4)$ in such a way that $R=12/\gamma^2+O(1/\tau)$, then, if*

$$M\simeq(-\infty,0)\times\mathbb{R}^3\ ,\quad g=a^2(\tau)\left[-d\tau\otimes d\tau+dr\otimes dr+r^2d\mathbb{S}^2(\theta,\varphi)\right],$$

$\tau\in(-\infty,0)$ and r,θ,ϕ standard spherical coordinates on $\mathbb{R}^3$, $X=\partial_\tau$ and $\Omega=a(\tau)=\gamma/\tau+O(1/\tau^2)$ as $\tau\to-\infty$ for some constant $\gamma<0$.), whenever $m^2\gamma^2+12\xi>2$, every $\varphi\in\mathcal{S}(M)$ extends smoothly to some $\Gamma\phi\in\mathsf{S}(\Im^-)$ and (30) holds true.

Remark 4.5. **(1)** Theorem 4.5 is also valid relaxing the hypothesis to the case $\xi=1/6$ and $m=0$. In this case the proof is similar to that of the case studied in [DMP06, Mo06].
(2) The validity of Hadamard property for the states λ_M will be investigated in a forthcoming paper. However, a first scrutiny shows that it does hold for the states λ_M considered in Theorem 4.5 provided the two-point function of such a state is a distribution of $\mathcal{D}'(M\times M)$. The proof is similar to the one in

[Mo07]. The distributional requirement is fulfilled if the functions $\Gamma\varphi$, $\varphi \in \mathcal{E}(M)$, satisfy a suitable decay property as $\ell \to -\infty$.

5 Conclusions and open issues.

In this manuscript, we were able to prove that, imposing some suitable constraints on the expansion factor $a(t)$, the FRW background can be extended to a larger spacetime which encompasses the cosmological horizon. Such structure is later generalised in definition 3.1 where we introduce a novel notion of an expanding universe (M, g) with geodesically complete cosmological past horizon $\Im^-$. It is worth to stress that, in the set of backgrounds we are taking into account, besides the conformal factor Ω, a relevant role is played by a future oriented timelike vector X which is a conformal Killing vector for the metric g. As a byproduct of these geometric properties, we were able to construct explicitly the structure of the subgroup $SG_{\Im^-}$ of the isometry group of $\Im^-$, *i.e.*, the iterated semidirect product $SO(3) \ltimes \left(C^\infty(\mathbb{S}^2) \ltimes C^\infty(\mathbb{S}^2)\right)$. Such a result suggests us that one could hope to readapt in this framework some of the properties of a scalar quantum field theory as discussed in [DMP06, Mo06, Mo07].

In fact, using only the universal structure of $\Im^-$, we was able to select, for the theory on the horizon, a preferred state λ which is quasi-free and pure. λ is the unique state which, besides the previous properties, is also invariant under the action of the horizon symmetry group; actually, uniqueness for pure quasifree states on $\mathcal{W}(\Im^-)$ holds with the only hypotheses of invariance with respect to the one-parameter group generated by ∂_ℓ and a more general uniqueness property is valid as discussed in Theorem 4.1. Moreover, for any future oriented timelike vector field Y in the bulk such that it projects on the horizon to $\widetilde{Y}$, *i. e.* a generator of the Lie algebra of $SG_{\Im^-}$, then the unitary group of operators implementing the action of $\widetilde{Y}$ on the GNS representation of λ is strongly continuous with a non negative self-adjoint generator. Finally the one-particle space in the GNS representation of the state λ turns out to be an irreducible representation of the group of horizon symmetries $SG_{\Im^-}$.

In section 4, we considered a generic massive scalar Klein-Gordon equation with an arbitrary coupling to curvature. Under the assumption that each solution of such an equation for compactly supported initial data projects on the horizon to a rapidly decreasing smooth function - say ψ - and that such a projection preserves a suitable symplectic form, then we were able to draw some interesting conclusions. As a first step the projection map between classical fields extends also at a level of Weyl algebras, namely we can embed the bulk Weyl C^*-algebra as a C^*-subalgebra of the horizon counterpart. Furthermore such an embedding between Weyl algebras can be exploited in order to pull-back λ to a bulk state λ_M which is still quasi-free and invariant under the action of any bulk isometry which preserves the cosmological horizon. Furthermore, whenever the Killing vector is everywhere future oriented and timelike, than the one-parameter group of unitary operators implementing such an action is positive with self-adjoint generator.

As previously mentioned these results hold true under certain hypotheses which we tested in section 4.6 where we studied the behaviour of solutions for the Klein-Gordon equation of motion with an arbitrary coupling to curvature both in the de-Sitter and in the FRW background. Our analysis shows – see theorem 4.6 – that the hypotheses made at the beginning of section 4, hold true at least whenever certain conditions between the relevant parameters in the equation of motion are satisfied. In the deSitter case λ_M coincides with the well-known Euclidean Bunch-Davies vacuum.

On the overall we feel safe to claim that the analysis we performed proves that the investigation of a quantum field theory in a suitable cosmological background by means of an horizon counterpart is a viable option. Hence, as a future perspective, one would hope as a first step to extend the domain of applicability of theorem 4.6, and later to further discuss the properties for the bulk state. In particular our long-term aim is to prove both that λ_M is pure and that it is Hadamard so that it can be used in

renormalisation procedures, especially for the stress energy tensor [Wa94, Mo03, HW05]. Furthermore we should also investigate possible relations with the adiabatic states often exploited in the study of field theories on FRW backgrounds [JS02, LR90, Ol07, Pa69]. Concerning the validity of Hadamard property, it holds true for λ_M when M is deSitter spacetime since in this case λ_M is the Euclidean vacuum. However, a first scrutiny shows that it does hold for all the states λ_M considered in Theorem 4.5 provided the two-point function of such a state is a distribution of $\mathcal{D}'(M \times M)$. The proof is almost the same as that preformed in [Mo07].

At last but not at least, it would be interesting to extend our results to interacting fields. From a physical perspective this would be the most appealing scenario since, as mentioned in the introduction, nowadays cosmological models are often based upon a single scalar field whose dynamic is governed by a non trivial potential. It could also be worth to investigate possible applications of our results to the description of dark matter. Being weakly interacting, it is feasible to model it, at least in a first approximation, as a free quantum scalar field on a curved background. Although here we do not address all the above mentioned topics, we believe that this manuscript could be a nice first step towards this direction and we hope to discuss many if not all these mentioned points in a forthcoming manuscript.

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A Proof of some technical results.

Proof of Proposition 3.1. (a) If there were a smooth extension of X to $\overline{M}$ it would be unique by continuity, moreover, by continuity again, it would define a Killing vector for $\widehat{g}$ when restricting to the surface $\Im^-$, because the right-hand side of (7) vanishes there. We, in fact, will prove the existence of a smooth extension to the whole $\widehat{M}$. Coordinates $(\ell, \Omega, \theta, \phi)$ are defined in a neighbourhood $U \subset \widehat{M}$ of $\Im^- = \partial M$. Using the whole class of smooth curves $\gamma : t \to (\ell_0, t, \theta_0, \phi_0)$ where $(\ell_0, \theta_0, \phi_0) \in \mathbb{R} \times \mathbb{S}^2$ are fixed arbitrarily, and the transport equations [Ge77, Hal04]

$$\dot\gamma^a \widehat\nabla_a \widehat X_b = \dot\gamma^a \left(\widehat F_{ab} + \frac{1}{2} \widehat g_{ab} \widehat\varphi \right), \qquad \dot\gamma^a \widehat\nabla_a \widehat\varphi = \dot\gamma^a \widehat K_a$$
$$\dot\gamma^a \widehat\nabla_a \widehat F_{bc} = \dot\gamma^a \left(\widehat R_{bcad} \widehat X^d + \widehat K_{[b}\, \widehat g_{c]a} \right), \qquad \dot\gamma^a \widehat\nabla_a \widehat K_b = \dot\gamma^a \left(\widehat X^d \widehat\nabla_d \widehat L_{ab} + \widehat\varphi \widehat L_{ab} + 2 \widehat R_{d(a}\, \widehat F_{\ b)}^{\ d} \right) \quad (43)$$

(where $\widehat L_{ab} := \widehat R_{ab} - \frac{1}{6} \widehat g_{ab} \widehat R$) we can "transport" X, $F_{ab} = \widehat\nabla_a X_b - \widehat\nabla_b X_a$, $\varphi\widehat g := \frac{1}{2} \mathscr{L}_X(\widehat g)$, and $K_a := \widehat\nabla_a \varphi$ beyond $\Im^-$ in U. The transported fields $\widehat X$, $\widehat F$, $\widehat\varphi$, and $\widehat K_a$ are nothing but the solutions of the first order differential equations (43), with initial conditions given by the known fields X, F, φ, K evaluated on a fixed smooth surface $\Omega = \Omega(\ell, \theta, \phi)$ completely included in $M \cap U$. In M, $\widehat X$ coincides with X itself (and $\widehat F$ coincides with F itself and so on), since every conformal Killing vector field fulfils transport equations (43) [Ge77, Hal04] and uniqueness theorem holds for solutions of ordinary differential equations. Outside M one gets a smooth field $\widehat X$ anyway, due to the jointly dependence of solution of differential equations from the initial data (assigned on a smooth surface as well). Obviously the constructed field $\widehat X$ does not need to fulfil conformal Killing equations outside $\overline{M}$. In this way we have constructed a smooth extension $\widehat X$ of X on the open set $M \cup U$ inclosing $\Im^-$, the further extension to $\widehat M$ is now trivial, using

standard smoothing technology. By continuity, $\mathscr{L}_{\widehat{X}} = \Omega^{-1}X(\Omega)\widehat{g}$ must hold on $\Im^-$. This means that the right-hand side smoothly extends there (to zero by hypotheses). In particular, since $\Omega = 0$ on $\Im^-$, $\widehat{X}(\Omega) = 0$ on $\Im^-$. That is $\langle \widehat{X}\restriction_{\Im^-}, d\Omega\rangle = 0$, and thus $\widehat{X}\restriction_{\Im^-}$ is tangent to $\Im^-$ as wanted.
The set on $\Im^-$ of the points where $\widehat{X}$ vanishes is closed since $\widehat{X}$ is continuous. To conclude, we wish to prove that $\widehat{X}\restriction_{\Im^-}$ cannot vanish on every (nonempty) open set $A \subset \Im^-$ (otherwise it vanishes everywhere on $\Im^-$, but this case is not allowed by definition of X). Assume that there is such A where $\widehat{X}\restriction_A = 0$, take $p \in A$ and fix any other point $q \in \Im^-$, such that there is a $\widehat{g}$-geodesics, $\gamma \subset \Im^-$, joining p and q. We assume here that γ is either a space-like geodesics on $\mathbb{S}^2$ or a null-like geodesic at constant angular variables. We want to prove that $\widehat{X}(q) = 0$ when $\widehat{X}\restriction_A = 0$.
If $\widehat{X}\restriction_A = 0$, all the derivatives $\widehat{\nabla}_a\widehat{X}^b$ vanish, in A, when $a \neq \Omega$, that is referring to directions tangent to $\Im^-$. However, on $\Im^-$ it holds $\mathscr{L}_{\widehat{X}}\widehat{g} = 0$, by hypotheses. Writing down these equations explicitly, one finds that $\widehat{X} = 0$ on A implies $\widehat{\nabla}_\Omega\widehat{X}^b = 0$ if $b \neq \Omega$. However $\widehat{\nabla}_\Omega X^\Omega\restriction_{\Im^-} = 0$ holds since both $X^\Omega = X(\Omega)$ and $X(\Omega)/\Omega = X^\Omega/\Omega$ vanishes on $\Im^-$. We have found that, in A, $\widehat{F}_{ab} = 0$. Notice that $\varphi = 0$ in A, since it is proportional to the limit of $\Omega^{-1}X(\Omega)$ approaching $\Im^-$ which vanishes by hypotheses. This also entails that $\widehat{K}_a = 0$ when $a \neq \Omega$, in A, that is $\widehat{K}^a \neq 0$ for $a = \ell$ at most, in A. Let k denote the value $\widehat{K}(p)$ for the considered field $\widehat{X}$ with $\widehat{X}\restriction_A = 0$. Let us finally focus on the differential equations (43) referred to the mentioned geodesic $[0,1] \ni t \mapsto \gamma(t)$. We argue that a solution, and thus the unique solution, for initial data at p, $\widehat{X}(0) = 0$, $\widehat{F}_{ab}(0) = 0$, $\widehat{\varphi}(0) = 0$, $\widehat{K}(0) := k$ is $\widehat{X}(t) = 0$, $\widehat{F}_{ab}(t) = 0$, $\widehat{\varphi}(t) = 0$, $\widehat{K}(t)$, for all $t \in [0,1]$, where the last function is the unique satisfying $\dot{\gamma}^a\widehat{\nabla}_a\widehat{K}_b = 0$ with $\widehat{K}(0) := k$. To prove it notice that, inserting these functions in (43), the equations reduce to

$$\dot{\gamma}^a\widehat{K}_a = 0\,, \quad \dot{\gamma}^a\widehat{K}^b - \dot{\gamma}^b\widehat{K}^a = 0\,, \quad \dot{\gamma}^a\widehat{\nabla}_a\widehat{K}_b = 0, \tag{44}$$

The first two equations are certainly fulfilled at $t = 0$ by hypotheses, the third one determines K uniquely with the initial condition $\widehat{K}(0) := k$. However also the first two equations are fulfilled on this solution in view of the fact that they are fulfilled at $t = 0$ and that $\dot{\gamma}^a\widehat{\nabla}_a\dot{\gamma}^b = 0$ since we are dealing with a geodesic. We have found that, in particular, X vanishes at q as wanted, since $X(1) = 0$. With the same procedure, moving p and q about the original positions, we find that X vanishes in a open set A_q which enlarges A and it includes q. Iterating the procedure, we can enlarge A_q in order to include any third point $q' \in \Im^-$, joined to q by means of a second geodesics, so that X vanishes at q' too. In view of the form (8) of the metric on $\Im^-$, for every couple of points $p, q' \in \Im^-$, there is always a sequence of three consecutive geodesics, of the two above-mentioned types, joining p and q'. Therefore X vanishes everywhere on $\Im^-$.
(b) In a neighbourhood of $\Im^-$, referring to coordinates $\Omega, \ell, \theta, \phi$ one has

$$\widehat{X} = f^\Omega\partial_\Omega + f^\ell\partial_\ell + f^\theta\partial_\theta + f^\phi\partial_\phi\,.$$

Approaching $\Im^-$ (i.e. as $\Omega = 0$) one gets (1) $f^\Omega = 0$, since $\widehat{X}$ becomes tangent to $\Im^-$. However one also finds (2) $\partial_\Omega f^\Omega\restriction_{\Im^-} = 0$ as a consequence of $(f^\Omega - f^\Omega\restriction_{\Im^-})/\Omega = \Omega^{-1}X(\Omega) \to 0$ approaching $\Im^-$. Since $\widehat{X}\restriction_{\Im^-}$ is tangent to the null surface $\Im^-$ and it is the limit of a timelike vector, we also know that, at the points where it does not vanish, it must be light-like and future directed. Since $\widehat{X}\restriction_{\Im^-} = f^\ell\partial_\ell + f^\theta\partial_\theta + f^\phi\partial_\phi$, the requirement $\widehat{g}(\widehat{X}, \widehat{X})\restriction_{\Im^-} = 0$ implies that (3) $f^\theta = f^\phi = 0$ everywhere on $\Im^-$, in view of the Bondi form of the metric on $\Im^-$. Therefore (4) $\widehat{X}\restriction_{\Im^-} = f^\ell(0,\ell,\theta,\phi)\partial_\ell$. Using Bondi form of the metric again, the requirement $(\mathscr{L}_{\widehat{X}}\widehat{g})\restriction_{\Im^-} = 0$ produces immediately the constraints $\partial_\ell f^\ell\restriction_{\Im^-} = 0$ in view of (1),(2), (3), and (4), so that $\widehat{X}\restriction_{\Im^-} = f(\theta,\phi)\partial_\ell$. Since $\widehat{X}\restriction_{\Im^-}$ cannot vanish in any open set on $\Im^-$, f cannot vanish in any open set on $\mathbb{S}^2$. Since f is smooth and thus continuous, the set $f^{-1}(0)$ must be closed. Since, with our sign convention for the Bondi metric, both X and ∂_ℓ are future oriented, f cannot be negative. □

Proof of Proposition 3.2. We start from the proofs of (a) and (b). If there were a smooth extension of Y to $\overline{M} = M \cup \Im^-$ it would be unique by continuity and it would satisfy $\mathscr{L}_{\widehat{Y}}\widehat{g} = 0$ up to $\Im^-$ by continuity again. Therefore it is sufficient to establish the existence of a smooth extension to $\widehat{M}$ to get the most relevant part of (a) and (b). The proof is essentially the same as done in the proof of Proposition 3.1, concerning the existence of the extension of the field X. Now, Y is a proper conformal Killing field so that the transport equations (44) [Ge77, Hal04] reduces to

$$\dot{\gamma}^a \widehat{\nabla}_a \widehat{Y}_b = \dot{\gamma}^a \widehat{F}_{ab} \quad \text{and} \quad \dot{\gamma}^a \widehat{\nabla}_a \widehat{F}_{bc} = \widehat{R}_{bcad}\dot{\gamma}^a \widehat{Y}^d \,, \tag{45}$$

The procedure is exactly as that in the proof of Proposition 3.1 and, in this way, one obtains a smooth extention $\widehat{Y}$ of Y on $\widehat{M}$ and in particular on $\Im^-$. The condition that $\widehat{Y}$ is tangent to $\Im^-$ is $\langle \widehat{Y}, d\Omega\rangle = 0$ everywhere on $\Im^-$. However $g^{sb}\partial_b\Omega = (\partial_\ell)^s$ and $X \to f\partial_\ell$ approaching $\Im^-$, for some nonnegative function $f \in C^\infty(\mathbb{S}^2)$, as showed in Proposition 3.1. Therefore $\langle \widehat{Y}, d\Omega\rangle f = \lim_{\to \Im^-} g(\widehat{Y}, X)$. If the limit vanishes approaching $\Im^-$, $\langle \widehat{Y}, d\Omega\rangle = 0$ on the points $(\ell, s) \in \mathbb{R}\times\mathbb{S}^2$ where $f(s) \neq 0$. This happens on an open nonempty set $B \subset \mathbb{S}^2$. Therefore $\langle \widehat{Y}, d\Omega\rangle = 0$ on $\mathbb{R}\times B$. Let $(\ell_0, s_0) \notin \mathbb{R}\times B$. Since $\mathbb{S}^2 \setminus B$ has no interior (see Proposition 3.1), there is a sequence $\mathbb{R}\times B \ni (\ell_0, s_n) \to (\ell_0, s_0)$ as $n \to \infty$. Continuity of $(\ell, s) \mapsto \langle \widehat{Y}, d\Omega\rangle(\ell, s)$ implies $\langle \widehat{Y}, d\Omega\rangle = 0$ in $\mathbb{R}\times(\mathbb{S}^2 \setminus B)$ and, thus, everywhere. Conversely, if $\widehat{Y}$ is tangent to $\Im^-$, then $\langle \widehat{Y}, d\Omega\rangle = 0$ on $\Im^-$, and hence $\lim_{\to \Im^-} g(\widehat{Y}, X) = \langle \widehat{Y}, d\Omega\rangle f = 0$.
To conclude, we prove the last statements: (c) and (d). Since the map $Y \mapsto \widehat{Y}\restriction_{\Im^-}$ is linear by construction, (d) is a trivial consequence of (c). Let us prove (c). If the considered space is made of the zero vector only, the proof of (c) is trivial. Assume that it is not the case. To prove (c), it is sufficient to prove that the identity $\widehat{Y}\restriction_{\Im^-} = 0$ on a set $A \subset \Im^-$ which is nonempty and open with respect to the topology of $\Im^-$, entails $Y = 0$ in M (and thus $\widehat{Y} = 0$ in $M \cup \Im^-$ by continuity). Let us show it. Consider any fixed point $p \in M$ and a smooth path γ from some $q \in A$ to p (it exists because M is connected and $\Im^- = \partial M$). In view of the first order transport equations (45), $Y(p) = \widehat{Y}(p) = 0$ when both $\widehat{Y}(q)$ and $\widehat{F}_{ab}(q)$ vanish. Let us show that it is the case. Suppose that $\widehat{Y}\restriction_{\Im^-} = 0$ on A as above. Using coordinates $(\ell, \Omega, \theta, \phi)$ about $\Im^-$, one has that $\partial_a \widehat{Y}^b \restriction_A = 0$ if $a \neq \Omega$. On the other hand, the condition $\mathscr{L}_{\widehat{Y}}\widehat{g}_{ab} = 0$ computed on A, taking into account $\widehat{Y}\restriction_A = 0$ and $\partial_a \widehat{Y}^b \restriction_A = 0$ if $a \neq \Omega$, yields $\partial_\Omega \widehat{Y}^b \restriction_A = 0$, so that $\widehat{\nabla}_a \widehat{Y}^b \restriction_A = \partial_\Omega \widehat{Y}^b \restriction_A + \widehat{\Gamma}^b_{ac}\widehat{Y}^c \restriction_A = 0$. Therefore $F_{ab}\restriction_A = 0$ and it concludes the proof. □

Proof of Proposition 3.3. (a) If (s^1, s^2) are (local) coordinates of a point $s \in \mathbb{S}^2$, fix $\alpha, \beta \in C^\infty(\mathbb{S}^2)$ and real constants r_1, r_2, r_3. We wish to study the integral lines $t \mapsto (\ell(t), s(t)) \in \mathbb{R}\times\mathbb{S}^2$ of the field $Z(\ell, s) := (\alpha(s)\ell + \beta(s))\partial_\ell + \sum_{k=1}^3 r_k S^i_k \partial_{s^i}$ on $\mathbb{R}\times\mathbb{S}^2$, with initial condition (ℓ_0, s_0). By construction, the components referred to the sphere do not depend on ℓ and thus, the corresponding equations can be integrated separately. Since $\sum_{k=1}^3 r_k S^i_k \partial_{s^i}$ is smooth and $\mathbb{S}^2$ is compact, the integral lines $t \mapsto s(t|s_0)$ (here and henceforth $|s_0$ denotes the initial condition at $t = 0$) must be smooth and complete (i.e. defined for $t \in (-\infty, +\infty)$), in view of well-known theorems of differential equations on manifolds. Then assume that the smooth function $\mathbb{R} \ni t \to s(t|s_0)$ is known (computed as above). The remaining differential equation reads

$$\frac{d\ell}{dt} = \alpha(s(t|s_0))\ell + \beta(s(t|s_0)) \,.$$

It can be integrated and the right-hand side is defined for the values of t where the full integral ' converges:

$$\ell(t|s_0, \ell_0) = e^{\int_0^t dt_1 \alpha(s(t_1|s_0))}\ell_0 + e^{\int_0^t dt_1 \alpha(s(t_1|s_0))}\int_0^t dt_1 \beta(s(t_1|s_0)) e^{-\int_0^{t_1} dt_2 \alpha(s(t_2|s_0))} . \tag{46}$$

It is apparent that the parameter t ranges in the whole real axis due to smoothness of $\mathbb{R} \ni t \to \alpha(s(t|s_0))$ and $\mathbb{R} \ni t \to \beta(s(t|s_0))$, and that $\mathbb{R} \ni t \mapsto \ell(t|s_0, t_0)$ is smooth as well. We have established that

the integral lines of Z are complete and thus, in view of known theorems, the one-parameter group of diffeomorphisms generated by Z is global. Since $s = s(t)$ must necessarily describe a rotation of $SO(3)$, about the axis $(r_1, r_2, r_3)/\sqrt{r_1^2 + r_2^2 + r_3^2}$ with angle $t\sqrt{r_1^2 + r_2^2 + r_3^2}$, of the point on $\mathbb{S}^2$ initially individuated by s_0 and, taking (46) into account, it is evident that each diffeomorphism

$$\mathbb{R} \times \mathbb{S}^2 \ni (\ell_0, s_0) \mapsto (\ell(t|s_0, t_0), s(t|s_0)) \in \mathbb{R} \times \mathbb{S}^2 \,,$$

for every fixed $t \in \mathbb{R}$, has the form (11) and, thus, it belongs to $SG_{\Im^-}$.
(b) A fixed $(a, b, R) \in SG_{\Im^-}$ can be decomposed as

$$(R, a, b) = (I, a \circ R^{-1}, b \circ R^{-1})\,(R, 0, 0)\,.$$

Looking at (46), $(R, 0, 0)$ is an element of the one-parameter group generated by $\sum_{k=1}^{3} n_k S_k$, where (n_1, n_2, n_3) are the Cartesian components of the rotation axis of R; conversely the transformation $(I, a \circ R^{-1}, b \circ R^{-1})$ can be written as $\exp\{1Z\}$ where $Z = \ell a\left(R^{-1}(s)\right)\partial_\ell + b\left(R^{-1}(s)\right)\partial_\ell$. □

Proof of Theorem 3.1. Consider the *local* one-parameter group of diffeomorphisms generated by $\widehat{Y}$ in a sufficiently small neighbourhood (in $\widehat{M}$) of a point $q \in \Im^-$ and for $t \in (-\epsilon, \epsilon)$ with $\epsilon > 0$ sufficiently small. In local coordinates over $\Im^-$, $(\ell, s^1, s^2) \in (a, b) \times A$, such a set of transformations can be represented by

$$\ell \to \ell_t := f(\ell, s_1, s_2, t)\,, \quad (s^1, s^2) \to (s^1_t, s^2_t) := g(\ell, s^1, s^2, t) \quad \text{with } (\ell, s^1, s^2) \in (a, b) \times A. \tag{47}$$

Using the same argument as the one used to characterise the group $SG_{\Im^-}$ (after Definition 3.2), one finds that it must be $g(\ell, s_1, s_2, t) = R_t(s)$ for all ℓ, s and $f(\ell, s_1, s_2, t) = c(s_1, s_2, t)\ell + b(s_1, s_2, t)$, for all ℓ, s, for some $R_t \in O(3)$ depending on t smoothly, and where c, b are jointly smooth real functions. The requirement, that $t \mapsto R_t$ is a (local) one-parameter subgroup of $SO(3)$, implies that $\frac{dR_t}{dt}|_{t=0} = \sum_{k=1}^{3} r_k S_k(s_1, s_2)$. Similarly $\frac{df_t}{dt}|_{t=0} = \frac{\partial c(s_1, s_2, t)}{\partial t}|_{t=0}\ell + \frac{\partial b(s_1, s_2, t)}{\partial t}|_{t=0}$. We have found that, in local coordinates

$$\widehat{Y}\restriction_{\Im^-} = \sum_{k=1}^{3} r_k S_k\,(s_1, s_2) + \frac{\partial c(s_1, s_2, t)}{\partial t}|_{t=0}\ell\partial_\ell + \frac{\partial b(s_1, s_2, t)}{\partial t}|_{t=0}\partial_\ell\,,$$

and thus, about q, $\widehat{Y}\restriction_{\Im^-}$ takes the form of the vectors in $\mathfrak{g}_{\Im^-}$. However, since it holds true in a neighbourhood of each point on $\Im^-$, we have that $\widehat{Y}\restriction_{\Im^-} \in \mathfrak{g}_{\Im^-}$.
To conclude, (b) is an immediate consequence of (a) and of the last part of (a) in Proof of Proposition 3.3. □

Proof of Proposition 3.4. Since $\widetilde{Y} \in \mathfrak{g}_{\Im^-}$, in principle it has the form

$$\widetilde{Y}(\ell, s) = \sum_{i=1}^{3} c_i S_i(s) + (f(s) + \ell g(s))\partial_\ell\,.$$

Since $\widehat{g}(Y, Y) < 0$ about $\Im^-$ and its limit toward $\Im^-$, namely $\widetilde{Y}$, is tangent to $\Im^-$ it must satisfy $\widehat{g}(\widetilde{Y}, \widetilde{Y}) = 0$ by continuity (no timelike tangent vectors can be tangent to a null surface). Using the form (8) of $\widehat{g}$ one see that it must be: $\sum_{i=1}^{3} c_i S_i(s) = 0$ on $\Im^-$. Using the explicit form of S_1, S_2, S_3 referring to the base $\partial_\phi, \partial_\theta$ of $T\mathbb{S}^2$, one sees that this is equivalent to claim that, everywhere on the sphere,

$$(c_1 \sin\phi - c_2 \cos\phi) = 0\,, \quad c_1 \cot\theta \cos\phi + c_2 \cot\theta \sin\phi + c_3 = 0$$

As a consequence $c_1 = c_2 = c_3 = 0$. Therefore, everywhere on $\Im^-$

$$\widetilde{Y} = (f(s) + \ell g(s))\partial_\ell \, ,$$

for some functions $f, g \in C^\infty(\mathbb{S}^2)$. $\widetilde{Y}$ is the limit of a causal future-directed vector. Therefore, it has either to vanish or to be directed as ∂_ℓ at every point of $\Im^-$. Since $\ell g(s)$ may take every arbitrarily large, positive or negative, value (notice that g is bounded, it being smooth on a compact set), it must be $g(s) = 0$ and $f(s) \geq 0$. □

Proof of Theorem 4.3. As before, from now on, $(\mathcal{F}_+(\mathcal{H}), \Pi, \Upsilon)$ is the GNS triple of λ. First of all we notice that λ_M is in fact a well-defined state on $\mathcal{W}(M)$ since $\imath$ is a $*$-homomorphism. λ_M is quasifree associated with a real scalar product $\mu_M : \mathcal{S}(M) \times \mathcal{S}(M) \to \mathbb{R}$ defined as $\mu_M(\varphi, \varphi') := \mu(\gamma\Gamma\varphi, \gamma\Gamma\varphi')$. From this fact, it follows that the GNS triple of λ_M can be constructed as $(\mathcal{F}_+(\mathcal{H}_M), \Pi\restriction_{A_M}, \Upsilon)$, where $A_M \subset \mathcal{W}(\Im^-)$ is the sub C^*-algebra isomorphic to $\mathcal{W}(M)$ in view of Theorem 4.2, $\mathcal{H}_M$ is the Hilbert subspace of $\mathcal{H}$ given by the closure of the space of complex linear combinations of $K_\mu(\Gamma(\varphi))$, for every $\varphi \in \mathcal{S}(M)$ and, thus, $\mathcal{F}_+(\mathcal{H}_M)$ is a Fock subspace of $\mathcal{F}_+(\mathcal{H})$. In particular, the canonical $\mathbb{R}$-linear map $K_{\mu_M} : \mathcal{S}(M) \to \mathcal{H}_M$ is nothing but $K_{\mu_M} = K_\mu \circ \gamma\Gamma$.
(a) By construction, using the definition of λ_M, taking advantage of (33) as well as of the invariance property of λ under the action of $SG_{\Im^-}$, if $a \in \mathcal{W}(M)$, one has

$$\lambda_M\left(\beta_t^{(Y)}(a)\right) = \lambda\left(\imath\left(\beta_t^{(Y)}(a)\right)\right) = \lambda\left(\alpha_t^{(\widetilde{Y})}\imath(a)\right) = \lambda\left(\imath(a)\right) = \lambda_M(a) \, .$$

This proves the first part of (a). To conclude the proof of (a), let $V_t^{(\widetilde{Y})} : \mathcal{H} \to \mathcal{H}$ the one-parameter group of unitaries that implements $\alpha_t^{(\widetilde{Y})}$ in the one-particle space $\mathcal{H}$ for λ. From $K_{\mu_M} = K_\mu \circ \gamma\Gamma$, (33) and the construction of V one has:

$$V_t^{(\widetilde{Y})} K_{\mu_M}\varphi = V_t^{(\widetilde{Y})} K_\mu \gamma\Gamma(\varphi) = K_\mu\left(\gamma\Gamma\left(\varphi \circ \exp\{-tY\}\right)\right) = K_{\mu_M}\left(\varphi \circ \exp\{-tY\}\right) \, .$$

We have found that, for every $\varphi \in \mathcal{S}(M)$, $V_t^{(\widetilde{Y})} K_{\mu_M}\varphi = K_{\mu_M}\left(\varphi \circ \exp\{-tY\}\right)$, hence $V_t^{(\widetilde{Y})}$ leaves the one particle space of λ_M, $\mathcal{H}_M$, invariant and $V_t^{(\widetilde{Y})}\restriction_{\mathcal{H}_M}$ implements $\beta_t^{(Y)}$ in $\mathcal{H}_M$. As a consequence of the structure of the GNS triple of λ_M, if $U_t^{(\widetilde{Y})}$ implements $\beta_t^{(\widetilde{Y})}$ unitarily in $\mathfrak{H} = \mathcal{F}_+(\mathcal{H})$ leaving Υ invariant, it leaves also invariant the structure of the GNS-Fock space of λ_M and, therein, $U_t^{(\widetilde{Y})}\restriction_{\mathcal{F}_+(\mathcal{H}_M)}$ implements $\alpha_t^{(Y)}$ unitarily in $\mathfrak{H}_M = \mathcal{F}_+(\mathcal{H}_M)$ leaving the cyclic vector invariant. In other words

$$U_t^{(Y)} = U_t^{(\widetilde{Y})}\restriction_{\mathcal{F}_+(\mathcal{H}_M)} \, .$$

Notice that $\mathbb{R} \ni \mapsto U_t^{(\widetilde{Y})}\restriction_{\mathcal{F}_+(\mathcal{H}_M)}$ is strongly continuous since $\mathbb{R} \ni \mapsto U_t^{(\widetilde{Y})}$ is such. Moreover the self-adjoint generator of $U_t^{(\widetilde{Y})}\restriction_{\mathcal{F}_+(\mathcal{H}_M)}$ is obtained by restricting that of $U_t^{(\widetilde{Y})}\restriction_{\mathcal{F}_+(\mathcal{H}_M)}$ to $\mathcal{F}_+(\mathcal{H}_M)$. If the former generator is positive, the latter has to be so. In the considered case, the former is positive since Y is timelike and future directed and thus we can apply (a) of Proposition 4.1. The same argument shows that the self-adjoint generator of $V_t^{(\widetilde{Y})}\restriction_{\mathcal{H}_M}$ has no zero modes if $V_t^{(\widetilde{Y})}\restriction_{\mathcal{H}_M}$ has no zero modes. This last fact happens if $\widetilde{Y}$ vanishes on a zero-measure subset of $\Im^-$ due to (b) of Proposition 4.1. □
Proof of Theorem 4.4. (a) Consider a wavefunction $\varphi \in \mathcal{S}(M)$. It satisfies $\varphi = Ef$ where $E : C_0^\infty(M) \to \mathcal{S}(M)$ is the causal propagator and f is some real smooth and compactly supported function in M. Since the maximally extended de Sitter spacetime M' is globally hyperbolic and $M \subset M'$, – so that

$C_0^\infty(M) \subset C_0^\infty(M')$ – one can focus on the wavefunction $\varphi' := E'f$, where E' is the causal propagator in M'. By construction $\varphi'\restriction_M = \varphi$, so that φ' is a smooth extension of φ. Since $\Im^- \subset M'$, all that implies that φ extends to $\Im^-$ smoothly (and uniquely) and this extension is $\lim_{\to \Im^-} \varphi = \varphi'\restriction_{\Im^-}$. In this way, an $\mathbb{R}$-linear map $\Gamma : \mathcal{S}(M) \ni \varphi \to \varphi'\restriction_{\Im^-} \in C_0^\infty(\Im^-)$ is defined. To conclude (a), it is enough to prove both that $Ran\Gamma \subset \mathsf{S}(\Im^-)$ and that Γ preserves the symplectic forms. Let us prove them. Bearing in mind the previously discussed behaviour of $H_\nu^{(2)}(z)$ for large z (with $|argz| \leq \pi - \epsilon$), making use of (36) and (37), the identity (40) can be recast as

$$\varphi(\tau, \mathbf{x}) = \frac{e^{-i\frac{\pi}{4}}}{\gamma 4\pi^{3/2}} \int_{\mathbb{S}^2} \epsilon_{\mathbb{S}^2}(\theta,\phi) \int_0^{+\infty} dk k e^{i(kr\cos\lambda_{\mathbf{x}}(\theta,\phi) - k\tau)} \left[\tau + O\left(\frac{1}{k}\right)\right] \sqrt{k}\widetilde{\varphi}(k,\theta,\phi) + c.c.\,, \tag{48}$$

where $\lambda_{\mathbf{x}}(\theta,\phi) \in [0,\pi]$ is the angle between $\mathbf{x}$ and $\mathbf{k}$. The iterated integrations make sense and can be interchanged (via Fubini-Tonelli theorem) since both $\sqrt{k}\widetilde{\varphi}(k,\theta,\phi)$ and $(\sqrt{k}\widetilde{\varphi}(k,\theta,\phi)$ are integrable in the measure $d\mathbf{k}$. They are smooth everywhere but $\mathbf{k} = 0$, they vanish very fast at large $|\mathbf{k}|$ and, for $\mathbf{k} = 0$, $\widetilde{\varphi} \propto 1/|\mathbf{k}|^{-Re|\nu|}$ if $m^2 + \xi R > 0$ for ν. Now, calling $\tau = (u+v)/2$ and $r = (u-v)/2$, $\Im^-$ arises as the limit $v \to -\infty$. The contribution due to the factor of $O\left(\frac{1}{k}\right)$ vanishes due to the Riemann-Lebesgue lemma:

$$(\Gamma\varphi)(u,\theta_{\mathbf{x}},\phi_{\mathbf{x}}) = \lim_{s\to+\infty} \frac{e^{-i\frac{\pi}{4}}}{\gamma 4\pi^{3/2}} \int_0^{+\infty} dk \int_{\mathbb{S}^2} \epsilon_{\mathbb{S}^2}(\theta,\phi) \frac{ks}{2} e^{i\frac{ks}{2}[\cos\lambda_{\mathbf{x}}(\theta,\phi)+1]} e^{-iuk}\sqrt{k}\widetilde{\varphi}(k,\theta,\phi) + c.c.$$

That limit can be computed using integration by parts exactly as in the appendix A2 of [DMP06]. In detail, one rotates the axes so that the axis z coincides with $\mathbf{x}$ and, thinking of $\widetilde{\varphi}$ as a function of k, c, ϕ where $c := \cos\theta \in [-1,1]$, one re-arranges the expression above as

$$(\Gamma\varphi)(u,\theta_{\mathbf{x}},\phi_{\mathbf{x}}) = \lim_{s\to+\infty} \frac{-ie^{-i\frac{\pi}{4}}}{\gamma 4\pi^{3/2}} \int_0^{+\infty} dk \int_0^{2\pi} d\phi \int_{-1}^{1} dc \frac{\partial}{\partial c}\left(e^{i\frac{ks}{2}[c+1]}\right) e^{-iuk}\sqrt{k}\widetilde{\varphi}(k,c,\phi) + c.c.$$

where $\theta_{\mathbf{x}} = 0$ in our case. The right-hand side can be expanded using integration by parts and only the contribution for $c = -1$ (that is $\theta = -\pi$, i.e. $\mathbf{k}/|\mathbf{k}| = -\mathbf{x}/|\mathbf{x}|$) survives, the others vanish as $s \to +\infty$, due to Riemann-Lebesgue's lemma (interchanging various integrations using Fubini-Tonelli theorem and finally taking advantage of dominate convergence theorem). The integration over ϕ produces a trivial factor 2π since the dependence from ϕ of the involved functions disappears as $\theta = 0, \pi$. The final result reads, using the initial generic choice for the axes x, y, z:

$$(\Gamma\varphi)(u,\theta_{\mathbf{x}},\phi_{\mathbf{x}}) = \frac{i2\pi e^{-i\frac{\pi}{4}}}{\gamma 4\pi^{3/2}} \int_0^{+\infty} dk\, e^{-iuk}\ \sqrt{k}\widetilde{\varphi}(k,\eta(\theta_{\mathbf{x}},\phi_{\mathbf{x}})) + c.c.\,,$$

$\eta : \mathbb{S}^2 \to \mathbb{S}^2$ denoting the parity inversion $\mathbb{S}^2 \ni \mathbf{n} \mapsto -\mathbf{n} \in \mathbb{S}^2$. Dropping the index $\mathbf{x}$, and viewing θ, ϕ as the standard coordinates on $\Im^-$, the obtained result can be re-written as

$$(\gamma\Gamma\varphi)(\ell,\theta,\phi) = i\frac{e^{-i\frac{\pi}{4}}}{(-\gamma)} \int_0^{+\infty} dk\, \frac{e^{-i\ell k}}{\sqrt{2\pi}}\ \sqrt{\frac{k}{2(-\gamma)}}\widetilde{\varphi}\left(\frac{k}{(-\gamma)}, \eta(\theta,\phi)\right) + c.c.\,. \tag{49}$$

where we have passed to the standard Bondi coordinates on $\Im^-$, i.e. ℓ, θ, ϕ with $u = -\gamma\ell$. In our hypotheses on φ and ν, most notably $m^2 + \xi R > \frac{5}{48}R$, the functions $\sqrt{\frac{k}{2}}\widetilde{\varphi}(k,\eta(\theta,\phi))$ and $k\sqrt{\frac{k}{2}}\widetilde{\varphi}(k,\eta(\theta,\phi))$ belong also to $L^2(\mathbb{R}^+ \times \mathbb{S}^2; dk \wedge \epsilon_{\mathbb{S}^2}(\theta,\phi))$. This implies that both the functions $\Gamma\varphi, \partial_\ell\Gamma\varphi$ belong to $L^2(\mathbb{R} \times \mathbb{S}^2; d\ell \wedge \epsilon_{\mathbb{S}^2})$. In this way we have found that $Ran\Gamma \subset \mathsf{S}(\Im^-)$. Actually we have obtained much

more: by means of both (21) and the Fourier transformed expression of σ, (49) implies that

$$\sigma(\gamma\Gamma\varphi,\gamma\Gamma\varphi') = -2Im\left\{(-\gamma)^{-2}\int_{\mathbb{R}^+\times\mathbb{S}^2} dk\wedge\epsilon_{\mathbb{S}^2}2k\frac{k}{2(-\gamma)}\overline{\widetilde{\varphi}\left(\frac{k}{(-\gamma)},\eta(\theta,\phi)\right)}\widetilde{\varphi'}\left(\frac{k}{(-\gamma)},\eta(\theta,\phi)\right)\right\}$$
$$= -2Im\left\{\int_{\mathbb{R}^+\times\mathbb{S}^2} k^2dk\wedge\epsilon_{\mathbb{S}^2}\overline{\widetilde{\varphi}(k,\theta,\phi)}\widetilde{\varphi'}(k,\theta,\phi)\right\} = -2Im\left\{\int_{\mathbb{R}^3} d\mathbf{k}\overline{\widetilde{\varphi}(\mathbf{k})}\widetilde{\varphi'}(\mathbf{k})\right\} = \sigma_M(\varphi,\varphi')\,,$$

where in the last step we exploited (41). Hence $\gamma\Gamma$ preserves the symplectic form as requested.
(b) Exactly as in the last step of the proof of (a), since the functions $\sqrt{\frac{k}{2}}\widetilde{\varphi}(k,\eta(\theta,\phi))$ and $k\sqrt{\frac{k}{2}}\widetilde{\varphi}(k,\eta(\theta,\phi))$ are also in $L^2(\mathbb{R}^+\times\mathbb{S}^2;dk\wedge\epsilon_{\mathbb{S}^2}(\theta,\phi))$, (23) and (49) imply:

$$\mu(K_\lambda\gamma\Gamma\varphi,K_\lambda\gamma\Gamma\varphi) = (-\gamma)^{-2}\int_{\mathbb{R}^+\times\mathbb{S}^2} dk\wedge\epsilon_{\mathbb{S}^2}2k\frac{k}{2(-\gamma)}\overline{\widetilde{\varphi}\left(\frac{k}{(-\gamma)},\eta(\theta,\phi)\right)}\widetilde{\varphi}\left(\frac{k}{(-\gamma)},\eta(\theta,\phi)\right)$$
$$= \int_{\mathbb{R}^+\times\mathbb{S}^2} k^2dk\wedge\epsilon_{\mathbb{S}^2}\overline{\widetilde{\varphi}(k,\theta,\phi)}\widetilde{\varphi}(k,\theta,\phi) = \int_{\mathbb{R}^3} d\mathbf{k}\overline{\widetilde{\varphi}(\mathbf{k})}\widetilde{\varphi}(\mathbf{k})$$

Therefore, for every $\varphi\in\mathcal{S}(M)$, in view of (42),

$$\lambda_M(W_M(\varphi)) := \lambda(W(\gamma\Gamma\varphi)) = e^{-\mu(K_\lambda\gamma\Gamma\varphi,K_\lambda\gamma\Gamma\varphi)/2} = e^{-\frac{1}{2}\int_{\mathbb{R}^3}\overline{\widetilde{\varphi}(\mathbf{k})}\widetilde{\varphi}(\mathbf{k})\,d\mathbf{k}} = \omega_E(W_M(\varphi))\,,$$

and this concludes the proof. □

Proof of Theorem 4.5. Here, we exploit the same notation, *i.e.* $\mathbf{x},\mathbf{k}$, as in the proof of Theorem 4.4. In particular $\nu := \sqrt{\frac{9}{4}-(m^2\gamma^2+12\xi)}$, so that $\nu\geq 0$ when $\frac{9}{4}-(m^2\gamma^2+12\xi)\geq 0$ in the following. However the sign of ν could be fixed arbitrarily (and this applies for imaginary ν, in particular), since the functions we shall employ are invariant under $\nu\to-\nu$.
As a first step, we notice that if $\varphi\in\mathcal{S}(M)$, it extends to $\Im^-$ smoothly so that $\Gamma\varphi := \lim_{\to\Im^-}\varphi\in C^\infty(\Im^-)$ does exist. This is because, as found in the section 2.2, the spacetime (M,g) extends to a larger spacetime equipped with a metric $\widehat{g}$ obtained by multiplying the metric of the closed static Einstein universe with a strictly positive smooth factor. Since closed static Einstein universe is globally hyperbolic and global hyperbolicity does not depend on nonsingular conformal rescaling of the metric, (M,g) itself is included in a globally hyperbolic spacetime. With the same argument used for de Sitter spacetime in the proof of Theorem 4.4, one has that every $\varphi\in\mathcal{S}(M)$ extends to $\Im^-$ smoothly. We have now to show that $Ran\Gamma\subset\mathsf{S}(\Im^-)$ and that Γ preserves the symplectic forms.
First of all, analogously to what done in the de Sitter case, we determine a class of modes $\Psi_{\mathbf{k}}(\tau,\mathbf{x})$ that will be useful in decomposing the solutions of Klein-Gordon equation in order to take the limit of wavefunctions towards $\Im^-$.

$$\Psi_{\mathbf{k}}(\tau,\mathbf{x}) := \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\frac{\rho_{\mathbf{k}}(\tau)}{a(\tau)}\,, \tag{50}$$

where, taking the exponential factor into account, the Klein-Gordon equation reduces to the following equation for the functions $(-\infty,0)\ni\tau\mapsto\psi_{\mathbf{k}}(\tau)$,

$$\frac{d^2}{d\tau^2}\rho_{\mathbf{k}}(\tau) + (V_0(\mathbf{k},\tau)+V(\tau))\rho_{\mathbf{k}}(\tau) = 0,$$
$$\text{with}\quad V_0(\mathbf{k},\tau) := k^2+\left(\frac{\gamma}{\tau}\right)^2\left[m^2+\left(\xi-\frac{2}{\gamma^2}\right)\right],\quad V(\tau)=O(1/\tau^3)\,. \tag{51}$$

Comparing with Klein-Gordon equation, one sees that $V_0(\mathbf{k},\tau)+V(\tau)=k^2+a(\tau)^2[m^2+(\xi-1/6)R(\tau)]$ where V_0 is nothing but the the contribution of pure de Sitter metric and V is a perturbation. If we dropped the perturbation $V(\tau)$, the functions $\rho_{\mathbf{k}}$ would reduce to the functions $\chi_{\mathbf{k}}$ and the modes $\Psi_{\mathbf{k}}$ would reduce to the modes Φ_k used to construct ω_E beforehand. Notice that the curvature of the spacetime does not coincide with $12/\gamma^2$ as in de Sitter spacetime, but it reads $R(\tau)=12/\gamma^2+O(1/\tau)$ and $a(\tau)=\gamma/\tau+O(1/\tau^2)$. It follows that the added potential $V(\tau)=O(1/\tau^3)$ above. A formal solution of (51) is obtained in terms of the series:

$$\rho_{\mathbf{k}}(\tau)=\chi_{\mathbf{k}}(\tau)$$
$$+(-1)^n\sum_{n=1}^{+\infty}\int_{-\infty}^{\tau}dt_1\int_{-\infty}^{t_1}dt_2\cdots\int_{-\infty}^{t_{n-1}}dt_n S_{\mathbf{k}}(\tau,t_1)S_{\mathbf{k}}(t_1,t_2)\cdots S_{\mathbf{k}}(t_{n-1},t_n)V(t_1)V(t_2)\cdots V(t_n)\chi_{\mathbf{k}}(t_n), \qquad (52)$$

where

$$S_{\mathbf{k}}(t,t'):=-i\left(\overline{\chi_{\mathbf{k}}(t)}\chi_{\mathbf{k}}(t')-\overline{\chi_{\mathbf{k}}(t')}\chi_{\mathbf{k}}(t)\right)\,,\quad t,t'\in(-\infty,0)\,, \qquad (53)$$

satisfying, in view of antisymmetry and (38),

$$S_{\mathbf{k}}(t,t)=0\quad\text{and}\quad\left.\frac{\partial}{\partial t}S_{\mathbf{k}}(t,t')\right|_{t'=t}=1\,. \qquad (54)$$

By direct inspection and making use of (54), one sees that the right-hand side of (52) defines a solution of (51) if one is allowed to interchange the τ-derivative operator – up to the second order – with the sign of sum. This is always possible when the series itself and the series of the derivatives of first and second order converge τ-uniformly in a neighbourhood of every fixed $\tau\in(-\infty,0)$. Actually the locally τ-uniform convergence of the series of derivatives of second order directly follows from the uniform convergence of those of zero and first order, when one refers to the solutions $\chi_{\mathbf{k}}$ and the solutions $S_{\mathbf{k}}$. Using the expression (37) of the modes $\chi_{\mathbf{k}}$, expanding $H^{(2)}_\nu$ in terms of Bessel functions $J_{\pm\nu}$ [GR95] and, finally, exploiting standard integral representations valid for $Re\nu>-1/2$ (formula 5 in 8.411 in [GR95]) of J_ν, one achieves the following bounds for $Re\nu<1/2$ (that is $m^2\gamma^2+12\xi>2$), for $\tau<-1$, and for some constant $C_\nu\geq 0$

$$|\chi_{\mathbf{k}}(\tau)|\leq C_\nu(-\tau)^{Re\nu+1/2}\left(k^{Re\nu}+k^{-Re\nu}\right)\qquad\left|\tfrac{d\chi_{\mathbf{k}}(\tau)}{d\tau}\right|\leq C_\nu(-\tau)^{Re\nu+1/2}\left(k^{Re\nu}+k^{-Re\nu}\right)(1+k), \qquad (55)$$

where $k=|\mathbf{k}|$. Furthermore, for the same reasons it is possible to obtain the following (non optimal) $\mathbf{k}$-uniform bound for $Re\nu<1/2$, for $t_2\leq t_1<-1$, and for some other constant $C'_\nu\geq 0$

$$|S_{\mathbf{k}}(t_1,t_2)|\leq C'_\nu(t_1t_2)^{Re\nu+1/2}\,. \qquad (56)$$

Now fix any $T<-1$ and consider $\tau\in(-\infty,T]$, so that $|V(\tau)|\leq K_T/(-\tau)^3$, for some constant $K_T\geq 0$. From (55), one sees with a few of trivial computations, that the series in the right-hand side of (52) and that of the τ-derivatives are τ-uniformly dominated, respectively, by

$$\left(k^{Re\nu}+k^{-Re\nu}\right)\,S_{\nu,T}\,,\qquad\left(k^{Re\nu}+k^{-Re\nu}\right)(1+k)\,S_{\nu,T}\,, \qquad (57)$$

where $S_{\nu,T}$ is the following convergent series of positive constants

$$S_{\nu,T}:=C_\nu\sum_{n=1}^{+\infty}\left(\frac{2C'_\nu K_T}{1-2Re\nu}\right)^n\frac{1}{n!}\frac{1}{((-T)^{1-2Re\nu})^{n-1/2}}\,. \qquad (58)$$

Summarising, we can conclude that (52) defines a solution of (51) and that, the same equation entails the solution to be smooth. As a straightforward consequence we also have the following τ-uniform bound valid on $(-\infty, T]$

$$|\rho_{\mathbf{k}}(\tau) - \chi_{\mathbf{k}}(\tau)| \leq \left(k^{Re\nu} + k^{-Re\nu}\right) S_{\nu,T}\,, \qquad \left|\frac{d\rho_{\mathbf{k}}(\tau)}{d\tau} - \frac{d\chi_{\mathbf{k}}(\tau)}{d\tau}\right| \leq 2\left(k^{Re\nu} + k^{-Re\nu}\right)(1+k) S_{\nu,T}. \quad (59)$$

This implies that, at fixed τ, the measurable (since limit of measurable functions) functions $\mathbb{R}^3 \ni \mathbf{k} \mapsto \rho_{\mathbf{k}}(\tau)$ and $\mathbb{R}^3 \ni \mathbf{k} \mapsto \frac{d\rho_{\mathbf{k}}(\tau)}{d\tau}$ do not grow, for large $|\mathbf{k}|$, fast than $|\mathbf{k}|^{Re\nu}$ and $|\mathbf{k}|^{1+Re\nu}$ respectively. Moreover, their divergence at $\mathbf{k} = 0$ cannot be worse than that of $\mathbb{R}^3 \ni \mathbf{k} \mapsto \chi_{\mathbf{k}}(\tau)$ and $\mathbb{R}^3 \ni \mathbf{k} \mapsto \frac{d\chi_{\mathbf{k}}(\tau)}{d\tau}$, that is $k^{-|Re\nu|}$.
Finally, notice that each term in the series in the right-hand side of (52) and in the analogy for $d\rho_{\mathbf{k}}/d\tau$ vanishes as $\tau \to -\infty$ by construction. In view of the fact that, τ-uniformly, the series in (57) dominates both the series in the right-hand side of (52) and the series of τ-derivatives, we are allowed to interchange the operations of limit with that of sum, obtaining

$$\lim_{\tau\to-\infty}(\rho_{\mathbf{k}}(\tau) - \chi_{\mathbf{k}}(\tau)) = 0 \quad \text{and} \quad \lim_{\tau\to-\infty}\left(\frac{d\rho_{\mathbf{k}}(\tau)}{d\tau} - \frac{d\chi_{\mathbf{k}}(\tau)}{d\tau}\right) = 0\,. \quad (60)$$

This result has a first important consequence. Using equation (51), one sees that the function $\tau \mapsto \frac{d\overline{\rho_{\mathbf{k}}(\tau)}}{d\tau}\rho_{\mathbf{k}}(\tau) - \overline{\rho_{\mathbf{k}}(\tau)}\frac{d\rho_{\mathbf{k}}(\tau)}{d\tau}$ is actually a constant. The value of this constant can be computed by taking the limit as $\tau \to -\infty$, making use of (38), (60) and taking into account the fact that, for $\mathbf{k}$ fixed, $\frac{d\rho_{\mathbf{k}}(\tau)}{d\tau}$ and $\rho_{\mathbf{k}}(\tau)$ are bounded on $(-\infty, T]$ (notice that these functions have no limit for $\tau \to -\infty$), as one can show employing the asymptotic behaviour of $H^{(2)}_\nu(z)$ for large values of the argument z. In this way one finds

$$\frac{d\overline{\rho_{\mathbf{k}}(\tau)}}{d\tau}\rho_{\mathbf{k}}(\tau) - \overline{\rho_{\mathbf{k}}(\tau)}\frac{d\rho_{\mathbf{k}}(\tau)}{d\tau} = i\,. \quad (61)$$

Now, to analyse the behaviour of $\Gamma\varphi$, we can follow the same way as that followed in de Sitter space. Take any (real by definition) $\varphi \in \mathcal{S}(M)$ and fix a Cauchy surface Σ_τ in (M, g) individuated by the points in M with the fixed value of τ; eventually define

$$\widetilde{\varphi}(\mathbf{k}) := -i\int_{\mathbb{R}^3}\left[\frac{\partial\overline{\Psi_{\mathbf{k}}(\tau,\mathbf{x})}}{\partial\tau}\varphi(\tau,\mathbf{x}) - \overline{\Psi_{\mathbf{k}}(\tau,\mathbf{x})}\frac{\partial\varphi(\tau,\mathbf{x})}{\partial\tau}\right]a(\tau)^2 d\mathbf{x}\,. \quad (62)$$

The right-hand side of (62) does not depend on the choice of τ, as it follows from direct inspection, exploiting (51). Remembering that $\varphi \in \mathcal{S}(M)$, so that its Cauchy data are real, smooth and compactly supported, we have that their Fourier transform are of Schwartz class. Afterwards, exploiting the fact that both the measurable functions $\mathbb{R}^3 \ni \mathbf{k} \mapsto \rho_{\mathbf{k}}(\tau)$ and $\mathbb{R}^3 \ni \mathbf{k} \mapsto \frac{d\rho_{\mathbf{k}}(\tau)}{d\tau}$ grows at most as a polynomial with degree two for large $|\mathbf{k}|$, and that their divergence at $\mathbf{k} = 0$ is at most of order $k^{-|Re\nu|}$ with $Re\nu < 1/2$, we find that $\widetilde{\varphi} \in C^\infty(\mathbb{R}^3 \setminus \{0\})$ and it vanishes for $|\mathbf{k}| \to \infty$ faster than every power $|\mathbf{k}|^{-n}$, $n = 1, 2, \ldots$. In particular $\widetilde{\varphi} \in L^2(\mathbb{R}^3; d\mathbf{k}) \cap L^1(\mathbb{R}^3; d\mathbf{k})$. Once one knows $\widetilde{\varphi}$ by (62), the associated φ can be constructed out of a decomposition in terms of modes $\Psi_{\mathbf{k}}$:

$$\varphi(\tau,\mathbf{x}) = \int_{\mathbb{R}^3}\left[\Psi_{\mathbf{k}}(\tau,\mathbf{x})\widetilde{\varphi}(\mathbf{k}) + \overline{\Psi_{\mathbf{k}}(\tau,\mathbf{x})\widetilde{\varphi}(\mathbf{k})}\right]d\mathbf{k}\,. \quad (63)$$

This is a trivial consequence of (62), (50), (61), and of the standard properties for the Fourier transform of smooth compactly supported functions on $\mathbb{R}^3$. Eventually, per direct computation, one verifies that, if $\varphi_1, \varphi_2 \in \mathcal{S}(M)$,

$$-2Im\left\{\int_{\mathbb{R}^3} \overline{\widetilde{\varphi}_1}(\mathbf{k})\widetilde{\varphi}_2(\mathbf{k})d\mathbf{k}\right\} = \int_{\mathbb{R}^3} (\varphi_2\partial_\tau\varphi_1 - \varphi_1\partial_\tau\varphi_2)\, a^2(\tau)d\mathbf{x} =: \sigma_M(\varphi_1,\varphi_2)\,. \tag{64}$$

We are now in position to draw some conclusions. Indeed, if $\varphi \in \mathcal{S}(M)$, $p \in \Im^-$ and $(\tau_q, \mathbf{x}_q)$ are the coordinates of $q \in M$, we can write down

$$(\Gamma\varphi)(p) = \lim_{q\to p}\int d\mathbf{k}\, \frac{e^{i\mathbf{k}\cdot\mathbf{x}_q}}{(2\pi)^{3/2}}\left(\rho_{\mathbf{k}}(\tau_q) - \chi_{\mathbf{k}}(\tau_q)\right)\widetilde{\varphi}(\mathbf{k}) + \lim_{q\to p}\int d\mathbf{k}\, \frac{e^{i\mathbf{k}\cdot\mathbf{x}_q}}{(2\pi)^{3/2}}\chi_{\mathbf{k}}(\tau)\widetilde{\varphi}(\mathbf{k}) \quad + c.c. \tag{65}$$

As $q \to p \in \Im^-$, $\tau_q \to -\infty$ so that $(\rho_{\mathbf{k}}(\tau_q) - \chi_{\mathbf{k}}(\tau_q)) \to 0$ due to (60). Moreover, since (57) is valid, we have the τ-uniform bound

$$\left|\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\left(\rho_{\mathbf{k}}(\tau) - \chi_{\mathbf{k}}(\tau)\right)\widetilde{\varphi}(\mathbf{k})\right| \leq \frac{S_{\nu,T}}{(2\pi)^{3/2}}\left(|\mathbf{k}|^{Re\nu} + |\mathbf{k}|^{-Re\nu}\right)|\widetilde{\varphi}(\mathbf{k})|\,,$$

where the right hand side is integrable because $Re\nu < 1/2$, $\widetilde{\varphi} \in L^1(\mathbb{R}^3; d\mathbf{k}) \cap L^2(\mathbb{R}^3; d\mathbf{k})$ and it vanishes faster than any power for $|\mathbf{k}| \to +\infty$. Lebesgue's dominate convergence theorem implies that the former limit in (65) vanishes. The remaining limit has been computed in the proof of (a) in Theorem 4.4. The final result reads as follows: if (ℓ, θ, ϕ) are Bondi coordinates of $p \in \Im^-$ and $\eta : \mathbb{S}^2 \to \mathbb{S}^2$ is the inversion $\mathbf{n} \mapsto -\mathbf{n}$ on the sphere,

$$(\gamma\Gamma\varphi)(\ell,\theta,\phi) = i\frac{e^{-i\frac{\pi}{4}}}{(-\gamma)}\int_0^{+\infty} dk\, \frac{e^{-i\ell k}}{\sqrt{2\pi}}\,\sqrt{\frac{k}{2(-\gamma)}}\widetilde{\varphi}\left(\frac{k}{(-\gamma)}, \eta(\theta,\phi)\right) + c.c.\,. \tag{66}$$

From this point on the proof carries on up to the conclusions exactly as in the proof of (a) in Theorem 4.4, since (41) holds also in our generalised case, as (64) shows. □

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