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Perturbative determination of c_{SW} for plaquette and Symanzik gauge action and stout link clover fermions

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Abstract

Using plaquette and Symanzik improved gauge action and stout link clover fermions we determine the improvement coefficient c_{SW} in one-loop lattice perturbation theory from the off-shell quark-quark-gluon three-point function. In addition, we compute the coefficients needed for the most general form of quark field improvement and present the one-loop result for the critical hopping parameter κ_c. We discuss mean field improvement for c_{SW} and κ_c and the choice of the mean field coupling for the actions we have considered.

1 Introduction

Simulations of Wilson-type fermions at realistic quark masses require an improved action with good chiral properties and scaling behavior. A systematic improvement scheme that

removes discretization errors order by order in the lattice spacing a has been proposed by Symanzik [1] and developed for on-shell quantities in [2, 3]. $\mathcal{O}(a)$ improvement of the Wilson fermion action is achieved by complementing it with the so-called clover term [3], provided the associated clover coefficient is tuned properly.

Wilson-type fermions break all chiral symmetries. This introduces an additive negative mass renormalization term in the action, which gives rise to singularities in the quark propagator at small quark masses and makes the approach to the chiral regime difficult. A chiral improvement of the action is expected to reduce the additive mass renormalization and the spread of negative eigenvalues. Surprisingly, this is not accomplished by the clover action.

While the magnitude of the additive mass term decreases with increasing clover term, the problem of negative eigenvalues is more severe for the clover than for the standard Wilson action. It is well known that via a combination of link fattening and tuning of the clover coefficient, it is possible to reduce both the negative mass term and the spread of negative eigenvalues [4, 5, 6].

The focus of this investigation is to determine the clover coefficient and the additive mass renormalization for plaquette and Symanzik improved gauge action and stout link clover fermions in one-loop lattice perturbation theory.

The Symanzik improved gauge action reads [1]

$$S_G^{\rm Sym} = \frac{6}{g^2} \left\{ c_0 \sum_{\rm Plaquette} \frac{1}{3} \,{\rm Re}\,{\rm Tr}\,(1 - U_{\rm Plaquette}) \, + c_1 \sum_{\rm Rectangle} \frac{1}{3} \,{\rm Re}\,{\rm Tr}\,(1 - U_{\rm Rectangle}) \right\} \quad (1)$$

with $c_0 + 8c_1 = 1$ and

$$c_0 = \frac{5}{3}\,, \quad c_1 = -\frac{1}{12}\,. \quad (2)$$

This reduces to the standard plaquette action $S_G^{\rm Plaq}$ for $c_1 = 0$.

Clover fermions have the action for each quark flavor [3]

$$\begin{aligned} S_F \;\; = \;\; a^4 \sum_x \Big\{ & -\frac{1}{2a} \Big[\bar\psi(x) \widetilde{U}_\mu(x) \,(1-\gamma_\mu)\, \psi(x+a\hat\mu) \\ & + \bar\psi(x) \widetilde{U}^\dagger_\mu(x-a\hat\mu) \,(1+\gamma_\mu)\, \psi(x-a\hat\mu) \Big] \\ & + \frac{1}{a}\,(4 + am_0 + am)\, \bar\psi(x)\psi(x) - c_{SW}\, g\, \frac{a}{4}\, \bar\psi(x)\, \sigma_{\mu\nu} F_{\mu\nu}(x)\, \psi(x) \Big\}\,, \end{aligned} \quad (3)$$

where

$$am_0 = \frac{1}{2\kappa_c} - 4\,, \tag{4}$$

κ_c being the critical hopping parameter, is the additive mass renormalization term, and $F_{\mu\nu}(x)$ is the field strength tensor in clover form with $\sigma_{\mu\nu} = (i/2)\,(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$. We consider a version of clover fermions in which we do not smear links in the clover term, but the link variables U_μ in the next neighbor terms have been replaced by (uniterated) stout links [7]

$$\widetilde{U}_\mu(x) = e^{i\,Q_\mu(x)}\,U_\mu(x) \tag{5}$$

with

$$Q_\mu(x) = \frac{\omega}{2\,i}\left[V_\mu(x)U_\mu^\dagger(x) - U_\mu(x)V_\mu^\dagger(x) - \frac{1}{3}\mathrm{Tr}\,\left(V_\mu(x)U_\mu^\dagger(x) - U_\mu(x)V_\mu^\dagger(x)\right)\right]\,. \tag{6}$$

$V_\mu(x)$ denotes the sum over all staples associated with the link and ω is a tunable weight factor. Stout smearing is preferred because (5) is expandable as a power series in g^2, so we can use perturbation theory. Many other forms of smearing do not have this nice property. Because both the unit matrix and the γ_μ terms are smeared, each link is still a projection operator in the Dirac spin index.

The reason for not smearing the clover term is that we want to keep the physical extent in lattice units of the fermion matrix small which is relevant for non-perturbative calculations. In that respect we refer to these fermions as SLiNC fermions, from the phrase **S**tout **Li**nk**N**on-perturbative **C**lover.

The improvement coefficient c_{SW} as well as the additive mass renormalization am_0 are associated with the chiral limit. So we will carry out the calculations for massless quarks, which simplifies things, though it means that we cannot present values for the mass dependent corrections.

For complete $\mathcal{O}(a)$ improvement of the action there are five terms which would have to be added to the $\mathcal{O}(a)$ effective action, they are listed, for example, in [8]. Fortunately, in the massless case only two remain,

$$\mathcal{O}_1 \;=\; \bar{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi\,, \tag{7}$$

$$\mathcal{O}_2 \;=\; \bar{\psi}\,\overleftrightarrow{D}\overleftrightarrow{D}\,\psi\,. \tag{8}$$

The first is the clover term, the second is the Wilson mass term. We have both in our action, there is no need to add any other terms to the action.

In perturbation theory

$$c_{SW} = 1 + g^2\,c_{SW}^{(1)} + \mathcal{O}(g^4)\,. \tag{9}$$

The one-loop coefficient $c_{SW}^{(1)}$ has been computed for the plaquette action using twisted antiperiodic boundary conditions [9] and Schrödinger functional methods [10]. Moreover, using conventional perturbation theory, Aoki and Kuramashi [11] have computed $c_{SW}^{(1)}$ for certain improved gauge actions. All calculations were performed for non-smeared links and limited to on-shell quantities.

We extend previous calculations of $c_{SW}^{(1)}$ to include stout links. This is done by computing the one-loop correction to the off-shell quark-quark-gluon three-point function. The improvement of the action is not sufficient to remove discretization errors from Green functions. To achieve this, one must also improve the quark fields. The most general form consistent with BRST symmetry is [12][1]

$$\psi_\star(x) = \left(1 + a\, c_D \overrightarrow{\not{D}} + a\, i\, g\; c_{NGI} \not{A}(x)\right)\, \psi(x)\,. \tag{10}$$

From now we denote improved quark fields and improved Green functions by an index $\star$. These are made free of $\mathcal{O}(a)$ effects by fixing the relevant improvement coefficients.

There is no *a priori* reason that the gauge variant contribution $c_{NGI}\not{A}(x)$ vanishes. The perturbative expansion of c_{NGI} has to start with the one-loop contribution [12]. As a byproduct of our calculation we determine that coefficient $c_{NGI}^{(1)}$

$$c_{NGI} = g^2\, c_{NGI}^{(1)} + \mathcal{O}(g^4) \tag{11}$$

and find that it is indeed nonvanishing.

2 Off-shell improvement

It is known [11] that the one-loop contribution of the Sheikoleslami-Wohlert coefficient in conventional perturbation theory can be determined using the quark-quark-gluon vertex $\Lambda_\mu(p_1, p_2, c_{SW})$ sandwiched between *on-shell* quark states. p_1 (p_2) denotes the incoming (outgoing) quark momentum. In general that vertex is an *amputated* three-point Green function.

Let us look at the $\mathcal{O}(a)$ expansion of tree-level $\Lambda_\mu^{(0)}(p_1, p_2, c_{SW})$ which is derived from action (3)

$$\Lambda_\mu^{(0)}(p_1, p_2, c_{SW}) = -i\, g\, \gamma_\mu - g\, \tfrac{1}{2}\, a\, \mathbf{1}(p_1 + p_2)_\mu + c_{SW}\, i\, g\, \tfrac{1}{2}\, a\, \sigma_{\mu\alpha}(p_1 - p_2)_\alpha + \mathcal{O}(a^2)\,. \tag{12}$$

[1] In [12] the authors use $\overrightarrow{\not{D}}$ and $\not{\partial}$ instead of $\overrightarrow{\not{D}}$ and $\not{A}$ - both choices are equivalent. Our choice is motivated by the discussion of off-shell improvement in the next section.

For simplicity we omit in all three-point Green functions the common overall color matrix T^a. That tree-level expression between on-shell quark states is free of order $\mathcal{O}(a)$ if the expansion of c_{SW} starts with one, as indicated in (9)

$$\bar{u}(p_2)\,\Lambda^{(0)}_{\star\mu}(p_1,p_2)\,u(p_1) = \bar{u}(p_2)\,(-i\,g\,\gamma_\mu)\,u(p_1)\,. \tag{13}$$

Therefore, at least a one-loop calculation of the $\Lambda_\mu(p_1,p_2,c^{(1)}_{SW})$ is needed as necessary condition to determine $c^{(1)}_{SW}$.

The *off-shell* improvement condition states that the *non-amputated* improved quark-quark-gluon Green function $G_{\star\mu}(p_1,p_2,q)$ has to be free of $\mathcal{O}(a)$ terms in one-loop accuracy. In position space that non-amputated improved quark-quark-gluon Green functions is defined via expectation values of improved quark fields and gauge fields as

$$G_{\star\mu}(x,y,z) = \langle \psi_\star(x)\,\overline{\psi}_\star(y)\,A_\mu(z)\rangle\,. \tag{14}$$

Since the gluon propagator is $\mathcal{O}(a)$-improved already, we do not need to improve gauge fields. Using relation (10) we can express the function $G_{\star\mu}$ by the unimproved quark fields ψ

$$\begin{aligned} G_{\star\mu}(x,y,z) &= G_\mu(x,y,z) + a\,c_D\,\left\langle \left(\not{D}\not{D}^{-1} + \not{D}^{-1}\not{D} \right) A_\mu \right\rangle \\ &\quad + i\,a\,g\,c_{NGI}\,\left\langle \left(\not{A}\not{D}^{-1} + \not{D}^{-1}\not{A} \right) A_\mu \right\rangle\,, \end{aligned} \tag{15}$$

where $G_\mu(x,y,z)$ is the unimproved Green function.

Taking into account

$$\left\langle \left(\not{A}\not{D}^{-1} + \not{D}^{-1}\not{A} \right) A_\mu \right\rangle = 2\,a\,c_D\,\delta(x-y)\,\langle A_\mu(z)\rangle \tag{16}$$

and setting $\langle A_\mu(z)\rangle = 0$ (unless there is an unexpected symmetry breaking), we obtain the following relation between the improved and unimproved Green function

$$G_{\star\mu}(x,y,z) = G_\mu(x,y,z) + i\,a\,g\,c_{NGI}\,\left\langle \left(\not{A}\not{D}^{-1} + \not{D}^{-1}\not{A} \right) A_\mu \right\rangle\,. \tag{17}$$

From (17) it is obvious that tuning only c_{SW} to its optimal value in $G_\mu(x,y,z)$, there would be an $\mathcal{O}(a)$ contribution left in the improved Green function. The requirement that $G_{\star\mu}(x,y,z)$ should be free of $\mathcal{O}(a)$ terms leads to an additional condition which determines the constant c_{NGI}. It has not been calculated before.

Taking into account the expansion (11) of c_{NGI} we get in momentum space ($\mathcal{F}[\cdot]$ denotes the Fourier transform)

$$i\,a\,g\,c_{NGI}\,\mathcal{F}\Big[\,\Big\langle\Big(\not{A}\not{D}^{-1}+\not{D}^{-1}\not{A}\Big)\,A_\mu\Big\rangle^{\rm tree}\,\Big]=i\,a\,g^3\,c^{(1)}_{NGI}\left(\gamma_\nu\frac{1}{i\not{p}_1}+\frac{1}{i\not{p}_2}\gamma_\nu\right)\,K^{\rm tree}_{\nu\mu}(q)\,, \quad (18)$$

or its amputated version

$$i\,a\,g\,c_{NGI}\,\mathcal{F}\Big[\,\Big\langle\Big(\not{A}\not{D}^{-1}+\not{D}^{-1}\not{A}\Big)\,A_\mu\Big\rangle^{\rm tree}_{\rm amp}\,\Big]=-a\,g^3\,c^{(1)}_{NGI}\left(\not{p}_2\,\gamma_\mu+\gamma_\mu\not{p}_1\right)\,. \quad (19)$$

The relation between non-amputated and amputated unimproved and improved three-point Green functions are defined by

$$\begin{aligned} G_\mu(p_1,p_2,q) &= S(p_2)\,\Lambda_\nu(p_1,p_2,q,c^{(1)}_{SW})\,S(p_1)\,K_{\nu\mu}(q)\,, & (20)\\ G_{\star\mu}(p_1,p_2,q) &= S_\star(p_2)\,\Lambda_{\star\nu}(p_1,p_2,q)\,S_\star(p_1)\,K_{\nu\mu}(q)\,, & (21)\end{aligned}$$

$K_{\nu\mu}(q)$ denotes the full gluon propagator which is $\mathcal{O}(a)$-improved already, $S(p)$ and $S_\star(p)$ the corresponding quark propagators.

With the definition of the quark self energy

$$\Sigma(p)=\frac{1}{a}\Sigma_0+i\not{p}\,\Sigma_1(p)+\frac{a\,p^2}{2}\Sigma_2(p) \quad (22)$$

the unimproved and improved inverse quark propagators are given by

$$\begin{aligned} S^{-1}(p) &= i\not{p}\,\Sigma_1(p)+\frac{a\,p^2}{2}\Sigma_2(p)=i\not{p}\,\Sigma_1(p)\left(1-\frac{1}{2}a\,i\not{p}\,\frac{\Sigma_2(p)}{\Sigma_1(p)}\right)\,, & (23)\\ S^{-1}_\star(p) &= i\not{p}\,\Sigma_1(p)\,. & (24)\end{aligned}$$

Using the Fourier transformed (17) with (19) and amputating the Green function (20), taking into account the inverse quark propagators (23), we get the off-shell improvement condition in momentum space

$$\begin{aligned}\Lambda_\mu(p_1,p_2,q,c^{(1)}_{SW}) &= \Lambda_{\star\mu}(p_1,p_2,q)+a\,g^3c^{(1)}_{NGI}(\not{p}_2\,\gamma_\mu+\gamma_\mu\not{p}_1)\\ &\quad-\frac{a}{2}\,i\not{p}_2\,\frac{\Sigma_2(p_2)}{\Sigma_1(p_2)}\,\Lambda_{\star\mu}(p_1,p_2,q)-\frac{a}{2}\,\Lambda_{\star\mu}(p_1,p_2,q)\,i\not{p}_1\,\frac{\Sigma_2(p_1)}{\Sigma_1(p_1)}\,. \quad (25)\end{aligned}$$

This expression should hold to order $\mathcal{O}(g^3)$ by determining both $c^{(1)}_{NGI}$ and $c^{(1)}_{SW}$ correctly. It is clear from (25) that the improvement term $\propto c^{(1)}_{NGI}$ does not contribute if both quarks are on-shell.

3 The one-loop lattice quark-quark-gluon vertex

The diagrams contributing to the amputated one-loop three-point function are shown in Fig. 1. The calculation is performed with a mixture of symbolic and numerical tech-

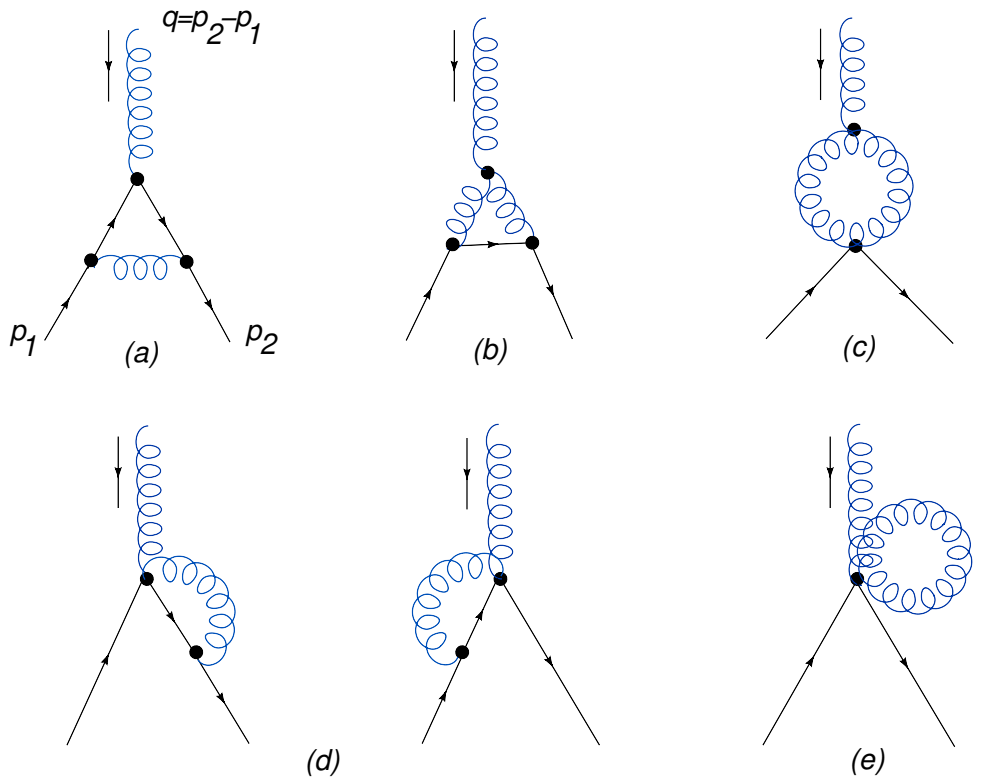

Figure 1: One-loop diagrams contributing to the amputated quark-quark-gluon vertex.

niques. For the symbolic computation we use a *Mathematica* package that we developed for one-loop calculations in lattice perturbation theory (for a more detailed description see [14]). It is based on an algorithm of Kawai et al. [15]. The symbolic treatment has several advantages: one can extract the infrared singularities exactly and the results are given as functions of lattice integrals which can be determined with high precision. The disadvantage consists in very large expressions especially for the problem under consideration. In the symbolic method the divergences are isolated by differentiation with respect to external momenta.

Looking at the general analytic form of the gluon propagator for improved gauge

actions [16] one easily recognizes that a huge analytic expression would arise. As discussed in [16] we split the full gluon propagator $D^{\rm Sym}_{\mu\nu}(k,\xi)$

$$D^{\rm Sym}_{\mu\nu}(k,\xi) = D^{\rm Plaq}_{\mu\nu}(k,\xi) + \Delta D_{\mu\nu}(k)\,, \tag{26}$$

where ξ is the covariant gauge parameter ($\xi = 0$ corresponds to the Feynman gauge). The diagrams with $D^{\rm Plaq}_{\mu\nu}(k,\xi)$ only contain the logarithmic parts and are treated with our *Mathematica* package. The diagrams with at least one $\Delta D_{\mu\nu}(k)$ are infrared finite and can be determined safely with numerical methods. The decomposition (26) means that we always need to calculate the plaquette action result, as part of the calculation for the improved gauge action. Therefore, we will give the results for both plaquette gauge action and Symanzik improved gauge action using the corresponding gluon propagators $D^{\rm Plaq}_{\mu\nu}$ and $D^{\rm Sym}_{\mu\nu}$, respectively.

Because the numerical part determines the accuracy of the total result we discuss it in more detail. There are several possibilities to combine the various contributions of the one-loop diagrams as given in Fig. 1. In view of a later analysis we have decided to group all coefficients in front of the independent color factors C_F and N_c and the powers of the stout parameter ω

$$\Lambda^{\rm num.}_{\mu} \;=\; C_F\,\left(C_0 + C_1\,\omega + C_2\,\omega^2 + C_3\,\omega^3\right) + N_c\,\left(C_4 + C_5\,\omega + C_6\,\omega^2 + C_7\,\omega^3\right)\,, \tag{27}$$

where the C_i have to be computed numerically. In order to obtain C_i we first add all contributions of the diagrams shown in (1) and integrate afterwards. We have used a Gauss-Legendre integration algorithm in four dimensions (for a description of the method see [14]) and have chosen a sequence of small external momenta (p_1, p_2) to perform an extrapolation to vanishing momenta.

Let us illustrate this by an example: the calculation of the coefficient C_4. We know the general structure of the one-loop amputated three-point function as (we set $a = 1$)

$$\begin{aligned} M_\mu(p_1,p_2) \;&=\; \gamma_\mu\,A(p_1,p_2) + \mathbf{1}\,p_{1,\mu}\,B(p_1,p_2) + \mathbf{1}\,p_{2,\mu}\,C(p_1,p_2) \\ &\quad + \sigma_{\mu\alpha}\,p_{1,\alpha}\,D(p_1,p_2) + \sigma_{\mu\alpha}\,p_{2,\alpha}\,E(p_1,p_2)\,. \end{aligned} \tag{28}$$

From this we can extract the coefficients by the following projections

$$\begin{aligned} \mathrm{Tr}\,\gamma_\mu M_\mu \;&=\; 4\,A(p_1,p_2), \qquad \mu \quad \text{fixed}\,, \\ \mathrm{Tr}\,M_\mu \;&=\; 4\,p_{1,\mu}\,B(p_1,p_2) + 4\,p_{2,\mu}\,C(p_1,p_2)\,, \\ \sum_\mu \mathrm{Tr}\,\sigma_{\nu\mu}\,M_\mu \;&=\; 12\,p_{1,\nu}\,D(p_1,p_2) + 12\,p_{2,\nu}\,E(p_1,p_2)\,. \end{aligned} \tag{29}$$

Relations (30) show that one has to compute the three-point function for all four values of μ. Further they suggest choosing the external momenta orthogonal to each other: $p_1 \cdot p_2 = 0$. A simple choice is $p_{1,\mu} = (0, 0, 0, p_{1,4})$ and $p_{2,\mu} = (0, 0, p_{2,3}, 0)$.

We discuss the determination of $B(p_1, p_2)$ and $C(p_1, p_2)$ in more detail. For small momenta they can be described by the ansatz

$$\begin{aligned} B(p_1, p_2) &= B_0 + B_1\, p_1^2 + B_2\, p_2^2\,, \\ C(p_1, p_2) &= C_0 + C_1\, p_1^2 + C_2\, p_2^2\,. \end{aligned} \tag{30}$$

The choice of the momenta is arbitrary except for two points. First, they should be sufficiently small in order to justify ansatz (30). Second, they should not be integer multiples of each other in order to avoid accidental symmetric results. The symmetry of the problem demands the relation $B_0 = C_0$ which must result from the numerical integration also. Performing the integration at fixed p_1 and p_2 we obtain complex 4×4 matrices for $M_3(p_1, p_2)$ and $M_4(p_1, p_2)$ from which the quantities $B(p_1, p_2)$ and $C(p_1, p_2)$ are extracted via (30).

A nonlinear regression fit with ansatz (30) gives

$$\begin{aligned} B_0 &= 0.00553791 \quad \text{with fit error } \delta B_0 = 7 \times 10^{-8}\,, \\ C_0 &= 0.00553789 \quad \text{with fit error } \delta C_0 = 6 \times 10^{-8}\,. \end{aligned} \tag{31}$$

It shows that the symmetry is fulfilled up to an error of $\mathcal{O}(10^{-7})$ which sets one scale for the overall error of our numerical calculations. In Fig. 2 we show the almost linear dependence of $B(p_1, p_2)$ and $C(p_1, p_2)$ on p_1^2. (In the integration we haven chosen $p_{1,\mu} = 0.87\, p_{2,\mu}$ so that we can restrict the plot to one variable.)

Another source of errors is the numerical Gauss-Legendre integration routine itself. We have chosen a sequence of $n^4 = 14^4$, 18^4, 22^4, 26^4 and 30^4 nodes in the four-dimensional hypercube and have performed an extrapolation to infinite nodes with an $1/n^4$ fit ansatz. Both procedures, Gauss-Legendre integration and the fit $p \to 0$, give a combined final error of 10^{-6}.

The third error source are the errors of the lattice integrals of our *Mathematica* calculation for the terms containing the plaquette propagator $D_{\mu\nu}^{\mathrm{Plaq}}$ only. These integrals have been calculated up to a precision of $\mathcal{O}(10^{-10})$. Therefore, their errors can be neglected in comparison with the others.

Summarizing we find that the error of our numerical procedure is of $\mathcal{O}(10^{-6})$. Additionally, we have checked our results by an independent code which completely numerically

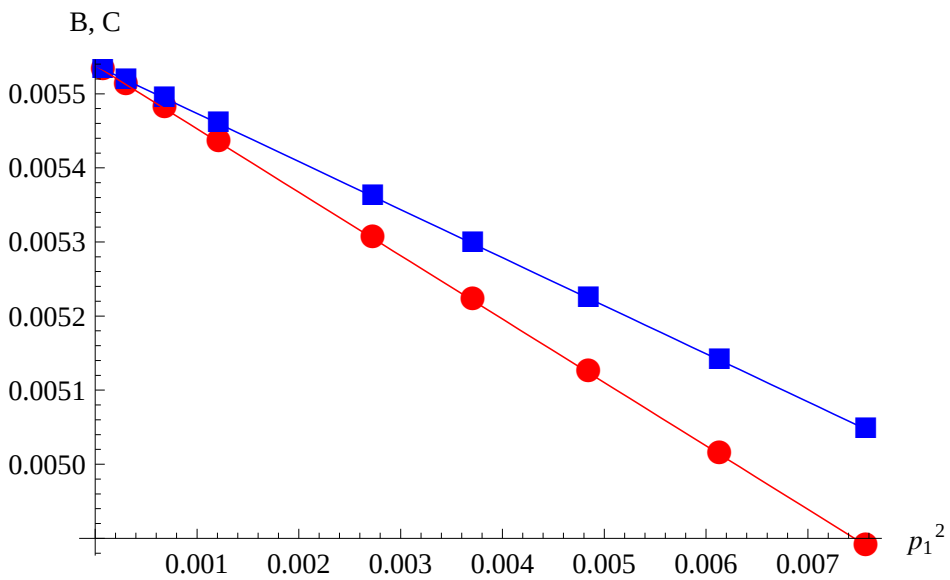

Figure 2: $B(p_1, p_2)$ (circles) and $C(p_1, p_2)$ (squares) as function of p_1^2 together with their corresponding linear fits in p_1^2.

computes the one-loop contributions for each diagram including the infrared logarithms. Both methods agree within errors.

The Feynman rules for non-smeared Symanzik gauge action have been summarized in [11]. For the stout smeared gauge links in the clover action the rules restricted to equal initial and final quark momenta are given in [6]. As mentioned in the introduction we perform a one-level smearing of the Wilson part in the clover action. The corresponding Feynman rules needed for the one-loop quark-quark-gluon vertex are much more complicated than those in [6]. The qqgg-vertex needed in diagrams (c) and (d) of Fig. 1 receives an additional antisymmetric piece. The qqggg-vertex in diagram (e) does not even exist in the forward case. The Feynman rules are given in Appendix A. The diagrams which are needed for the calculation of the quark propagator are shown in Fig. 3. We have performed our calculation in general covariant gauge.

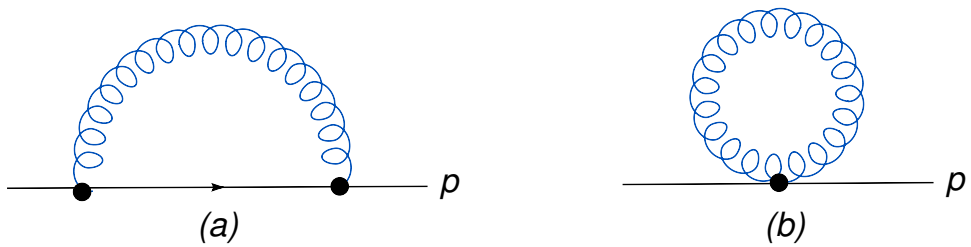

Figure 3: One-loop diagrams contributing to the quark self energy.

4 Results for the improvement coefficients and critical hopping parameter

The anticipated general structure for the amputated three-point function in one-loop is

$$\begin{aligned}\Lambda_\mu(p_1,p_2,q) &= \Lambda_\mu^{\overline{MS}}(p_1,p_2,q) + A_{\rm lat}\, i\, \frac{g^3}{16\pi^2}\,\gamma_\mu \\ &\quad + B_{\rm lat}\,\frac{a}{2}\,\frac{g^3}{16\pi^2}\,\left(\not{p}_2\,\gamma_\mu + \gamma_\mu\,\not{p}_1\right) + C_{\rm lat}\, i\,\frac{a}{2}\,\frac{g^3}{16\pi^2}\,\sigma_{\mu\alpha}\,q_\alpha\,. \end{aligned} \tag{32}$$

$\Lambda_\mu^{\overline{MS}}(p_1,p_2,q)$ is the universal part of the three-point function, independent of the chosen gauge action, computed in the $\overline{MS}$-scheme

$$\begin{aligned}\Lambda_\mu^{\overline{MS}}(p_1,p_2,q) &= -i\,g\,\gamma_\mu - g\,\frac{a}{2}\,\mathbf{1}\,(p_{1,\mu}+p_{2,\mu}) - c_{SW}\,i\,g\,\frac{a}{2}\sigma_{\mu\alpha}\,q_\alpha \\ &\quad + i\,\frac{1}{2}\,\frac{g^3}{16\pi^2}\,\Lambda_{1,\mu}^{\overline{MS}}(p_1,p_2,q) + \frac{a}{2}\frac{g^3}{16\pi^2}\,\Lambda_{2,\mu}^{\overline{MS}}(p_1,p_2,q)\,. \end{aligned} \tag{33}$$

We have calculated the complete expressions for $\Lambda_{1,\mu}^{\overline{MS}}(p_1,p_2,q)$ and $\Lambda_{2,\mu}^{\overline{MS}}(p_1,p_2,q)$.

The $\mathcal{O}(a)$ contribution, $\Lambda_{2,\mu}^{\overline{MS}}(p_1,p_2,q)$, simplifies if we set $c_{SW} = 1+\mathcal{O}(g^2)$ as in (9). After some algebra we find

$$\begin{aligned}\Lambda_{2,\mu}^{\overline{MS}}(p_1,p_2,q) &= \frac{1}{2}\left(\not{p}_2\,\Lambda_{1,\mu}^{\overline{MS}}(p_1,p_2,q) + \Lambda_{1,\mu}^{\overline{MS}}(p_1,p_2,q)\,\not{p}_1\right) \\ &\quad - C_F\left(\not{p}_2\,\gamma_\mu\,(1-\xi)(1-\log(p_2^2/\mu^2))\right. \\ &\qquad \left. +\gamma_\mu\,\not{p}_1\,(1-\xi)(1-\log(p_1^2/\mu^2))\right)\,, \end{aligned} \tag{34}$$

where μ^2 is the $\overline{MS}$ mass scale (not to be confused with the index μ). Therefore, we only need $\Lambda_{1,\mu}^{\overline{MS}}(p_1,p_2,q)$ to present the one-loop result (33). $\Lambda_{1,\mu}^{\overline{MS}}(p_1,p_2,q)$ is given in Appendix B.

If we insert (32) and (33) with (34) into the off-shell improvement relation (25) we get the following conditions that all terms of order $\mathcal{O}(ag^3)$ have to vanish

$$\left(c_{SW}^{(1)} - \frac{C_{\rm lat}}{16\pi^2}\right)\,\sigma_{\mu\alpha}\,q_\alpha \;=\; 0\,, \tag{35}$$

$$\left(c_{NGI}^{(1)} - \frac{1}{32\pi^2}\,(A_{\rm lat} - B_{\rm lat} - \Sigma_{21})\right)\left(\not{p}_2\,\gamma_\mu + \gamma_\mu\,\not{p}_1\right) \;=\; 0\,, \tag{36}$$

with Σ_{21} defined from (23) as

$$\begin{aligned}\frac{\Sigma_2(p)}{\Sigma_1(p)} &= 1 + \frac{g^2\,C_F}{16\pi^2}\left((1-\xi)(1-\log(a^2p^2)) + \Sigma_{21,0}\right)\\ &\equiv 1 + \frac{g^2\,C_F}{16\pi^2}\left((1-\xi)(1-\log(p^2/\mu^2))\right) + \frac{g^2}{16\pi^2}\Sigma_{21}\end{aligned} \tag{37}$$

and

$$\Sigma_{21} = C_F\,\left(-(1-\xi)\log(a^2\mu^2) + \Sigma_{21,0}\right)\,. \tag{38}$$

The constant $\Sigma_{21,0}$ depends on the chosen lattice action.

It should be noted that equations (35) and (36) are obtained by using the general structure (34) only – we do not need to insert the complete calculated result for $\Lambda_{1,\mu}^{\overline{MS}}(p_1,p_2,q)$. In order to get momentum independent and gauge invariant improvement coefficients we see from (35) that $C_{\rm lat}$ itself has to be constant and gauge invariant. From (36) and (38) we further conclude that the $\log(a^2\mu^2)$-terms from $A_{\rm lat}$ and $B_{\rm lat}$ have to cancel those from Σ_{21}. The same is true for the corresponding gauge terms. The terms $\propto (1-\xi)(1-\log(p_i^2/\mu^2))$ $(i=1,2)$ coming from (37) are canceled by the corresponding terms in (34).

Therefore, the relation between $\Lambda_{1,\mu}^{\overline{MS}}(p_1,p_2,q)$ and $\Lambda_{2,\mu}^{\overline{MS}}(p_1,p_2,q)$ as given in (34) is a nontrivial result. Once more, it should be emphasized that this relation only holds if we use $c_{SW}=1$ at leading order in g^2. If (34) were not true, we would not be able to improve the Green functions by adding the simple $\mathcal{O}(a)$ terms we have considered.

For completeness we also give the corresponding one-loop values for the quark field improvement coefficient c_D as defined in (10). They can be derived from the $\mathcal{O}(a)$ improvement of the quark propagator. The one-loop improvement coefficient $c_D^{(1)}$ is related to the quark self energy by

$$c_D = -\frac{1}{4}\,\left(1 + \frac{g^2\,C_F}{16\pi^2}\,(2\,\Sigma_1 - \Sigma_2)\right) + \mathcal{O}(g^4) \equiv -\frac{1}{4}\,\left(1 + g^2\,c_D^{(1)}\right) + \mathcal{O}(g^4)\,. \tag{39}$$

$c_D^{(1)}$ has been calculated for ordinary clover fermions and plaquette gauge action in [13].

Now we present our numerical results for general covariant gauge ξ and as function of the stout parameter ω. For the plaquette action with stout smearing the quantities $A_{\rm lat}$, $B_{\rm lat}$ and $C_{\rm lat}$ are obtained as

$$\begin{aligned}
A_{\rm lat}^{\rm Plaq} &= C_F \Big(9.206269 + 3.792010\,\xi - 196.44601\,\omega + 739.683641\,\omega^2 \\
&\qquad +(1-\xi)\log(a^2\mu^2)\Big) \\
&\quad + N_c \left(-4.301720 + 0.693147\,\xi + (1-\xi/4)\log(a^2\mu^2)\right) , \\
B_{\rm lat}^{\rm Plaq} &= C_F \Big(9.357942 + 5.727769\,\xi - 208.583208\,\omega + 711.565256\,\omega^2 \\
&\qquad +2\,(1-\xi)\log(a^2\mu^2)\Big) \\
&\quad + N_c \left(-4.752081 + 0.693147\,\xi + 3.683890\,\omega + (1-\xi/4)\log(a^2\mu^2)\right) , \\
C_{\rm lat}^{\rm Plaq} &= C_F \left(26.471857 + 170.412296\,\omega - 582.177099\,\omega^2\right) \\
&\quad + N_c \left(2.372649 + 1.518742\,\omega - 44.971612\,\omega^2\right) .
\end{aligned} \tag{40}$$

For the stout smeared Symanzik action we get

$$\begin{aligned}
A_{\rm lat}^{\rm Sym} &= C_F \Big(5.973656 + 3.792010\,\xi - 147.890719\,\omega + 541.380348\,\omega^2 \\
&\qquad +(1-\xi)\log(a^2\mu^2)\Big) \\
&\quad + N_c \left(-3.08478 + 0.693159\,\xi - 0.384236\,\omega + (1-\xi/4)\log(a^2\mu^2)\right) , \\
B_{\rm lat}^{\rm Sym} &= C_F \Big(6.007320 + 5.727769\,\xi - 163.833410\,\omega + 542.892478\,\omega^2 \\
&\qquad +2\,(1-\xi)\log(a^2\mu^2)\Big) \\
&\quad + N_c \left(-13.841082 + 0.693179\,\xi + 3.039641\,\omega + (1-\xi/4)\log(a^2\mu^2)\right) , \\
C_{\rm lat}^{\rm Sym} &= C_F \left(18.347163 + 130.772885\,\omega - 387.690744\,\omega^2\right) \\
&\quad + N_c \left(2.175560 + 2.511657\,\omega - 50.832203\,\omega^2\right) .
\end{aligned} \tag{41}$$

As shown in (25) (or equivalently (36)) we need the self energy parts $\Sigma_1(p)$ and $\Sigma_2(p)$ as defined in (23) to solve the off-shell improvement condition. They have the general form

$$\begin{aligned}
\Sigma_1(p) &= 1 - \frac{g^2\,C_F}{16\pi^2}\,\left[(1-\xi)\log(a^2p^2) + \Sigma_{1,0}\right] , \\
\Sigma_2(p) &= 1 - \frac{g^2\,C_F}{16\pi^2}\,\left[2\,(1-\xi)\log(a^2p^2) + \Sigma_{2,0}\right] .
\end{aligned} \tag{42}$$

For the plaquette and Symanzik actions we obtain

$$\begin{aligned}
\Sigma_{1,0}^{\mathrm{Plaq}} &= 8.206268 - 196.446005\,\omega + 739.683641\,\omega^2 + 4.792010\,\xi\,, \\
\Sigma_{2,0}^{\mathrm{Plaq}} &= 7.357942 - 208.583208\,\omega + 711.565260\,\omega^2 + 7.727769\,\xi\,, \\
\Sigma_{1,0}^{\mathrm{Sym}} &= 4.973689 - 147.890720\,\omega + 541.380518\,\omega^2 + 4.792010\,\xi\,, \\
\Sigma_{2,0}^{\mathrm{Sym}} &= 4.007613 - 163.833419\,\omega + 542.892535\,\omega^2 + 7.727769\,\xi\,.
\end{aligned} \tag{43}$$

This results in the following expressions for Σ_{21} as defined in (38)

$$\begin{aligned}
\Sigma_{21}^{\mathrm{Plaq}} &= C_F \Big(-0.151673 - 1.935759\,\xi + 12.137203\,\omega + 28.118384\,\omega^2 \\
&\qquad - (1-\xi)\,\log(a^2\mu^2)\Big)\,, \\
\Sigma_{21}^{\mathrm{Sym}} &= C_F \Big(-0.033924 - 1.935759\,\xi + 15.942699\,\omega - 1.512017\,\omega^2 \\
&\qquad - (1-\xi)\,\log(a^2\mu^2)\Big)\,.
\end{aligned} \tag{44}$$

Inserting the corresponding numbers into (35), (36) and (39), we obtain the one-loop contributions of the clover improvement coefficient

$$\begin{aligned}
c_{SW}^{(1),\mathrm{Plaq}} &= C_F\,\left(0.167635 + 1.079148\,\omega - 3.697285\,\omega^2\right) \\
&\quad + N_c\,\left(0.015025 + 0.009617\,\omega - 0.284786\,\omega^2\right)\,, \qquad (45) \\
c_{SW}^{(1),\mathrm{Sym}} &= C_F\,\left(0.116185 + 0.828129\,\omega - 2.455080\,\omega^2\right) \\
&\quad + N_c\,\left(0.013777 + 0.015905\,\omega - 0.321899\,\omega^2\right)\,, \qquad (46)
\end{aligned}$$

the off-shell quark field improvement coefficient

$$\begin{aligned}
c_{NGI}^{(1),\mathrm{Plaq}} &= N_c\,\left(0.001426 - 0.011664\,\omega\right)\,, \qquad (47) \\
c_{NGI}^{(1),\mathrm{Sym}} &= N_c\,\left(0.002395 - 0.010841\,\omega\right)\,, \qquad (48)
\end{aligned}$$

and the on-shell quark field improvement coefficient

$$\begin{aligned}
c_{D}^{(1),\mathrm{Plaq}} &= C_F\,\left(0.057339 + 0.011755\,\xi - 1.167149\,\omega + 4.862163\,\omega^2\right)\,, \qquad (49) \\
c_{D}^{(1),\mathrm{Sym}} &= C_F\,\left(0.037614 + 0.011755\,\xi - 0.835571\,\omega + 3.418757\,\omega^2\right)\,, \qquad (50)
\end{aligned}$$

for the plaquette and Symanzik action, respectively. For $\omega = 0$ both the plaquette result (45) and the Symanzik result (46) agree, within the accuracy of our calculations, with the numbers quoted in [9, 10] and [11].

From Ward identity considerations it is known that the coefficient c_{NGI} has to be proportional to N_c only. Additionally, c_{NGI} and c_{SW} should be gauge invariant. Both conditions are fulfilled within the errors which have been discussed in the previous section. It should be noted that (47) and (48) are the first one-loop results for the quark field improvement coefficient c_{NGI}. The gauge dependent improvement coefficient c_D depends only on the color factor C_F because it is determined by $\mathcal{O}(a)$ improvement of the quark propagator.

The additive mass renormalization is given by

$$am_0 = \frac{g^2\, C_F}{16\pi^2}\,\frac{\Sigma_0}{4}\,. \tag{51}$$

This leads to the critical hopping parameter κ_c, at which chiral symmetry is approximately restored,

$$\kappa_c = \frac{1}{8}\left(1 - \frac{g^2\, C_F}{16\pi^2}\,\frac{\Sigma_0}{4}\right)\,. \tag{52}$$

Using the plaquette or Symanzik gauge actions, we obtain

$$\Sigma_0^{\mathrm{Plaq}} = -31.986442 + 566.581765\,\omega - 2235.407087\,\omega^2\,, \tag{53}$$

$$\Sigma_0^{\mathrm{Sym}} = -23.832351 + 418.212508\,\omega - 1685.597405\,\omega^2\,. \tag{54}$$

This leads to the perturbative expression for κ_c

$$\kappa_c^{\mathrm{Plaq}} = \frac{1}{8}\left[1 + g^2\, C_F\left(0.050639 - 0.896980\,\omega + 3.697285\,\omega^2\right)\right]\,, \tag{55}$$

$$\kappa_c^{\mathrm{Sym}} = \frac{1}{8}\left[1 + g^2\, C_F\left(0.037730 - 0.662090\,\omega + 2.668543\,\omega^2\right)\right]\,. \tag{56}$$

For both actions am_0 can be tuned to zero for admissible values of ω. Using the smaller possible value we find $\omega = 0.089396$ for the plaquette action and $\omega = 0.088689$ for the Symanzik gauge action.

5 Mean field improvement

It is well known that one-loop perturbation theory in the bare coupling constant g^2 leads to a poor approximation. The coefficient of g^2 is large in most quantities, and the series converges poorly. One traditional way to reduce this problem is by mean field improvement, which consists of two ideas.

The first is that we calculate each quantity in a simple mean field approximation, and then re-express the perturbative result as the mean field result multiplied by a perturbative correction factor. If the mean field approximation is good, the correction factor will be close to 1, and we have resolved the problem of the large one-loop coefficient. As a good internal test of this part, we can simply look to see how large the coefficient in this correction factor is (the "tadpole improved coefficient"), compared with the initial unimproved coefficient.

The second part of the mean field approximation is that we change our expansion parameter from the bare coupling g^2 to some "boosted" coupling constant, g^2_{MF}, which we hope represents physics at a more relevant scale, and leads to a more rapidly convergent series. A well-chosen boosted coupling would reduce the two-loop coefficient. Unfortunately we usually cannot test this part of the improvement procedure, because the two-loop coefficient is unknown. Fortunately, if the mean field approximation is good, the exact choice of boosted coupling constant will not be too crucial, because the lowest order improved coefficient will be a small number.

5.1 Mean field approximation for smeared fermions

In the mean field approximation we typically assume that the gauge fields on each link are independently fluctuating variables, and that we can simply represent the links by an average value u_0. Typical choices for u_0 would be to choose u_0^4 to be the average plaquette value, or to choose u_0 to be the average link value in the Landau gauge.

A natural question is how we should extend the mean field approximation if we employ smearing. One possibility is to express everything in terms of two quantities, u_0, a mean value for the unsmeared link, and u_S, a mean value for smeared links[2]. We will discuss the relation between these two quantities later, first we want to make a general point about mean field approximations and smearing.

The reason we smear our gauge links is to suppress very short range fluctuations in the gauge field, which is justified by the argument that these short range fluctuations are very lattice-dependent, rather than physical. However, put another way, suppressing short range fluctuations means that we are correlating nearby gauge links. So there is a certain tension between smearing and the mean field notion that each link is fluctuating independently. We will take the attitude that it does still make sense to use the mean field approximation if smearing is mild – but we should treat the results with some degree

[2]PR would like to thank Colin Morningstar for conversations on this point.

of caution if extreme smearing is used.

Applying this double-u mean field approximation to the SLiNC fermion matrix we find the following results for the principal fermion quantities,

$$\Sigma_1(p) \approx u_S\,, \quad \Sigma_2(p) \approx u_S\,, \quad Z_\psi \approx u_S\,, \quad \kappa_c \approx \frac{1}{8u_S}\,, \quad c_{SW} \approx \frac{u_S}{u_0^4} \tag{57}$$

(we define Z_ψ by the relation $S^{\rm ren} = Z_\psi S^{\rm lat}$). For reasonable smearing we expect the smeared link u_S to be closer to 1 than the bare link u_0, so most quantities will lie closer to their tree-level values with smearing. However, the clover coefficient c_{SW} is an exception; it will be further from 1 with smearing than without, because we construct our clover term from unsmeared links.

As a result, we obtain the mean field expressions for κ_c and c_{SW} by performing the following replacements

$$\kappa_c(g^2) \to \kappa_c^{MF}(g^2_{MF}, u_S) = \frac{1}{8}\,\frac{u_S^{\rm pert}(g^2_{MF})}{u_S}\,\kappa_c(g^2_{MF}) \tag{58}$$

and

$$c_{SW}(g^2) \to c_{SW}^{MF}(g^2_{MF}, u_S, u_0) = \frac{u_S}{u_0^4}\,\frac{u_0^{\rm pert}(g^2_{MF})^{\,4}}{u_S^{\rm pert}(g^2_{MF})}\,c_{SW}(g^2_{MF})\,. \tag{59}$$

Here u_S and u_0 are the measured smeared and unsmeared links at the given coupling and $u_S^{\rm pert}$ and $u_0^{\rm pert}$ denote the corresponding expressions in lattice perturbation theory.

5.2 The smeared plaquette in perturbation theory

We will use $u_S^{\rm pert}$ derived from the smeared perturbative plaquette P_S

$$u_S^{\rm pert} \equiv P_S^{1/4}. \tag{60}$$

To one-loop order we have

$$u_S^{\rm pert} = 1 - \frac{g^2\,C_F}{16\pi^2}\,k_S\,, \tag{61}$$

with[3]

$$k_S = 8\pi^2 a^4 \int \frac{d^4k}{(2\pi)^4}\,D_{\alpha\beta}(k)\Big[\,V_{\alpha1}(k,\omega)V_{\beta1}(k,\omega)s_2^2(k) + V_{\alpha2}(k,\omega)V_{\beta2}(k,\omega)s_1^2(k) - \left(V_{\alpha1}(k,\omega)V_{\beta2}(k,\omega) + V_{\beta1}(k,\omega)V_{\alpha2}(k,\omega)\right)s_1(k)s_2(k)\Big] \tag{62}$$

[3]We have written this integral for the case of a plaquette in the 1-2 plane, any orientation gives the same result.

where $D_{\alpha\beta}(k)$ the gluon propagator for the action in question. The smearing function $V_{\alpha\mu}(k,\omega)$ is defined in (A.5) in Appendix A, $s_\mu(k)$ and $s^2(k)$ used below are given in (A.2). Using symmetry and the definition of V, the expression simplifies to

$$k_S = 16\pi^2 a^4 \int \frac{d^4k}{(2\pi)^4} \left[D_{11}(k)s_2(k)s_2(k) - D_{12}(k)s_1(k)s_2(k)\right] \left(1 - 4\,\omega\, s^2(k)\right)^2 . \qquad (63)$$

We can see from this form that mild smearing has the effect of suppressing the contribution from large k. Setting $\omega = 0$ in k_S, we recover the unsmeared link in perturbation theory

$$u_0^{\rm pert} = 1 - \frac{g^2\, C_F}{16\pi^2}\, k_S(\omega = 0)\,. \qquad (64)$$

For the plaquette action propagator we can calculate the integral exactly. The result is

$$k_S^{\rm Plaq} = \pi^2 \left(1 - 16\,\omega + 72\,\omega^2\right) . \qquad (65)$$

Let us see how well this improves the expressions for κ_c and c_{SW}. Using the result (55) we find

$$\kappa_c^{{\rm Plaq},MF} = \frac{1}{8u_S}\left[1 + g_{MF}^2\, C_F\, \left(-0.011861 + 0.103020\,\omega - 0.802715\,\omega^2\right)\right] \qquad (66)$$

which successfully reduces the perturbative coefficients for every power of ω. Trying the same thing with the clover coefficient (45) gives

$$c_{SW}^{{\rm Plaq},MF} = \frac{u_S}{u_0^4}\Big\{ 1 + g_{MF}^2 \Big[C_F\, \left(-0.019865 + 0.079148\,\omega + 0.813321\,\omega^2\right) + N_c\, \left(0.015025 + 0.009617\,\omega - 0.284786\,\omega^2\right) \Big]\Big\}. \qquad (67)$$

Again, mean field improvement works well.

For the Symanzik action we calculate the integral in (63) numerically, and get the result

$$k_S^{\rm Sym} = \pi^2 \left(0.732525 - 11.394696\,\omega + 50.245225\,\omega^2\right) . \qquad (68)$$

The corresponding mean field improved expressions for κ_c (56) and c_{SW} (46) are

$$\kappa_c^{{\rm Sym},MF} = \frac{1}{8u_S}\left[1 + g_{MF}^2\, C_F\, \left(-0.008053 + 0.0500781\,\omega - 0.471784\,\omega^2\right)\right] , \qquad (69)$$

$$c_{SW}^{{\rm Sym},MF} = \frac{u_S}{u_0^4}\Big\{ 1 + g_{MF}^2 \Big[C_F\, \left(-0.0211635 + 0.115961\,\omega + 0.685247\,\omega^2\right) + N_c\, \left(0.013777 + 0.015905\,\omega - 0.321899\,\omega^2\right) \Big]\Big\}. \qquad (70)$$

5.3 Choice of g^2_{MF}

In this section we discuss the boosted coupling for $SU(3)$, we have set $N_c = 3$, $C_F = 4/3$ throughout.

From higher order continuum calculations we know that $g^2_{\overline{MS}}(\mu)$ is a good expansion parameter if μ is close to the appropriate physical scale. On the other hand, series in the bare lattice coupling $g^2(a)$ usually converge poorly. To understand this difference let us compare the two couplings. To one-loop order we have

$$\frac{1}{g^2_{\overline{MS}}(\mu)} - \frac{1}{g^2(a)} = 2b_0 \left(\log \frac{\mu}{\Lambda_{\overline{MS}}} - \log \frac{1}{a\Lambda_{\rm lat}} \right) = 2b_0 \log(a\mu) + d_g + N_f\, d_f \,, \tag{71}$$

where $b_0 = (11 - 2N_f/3)/(4\pi)^2$, and N_f is the number of flavors. The ratio of Λ parameters is thus given by

$$\frac{\Lambda_{\rm lat}}{\Lambda_{\overline{MS}}} = \exp\left(\frac{d_g + N_f\, d_f}{2b_0} \right) \,. \tag{72}$$

The coefficient d_g is known for the plaquette and Symanzik gauge action [17]:

$$d_g^{\rm Plaq} = -0.4682 \,, \quad d_g^{\rm Sym} = -0.2361 \,. \tag{73}$$

In Appendix C we show that d_f is independent of the stout smearing parameter ω. Therefore, we can use the value for clover fermions computed in [18]

$$d_f = 0.0314917 \,. \tag{74}$$

For $N_f = 3$ this leads to

$$\frac{\Lambda_{\rm lat}}{\Lambda_{\overline{MS}}} = 0.038 \quad \text{Plaquette} \,, \tag{75}$$

$$\frac{\Lambda_{\rm lat}}{\Lambda_{\overline{MS}}} = 0.289 \quad \text{Symanzik} \,. \tag{76}$$

These ratios are far from 1, especially for the plaquette action, which explains the poor convergence of series in $g^2(a)$.

Now let us see what happens to the Lambda ratio if we make the popular choice of boosted coupling

$$g^2_{MF} = \frac{g^2}{u_0^4} \,. \tag{77}$$

Upon inserting (64) and (77) in (71), we obtain

$$\frac{1}{g^2_{\overline{MS}}(\mu)} - \frac{1}{g^2_{MF}(a)} = 2b_0 \left(\log \frac{\mu}{\Lambda_{\overline{MS}}} - \log \frac{1}{a\Lambda^{MF}_{\rm lat}} \right) = 2b_0 \log(a\mu) + d_g + N_f\, d_f + \frac{k_u}{3\pi^2}\,, \quad (78)$$

which gives

$$\frac{\Lambda^{MF}_{\rm lat}}{\Lambda_{\overline{MS}}} = \exp\left(\frac{d_g + N_f\, d_f + k_u/3\pi^2}{2b_0} \right)\,. \quad (79)$$

For $N_f = 3$ the numerical values of this ratio are

$$\frac{\Lambda^{MF}_{\rm lat}}{\Lambda_{\overline{MS}}} = 0.702 \quad \text{Plaquette}\,, \quad (80)$$

$$\frac{\Lambda^{MF}_{\rm lat}}{\Lambda_{\overline{MS}}} = 2.459 \quad \text{Symanzik}\,. \quad (81)$$

We see that mean field improvement drives $\Lambda_{\rm lat}$ towards $\Lambda_{\overline{MS}}$ for both the plaquette and Symanzik gauge action, giving $g^2_{MF} \approx g^2_{\overline{MS}}$, so that g^2_{MF} appears to be a good expansion parameter in both cases. A perfect match is obtained for $\mu = 1/0.702a$ ($\mu = 1/2.459a$) for the plaquette (Symanzik) action.

6 Concluding remarks

In the present paper we have computed the improvement coefficient c_{SW} and the additive mass renormalization/critical hopping parameter in one-loop perturbation theory for general stout parameter ω performing a single smearing. To separate the effect of improving the gauge action from the effect of tuning the fermion action, we have done the calculation for both the plaquette action and the tree-level Symanzik gauge action. In addition we also present the $\mathcal{O}(g^2)$ corrections to the coefficients c_{NGI} and c_D needed to $\mathcal{O}(a)$ improve the quark fields in the most general case.

We give mean field (tadpole) improved results for κ_c and c_{SW}. For both the plaquette and the Symanzik action the boosted coupling g^2_{MF} turns out to be close to $g^2_{\overline{MS}}$, which makes g^2_{MF} a good expansion parameter. We thus may expect that the perturbative series converges rapidly.

For $N_f = 3$ flavors of dynamical quarks it turns out that the one-loop improved Symanzik gauge action [2] coincides largely with its tree-level counterpart, with coefficients $c_0 \approx 5/3$, $c_1 \approx -1/12$ and $c_2 \approx 0$ [19]. This makes the tree-level Symanzik action (1)

stand out against other improved gauge actions, at least from the perturbative point of view. SLiNC fermions represent a family of ultralocal, ultraviolet filtered clover fermions. While they share all prominent features of clover fermions, among them $\mathcal{O}(a)$ improvement and flavor symmetry, they allow to further optimize the chiral properties of the action by tuning the fattening of the links. In our forthcoming simulations with $N_f = 2 + 1$ and $2 + 1 + 1$ flavors of dynamical quarks at realistic pion masses we shall employ this combination of gauge and fermion actions.

Knowing the perturbative (asymptotic) value of c_{SW}, we can derive a closed expression for c_{SW} that covers the whole range of g^2. We will do so in a subsequent paper employing the Schrödinger functional method. The one-loop coefficient $c_{SW}^{(1)}$ varies only slightly within the interval $0 \leq \omega \leq 0.2$ for both the plaquette and Symanzik action. For $\omega = 0.1$, which is our favorite value, the tadpole improved one-loop coefficient becomes $c_{SW}^{(1)} \approx 0$, indicating that mean field approximation works well. The final result is $c_{SW}^{MF} \approx u_S/u_0^4$ to a very good approximation for both gauge actions, where u_S is the average smeared link, found by measuring the smeared plaquette, and u_0 the average unsmeared link, found by measuring the unsmeared plaquette.

This is to be compared with $c_{SW}^{MF} \approx 1/u_0^3$ over fermions with no smearing. We therefore expect c_{SW} to be a steeper function of g^2 in the case of SLiNC fermions than for clover fermions.

Stout link fattening reduces the additive mass renormalization considerably, with and without tadpole improvement, as expected. In fact, the critical hopping parameter κ_c can be tuned to its continuum value of 1/8 for an appropriate choice of ω. We also confirm by early simulations with this action [20] that the spread of the negative eigenvalues is reduced by a factor of ≈ 2 for $\omega = 0.1$ and non-perturbative c_{SW}, as compared to ordinary clover fermions. SLiNC fermions have many other appealing features as well. The renormalization factors of quark bilinear operators, for example, come out to be very close to unity, which hints at virtually continuum-like behavior.

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Appendix A: Feynman rules

In this Appendix we give the Feynman rules for quark-gluon vertices derived from action (3) with single stout smeared gauge link variables in the Wilson part and general Wilson parameter r. The pieces in the vertices proportional to c_{SW} are denoted with $\widetilde{V}$. They have been rederived using our notations and they agree with the Feynman rules given in [11]. In the vertices we denote the incoming/outgoing quark momenta by p_1/p_2. The incoming gluons are described by momenta k_i, Lorentz indices α, β, γ and color indices $a, b, c = 1, \ldots, N_c^2 - 1$.

For the color matrices we have:

$$\begin{aligned} & T^a T^b = \frac{1}{2N_c}\delta^{ab} I_{N_c} + \frac{1}{2}(d^{abc} + i\, f^{abc}) T^c \\ & C_F = \frac{N_c^2 - 1}{2N_c}\,, \quad [T^a, T^b] = T^a T^b - T^b T^a\,, \quad \{T^a, T^b\} = T^a T^b + T^b T^a \\ & T^{abc}_{ss} = \{T^a, \{T^b, T^c\}\}\,, \quad T^{abc}_{aa} = [T^a, [T^b, T^c]]\,, \quad T^{abc}_{sa} = \{T^a, [T^b, T^c]\}\,. \end{aligned} \tag{A.1}$$

We use the abbreviations

$$\begin{aligned} & \mathrm{s}_\mu(k) = \sin\left(\frac{a}{2}k_\mu\right)\,, \quad \mathrm{c}_\mu(k) = \cos\left(\frac{a}{2}k_\mu\right)\,, \quad s^2(k) = \sum_\mu \mathrm{s}_\mu^2(k)\,, \\ & s^2(k_1, k_2) = \sum_\mu \mathrm{s}_\mu(k_1 + k_2)\, \mathrm{s}_\mu(k_1 - k_2) \equiv s^2(k_1) - s^2(k_2)\,. \end{aligned} \tag{A.2}$$

For later use we give the bare massless quark propagator

$$S(k) = \frac{a}{i\sum_\mu \gamma_\mu \mathrm{s}_\mu(2k) + r\sum_\mu (1 - \mathrm{c}_\mu(2k))}\,. \tag{A.3}$$

The structure of the Wilson quark-gluon vertices is

$$\begin{aligned} W_{1\mu}(p_2, p_1) &= i\, \mathrm{c}_\mu(p_2 + p_1)\, \gamma_\mu + r\, \mathrm{s}_\mu(p_2 + p_1) \\ W_{2\mu}(p_2, p_1) &= i\, \mathrm{s}_\mu(p_2 + p_1)\, \gamma_\mu - r\, \mathrm{c}_\mu(p_2 + p_1)\,. \end{aligned} \tag{A.4}$$

Let us introduce the following functions to be useful in the definitions of the improved

vertices

$$
\begin{aligned}
V_{\alpha\mu}(k,\omega) &= \delta_{\alpha\mu} + 4\,\omega\, v_{\alpha\mu}(k) && \text{(A.5)}\\
v_{\alpha\mu}(k) &= \mathrm{s}_\alpha(k)\,\mathrm{s}_\mu(k) - \delta_{\alpha\mu}\, s^2(k) \\
g_{\alpha\beta\mu}(k_1,k_2) &= \delta_{\alpha\beta}\,\mathrm{c}_\alpha(k_1+k_2)\,\mathrm{s}_\mu(k_1-k_2) \\
&\quad - \delta_{\alpha\mu}\,\mathrm{c}_\alpha(k_2)\,\mathrm{s}_\beta(2k_1+k_2) + \delta_{\beta\mu}\,\mathrm{c}_\beta(k_1)\,\mathrm{s}_\alpha(2k_2+k_1) && \text{(A.6)}\\
w_{\alpha\mu}(k_1,k_2) &= \mathrm{s}_\alpha(k_1+k_2)\,\mathrm{s}_\mu(k_1-k_2) - \delta_{\alpha\mu}\, s^2(k_1,k_2)\,, && \text{(A.7)}\\
w_{\alpha\mu}(k,0) &= v_{\alpha\mu}(k)
\end{aligned}
$$

The qqg-vertex: $V^a_\alpha(p_2,p_1,k_1;c_{SW},\omega)$

The qqg-vertex including stout smeared links and clover contribution is given by the expression ($p_1+k_1=p_2$)

$$
V^a_\alpha(p_2,p_1,k_1;c_{SW},\omega) = -g\,T^a \sum_\mu V_{\alpha\mu}(k_1,\omega)\,W_{1\mu}(p_2,p_1) + c_{SW}\,\widetilde{V}^a_\alpha(k_1)\,. \qquad \text{(A.8)}
$$

The stout smeared part shows the separation property mentioned in [6]. The clover part is given by

$$
\widetilde{V}^a_\alpha(k_1) = -i\,g\,T^a\,\frac{r}{2}\sum_\mu \sigma_{\alpha\mu}\,\mathrm{c}_\alpha(k_1)\,\mathrm{s}_\mu(2k_1)\,. \qquad \text{(A.9)}
$$

The qqgg-vertex: $V^{ab}_{\alpha\beta}(p_2,p_1,k_1,k_2;c_{SW},\omega)$

We define the qqgg-vertex as follows ($p_1+k_1+k_2=p_2$):

$$
V^{ab}_{\alpha\beta}(p_2,p_1,k_1,k_2;c_{SW},\omega) = V^{\{a,b\}}_{\alpha\beta} + V^{[a,b]}_{\alpha\beta} + c_{SW}\,\widetilde{V}^{ab}_{\alpha\beta}(k_1,k_2)\,. \qquad \text{(A.10)}
$$

The stout smeared part is separated into two parts proportional to $\{T^a,T^b\}$ and $[T^a,T^b]$. The anticommutator part shows the factorization property mentioned for two and four quark operators

$$
V^{\{a,b\}}_{\alpha\beta} = \frac{1}{2}a\,g^2\,\{T^a,T^b\}\sum_\mu V_{\alpha\mu}(k_1,\omega)\,V_{\beta\mu}(k_2,\omega)\,W_{2\mu}(p_2,p_1)\,. \qquad \text{(A.11)}
$$

The commutator part is given by

$$V^{[a,b]}_{\alpha\beta} \;=\; \frac{1}{2} a\, g^2\, [T^a, T^b]\, 4\, \omega \sum_\mu g_{\alpha\beta\mu}(k_1, k_2)\; W_{1\mu}(p_2, p_1)\,. \tag{A.12}$$

Note that this part is proportional to ω. The part $\propto c_{SW}$ has been used in the form

$$\begin{aligned}\widetilde{V}^{ab}_{\alpha\beta}(k_1, k_2) \;=\; & i\, \frac{r}{4} a\, g^2\, [T^a, T^b] \Big\{ 2\, \sigma_{\alpha\beta} \big[2\, \mathrm{c}_\beta(k_1)\; \mathrm{c}_\alpha(k_2)\; \mathrm{c}_\alpha(k_1+k_2)\; \mathrm{c}_\beta(k_1+k_2) \\ & - \mathrm{c}_\alpha(k_1)\; \mathrm{c}_\beta(k_2) \big] + \delta_{\alpha\beta} \sum_\mu \sigma_{\alpha\mu}\, \mathrm{s}_\alpha(k_1+k_2)\; [\, \mathrm{s}_\mu(2k_2) \,-\, \mathrm{s}_\mu(2k_1)\,] \Big\}\,.\end{aligned} \tag{A.13}$$

Both (A.12) and (A.13) vanish for tadpole diagrams along quark lines.

The qqggg-vertex: $V^{abc}_{\alpha\beta\gamma}(p_2, p_1, k_1, k_2, k_3; c_{SW}, \omega)$

We present that vertex contribution in the following form ($p_1 + k_1 + k_2 + k_3 = p_2$)

$$\begin{aligned}V^{abc}_{\alpha\beta\gamma}(p_2, p_1, k_1, k_2, k_3; c_{SW}, \omega) = & \frac{1}{6}\, a^2 g^3 \,\times \\ & \sum_\mu \Big\{ W_{1\mu}(p_2, p_1) \Big[F^{abc}_{\alpha\beta\gamma\mu}(k_1, k_2, k_3) + \text{cyclic perm.} \Big] \\ & - 6\, \omega\, W_{2\mu}(p_2, p_1) \Big[T^{abc}_{sa}\, V_{\alpha\mu}(k_1)\, g_{\beta\gamma\mu}(k_2, k_3) + \text{cyclic perm.} \Big] \Big\} \\ & + c_{SW}\, \widetilde{V}^{abc}_{\alpha\beta\gamma}(k_1, k_2, k_3)\,.\end{aligned} \tag{A.14}$$

Cyclic permutations have to be performed in the gluon momenta as well as in the color and Lorentz indices of the three gluons. Note that the general stout smeared part is proportional both to $W_{1\mu}$ and $W_{2\mu}$.

The coefficient $F^{abc}_{\alpha\beta\gamma\mu}(k_1, k_2, k_3)$ is decomposed into its different color structures:

$$\begin{aligned}F^{abc}_{\alpha\beta\gamma\mu}(k_1, k_2, k_3) \;=\; & T^{abc}_{ss}\, f^{(1)}_{\alpha\beta\gamma\mu}(k_1, k_2, k_3) + T^{abc}_{aa} \left(f^{(2)}_{\alpha\beta\gamma\mu}(k_1, k_2, k_3) - f^{(2)}_{\alpha\gamma\beta\mu}(k_1, k_3, k_2) \right) \\ & + \left(T^{abc}_{ss} - \frac{1}{N_c} d^{abc} \right) f^{(3)}_{\alpha\beta\gamma\mu}(k_1, k_2, k_3)\,,\end{aligned} \tag{A.15}$$

where the $f^{(i)}_{\alpha\beta\gamma\mu}$ are given as

$$
\begin{aligned}
f^{(1)}_{\alpha\beta\gamma\mu}(k_1,k_2,k_3) &= \frac{1}{2}\,V_{\alpha\mu}(k_1,\omega)\,V_{\beta\mu}(k_2,\omega)\,V_{\gamma\mu}(k_3,\omega)\,, \\
f^{(2)}_{\alpha\beta\gamma\mu}(k_1,k_2,k_3) &= \frac{1}{2}\,V_{\alpha\mu}(k_1,\omega)\,V_{\beta\mu}(k_2,\omega)\,\delta_{\gamma\mu} - \frac{1}{2}\,\delta_{\alpha\mu}\delta_{\beta\mu}\,V_{\gamma\mu}(k_3,\omega) \\
&\quad + 6\,\omega\,\delta_{\alpha\beta}\Big[\,\mathrm{c}_\mu(k_1-k_2)\;\mathrm{c}_\beta(2k_3+k_1+k_2)\,\delta_{\gamma\mu} + \mathrm{s}_\mu(k_3)\;\mathrm{s}_\gamma(k_3+2k_1)\;\delta_{\beta\mu}\Big]\,, \\
f^{(3)}_{\alpha\beta\gamma\mu}(k_1,k_2,k_3) &= 2\,\omega\,\delta_{\beta\gamma}\Big[\big(3\,w_{\alpha\mu}(k_1,k_2+k_3)+v_{\alpha\mu}(k_1+k_2+k_3)\big)\,\delta_{\alpha\beta} \\
&\quad + 12\,\mathrm{s}_\beta(k_1)\;\mathrm{s}_\alpha(k_2)\;\mathrm{s}_\alpha(k_3)\,\big(\mathrm{s}_\beta(k_1+k_2+k_3)\;\delta_{\alpha\mu} - \mathrm{s}_\alpha(k_1+k_2+k_3)\,\delta_{\beta\mu}\big)\Big]\,.
\end{aligned}
\tag{A.16}
$$

The clover part of the qqggg-vertex is given by

$$
\widetilde{V}^{abc}_{\alpha\beta\gamma}(k_1,k_2,k_3) = \frac{1}{6}\left\{\widetilde{\widetilde{V}}^{abc}_{\alpha\beta\gamma}(k_1,k_2,k_3) + \text{total perm.}\right\}
\tag{A.17}
$$

with

$$
\begin{aligned}
\widetilde{\widetilde{V}}^{abc}_{\alpha\beta\gamma}(k_1,k_2,k_3) &= -3\,i\,g^3\,a^2\,r\,\times \\
&\Bigg[T^aT^bT^c\delta_{\alpha\beta}\delta_{\alpha\gamma}\sum_\mu\,\sigma_{\alpha\mu}\Bigg\{-\frac{1}{6}\,\mathrm{c}_\alpha(k_1+k_2+k_3)\;\mathrm{s}_\mu(2(k_1+k_2+k_3)) \\
&\quad + \mathrm{c}_\alpha(k_1+k_2+k_3)\;\mathrm{c}_\mu(k_1+k_2+k_3)\;\mathrm{c}_\mu(k_3-k_1)\;\mathrm{s}_\mu(k_2)\Bigg\} \\
&\quad - \frac{1}{2}\Big[T^aT^bT^c+T^cT^bT^a\Big]\,\sigma_{\alpha\beta}\,\times \\
&\quad \Bigg\{2\,\delta_{\beta\gamma}\,\mathrm{c}_\alpha(k_1+k_2+k_3)\;\mathrm{c}_\beta(k_1+k_2+k_3)\;\mathrm{c}_\alpha(k_3+k_2)\;\mathrm{s}_\beta(k_1) \\
&\quad + \delta_{\beta\gamma}\,\mathrm{s}_\beta(k_3+k_2)\;\mathrm{c}_\alpha(k_1+2k_2) \\
&\quad + \delta_{\alpha\gamma}\,\mathrm{s}_\alpha(k_1+2k_2+k_3)\;\mathrm{c}_\beta(k_1+k_2+k_3)\;\mathrm{c}_\beta(k_3-k_1)\Bigg\}\Bigg]\,.
\end{aligned}
\tag{A.18}
$$

In (A.17) the total permutation has to be performed in the gluon momenta, color and Lorentz indices.

We only need this vertex for the gluon tadpole diagram of Fig. 1, which simplifies the expressions. In the tadpole contribution to the vertex (A.14) we denote the external gluon momentum by $q = p_2 - p_1$, the color index of the gluon by a and the internal momenta by k

and $-k$. The color indices (b,c) of the remaining gluons forming the tadpole are summed up using the color diagonality δ^{bc} of the gluon propagator, k is the gluon momentum in the tadpole loop. So the stout smeared tadpole contribution is defined from the general qqggg-vertex (explicitly symmetrized in the three gluons) as

$$\begin{aligned} V^a_{\alpha\beta\gamma}(p_2,p_1,k) &= \sum_{b=1}^{N_c^2-1}\left\{V^{abb}_{\alpha\beta\gamma}(p_2,p_1,q,k,-k)+c_{SW}\,\widetilde{V}^{abb}_{\alpha\beta\gamma}(p_2,p_1,q,k,-k)\right\} \\ &= \frac{1}{6}a^2\,g^3\,T^a\sum_{\mu}W_{1\mu}(p_2,p_1)V_{\alpha\beta\gamma\mu}(q,k) \\ &\quad + c_{SW}\sum_{b=1}^{N_c^2-1}\widetilde{V}^{abb}_{\alpha\beta\gamma}(p_2,p_1,q,k,-k)\,. \end{aligned} \tag{A.19}$$

Using that definition we obtain for the stout smeared part

$$\begin{aligned} V_{\alpha\beta\gamma\mu}(q,k) &= \Big\{(6\,C_F-N_c)\;f^{(1)}_{\alpha\beta\gamma\mu}(q,k,-k)+\frac{N_c}{2}\Big[f^{(2)}_{\beta\gamma\alpha\mu}(k,-k,q)-f^{(2)}_{\beta\alpha\gamma\mu}(k,q,-k) \\ &\quad -f^{(2)}_{\gamma\alpha\beta\mu}(-k,q,k)+f^{(2)}_{\gamma\beta\alpha\mu}(-k,k,q)\Big]+4\,C_F\,f^{(3)}_{\alpha\beta\gamma\mu}(q,k,-k) \\ &\quad +(4\,C_F-N_c)\Big[f^{(3)}_{\beta\gamma\alpha\mu}(k,-k,q)+f^{(3)}_{\gamma\alpha\beta\mu}(-k,q,k)\Big]\Big\}\,. \end{aligned} \tag{A.20}$$

From that expression a convenient representation is found in the form

$$\begin{aligned} V_{\alpha\beta\gamma\mu}(q,k) &= \Big\{(6\,C_F-N_c)\;V_{\alpha\mu}(q,\omega)\,V_{\beta\mu}(k,\omega)\,V_{\gamma\mu}(k,\omega) \\ &\quad +\frac{N_c}{2}\Big[2\,\delta_{\alpha\mu}V_{\beta\mu}(k,\omega)\,V_{\gamma\mu}(k,\omega)-V_{\alpha\mu}(q,\omega)\,\big(\delta_{\beta\mu}\,V_{\gamma\mu}(k,\omega)+\delta_{\gamma\mu}\,V_{\beta\mu}(k,\omega)\big)\Big] \\ &\quad +2\,\omega\,\Big[3\,(4\,C_F-N_c)\;C_{\alpha\beta\gamma\mu}(q,k)+N_c\,D_{\alpha\beta\gamma\mu}(q,k)\Big]\Big\}\,. \end{aligned} \tag{A.21}$$

The structures $C_{\alpha\beta\gamma\mu}$ and $D_{\alpha\beta\gamma\mu}$, additionally contributing to $O(\omega)$, are

$$\begin{aligned} C_{\alpha\beta\gamma\mu}(q,k) &= -4\left[\delta_{\alpha\mu}\,\mathrm{s}^2_\gamma(p) - \delta_{\alpha\gamma}\,\mathrm{s}_\alpha(p)\,\mathrm{s}_\mu(p)\right]\left[\delta_{\beta\gamma}\,\mathrm{s}^2_\mu(k) - \delta_{\beta\mu}\,\mathrm{s}_\beta(k)\,\mathrm{s}_\gamma(k)\right] \\ &\quad - 4\,\delta_{\gamma\mu}\,\mathrm{s}_\beta(p)\,\mathrm{s}_\alpha(k)\left[\delta_{\alpha\beta}\,\mathrm{s}_\mu(p)\,\mathrm{s}_\mu(k) - \delta_{\alpha\mu}\,\mathrm{s}_\beta(p)\,\mathrm{s}_\beta(k) - \delta_{\beta\mu}\,\mathrm{s}_\alpha(p)\,\mathrm{s}_\alpha(k)\right] \\ &\quad - \delta_{\alpha\mu}\delta_{\beta\mu}\delta_{\gamma\mu}\left[2s^2(p) + 2s^2(k) - s^2(p+k) - s^2(p-k)\right], \end{aligned} \tag{A.22}$$

$$\begin{aligned} D_{\alpha\beta\gamma\mu}(q,k) &= -3\,\delta_{\alpha\gamma}\delta_{\beta\mu}\,\mathrm{c}_\beta(p+k)\,\mathrm{c}_\gamma(p+k) - 3\,\delta_{\alpha\beta}\delta_{\gamma\mu}\,\mathrm{c}_\beta(p-k)\,\mathrm{c}_\gamma(p-k) \\ &\quad + 4\,\delta_{\beta\gamma}(\delta_{\alpha\beta}+\delta_{\beta\mu})\,\mathrm{s}_\alpha(p)\,\mathrm{s}_\mu(p) + 4\,\delta_{\alpha\mu}(\delta_{\beta\mu}+\delta_{\gamma\mu})\,\mathrm{s}_\beta(k)\,\mathrm{s}_\gamma(k) \\ &\quad - 2\,\delta_{\alpha\mu}\delta_{\beta\mu}\delta_{\gamma\mu}\left[s^2(p)+s^2(k)\right] + 6\,\delta_{\alpha\mu}\delta_{\beta\gamma}\left[2\,\mathrm{c}^2_\gamma(p)\,\mathrm{c}^2_\alpha(k) - 1\right]. \end{aligned}$$

Appendix B: Three-point function - universal part

As discussed above, the universal part of the three-point function has the form (34) when $c_{SW} = 1 + \mathcal{O}(g^2)$. Therefore, it is sufficient to give only the one-loop result for $\Lambda^{\overline{MS}}_{1,\mu}(p_1,p_2,q)$. It is cast into the following form ($q = p_2 - p_1$)

$$\begin{aligned} \Lambda^{\overline{MS}}_{1,\mu}(p_1,p_2,q) &= F_1(p_1,p_2)\,\gamma_\mu + F_2(p_1,p_2)\,\not{p}_2\,\gamma_\mu\not{p}_1 \\ &\quad + \left[F_3(p_1,p_2)\,p_{1,\mu} + F_4(p_1,p_2)\,p_{2,\mu}\right]\not{p}_1 \\ &\quad + \left[F_5(p_1,p_2)\,p_{2,\mu} + F_6(p_1,p_2)\,p_{1,\mu}\right]\not{p}_2\,. \end{aligned} \tag{B.1}$$

Due to the symmetries $F_5(p_1,p_2) = F_3(p_2,p_1)$ and $F_6(p_1,p_2) = F_4(p_2,p_1)$ we have four independent functions $F_i(p_1,p_2)$ only. We represent them as follows:

$$\begin{aligned} F_1(p_1,p_2) &= 4\,C_F\,\xi - \frac{N_c}{2}(12 + 2\xi - \xi^2) + 2\,\Theta\left(\mathcal{C}_1\,\mathcal{S} + N_c\,p_1.p_2 + C_F\,q^2\right) \\ &\quad + \left(C_F(1-\xi) + \frac{N_c}{4}(4-\xi)\right)\log\left(\frac{p_1^2 p_2^2}{(\mu^2)^2}\right) \\ &\quad + V_1(p_1,p_2)\log\left(\frac{p_1^2}{q^2}\right) + V_1(p_2,p_1)\log\left(\frac{p_2^2}{q^2}\right), \end{aligned} \tag{B.2}$$

$$F_2(p_1,p_2) = \frac{\Theta}{8}\left(2N_c(6-\xi) + \mathcal{C}_2\,\frac{p_1.p_2\,q^2}{\Delta}\right) + \frac{\mathcal{C}_2}{4\Delta}\left[p_1.q\log\left(\frac{p_1^2}{q^2}\right) - p_2.q\log\left(\frac{p_2^2}{q^2}\right)\right], \tag{B.3}$$

$$\begin{aligned} F_3(p_1,p_2) &= \mathcal{C}_3\,\frac{p_2^2}{2\,\Delta} + \frac{2\,N_c\,\xi}{q^2} + \frac{\Theta}{8\,\Delta}\Big[4N_c\,\xi(p_1.p_2)^2 + \left(2\,\mathcal{C}_3\,(6\,\mathcal{S} + p_2^2) - \mathcal{C}_4\,p_1.q\right)\,p_2^2\Big] \\ &\quad + \frac{1}{q^2}\left[V_2(p_1,p_2)\log\left(\frac{p_1^2}{q^2}\right) + V_3(p_1,p_2)\log\left(\frac{p_2^2}{q^2}\right)\right], \end{aligned} \tag{B.4}$$

$$
\begin{aligned}
F_4(p_1,p_2) &= -\mathcal{C}_3\,\frac{p_1.p_2}{2\,\Delta} - \frac{2\,N_c\,\xi}{q^2} + \frac{\Theta}{8\,\Delta}\Big[4(8\,C_F - N_c(4-\xi))\,(p_1.p_2)^2 \\
&\quad - (12\,\mathcal{C}_3\,\mathcal{S} + 4\,\mathcal{C}_6\;p_1^2 + (\,\mathcal{C}_5\;+8\,C_F(2+\xi))\,p_2^2)\,p_1.p_2 + \;\mathcal{C}_7\;p_1^2\,p_2^2\Big] \\
&\quad + \frac{1}{q^2}\left[V_4(p_1,p_2)\log\left(\frac{p_1^2}{q^2}\right) + V_5(p_1,p_2)\log\left(\frac{p_2^2}{q^2}\right)\right].
\end{aligned}
\tag{B.5}
$$

The function V_i in front of the logarithms are found as follows

$$
\begin{aligned}
V_1(p_1,p_2) &= C_F(3+\xi) - \frac{N_c}{4}(4-\xi) + \;\mathcal{C}_1\,\frac{p_2.q\,p_1^2}{\Delta}\,, \\
V_2(p_1,p_2) &= \frac{1}{4\,\Delta}\Big[(4\,\mathcal{C}_3\;-\;\mathcal{C}_4\;-4N_c\,\xi)\,p_2^2\,q^2 \\
&\quad + (12\,\mathcal{C}_3\,\mathcal{S} + 4N_c\,\xi\,p_1.p_2 + (\,\mathcal{C}_5\;+8\,C_F)\,q^2)\,p_2.q\Big]\,, \\
V_3(p_1,p_2) &= \frac{1}{4\,\Delta\,p_1^2}\left[-4N_c\,\xi\,p_1.p_2\,p_2.q\,p_1^2 + \left(-12\,\mathcal{C}_3\,\mathcal{S}\,p_1.q + \mathcal{C}_4\,p_1^2\,q^2\right)\;p_2^2\right]\,, \\
V_4(p_1,p_2) &= V_2(p_2,p_1) + \frac{1}{4\,\Delta}\left[-8\,C_F(1+\xi)\,p_1.q + (4\,C_F(1-3\xi) + N_c(5-\xi)\xi)\,p_1^2\right]\,, \\
V_5(p_1,p_2) &= V_2(p_1,p_2) + \frac{1}{4\,\Delta}\left[(8\,C_F + N_c(2-\xi)\xi)\,p_2.q + (1+\xi)(4\,C_F + N_c\,\xi)p_2^2\right]\,.
\end{aligned}
\tag{B.6}
$$

We have introduced the kinematic functions

$$
\begin{aligned}
\Delta &= (p_1.p_2)^2 - p_1^2\,p_2^2\,, \quad \mathcal{S} = \frac{p_1^2\,p_2^2\,q^2}{4\,\Delta}\,, \\
\Theta &= \frac{4}{\pi^2\,\sqrt{\Delta}}\left(\mathrm{Sp}\left(\frac{p_2.q+\sqrt{\Delta}}{p_2^2}\right) - \mathrm{Sp}\left(\frac{p_2.q-\sqrt{\Delta}}{p_2^2}\right)\right. \\
&\quad \left. + \frac{1}{2}\log\left(\frac{p_1.p_2-\sqrt{\Delta}}{p_1.p_2+\sqrt{\Delta}}\right)\log\left(\frac{q^2}{p_2^2}\right)\right),
\end{aligned}
\tag{B.7}
$$

with $\mathrm{Sp}(x)$ being the Spence function:

$$
\mathrm{Sp}(x) = -\int_0^x dy\frac{\log(1-y)}{y}\,.
$$

The quantities $\mathcal{C}_i$ depend on the color factors and gauge parameter and have the values

$$\begin{aligned}
\mathcal{C}_1 &= C_F\,(3+\xi) - \frac{1}{2}N_c\,(1-\xi)\,, \\
\mathcal{C}_2 &= 8\,C_F + N_c\,(2+(3-\xi)\xi))\,, \\
\mathcal{C}_3 &= 4\,C_F\,(1+\xi) - N_c\,(4+(1-\xi)\xi)\,, \\
\mathcal{C}_4 &= 8\,C_F\,(2+\xi) - N_c\,(12+(4-3\xi)\xi)\,, \\
\mathcal{C}_5 &= -N_c\,(4-(2+\xi)\xi)\,, \\
\mathcal{C}_6 &= 4\,C_F - N_c\,(1-\xi)\,, \\
\mathcal{C}_7 &= 8\,C_F - N_c\,(16-\xi^2)\,.
\end{aligned} \tag{B.8}$$

In order to express the one-loop result (B.1) in terms of Spence functions, logarithms and rational functions of external momenta we have proceeded in two steps. First we have expanded all tensor integrals over the internal momentum into scalar three-point integrals times tensor functions of the external momenta [21]. Then we used recursion relations of Davydychev [22] to reduce these scalar three-point integrals into scalar two-point integrals and Θ.

Appendix C: ω-independence of d_f

We find d_f, the coefficient which tells us the fermionic shift in $\Lambda_{\rm lat}$, by calculating the massless quark vacuum polarization in a gluon with $a^2q^2 \ll 1$:

$$\begin{aligned}
&\Pi^{ab}_{\alpha\beta}(q;c_{SW},\omega) = \\
&\quad - \quad N_f \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[V^a_\alpha(q+k,k,q;c_{SW},\omega)S(k)V^b_\beta(k,q+k,-q;c_{SW},\omega)S(k+q)\right] \\
&\quad - \quad N_f \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[V^{\{a,b\}}_{\alpha\beta}(k,k,q,-q;c_{SW},\omega)S(k)\right]\,.
\end{aligned} \tag{C.1}$$

The quark propagator S and the vertices V are defined in Appendix A, the trace here is over both spin and color. The corresponding one-loop diagrams are shown in Fig. 4.

In the required limit of small a^2q^2 we can expand in q^2 and drop any terms $\mathcal{O}(a^2q^4)$.

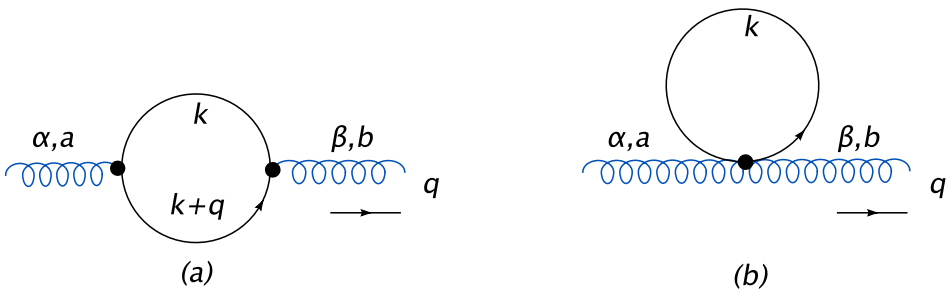

Figure 4: One-loop quark vacuum polarization diagrams.

We then get

$$\begin{aligned}
\Pi^{ab}_{\alpha\beta}(q;c_{SW},\omega) &= \Pi^{ab}_{\alpha\beta}(q;c_{SW},0) - 2\,\omega\,N_f\,\delta^{ab}g^2a^2\,\times \\
&\Bigg\{\sum_\mu \left(q_\alpha q_\mu - q^2\delta_{\alpha\mu}\right)\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[W_{1\mu}(k,k)S(k)W_{1\beta}(k,k)S(k)\right] \\
&+ a\left(q_\alpha q_\beta - q^2\delta_{\alpha\beta}\right)\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[W_{2\beta}(k,k)S(k)\right] \qquad \text{(C.2)} \\
&+ \sum_\mu \left(q_\beta q_\mu - q^2\delta_{\beta\mu}\right)\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[W_{1\alpha}(k,k)S(k)W_{1\mu}(k,k)S(k)\right] \\
&+ a\left(q_\alpha q_\beta - q^2\delta_{\alpha\beta}\right)\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[W_{2\alpha}(k,k)S(k)\right]\Bigg\} + \mathcal{O}(a^2q^4)
\end{aligned}$$

where $\Pi^{ab}_{\alpha\beta}(q;c_{SW},0)$ is the vacuum polarization tensor with no smearing, W_1 and W_2 are the Wilson quark gluon vertices defined in (A.4), and the trace is now only over the spin index. All ω^2 terms have dropped out because they first appear at $\mathcal{O}(a^2q^4)$. Calculating $\Pi^{ab}_{\alpha\beta}(q;c_{SW},0)$ in one loop for $c_{SW}=1$ leads to the value of d_f given in Eq. (74).

From power counting we would at first expect the integrals $\propto\omega$ in (C.2) to have values proportional to $1/a^2$ or $1/a^3$, and to make a finite contribution to d_f. However we show now that there is a perfect cancellation between the continuum-like diagram Fig. 4(a) (the integrals involving W_1) and the tadpole contribution Fig. 4(b) (those with W_2). To do this we use the identities

$$\frac{\partial}{\partial k_\mu}S(k) = -S(k)W_{1\mu}(k,k)S(k)\,, \qquad \text{(C.3)}$$

$$\frac{\partial}{\partial k_\mu}W_{1\nu}(k,k) = -a\,\delta_{\mu\nu}W_{2\mu}(k,k) \qquad \text{(C.4)}$$

which follow immediately from the definitions. Eq. (C.2) becomes

$$\begin{aligned}\Pi^{ab}_{\alpha\beta}(q;c_{SW},\omega) &= \Pi^{ab}_{\alpha\beta}(q;c_{SW},0) + 2\,\omega\,N_f\,\delta^{ab}g^2a^2\,\times \\ &\left\{\sum_\mu \left(q_\alpha q_\mu - q^2\delta_{\alpha\mu}\right)\int\frac{d^4k}{(2\pi)^4}\,\frac{\partial}{\partial k_\beta}\mathrm{Tr}\left[W_{1\mu}(k,k)S(k)\right]\right. \\ &\left.+\sum_\mu \left(q_\beta q_\mu - q^2\delta_{\beta\mu}\right)\int\frac{d^4k}{(2\pi)^4}\,\frac{\partial}{\partial k_\alpha}\mathrm{Tr}\left[W_{1\mu}(k,k)S(k)\right]\right\} + \mathcal{O}(a^2q^4)\,.\end{aligned} \tag{C.5}$$

The integrals are now zero because W_1 and S are periodic,

$$\int_{-\pi/a}^{\pi/a} dk_\alpha \frac{\partial}{\partial k_\alpha}\mathrm{Tr}\left[W_{1\mu}(k,k)S(k)\right] = \mathrm{Tr}\left[W_{1\mu}(k,k)S(k)\right]\Big|_{k_\alpha=-\pi/a}^{k_\alpha=\pi/a} = 0\,. \tag{C.6}$$

Thus we have proved that the vacuum polarization is independent of smearing the one-link part of the fermion action,

$$\Pi^{ab}_{\alpha\beta}(q;c_{SW},\omega) = \Pi^{ab}_{\alpha\beta}(q;c_{SW},0) + \mathcal{O}(a^2q^4) \tag{C.7}$$

which implies that d_f depends on r and c_{SW}, but not on ω.

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