Revised Version

DESY 08-098
TUM-HEP 712/09
SISSA 07/2009/EP

DESY 08-098
TUM-HEP 712/09
SISSA 07/2009/EP

Enhanced Symmetries of Orbifolds from Moduli Stabilization

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arXiv:0902.4512v3 [hep-th] 17 Jun 2009

Abstract

We study a supersymmetric field theory in six dimensions compactified on the orbifold $T^2/\mathbb{Z}_2$ with two Wilson lines. After supersymmetry breaking, the Casimir energy fixes the shape moduli at fixed points in field space where the symmetry of the torus lattice is enhanced. Localized Fayet-Iliopoulos terms stabilize the volume modulus at a size much smaller than the inverse supersymmetry breaking scale. All moduli masses are smaller than the gravitino mass.

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1 Introduction

Higher-dimensional theories provide a natural framework for extensions of the supersymmetric standard model which unify gauge interactions with gravity [1]. In recent years, phenomenologically attractive examples have been constructed in five and six dimensions compactified on orbifolds, and it has become clear how to embed such orbifold GUTs into the heterotic string [2].

An important problem in orbifold compactifications is the stabilization of moduli. In the following we study this question for an SO(10) model in six dimensions (6D) [3, 4] which, compared to models derived from the heterotic string [5, 6], has considerably simpler field content. The paper extends previous work which demonstrated that the compact dimensions can be stabilized at small radii, $R \sim 1/M_{\rm GUT}$, much smaller than the inverse supersymmetry breaking scale $1/\mu$ [7].

A crucial ingredient for the stabilization of compact dimensions is the Casimir energy of bulk fields [8]. Various aspects of the Casimir energy for 6D orbifolds have already been studied in [9–11]. Stabilization of the volume modulus can be achieved by means of massive bulk fields, brane localized kinetic terms or bulk and brane cosmological terms [9]. Alternatively, the interplay of one- and two-loop contributions to the Casimir energy can lead to a stabilization at the length scale of higher-dimensional couplings [12]. Furthermore, fluxes and gaugino condensates play an important role [13, 14]. The mechanism studied in this paper is based on expectation values $\mathcal{O}(M_{\rm GUT})$ of bulk fields, induced by local Fayet-Iliopoulos (FI) terms, and localized supersymmetry breaking leading to gaugino mediation [15, 16]. In the framework of moduli mediation this mechanism is discussed in [17].

For a rectangular torus it has been shown in [7] that the interplay of 'classical' and one-loop contributions to the vacuum energy density can stabilize the compact dimensions at $R \sim 1/M_{\rm GUT}$. Here we study the stabilization of all three shape and volume moduli of the torus. Remarkably, it turns out that the minimum occurs at a point with 'enhanced symmetry', where the torus lattice corresponds to the root lattice of SO(5). Tori defined by Lie lattices are the starting point for orbifold compactifications in string theory, which lead to large discrete symmetries [18]. These restrict Yukawa couplings and can forbid or strongly suppress the μ-term of the supersymmetric standard model [6, 19]. Enhanced discrete symmetries have previously been discussed in connection with string vacua [20].

The paper is organized as follows. In Section 2 we discuss symmetries of the compact space and the associated moduli fields, whereas the relevant features of the considered 6D orbifold GUT model are briefly described in Section 3. The Casimir energies of scalar fields with different boundary conditions are analyzed in Section 4. These results are the basis for the moduli stabilization discussed in Section 5. The Appendix deals with the evaluation of Casimir sums.

2 Modular Symmetries of Orbifolds

In this section we briefly discuss the geometry of the compact space and the associated three moduli fields. The torus T^2, and also the $T^2/\mathbb{Z}_2$ orbifold, can be parameterized by the volume parameter $\mathcal{A}$ and the complex shape parameter $\tau = \tau_1 + i\tau_2$. Following [9], we choose the following metric for $M^4 \times T^2$,

$$ds^2 = \mathcal{A}^{-1}\ g_{\mu\nu}dx^\mu dx^\nu + \mathcal{A}\ \gamma_{ij}dy^i dy^j\ , \tag{1}$$

where $y^i \in [0, L]$, and the metric γ_{ij} on the torus is given by

$$\gamma_{ij} = \frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}\ . \tag{2}$$

4D Minkowski space corresponds to $g_{\mu\nu} = \eta_{\mu\nu}$, and the induced metric at the orbifold fixed points is $\tilde{g}_{\mu\nu} = \mathcal{A}^{-1}\eta_{\mu\nu}$. The kinetic terms of the moduli fields are obtained by dimensional reduction from the 6D Einstein-Hilbert action,

$$\begin{aligned} S &= \frac{M_6^4}{2}\int d^6x\sqrt{G}R(G) \\ &= \frac{M_6^4 L^2}{2}\int d^4x\sqrt{g}\left(R(g) + \frac{g^{\mu\nu}\partial_\mu\mathcal{A}\partial_\nu\mathcal{A}}{\mathcal{A}^2} + \frac{g^{\mu\nu}\partial_\mu\tau\partial_\nu\tau^*}{2\tau_2^2}\right)\ . \end{aligned} \tag{3}$$

Here $\tau^* = \tau_1 - i\tau_2$, and the Ricci scalar R is evaluated with the metric indicated in parenthesis. Note, that $M_6^2 L$ is not the 4D Planck mass, since $g_{\mu\nu}$ does not determine the physical distance in the non-compact dimensions, cf. (1). Once the area modulus $\mathcal{A}$ is stabilized at $\mathcal{A}_0$, a constant Weyl rescaling $g_{\mu\nu} = \mathcal{A}_0\bar{g}_{\mu\nu}$ yields the physical Planck mass $M_4 = \sqrt{\mathcal{A}_0 L^2}M_6^2$, with $\mathcal{A}_0 L^2$ being the area of the torus.

In this paper we extend previous work [7], where only a rectangular torus lattice was considered. Note, that the torus can alternatively be described by the two radii $R_{1,2}$ of the torus lattice and the angle θ between them. The relation between the two sets of parameters is given by

$$2\pi R_1 = L\sqrt{\frac{\mathcal{A}}{\tau_2}}\ , \quad 2\pi R_2 = |\tau|L\sqrt{\frac{\mathcal{A}}{\tau_2}}\ , \quad \theta = \arccos\frac{\tau_1}{|\tau|}\ . \tag{4}$$

The rectangular torus in [7] has been parameterized in terms of the two radii $R_{1,2}$, corresponding to $\tau_1 = 0$ and $\tau_2 = R_2/R_1$.

The group $\mathrm{SL}(2,\mathbb{Z})$ of modular transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}\ , \quad a, b, c, d \in \mathbb{Z}\ , \quad ad - bc = 1\ , \tag{5}$$

relates modular parameters of diffeomorphic tori. Distinct tori have modular parameters τ taking values in the fundamental region $|\tau| \geq 1$, $-1/2 \leq \tau_1 \leq 1/2$ and $\tau_2 > 0$ (cf. Figure 1).

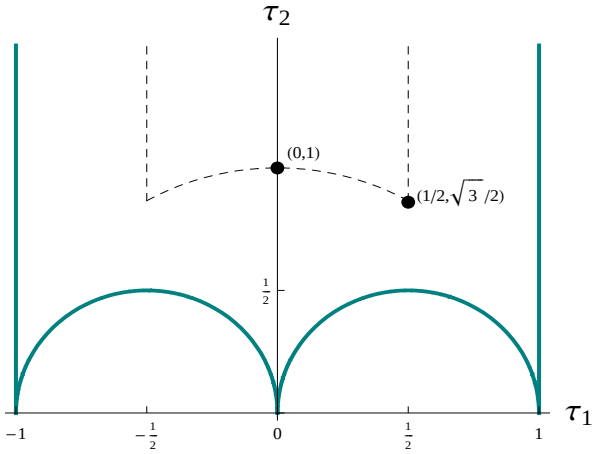

Figure 1: Fundamental domain for the modular groups $SL(2,\mathbb{Z})$ and $\Gamma(2)$.

The Kaluza-Klein mode expansion of bulk fields on the torus can be written as

$$\Phi(x,y) = \frac{1}{\sqrt{\mathcal{A}}L} \sum_{m,n=-\infty}^{\infty} \phi_{m,n}(x) \exp\left\{ \frac{2\pi i}{L\sqrt{\mathcal{A}\tau_2}} \left[m\left(\tau_2 y_1 - \tau_1 y_2\right) + n y_2 \right] \right\} \ , \tag{6}$$

with the corresponding Kaluza-Klein (KK) masses

$$\mathcal{M}_{m,n}^2 = \frac{(2\pi)^2}{\mathcal{A}L^2\tau_2} |m\tau - n|^2 \ . \tag{7}$$

Note that the sum over all KK modes is modular invariant: the transformations associated with the two SL$(2,\mathbb{Z})$ generators, $\tau \to \tau + 1$ and $\tau \to -1/\tau$, correspond to the relabeling of terms $(m,n) \to (m, n-m)$ and $(m,n) \to (-n, m)$, respectively.

In the case of non-zero Wilson lines the KK masses take the values [11]

$$\mathcal{M}_{m,n}^2 = \frac{(2\pi)^2}{\mathcal{A}L^2\tau_2} |n + \beta - \tau(m+\alpha)|^2 \ , \tag{8}$$

where (α,β) are real numbers. For a $T^2/\mathbb{Z}_2$ orbifold, α and β are restricted, $\alpha,\beta \in \{0, 1/2\}$. The modular transformation (5), with $\tau_2 \to \tau_2/(|c\tau + d|^2)$, now corresponds to the relabeling of KK modes

$$m + \alpha \to a(m+\alpha) - c(n+\beta) \ , \quad n + \beta \to d(n+\beta) - b(m+\alpha) \ . \tag{9}$$

Depending on the values of the discrete Wilson lines, the sum over KK modes is invariant under the full modular group SL$(2,\mathbb{Z})$ or some subgroup [21]. For $\alpha = \beta = 0$, the Wilson lines are zero and SL$(2,\mathbb{Z})$ remains unbroken. In the case $\alpha = 0$ and $\beta = 1/2$, modular invariance yields the additional restriction $c = 0 \mod 2$ and $d = 1 \mod 2$. Correspondingly, for $\alpha = 1/2$ and $\beta = 0$ one finds the restriction $a = 1 \mod 2$ and

$b = 0 \mod 2$, while for $\alpha = \beta = 1/2$ one has $a, d = 1 \mod 2$, $b, c = 0 \mod 2$ or $a, d = 0 \mod 2$ and $b, c = 1 \mod 2$. The largest common subgroup corresponds to $a, d = 1 \mod 2$ and $b, c = 0 \mod 2$, which corresponds to $\Gamma(2)$ [22]. The fundamental domain of the groups $\Gamma(2)$ and $\mathrm{SL}(2, \mathbb{Z})$ are compared in Figure 1.

We are interested in fixed points of the modular group in the upper half plane, because the effective potential $V(\tau_1, \tau_2)$ has extrema at these fixed points. To this end notice that a matrix $M \in \mathrm{SL}(2, \mathbb{Z}), M \neq \pm\mathbb{1}$, has a fixed point in the upper half plane if and only if $\mathrm{Tr}\, M < 2$. This can be seen from the fixed point equation $Mz = z$ which implies

$$cz^2 + (d - a)z - b = 0 \ . \tag{10}$$

Using the property $ad - bc = 1$, one obtains for the solutions of this equation

$$z = \frac{a - d \pm \sqrt{(a + d)^2 - 4}}{2c} \ . \tag{11}$$

We see that only for $(a + d)^2 < 4$ we have complex solutions in the upper half plane, whereas for $(a + d)^2 \geq 4$ there are only real solutions. Clearly, only points on the edge of the fundamental domain can be fixed points, because points within the fundamental domain are inequivalent and therefore cannot be mapped onto each other by a modular transformation.

It is well known that $\mathrm{SL}(2, \mathbb{Z})$ has two fixed points at $(\tau_1, \tau_2) = (0, 1)$ and $(\tau_1, \tau_2) = (1/2, \sqrt{3}/2)$, respectively. For the case $c = 0 \mod 2$ and $d = 1 \mod 2$ there is a fixed point at $(\tau_1, \tau_2) = (1/2, 1/2)$, while for $a = 1 \mod 2$ and $b = 0 \mod 2$ there is a fixed point at $(\tau_1, \tau_2) = (1, 1)$. Finally, in the case $a, d = 0 \mod 2$ and $b, c = 1 \mod 2$ there is a fixed point at $(0, 1)$. The subgroup $\Gamma(2)$ has no fixed points in the upper half plane.

3 An Orbifold GUT Model

As an example, we consider a 6D $\mathcal{N} = 1$ SO(10) gauge theory compactified on an orbifold $T^2/\mathbb{Z}_2^3$, corresponding to $T^2/\mathbb{Z}_2$ with two Wilson lines [3]. The model has four inequivalent fixed points ('branes') with the unbroken gauge groups SO(10), the Pati-Salam group $\mathrm{G_{PS}} = \mathrm{SU}(4) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$, the extended Georgi-Glashow group $\mathrm{G_{GG}} = \mathrm{SU}(5) \times \mathrm{U}(1)_X$ and flipped SU(5), $\mathrm{G_{fl}} = \mathrm{SU}(5)' \times \mathrm{U}(1)'$, respectively. The intersection of these GUT groups yields the standard model group with an additional U(1) factor, $\mathrm{G'_{SM}} = \mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \times \mathrm{U}(1)_X$, as unbroken gauge symmetry below the compactification scale.

The model has three **16**-plets of matter fields, localized at the Pati-Salam, the Georgi-Glashow, and the flipped SU(5) branes. Further, there are two **16**-plets, ϕ and ϕ^c, and two **10**-plets, H_5 and H_6, of bulk matter fields. Their mixing with the brane fields yields the characteristic flavor structure of the model [3].

The Higgs sector consists of two **16**-plets, Φ and Φ^c, and four **10**-plets, $H_1, \ldots, H_4$, of bulk hypermultiplets. The hypermultiplets H_1 and H_2 contain the two Higgs doublets of the supersymmetric standard model as zero modes, whereas the zero modes of H_3 and H_4 are color triplets. The zero modes of the **16**-plets are singlets and color triplets,

$$\Phi: \;\; N^c, \; D^c \; ; \qquad \Phi^c: \;\; N, \; D \; . \tag{12}$$

The color triplets D^c and D, together with the zero modes of H_3 and H_4, acquire masses through brane couplings.

Equal vacuum expectation values of Φ and Φ^c form a flat direction of the classical potential,

$$\langle \Phi \rangle = \langle N^c \rangle = \langle N \rangle = \langle \Phi^c \rangle \; . \tag{13}$$

Non-zero expectation values can be enforced by a brane superpotential term or by an FI-term localized at the GG-brane where the U(1) factor commutes with the standard model gauge group.

The expectation values (13) break SO(10) to SU(5), and therefore also the additional $\mathrm{U}(1)_X$ symmetry, leading to bulk masses[1]

$$\mathcal{M}^2 \simeq g_6^2 \langle \Phi^c \rangle^2 \; , \tag{14}$$

where g_6 is the 6D gauge coupling, which is related to the 4D gauge coupling by a volume factor, $g_4 = g_6/\sqrt{\mathcal{A}L^2}$.

Supersymmetry breaking is naturally incorporated via gaugino mediation [4]. The non-vanishing F-term of a brane field S generates mass terms for vector- and hypermultiplets. In the considered model, S is localized at the SO(10) preserving brane, which yields the same mass for all members of an SO(10) multiplet. For the **45** vector multiplet and the **10** and **16** hypermultiplets of the Higgs sector one has

$$\begin{aligned} \Delta S \;\; = \;\; & \int \mathrm{d}^4x \mathrm{d}^2y \; \sqrt{\widetilde{g}} \; \delta^2(y) \left\{ \int \mathrm{d}^2\theta \frac{h}{2\Lambda^3} S \mathrm{Tr}[W^\alpha W_\alpha] + \mathrm{h.c.} \right. \\ & + \int \mathrm{d}^4\theta \left(\frac{\lambda}{\Lambda^4} S^\dagger S \left(H_1^\dagger H_1 + H_2^\dagger H_2 \right) + \frac{\lambda'}{\Lambda^4} S^\dagger S \left(H_3^\dagger H_3 + H_4^\dagger H_4 \right) \right. \\ & \left. \left. + \frac{\lambda''}{\Lambda^4} S^\dagger S \left(\Phi^\dagger \Phi + \Phi^{c\dagger} \Phi^c \right) \right) \right\} \; . \end{aligned} \tag{15}$$

Here $\widetilde{g}_{\mu\nu}$ is the metric induced at the fixed point, and $W^\alpha(V)$, $H_1, \ldots, H_4$, Φ, Φ^c are the 4D $\mathcal{N} = 1$ multiplets contained in the 6D $\mathcal{N} = 1$ multiplets, which have positive parity at $y = 0$; Λ is the UV cutoff of the model, which is much larger than the inverse size

[1] For more details concerning the parity assignments and gauge symmetry breaking, see [7].

of the compact dimensions. For the zero modes, the corresponding gaugino and scalar masses are given by

$$m_g = \frac{h\mu}{\mathcal{A}L^2\Lambda^2}\,, \quad m^2_{H_{1,2}} = -\frac{\lambda\mu^2}{\mathcal{A}L^2\Lambda^2}\,, \quad m^2_{H_{3,4}} = -\frac{\lambda'\mu^2}{\mathcal{A}L^2\Lambda^2}\,, \quad m^2_{\Phi} = -\frac{\lambda''\mu^2}{\mathcal{A}L^2\Lambda^2}\,, \tag{16}$$

where $\mathcal{A}L^2$ is the volume of the compact dimensions and $\mu = F_S/\Lambda$. Note that the gaugino mass is stronger volume suppressed than the scalar masses. This implies that the contribution of the vector multiplet to the Casimir energy relative to the one of the hypermultiplets is also suppressed, as shown in the appendix.

4 Casimir Energy on $T^2/\mathbb{Z}_2^3$

The Casimir energy of a real scalar field on the given orbifold background can be written as

$$V_M = \frac{1}{2}\left[\sum\right]_{m,n}\int\frac{\mathrm{d}^4k_E}{(2\pi)^4}\log\left(k_E^2 + \frac{\mathcal{M}^2_{m,n}}{\mathcal{A}} + \frac{M^2}{\mathcal{A}}\right)\,, \tag{17}$$

with $[\sum]_{m,n}$ shorthand for the double sum and $\mathcal{M}^2_{m,n}$ denoting the Kaluza-Klein masses, which are given by (8) except for a factor of four due to the two additional $\mathbb{Z}_2$ symmetries, which have been modded out. The mass M stands for bulk and brane mass terms.

The expression (17) for the Casimir energy is divergent. Following [9], we extract a finite piece using zeta function regularization,

$$V = -\left.\frac{\mathrm{d}\zeta(s)}{\mathrm{d}s}\right|_{s=0}\,, \tag{18}$$

where

$$\zeta(s) = \frac{1}{2}\left[\sum\right]_{m,n}\mu_r^{2s}\int\frac{\mathrm{d}^4k_E}{(2\pi)^4}\left(k_E^2 + \frac{4(2\pi)^2}{\mathcal{A}^2L^2\tau_2}|n+\beta-\tau(m+\alpha)|^2 + \frac{M^2}{\mathcal{A}}\right)^{-s}\,. \tag{19}$$

Note that, as in dimensional regularization, a mass scale μ_r is introduced. The momentum integration can be performed, which yields

$$\begin{aligned}\zeta(s) &= \frac{1}{2}\frac{1}{(2\pi)^4}\pi^2\frac{\Gamma(s-2)}{\Gamma(s)}\left[\sum\right]_{m,n}\mu_r^{2s}\left(\frac{4(2\pi)^2}{\mathcal{A}^2L^2\tau_2}|n+\beta-\tau(m+\alpha)|^2 + \frac{M^2}{\mathcal{A}}\right)^{2-s}\\ &= \frac{\mu_r^{2s}4^{2-s}(2\pi)^{-2s}\pi^2}{2\mathcal{A}^{4-2s}L^{4-2s}\tau_2^{2-s}(s-2)(s-1)}\\ &\quad\left[\sum\right]_{m,n}\left[(n+\beta-(m+\alpha)\tau_1)^2 + (m+\alpha)^2\tau_2^2 + \frac{\mathcal{A}L^2\tau_2}{4(2\pi)^2}M^2\right]^{2-s}\,.\end{aligned} \tag{20}$$

Carrying out the summations (cf. Appendix) we find for the Casimir energy

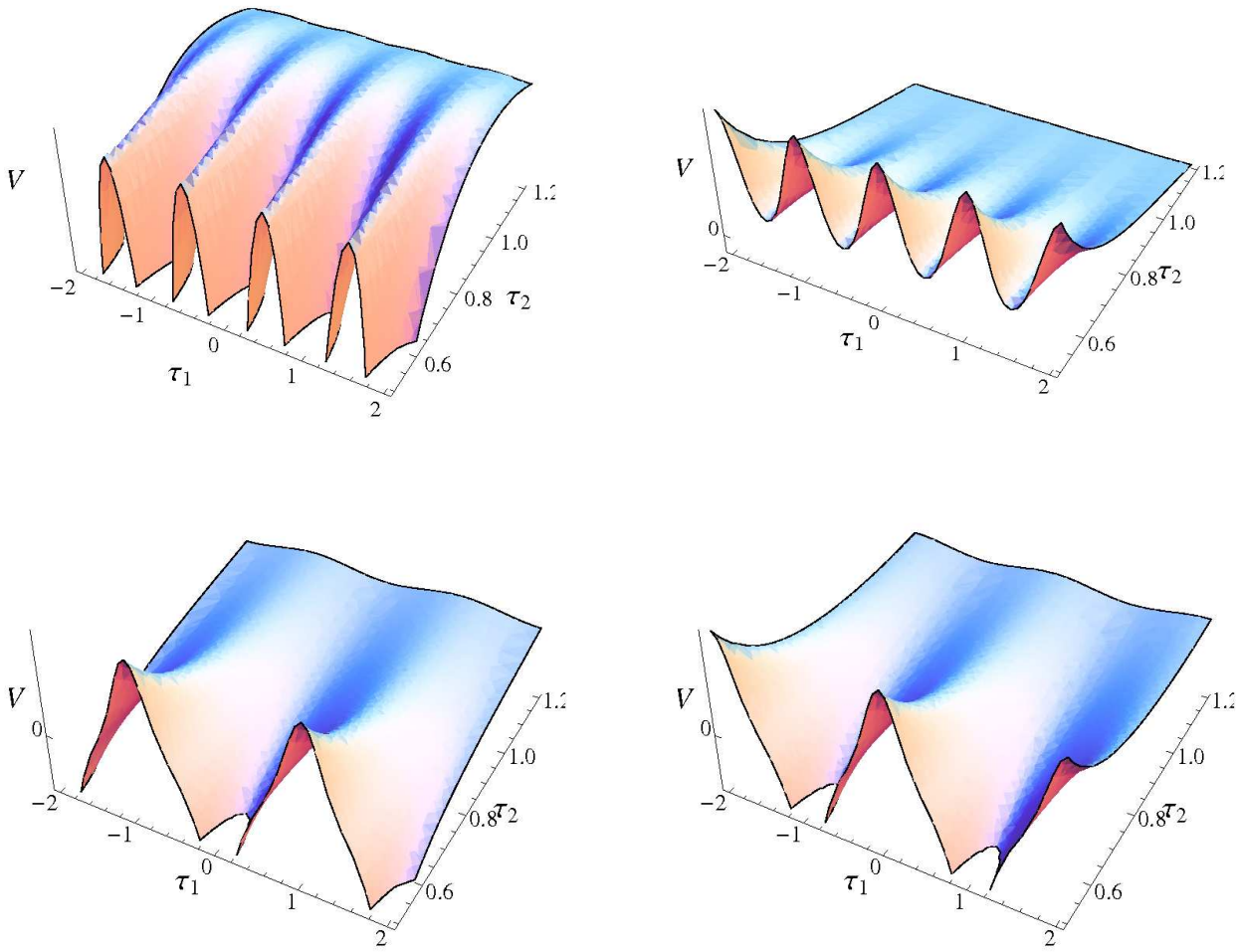

Figure 2: The different contributions to the Casimir energy for the different boundary conditions. Note that the potential is periodic in τ_1 with period 1 for $\alpha = 0$ and period 2 for $\alpha = 1/2$.

$$
\begin{aligned}
V_M^{\alpha,\beta} =& \frac{M^6 L^2}{3072\pi^3 \mathcal{A}} \left[\frac{11}{12} - \log\left(\frac{M}{\sqrt{\mathcal{A}}\mu_r}\right)\right] \\
&- \frac{M^4}{64\pi^2\mathcal{A}^2}\left[\frac{3}{4} - \log\left(\frac{M}{\sqrt{\mathcal{A}}\mu_r}\right)\right]\delta_{\alpha 0}\delta_{\beta 0} \\
&- \frac{M^3 \tau_2^{3/2}}{4\pi^3 \mathcal{A}^{5/2} L} \sum_{p=1}^{\infty} \frac{\cos(2\pi p\alpha)}{p^3} K_3\left(p\frac{\sqrt{\mathcal{A}}LM}{2\sqrt{\tau_2}}\right) \\
&- \frac{32}{\mathcal{A}^4 L^4 \tau_2^2} \sum_{p=1}^{\infty}\sum_{m=0}^{\infty} \frac{1}{2^{\delta_{\alpha 0}\delta_{m0}}} \frac{\cos(2\pi p(\beta - (m+\alpha)\tau_1))}{p^{5/2}} \\
&\times \left(\tau_2^2(m+\alpha)^2 + \frac{\mathcal{A}L^2\tau_2 M^2}{(4\pi)^2}\right)^{\frac{5}{4}} K_{5/2}\left(2\pi\, p\sqrt{\tau_2^2(m+\alpha)^2 + \frac{\mathcal{A}L^2\tau_2 M^2}{(4\pi)^2}}\right). \quad (21)
\end{aligned}
$$

The different contributions to the Casimir energy are displayed in Figure 2 as function of the shape moduli τ_1 and τ_2 for fixed volume modulus $\mathcal{A}$.

In supersymmetric theories there is a cancellation between bosonic and fermionic contributions, and the expression for the Casimir energy is given by

$$
\begin{aligned}
V &= A\left(V_{M'}^{0,0} - V_M^{0,0}\right) + B\left(V_{M'}^{0,1/2} - V_M^{0,1/2}\right) \\
&\quad + C\left(V_{M'}^{1/2,0} - V_M^{1/2,0}\right) + D\left(V_{M'}^{1/2,1/2} - V_M^{1/2,1/2}\right), \quad (22)
\end{aligned}
$$

where $M' = \sqrt{M^2 + m^2}$, with supersymmetric mass M and supersymmetry breaking mass m; the coefficients A,B,C,D depend on the field content of the model. Note that even in the supersymmetric framework there are divergent bulk and brane terms, which are proportional to the supersymmetry breaking mass m^2, unlike the case in Scherk-Schwarz breaking. These divergencies have to be subtracted from the unrenormalized Casimir energy to obtain a finite result, and to tune the four-dimensional cosmological constant to zero.

5 Stabilization

5.1 Shape Moduli

Before discussing moduli stabilization for our particular orbifold GUT model, it is instructive to consider the shape moduli potential for varying field content, i.e., for different coefficients A,B,C,D. The modular symmetries (5) of the four different contributions are given in Table 1. They are obtained by requiring invariance of the Kaluza-Klein sums under the corresponding modular transformation, as discussed in Section 2. Naively, one would expect that adding two different contributions with different symmetries would lead to the largest common subgroup, which is given by $\Gamma(2)$. However, for certain relations between the coefficients A,B,C,D there can be non-trivial cancellations, which

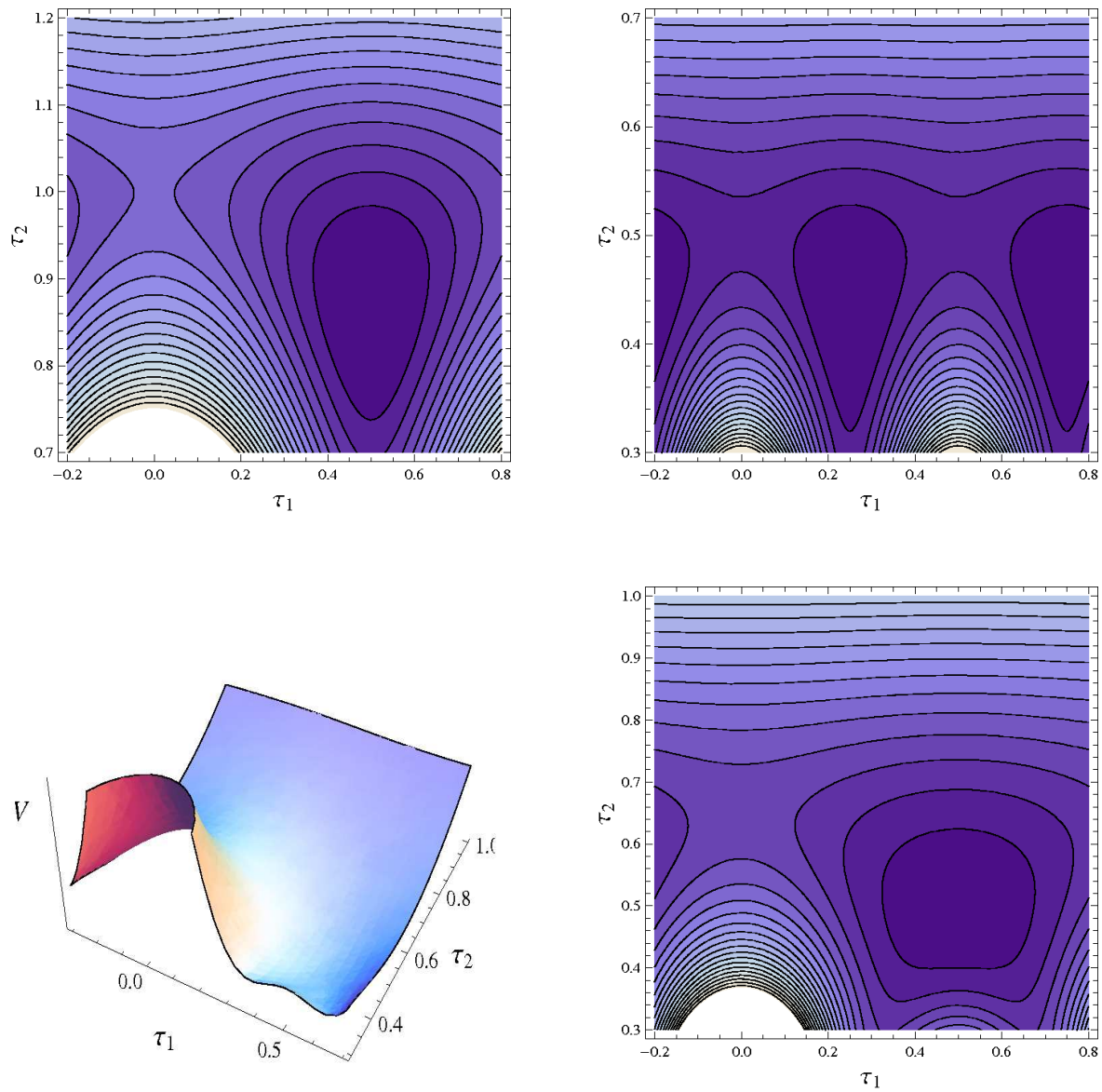

Figure 3: Effective potential for the shape moduli τ_1 and τ_2. In the upper left (right) panel we plot the potential for $V_M^{0,0}$ ($V_M^{0,0}+V_M^{0,1/2}$). In the lower panels we show the potential for the shape moduli in the given model. Note that the scaling in the τ_2 direction is different. The different periodicities of the potential in the τ_1 direction correspond to different values of the parameter b in the modular transformations. For the given model there is a local minimum of the full potential at $(\tau_1, \tau_2) = (1/2, 1/2)$ and a saddle point at $(\tau_1, \tau_2) = (0, 1/\sqrt{2})$.

	a	b	c	d	Fixed Points (τ_1, τ_2)
$V^{0,0}$	0 mod 1	0 mod 1	0 mod 1	0 mod 1	$(0,1)$, $(1/2, \sqrt{3}/2)$
$V^{0,1/2}$	0 mod 1	0 mod 1	0 mod 2	1 mod 2	$(1/2, 1/2)$
$V^{1/2,0}$	1 mod 2	0 mod 2	0 mod 1	0 mod 1	$(1,1)$
$V^{1/2,1/2}$	1 mod 2	0 mod 2	0 mod 2	1 mod 2	
	0 mod 2	1 mod 2	1 mod 2	0 mod 2	$(0,1)$
$\Gamma(2)$	1 mod 2	0 mod 2	0 mod 2	1 mod 2	–
$V^{0,0}+V^{0,1/2}$	0 mod 1	0 mod 1/2	0 mod 2	0 mod 1	$(0,1/2)$, $(1/4, \sqrt{3}/4)$
V_{casimir}	0 mod 1	0 mod 1	0 mod 2	1 mod 2	$(1/2, 1/2)$

Table 1: Modular symmetries (cf. (5)) of different contributions to the Casimir energy and the fixed points under those symmetries. For general coefficients A, B, C, D in (22), the symmetry corresponds to the largest common subgroup, which is known as $\Gamma(2)$. However, if the coefficients fulfill certain relations, the modular symmetry can be enhanced as shown in the last two lines.

lead to a larger modular symmetry. For example, if the field content is such that $A = B$ and $D = C = 0$, the parameters of the modular group are restricted to $b = 0$ mod $1/2$ and $c = 0$ mod 2. Surprisingly, the resulting symmetry is not only larger than the symmetry of $V^{0,1/2}$, it is not even a subgroup of $\mathrm{SL}(2, \mathbb{Z})$.

Fixed points under the modular symmetry are extrema of the effective potential, assuming that the volume is stabilized. Hence, minima of the effective potential may correspond to such fixed points. For fields with boundary condition $(+,+)$, this is indeed the case. The Casimir energy then has a minimum at $(\tau_1, \tau_2) = (1/2, \sqrt{3}/2)$ and a saddle point at $(\tau_1, \tau_2) = (0, 1)$ [9]. This implies that the shape moduli are stabilized at a torus lattice with $R_1 = R_2$ and $\theta = \pi/3$, which corresponds to the root lattice of the Lie algebras SU(3) or G_2. For our example with $A = B$ and $D = C = 0$ on the other hand, there is a minimum at $(\tau_1, \tau_2) = (1/4, \sqrt{3}/4)$ and a saddle point at $(\tau_1, \tau_2) = (0, 1/2)$. The minimum corresponds to the lattice with $R_1 = 2R_2$ and $\theta = \pi/3$.

Let us now turn to our model. The wanted repulsive behavior of the Casimir energy at small volume can be obtained if the contribution of particular bulk hypermultiplets dominates [7],

$$\begin{aligned} V_{\rm casimir} &= 12\left(V_{m_H}^{0,0}-V^{0,0}\right)+12\left(V_{m_H}^{0,1/2}-V^{0,1/2}\right) \\ &\quad +8\left(V_{m_H}^{1/2,0}-V^{1/2,0}\right)+8\left(V_{m_H}^{1/2,1/2}-V^{1/2,1/2}\right)\ , \end{aligned} \tag{23}$$

with $m_H^2=-\lambda'\mu^2/(\mathcal{A}L^2\Lambda^2)$ and $\lambda'<0$, $|\lambda'|>|\lambda|, |\lambda''|$ (cf. (16)). Remarkably, the potential has an enhanced modular symmetry compared to $\Gamma(2)$. The allowed transformations have $c=0 \mod 2$ and $d=1 \mod 2$, with a and $b\in\mathbb{Z}$.

Solving the fixed point equation (11), one finds a fixed point in the upper half-plane: $(\tau_1,\tau_2)=(1/2,1/2)$ with $a=-b=-d=1$ and $c=2$. It corresponds to a minimum in the effective potential. There is also a saddle point at $(\tau_1,\tau_2)=(0,1/\sqrt{2})$. For the minimum, the torus lattice again has an enhanced symmetry: $R_1=\sqrt{2}R_2$ and $\theta=\pi/4$, which corresponds to the root lattice of SO(5). Its discrete symmetry is $\mathbb{Z}_4$.

5.2 Volume Modulus

In [7] it has been shown that spontaneous gauge symmetry breaking by bulk Higgs fields together with supersymmetry breaking can stabilize the compact dimensions at the GUT scale. The detailed mechanism of supersymmetry breaking is discussed in [17]. Consider the breaking of $U(1)_X$ as discussed in Section 3. In orbifold compactifications of the heterotic string a vacuum expectation value $\langle\Phi^c\rangle$ can be induced by localized Fayet-Iliopoulos (FI) terms. Vanishing of the D-terms then implies

$$\langle\Phi^c\rangle^2=\frac{C\Lambda^2}{\mathcal{A}L^2}\ , \tag{24}$$

where $C\ll 1$ is a loop factor and Λ is the string scale or, more generally, the UV cutoff of the model. The expectation value is volume suppressed because Φ^c is a bulk field and the FI-terms are localized at fixed points. In terms of the bulk Higgs mass (14) one obtains, using $g_6^2/(\mathcal{A}L^2)=g_4^2\simeq 1/2$,

$$\mathcal{M}^2\simeq g_6^2\langle\Phi^c\rangle^2\simeq\frac{1}{2}C\Lambda^2\ . \tag{25}$$

For orbifold compactifications of the heterotic string one finds $\mathcal{M}\sim M_{\rm GUT}$.

Supersymmetry breaking by a brane field S, with $\mu=F_S/\Lambda$, leads to a 'classical' contribution to the vacuum energy density,

$$\begin{aligned} V_{\rm cl} &= \int {\rm d}^2y\int {\rm d}^4\theta\sqrt{\widetilde{g}}\,\delta^2(y)\left\langle S^\dagger S\left(1-\frac{\lambda''}{\Lambda^4}(\Phi^\dagger\Phi+\Phi^{c\dagger}\Phi^c)\right)\right\rangle \\ &= \frac{F_S^2}{\mathcal{A}^2}-\frac{2\lambda''\mu^2\mathcal{M}^2}{\mathcal{A}^3L^2\Lambda^2}+\dots\ , \end{aligned} \tag{26}$$

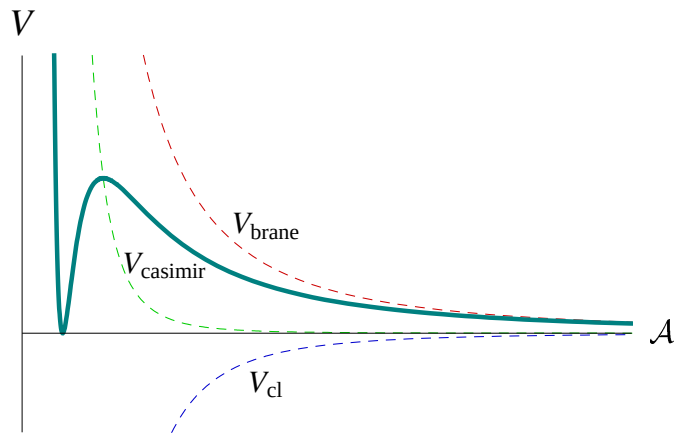

Figure 4: Effective potential for the volume modulus (full line). The different contributions to the potential are also shown separately; $V_{\rm brane}$ represents the brane counterterm.

where the first term is a tree level potential [17], and $\tilde{g}_{\mu\nu} = \mathcal{A}^{-1}\eta_{\mu\nu}$ is the induced metric at the fixed point $y = 0$. The first term proportional to F_S^2 will be absorbed into the brane tension.

In the vicinity of $\tau_1 = 1/2$, the Casimir energy is to a good approximation given by

$$\begin{aligned} V_M^{\alpha,\beta} = &+ \frac{M^6 L^2}{3072\pi^3\mathcal{A}}\left[\frac{11}{12} - \log\left(\frac{M}{\sqrt{\mathcal{A}}\mu_r}\right)\right] \\ &- \frac{M^4}{64\pi^2\mathcal{A}^2}\left[\frac{3}{4} - \log\left(\frac{M}{\sqrt{\mathcal{A}}\mu_r}\right)\right]\delta_{\alpha 0}\delta_{\beta 0} \\ &- \frac{M^3\tau_2^{3/2}}{4\pi^3\mathcal{A}^{5/2}L}\sum_{p=1}^{\infty}\frac{\cos(2\pi p\alpha)}{p^3}K_3\left(p\tfrac{\sqrt{\mathcal{A}}LM}{2\sqrt{\tau_2}}\right) \\ &- \frac{M^3\tau_2^{-3/2}}{32\pi^3\mathcal{A}^{5/2}L}\sum_{p=1}^{\infty}\frac{\cos(2\pi p\alpha)}{p^3}K_3\left(p\sqrt{\mathcal{A}\tau_2}LM\right) \\ &+ \frac{M^4}{64\pi^3\tau_2^3\mathcal{A}^2}\sum_{p=1}^{\infty}\frac{\cos(2\pi p\alpha)}{p^2}K_4\left(p\sqrt{\mathcal{A}\tau_2}LM\right)\left(\tau_1 - \tfrac{1}{2}\right)^2 \ , \end{aligned} \tag{27}$$

where we have performed a Taylor expansion around $\tau_1 = 1/2$. This approximation is valid for small τ_2, and we have dropped additional terms in $V_M^{0,\beta}$, which cancel in the sum of $V_M^{0,0}$ and $V_M^{0,1/2}$.

Expanding the Bessel functions for small arguments and performing the summations over p, we obtain for the four different contributions

$$\begin{aligned} V_M^{0,0}(\mathcal{A},\tau_1,\tau_2) = &- \frac{16\pi^3\tau_2^3}{945\mathcal{A}^4L^4} - \frac{\pi^3}{3780\mathcal{A}^4L^4\tau_2^3} + \frac{\pi^3(\tau_1-1/2)^2}{1260\mathcal{A}^4L^4\tau_2^5} \\ &+ \frac{\pi M^2\tau_2^2}{180\mathcal{A}^3L^2} + \frac{\pi M^2}{2880\mathcal{A}^3L^2\tau_2^2} - \frac{\pi M^2(\tau_1-1/2)^2}{5760\mathcal{A}^3L^2\tau_2^2} \ , \end{aligned} \tag{28}$$

$$V_M^{0,1/2}(\mathcal{A},\tau_1,\tau_2) = -\frac{16\pi^3\tau_2^3}{945\mathcal{A}^4L^4} - \frac{\pi^3}{3780\mathcal{A}^4L^4\tau_2^3} + \frac{\pi^3(\tau_1-1/2)^2}{1260\mathcal{A}^4L^4\tau_2^5} + \frac{\pi M^2\tau_2^2}{180\mathcal{A}^3L^2} + \frac{\pi M^2}{2880\mathcal{A}^3L^2\tau_2^2} - \frac{\pi M^2(\tau_1-1/2)^2}{5760\mathcal{A}^3L^2\tau_2^2} \ , \tag{29}$$

$$V_M^{1/2,0}(\mathcal{A},\tau_1,\tau_2) = +\frac{31\pi^3\tau_2^3}{1890\mathcal{A}^4L^4} + \frac{31\pi^3}{120960\mathcal{A}^4L^4\tau_2^3} - \frac{31\pi^3(\tau_1-1/2)^2}{40320\mathcal{A}^4L^4\tau_2^5} - \frac{7\pi M^2\tau_2^2}{1440\mathcal{A}^3L^2} - \frac{7\pi M^2}{23040\mathcal{A}^3L^2\tau_2^2} + \frac{7\pi M^2(\tau_1-1/2)^2}{46080\mathcal{A}^3L^2\tau_2^2} \ , \tag{30}$$

$$V_M^{1/2,1/2}(\mathcal{A},\tau_1,\tau_2) = +\frac{31\pi^3\tau_2^3}{1890\mathcal{A}^4L^4} + \frac{31\pi^3}{120960\mathcal{A}^4L^4\tau_2^3} - \frac{31\pi^3(\tau_1-1/2)^2}{40320\mathcal{A}^4L^4\tau_2^5} - \frac{7\pi M^2\tau_2^2}{1440\mathcal{A}^3L^2} - \frac{7\pi M^2}{23040\mathcal{A}^3L^2\tau_2^2} + \frac{7\pi M^2(\tau_1-1/2)^2}{46080\mathcal{A}^3L^2\tau_2^2} \ . \tag{31}$$

The total effective potential is now given by the sum of the Casimir energy (23), the classical energy density (26) and a brane tension,

$$V_{\rm tot}(\mathcal{A},\tau_1,\tau_2) = V_{\rm casimir}(\mathcal{A},\tau_1,\tau_2) + V_{\rm cl}(\mathcal{A}) + V_{\rm brane}(\mathcal{A}) \ . \tag{32}$$

Inserting the expansions (28)-(31) into the expression for the Casimir energy, one finally obtains

$$V_{\rm tot}(\mathcal{A},\tau_1,\tau_2) = -\frac{\pi\lambda'\mu^2}{288\mathcal{A}^4L^4\Lambda^2}\left(16\tau_2^2+\tau_2^{-2}-\frac{2(\tau_1-1/2)^2}{\tau_2^4}\right) - \frac{2\lambda''\mu^2\mathcal{M}^2}{\mathcal{A}^3L^2\Lambda^2} + \frac{\kappa}{\mathcal{A}^2} \ , \tag{33}$$

where

$$\kappa = -\frac{36\lambda''^2\mu^2\mathcal{M}^4}{\pi\lambda'\Lambda^2} > 0 \ . \tag{34}$$

The brane tension κ has been adjusted such that the potential $V_{\rm tot}$ vanishes at the local minimum. The different contributions to the effective potential are shown in Figure 4.

As discussed in the previous section, the Casimir energy, and therefore $V_{\rm tot}$, has a local minimum at $\tau_1 = \tau_2 = 1/2$. The volume modulus is then fixed at

$$\mathcal{A}_0 L^2 = -\frac{\pi\lambda'}{36\lambda''}\frac{1}{\mathcal{M}^2} \ . \tag{35}$$

For $|\lambda'| > \lambda''$, as required by a repulsive Casimir energy at small volume, one then obtains stabilization of the compact dimensions at the inverse GUT scale, $\sqrt{\mathcal{A}L^2} \sim 1/\mathcal{M} \sim 1/M_{\rm GUT}$.

5.3 Moduli Masses

The moduli fields $\mathcal{A}, \tau_1$ and τ_2 have masses much smaller than the inverse size of the compact dimensions. Their Lagrangian is obtained by dimensional reduction (cf. [9])

and from the effective potential (32),

$$\mathcal{L} = \sqrt{g}\left\{\frac{M_6^4 L^2}{2}\left(R(g) + \frac{g^{\mu\nu}\partial_\mu \mathcal{A}\partial_\nu \mathcal{A}}{\mathcal{A}^2} + \frac{g^{\mu\nu}\partial_\mu \tau \partial_\nu \tau^*}{2\tau_2^2}\right) - V_{\rm tot}(\mathcal{A},\tau_1,\tau_2)\right\} \ . \tag{36}$$

After a constant Weyl rescaling, $g_{\mu\nu} = \mathcal{A}_0 \bar{g}_{\mu\nu}$ (cf. (35)), the Lagrangian for the moduli depends on $\mathcal{A}_0$ and the 4D Planck mass $M_4 = \sqrt{\mathcal{A}_0 L^2} M_6^2$,

$$\mathcal{L}_{\mathcal{M}} = \sqrt{\bar{g}}\left\{\frac{M_4^2}{2}\left(\frac{\bar{g}^{\mu\nu}\partial_\mu \mathcal{A}\partial_\nu \mathcal{A}}{\mathcal{A}^2} + \frac{\bar{g}^{\mu\nu}\partial_\mu \tau \partial_\nu \tau^*}{2\tau_2^2}\right) - \mathcal{A}_0^2\ V_{\rm tot}(\mathcal{A},\tau_1,\tau_2)\right\} \ . \tag{37}$$

Expanding the moduli fields around the minimum,

$$\mathcal{A} = \mathcal{A}_0 + \frac{\mathcal{A}_0}{M_4}\ \bar{\mathcal{A}}\ , \quad \tau_{1,2} = \frac{1}{2} + \frac{1}{\sqrt{2}M_4}\ \bar{\tau}_{1,2}\ , \tag{38}$$

yields the Lagrangian for the canonically normalized fluctuations,

$$\begin{aligned}\mathcal{L}_{\mathcal{M}} \ = \ & \sqrt{\bar{g}}\Bigg\{\frac{1}{2}\left(\bar{g}^{\mu\nu}\partial_\mu \bar{\mathcal{A}}\partial_\nu \bar{\mathcal{A}} + \bar{g}^{\mu\nu}\partial_\mu \bar{\tau}_1 \partial_\nu \bar{\tau}_1 + \bar{g}^{\mu\nu}\partial_\mu \bar{\tau}_2 \partial_\nu \bar{\tau}_2\right) \\ & -\frac{\mathcal{A}_0^2}{M_4^2}\left(\frac{\mathcal{A}_0^2}{2}\widehat{\frac{\partial^2 V_{\rm tot}}{\partial \mathcal{A}^2}}\bar{\mathcal{A}}^2 + \frac{1}{4}\widehat{\frac{\partial^2 V_{\rm tot}}{\partial \tau_1^2}}\bar{\tau}_1^2 + \frac{1}{4}\widehat{\frac{\partial^2 V_{\rm tot}}{\partial \tau_2^2}}\bar{\tau}_2^2\right) + \dots\Bigg\}\ ,\end{aligned} \tag{39}$$

where the hat denotes that the second derivatives of $V_{\rm tot}$ are evaluated at the minimum. Together with Eqs. (33) and (35) we now obtain the moduli masses

$$m_{\mathcal{A}}^2 = \frac{\mathcal{A}_0^4}{M_4^2}\widehat{\frac{\partial^2 V_{\rm tot}}{\partial \mathcal{A}^2}} = \frac{\lambda''}{\mathcal{A}_0 L^2}\frac{2\mathcal{M}^2\mu^2}{\Lambda^2 M_4^2}\ , \tag{40}$$

$$m_{\tau_2}^2 = \frac{\mathcal{A}_0^2}{2M_4^2}\widehat{\frac{\partial^2 V_{\rm tot}}{\partial \tau_2^2}} = 4m_{\mathcal{A}}^2\ , \tag{41}$$

$$m_{\tau_1}^2 \simeq m_{\tau_2}^2\ , \tag{42}$$

which depend on the scale of supersymmetry breaking μ, the cutoff Λ and the size of the compact dimensions $\sqrt{\mathcal{A}_0 L^2} \sim 1/\mathcal{M} > 1/\Lambda$. The mass m_{τ_1} has been obtained numerically, based on the complete expression (23) for $V_{\rm casimir}$, since the analytical result (33) away from $\tau_1 = 1/2$ only holds for small τ_2 and not at the minimum $\tau_1 = \tau_2 = 1/2$.

The moduli masses can be related to the gravitino mass using $\mu = F_S/\Lambda$ and $m_{3/2} = F_S/(\sqrt{3}M_4)$, which yields

$$m_{\mathcal{A}}^2 = \frac{6\lambda''\mathcal{M}^2}{\mathcal{A}_0 L^2 \Lambda^4} m_{3/2}^2\ . \tag{43}$$

For a compactification scale $\sqrt{\mathcal{A}_0 L^2} \sim 1/\mathcal{M}$, one obtains

$$m_{\mathcal{A}}^2 = \frac{6\lambda''}{\mathcal{A}_0^2 L^4 \Lambda^4} m_{3/2}^2\ , \tag{44}$$

i.e., the moduli masses are volume suppressed compared to the gravitino mass [17].

An upper bound on the coupling λ'' of the brane field $S(x)$ to the bulk field $\Phi(x,y)$, and therefore on the moduli masses, can be obtained by naive dimensional analysis (NDA) [23]. For this purpose, one rewrites the relevant part of the 6D Lagrangian

$$\mathscr{L} = \hat{\mathscr{L}}_{\rm bulk}(\Phi(x,y)) + \delta^2(y-y_S)\,\hat{\mathscr{L}}_S(\Phi(x,y),S(x)) \tag{45}$$

in terms of dimensionless fields $\hat{\Phi}(x,y)$ and $\hat{S}(x)$, and the cutoff Λ,

$$\mathscr{L} = \frac{\Lambda^6}{\ell_6/C}\,\hat{\mathscr{L}}_{\rm bulk}(\hat{\Phi}(x,y)) + \delta^2(y-y_S)\,\frac{\Lambda^4}{\ell_4/C}\,\hat{\mathscr{L}}_S(\hat{\Phi}(x,y),\hat{S}(x))\ , \tag{46}$$

where $\ell_6 = 128\pi^3$ and $\ell_4 = 16\pi^2$; the factor C accounts for the multiplicity of fields in loop diagrams, with $C=8$ in the present model (cf. [4]). The rescaling of chiral bulk and brane superfields reads

$$\Phi(x,y) = \frac{\Lambda}{\sqrt{\ell_6/C}}\hat{\Phi}(x,y) \quad , \quad S(x) = \frac{\Lambda}{\sqrt{\ell_4/C}}\hat{S}(x)\ . \tag{47}$$

The ratio C/ℓ_D gives the typical suppression of loop diagrams. This suppression is canceled by the factors ℓ_6/C and ℓ_4/C in front of the Lagrangians $\hat{\mathscr{L}}$ in Eq. (46). Consequently, all loops will be of the same order of magnitude, provided that all couplings are $\mathcal{O}(1)$. Thus, according to the NDA recipe the effective 6D theory remains weakly coupled up to the cutoff Λ, if the dimensionless couplings in Eq. (46) are smaller than one.

Let us now apply the NDA recipe to the coupling λ''. Using Eq. (47), we obtain

$$\mathscr{L}_S \supset \frac{\Lambda^4}{\ell_4/C}\int\frac{\mathrm{d}^4\theta}{\Lambda^2}\,\frac{\lambda''C}{\ell_6}\hat{S}^\dagger\hat{S}\left(\hat{\Phi}^\dagger\hat{\Phi}+\hat{\Phi}^{c\dagger}\hat{\Phi}^c\right)\ . \tag{48}$$

The NDA requirement that all couplings be smaller than one implies $\lambda'' \lesssim \ell_6/C = 16\pi^3$. This translates into

$$m_{\mathcal{A}}^2 \lesssim \frac{96\pi^3}{\mathcal{A}_0^2 L^4\Lambda^4} m_{3/2}^2\ . \tag{49}$$

However, this bound cannot be saturated, since the same bound holds for $|\lambda'| > \lambda''$. Further, one has $\Lambda \simeq M_6 \simeq 10\ M_{\rm GUT}$ in the model under consideration [4]. This, together with the bound on λ'', leads to the estimate

$$m_{\mathcal{A}}^2 \lesssim 0.1\ m_{3/2}^2\ . \tag{50}$$

Hence, all moduli masses are smaller than the gravitino mass.

It is instructive to compare the upper bound on the moduli masses with the upper bound on the gaugino mass (16),

$$m_g = \frac{h\mu}{\mathcal{A}L^2\Lambda^2} \simeq \frac{\sqrt{3}h}{\sqrt{\mathcal{A}_0 L^2}\Lambda} m_{3/2}\ . \tag{51}$$

Compared to the moduli masses (44), the gaugino mass is weaker volume suppressed. Correspondingly, the NDA analysis allows the gaugino mass to be larger or smaller than the gravitino mass [24].

6 Conclusions

We have studied a 6-dimensional orbifold GUT model, compactified on a $T^2/\mathbb{Z}_2$ orbifold with two Wilson lines. The Casimir energy depends on the boundary conditions of the various bulk fields and is a function of the shape moduli. It is remarkable that the minimum of the effective potential occurs at a point in field space where the torus lattice has an 'enhanced symmetry' corresponding to the root lattice of SO(5).

The SO(5) lattice has a discrete $\mathbb{Z}_4$ symmetry which is larger than the $\mathbb{Z}_2$ symmetry of a generic torus. Vacua with unbroken discrete symmetries are phenomenologically desirable since they can explain certain features of the supersymmetric standard model, in particular the difference between Higgs and matter fields. Our analysis suggests that such discrete symmetries may arise dynamically in the compactification of higher-dimensional field and string theories.

The interplay of a repulsive Casimir force at small volume and an attractive interaction generated by the coupling of a bulk Higgs field to a supersymmetry breaking brane field stabilizes the volume modulus at the GUT scale, which is determined by the size of localized Fayet-Iliopoulos terms. The masses of shape and volume moduli are smaller than the gravitino mass.

A full supergravity treatment of the described stabilization mechanism still remains to be worked out. Also the phenomenological and cosmological consequences of moduli fields lighter than the gravitino require further investigations.

Acknowledgments

We would like to thank A. Hebecker, M. Klaput, O. Lebedev, J. Louis, J. Möller, C. Paleani, M. Ratz, J. Schmidt and J. Teschner for helpful discussions. This work has been supported by the SFB-Transregio 27 "Neutrinos and Beyond" and by the DFG cluster of excellence "Origin and Structure of the Universe".

A Evaluation of Casimir Sums

Our evaluation of the Casimir double sums requires two single sums which we shall now consider. The first sum reads

$$\widetilde{F}(s;a,c) \equiv \sum_{m=0}^{\infty} \frac{1}{\left[(m+a)^2+c^2\right]^s} \ . \tag{52}$$

This is a series of the generalized Epstein-Hurwitz zeta type. The result can be found in [25] and is given by

$$\begin{aligned}\widetilde{F}(s;a,c) =& \frac{c^{-2s}}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(m+s)}{m!} c^{-2m} \zeta_H(-2m,a) + \sqrt{\pi} \frac{\Gamma(s-\frac{1}{2})}{2\Gamma(s)} c^{1-2s} \\ &+ \frac{2\pi^s}{\Gamma(s)} c^{1/2-s} \sum_{p=1}^{\infty} p^{s-1/2} \cos(2\pi p a) K_{s-1/2}(2\pi p c) \ ,\end{aligned} \tag{53}$$

where $\zeta_H(s,a)$ is the Hurwitz zeta-function. Note that this is not a convergent series but an asymptotic one. In the following it will be important that $\zeta_H(-2m,0) = \zeta_H(-2n,1/2) = 0$ for $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. In our case, the first sum in $\widetilde{F}(s;a,c)$ thus reduces to a single term. For $a = 1/2$ the sum vanishes, and for $a = 0$ only the first term contributes; with $\zeta_H(0,0) = 1/2$ one obtains $c^{-2s}/2$.

The second, related sum is given by

$$F(s;a,c) \equiv \sum_{m=-\infty}^{\infty} \frac{1}{\left[(m+a)^2+c^2\right]^s} \ . \tag{54}$$

Using the two identities ($m \in \mathbb{N}$)

$$\zeta_H(-2m,a) = -\zeta_H(-2m,1-a) \ , \tag{55}$$

$$F(s;a,c) = \widetilde{F}(s;a,c) + \widetilde{F}(s;1-a,c) \ , \tag{56}$$

one easily obtains, in agreement with [9],

$$F(s;a,c) = \frac{\sqrt{\pi}}{\Gamma(s)} |c|^{1-2s} \left[\Gamma\left(s-\tfrac{1}{2}\right) + 4 \sum_{p=1}^{\infty} \cos(2\pi p a) (\pi\, p\, |c|)^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi\, p\, |c|) \right] \ . \tag{57}$$

These two sums provide the basis for our evaluation of the Casimir sums.

A.1 Casimir Sum (I) on $\mathbf{T^2}/\mathbb{Z}_2^3$

We first consider the summation

$$\Big[\sum\Big]_{m,n} = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \ . \tag{58}$$

In this case the Casimir energy is obtained from

$$\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\left[(n+\beta-(m+\alpha)\tau_1)^2+(m+\alpha)^2\tau_2^2+\kappa^2\right]^{-s}\ . \tag{59}$$

where we have shifted $s\to s+2$ and defined $\kappa^2=\frac{\mathcal{A}L^2\tau_2}{4(2\pi)^2}M^2$. Using the expression for $F(s;a,c)$ we can perform the sum over n,

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\left[(n+\beta-(m+\alpha)\tau_1)^2+(m+\alpha)^2\tau_2^2+\kappa^2\right]^{-s}\\
=&\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\sum_{m=0}^{\infty}(\tau_2^2(m+\alpha)^2+\kappa^2)^{1/2-s}\\
&+\frac{4\sqrt{\pi}}{\Gamma(s)}\sum_{p=1}^{\infty}\cos(2\pi p(\beta-(m+\alpha)\tau_1))\sum_{m=0}^{\infty}(\pi\,p)^{s-\frac{1}{2}}\left(\sqrt{\tau_2^2(m+\alpha)^2+\kappa^2}\right)^{\frac{1}{2}-s}\\
&\times\ K_{s-\frac{1}{2}}\left(2\pi\,p\sqrt{\tau_2^2(m+\alpha)^2+\kappa^2}\right)\\
\equiv&f_1(s)+f_2(s)\ .
\end{aligned}\tag{60}$$

Let us consider $f_1(s)$ first. The sum over m can be performed with the help of $\widetilde{F}(s;a,c)$,

$$\begin{aligned}
f_1(s)=&\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\sum_{m=0}^{\infty}(\tau_2^2(m+\alpha)^2+\kappa^2)^{1/2-s}\\
=&\sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\kappa^{1-2s}\zeta_H(0,\alpha)+\frac{\pi}{2(s-1)}\frac{\kappa^{2-2s}}{\tau_2}\\
&+\ \frac{2\pi^s}{\Gamma(s)}\tau_2^{-s}\kappa^{1-s}\sum_{p=1}^{\infty}p^{s-1}\cos(2\pi p\alpha)K_{s-1}\left(2\pi p\left(\tfrac{\kappa}{\tau_2}\right)\right)
\end{aligned}\tag{61}$$

Recalling the shift in s, we can now write $\zeta(s)$ as

$$\begin{aligned}
\zeta(s)=&\frac{\mu_r^{2s+4}4^{-s}(2\pi)^{-2s-4}\pi^2}{2\mathcal{A}^{-2s}L^{-2s}\tau_2^{-s}s(s+1)}\left\{\sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\kappa^{1-2s}\zeta_H(0,\alpha)\right.\\
&+\frac{\pi}{2(s-1)}\frac{\kappa^{2-2s}}{\tau_2}\\
&+\frac{2\pi^s}{\Gamma(s)}\tau_2^{-s}\kappa^{1-s}\sum_{p=1}^{\infty}p^{s-1}\cos(2\pi p\alpha)K_{s-1}\left(2\pi p\left(\tfrac{\kappa}{\tau_2}\right)\right)\\
&+\frac{4\sqrt{\pi}}{\Gamma(s)}\sum_{p=1}^{\infty}\cos(2\pi p(\beta-(m+\alpha)\tau_1))\sum_{m=0}^{\infty}(\pi\,p)^{s-\frac{1}{2}}\left(\sqrt{\tau_2^2(m+\alpha)^2+\kappa^2}\right)^{\frac{1}{2}-s}\\
&\left.K_{s-\frac{1}{2}}\left(2\pi\,p\sqrt{\tau_2^2(m+\alpha)^2+\kappa^2}\right)\right\}\ .
\end{aligned}\tag{62}$$

Now we have to differentiate with respect to s and set $s=-2$. Since $\Gamma(-2)=\infty$, the derivative has only to act on $\Gamma(s)$ if the corresponding term is inversely proportional to $\Gamma(s)$. Performing the differentiation, using

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{\Gamma(s)}\bigg|_{s=-2} = -\frac{\Gamma^{'}(s)}{\Gamma(s)^2}\bigg|_{s=-2} = +2\,, \tag{63}$$

as well as $K_a(z)=K_{-a}(z)$ and substituting again $\kappa=\frac{\sqrt{\mathcal{A}\tau_2}ML}{4\pi}$ we finally obtain for the Casimir energy,

$$\begin{aligned}
V_M^{\alpha,\beta(I)} = &-\frac{4\pi^2}{\mathcal{A}^4L^4\tau_2^2}\Bigg\{-\frac{16\pi}{15}\frac{\mathcal{A}^{5/2}L^5\tau_2^{5/2}M^5}{(4\pi)^5}\zeta_H(0,\alpha)\\
&+\frac{\pi\mathcal{A}^3L^6\tau_2^2M^6}{36(4\pi)^6}\left[-11+12\log\left(\frac{M}{\sqrt{\mathcal{A}}\mu_r}\right)\right]\\
&+\frac{4}{\pi^2}\tau_2^2\frac{\mathcal{A}^{3/2}L^3\tau_2^{3/2}M^3}{(4\pi)^3}\sum_{p=1}^{\infty}\frac{\cos(2\pi p\alpha)}{p^3}K_3(2\pi p\left(\tfrac{\sqrt{\mathcal{A}}LM}{4\pi\sqrt{\tau_2}}\right))\\
&+\frac{8}{\pi^2}\sum_{p=1}^{\infty}\frac{\cos(2\pi p(\beta-(m+\alpha)\tau_1))}{p^{5/2}}\sum_{m=0}^{\infty}\left(\tau_2^2(m+\alpha)^2+\frac{\mathcal{A}L^2\tau_2M^2}{(4\pi)^2}\right)^{\frac{5}{4}}\\
&K_{5/2}(2\pi\,p\sqrt{\tau_2^2(m+\alpha)^2+\frac{\mathcal{A}L^2\tau_2M^2}{(4\pi)^2}})\Bigg\}\,.
\end{aligned} \tag{64}$$

A.2 Casimir Sum (II) on $\mathbf{T^2}/\mathbb{Z}_2^3$

The second relevant summation is

$$\Big[\sum\Big]_{m,n} = \left[\delta_{0,m}\sum_{n=0}^{\infty}+\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\right]\,. \tag{65}$$

For the corresponding boundary conditions one has $\alpha=0$. The Casimir sum can then be written as

$$\left[\delta_{0,m}\sum_{n=0}^{\infty}+\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}-\delta_{m,0}\sum_{n=-\infty}^{\infty}\right]\left[(n+\beta-m\tau_1)^2+m^2\tau_2^2+\kappa^2\right]^{-s}\,, \tag{66}$$

where we again shifted $s\to s+2$. The double sum is the sum (I) which we have already calculated. Using

$$\sum_{n=-\infty}^{-1}\left[(n+\beta)^2+\kappa^2\right]^{-s}=\sum_{n=0}^{\infty}\left[(n+1-\beta)^2+\kappa^2\right]^{-s} \tag{67}$$

one easily finds for the remaining piece[2]

$$
\begin{aligned}
f_3(s) &= -\sum_{n=0}^{\infty}\left[(n+1-\beta)^2+\kappa^2\right]^{-s} \\
&= -\kappa^{-2s}\zeta_H(0,1-\beta) - \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{2\Gamma(s)}\kappa^{1-2s} \\
&- \frac{2\pi^s}{\Gamma(s)}\kappa^{1/2-s}\sum_{p=1}^{\infty} p^{s-1/2}\cos(2\pi p(1-\beta))K_{s-1/2}\left(2\pi p\kappa\right) \ . \qquad (68)
\end{aligned}
$$

Differentiating the corresponding contribution to $\zeta(s)$, setting $s=-2$, and substituting κ yields the Casimir energy,

$$
\begin{aligned}
V_M^{\alpha,\beta(II)} =& V_M^{\alpha,\beta(I)} \\
&+ \frac{4\pi^2}{\mathcal{A}^4L^4\tau_2^2}\left\{\frac{\mathcal{A}^2L^4\tau_2^2M^4}{(4\pi)^4}\left[\frac{3}{2}-2\log\left(\frac{M}{\sqrt{\mathcal{A}}\mu_r}\right)\right]\zeta_H(0,1-\beta)\right. \\
&- \frac{8\pi}{15}\frac{\mathcal{A}^{5/2}L^5\tau_2^{5/2}M^5}{(4\pi)^5} \\
&\left.+ \frac{4}{\pi^2}\left(\frac{(\mathcal{A}\tau_2)^{1/2}LM}{(4\pi)}\right)^{5/2}\sum_{p=1}^{\infty}\frac{\cos(2\pi p(1-\beta))}{p^{5/2}}K_{5/2}\left(2\pi p\frac{\sqrt{\mathcal{A}\tau_2}LM}{4\pi}\right)\right\}. \qquad (69)
\end{aligned}
$$

A.3 Result

Putting everything together the Casimir energy can be written as

$$
\begin{aligned}
V_M^{\alpha,\beta} =& + \frac{M^6L^2}{3072\pi^3\mathcal{A}}\left[\frac{11}{12}-\log\left(\frac{M}{\sqrt{\mathcal{A}}\mu_r}\right)\right] \\
&- \frac{M^4}{64\pi^2\mathcal{A}^2}\left[\frac{3}{4}-\log\left(\frac{M}{\sqrt{\mathcal{A}}\mu_r}\right)\right]\delta_{\alpha0}\delta_{\beta0} \\
&- \frac{M^3\tau_2^{3/2}}{4\pi^3\mathcal{A}^{5/2}L}\sum_{p=1}^{\infty}\frac{\cos(2\pi p\alpha)}{p^3}K_3\left(p\frac{\sqrt{\mathcal{A}}LM}{2\sqrt{\tau_2}}\right) \\
&- \frac{32}{\mathcal{A}^4L^4\tau_2^2}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{1}{2^{\delta_{\alpha0}\delta_{m0}}}\frac{\cos(2\pi p(\beta-(m+\alpha)\tau_1))}{p^{5/2}}\left(\tau_2^2(m+\alpha)^2+\frac{\mathcal{A}L^2\tau_2M^2}{(4\pi)^2}\right)^{\frac{5}{4}} \\
&K_{5/2}\left(2\pi\, p\sqrt{\tau_2^2(m+\alpha)^2+\frac{\mathcal{A}L^2\tau_2M^2}{(4\pi)^2}}\right) \qquad (70)
\end{aligned}
$$

[2] Note that $\zeta_H(0,1)=-1/2$ and $\zeta_H(-2m,1)=0$

or in terms of the moduli R_1, R_2 and θ (and in the frame used in [7])

$$\begin{aligned} V_M^{\alpha\beta} = &+ \frac{M^6 R_1 R_2}{768\pi} \sin\theta \left(\frac{11}{12} - \log\left(\frac{M}{\mu_r}\right)\right) \\ &- \delta_{\alpha 0}\delta_{\beta 0} \frac{M^4}{64\pi^2}\left(\frac{3}{4} - \log\left(\frac{M}{\mu_r}\right)\right) \\ &- \frac{1}{8\pi^4}\frac{M^3 R_2}{R_1^2}\sin\theta \sum_{p=1}^{\infty} \frac{\cos(2\pi p\alpha)}{p^3} K_3(\pi p M R_1) \\ &- \frac{2}{\pi^4}\frac{1}{R_2^4}\frac{1}{\sin^4\theta}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty} \frac{1}{2^{\delta_{\alpha 0}\delta_{m0}}} \frac{\cos(2\pi p[\beta - (m+\alpha)R_2/R_1\cos\theta])}{p^{5/2}} \\ &\left(\tfrac{R_2}{R_1}\sin\theta\sqrt{(m+\alpha)^2 + \tfrac{M^2R_1^2}{4}}\right)^{5/2} K_{5/2}\left(2\pi\, p\, \tfrac{R_2}{R_1}\sin\theta\sqrt{(m+\alpha)^2 + \tfrac{M^2R_1^2}{4}}\right) \ . \end{aligned} \tag{71}$$

For $\theta = \frac{\pi}{2}$ this agrees with the expression for a rectangular torus [7], as expected.

B Contributions from Vector- and Hypermultiplets

In this appendix we compare the leading contributions to the Casimir energy from vector and hypermultiplets, respectively. We will see that the contribution of the vector multiplets is generically suppressed compared to the one of the hypermultiplets, and hence it was justified to neglect this contribution in Eq. (23).

The relative suppression can be seen by an explicit investigation of the mass matrices of the gauginos and the hyperscalars, respectively. For simplicity we only focus on one single scalar Φ and one gaugino ψ. For the present discussion, the relevant part of the 4D Lagrangian reads

$$\mathcal{L} = -\sum_i \Phi_i M_i^{s2} \Phi_i^* - \sum_i \psi_i M_i^f \chi_i + \frac{\lambda\mu^2}{\Lambda^2}\sum_{ij}\Phi_i C_{ij}^{\Phi}\Phi_j^* - \frac{h\mu}{2\Lambda^2}\sum_{ij}\psi_i C_{ij}^{\psi}\psi_j \tag{72}$$

where for the mode expansion we used the notation

$$\Phi(x,y) = \sum_i \Phi_i(x)\xi_i(y) \ , \quad \int \mathrm{d}^2y\ \xi_i(y)\xi_j(y) = \delta_{ij} \ , \tag{73}$$

with Kaluza-Klein mass M_i^s and

$$C_{ij}^{\Phi} = \xi_i(0)\xi_j(0) \ . \tag{74}$$

In the fermionic case the notation is analogous. Here χ is the Weyl fermion which, together with the gaugino, forms the four-component spinor of the six-dimensional vector multiplet. For simplicity we did not consider any mass terms coming from gauge

symmetry breaking, although to include also these terms would be straightforward. The first two terms in Eq. (72) follow directly from the KK mode expansion and dimensional reduction, whereas the last two terms come from supersymmetry breaking with μ the supersymmetry breaking mass. From Eq. (72) one can read off the scalar as well as the fermionic mass matrix squared. The fermionic mass matrix squared reads explicitly (in the basis (ψ, χ))

$$\mathcal{M}^{f2} = \begin{pmatrix} M_i^{f2}\delta_{ik} + \left(\frac{h}{2\Lambda^2}\right)^2 \sum_j C_{ij}^{\psi} C_{jk}^{\psi} \mu^2 & \frac{h}{2\Lambda^2} C_{jk}^{\psi} \mu M_k^f \\ \\ \frac{h}{2\Lambda^2} C_{jk}^{\psi} \mu M_k^f & M_i^{f2}\delta_{ik} \end{pmatrix} . \tag{75}$$

Both, the scalars and the gauginos give a contribution to the Casimir energy which is proportional to the Trace-Log operator $\mathrm{Tr}\log(k^2 + \mathcal{M}^2)$. One can decompose the matrix $\mathcal{M}^2$ as the sum of two terms $\mathcal{M}^2 = \mathcal{M}_0^2 + \Delta\mathcal{M}^2$ (diagonal plus corrections). In the fermionic case this reads

$$\mathcal{M}_0^{f2} = \begin{pmatrix} M_i^{f2}\delta_{ik} & 0 \\ \\ 0 & M_i^{f2}\delta_{ik} \end{pmatrix} ; \Delta\mathcal{M}^{f2} = \begin{pmatrix} \left(\frac{h}{2\Lambda^2}\right)^2 \sum_j C_{ij}^{\psi} C_{jk}^{\psi} \mu^2 & \frac{h}{2\Lambda^2} C_{jk}^{\psi} \mu M_k^f \\ \\ \frac{h}{2\Lambda^2} C_{jk}^{\psi} \mu M_k^f & 0 \end{pmatrix} \tag{76}$$

Expanding the Trace-Log operator in powers of $\Delta\mathcal{M}^2/(k^2 + \mathcal{M}_0^2)$ leads to the expression

$$\begin{aligned} \mathrm{Tr}\log(k^2 + \mathcal{M}^2) &= \mathrm{Tr}\log(k^2 + \mathcal{M}_0^2) + \mathrm{Tr}\left(\frac{1}{k^2 + \mathcal{M}_0^2}\Delta\mathcal{M}^2\right) \\ &- \frac{1}{2}\mathrm{Tr}\left(\frac{1}{k^2 + \mathcal{M}_0^2}\Delta\mathcal{M}^2 \frac{1}{k^2 + \mathcal{M}_0^2}\Delta\mathcal{M}^2\right) + \ldots \end{aligned} \tag{77}$$

which in the fermionic case reads

$$\begin{aligned} \mathrm{Tr}\log(k^2 + \mathcal{M}^{f2}) &= \mathrm{Tr}\log(k^2 + \mathcal{M}_0^{f2}) + \left(\frac{h}{2\Lambda^2}\right)^2 \sum_i \frac{1}{k^2 + M_i^{f2}} \sum_k C_{ik}^{\psi} C_{ki}^{\psi} \mu^2 \\ &- \frac{1}{2}\sum_{ij} \frac{1}{k^2 + M_i^{f2}} \frac{1}{k^2 + M_j^{f2}} \left(\frac{h}{2\Lambda^2}\right)^2 \sum_k C_{ik}^{\psi} C_{kj}^{\psi} \mu^2 M_i^f M_j^f + \ldots \end{aligned} \tag{78}$$

Performing the analogous steps in the scalar case we obtain

$$\begin{aligned} \mathrm{Tr}\log(k^2 + \mathcal{M}^{s2}) &= \mathrm{Tr}\log(k^2 + \mathcal{M}_0^{s2}) + \frac{\lambda}{\Lambda^2} \sum_i \frac{1}{k^2 + M_i^{s2}} C_{ii}^{\Phi} \mu^2 \\ &- \frac{1}{2}\sum_{ij} \frac{1}{k^2 + M_i^{s2}} \frac{1}{k^2 + M_j^{s2}} \left(\frac{\lambda}{\Lambda^2}\right)^2 \sum_k C_{ik}^{\Phi} C_{kj}^{\Phi} \mu^4 + \ldots \end{aligned} \tag{79}$$

Since $C^{\psi,\Phi} \sim 1/\mathcal{A}L^2$, we see that the leading contribution to the Casimir energy from the vector multiplet is generically volume and cutoff suppressed compared to the one from the hypermultiplets.

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