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Nucleon structure in terms of OPE with non-perturbative Wilson coefficients

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Lattice calculations could boost our understanding of Deep Inelastic Scattering by evaluating moments of the Nucleon Structure Functions. To this end we study the product of electromagnetic currents between quark states. The Operator Product Expansion (OPE) decomposes it into matrix elements of local operators (depending on the quark momenta) and Wilson coefficients (as functions of the larger photon momenta). For consistency with the matrix elements, we evaluate a set of Wilson coefficients non-perturbatively, based on propagators for numerous momentum sources, on a $24^3 \times 48$ lattice. The use of overlap quarks suppresses unwanted operator mixing and lattice artifacts. Results for the leading Wilson coefficients are extracted by means of Singular Value Decomposition.

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1. Motivation

The computation of moments of the *nucleon structure functions* is a fascinating challenge: it is known to be difficult, but it is a point where lattice results could contribute much to the understanding and interpretation of phenomenological data from Deep Inelastic Scattering.

The continuum formulation of hadron structure functions is plagued by renormalon ambiguities, *i.e.* power-like IR contributions, see *e.g.* Refs. [1]. Here we refer to the lattice regularisation, where a general moment of the nucleon structure function can be expanded as

$$\mathscr{M}(q^2) = c^{(2)}(aq)A_2(a) + c^{(4)}(aq)\frac{1}{q^2}A_4(a) + \dots \ \{ \text{ higher twists } \} \ . \tag{1.1}$$

It depends on the transfer momentum q, while a is the lattice spacing, $c^{(2)}$, $c^{(4)}\dots$ are Wilson coefficients (where the superscript is the twist), and $A_2, A_4\dots$ are reduced matrix elements ("reduced" in the sense that the Lorentz structure is factored out).

For the evaluation of A_2 there is an established procedure, which employs the ratio between two-point and three-point correlation functions [2]. On the other hand, the Wilson coefficients have usually been evaluated in continuum perturbation theory. However, we need a cancellation of singularities in the terms $c^{(2)}$, $A_4 \propto 1/a^2$, which are again a facet of the renormalon problem. This requires a strictly consistent treatment [3]. Therefore we evaluate the Wilson coefficients non-perturbatively as well. This method is particularly suitable for disentangling higher twist contributions. The use of overlap quarks suppresses undesired operator mixings.

In this report we present new numerical results for $c^{(2)}$. For earlier results with Wilson fermions we refer to Ref. [4]. In a previous study with overlap quarks on a $16^3 \times 32$ lattice [5] some problems persisted, which motivated us to enlarge the lattice to the size $24^3 \times 48$.

2. Operator Product Expansion on the lattice

To be explicit, we apply the Operator Product Expansion (OPE) to a product of electromagnetic currents J_μ between quark states $|\psi(p)\rangle$,

$$\begin{aligned} W_{\mu\nu}(p,q) &= \langle\psi(p)|J_\mu(q)J_\nu^\dagger(q)|\psi(p)\rangle \\ &= \sum_m C^{(m)}_{\mu\nu,i,\mu_1\dots\mu_n}(q)\ \langle\psi(p)|\mathscr{O}^{(m)}_{i,\mu_1\dots\mu_n}|\psi(p)\rangle \ . \end{aligned} \tag{2.1}$$

The lower line is a decomposition in terms of *local* operators $\mathscr{O}^{(m)}$, which characterise the nucleon structure. The index $i = 1\dots16$ specifies the Dirac structure, $\mu_1\dots\mu_n$ indicate the momentum components involved, and m distinguishes operators with the same symmetry. Note that the corresponding Wilson coefficients $C^{(m)}$ depend solely on the transfer momentum q.

A truncation of this expansion at some low operator dimension requires the following *scale separation,*

$$p^2 \ll q^2 \ll \left(\frac{\pi}{a}\right)^2 \ . \tag{2.2}$$

So far we have been using periodic boundary conditions for the gauge field. Thus a sizable lattice is required to have a set of small momenta p^2 available. If this scale separation is realised, it justifies

a truncation of the OPE (2.1), which cuts off operators with high derivatives, *i.e.* high powers of p. Here we consider quark bilinears with up to 3 derivatives,

$$\bar{\psi}\Gamma_i\psi\ , \quad \bar{\psi}\Gamma_i D_{\mu_1}\psi\ , \quad \bar{\psi}\Gamma_i D_{\mu_1}D_{\mu_2}\psi\ , \quad \bar{\psi}\Gamma_i D_{\mu_1}D_{\mu_2}D_{\mu_3}\psi\ , \tag{2.3}$$

where Γ_i runs over a basis of the Clifford algebra. This still amounts to an apparently frightening set of $16\cdot\sum_{d=0}^{3}4^d = 1360$ operators. However, we choose the *isotropic* transfer momenta

$$\begin{aligned} q_a &= \frac{\pi}{4a}(1,1,1,1) \qquad (|q_a| \simeq 4.1 \text{ GeV})\ , \\ q_b &= \frac{\pi}{3a}(1,1,1,1) \qquad (|q_b| \simeq 5.5 \text{ GeV})\ . \end{aligned} \tag{2.4}$$

In this case the symmetry reduces the set of operators to only 67 equivalence classes [5]. We denote them as $C_1 \ldots C_{67}$, which refer to an ascending number of derivatives:

$$\begin{aligned} C_1 &: \text{no derivative, vanishes in the chiral limit, multiplies } \bar{\psi}\mathbb{1}\psi \\ C_2 \ldots C_6 &: \text{one derivative, Bjorken scaling} \propto 1/q^2 \\ C_7 \ldots C_{16} &: \text{two derivatives, vanish in the chiral limit} \\ C_{17} \ldots C_{67} &: \text{three derivatives, Bjorken scaling} \propto 1/(q^2)^2\ . \end{aligned}$$

The coefficients of terms with an even number of derivatives vanish at quark mass $m = 0$ due to chiral symmetry. In the Bjorken limit of large q^2 the coefficients of terms with one (three) derivative(s) are expected to scale as $C_m \propto 1/q^2$ ($C_m \propto 1/(q^2)^2$).

To separate the scales even better, we have now implemented twisted boundary conditions: thus very small p^2 become accessible, which enables us to use q^2 further below the momentum cutoff squared [6]. This allows us to consider for instance $q = \frac{\pi}{6a}(1,1,1,1)$.

3. Results for the Wilson coefficients

In our numerical study we analysed configurations that were generated in the quenched approximation with the Lüscher-Weisz gauge action in a volume $V = 24^3 \times 48$, with a physical lattice spacing $a \simeq 0.075$ fm. To provide finite OPE matrix elements, as they occur in eq. (2.1), we fixed the lattice Landau gauge; this is also favourable to reduce the statistical noise.

For the valence quarks we used overlap fermions with the parameter $\rho = 1.4$ (negative mass of the Wilson kernel) at a bare quark mass of 0.028 in lattice units, which corresponds to about 73 MeV. The application of chiral lattice fermions suppresses undesired operator mixings, as well as $O(a)$ lattice artifacts.

We denote the number of p-sources as M, and the number of Wilson coefficients to be considered as $C_1 \ldots C_N$, with $N = 67$ in our case. So far we evaluated $W_{\mu\nu}$ off-shell for $M = 25$ p-momentum sources at transfer momentum q_a (on different configurations), and for $M = 10$ p-momentum sources at q_b. In each case this yields a system constraining the Wilson coefficients,

$$\begin{pmatrix} W^{(p_1)} \\ \cdot \\ \cdot \\ \cdot \\ W^{(p_M)} \end{pmatrix} = \begin{pmatrix} \mathcal{O}_1^{(p_1)} & \ldots & \mathcal{O}_N^{(p_1)} \\ \cdot & \ldots & \cdot \\ \cdot & \ldots & \cdot \\ \cdot & \ldots & \cdot \\ \mathcal{O}_1^{(p_M)} & \ldots & \mathcal{O}_N^{(p_M)} \end{pmatrix} \begin{pmatrix} C_1 \\ \cdot \\ \cdot \\ C_N \end{pmatrix} \tag{3.1}$$

where the elements $W^{(p_i)}$ and $\mathcal{O}_k^{(p_i)}$ are 4×4 matrices.

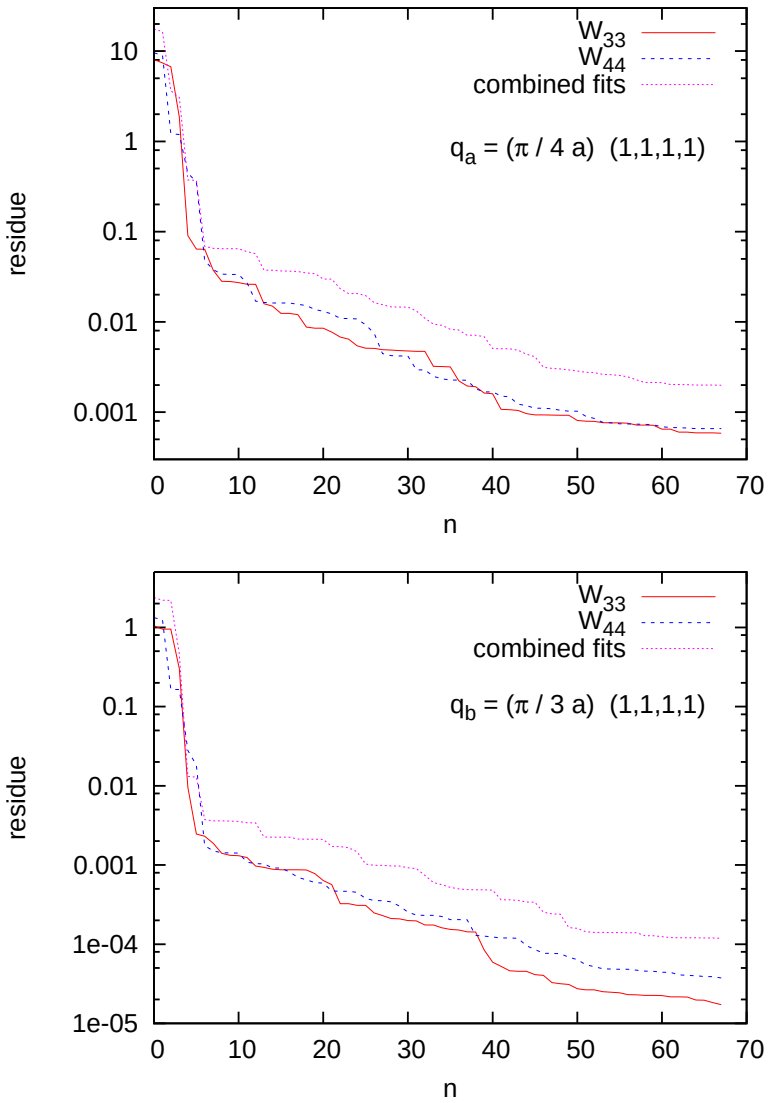

Figure 1: The convergence of the Wilson coefficients as the number of the n most powerful constraints is increased. We show the residues as a function of n for the transfer momentum q_a with 25 p-sources (upper plot), and for q_b with 10 p-sources (lower plot). In both cases we see a convincing convergence before n reaches the deterministic number 67, which confirms that the extracted Wilson coefficients are trustworthy.

In both cases the system is over-determined since $16M > N$. We apply *Singular Value Decomposition* as an established method to analyse such systems [7]. Roughly speaking, this methods selects the $n \leq N$ conditions with the "maximal impact" on the solution $C_1, \ldots, C_N$. A rapid convergence for increasing n approves a reliable result (*i.e.* the remaining conditions are negligible). With 12 singular values (analogues of eigenvalues in rectangular matrices [7]) we do observe this feature. As an example Figure 1 shows the convergence in n for the Wilson coefficients under consideration, which saturates around $n \approx 50$ for q_a and for q_b. In the latter case, the total number of

constraints is lower, hence it is easier to satisfy them to a good precision. Therefore the residues[1] are smaller, but the result is less reliable. Note also that $q_a^2 = \frac{1}{4}(\pi/a)^2$ is more promising than $q_b^2 = \frac{4}{9}(\pi/a)^2$ in view of the required scale separation (2.2).

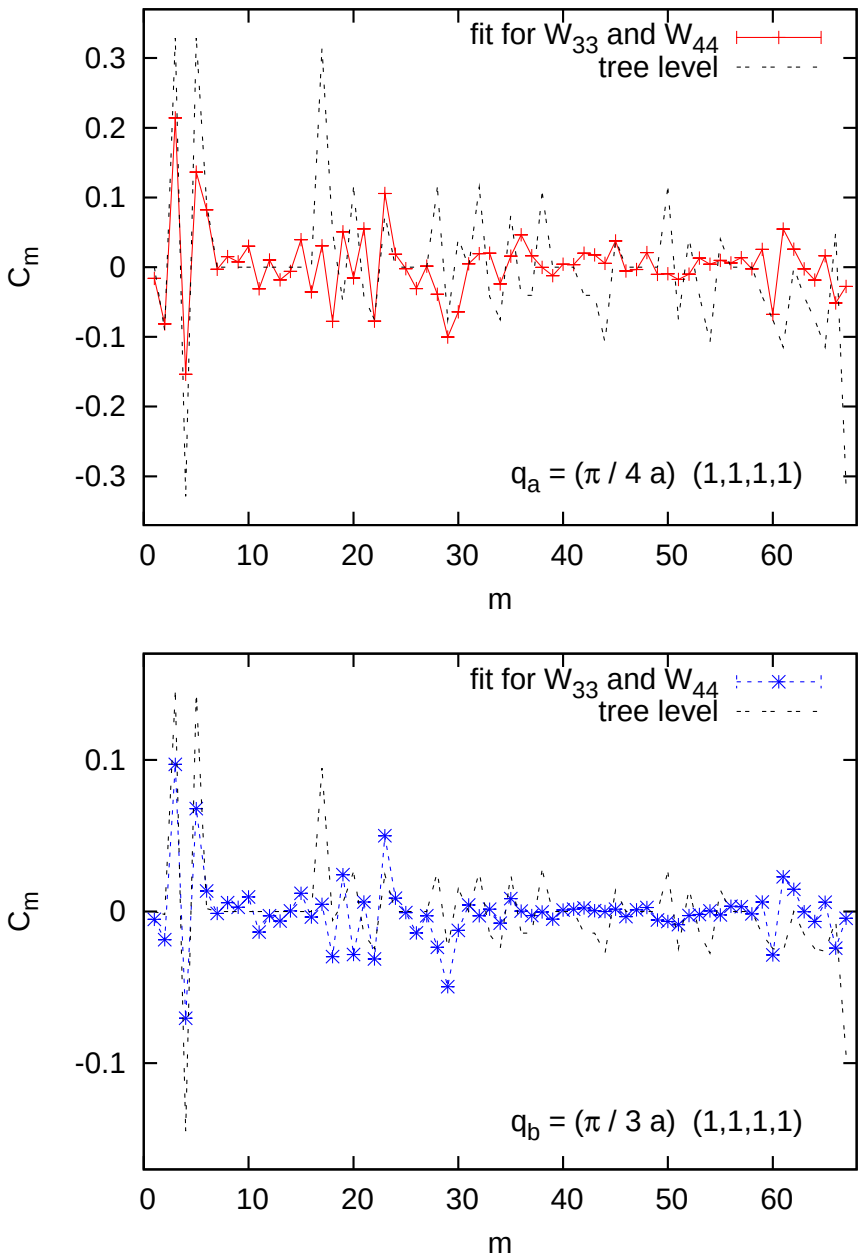

Figure 2: The Wilson coefficients evaluated at momentum transfer q_a (upper plot) and q_b (lower plot). The results are compared to the corresponding tree level values (with the same overlap Dirac operator). We observe the same pattern, but a significant non-perturbative correction. In the chiral limit C_1 and $C_7 \ldots C_{16}$ vanish. The small values that we obtain (with light quarks) for these coefficients provide a consistent picture.

[1] The residue is the norm of the difference between the two sides of eq. (3.1) in the given approximation.

There are a number of discrete rotations and reflections which leave our isotropic q-momenta (2.4) invariant. For example this allows us to exchange W_{33} with W_{44}. As a consequence, specific pairs of Wilson coefficients belonging to different operators have to coincide; for instance the coefficient of $\mathcal{O}_{i,33}$ in W_{33} is equal to the coefficient of $\mathcal{O}_{i,44}$ in W_{44}. We implemented this property in combined fits, which have of course somewhat larger residues, see Figure 1.

We now proceed to the actual results for the Wilson coefficients obtained from W_{33} and W_{44}. The best results emerge from combined fits. They are shown in Figure 2, which also displays the corresponding tree level values for comparison.

In the limit of zero quark mass the coefficients C_1 and $C_7 \dots C_{16}$ vanish due to chiral symmetry, as we anticipated in Section 2. Since we are dealing with light quarks represented by chiral lattice fermions, we are fairly close to chirality. Therefore it is a stringent consistency test that we do obtain particularly small values for these coefficients. That property had not been observed with Wilson fermions [4].

Finally we also test the Bjorken scaling property, which we mentioned in Section 2. We show again the C_m obtained at q_a. We compare them to the Wilson coefficients evaluated at q_b, where $C_2 \dots C_6$ are enhanced with a factor $q_b^2/q_a^2 = 16/9$, and $C_{17} \dots C_{67}$ are amplified with the square of this factor. The Bjorken re-scaled results are confronted in Figure 3, and we see an impressive similarity, in particular for cases with relatively large $|C_m|$.

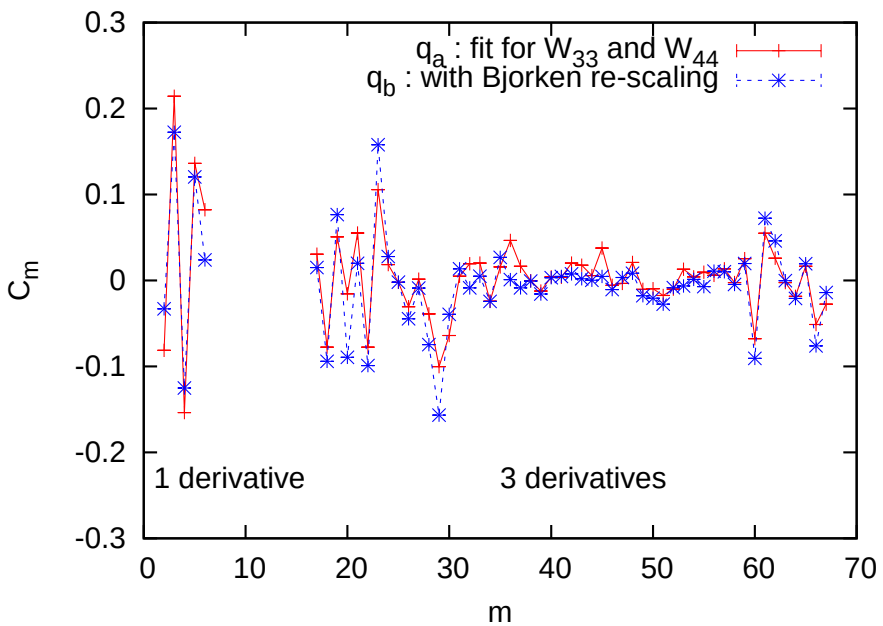

Figure 3: The Wilson coefficients evaluated at momentum transfer q_a and q_b, where in the latter case the coefficients of terms with one or three derivatives are amplified according to the expected scaling in the Bjorken limit. In fact the C_m are now in the same range, and we observe a very satisfactory level of agreement.

4. Conclusions and outlook

Our method to evaluate the Wilson coefficients non-perturbatively works successfully. We arrived at conclusive non-perturbative results for a considerable set of Wilson coefficients. We are continuing to evaluate constraints for additional q and p momenta to further tighten the control over possible artifacts. The use of twisted boundary conditions enables us now to achieve an even better scale separation.

In a final step, the nucleon structure function $\mathscr{M}$ is obtained by *Nachtmann integration* [8] over $W_{\mu\nu}$. For instance, for the second moment this integral takes the form

$$\begin{aligned} \mathscr{M}_2(q) &= \frac{3q^2}{(4\pi)^2} \int d\Omega_q \, n_\mu \Big[W_{\mu\nu}(q) - \frac{1}{4} \delta_{\mu\nu} W_{\rho\rho} \Big] n_\nu \\ &\to \int_0^1 dx \Big[F_2(x,q^2) + \frac{1}{6} F_L(x,q^2) \Big] \,, \end{aligned} \tag{4.1}$$

where the Bjorken limit is taken in the lower line. The projection vector has length $n^2 = 1$. For a different projection one obtains $\int_0^1 dx [F_2 - \frac{3}{2} F_L]$ instead. Thus the combination of different projections determines the *longitudinal structure function* $F_L = F_2 - 2xF_1$.

With completed data, we will obtain a fully non-perturbative and consistent evaluation of the Nucleon Structure Functions [6].

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