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On-shell renormalization of the mixing matrices in Majorana neutrino theories

Andrea A. Almasy*, Bernd A. Kniehl* and Alberto Sirlin†

* *II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany*

† *Department of Physics, New York University, 4 Washington Place, New York, New York 10003, USA*

Abstract

We generalize a recently proposed on-shell approach to renormalize the Cabibbo-Kobayashi-Maskawa quark-mixing matrix to the case of an extended leptonic sector that includes Dirac and Majorana neutrinos in the framework of the seesaw mechanism. An important property of this formulation is the gauge independence of both the renormalized and bare lepton mixing matrices. Also, the texture zero in the neutrino mass matrix is preserved.

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1 Introduction

Renormalizability endows the Standard Model (SM) with enhanced predictive power due to the fact that ultraviolet (UV) divergences from quantum effects can be eliminated by a redefinition of a finite number of independent parameters, such as masses and coupling constants. Furthermore, it has been known for a long time that, in the most frequently employed formulations in which the complete bare mass matrices of quarks are diagonalized, the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix [1] must be also renormalized [2]. In fact, this problem has been the object of several interesting studies over the last two decades [3, 4].

A matter of considerable interest is the generalization of these considerations to minimal renormalizable extensions of the SM. In particular, in Refs. [5, 6] the mixing-matrix renormalization was extended to theories that include isosinglet neutrinos and admit the presence of lepton-number-violating Majorana masses. A minimal realization of such a theory is the SM with right-handed Dirac and Majorana neutrinos [5, 7], an appealing scenario that may explain the smallness of the observed neutrino masses and may lead to neutrino-less double beta decays. Furthermore, this minimal extension may give rise to a number of observable phenomena, such as lepton-flavor and/or lepton-number violation in μ, τ [8] and Z-boson decays [9], or to possible lepton-number-violating signals at high-energy colliders [10].

The aim of this paper is to generalize the on-shell renormalization of the CKM matrix recently proposed in Ref. [4] to extensions of the SM in which the lepton sector contains Majorana neutrinos. An important property is that this formulation complies with UV finiteness and gauge independence,[1] and also preserves the basic structure of the theory. In particular, the texture zero ($m_L'^0 = 0$) in the neutrino mass matrix is preserved by renormalization.

This paper is organized as follows. After briefly reviewing in Section 2 the basic formalism of the seesaw mechanism in the minimal extension of the SM neutral-lepton sector, we evaluate in Section 3 the one-loop self-energy insertions (see Figs. 1 and 3) in an external charged-lepton or Majorana-neutrino leg, perform the separation into wave-function renormalization (wfr) and self-mass (sm) amplitudes, and show explicitly the cancellation of gauge dependences in the latter. As in the quark case [4], the mass counterterm matrix, to be discussed in Section 4, is chosen to cancel, as much as possible, the sm contributions. In Section 5, we discuss the diagonalization of the complete mass matrix and show explicitly how this procedure generates mixing counterterm matrices in a manner that preserves the basic structure of the theory, as well as gauge independence and UV finiteness. Finally, our conclusions are summarized in Section 6.

[1] Throughout this paper, the term *gauge independence* is used as an abbreviation for *gauge parameter independence*.

2 Neutrino see-saw mechanism

We consider a minimal, renormalizable extension of the SM, based on the $SU(2)_I \otimes U(1)_Y$ gauge group, that can naturally accommodate heavy Majorana neutrinos. We allow for an arbitrary number N_G of fermion generations. Similarly to the SM, each lepton family contains one weak-isospin (I) doublet $(\nu'^0_{L,i}, l'^0_{L,i})$ of left-handed states with weak hypercharge $Y = -1$ and one right-handed charged-lepton state $l'^0_{R,i}$ with $I = 0$ and $Y = -2$ ($i = 1, 2, \ldots, N_G$). In addition, there is a total of N_R right-handed neutrinos $\nu'^0_{R,i}$ with $I = Y = 0$ ($i = 1, 2, \ldots, N_R$). The superscript 0 denotes bare quantities, while the primes are to remind us that we are dealing with weak-interaction eigenstates.

The bare Lagrangian density contains the neutrino mass terms

$$\mathcal{L}'^{0,\nu} = -\frac{1}{2} \left(\overline{\nu}'^0_L, \overline{\nu}'^{0C}_R \right) m'^{0,\nu} \begin{pmatrix} \nu'^{0C}_L \\ \nu'^0_R \end{pmatrix} + \text{h.c.}, \tag{1}$$

where $\nu'^0_L = \left(\nu'^0_{L,1}, \ldots, \nu'^0_{L,N_G} \right)^T$, $\nu'^0_R = \left(\nu'^0_{R,1}, \ldots, \nu'^0_{R,N_R} \right)^T$, the superscript C denotes charge conjugation, T means transpose, and $m'^{0,\nu}$ is a complex, symmetric matrix of the form

$$m'^{0,\nu} = \begin{pmatrix} m'^0_L & m'^0_D \\ m'^{0T}_D & m'^0_M \end{pmatrix}. \tag{2}$$

Unless the SM Higgs sector is supplemented by additional weak-isospin singlets and/or triplets of Higgs fields, invariance under $SU(2)_I \times U(1)_Y$ leads to $m'^0_L = 0$. In the following, we do assume that $m'^0_L = 0$. This allows for the implementation of the seesaw mechanism.

The neutrino mass matrix (2) can always be diagonalized through a unitary transformation. For the reader's convenience, we present a simple proof in Appendix B. The non-negative diagonal matrix then contains the bare neutrino mass eigenvalues. The corresponding mass eigenstates are given by

$$\begin{pmatrix} \nu'^0_L \\ \nu'^{0C}_R \end{pmatrix}_a = \sum_b U^{0,\nu *}_{ab} \nu^0_{L,b}, \qquad \begin{pmatrix} \nu'^{0C}_L \\ \nu'^0_R \end{pmatrix}_a = \sum_b U^{0,\nu}_{ab} \nu^0_{R,b}, \tag{3}$$

with $a, b, c = 1, 2, \ldots, N_G + N_R$. It is important to note that Eq. (3) leads to the relation

$$\nu^0_R = \nu^{0C}_L. \tag{4}$$

This implies that the bare neutrino mass eigenstates ν^0_L and ν^0_R can be identified with the left and right-handed components of the Majorana fields.

$$\nu^0 = \nu^0_L + \nu^{0C}_L = \nu^{0C}_R + \nu^0_R. \tag{5}$$

In Eqs. (3)–(5) the first N_G mass eigenstates are identified with the ordinary light neutrinos (assuming that $N_G = 3$), and the remaining N_R states represent the new neutral leptons predicted by the theory. For convenience, in what follows we denote the charged-lepton mass eigenstates using indices $i, j, k, \ldots$ and the Majorana-neutrino mass eigenstates using indices from the beginning of the alphabet $a, b, c, \ldots$. Accordingly, sums over

repeated charged-lepton indices $i, j, k, \ldots$ run from 1 to N_G, while those over the neutrino indices $a, b, c, \ldots$ extend from 1 to $N_G + N_R$.

The parts of the bare Lagrangian describing the couplings of the $W^{\pm}$, Z, and Higgs (H) bosons to the charged-lepton mass eigenstates, l_i^0, and Majorana-neutrino mass eigenstates, ν_a^0, are given by:[2]

$$\begin{aligned}
\mathcal{L}_W^0 &= -\frac{g^0}{\sqrt{2}}(W_\mu^-)^0 \sum_{i,a} \overline{l}_i^0 B_{ia}^0 \gamma^\mu a_- \nu_a^0 + \text{h.c.}, \\
\mathcal{L}_{\phi^\pm}^0 &= -\frac{g^0}{\sqrt{2}m_W^0}(\phi^-)^0 \sum_{i,a} \overline{l}_i^0 B_{ia}^0 (m_i^0 a_- - m_a^0 a_+)\nu_a^0 + \text{h.c.}, \\
\mathcal{L}_Z^0 &= -\frac{g^0}{4c_w^0} Z_\mu^0 \sum_{a,b} \overline{\nu}_a^0 \gamma^\mu (C_{ab}^0 a_- - C_{ab}^{0*} a_+)\nu_b^0, \\
\mathcal{L}_{\phi^0}^0 &= \frac{ig^0}{4m_W^0}(\phi^0)^0 \sum_{a,b} \overline{\nu}_a^0 \left[(m_a^0 C_{ab}^{0*} + m_b^0 C_{ab}^0)a_+ - (m_a^0 C_{ab}^0 + m_b^0 C_{ab}^{0*})a_-)\right] \nu_b^0, \\
\mathcal{L}_H^0 &= -\frac{g^0}{4m_W^0} H^0 \sum_{a,b} \overline{\nu}_a^0 \left[(m_a^0 C_{ab}^{0*} + m_b^0 C_{ab}^0)a_+ + (m_a^0 C_{ab}^0 + m_b^0 C_{ab}^{0*})a_-)\right] \nu_b^0,
\end{aligned} \tag{6}$$

where g is the $SU(2)_L$ gauge coupling, c_w the cosine of the electroweak mixing angle, $\phi^\pm$ and ϕ^0 are the charged and neutral Higgs-Kibble ghosts, respectively, and $a_\pm = (1 \pm \gamma_5)/2$ are the chiral projectors. B and C are $N_G \times (N_G + N_R)$ and $(N_G + N_R) \times (N_G + N_R)$ non-unitary matrices, respectively. The bare matrices are defined by the expressions

$$B_{ia}^0 = \sum_k V_{ik}^{0,l} U_{ka}^{0,\nu *}, \qquad C_{ab}^0 = \sum_c U_{ac}^{0,\nu T} U_{cb}^{0,\nu *}, \tag{7}$$

where $V^{0,l}$ is the unitary $N_G \times N_G$ matrix relating the weak-interaction and mass eigenstates of the charged leptons and $U^{0,\nu}$ is the unitary $(N_G + N_R) \times (N_G + N_R)$ matrix relating the corresponding neutrino eigenstates, defined in Eq. (3). They obey a number of basic identities, which ensure the renormalizability of the theory, namely [5, 7]

$$\sum_c B_{ic}^0 B_{jc}^{0*} = \delta_{ij}, \qquad \sum_i B_{ia}^{0*} B_{ib}^0 = C_{ab}^0, \tag{8}$$

$$\sum_c B_{ic}^0 C_{ca}^0 = B_{ia}^0, \qquad \sum_c C_{ac}^0 C_{cb}^0 = C_{ab}^0 = C_{ab}^{0\dagger}, \tag{9}$$

$$\sum_c m_c^0 B_{ic}^0 B_{jc}^0 = 0, \qquad \sum_c m_c^0 B_{ic}^0 C_{ac}^0 = 0, \qquad \sum_c m_c^0 C_{ac}^0 C_{bc}^0 = 0. \tag{10}$$

The last three relations are manifestations of the presence of lepton-number violation in the neutrino sector.

[2] In Eq. (6) we have not included the terms describing the interactions of the neutral bosons with the charged leptons, since they are the same as in the SM.

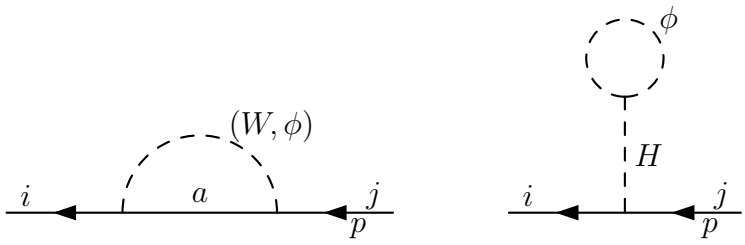


Figure 1: Charged-lepton self-energy diagrams at one loop.

3 Self-energy corrections to an external leg

Following the approach of Ref. [4], the analysis of external-leg corrections leads to two classes of contributions:

(i) terms proportional to the virtual-fermion propagator $i/(\not{p}-m_{f'})$ with gauge-independent cofactors not involving $\not{p}$, where $m_{f'}$ stands generically for the mass of the virtual fermion;

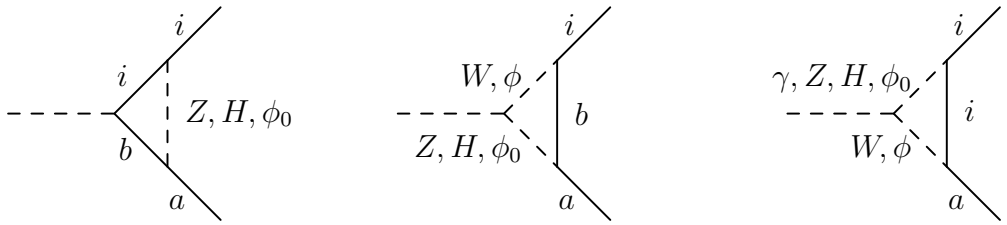
(ii) terms in which the virtual propagator is cancelled in both the diagonal and off-diagonal amplitudes.

The gauge-independent cofactors of class (i) and the contributions of class (ii) are identified with the sm and wfr contributions, respectively. In analogy with QED, the latter contain both gauge-dependent and UV-divergent parts but, in the evaluation of physical amplitudes, these pieces cancel the corresponding contributions from the proper vertex diagrams. On the other hand, also in analogy with QED, the UV-divergent sm contributions are cancelled by the UV-divergent parts of the mass counterterms.

In order to implement the analysis of the external-leg corrections, we evaluate the contributions of Figs. 1 and 3 in the R_ξ gauges and, applying the algorithm developed in Ref. [4], we separate them into sm and wfr amplitudes. We do not enter into details, but rather present the results and emphasize the differences with respect to the quark case. We first treat the case of an outgoing on-shell charged lepton in Section 3.1 and then that of an outgoing on-shell Majorana neutrino in Section 3.2. We have chosen to do so, since the charged-lepton case is very similar to that of quarks, while in the Majorana-neutrino case additional interactions involving flavor mixing appear. For completeness, in Section 3.3 we discuss also the case of incoming charged leptons and Majorana neutrinos.

3.1 Outgoing charged lepton

If i is an outgoing on-shell charged lepton, the external-leg amplitude is obtained by multiplying the diagrams in Fig. 1 on the left by $\overline{u}_i(p)$, the spinor of the outgoing charged lepton, and on the right by $i(\not{p} - m_j)^{-1}$, the propagator of the virtual charged lepton.



(a) $\sim B \cdot C$ (b) $\sim B \cdot C$ (c) $\sim B$

Figure 2: Proper $Wl\nu$ vertex diagrams at one loop.

Thus, the relevant amplitude associated with the external leg is:

$$\Delta\mathcal{M}_{ij}^{\text{leg}} = \overline{u}_i(p) M_{ij}^{(1)} \frac{i}{\not{p} - m_j}, \tag{11}$$

where $M_{ij}^{(1)}$ denotes the contributions of Fig. 1.

The sm contributions to the external-leg corrections for an outgoing on-shell charged lepton are:

$$\begin{aligned}
\Delta\mathcal{M}_{ij}^{\text{leg,sm}} = \frac{g^2}{32\pi^2} \overline{u}_i(p) \sum_a B_{ia} B_{aj}^\dagger \Bigg\{ & m_i \left(1 + \frac{m_i^2}{2m_W^2} \Delta_W\right) \\
& + \left[m_i a_- + m_j a_+ + \frac{m_i m_j}{2m_W^2} (m_i a_+ + m_j a_-)\right] \\
& \times \left[I(m_i^2, m_W, m_a) - J(m_i^2, m_W, m_a)\right] \\
& - \frac{m_a^2}{2m_W^2} (m_i a_- + m_j a_+) \\
& \times \left[3\Delta_W + I(m_i^2, m_W, m_a) + J(m_i^2, m_W, m_a)\right] \Bigg\} \frac{1}{\not{p} - m_j}.
\end{aligned} \tag{12}$$

The functions I and J as well as the UV divergence Δ_W are defined in Appendix A. Note that Eq. (12) is a multiple of the virtual charged-lepton propagator $i(\not{p} - m_j)^{-1}$ with a cofactor that is gauge and momentum independent. As expected in a chiral theory, it involves the chiral projectors. At this point, we should emphasize that Eq. (12) is the same as that for up-type quarks, given in Eq. (29) of Ref. [4], up to particle changes. The only difference is that now no complications due to imaginary parts appear. The amplitudes $I(p^2, m_1, m_2)$ and $J(p^2, m_1, m_2)$ may have absorptive parts only when their arguments fulfill the condition $p^2 > (m_1 + m_2)^2$. In the present case, we have $p^2 = m_i^2$, $m_1 = m_W$, and $m_2 = m_a$, which ensures that the above inequality can not be satisfied, since the external-charged-lepton mass is much smaller than that of the W boson.

The wfr contributions to the external-leg correction for an outgoing on-shell charged

lepton are given by

$$\begin{aligned}\Delta\mathcal{M}_{ij}^{\rm leg,wfr} = \frac{g^2}{32\pi^2}\overline{u}_i(p)\sum_a B_{ia}B_{aj}^\dagger\Bigg\{ & \left[I(m_i^2,m_W,m_a) - J(m_i^2,m_W,m_a)\right] a_+ \\ & + \frac{1}{2m_W^2}\left[m_i m_j a_- + m_a^2 a_+\right]\left[\Delta_W + I(m_i^2,m_W,m_a) - J(m_i^2,m_W,m_a)\right] \\ & - \delta_{ij}\frac{m_i^2 m_a^2}{2m_W^2}\left[I'(m_i^2,m_W,m_a) + J'(m_i^2,m_W,m_a)\right] \\ & + \delta_{ij}m_i^2\left(1+\frac{m_i^2}{2m_W^2}\right)\left[I'(m_i^2,m_W,m_a) - J'(m_i^2,m_W,m_a)\right] \\ & + \left[1+\xi_W\left(\Delta_W - \frac{1}{2} + \frac{1}{2}\ln\xi_W\right)\right] a_+ - N(m_W,m_i,m_a,\xi_W)a_+\Bigg\}. \end{aligned} \tag{13}$$

Here $I'(m_i^2,m_W,m_a)$ and $J'(m_i^2,m_W,m_a)$ are the first derivatives of $I(p^2,m_W,m_a)$ and $J(p^2,m_W,m_a)$ with respect to p^2, evaluated at $p^2 = m_i^2$, and the function $N(m_W,m_i,m_a,\xi_W)$ is defined in Appendix A.

The UV-divergent part of Eq. (13) is then

$$\Delta\mathcal{M}_{ij}^{\rm leg,wfr,div} = \frac{g^2}{32\pi^2}\overline{u}_i(p)\sum_a B_{ia}B_{aj}^\dagger\left[\frac{m_i m_j}{2m_W^2}a_- + \left(\xi_W + \frac{m_a^2}{2m_W^2}\right)a_+\right]\Delta_W. \tag{14}$$

If Eq. (13) is inserted in the leptonic W-boson decay amplitude, important simplifications take place, in analogy with the analysis in Ref. [4]. In fact, using Eqs. (8)–(10) one readily finds that the contributions of the terms not involving I' and J' reduce to expressions that combine naturally with the proper vertex diagrams of Fig. 2, an important property to ensure the cancellation of UV divergences and gauge dependences in the full physical amplitude. Although the corresponding contributions from the terms involving I' and J' do not simplify, we note that they are UV finite and gauge independent.

3.2 Outgoing Majorana neutrino

While the charged leptons could be treated analogously to the quarks, the Majorana neutrinos require a more comprehensive analysis. In this case, mixing occurs not only in charged-current but also in neutral-current interactions. For this reason, it is necessary to consider the corrections in Fig. 3 induced by neutral currents, as well as those emerging from the charged currents, already present in the charged-lepton case. Note that, if neutrinos were Dirac particles, the second diagram would be absent.

As before, we evaluate the contributions in Fig. 3 in the R_ξ gauges. The tadpole diagrams are needed to remove the gauge dependence in the diagonal (aa) and in parts of the non-diagonal (ab) contributions to the self-energy diagrams.

We consider the case in which a is an outgoing on-shell Majorana neutrino and b is a virtual Majorana neutrino. In analogy to the charged-lepton case, the sm contributions

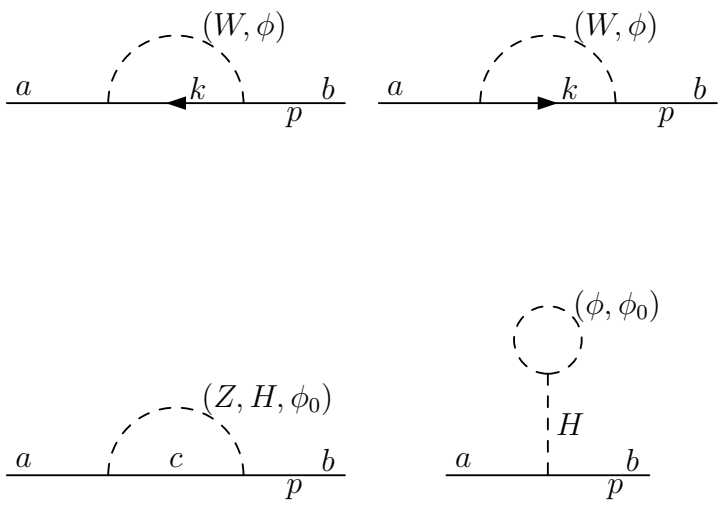


Figure 3: Majorana neutrino self-energy diagrams.

$\Delta\mathcal{M}_{ab}^{\rm leg,sm}$ to the external-leg correction for an outgoing on-shell Majorana neutrino a read:

$$\Delta\mathcal{M}_{ab}^{\rm leg,sm} = \overline{u}_a(p)\left\{\frac{g^2}{32\pi^2}T_{ab}^{\rm sm}\right\}\frac{1}{\not{p}-m_b}, \tag{15}$$

with

$$\begin{aligned}
T_{ab}^{\rm sm} = {} & (m_a C_{ab}^* + m_b C_{ab})\left(a_+ + \frac{m_a m_b}{2m_W^2}\Delta_W a_-\right) \\
& + (m_a C_{ab} + m_b C_{ab}^*)\left(a_- + \frac{m_a m_b}{2m_W^2}\Delta_W a_+\right) \\
& - \sum_k \frac{m_k^2}{2m_W^2}\left[(m_a B_{ka} B_{kb}^* + m_b B_{ka}^* B_{kb})\, a_+ + (m_a B_{ka}^* B_{kb} + m_b B_{ka} B_{kb}^*)\, a_-\right] \\
& \times \left[3\Delta_W + I(m_a^2, m_W, m_k) + J(m_a^2, m_W, m_k)\right] \\
& + \sum_k \left[(m_a B_{ka} B_{kb}^* + m_b B_{ka}^* B_{kb})\left(a_+ + \frac{m_a m_b}{2m_W^2}a_-\right)\right. \\
& \left. + (m_a B_{ka}^* B_{kb} + m_b B_{ka} B_{kb}^*)\left(a_- + \frac{m_a m_b}{2m_W^2}a_+\right)\right] \\
& \times \left[I(m_a^2, m_W, m_k) - J(m_a^2, m_W, m_k)\right] \\
& + \frac{1}{4m_W^2}\sum_c \left[(m_a a_- + m_b a_+)\left(m_a m_b C_{ab}^* + 5m_c^2 C_{ac} C_{cb}\right)\right. \\
& \left. + (m_a a_+ + m_b a_-)\left(m_a m_b C_{ab} + 5m_c^2 C_{ac}^* C_{cb}^*\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+ 4m_c^3 \left(C_{ac}C_{cb}^* a_+ + C_{ac}^* C_{cb} a_-\right)\Big] \Delta_H \\
&+ \frac{1}{4m_W^2} \sum_c \Big[\left(m_a C_{ac}^* + m_c C_{ac}\right)\left(m_c C_{cb} + m_b C_{cb}^*\right)\left(m_a a_- + m_b a_+\right) \\
&+ \left(m_a C_{ac} + m_c C_{ac}^*\right)\left(m_c C_{cb}^* + m_b C_{cb}\right)\left(m_a a_+ + m_b a_-\right)\Big] \\
&\times \left[I(m_a^2, m_H, m_c) - J(m_a^2, m_H, m_c)\right] \\
&+ \sum_c \frac{m_c}{2m_W^2}\Big[\left(m_a C_{ac}^* + m_c C_{ac}\right)\left(m_c C_{cb}^* + m_b C_{cb}\right) a_+ \\
&+ \left(m_a C_{ac} + m_c C_{ac}^*\right)\left(m_c C_{cb} + m_b C_{cb}^*\right) a_-\Big] I(m_a^2, m_H, m_c) \\
&+ \frac{1}{2c_w^2}\left[C_{ab}^* + \frac{m_a m_b}{2m_Z^2} C_{ab}\Delta_Z\right]\left(m_a a_+ + m_b a_-\right) \\
&+ \frac{1}{2c_w^2}\left[C_{ab} + \frac{m_a m_b}{2m_Z^2} C_{ab}^*\Delta_Z\right]\left(m_a a_- + m_b a_+\right) \\
&- \frac{m_c^3}{m_W^2}\Delta_Z \left(C_{ac}C_{cb}^* a_+ + C_{ac}^* C_{cb} a_-\right) \\
&+ \frac{1}{2c_w^2}\sum_c \Bigg\{ \left(1 - \frac{m_c^2}{2m_Z^2}\right)\left[C_{ac}C_{cb}(m_a a_- + m_b a_+) + C_{ac}^* C_{cb}^*(m_a a_+ + m_b a_-)\right] \\
&+ \frac{1}{2m_Z^2}(m_a a_- + m_b a_+)(m_a C_{ac}^* + m_c C_{ac})(m_b C_{cb}^* + m_c C_{cb}) \\
&+ \frac{1}{2m_Z^2}(m_a a_+ + m_b a_-)(m_a C_{ac} + m_c C_{ac}^*)(m_b C_{cb} + m_c C_{cb}^*)\Bigg\} \\
&\times \left[I(m_a^2, m_Z, m_c) - J(m_a^2, m_Z, m_c)\right] \\
&- \sum_c \frac{m_c^2}{4m_W^2}\left[C_{ac}C_{cb}\left(m_a a_- + m_b a_+\right) + C_{ac}^* C_{cb}^*\left(m_a a_+ + m_b a_-\right)\right] \\
&\times \left[3\Delta_Z + I(m_a^2, m_Z, m_c) + J(m_a^2, m_Z, m_c)\right] \\
&+ \sum_c \frac{m_c}{2c_w^2}\Bigg[\left(4 - \frac{m_c^2}{m_Z^2}\right)\left(C_{ac}C_{cb}^* a_+ + C_{ac}^* C_{cb} a_-\right) \\
&- \frac{m_a m_b}{m_Z^2}\left(C_{ac}^* C_{cb} a_+ + C_{ac} C_{cb}^* a_-\right)\Bigg] I(m_a^2, m_Z, m_c). \qquad (16)
\end{aligned}
$$

The I and J functions, together with the UV divergences Δ_W, Δ_Z, and Δ_H are defined in Appendix A.

Equations (15)–(16) are much lengthier than Eq. (12) because of the additional class of diagrams considered. However, their structure is similar.

The wfr contributions $\Delta\mathcal{M}_{ab}^{\rm leg,wfr}$ to the external-leg correction for an outgoing on-shell Majorana neutrino a read:

$$
\Delta\mathcal{M}_{ab}^{\rm leg,wfr} = \overline{u}_a(p)\left\{\frac{g^2}{32\pi^2} T_{ab}^{\rm wfr}\right\}, \qquad (17)
$$

with

$$
\begin{aligned}
T_{ab}^{\rm wfr} = {} & (C_{ab}a_+ + C_{ab}^* a_-) \left[1 + \xi_W \left(\Delta_W - \frac{1}{2} + \frac{1}{2}\ln\xi_W\right)\right] \\
& + \frac{1}{2c_w^2}(C_{ab}a_+ + C_{ab}^* a_-) \left[1 + \xi_Z \left(\Delta_Z - \frac{1}{2} + \frac{1}{2}\ln\xi_Z\right)\right] \\
& - \sum_k \left(B_{ka}^* B_{kb} a_+ + B_{ka} B_{kb}^* a_-\right) N(m_W, m_a, m_k, \xi_W) \\
& - \frac{1}{2c_w^2} \sum_c (C_{ac}C_{cb}a_+ + C_{ac}^* C_{cb}^* a_-) N(m_Z, m_a, m_c, \xi_Z) \\
& + \frac{1}{2c_w^2} \sum_c m_a m_c \left(C_{ac}C_{cb}^* a_- + C_{ac}^* C_{cb} a_+\right) M(m_Z, m_a, m_c, \xi_Z) \\
& + \frac{1}{2m_W^2} \sum_k \left[B_{ka}^* B_{kb} \left(m_a m_b a_- + m_k^2 a_+\right) + B_{ka} B_{kb}^* \left(m_a m_b a_+ + m_k^2 a_-\right)\right] \\
& \times \left[\Delta_W + I(m_a^2, m_W, m_k) - J(m_a^2, m_W, m_k)\right] \\
& + \sum_k \left[B_{ka}^* B_{kb} a_+ + B_{ka} B_{kb}^* a_-\right] \left[I(m_a^2, m_W, m_k) - J(m_a^2, m_W, m_k)\right] \\
& + \frac{1}{4m_W^2} \left(m_a m_b C_{ab}^* + \sum_c m_c^2 C_{ac} C_{cb}\right) \Delta_H a_+ \\
& + \frac{1}{4m_W^2} \left(m_a m_b C_{ab} + \sum_c m_c^2 C_{ac}^* C_{cb}^*\right) \Delta_H a_- \\
& + \frac{1}{4m_W^2} \sum_c \left[(m_a C_{ac}^* + m_c C_{ac})(m_c C_{cb} + m_b C_{cb}^*) a_+\right. \\
& \left. + (m_a C_{ac} + m_c C_{ac}^*)(m_c C_{cb}^* + m_b C_{cb}) a_-\right] \\
& \times \left[I(m_a^2, m_H, m_c) - J(m_a^2, m_H, m_c)\right] \\
& + \frac{1}{2c_w^2} \sum_c (C_{ac}C_{cb}a_+ + C_{ac}^* C_{cb}^* a_-) \left[I(m_a^2, m_Z, m_c) - J(m_a^2, m_Z, m_c)\right] \\
& + \frac{1}{4m_W^2} \sum_c \left[(m_a C_{ac} + m_c C_{ac}^*)(m_c C_{cb}^* + m_b C_{cb}) a_-\right. \\
& \left. + (m_a C_{ac}^* + m_c C_{ac})(m_c C_{cb} + m_b C_{cb}^*) a_+\right] \\
& \times \left[\Delta_Z + I(m_a^2, m_Z, m_c) - J(m_a^2, m_Z, m_c)\right] \\
& - \delta_{ab} \sum_k \frac{m_a^2 m_k^2}{m_W^2} B_{ka}^* B_{kb} \left[I'(m_a^2, m_W, m_k) + J'(m_a^2, m_W, m_k)\right] \\
& + 2\delta_{ab} m_a^2 \left(1 + \frac{m_a^2}{2m_W^2}\right) \sum_k B_{ka}^* B_{kb} \left[I'(m_a^2, m_W, m_k) - J'(m_a^2, m_W, m_k)\right] \\
& + \delta_{ab} \frac{m_a^2}{2m_W^2} \sum_c (m_a C_{ac} + m_c C_{ac}^*)(m_c C_{cb}^* + m_b C_{cb})
\end{aligned}
$$

$$
\begin{aligned}
&\times \left[I'(m_a^2, m_H, m_c) - J'(m_a^2, m_H, m_c)\right] \\
&+ \delta_{ab} \sum_c \frac{m_a m_c}{2m_W^2} \left[(m_a C_{ac}^* + m_c C_{ac})(m_c C_{cb}^* + m_b C_{cb}) a_+ \right. \\
&+ \left. (m_a C_{ac} + m_c C_{ac}^*)(m_c C_{cb} + m_b C_{cb}^*) a_- \right] I'(m_a^2, m_H, m_c) \\
&+ \delta_{ab} \frac{m_a^2}{c_w^2} \sum_c \left[\frac{1}{2m_Z^2} (m_a C_{ac} + m_c C_{ac}^*)(m_b C_{cb} + m_c C_{cb}^*) \right. \\
&+ \left. C_{ac} C_{cb} \left(1 - \frac{m_c^2}{2m_Z^2}\right) \right] \left[I'(m_a^2, m_Z, m_c) - J'(m_a^2, m_Z, m_c)\right] \\
&- \delta_{ab} \sum_c \frac{m_a^2 m_c^2}{2m_W^2} C_{ac} C_{cb} \left[I'(m_a^2, m_Z, m_c) + J'(m_a^2, m_Z, m_c)\right] \\
&+ \delta_{ab} \sum_c \frac{m_a m_c}{2c_w^2} \left[\left(4 - \frac{m_c^2}{m_Z^2}\right) (C_{ac} C_{cb}^* a_+ + C_{ac}^* C_{cb} a_-) \right. \\
&- \left. \frac{m_a^2}{m_Z^2} (C_{ac} C_{cb}^* a_- + C_{ac}^* C_{cb} a_+) \right] I'(m_a^2, m_Z, m_c). \qquad (18)
\end{aligned}
$$

The UV-divergent part of Eq. (17) is:

$$
\begin{aligned}
\Delta \mathcal{M}_{ab}^{\text{leg,wfr,div}} = \frac{g^2}{32\pi^2} \overline{u}_a(p) \Bigg\{ & (C_{ab} a_+ + C_{ab}^* a_-) \left(\xi_W \Delta_W + \frac{1}{2c_w^2} \xi_Z \Delta_Z \right) \\
& + \frac{m_a m_b}{2m_W^2} (C_{ab}^* a_+ + C_{ab} a_-) (\Delta_W + \Delta_H) \\
& + \sum_k \frac{m_k^2}{2m_W^2} (B_{ka}^* B_{kb} a_+ + B_{ka} B_{kb}^* a_-) \Delta_W \\
& + \sum_c \frac{m_c^2}{4m_W^2} (C_{ac} C_{cb} a_+ + C_{ac}^* C_{cb}^* a_-) (\Delta_H + \Delta_Z) \Bigg\}. \qquad (19)
\end{aligned}
$$

The discussion of the wfr contributions at the end of Section 3.1 remains valid. In fact, when inserted into the leptonic W-boson decay amplitude, Eq. (17) is also subject to important simplifications. One then finds that the wfr contributions to the external leg involving an outgoing Majorana neutrino can be combined naturally with the proper vertex diagrams of Fig. 2. To prove this, one needs to multiply Eq. (17) on the right by $\left(-ig/\sqrt{2}\right) B_{ib}^* \gamma^\mu a_- v_i \varepsilon_\mu$, where v_i is the spinor associated with the charged lepton l_i and ε_μ is the polarization four-vector of the W boson, and to perform the summation over the index b. Making use of Eqs. (8)–(10), it can be verified that the terms in Eq. (17) not involving derivatives of the amplitudes I and J lead to the structures

(i) $B_{ia}^* f(m_i, m_a)$;

(ii) $\sum_b B_{ib}^* \left(C_{ba} f_1(m_i, m_a, m_b) + C_{ba}^* f_2(m_i, m_a, m_b)\right)$.

The terms having the structure (i) combine naturally with the proper vertex diagrams of Fig. 2(c), while those having the structure (ii) are to be combined with the diagrams

depicted in Figs. 2(a) and (b). We emphasize that, also here, these terms include all the gauge-dependent and UV-divergent contributions in Eq. (17). However, in Eq. (17) there are also terms proportional to derivatives of the functions I and J, which cannot be further simplified, but are UV finite and gauge independent.

3.3 Incoming leptons

Equations (12), (13), (15), and (17) exhibit the sm and wfr contributions to the external-leg corrections in the case of an outgoing on-shell lepton. For the purpose of the following discussion, it is convenient to call x and y the flavors of the outgoing and virtual leptons. Thus, in Fig. 1, $x = i$ and $y = j$, while in Fig. 3, $x = a$ and $y = b$.

The corresponding expressions for an incoming lepton of flavor y is obtained by multiplying the diagrams in Figs. 1 and 3, by $u_y(p)$ on the right and $(\not{p} - m_x)^{-1}$ on the left, where x now denotes the virtual lepton. It is then easy to see that the sm contributions in the incoming case are obtained by interchanging $a_+ \leftrightarrow a_-$ and $m_x \leftrightarrow m_y$ between the curly brackets of Eqs. (12) and (15), and multiplying the resulting expression by $u_y(p)$ on the right and $(\not{p} - m_x)^{-1}$ on the left. Similarly, the wfr contributions for an incoming lepton of flavor y are obtained by interchanging $a_+ \leftrightarrow a_-$ and $m_x \leftrightarrow m_y$ between the curly brackets of Eqs. (13) and (17), and multiplying the resulting expression by $u_y(p)$ on the right.

4 Mass renormalization

In this section, we study the cancellation of the sm contributions by suitably adjusting the mass counterterms. We start with the simpler case of the charged leptons, which is, up to the particle content, identical to that of quarks [4]. When treating the Majorana-neutrino case, a new feature appears. One needs to keep in mind the fact that we are dealing with Majorana particles, *i.e.* particles and antiparticles are identical. As a consequence, a new condition for the mass counterterms arises.

4.1 Charged-lepton mass counterterm matrix

In order to generate mass counterterms suitable for the renormalization of the sm contributions shown in Eq. (12), we proceed as in Ref. [4], where the case of quark mixing was considered. Decomposing the mass matrix as $m_0^{ll} = m^{ll} + \delta m^{ll}$, where m^{ll} and δm^{ll} denote the renormalized and mass counterterm matrices, and considering a bi-unitary transformation of the charged-lepton fields $l'_{L,R}$ that diagonalizes m^{ll}, the mass term in the Lagrangian density takes the form

$$-\bar{l}(m^l + \delta m^{l(-)} a_- + \delta m^{l(+)} a_+) l, \tag{20}$$

where m^l is real, diagonal, and positive, and $\delta m^{l(-)}$ and $\delta m^{l(+)}$ are arbitrary non-diagonal matrices subject to the Hermiticity condition

$$\delta m^{l(+)} = \delta m^{l(-)\dagger}. \tag{21}$$

The contribution of the mass counterterm to the external-leg amplitude is given by

$$- i\overline{u}_i(\delta m_{ij}^{l(-)} a_- + \delta m_{ij}^{l(+)} a_+)\frac{i}{\not{p} - m_j}. \tag{22}$$

We now adjust $\delta m_{ij}^{l(-)}$ and $\delta m_{ij}^{l(+)}$ to cancel, as much as possible, the sm contributions in Eq. (12). The cancellation of the UV-divergent part is achieved by choosing

$$\begin{aligned}
\left(\delta m_{\rm div}^{l(-)}\right)_{ij} &= -\frac{g^2 m_i}{64\pi^2 m_W^2}\Delta_W \left(\delta_{ij} m_i^2 - 3\sum_a B_{ia} B_{aj}^\dagger m_a^2\right), \\
\left(\delta m_{\rm div}^{l(+)}\right)_{ij} &= -\frac{g^2 m_j}{64\pi^2 m_W^2}\Delta_W \left(\delta_{ij} m_i^2 - 3\sum_a B_{ia} B_{aj}^\dagger m_a^2\right).
\end{aligned} \tag{23}$$

Note that

$$\left(\delta m_{\rm div}^{l(+)}\right)_{ij} = \left(\delta m_{\rm div}^{l(-)}\right)_{ji}^*, \tag{24}$$

so that the Hermiticity condition is fulfilled.

We call ij channel the amplitude in which i labels the outgoing on-shell charged lepton and j the virtual one. The ji channel is then the amplitude in which the roles are reversed: j is the outgoing on-shell charged lepton, while i is the virtual one. On the basis of Eq. (22), we define the mass counterterms $\delta m_{ij}^{l(\pm)}$ such that they completely cancel the sm corrections in Eq. (12) for an outgoing charged lepton in the ij channel. As a consequence, we may write

$$\begin{aligned}
\delta m_{ij}^{l(-)} = -\frac{g^2 m_i}{32\pi^2}\Bigg\{ & \delta_{ij}\left(1 + \frac{m_i^2}{2m_W^2}\Delta_W\right) \\
& + \sum_a B_{ia} B_{aj}^\dagger \left(1 + \frac{m_j^2}{2m_W^2}\right)\left[I(m_i^2, m_W, m_a) - J(m_i^2, m_W, m_a)\right] \\
& - \sum_a B_{ia} B_{aj}^\dagger \frac{m_a^2}{2m_W^2}\left[3\Delta_W + I(m_i^2, m_W, m_a) + J(m_i^2, m_W, m_a)\right]\Bigg\}, \\
\delta m_{ij}^{l(+)} = -\frac{g^2 m_j}{32\pi^2}\Bigg\{ & \delta_{ij}\left(1 + \frac{m_i^2}{2m_W^2}\Delta_W\right) \\
& + \sum_a B_{ia} B_{aj}^\dagger \left(1 + \frac{m_i^2}{2m_W^2}\right)\left[I(m_i^2, m_W, m_a) - J(m_i^2, m_W, m_a)\right] \\
& - \sum_a B_{ia} B_{aj}^\dagger \frac{m_a^2}{2m_W^2}\left[3\Delta_W + I(m_i^2, m_W, m_a) + J(m_i^2, m_W, m_a)\right]\Bigg\}.
\end{aligned} \tag{25}$$

Once $\delta m_{ij}^{l(-)}$ and $\delta m_{ij}^{l(+)}$ are fixed, the mass counterterms for the reverse ji channel are determined by the Hermiticity conditions

$$\delta m_{ji}^{l(-)} = \delta m_{ij}^{l(+)*}, \qquad \delta m_{ji}^{l(+)} = \delta m_{ij}^{l(-)*}. \tag{26}$$

We note that the functions I and J in Eq. (25) are evaluated at $p^2 = m_i^2$ in the ij channel and at $p^2 = m_j^2$ in the ji channel. As a consequence, the mass counterterms cannot completely remove the sm contributions in both amplitudes. Due to this restriction, we choose δm^l_{ii} to cancel, as is customary, all the diagonal contributions in Eq. (12), while for the non-diagonal entries, we choose δm^l_{ij} with $i < j$ to cancel the corresponding sm contributions. Once δm^l_{ij} with $i < j$ are fixed, the mass counterterms for the ji channel, *i.e.* δm^l_{ji}, are fixed by the hermiticity conditions in Eq. (21).

4.2 Majorana-neutrino mass counterterm matrix

In the weak-eigenstate basis, the bare mass matrix $m'^{0,\nu}$ for the neutrinos is symmetric and non-diagonal, and the corresponding terms in the Lagrangian density are given in Eq. (1). Decomposing $m'^{0,\nu} = m'^{\nu} + \delta m'^{\nu}$, where m'^{ν} and $\delta m'^{\nu}$ denote the renormalized and counterterm mass matrices, we envisage a unitary transformation of the Majorana-neutrino fields that diagonalizes m'^{ν}, leading to a renormalized mass matrix m^{ν} that is diagonal, real, and positive. As shown in Appendix B, this can be achieved by the following transformation:

$$W^T m'^{\nu} W = m^{\nu}, \tag{27}$$

where W is unitary. This also transforms $\delta m'^{\nu}$ into a new symmetric matrix δm^{ν}, which, in general, is non-diagonal. In the new framework, the mass term becomes

$$-\frac{1}{2}\overline{\nu}(m^{\nu} + \delta m^{\nu(-)} a_- + \delta m^{\nu(+)} a_+)\nu, \tag{28}$$

where m^{ν} is real, diagonal, and positive, and $\delta m^{\nu(-)}$ and $\delta m^{\nu(+)}$ are symmetric non-diagonal matrices subject to the constraint

$$\delta m^{\nu(+)} = \delta m^{\nu(+)T} = \delta m^{\nu(-)*} = \delta m^{\nu(-)\dagger}. \tag{29}$$

As is customary, the mass counterterms are included in the interaction Lagrangian density. Their contribution to the external-leg amplitude reads:

$$-i\overline{u}_a(p)(\delta m^{\nu(-)}_{ab} a_- + \delta m^{\nu(+)}_{ab} a_+)\frac{i}{\not{p} - m_b}. \tag{30}$$

We now adjust $\delta m^{\nu(-)}_{ab}$ and $\delta m^{\nu(+)}_{ab}$ to cancel, as much as possible, the sm contributions given in Eq. (15). The cancellation of the UV-divergent parts is achieved by choosing

$$\begin{aligned}\left(\delta m^{\nu(-)}_{\rm div}\right)_{ab} = -\frac{g^2}{64\pi^2 m_W^2}\Bigg\{ & m_a m_b (m_a C^*_{ab} + m_b C_{ab})\left(\Delta_W + \frac{1}{2}\Delta_H + \frac{1}{2}\Delta_Z\right) \\ & - 3\sum_k m_k^2 (m_a B^*_{ka} B_{kb} + m_b B_{ka} B^*_{kb})\Delta_W \\ & + \sum_c m_c^2 (m_a C_{ac} C_{cb} + m_b C^*_{ac} C^*_{cb})\left(\frac{5}{2}\Delta_H - \frac{3}{2}\Delta_Z\right)\end{aligned}$$

$$
\begin{aligned}
&+ 2\sum_c m_c^3 C_{ac}^* C_{cb} (\Delta_H - \Delta_Z) \Big\}, \\
\left(\delta m_{\rm div}^{\nu(+)}\right)_{ab} = &-\frac{g^2}{64\pi^2 m_W^2} \Big\{ m_a m_b (m_a C_{ab} + m_b C_{ab}^*) \left(\Delta_W + \frac{1}{2}\Delta_H + \frac{1}{2}\Delta_Z\right) \\
&- 3\sum_k m_k^2 (m_a B_{ka} B_{kb}^* + m_b B_{ka}^* B_{kb}) \Delta_W \\
&+ \sum_c m_c^2 (m_a C_{ac}^* C_{cb}^* + m_b C_{ac} C_{cb}) \left(\frac{5}{2}\Delta_H - \frac{3}{2}\Delta_Z\right) \\
&+ 2\sum_c m_c^3 C_{ac} C_{cb}^* (\Delta_H - \Delta_Z) \Big\}. \qquad (31)
\end{aligned}
$$

It is easy to check that

$$
\left(\delta m_{\rm div}^{\nu(+)}\right)_{ab} = \left(\delta m_{\rm div}^{\nu(+)}\right)_{ba} = \left(\delta m_{\rm div}^{\nu(-)}\right)_{ab}^* = \left(\delta m_{\rm div}^{\nu(-)}\right)_{ba}^*, \qquad (32)
$$

so that $\delta m_{\rm div}^{\nu(-)}$ and $\delta m_{\rm div}^{\nu(+)}$ satisfy the requirements in Eq. (29).

In order to discuss the cancellation of the UV-finite parts, as we did in the charged-lepton case, we call ab channel the amplitude in which a labels the outgoing on-shell Majorana neutrino and b the virtual one. In the ab channel, we define then the mass counterterms $\delta m_{ab}^{\nu(\pm)}$ such that they fully cancel the sm contributions of Eq. (15) and obtain:

$$
\begin{aligned}
-\frac{32\pi^2}{g^2}\delta m_{ab}^{\nu(-)} = &-\frac{32\pi^2}{g^2}\left(\delta m_{\rm div}^{\nu(-)}\right)_{ab} + (m_a C_{ab} + m_b C_{ab}^*)\left(1 + \frac{1}{2c_w^2}\right) \\
&- \sum_k \frac{m_k^2}{2m_W^2}\left(m_a B_{ka}^* B_{kb} + m_b B_{ka} B_{kb}^*\right) \\
&\times \left[I(m_a^2, m_W, m_k) + J(m_a^2, m_W, m_k)\right] \\
&+ \sum_k \left[\frac{m_a m_b}{2m_W^2}\left(m_a B_{ka} B_{kb}^* + m_b B_{ka}^* B_{kb}\right) + \left(m_a B_{ka}^* B_{kb} + m_b B_{ka} B_{kb}^*\right)\right] \\
&\times \left[I(m_a^2, m_W, m_k) - J(m_a^2, m_W, m_k)\right] \\
&+ \frac{1}{4m_W^2}\sum_c \Big[m_a \left(m_a C_{ac}^* + m_c C_{ac}\right)\left(m_c C_{cb} + m_b C_{cb}^*\right) \\
&+ m_b \left(m_a C_{ac} + m_c C_{ac}^*\right)\left(m_c C_{cb}^* + m_b C_{cb}\right)\Big] \\
&\times \left[I(m_a^2, m_H, m_c) - J(m_a^2, m_H, m_c)\right] \\
&+ \sum_c \frac{m_c}{2m_W^2}\left(m_a C_{ac} + m_c C_{ac}^*\right)\left(m_c C_{cb} + m_b C_{cb}^*\right) I(m_a^2, m_H, m_c) \\
&+ \frac{1}{2c_w^2}\sum_c \Big\{\left(1 - \frac{m_c^2}{2m_Z^2}\right)\left(m_a C_{ac} C_{cb} + m_b C_{ac}^* C_{cb}^*\right) \\
&+ \frac{m_a}{2m_Z^2}(m_a C_{ac}^* + m_c C_{ac})(m_b C_{cb}^* + m_c C_{cb})
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{m_b}{2m_Z^2}(m_a C_{ac} + m_c C_{ac}^*)(m_b C_{cb} + m_c C_{cb}^*)\Big\} \\
&\times \left[I(m_a^2, m_Z, m_c) - J(m_a^2, m_Z, m_c)\right] \\
&- \sum_c \frac{m_c^2}{4m_W^2}\left(m_a C_{ac} C_{cb} + m_b C_{ac}^* C_{cb}^*\right) \\
&\times \left[I(m_a^2, m_Z, m_c) + J(m_a^2, m_Z, m_c)\right] \\
&+ \sum_c \frac{m_c}{2c_w^2}\left[\left(4 - \frac{m_c^2}{m_Z^2}\right) C_{ac}^* C_{cb} - \frac{m_a m_b}{m_Z^2} C_{ac} C_{cb}^*\right] I(m_a^2, m_Z, m_c), \\
-\frac{32\pi^2}{g^2}\delta m_{ab}^{\nu(+)} &= -\frac{32\pi^2}{g^2}\left(\delta m_{\rm div}^{\nu(+)}\right)_{ab} + \left(m_a C_{ab}^* + m_b C_{ab}\right)\left(1 + \frac{1}{2c_w^2}\right) \\
&- \sum_k \frac{m_k^2}{2m_W^2}\left(m_a B_{ka} B_{kb}^* + m_b B_{ka}^* B_{kb}\right) \\
&\times \left[I(m_a^2, m_W, m_k) + J(m_a^2, m_W, m_k)\right] \\
&+ \sum_k \left[\frac{m_a m_b}{2m_W^2}\left(m_a B_{ka}^* B_{kb} + m_b B_{ka} B_{kb}^*\right) + \left(m_a B_{ka} B_{kb}^* + m_b B_{ka}^* B_{kb}\right)\right] \\
&\times \left[I(m_a^2, m_W, m_k) - J(m_a^2, m_W, m_k)\right] \\
&+ \frac{1}{4m_W^2}\sum_c \Big[m_b\left(m_a C_{ac}^* + m_c C_{ac}\right)\left(m_c C_{cb} + m_b C_{cb}^*\right) \\
&+ m_a\left(m_a C_{ac} + m_c C_{ac}^*\right)\left(m_c C_{cb}^* + m_b C_{cb}\right)\Big] \\
&\times \left[I(m_a^2, m_H, m_c) - J(m_a^2, m_H, m_c)\right] \\
&+ \sum_c \frac{m_c}{2m_W^2}\left(m_a C_{ac}^* + m_c C_{ac}\right)\left(m_c C_{cb}^* + m_b C_{cb}\right) I(m_a^2, m_H, m_c) \\
&+ \frac{1}{2c_w^2}\sum_c \Bigg\{\left(1 - \frac{m_c^2}{2m_Z^2}\right)\left(m_b C_{ac} C_{cb} + m_a C_{ac}^* C_{cb}^*\right) \\
&+ \frac{m_b}{2m_Z^2}(m_a C_{ac}^* + m_c C_{ac})(m_b C_{cb}^* + m_c C_{cb}) \\
&+ \frac{m_a}{2m_Z^2}(m_a C_{ac} + m_c C_{ac}^*)(m_b C_{cb} + m_c C_{cb}^*)\Bigg\} \\
&\times \left[I(m_a^2, m_Z, m_c) - J(m_a^2, m_Z, m_c)\right] \\
&- \sum_c \frac{m_c^2}{4m_W^2}\left(m_b C_{ac} C_{cb} + m_a C_{ac}^* C_{cb}^*\right) \\
&\times \left[I(m_a^2, m_Z, m_c) + J(m_a^2, m_Z, m_c)\right] \\
&+ \sum_c \frac{m_c}{2c_w^2}\left[\left(4 - \frac{m_c^2}{m_Z^2}\right) C_{ac} C_{cb}^* - \frac{m_a m_b}{m_Z^2} C_{ac}^* C_{cb}\right] I(m_a^2, m_Z, m_c). \qquad (33)
\end{aligned}
$$

Also here, the functions I and J are evaluated at $p^2 = m_a^2$ in the ab channel and at $p^2 = m_b^2$ in the ba channel. Therefore, the mass counterterms in Eq. (33) cannot

completely remove the sm contributions of Eq. (15) in both channels. We then choose δm^{ν}_{aa} to cancel all the diagonal contributions in Eq. (15) and δm^{ν}_{ab} with $a < b$ to fully cancel the corresponding sm contributions. Once δm^{ν}_{ab} with $a < b$ are fixed, the mass counterterms for the ba channel, *i.e.* δm^{ν}_{ba}, are determined by the conditions in Eq. (29).

5 Renormalization of mixing matrices

In the previous section, we have shown how one can define mass counterterms on the basis of the sm contributions calculated in Section 3. In particular, in both charged-lepton and Majorana-neutrino cases, the UV-divergent parts in the sm contributions of Eqs. (12) and (15) are completely canceled by the mass counterterms. In addition, also UV-finite parts get canceled, up to the Hermiticity conditions (21) and (29). We wish to emphasize that the mass counterterms constructed in this way are explicitly gauge independent.

In what follows, we proceed with the diagonalization of the complete mass matrices, which include the renormalized and counterterm mass matrices. Similar to the quark case [4], this procedure leads to mixing matrix counterterms which automatically satisfy the basic properties (8)–(10) and are gauge independent. As before, we first discuss the case of charged leptons followed by that of Majorana neutrinos.

The renormalized fermion masses thus resulting are the familiar on-shell masses, which coincide with the pole masses [11] to the order of our calculation.

5.1 Diagonalization of the charged-lepton mass matrix

Following Ref. [4], we consider a bi-unitary transformation that diagonalizes the complete charged-lepton mass matrix in Eq. (20) through terms of $\mathcal{O}(g^2)$. Calling U^l_L and U^l_R the unitary matrices in this transformation and writing

$$U^l_L = 1 + ih^l_L, \qquad U^l_R = 1 + ih^l_R, \tag{34}$$

where h^l_L and h^l_R are hermitian matrices of $\mathcal{O}(g^2)$, one finds that the off-diagonal elements $(i \neq j)$ are given by

$$\begin{aligned} i(h^l_L)_{ij} &= -\frac{m^l_i \delta m^{l(-)}_{ij} + \delta m^{l(+)}_{ij} m^l_j}{(m^l_i)^2 - (m^l_j)^2}, \\ i(h^l_R)_{ij} &= -\frac{m^l_i \delta m^{l(+)}_{ij} + \delta m^{l(-)}_{ij} m^l_j}{(m^l_i)^2 - (m^l_j)^2}, \end{aligned} \tag{35}$$

while the diagonal elements can be chosen to vanish, namely $(h^l_{L,R})_{ii} = 0$. As shown in Appendix B in Ref. [4], the alternative choice $(h^l_{L,R})_{ii} \neq 0$ has no physical effect on the $Wl\nu$ interactions through $\mathcal{O}(g^2)$.

5.2 Diagonalization of the Majorana-neutrino mass matrix

The situation in the case of Majorana neutrinos is similar to the one of the charged leptons, except that now one needs only one unitary matrix for the diagonalization of the complete mass matrix. Writing

$$U^\nu = 1 + ih^\nu, \tag{36}$$

where h^ν is again a Hermitian matrix of $\mathcal{O}(g^2)$, one finds that the off-diagonal elements $(a \neq b)$ are given by

$$i(h^\nu)_{ab} = -\frac{m_a^\nu \delta m_{ab}^{\nu(-)} + \delta m_{ab}^{\nu(+)} m_b^\nu}{(m_a^\nu)^2 - (m_b^\nu)^2}, \tag{37}$$

and, in analogy with the charged-lepton case, we choose $(h^\nu)_{aa} = 0$.

5.3 Mixing counterterm matrices

We analyze next the effect of the transformations of Eqs. (34)–(37) on the $Wl\nu$ coupling in Eq. (6). Performing the above transformations, we find through terms of $\mathcal{O}(g^2)$ that

$$\mathcal{L}_W = -\frac{g}{\sqrt{2}} W_\mu^- \bar{l}(B + \delta B)\gamma^\mu a_- \nu + \text{h.c.}, \tag{38}$$

where

$$\delta B = i(Bh^\nu - h_L^l B). \tag{39}$$

It is easy to verify that both the renormalized and bare mixing matrices satisfy the first condition in Eq. (8) while, due to the second condition, once δB is fixed, δC is fixed as well, leading to

$$\delta C = i(Ch^\nu - h^\nu C). \tag{40}$$

One can further check that all the other conditions in Eqs. (9) and (10) for the two mixing matrices are satisfied. Of course, all the equalities hold through the order of the calculation, namely $\mathcal{O}(g^2)$.

For completeness, we give the two counterterm matrices in component form:

$$\begin{aligned}
\delta B_{ia} &= i\left[\sum_b B_{ib}\,(h^\nu)_{ba} - \sum_j \left(h_L^l\right)_{ij} B_{ja}\right] \\
&= -\sum_b B_{ib} \frac{m_b^\nu \delta m_{ba}^{\nu(-)} + \delta m_{ba}^{\nu(+)} m_a^\nu}{(m_b^\nu)^2 - (m_a^\nu)^2} + \sum_j \frac{m_i^l \delta m_{ij}^{l(-)} + \delta m_{ij}^{l(+)} m_j^l}{(m_i^l)^2 - (m_j^l)^2} B_{ja}, \qquad (41) \\
\delta C_{ab} &= i\sum_c \left[C_{ac}\,(h^\nu)_{cb} - (h^\nu)_{ac}\,C_{cb}\right] \\
&= -\sum_c C_{ac} \frac{m_c^\nu \delta m_{cb}^{\nu(-)} + \delta m_{cb}^{\nu(+)} m_b^\nu}{(m_c^\nu)^2 - (m_b^\nu)^2} + \sum_c \frac{m_a^\nu \delta m_{ac}^{\nu(-)} + \delta m_{ac}^{\nu(+)} m_c^\nu}{(m_a^\nu)^2 - (m_c^\nu)^2} C_{cb}, \qquad (42)
\end{aligned}$$

where $\delta m_{ij}^{l(\pm)}$ and $\delta m_{ab}^{\nu(\pm)}$ are the off-diagonal mass counterterms determined in Sections 4.1 and 4.2, respectively, and it is understood that $b \neq a$ in the first and $j \neq i$ in the second term of Eq. (41) and that $c \neq b$ in the first and $c \neq a$ in the second term of Eq. (42).

Since the mass counterterms are adjusted to cancel the off-diagonal sm contributions to the extent allowed by the properties of the mass matrices, the same is true of the mixing counterterm matrices δB and δC. In particular, they fully cancel the UV-divergent parts of the off-diagonal sm contributions.

6 Conclusions

In this paper, we have generalized the on-shell framework to renormalize the CKM matrix at the one-loop level proposed in Ref. [4] to extensions of the SM that include Majorana neutrinos, an appealing scenario that may explain the smallness of the observed neutrino masses and may lead to neutrino-less double beta decays. The presence of Majorana neutrinos requires a separate analysis, due to modified interactions and symmetry factors leading to a generically different set of Feynman rules. Here, the mixing generally also occurs in neutral-current interactions. However, once the Feynman rules are established, the procedure is similar to the case of the CKM matrix.

We showed how gauge-independent mass counterterms can be fixed by means of the sm contributions and how they lead to mixing counterterm matrices. We gave explicit expressions for δB and δC. They are consistent with the properties satisfied by the two mixing matrices and are explicitly gauge independent. We saw that once δB is fixed, δC is fixed as well, as a consequence of the second property in Eq. (8). However, one could also choose to fix the δC counterterm separately, *e.g.* by choosing to study the $Z\nu\nu$ coupling, with the same result.

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A Definitions

In this appendix, we gather important definitions used throughout this work. The UV divergences which appear in the expressions of the sm and wfr contributions and later in

the mass and mixing counterterm matrices are defined by

$$\Delta_B = \frac{1}{n-4} + \frac{1}{2}\left(\gamma_E - \ln 4\pi\right) + \ln\frac{m_B}{\mu} \equiv \Delta + \ln\frac{m_B}{\mu}, \tag{43}$$

where n is the space-time dimension, γ_E the Euler's constant, μ is the 't Hooft mass scale, and m_B is the mass of boson $B = W, Z, H$.

The functions I, J, N, and M are defined through the integrals:

$$\begin{aligned}
\{I;J\}(p^2,m_1,m_2) &= \int_0^1 \mathrm{d}x\{1;x\}\ln\frac{m_2^2x + m_1^2(1-x) - p^2x(1-x) - i\varepsilon}{m_1^2},\\
N(m_1,m_2,m_3,\xi_1) &= \frac{1}{m_1^2}\int_0^1 \mathrm{d}x\left[m_2^2(1-x)+m_3^2\right]\\
&\quad\times\ln\frac{m_3^2x + m_1^2\xi_1(1-x) - m_2^2x(1-x) - i\varepsilon}{m_3^2x + m_1^2(1-x) - m_2^2x(1-x) - i\varepsilon},\\
M(m_1,m_2,m_3,\xi_1) &= \frac{1}{m_1^2}\int_0^1 \mathrm{d}x\,x\ln\frac{m_3^2x + m_1^2\xi_1(1-x) - m_2^2x(1-x) - i\varepsilon}{m_3^2x + m_1^2(1-x) - m_2^2x(1-x) - i\varepsilon}.
\end{aligned} \tag{44}$$

The above integrals are not all independent. In fact, the integrals J, N, and M can be expressed by means of I as

$$\begin{aligned}
J(p^2,m_1,m_2) &= \frac{1}{2p^2}\left[-m_2^2 + m_1^2 + m_2^2\ln\frac{m_2^2}{m_1^2} + (p^2 - m_2^2 + m_1^2)\,I(p^2,m_1,m_2)\right],\\
M(m_1,m_2,m_3,\xi_1) &= \frac{1}{m_1^2}\left[J(m_2^2,m_1\sqrt{\xi}_1,m_3) - J(m_2^2,m_1,m_3) + \ln\xi_1\right],\\
N(m_1,m_2,m_3,\xi_1) &= \frac{m_2^2+m_3^2}{m_1^2}\left[I(m_2^2,m_1\sqrt{\xi}_1,m_3) - I(m_2^2,m_1,m_3) + \ln\xi_1\right]\\
&\quad - m_2^2\,M(m_1,m_2,m_3,\xi_1).
\end{aligned} \tag{45}$$

Note that these integrals represent UV-finite parts of the standard scalar one-loop integrals [12]. In fact, we have

$$I(p^2,m_1,m_2) = -2\Delta_1 - B_0(p^2,m_1,m_2), \tag{46}$$

where Δ_1 is defined by Eq. (43) and B_0 is defined as in Ref. [13]. According to Eq. (45), the integrals J, N, and M can be written in terms of the scalar two-point function B_0 as well.

B Majorana-neutrino mass matrix diagonalization

According to the singular-value decomposition theorem, any complex matrix M' can be diagonalized by a bi-unitary transformation of the form

$$M = S^\dagger M' U, \tag{47}$$

where S and U are unitary matrices and M is real and diagonal with non-negative eigenvalues. In the proof of Eq. (47), S is chosen such that

$$S^\dagger M' M'^\dagger S = M^2. \tag{48}$$

For a symmetric matrix M', this becomes:

$$S^\dagger M' M'^* S = M^2. \tag{49}$$

We now take the hermitian adjoint of Eq. (47),

$$U^\dagger M'^\dagger S = M, \tag{50}$$

multiply Eq. (47) on the left by Eq. (50), and take the complex conjugate,

$$U^T M' M'^* U^* = M^2. \tag{51}$$

Comparing Eqs. (49) and (51), we see that we can identify

$$S = U^*, \qquad S^\dagger = U^T. \tag{52}$$

Inserting Eq. (52) in Eq. (47) we obtain

$$M = U^T M' U, \tag{53}$$

which tells us that, for the diagonalization of a complex symmetric matrix, one needs only one unitary transformation [14].

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