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A complete reduction of one-loop tensor 5- and 6-point integrals

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ABSTRACT: We perform a complete analytical reduction of general one-loop Feynman integrals with five and six external legs for tensors up to rank $R = 3$ and 4, respectively. An elegant formalism with extensive use of signed minors is developed for the cancellation of inverse Gram determinants. The 6-point tensor functions of rank R are expressed in terms of 5-point tensor functions of rank $R - 1$, and the latter are reduced to scalar four-, three-, and two-point functions. The resulting compact formulae allow both for a study of analytical properties and for efficient numerical programming. They are implemented in Fortran and Mathematica.

KEYWORDS: NLO Computations, QCD, QED, Feynman Integrals.

Contents

1. Introduction **2**

2. Representing tensor integrals by scalar integrals in shifted space-time dimensions **3**

3. Pentagons **5**
- 3.1 Scalar and vector integrals 5
- 3.2 $R = 2$ tensor integrals 6
- 3.3 $R = 3$ tensor integrals 7

4. Hexagons **14**
- 4.1 Scalar and vector integrals 14
- 4.2 $R = 2$ tensor integrals 15
- 4.3 $R = 3$ tensor integrals 16
- 4.4 $R = 4$ tensor integrals 18

5. Numerical results and discussion **22**

A. Gram determinants and algebra of signed minors **32**

B. Reduction of dimensionally shifted five- and four-point integrals **35**

References **37**

1. Introduction

At the proton-proton collider LHC and the planned e^+e^- collider ILC, a large number of particles per event may be produced. The hope is to discover one or several Higgs bosons or supersymmetric particles, which are typically expected to be quite heavy. The interest is also directed to the study of known massive particles like the W and Z bosons or the top quark. Since the production rates are large, a proper description of the cross-sections will typically include one-loop corrections to n-particle reactions, where some of the final state particles may be massive.

The Feynman integrals for reactions with up to four external particles have been systematically studied and evaluated in numerous studies. We just want to mention here the seminal papers [1] and [2] and the Fortran packages `FF` [3] and `LoopTools` [4], which represent the state of the art until now. The treatment of Feynman integrals with a higher multiplicity than four becomes quite involved if questions of efficiency and stability become vital, as it happens with the calculational problems related to high-dimensional phase space integrals over sums of thousands of Feynman diagrams with internal loops.

In this article, we will concentrate on the evaluation of massive one-loop Feynman integrals with n external legs and some tensor structure,

$$I_n^{\mu_1\cdots\mu_R} = \int \frac{d^dk}{i\pi^{d/2}} \frac{\prod_{r=1}^R k^{\mu_r}}{\prod_{j=1}^n c_j^{\nu_j}}, \tag{1.1}$$

where the denominators c_j have *indices* ν_j and *chords* q_j,

$$c_j = (k-q_j)^2 - m_j^2 + i\varepsilon. \tag{1.2}$$

We will study in the following the cases $n = 5$ with $R \leq 3$ and $n = 6$ with $R \leq 4$, and we will conventionally assume $q_n = 0$. The space-time dimension is $d = 4 - 2\epsilon$.

There are several strategies one might follow. One is the reduction of higher-point tensor integrals to tensor integrals with less external lines and/or lower tensor rank [5, 6, 7, 8]; a second approach is essentially numerical [9, 10] or semi-numerical [11, 12, 13]. A third one rests on the unitarity cut method [14, 15, 16, 17]. In this case, a one-loop amplitude is evaluated as a whole, by using Cutkosky rules, instead of computing loop integrals from each of the Feynman diagrams. It is impossible to give here a comprehensive survey of recent activities, and we would like to refer to e.g. [18, 19, 20, 21] for recent overviews on the subject.

Here, we will advocate yet another approach and reduce the tensor integrals algebraically to sums over a small set of scalar two-, three- and four-point functions, which we assume to be known. Whether such a complete reduction is competitive with the other approaches might be disputed. Evidently, this depends on the specific problem under investigation. For a study of gauge invariance and of the ultraviolet (UV) and infrared (IR) singularity structure of a set of Feynman diagrams, it is evident that a complete reduction is advantageous, and it may also be quite useful for a tuned, analytical study of certain regions of potential numerical instabilities.

We have chosen a strictly algebraic approach and will rely heavily on the algebra of signed minors which was worked out in detail by Melrose in [22]. One of the basic observations of Melrose was that in four dimensions all the scalar integrals can be reduced to scalar 4-point functions and simpler ones. In [23], a representation of arbitrary one-loop tensor integrals in terms

of scalar integrals was derived. The representation includes, however, scalar integrals with higher indices ν_j and higher space-time dimensions $d + 2l$. The subsequent reduction to scalar integrals with only the original indices and the generic space-time dimension d is possible with the use of integration-by-parts identities [24] and generalizations of them with dimensional shifts. The latter have been derived in [25], and a systematical application to one-loop integrals may be found in [26].[1] Basically, the reduction problem has been solved this way for n-point functions. There was one attempt to use the Davydychev-Tarasov reduction for the description of one-loop contributions to the process $e^+e^- \to H\nu\bar{\nu}$ [27], and the numerical problems due to the five-point functions were discussed in some detail. To a large extent they root in the appearance of inverse powers of Gram determinants. This feature of the Davydychev-Tarasov reduction was identified as disadvantageous soon after its derivation, e.g. in [28], where a strategy for avoiding these problems was developed. Besides the problem of inverse powers of the Gram determinant of the corresponding Feynman diagram, there are additional kinematical singularities related to sub-diagrams. This will not be discussed here; we refer to e.g. [28, 29, 5, 6, 7, 8, 13, 17] and references therein.

In this article, we investigate the reduction of tensor integrals with five and six external legs which are of immediate importance in applications at the LHC. In Section 2 we represent tensor integrals by scalar integrals in shifted space-time dimensions with shifted indices. Section 3 and Section 4 contain our main result. In Section 3 we go one step further in the reduction of five-point tensors compared to [26] and demonstrate how to cancel all inverse powers of the Gram determinant appearing in the Davydychev-Tarasov reduction. Earlier results for tensors of rank two may be found in [30]. Section 4 contains the reduction of tensorial six-point functions to tensorial 5-point functions. The corresponding Gram determinant is identically zero [26, 6, 8], and the reduction becomes quite compact. Some numerical results and a short discussion are given in Section 5. The numerics is obtained with two independent implementations, one made in Mathematica, and another one in Fortran. The Mathematica program `hexagon.m` with the reduction formulae is made publicly available [31], see also [32] for a short description. For numerical applications, one has to link the package with a program for the evaluation of scalar one- to four-point functions, e.g. with `LoopTools` [4, 33, 3], `CutTools` [34, 12], `QCDLoop` [35]. Appendices are devoted to some known, but necessary details on Gram determinants and the algebra of signed minors and to a short summary about the reduction of dimensionally shifted four- and five-point integrals.

2. Representing tensor integrals by scalar integrals in shifted space-time dimensions

At first we give the reduction of tensor integrals to a set of scalar integrals for arbitrary n-point functions. Following [23, 26], assuming here the indices of propagators to be equal to one, $\nu_r = 1$, one has:

$$\begin{aligned} I_n^\mu &= \int^d k^\mu \prod_{r=1}^{n} c_r^{-1} \\ &= -\sum_{i=1}^{n-1} q_i^\mu \, I_{n,i}^{[d+]}, \end{aligned} \tag{2.1}$$

[1] We will extensively quote from article [26], so we introduce here the notation (I.num) for a reference to equation (num) there.

$$
\begin{aligned}
I_n^{\mu\,\nu} &= \int^d k^\mu\, k^\nu \prod_{r=1}^{n} c_r^{-1} \\
&= \sum_{i,j=1}^{n-1} q_i^\mu\, q_j^\nu\, n_{ij}\; I_{n,ij}^{[d+]^2} - \frac{1}{2}\, g^{\mu\nu}\, I_n^{[d+]}\,, \qquad (2.2)\\
I_n^{\mu\,\nu\,\lambda} &= \int^d k^\mu\, k^\nu\, k^\lambda \prod_{r=1}^{n} c_r^{-1} \\
&= -\sum_{i,j,k=1}^{n-1} q_i^\mu\, q_j^\nu\, q_k^\lambda\, n_{ijk}\; I_{n,ijk}^{[d+]^3} + \frac{1}{2}\sum_{i=1}^{n-1}(\, g^{\mu\nu}\, q_i^\lambda\, + g^{\mu\lambda}\, q_i^\nu\, + g^{\nu\lambda}\, q_i^\mu\,) I_{n,i}^{[d+]^2}\,, \qquad (2.3)\\
I_n^{\mu\nu\lambda\rho} &= \int^d k^\mu k^\nu k^\lambda k^\rho \prod_{r=1}^{n} c_r^{-1} \\
&= \sum_{i,j,k,l=1}^{n-1} q_i^\mu\, q_j^\nu\, q_k^\lambda\, q_l^\rho\, n_{ijkl}\; I_{n,ijkl}^{[d+]^4} \\
&\quad - \frac{1}{2}\sum_{i,j=1}^{n-1}(g^{\mu\nu}\, q_i^\lambda q_j^\rho + g^{\mu\lambda} q_i^\nu q_j^\rho + g^{\nu\lambda} q_i^\mu q_j^\rho + g^{\mu\rho} q_i^\nu q_j^\lambda + g^{\nu\rho} q_i^\mu q_j^\lambda + g^{\lambda\rho} q_i^\mu q_j^\nu) n_{ij} I_{n,ij}^{[d+]^3} \\
&\quad + \frac{1}{4}\left(\, g^{\mu\nu}\, g^{\lambda\rho}\, + g^{\mu\lambda}\, g^{\nu\rho}\, + g^{\mu\rho}\, g^{\nu\lambda}\,\right) I_n^{[d+]^2}\,, \qquad (2.4)
\end{aligned}
$$

where $[d+]$ is an operator shifting the space-time dimension by two units and

$$
\begin{aligned}
I_{p,\,i\,j\,k\cdots}^{[d+]^l,stu\cdots} &= \int^{[d+]^l} \prod_{r=1}^{n} \frac{1}{c_r^{1+\delta_{ri}+\delta_{rj}+\delta_{rk}+\cdots-\delta_{rs}-\delta_{rt}-\delta_{ru}-\cdots}}, \\
\int^d &\equiv \int \frac{d^d k}{i\pi^{d/2}}, \qquad (2.5)
\end{aligned}
$$

where $[d+]^l = 4 + 2l - 2\epsilon$ (observe that p is the number of scalar propagators of the "p-point function" and that equal lower and upper indices cancel, $p \le n$). In (2.2–2.4), the coefficients n_{ij}, n_{ijk} and n_{ijkl} were introduced. These stand for the product of factorials of the number of equal indices: e.g. $n_{iiii} = 4!, n_{ijii} = 3!, n_{iijj} = 2!2!, n_{ijkk} = 2!, n_{ijkl} = 1!$ (indices i, j, k, l all different from each other). Of particular relevance are the following relations for the successive application of recurrence relations to reduce higher dimensional integrals:

$$
\begin{aligned}
n_{ij} &= \nu_{ij}, \\
n_{ijk} &= \nu_{ij}\nu_{ijk}, \\
n_{ijkl} &= \nu_{ij}\nu_{ijk}\nu_{ijkl}, \qquad (2.6)
\end{aligned}
$$

where

$$
\begin{aligned}
\nu_{ij} &= 1 + \delta_{ij}, \\
\nu_{ijk} &= 1 + \delta_{ik} + \delta_{jk}, \\
\nu_{ijkl} &= 1 + \delta_{il} + \delta_{jl} + \delta_{kl}. \qquad (2.7)
\end{aligned}
$$

In the next step the integrals in higher dimension have to be reduced to integrals in generic dimension. Here particular attention has to be paid to $I_5^{[d+]}$. Reducing the tensor integrals, this term drops out in general [36, 7].

3. Pentagons

We start with the reduction of the pentagons. This will also provide the basis for calculating the hexagons as we shall see.

3.1 Scalar and vector integrals

For the **scalar** 5-point function the recursion relation (I.31) reads for $n = 5$

$$(d-4)\left(\ \right)_5 I_5^{[d+]} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}_5 I_5 - \sum_{s=1}^{5} \begin{pmatrix} 0 \\ s \end{pmatrix}_5 I_4^s \tag{3.1}$$

With $I_5^{[d+]}$ finite for $d = 4$, we have in this limit

$$E \equiv I_5 = \frac{1}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}_5} \sum_{s=1}^{5} \begin{pmatrix} 0 \\ s \end{pmatrix}_5 I_4^s, \tag{3.2}$$

i.e. the scalar five-point function is expressed in the limit $d \to 4$ in terms of scalar four-point functions, which are obtained by scratching in the five terms of the sum the s^{th} scalar propagator, respectively. This was already derived in [22], see eq. (6.1) there.

Similarly, for the tensor integral of rank $R = 1$ (vector) in (2.1) we obtain:

$$I_5^\mu = \sum_{i=1}^{4} q_i^\mu I_{5,i}, \tag{3.3}$$

with

$$\begin{aligned} I_{5,i} \equiv E_i &= -I_{5,i}^{[d+]} \\ &= (d-4)\frac{\begin{pmatrix} 0 \\ i \end{pmatrix}_5}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}_5} I_5^{[d+]} - \frac{1}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}_5} \sum_{s=1}^{5} \begin{pmatrix} 0 & i \\ 0 & s \end{pmatrix}_5 I_4^s, \end{aligned} \tag{3.4}$$

where again in the limit $d \to 4$ the $I_5^{[d+]}$ disappears.[2] These two cases are simple and lead to a direct reduction to scalar integrals, without the Gram determinant $()_5$ occurring anyway. In the following we want to reduce tensor integrals of higher rank and show, like in [5, 8], that also in these cases the Gram determinant can be cancelled.

[2]The $I_{5,i}$ should not be confused with quantities introduced in Equation (2.5).

3.2 $R = 2$ tensor integrals

The tensor integral of **rank 2** can be written without a $g_{\mu\nu}$-term:

$$I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} q_i^\mu\, q_j^\nu I_{5,ij}, \tag{3.5}$$

which is obtained by replacing $g_{\mu\nu}$ by

$$g^{\mu\,\nu} = 2\sum_{i,j=1}^{4} \frac{\begin{pmatrix} i \\ j \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5}\, q_i^\mu\, q_j^\nu \tag{3.6}$$

(assuming $q_1 \cdots q_4$ 4-dimensional and independent) and further by reducing the integrals in (2.2) to generic dimension. This applies in the same manner also for the tensor integrals of higher rank. Reducing the integrals of highest dimension, recursion relation (I.30) is used. For the 5-point function several cases have been worked out in : (I.41), (I.42) and (I.43). For completeness we give in the Appendix explicitly the cases needed in the present work.

One remark is needed concerning the integral $I_5^{[d+]}$. It is known [36, 7] that it always cancels in the end. This provides a very useful check on our calculations, which we have performed in every particular case under consideration. Anticipating this cancellation, we will, for the ease of our discussion, drop terms proportional to $I_5^{[d+]}$ wherever they appear in the following derivation. With this in mind we can write for $I_{5,ij}$ in (2.2) with (B.2):

$$\begin{aligned} I_{5,ij} &= \nu_{ij} I_{5,ij}^{[d+]^2} \\ &= -\frac{\begin{pmatrix} 0 \\ j \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} I_{5,i}^{[d+]} + \sum_{s=1,s\neq i}^{5} \frac{\begin{pmatrix} s \\ j \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} I_{4,i}^{[d+],s} \\ &= \frac{\begin{pmatrix} 0 \\ j \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} I_{5,i} + \sum_{s=1,s\neq i}^{5} \frac{\begin{pmatrix} s \\ j \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} I_{4,i}^{[d+],s}, \end{aligned} \tag{3.7}$$

and by means of (B.3) we obtain:

$$\begin{aligned} I_{5,ij} = \; & \frac{1}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}_5 \begin{pmatrix} \\ \end{pmatrix}_5} \sum_{s=1,s\neq i}^{5} \frac{1}{\begin{pmatrix} s \\ s \end{pmatrix}_5} \left\{ -\begin{pmatrix} 0 \\ j \end{pmatrix}_5 \begin{pmatrix} 0 & s \\ 0 & i \end{pmatrix}_5 \begin{pmatrix} s \\ s \end{pmatrix}_5 - \begin{pmatrix} s \\ j \end{pmatrix}_5 \begin{pmatrix} 0 & s \\ i & s \end{pmatrix}_5 \begin{pmatrix} 0 \\ 0 \end{pmatrix}_5 \right. \\ & \left. + \begin{pmatrix} s \\ 0 \end{pmatrix}_5 \begin{pmatrix} 0 & s \\ 0 & s \end{pmatrix}_5 \begin{pmatrix} i \\ j \end{pmatrix}_5 \right\} I_4^s - \frac{\begin{pmatrix} i \\ j \end{pmatrix}_5}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}_5 \begin{pmatrix} \\ \end{pmatrix}_5} \sum_{s=1,s\neq i}^{5} \frac{1}{\begin{pmatrix} s \\ s \end{pmatrix}_5} \begin{pmatrix} s \\ 0 \end{pmatrix}_5 \begin{pmatrix} 0 & s \\ 0 & s \end{pmatrix}_5 I_4^s \end{aligned}$$

$$- \frac{1}{\binom{0}{0}_5 \binom{}{}_5} \sum_{s,t=1,s\neq i,t}^{5} \frac{1}{\binom{s}{s}_5} \left\{ -\binom{0}{j}_5 \binom{t\ s}{0\ i}_5 \binom{s}{s}_5 - \binom{s}{j}_5 \binom{t\ s}{i\ s}_5 \binom{0}{0}_5 \right.$$

$$\left. + \binom{s}{0}_5 \binom{t\ s}{0\ s}_5 \binom{i}{j}_5 \right\} I_3^{st} + \frac{\binom{i}{j}_5}{\binom{0}{0}_5 \binom{}{}_5} \sum_{s,t=1,s\neq i,t}^{5} \frac{1}{\binom{s}{s}_5} \binom{s}{0}_5 \binom{t\ s}{0\ s}_5 I_3^{st}. \quad (3.8)$$

Using (3.6) again, we find

$$I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} q_i^\mu\, q_j^\nu E_{ij} + g^{\mu\nu} E_{00}, \quad (3.9)$$

$$E_{ij} = \sum_{s=1}^{5} S_{ij}^{4,s} I_4^s + \sum_{s,t=1}^{5} S_{ij}^{3,st} I_3^{st}, \quad (3.10)$$

where

$$S_{ij}^{4,s} = \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \frac{1}{\binom{s}{s}_5} X_{ij}^{s0}, \quad (3.11)$$

$$S_{ij}^{3,st} = -\frac{1}{\binom{0}{0}_5} \sum_{s,t=1}^{5} \frac{1}{\binom{s}{s}_5} X_{ij}^{st} \quad (3.12)$$

and X_{ij}^{s0} and X_{ij}^{st} are defined in (A.22). Finally,

$$E_{00} = -\frac{1}{2} \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \frac{\binom{s}{0}_5}{\binom{s}{s}_5} \left[\binom{0\ s}{0\ s}_5 I_4^s - \sum_{t=1}^{5} \binom{t\ s}{0\ s}_5 I_3^{st} \right]. \quad (3.13)$$

In this way we have cancelled the Gram determinant for the tensor of rank 2. For later reference, we note that, by taking into account (B.4), we can also write

$$E_{00} = -\frac{1}{2} \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \binom{s}{0}_5 I_4^{[d+],s}. \quad (3.14)$$

3.3 $R = 3$ tensor integrals

The tensor integral of **rank 3** can be written as:

$$I_5^{\mu\,\nu\,\lambda} = \sum_{i,j,k=1}^{4} q_i^\mu\, q_j^\nu\, q_k^\lambda I_{5,ijk}. \quad (3.15)$$

We will now rewrite this into another representation, thereby avoiding Gram determinants $()_5$ in the denominators of the new tensor coefficients E_{ijk}, E_{00k}:

$$I_5^{\mu\,\nu\,\lambda} = \sum_{i,j,k=1}^{4} q_i^\mu\, q_j^\nu\, q_k^\lambda E_{ijk} + \sum_{k=1}^{4} g^{[\mu\nu} q_k^{\lambda]} E_{00k}, \tag{3.16}$$

$$E_{ijk} = \sum_{s=1}^{5} S_{ijk}^{4,s} I_4^s + \sum_{s,t=1}^{5} S_{ijk}^{3,st} I_3^{st} + \sum_{s,t,u=1}^{5} S_{ijk}^{2,stu} I_2^{stu}. \tag{3.17}$$

According to (2.3) we have with (3.6):

$$I_{5,ijk} = -\,\nu_{ij}\nu_{ijk} I_{5,ijk}^{[d+]^3} + \frac{\begin{pmatrix} j \\ k \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} I_{5,i}^{[d+]^2} + \frac{\begin{pmatrix} i \\ k \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} I_{5,j}^{[d+]^2} + \frac{\begin{pmatrix} i \\ j \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} I_{5,k}^{[d+]^2}. \tag{3.18}$$

By means of recursion (B.1), taking into account (3.7) and keeping in mind to drop $I_5^{[d+]}$, we have:

$$\begin{aligned} I_{5,ijk} &= \frac{\begin{pmatrix} 0 \\ k \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} \nu_{ij} I_{5,ij}^{[d+]^2} - \sum_{s=1, s\neq i,j}^{5} \frac{\begin{pmatrix} s \\ k \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} \nu_{ij} I_{4,ij}^{[d+]^2,s} + \frac{\begin{pmatrix} i \\ j \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} \sum_{s=1}^{5} \frac{\begin{pmatrix} s \\ k \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} I_4^{[d+],s} \\ &= \frac{\begin{pmatrix} 0 \\ k \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} I_{5,ij} + \frac{\begin{pmatrix} i \\ j \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} \sum_{s=1}^{5} \frac{\begin{pmatrix} s \\ k \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} I_4^{[d+],s} - \sum_{s=1, s\neq i,j}^{5} \frac{\begin{pmatrix} s \\ k \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} \nu_{ij} I_{4,ij}^{[d+]^2,s}. \end{aligned} \tag{3.19}$$

Collecting the terms proportional to $()_5^{-2}$ we have with $I_{5,ij} = \cdots + 2\dfrac{\begin{pmatrix} i \\ j \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} E_{00}$ and (3.14):

$$\frac{\begin{pmatrix} i \\ j \end{pmatrix}_5}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}_5} \frac{1}{\begin{pmatrix} \\ \end{pmatrix}_5^2} \sum_{s=1}^{5} \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}_5 \begin{pmatrix} s \\ k \end{pmatrix}_5 - \begin{pmatrix} 0 \\ k \end{pmatrix}_5 \begin{pmatrix} s \\ 0 \end{pmatrix}_5 \right] I_4^{[d+],s} = \frac{\begin{pmatrix} i \\ j \end{pmatrix}_5}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}_5} \frac{1}{\begin{pmatrix} \\ \end{pmatrix}_5} \sum_{s=1}^{5} \begin{pmatrix} 0 & s \\ 0 & k \end{pmatrix}_5 I_4^{[d+],s}, \tag{3.20}$$

i.e. we have already cancelled one Gram determinant. We multiply (3.19) by $\begin{pmatrix} 0 \\ 0 \end{pmatrix}_5$ such that we can make use of

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}_5 \begin{pmatrix} s \\ k \end{pmatrix}_5 = \begin{pmatrix} 0 & s \\ 0 & k \end{pmatrix}_5 \begin{pmatrix} \\ \end{pmatrix}_5 + \begin{pmatrix} s \\ 0 \end{pmatrix}_5 \begin{pmatrix} 0 \\ k \end{pmatrix}_5, \tag{3.21}$$

which will give us another factor $()_5$. Adding all contributions, we obtain

$$
\begin{aligned}
3\binom{0}{0}_5 I_{5,ijk} \;=\; & \sum{}' \frac{1}{\left(\right)_5 \binom{s}{s}_5^2} \Bigg\{ \binom{0}{k}_5 \binom{s}{s}_5 \left[X^s_{ij} I^s_4 - X^{st}_{ij} I^{st}_3\right] \\
& + \binom{i}{j}_5 \binom{s}{s}_5 \binom{0\ s}{0\ k}_5 \left[\binom{0\ s}{0\ s}_5 I^s_4 - \binom{t\ s}{0\ s}_5 I^{st}_3 \right] \\
& - \binom{0}{0}_5 \binom{s}{k}_5 \Bigg(\left[\binom{0\ s}{i\ s}_5 \binom{0\ s}{j\ s}_5 + \binom{i\ s}{j\ s}_5 \binom{0\ s}{0\ s}_5 \right] I^s_4 \\
& - \left[\binom{0\ s}{j\ s}_5 \binom{t\ s}{i\ s}_5 + \binom{i\ s}{j\ s}_5 \binom{t\ s}{0\ s}_5 \right] I^{st}_3 \Bigg) \Bigg\} \\
& + \sum{}' \frac{1}{\left(\right)_5} \frac{\binom{0}{0}_5 \binom{s}{k}_5}{\binom{s}{s}_5 \binom{s\ t}{s\ t}_5} \binom{t\ s}{j\ s}_5 \left[\binom{0\ s\ t}{i\ s\ t}_5 I^{st}_3 - \binom{u\ s\ t}{i\ s\ t}_5 I^{stu}_2 \right] \\
& + (i \leftrightarrow k) + (j \leftrightarrow k) \\
\equiv\; A \;\; & + \sum{}' \Bigg\{ \frac{1}{\left(\right)_5 \binom{s}{s}_5^2} \binom{i}{j}_5 \binom{s}{s}_5 \binom{0\ s}{0\ k}_5 \left[\binom{0\ s}{0\ s}_5 I^s_4 - \binom{t\ s}{0\ s}_5 I^{st}_3 \right] \\
& + \frac{1}{\left(\right)_5} \frac{\binom{0}{0}_5 \binom{s}{k}_5}{\binom{s}{s}_5 \binom{s\ t}{s\ t}_5} \binom{t\ s}{j\ s}_5 \left[\binom{0\ s\ t}{i\ s\ t}_5 I^{st}_3 - \binom{u\ s\ t}{i\ s\ t}_5 I^{stu}_2 \right] \Bigg\} \\
& + (i \leftrightarrow k) + (j \leftrightarrow k)
\end{aligned}
\tag{3.22}
$$

The symbol $\sum'$ in these equations denotes a sum $\sum_{s,t,u=1}^5$ in terms proportional to I^{stu}_2, $\sum_{s,t=1}^5$ in terms proportional to I^{st}_3, and $\sum_{s=1}^5$ in terms proportional to I^s_4. Concerning the symmetrization in (3.22), we point out that the original expression (3.18) is obviously symmetric under $(i \leftrightarrow j)$, while this is not explicitly seen in (3.22) anymore. Later on, however, this symmetry will become apparent again.

All terms with factors of the type $\binom{i}{j}_5$ can be considered, due to (3.6), as belonging to some $g_{\mu\nu}$ term. For other terms we have to use (3.21), which yields terms with $()_5$ to be cancelled. These are explicitly given in the coefficients of I^s_4, I^{st}_3 and I^{stu}_2, i.e. (3.30, 3.32, 3.33). Apart from the terms in the last line of (3.22) and the $\binom{i}{j}_5$ term, the remaining contributions to the coefficients

of I_4^s and I_3^{st}, inserting X_{ij}^s and X_{ij}^{st}, can be written as

$$
\begin{aligned}
A = & -\sum_{s=1}^{5} \frac{1}{\left(\right)_5 \binom{s}{s}_5^2} \binom{0}{k}_5 \left\{ \binom{s}{s}_5 \left[\binom{0\ s}{0\ i}_5 \binom{0\ s}{j\ s}_5 - \binom{0\ j}{s\ i}_5 \binom{0\ s}{0\ s}_5 \right] \right. \\
& \left. + \binom{s}{0}_5 \left[\binom{0\ s}{i\ s}_5 \binom{0\ s}{j\ s}_5 + \binom{i\ s}{j\ s}_5 \binom{0\ s}{0\ s}_5 \right] \right\} I_4^s \\
& + \sum_{s,t=1}^{5} \frac{1}{\left(\right)_5 \binom{s}{s}_5^2} \binom{0}{k}_5 \left\{ \binom{s}{s}_5 \left[\binom{0\ s}{0\ j}_5 \binom{t\ s}{i\ s}_5 - \binom{0\ i}{s\ j}_5 \binom{t\ s}{0\ s}_5 \right] \right. \\
& \left. + \binom{s}{0}_5 \left[\binom{0\ s}{j\ s}_5 \binom{t\ s}{i\ s}_5 + \binom{j\ s}{i\ s}_5 \binom{t\ s}{0\ s}_5 \right] \right\} I_3^{st}. \qquad (3.23)
\end{aligned}
$$

Here the following "master formula" (Equation (A.13) of [22]) is of great help:

$$
\binom{s}{i}_5 \binom{s\ \tau}{0\ s}_5 = \binom{s}{0}_5 \binom{s\ \tau}{i\ s}_5 + \binom{s}{s}_5 \binom{s\ \tau}{0\ i}_5, \quad \tau = 0, 1, \ldots 5, \qquad (3.24)
$$

which yields explicitly:

$$
\binom{s}{s}_5 \binom{0\ s}{0\ i}_5 + \binom{s}{0}_5 \binom{0\ s}{i\ s}_5 = \binom{s}{i}_5 \binom{0\ s}{0\ s}_5, \qquad (3.25)
$$

and:

$$
\binom{s}{s}_5 \binom{0\ j}{s\ i}_5 - \binom{s}{0}_5 \binom{i\ s}{j\ s}_5 = -\binom{s}{j}_5 \binom{0\ s}{i\ s}_5, \qquad (3.26)
$$

so that (3.23) reads:

$$
\begin{aligned}
A = & -\sum_{s=1}^{5} \frac{1}{\left(\right)_5 \binom{s}{s}_5^2} \binom{0}{k}_5 \cdot \left\{ \left[\binom{s}{i}_5 \binom{0\ s}{j\ s}_5 + \binom{s}{j}_5 \binom{0\ s}{i\ s}_5 \right] \binom{0\ s}{0\ s}_5 I_4^s \right. \\
& \left. - \sum_{t=1}^{5} \left[\binom{s}{j}_5 \binom{0\ s}{0\ s}_5 \binom{t\ s}{i\ s}_5 + \binom{s}{i}_5 \binom{0\ s}{j\ s}_5 \binom{t\ s}{0\ s}_5 \right] I_3^{st} \right\}. \qquad (3.27)
\end{aligned}
$$

Next we will use :

$$
\binom{0}{k}_5 \binom{s}{i}_5 = -\binom{0\ i}{s\ k}_5 \left(\right)_5 + \binom{i}{k}_5 \binom{s}{0}_5. \qquad (3.28)
$$

As trivial as this relation may look, it plays the crucial role of splitting off $\binom{i}{k}_5$ in order to produce $g^{\mu\nu}$ terms. It might also have been written as:

$$
\binom{0}{k}_5 \binom{s}{i}_5 = \binom{0\ s}{k\ i}_5 \left(\right)_5 + \binom{0}{i}_5 \binom{s}{k}_5, \qquad (3.29)
$$

but then it would not fulfill its purpose.

The first term at the rhs. of (3.28) cancels a $()_5$, while the second term enters the $g_{\mu\nu}$-terms, all of which are collected in (3.36). The complete coefficient of I_4^s in (3.16) is thus given by:

$$S_{ijk}^{4,s} = \frac{1}{3\begin{pmatrix}0\\0\end{pmatrix}_5\begin{pmatrix}s\\s\end{pmatrix}_5^2}\Bigg\{-\begin{pmatrix}0&s\\0&k\end{pmatrix}_5\left[\begin{pmatrix}0&s\\i&s\end{pmatrix}_5\begin{pmatrix}0&s\\j&s\end{pmatrix}_5+\begin{pmatrix}i&s\\j&s\end{pmatrix}_5\begin{pmatrix}0&s\\0&s\end{pmatrix}_5\right]$$
$$+\left[\begin{pmatrix}0&i\\s&k\end{pmatrix}_5\begin{pmatrix}0&s\\j&s\end{pmatrix}_5+\begin{pmatrix}0&j\\s&k\end{pmatrix}_5\begin{pmatrix}0&s\\i&s\end{pmatrix}_5\right]\begin{pmatrix}0&s\\0&s\end{pmatrix}_5+(i\leftrightarrow k)+(j\leftrightarrow k)\Bigg\}. \quad (3.30)$$

Finally we have to investigate the last line of (3.22), being left with the factor $\begin{pmatrix}0\\k\end{pmatrix}_5\begin{pmatrix}s\\0\end{pmatrix}_5$ as before in (3.21). The master formula (3.24) then yields:

$$\begin{pmatrix}s\\0\end{pmatrix}_5\begin{pmatrix}t&s\\j&s\end{pmatrix}_5=\begin{pmatrix}s\\j\end{pmatrix}_5\begin{pmatrix}t&s\\0&s\end{pmatrix}_5-\begin{pmatrix}s\\s\end{pmatrix}_5\begin{pmatrix}t&s\\0&j\end{pmatrix}_5. \quad (3.31)$$

The $\begin{pmatrix}s\\s\end{pmatrix}_5$ in the second term of (3.31) cancels and the remaining factor is antisymmetric in s and t, i.e. this term drops out after summation over s,t. Using again (3.28) and dropping for the time being the contribution to $g_{\mu\nu}$ terms, we finally write the coefficients of I_3^{st} and I_2^{stu} in the following way, taking care of the original $(i\leftrightarrow j)$ symmetry in (3.22):

$$S_{ijk}^{3,st} = \frac{1}{3\begin{pmatrix}0\\0\end{pmatrix}_5\begin{pmatrix}s\\s\end{pmatrix}_5^2}\Bigg\{\begin{pmatrix}0&s\\0&k\end{pmatrix}_5\left[\begin{pmatrix}t&s\\i&s\end{pmatrix}_5\begin{pmatrix}0&s\\j&s\end{pmatrix}_5+\begin{pmatrix}i&s\\j&s\end{pmatrix}_5\begin{pmatrix}t&s\\0&s\end{pmatrix}_5\right.$$
$$\left.+\frac{\begin{pmatrix}s\\s\end{pmatrix}_5\begin{pmatrix}0&s&t\\i&s&t\end{pmatrix}_5}{\begin{pmatrix}s&t\\s&t\end{pmatrix}_5}\begin{pmatrix}t&s\\j&s\end{pmatrix}_5\right]-\left[\begin{pmatrix}0&i\\s&k\end{pmatrix}_5\begin{pmatrix}0&s\\j&s\end{pmatrix}_5+\begin{pmatrix}0&j\\s&k\end{pmatrix}_5\begin{pmatrix}0&s\\i&s\end{pmatrix}_5\right]\begin{pmatrix}t&s\\0&s\end{pmatrix}_5$$
$$-\left[\begin{pmatrix}0&i\\s&k\end{pmatrix}_5\begin{pmatrix}t&s\\j&s\end{pmatrix}_5+\begin{pmatrix}0&j\\s&k\end{pmatrix}_5\begin{pmatrix}t&s\\i&s\end{pmatrix}_5\right]\frac{\begin{pmatrix}s\\s\end{pmatrix}_5\begin{pmatrix}0&s&t\\0&s&t\end{pmatrix}_5}{2\begin{pmatrix}s&t\\s&t\end{pmatrix}_5}$$
$$+(i\leftrightarrow k)+(j\leftrightarrow k)\Bigg\}, \quad (3.32)$$

and

$$S_{ijk}^{2,stu} = -\frac{1}{3\begin{pmatrix}0\\0\end{pmatrix}_5\begin{pmatrix}s\\s\end{pmatrix}_5\begin{pmatrix}s&t\\s&t\end{pmatrix}_5}\Bigg\{\begin{pmatrix}0&s\\0&k\end{pmatrix}_5\begin{pmatrix}t&s\\j&s\end{pmatrix}_5\begin{pmatrix}u&s&t\\i&s&t\end{pmatrix}_5-$$

$$\frac{1}{2}\left[\binom{0\ j}{s\ k}_5\binom{u\ s\ t}{i\ s\ t}_5+\binom{0\ i}{s\ k}_5\binom{u\ s\ t}{j\ s\ t}_5\right]\binom{t\ s}{0\ s}_5+(i\leftrightarrow k)+(j\leftrightarrow k)\Bigg\}. \quad (3.33)$$

At the end we can determine the $g_{\mu\nu}$ terms from the above by collecting all terms containing factors of the type $\binom{i}{j}_5$:

$$\sum_{j=1}^{4} g^{[\mu\nu}q_j^{\lambda]}E_{00j}=\frac{2}{\binom{}{}_5}\sum_{ijk=1}^{4}\left[\binom{j}{k}_5E_{00i}+\binom{i}{k}_5E_{00j}+\binom{i}{j}_5E_{00k}\right]q_i^{\mu}q_j^{\nu}q_k^{\lambda}, \quad (3.34)$$

where the square bracket means symmetrization of the included indices,

$$g^{[\mu\nu}q_k^{\lambda]}=g^{\mu\nu}\,q_k^{\lambda}+g^{\mu\lambda}\,q_k^{\nu}+g^{\nu\lambda}\,q_k^{\mu}, \quad (3.35)$$

and use has been made of (3.6). Collecting all terms of type $\binom{i}{j}_5$ in (3.22) we have:

$$\begin{aligned}
3\binom{0}{0}_5E_{00j}=&-\frac{1}{2}\sum_{s=1}^{5}\frac{1}{\binom{s}{s}_5^2}\left[2\binom{s}{0}_5\binom{0\ s}{j\ s}_5-\binom{s}{s}_5\binom{0\ s}{0\ j}_5\right]\binom{0\ s}{0\ s}_5I_4^s\\
&+\frac{1}{2}\sum_{s,t=1}^{5}\Bigg\{\frac{1}{\binom{s}{s}_5^2}\left[\binom{s}{0}_5\binom{0\ s}{j\ s}_5-\binom{s}{s}_5\binom{0\ s}{0\ j}_5\right]\binom{t\ s}{0\ s}_5\\
&+\frac{1}{\binom{s}{s}_5^2}\binom{s}{0}_5\binom{t\ s}{j\ s}_5\binom{0\ s}{0\ s}_5\\
&+\frac{1}{\binom{s}{s}_5\binom{s\ t}{s\ t}_5}\binom{s}{0}_5\binom{t\ s}{0\ s}_5\binom{0\ s\ t}{j\ s\ t}_5\Bigg\}I_3^{st}\\
&-\frac{1}{2}\sum_{s,t,u=1}^{5}\frac{1}{\binom{s}{s}_5\binom{s\ t}{s\ t}_5}\binom{s}{0}_5\binom{t\ s}{0\ s}_5\binom{u\ s\ t}{j\ s\ t}_5I_2^{stu}.
\end{aligned} \quad (3.36)$$

The following relation can be proven by multiplication with $\binom{s}{s}_5$, transforming it into the relation for an extensional of Equation (A.8) of [22]:

$$\binom{s}{0}_5\binom{\mu\ s\ t}{j\ s\ t}_5=\binom{s}{j}_5\binom{\mu\ s\ t}{0\ s\ t}_5-\binom{\mu\ s}{0\ j}_5\binom{s\ t}{s\ t}_5+\binom{t\ s}{0\ j}_5\binom{t\ s}{\mu\ s}_5, \quad \mu=0,1,\cdots,4.$$

(3.37)

It turns out to be useful for the simplification of the coefficients of I_3^{st} and I_2^{stu} in (3.36). For the coefficient of I_3^{st}, we apply relation (3.37) with $\mu = 0$. The last term on the r.h.s. of (3.37) is combined with the term on the third line of (3.36) using (3.31):

$$\begin{aligned} & \frac{1}{\binom{s}{s}_5^2}\left\{\frac{\binom{s}{s}_5}{\binom{s\ t}{s\ t}_5}\binom{t\ s}{0\ j}_5\binom{t\ s}{0\ s}_5^2+\binom{s}{0}_5\binom{t\ s}{j\ s}_5\binom{0\ s}{0\ s}_5\right\} \\ = \ & \frac{1}{\binom{s}{s}_5^2}\binom{s}{j}_5\binom{t\ s}{0\ s}_5\binom{0\ s}{0\ s}_5-\frac{\binom{t\ s}{0\ j}_5\binom{0\ s\ t}{0\ s\ t}_5}{\binom{s\ t}{s\ t}_5}, \end{aligned} \tag{3.38}$$

After summation over s and t, the last term on the r.h.s. will vanish. Furthermore we apply (3.24) taking $\tau = 0$.

For the coefficient of I_2^{stu} in (3.36) we apply relation (3.37) with $\mu = u$. Since I_2^{stu} is symmetric in s, t and u, we consider the sum over all permutations of any fixed set of values of s, t and u. We find that

$$\sum_{permutations}\frac{1}{\binom{s}{s}_5\binom{s\ t}{s\ t}_5}\binom{t\ s}{0\ s}_5\left[\binom{u\ s}{0\ j}_5\binom{s\ t}{s\ t}_5-\binom{t\ s}{0\ j}_5\binom{t\ s}{u\ s}_5\right]=0, \tag{3.39}$$

so that the two last terms on the r.h.s. of (3.37) can be dropped in this case. Thus we have:

$$\begin{aligned} 3\binom{0}{0}_5 E_{00j} = & -\frac{1}{2}\sum_{s=1}^{5}\frac{1}{\binom{s}{s}_5^2}\left[3\binom{s}{0}_5\binom{0\ s}{j\ s}_5-\binom{s}{j}_5\binom{0\ s}{0\ s}_5\right]\binom{0\ s}{0\ s}_5 I_4^s \\ & +\frac{1}{2}\sum_{s,t=1}^{5}\frac{1}{\binom{s}{s}_5^2}\left[3\binom{s}{0}_5\binom{0\ s}{j\ s}_5-\binom{s}{j}_5\frac{\binom{t\ s}{0\ s}_5^2}{\binom{s\ t}{s\ t}_5}\right]\binom{t\ s}{0\ s}_5 I_3^{st} \\ & -\frac{1}{2}\sum_{s,t,u=1}^{5}\frac{1}{\binom{s}{s}_5^2}\binom{s}{j}_5\frac{\binom{s}{s}_5\binom{u\ s\ t}{0\ s\ t}_5}{\binom{s\ t}{s\ t}_5}\binom{t\ s}{0\ s}_5 I_2^{stu}. \end{aligned} \tag{3.40}$$

Collecting all contributions, our final result for the tensor of rank 3 can be written as:

$$I_5^{\mu\,\nu\,\lambda} = \sum_{i,j,k=1}^{4} q_i^\mu\, q_j^\nu\, q_k^\lambda E_{ijk} + \sum_{k=1}^{4} g^{[\mu\nu} q_k^{\lambda]} E_{00k}, \tag{3.41}$$

$$E_{ijk} = \sum_{s=1}^{5} S_{ijk}^{4,s} I_4^s + \sum_{s,t=1}^{5} S_{ijk}^{3,st} I_3^{st} + \sum_{s,t,u=1}^{5} S_{ijk}^{2,stu} I_2^{stu}, \tag{3.42}$$

and the coefficients $S_{ijk}^{4,s}, S_{ijk}^{3,st}, S_{ijk}^{2,stu}$ are given in (3.30), (3.32) and (3.33) and E_{00k} in (3.40).

4. Hexagons

The 6-point function has the nice property that the tensors of rank R can be reduced to a sum of six 5-point tensors of rank $R-1$. This property has also been derived in [5]; an earlier demonstration of this property, however, has been given already in [26]. The simplification in this case is due to the fact that $()_6 \equiv 0$, which has extensively been discussed in [26]. Beyond that, in our approach, the above results for the 5-point tensors can be directly used, thus reducing the 6-point tensors of up to rank $R = 4$ to scalar 4- and 3- and 2-point integrals. Particularly simple results are thus obtained for the 6-point tensors using the results of Appendix A and Sections 3.1 and 3.2. What was missing in [26] is exactly this simplification, which comes with the cancellation of the Gram determinant $()_5$; see Appendix A of that paper.

4.1 Scalar and vector integrals

According to (I.33) we write (see [22] and also (I.55)):

$$\begin{aligned} I_6 &= \sum_{r=1}^{6} \frac{\begin{pmatrix} 0 \\ r \end{pmatrix}_6}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}_6} E^{\,r} \\ &= \sum_{r=1}^{6} \frac{\begin{pmatrix} r \\ k \end{pmatrix}_6}{\begin{pmatrix} 0 \\ k \end{pmatrix}_6} E^{\,r}, \quad k = 1, \ldots, 6, \end{aligned} \tag{4.1}$$

and (3.2) now reads:

$$E^{\,r} \equiv I_5^r = \frac{1}{\begin{pmatrix} 0 & r \\ 0 & r \end{pmatrix}_6} \sum_{s=1, s\neq r}^{6} \begin{pmatrix} 0 & r \\ s & r \end{pmatrix}_6 I_4^{rs}. \tag{4.2}$$

Here we see already the general scheme of reducing 6-point functions to 5-point functions: In general, in any signed minor $(\cdots)_5$ a further column $\begin{pmatrix} r \\ r \end{pmatrix}$ is scratched, resulting in a $(\cdots)_6$ and in the scalar functions a further propagator is scratched.

As in (3.3) and (3.4), with the use of (I.57), we obtain:

$$I_6^\mu = \sum_{i=1}^{5} q_i^\mu I_{6,i}, \tag{4.3}$$

$$
\begin{aligned}
I_{6,i} &= -I_{6,i}^{[d+]} \\
&= (d-5)\frac{\binom{0}{i}_6}{\binom{0}{0}_6} I_6^{[d+]} - \frac{1}{\binom{0}{0}_6}\sum_{r=1}^{6}\binom{0\ i}{0\ r}_6 I_5^r .
\end{aligned} \tag{4.4}
$$

While in (3.4) the first part vanishes in the limit $d \to 4$, here its disappearance is due to (I.61):

$$
\sum_{i=1}^{5} q_i^\mu \binom{0}{i}_6 = 0. \tag{4.5}
$$

Indeed (4.5) will play a crucial role for the higher tensor reduction. The resulting form in (4.4) is already the generic form for the higher tensors too! Therefore it appears useful to introduce the vector, applying further (A.15) of [22] and (I.61):

$$
\begin{aligned}
v_r^\mu &= -\frac{1}{\binom{0}{0}_6}\sum_{i=1}^{5}\binom{0\ i}{0\ r}_6 q_i^\mu \\
&= -\frac{1}{\binom{0}{k}_6}\sum_{i=1}^{5}\binom{0\ r}{k\ i}_6 q_i^\mu, \quad k = 0,\ldots,6,
\end{aligned} \tag{4.6}
$$

summing over all 5 (dependent) vectors. v_r projected on these vectors reads:

$$
\begin{aligned}
v_r \cdot q_i &= -\frac{1}{2}\left[\delta_{ir} - (Y_{i6} - Y_{66})\frac{\binom{0}{r}_6}{\binom{0}{0}_6}\right] \\
&= -\frac{1}{2}\left[\delta_{ir} + (q_i^2 + m_6^2 - m_i^2)\frac{\binom{0}{r}_6}{\binom{0}{0}_6}\right].
\end{aligned} \tag{4.7}
$$

With this definition we can write in a compact way:

$$
I_6^\mu = \sum_{r=1}^{6} v_r^\mu E^{\,r}. \tag{4.8}
$$

4.2 $R = 2$ tensor integrals

The equation (2.2) reads in this case:

$$
I_6^{\mu\,\nu} = \sum_{i,j=1}^{5} q_i^\mu\, q_j^\nu\, \nu_{ij}\ I_{6,ij}^{[d+]^2} - \frac{1}{2}\, g^{\mu\nu}\, I_6^{[d+]}, \tag{4.9}
$$

and by using (I.59) we have:

$$\nu_{ij}\ I_{6,ij}^{[d+]^2} = -(d-4)\frac{\binom{0}{i}_6}{\binom{0}{0}_6} I_{6,i}^{[d+]^2} + \frac{\binom{0\ i}{0\ j}_6}{\binom{0}{0}_6} I_6^{[d+]} + \frac{1}{\binom{0}{0}_6} \sum_{r=1,r\neq i}^{6} \binom{0\ j}{0\ r}_6 I_{5,i}^{[d+],r}. \quad (4.10)$$

We consider the limit $d \to 4$ and use (I.67):

$$g^{\mu\nu} = \frac{2}{\binom{0}{0}_6} \sum_{i,j=1}^{5} \binom{0\ i}{0\ j}_6 q_i^\mu\, q_j^\nu. \quad (4.11)$$

Writing it like in (3.5),

$$I_6^{\mu\,\nu} = \sum_{i,j=1}^{5} q_i^\mu\, q_j^\nu I_{6,ij}, \quad (4.12)$$

we obtain by using (3.4):

$$I_{6,ij} = -\frac{1}{\binom{0}{0}_6} \sum_{r=1,r\neq i}^{6} \binom{0\ j}{0\ r}_6 E_i{}^r, \quad (4.13)$$

to be compared with (4.4). For completeness we specify $E_i{}^r$, which we read off from (3.4) to be:

$$E_i{}^r = -\frac{1}{\binom{0\ r}{0\ r}_6} \sum_{s=1}^{6} \binom{0\ i\ r}{0\ s\ r}_6 I_4^{rs}, \quad (4.14)$$

and finally:

$$I_6^{\mu\,\nu} = \sum_{i=1}^{5} q_i^\mu \sum_{r=1,r\neq i}^{6} v_r^\nu E_i{}^r. \quad (4.15)$$

We remark that due to (4.14), $E_i{}^r = 0$ for $r = i$ and correspondingly this will be the case for all higher tensors such that limitations like $r \neq i$ could be dropped but are convenient to keep in numerical programs.

4.3 $R = 3$ tensor integrals

Equation (2.3) reads in this case:

$$I_6^{\mu\,\nu\,\lambda} = -\sum_{i,j,k=1}^{5} q_i^\mu\, q_j^\nu\, q_k^\lambda\, \nu_{ij}\nu_{ijk}\ I_{6,ijk}^{[d+]^3} + \frac{1}{2}\sum_{i=1}^{5} (\, g^{\mu\nu}\, q_i^\lambda + g^{\mu\lambda}\, q_i^\nu + g^{\nu\lambda}\, q_i^\mu\,) I_{6,i}^{[d+]^2}, \quad (4.16)$$

and with (I.60) we have:

$$\nu_{ij}\nu_{ijk}\ I^{[d+]^3}_{6,ijk} = -\ (d-3)\frac{\binom{0}{k}_6}{\binom{0}{0}_6} I^{[d+]^2}_{6,ij} + \frac{\binom{0\ k}{0\ i}_6}{\binom{0}{0}_6} I^{[d+]^2}_{6,j} + \frac{\binom{0\ k}{0\ j}_6}{\binom{0}{0}_6} I^{[d+]^2}_{6,i}$$
$$+ \frac{1}{\binom{0}{0}_6} \sum_{r=1, r\neq i,j}^{6} \binom{0\ k}{0\ r}_6 \nu_{ij} I^{[d+]^2,r}_{5,ij}. \tag{4.17}$$

The first term on the r.h.s. is eliminated due to (4.5) and the next two terms cancel due to (4.11). Taking into account $I^{[d+]}_5$, relation (3.7) now reads:

$$I^r_{5,ij} = \nu_{ij} I^{[d+]^2,r}_{5,ij} - \frac{\binom{i\ r}{j\ r}_6}{\binom{r}{r}_6} I^{[d+],r}_5. \tag{4.18}$$

As a further representation of $g^{\mu\nu}$ we have (see (I.75)):

$$g^{\mu\nu} = \frac{2}{\binom{r}{r}_6} \sum_{i,j=1}^{5} \binom{i\ r}{j\ r}_6 q^\mu_i\, q^\nu_j, \quad r = 1\ldots 6\ . \tag{4.19}$$

Using again (I.57) and the definition

$$I^{\mu\,\nu\lambda}_6 = \sum_{i,j,k=1}^{5} q^\mu_i\, q^\nu_j q^\lambda_k I_{6,ijk}, \tag{4.20}$$

we obtain:

$$I_{6,ijk} = -\frac{1}{\binom{0}{0}_6} \sum_{r=1, r\neq i,j}^{6} \binom{0\ k}{0\ r}_6 I^r_{5,ij}. \tag{4.21}$$

From (3.10) and (4.19) $I^r_{5,ij}$ reads:

$$I^r_{5,ij} = E^{\ r}_{ij} + 2\frac{\binom{i\ r}{j\ r}_6}{\binom{r}{r}_6} E^r_{00}, \tag{4.22}$$

so that we get:

$$I_6^{\mu\,\nu\,\lambda} = \sum_{i,j=1}^{5} q_i^\mu q_j^\nu \sum_{r=1,r\neq i,j}^{6} v_r^\lambda E_{ij}^{\ r} + \sum_{i,j=1}^{5} q_i^\mu q_j^\nu \sum_{r=1}^{6} 2\frac{\begin{pmatrix} i & r \\ j & r \end{pmatrix}_6}{\begin{pmatrix} r \\ r \end{pmatrix}_6} v_r^\lambda E_{00}^{\ r}, \tag{4.23}$$

where in the second term we can drop the limitation $r \neq i,j$ since it is automatically fulfilled due to the numerator $\begin{pmatrix} i & r \\ j & r \end{pmatrix}_6$, vanishing for $r = i$ and $r = j$. Thus summation over i and j is possible, using (4.19), with a result:

$$I_6^{\mu\,\nu\,\lambda} = \sum_{i,j=1}^{5} q_i^\mu q_j^\nu \sum_{r=1,r\neq i,j}^{6} v_r^\lambda E_{ij}^{\ r} + g^{\mu\nu} \sum_{r=1}^{6} v_r^\lambda E_{00}^{\ r}, \tag{4.24}$$

or:

$$I_6^{\mu\,\nu\,\lambda} = \sum_{r=1}^{6} v_r^\lambda I_5^{\mu\,\nu\,,r}, \tag{4.25}$$

with

$$I_5^{\mu\,\nu\,,r} = \sum_{i,j=1,i,j\neq r}^{5} q_i^\mu q_j^\nu E_{ij}^{\ r} + g^{\mu\nu} E_{00}^{\ r}. \tag{4.26}$$

4.4 $R = 4$ tensor integrals

The tensor integral in (2.4) contains three different integrals in higher dimension, which have to be reduced or to be eliminated. We begin with $I^{[d+]^4}_{n,ijkl}$ using (I.26). For convenience we use x instead of 4:

$$\begin{aligned}
\begin{pmatrix} 0 \\ 0 \end{pmatrix}_n \nu_{ijkl}\, \mathbf{l}^+\ I^{[d+]^x}_{n,ijk} &\equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}_n \nu_{ijkl} I^{[d+]^x}_{n,ijkl} \\
&= \sum_{r=1}^{n} \begin{pmatrix} 0 & l \\ 0 & r \end{pmatrix}_n [d + 2x - (n+3)]\, I^{[d+]^x}_{n,ijk} \\
&\quad - \sum_{s=1}^{n} \begin{pmatrix} 0 & l \\ 0 & s \end{pmatrix}_n \nu_{ijks} I^{[d+]^x}_{n,ijk} - \sum_{r,s=1;r\neq s}^{n} \begin{pmatrix} 0 & l \\ 0 & r \end{pmatrix}_n \nu_{ijks}\ \mathbf{r}^-\ \mathbf{s}^+\ I^{[d+]^x}_{n,ijk} \\
&= \left\{ [n + 4 - (d + 2x)] \begin{pmatrix} 0 \\ l \end{pmatrix}_n - \begin{pmatrix} 0 & l \\ 0 & i \end{pmatrix}_n - \begin{pmatrix} 0 & l \\ 0 & j \end{pmatrix}_n - \begin{pmatrix} 0 & l \\ 0 & k \end{pmatrix}_n \right\} I^{[d+]^x}_{n,ijk} \\
&\quad - \sum_{r=1}^{n} \begin{pmatrix} 0 & l \\ 0 & r \end{pmatrix}_n \sum_{s=1;s\neq r}^{n} \nu_{ijks} I^{[d+]^x,r}_{n,ijks}.
\end{aligned} \tag{4.27}$$

The last double sum in (4.27), assuming all indices i, j, k to be different, reads:

$$- \begin{pmatrix} 0 & l \\ 0 & i \end{pmatrix}_n \sum_{s=1;s\neq i}^{n} \nu_{jks} I^{[d+]^x}_{n,jks} - \begin{pmatrix} 0 & l \\ 0 & j \end{pmatrix}_n \sum_{s=1;s\neq j}^{n} \nu_{iks} I^{[d+]^x}_{n,iks} - \begin{pmatrix} 0 & l \\ 0 & k \end{pmatrix}_n \sum_{s=1;s\neq k}^{n} \nu_{ijs} I^{[d+]^x}_{n,ijs}$$

$$- \sum_{r=1;r\neq i,j,k}^{n} \begin{pmatrix} 0 & l \\ 0 & r \end{pmatrix}_n \sum_{s=1;s\neq r}^{n} \nu_{ijks} I^{[d+]^x,r}_{n-1,ijks}. \tag{4.28}$$

Now adding corresponding terms in (4.27) and (4.28), e.g. for $r = i$, we get:

$$-\begin{pmatrix} 0 & l \\ 0 & i \end{pmatrix}_n \sum_{s=1;s\neq i}^{n} \nu_{jks} I^{[d+]^x}_{n,jks} - \begin{pmatrix} 0 & l \\ 0 & i \end{pmatrix}_n I^{[d+]^x}_{n,ijk} = -\begin{pmatrix} 0 & l \\ 0 & i \end{pmatrix}_n \sum_{s=1}^{n} \nu_{jks} I^{[d+]^x}_{n,jks} = \begin{pmatrix} 0 & l \\ 0 & i \end{pmatrix}_n I^{[d+]^{(x-1)}}_{n,jk}, \tag{4.29}$$

due to (I.29). In case two indices are equal, e.g. $i = j \neq k$, we have:

$$\begin{aligned} -\begin{pmatrix} 0 & l \\ 0 & i \end{pmatrix}_n \sum_{s=1;s\neq i}^{n} (1 + 2\delta_{is} + \delta_{ks}) I^{[d+]^x}_{n,iks} - 2\begin{pmatrix} 0 & l \\ 0 & i \end{pmatrix}_n I^{[d+]^x}_{n,iis} &= -\begin{pmatrix} 0 & l \\ 0 & i \end{pmatrix}_n \sum_{s=1}^{n} (1 + \delta_{is} + \delta_{ks}) I^{[d+]^x}_{n,iks} \\ &\equiv -\begin{pmatrix} 0 & l \\ 0 & i \end{pmatrix}_n \sum_{s=1}^{n} \nu_{iks} I^{[d+]^x}_{n,iks}, \end{aligned} \tag{4.30}$$

like (4.29), i.e. if two indices agree, this integral occurs only once. As final result we have:

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix}_n \nu_{ijkl}\, I^{[d+]^x}_{n,ijkl} &= [n + 4 - (d + 2x)] \begin{pmatrix} 0 \\ l \end{pmatrix}_n I^{[d+]^x}_{n,ijk} \\ &\quad + [ijk]^{(l)}_{\rm red} + \sum_{r=1;r\neq i,j,k}^{n} \begin{pmatrix} 0 & l \\ 0 & r \end{pmatrix}_n I^{[d+]^{(x-1)},r}_{n-1,ijk}, \end{aligned} \tag{4.31}$$

where according to (4.29) and the discussion thereafter:

$$[ijk]^{(l)} = \begin{pmatrix} 0 & l \\ 0 & i \end{pmatrix}_n I^{[d+]^{(x-1)}}_{n,jk} + \begin{pmatrix} 0 & l \\ 0 & j \end{pmatrix}_n I^{[d+]^{(x-1)}}_{n,ik} + \begin{pmatrix} 0 & l \\ 0 & k \end{pmatrix}_n I^{[d+]^{(x-1)}}_{n,ij} \tag{4.32}$$

and $[ijk]^{(l)}_{\rm red} = [ijk]^{(l)}$ without repetition, e.g. $[iii]^{(l)}_{\rm red} = \begin{pmatrix} 0 & l \\ 0 & i \end{pmatrix}_n I^{[d+]^{(x-1)}}_{n,ii}$.

Now, making use of $n_{ijkl} = \nu_{ij}\nu_{ijk}\nu_{ijkl}$, we see that due to (4.5) the first part in (4.31) drops out after insertion into (2.4). The second contribution of (4.31) yields:

$$\frac{1}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}_n} \sum_{l=1}^{n-1} q_l^\rho \sum_{i,j,k=1}^{n-1} \nu_{ij}\nu_{ijk}\, [ijk]^{(l)}_{\rm red}\, q_i^\mu q_j^\nu q_k^\lambda. \tag{4.33}$$

We have:

$$\nu_{ij}\nu_{ijk}\, [ijk]^{(l)}_{\rm red} = [ijk]^{(l)} + \begin{pmatrix} 0 & l \\ 0 & i \end{pmatrix}_n \delta_{jk} I^{[d+]^{(x-1)}}_{n,jk}$$

$$+\begin{pmatrix}0 & l\\ 0 & j\end{pmatrix}_n \delta_{ik} I^{[d+]^{(x-1)}}_{n,ik} + \begin{pmatrix}0 & l\\ 0 & k\end{pmatrix}_n \delta_{ij} I^{[d+]^{(x-1)}}_{n,ij}, \tag{4.34}$$

with the help of which (4.33) reads:

$$\frac{1}{\begin{pmatrix}0\\0\end{pmatrix}_n}\sum_{l=1}^{n-1} q_l^\rho \sum_{i,j,k=1}^{n-1}\left[q_i^\mu \begin{pmatrix}0 & l\\ 0 & i\end{pmatrix}_n (1+\delta_{jk}) I^{[d+]^{(x-1)}}_{n,jk} q_j^\nu q_k^\lambda + q_j^\nu \begin{pmatrix}0 & l\\ 0 & j\end{pmatrix}_n (1+\delta_{ik}) I^{[d+]^{(x-1)}}_{n,ik} q_i^\mu q_k^\lambda \right.$$
$$\left. + q_k^\lambda \begin{pmatrix}0 & l\\ 0 & k\end{pmatrix}_n (1+\delta_{ij}) I^{[d+]^{(x-1)}}_{n,ij} q_i^\mu q_j^\nu\right]. \tag{4.35}$$

Using (I.67) we have for $d = 4$:

$$\frac{1}{2}\left\{g^{\mu\rho} n_{jk} I^{[d+]^{(x-1)}}_{n,jk} q_j^\nu q_k^\lambda + g^{\nu\rho} n_{ik} I^{[d+]^{(x-1)}}_{n,ik} q_i^\mu q_k^\lambda + g^{\lambda\rho} n_{ij} I^{[d+]^{(x-1)}}_{n,ij} q_i^\mu q_j^\nu\right\}, \tag{4.36}$$

and we see that this contribution is canceled by the last three terms of the type $I^{[d+]^{(x-1)}}_{n,jk}$ in (2.4). The first three terms of this type are evaluated by means of (I.59) to yield:

$$\begin{aligned} n_{ij} I^{[d+]^{(x-1)}}_{n,jk} &= \frac{1}{\begin{pmatrix}0\\0\end{pmatrix}_n}\left\{[n+2(2-x)-d]\begin{pmatrix}0\\j\end{pmatrix}_n I^{[d+]^{(x-1)}}_{n,i} + \begin{pmatrix}0 & i\\ 0 & j\end{pmatrix}_n I^{[d+]^{(x-2)}}_{n}\right. \\ &\quad \left. + \sum_{r=1;r\neq i}^{n}\begin{pmatrix}0 & j\\ 0 & r\end{pmatrix}_n I^{[d+]^{(x-2)},r}_{n-1,i}\right\}. \end{aligned} \tag{4.37}$$

Inserting this into (2.4), the first part yields a vanishing contribution due to (4.5) . The second term yields, again due to (4.5):

$$\begin{aligned} &-\frac{1}{2\begin{pmatrix}0\\0\end{pmatrix}_n}\sum_{i,j=1}^{n-1}\left\{g^{\mu\nu}q_i^\lambda q_j^\rho + g^{\mu\lambda}q_i^\nu q_j^\rho + g^{\nu\lambda}q_i^\mu q_j^\rho\right\}\begin{pmatrix}0 & i\\ 0 & j\end{pmatrix}_n I^{[d+]^{(x-2)}}_{n} \\ &= -\frac{1}{4}\left(g^{\mu\nu}g^{\lambda\rho} + g^{\mu\lambda}g^{\nu\rho} + g^{\mu\rho}g^{\nu\lambda}\right) I^{[d+]^{(x-2)}}_{n}, \end{aligned} \tag{4.38}$$

which cancels the last term in (2.4) and the total contribution thus reads:

$$\begin{aligned} &\frac{1}{\begin{pmatrix}0\\0\end{pmatrix}_n}\left\{\sum_{i,j,k,l=1}^{n-1}\nu_{ij}\nu_{ijk}\, q_i^\mu\, q_j^\nu\, q_k^\lambda\, q_l^\rho \sum_{r=1;r\neq i,j,k}^{n}\begin{pmatrix}0 & l\\ 0 & r\end{pmatrix}_n I^{[d+]^{(x-1)},r}_{n-1,ijk}\right. \\ &\left. -\frac{1}{2}\sum_{i,j=1}^{n-1}\left(g^{\mu\nu}q_i^\lambda q_j^\rho + g^{\mu\lambda}q_i^\nu q_j^\rho + g^{\nu\lambda}q_i^\mu q_j^\rho\right)\sum_{r=1;r\neq i}^{n}\begin{pmatrix}0 & j\\ 0 & r\end{pmatrix}_n I^{[d+]^{(x-2)},r}_{n-1,i}\right\}, \end{aligned} \tag{4.39}$$

reducing the 6-point tensor to 5-point tensors in lower dimensions. For further reduction we put explicitly $n = 6$ and $x = 4$ and write (3.18) in the form

$$\nu_{ij}\nu_{ijk}I^{[d+]^3,r}_{5,ijk} = -I^r_{5,ijk} + \left[\frac{\binom{j\ r}{k\ r}_6}{\binom{r}{r}_6} I^{[d+]^2,r}_{5,i} + \frac{\binom{i\ r}{k\ r}_6}{\binom{r}{r}_6} I^{[d+]^2,r}_{5,j} + \frac{\binom{i\ r}{j\ r}_6}{\binom{r}{r}_6} I^{[d+]^2,r}_{5,k} \right]. \quad (4.40)$$

With (4.19) it is now easy to see that the square bracket in (4.40) cancels out the second part in (4.39) and using the definition:

$$I^{\mu\,\nu\,\lambda\,\rho}_6 = \sum_{i,j,k,l=1}^{5} q^\mu_i\, q^\nu_j q^\lambda_k q^\rho_l I_{6,ijkl}, \quad (4.41)$$

we obtain:

$$I_{6,ijkl} = -\frac{1}{\binom{0}{0}_6} \sum_{r=1,r\neq i,j,k}^{6} \binom{0\ l}{0\ r}_6 I^r_{5,ijk}. \quad (4.42)$$

Again, with (3.17) and (4.19) $I^r_{5,ijk}$ reads:

$$I^r_{5,ijk} = E^{\ r}_{ijk} + 2\frac{\binom{i\ r}{j\ r}_6}{\binom{r}{r}_6} E^r_{00k} + 2\frac{\binom{i\ r}{k\ r}_6}{\binom{r}{r}_6} E^r_{00j} + 2\frac{\binom{j\ r}{k\ r}_6}{\binom{r}{r}_6} E^r_{00i}, \quad (4.43)$$

so:

$$\begin{aligned} I^{\mu\,\nu\,\lambda\rho}_6 = & \sum_{i,j,k=1}^{5} q^\mu_i q^\mu_j q^\lambda_k \sum_{r=1,r\neq i,j,k}^{6} v^\rho_r E^{\ r}_{ijk} \\ & + \sum_{i,j,k=1}^{5} q^\mu_i q^\mu_j q^\lambda_k \sum_{r=1,r\neq i,j,k}^{6} v^\rho_r \left\{ 2\frac{\binom{i\ r}{j\ r}_6}{\binom{r}{r}_6} E^r_{00k} + 2\frac{\binom{i\ r}{k\ r}_6}{\binom{r}{r}_6} E^r_{00j} + 2\frac{\binom{j\ r}{k\ r}_6}{\binom{r}{r}_6} E^r_{00i} \right\}, \end{aligned} \quad (4.44)$$

and with the same argument like the one used after (4.23) we obtain the final result:

$$\begin{aligned} I^{\mu\,\nu\,\lambda\,\rho}_6 = & \sum_{i,j,k=1}^{5} q^\mu_i q^\mu_j q^\lambda_k \sum_{r=1,r\neq i,j,k}^{6} v^\rho_r E^{\ r}_{ijk} + g^{\mu\nu} \sum_{k=1}^{5} q^\lambda_k \sum_{r=1,r\neq k}^{6} v^\rho_r E^r_{00k} \\ & + g^{\mu\lambda} \sum_{j=1}^{5} q^\nu_j \sum_{r=1,r\neq j}^{6} v^\rho_r E^r_{00j} + g^{\nu\lambda} \sum_{i=1}^{5} q^\mu_i \sum_{r=1,r\neq i}^{6} v^\rho_r E^r_{00i}, \end{aligned} \quad (4.45)$$

or:

$$I_6^{\mu\,\nu\,\lambda\rho} = \sum_{r=1}^{6} v_r^\rho I_5^{\mu\,\nu\,\lambda,r}, \tag{4.46}$$

with:

$$I_5^{\mu\,\nu\,\lambda,r} = \sum_{i,j,k=1;i,j,k\neq r}^{5} q_i^\mu q_j^\nu q_k^\lambda E_{ijk}^{\ r} + \sum_{k=1,k\neq r}^{5} g^{[\mu\nu} q_k^{\lambda]} E_{00k}^{\ r}. \tag{4.47}$$

5. Numerical results and discussion

In order to illustrate the numerical results which can be obtained with the described approach, we will evaluate a representative collection of tensor coefficients. We rely on two implementations of the formalism, one has been established in Fortran, and the other one in the Mathematica package `hexagon.m`.

In the following, we denote the scalar five-point function by E_0 and the scalar six-point function by F_0. The tensor decompositions of pentagons E and hexagons F read:

$$E^\mu = \sum_{i=1}^{4} q_i^\mu E_i, \tag{5.1}$$

$$E^{\mu\nu} = \sum_{i,j=1}^{4} q_i^\mu q_j^\nu E_{ij} + g^{\mu\nu} E_{00}, \tag{5.2}$$

$$E^{\mu\nu\lambda} = \sum_{i,j,k=1}^{4} q_i^\mu q_j^\nu q_k^\lambda E_{ijk} + \sum_{i=1}^{4} g^{[\mu\nu} q_i^{\lambda]} E_{00i}, \tag{5.3}$$

$$F^\mu = \sum_{i=1}^{5} q_i^\mu F_i, \tag{5.4}$$

$$F^{\mu\nu} = \sum_{i,j=1}^{5} q_i^\mu q_j^\nu F_{ij}, \tag{5.5}$$

$$F^{\mu\nu\lambda} = \sum_{i,j,k=1}^{5} q_i^\mu q_j^\nu q_k^\lambda F_{ijk} + \sum_{i=1}^{5} g^{\mu\nu} q_i^\lambda F_{00i}, \tag{5.6}$$

$$F^{\mu\nu\lambda\rho} = \sum_{i,j,k,l=1}^{5} q_i^\mu q_j^\nu q_k^\lambda q_l^\rho F_{ijkl} + \sum_{i,j=1}^{5} q_i^\mu q_j^{[\nu} g^{\lambda\rho]} F_{00ij}. \tag{5.7}$$

Please observe the difference of E^0, F^0 and E_0, F_0 in the following. The kinematics is visualized in Figure 5.1. Deviating from the first sections, we have chosen here $q_0 = 0$ in order to stay close to common conventions of other numerical packages.

For the evaluation of the scalar two-, three- and four-point functions, which appear after the complete reduction, we have implemented two numerical libraries:

- For massive internal particles: `Looptools 2.2` [4, 33];

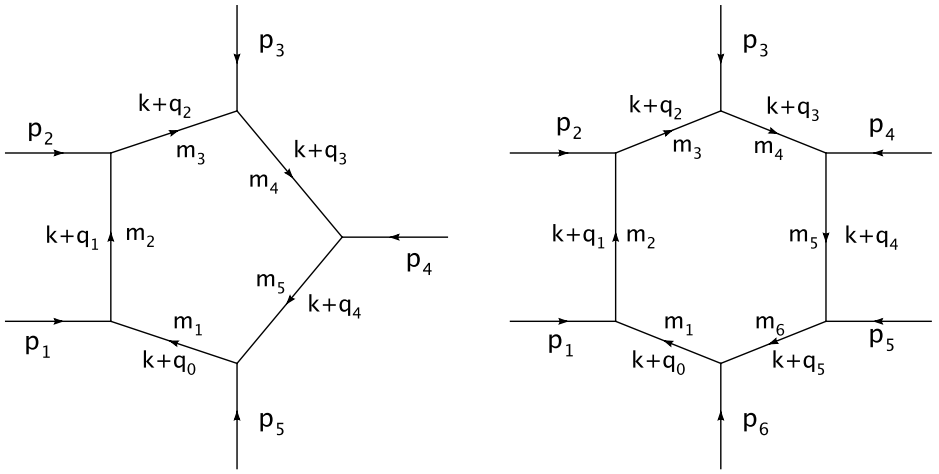

Figure 5.1: Momenta flow used in the numerical examples for six- and five-point integrals.

- If there are also massless internal particles: `QCDLoop-1.4` [35].

We observed that `Looptools` may become unstable in the presence of massless internal particles, while `QCDLoop` seems to be generally slower. Our Mathematica package has an implementation of only `Looptools`.

For completeness, we would like to mention also other publicly available Fortran packages for tensor functions, which we found useful for comparisons:

- Six-point tensors with massive internal particles: none;
- Five-point tensors with massive internal particles: `Looptools` [4, 33] ;
- Five-point tensors with both massive and massless particles: none;
- Five- and six-point tensors with only massless internal particles: `golem95` [37].

The two independent numerical implementations have been checked in several ways:

- By internal comparisons of the two codes, relying on the formulae presented in this article; With alternative, direct representations of the tensor integrals with sector decomposition [3] [38] and Mellin-Barnes representations [39, 40];
- By simplifying the numerator structures algebraically and subsequent evaluation of the resulting integrals of lower rank;
- By direct comparison with other tensor integral packages [33, 37].

Some of the comparisons were documented in [32].

[3] We used a Mathematica interface to the `GINAC` package `sector_decomposition` in order to have a convenient way to evaluate tensor Feynman integrals.

We restrict ourselves to a few phase-space points, see Tables 5.1 to 5.3. The first configuration corresponds to the reaction $gg \to t\bar{t}q\bar{q}$, with external momenta generated by `Madgraph` [41, 42]. The second configuration comes from [37], while the third is a slight modification of the first one. The kinematical input is completed by adding the masses of internal particles.

We begin with massive six-point tensors. For the kinematics introduced above, we determine the tensor components with our Fortran pacakge as shown in Tables 5.4 to 5.6. They are complex, finite numbers. Only independent components of the tensors are shown, all the remaining ones are obtained by permutations of indices.

Selected tensor coefficients of five-point tensors for the case of massive internal particles are shown in Table 5.7.[4] The coefficients have been compared with `LoopTools 2.2` and indeed we agree. For the massive six-point functions, there is no alternative package publicly available.

In presence of massless internal particles, we face potential infrared singularities. Then, the loop functions are Laurent series in ϵ, starting with a term proportional to $\frac{1}{\epsilon^2}$, and one has to care about re-normalizations compared to our basic definition 1.1. A popular measure is [35, 37]:

$$\mathcal{M} = (\mu)^{4-d} \frac{\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)} \int \frac{d^d k}{i\pi^{d/2}}. \tag{5.8}$$

When discussing Feynman integrals with a dependence on inverse powers of ϵ there appears a dependence of their constant terms on these conventions. For convenience of the reader, the tables are produced with a normalization as introduced in Equation 5.8, with the choice $\mu = 1$.

For the case of six-point and five-point functions with only massless internal particles, we show only a few sample coefficients in Table 5.9 and Table 5.8, which are produced with our Fortran package. The phase space point chosen here is defined in Table 5.2. We checked that, within double precision, we completely agree with corresponding numbers produced with `golem95`.

Finally, to complete the list of relevant results, we show also sample tensor coefficients for the case of both massive and massless internal particles, for five-point tensors in Table 5.10 and for six-point tensors in Table 5.11. For this case with mixed internal masses, there is no other publicly released code available.

[4] Please notice that we show here five-point tensor *coefficients*, while in the case of six-point tensors we have shown tensor *components*. The tensor components are representation independent and should be preferred as numerical output. For the five-point tensors with massive internal particles, however, we have arranged for a one-to-one correspondence with output of `LoopTools 2.2`, so it might be interesting to have, in this case, the tensor coefficients instead.

p_1	0.21774554 E+03	0.0	0.0	0.21774554 E+03
p_2	0.21774554 E+03	0.0	0.0	– 0.21774554 E+03
p_3	– 0.20369415 E+03	– 0.47579512 E+02	0.42126823 E+02	0.84097181 E+02
p_4	– 0.20907237 E+03	0.55215961 E+02	– 0.46692034 E+02	– 0.90010087 E+02
p_5	– 0.68463308 E+01	0.53063195 E+01	0.29698267 E+01	– 0.31456871 E+01
p_6	– 0.15878244 E+02	– 0.12942769 E+02	0.15953850 E+01	0.90585932 E+01
$m_1 = 110.0,\ m_2 = 120.0,\quad m_3 = 130.0,\quad m_4 = 140.0,\quad m_5 = 150.0,\quad m_6 = 160.0$				

Table 5.1: The components of external four-momenta for the six-point numerics; all internal particles are massive. For five-point functions, we shrink line 2 and fix $p_1 + p_2 \to p_1$ in order to retain momentum conservation.

p_1	0.5	0.0	0.0	0.5
p_2	0.5	0.0	0.0	– 0.5
p_3	– 0.19178191	– 0.12741180	– 0.08262477	– 0.11713105
p_4	– 0.33662712	0.06648281	0.31893785	0.08471424
p_5	– 0.21604814	0.20363139	– 0.04415762	– 0.05710657
$p_6 = -(p_1 + p_2 + p_3 + p_4 + p_5), \quad m_1 = \cdots = m_6 = 0.0$				

Table 5.2: The external four-momenta for the six-point numerics; all internal particles are massless. This set of momenta comes from [37]. For five-point functions, we shrink line 2 and fix $p_1 + p_2 \to p_1$ in order to retain momentum conservation.

To summarize, we have presented in this article tensor integrals of rank $R \leq 3$ for five-point functions and of rank $R \leq 4$ for six-point functions. This is sufficient for the calculation of e.g. four fermion production at the LHC with NLO QCD corrections.

There are further reactions of interest which will need higher-point functions and higher ranks of five- and six-point functions. The details of their reductions have been left for a later investigation.

p_1	0.21774554 E+01	0.0	0.0	0.21774554 E+01
p_2	0.21774554 E+01	0.0	0.0	– 0.21774554 E+01
p_3	– 0.20369415 E+01	– 0.47579512 E+00	0.42126823 E+00	0.84097181 E+00
p_4	– 0.20907237 E+01	0.55215961 E+00	– 0.46692034 E+00	– 0.90010087 E+00
p_5	– 0.68463308 E–01	0.53063195 E–01	0.29698267 E–01	– 0.31456871 E–01
p_6	– 0.15878244 E+00	– 0.12942769 E+00	0.15953850 E–01	0.90585932 E–01
$m_1 = 0.0, \; m_2 = 0.0, \quad m_3 = 0.0, \quad m_4 = 1.7430, \quad m_5 = 0.0, \quad m_6 = 0.0$				

Table 5.3: The external four-momenta for the six-point numerics; one internal mass is finite. For five-point functions, we shrink line 2 and fix $p_1 + p_2 \to p_1$ in order to retain momentum conservation.

		F_0
		– 0.223393 E–18 – i 0.396728 E–19
μ		F^{μ}
0		0.192487 E–17 + i 0.972635 E–17
1		– 0.363320 E–17 – i 0.11940 E–17
2		0.365514 E–17 + i 0.106928 E–17
3		0.239793 E–16 + i 0.341928 E–17
μ	ν	$F^{\mu\nu}$
0	0	0.599459 E–14 – i 0.114601 E–14
0	1	0.323869 E–15 + i 0.423754 E–15
0	2	– 0.294252 E–15 – i 0.375481 E–15
0	3	– 0.255450 E–14 – i 0.195640 E–14
1	1	– 0.164562 E–14 – i 0.993796 E–16
1	2	0.920944 E–16 + i 0.706487 E–17
1	3	0.347694 E–15 – i 0.127190 E–16
2	2	– 0.163339 E–14 – i 0.994148 E–16
2	3	– 0.341773 E–15 + i 0.818678 E–17
3	3	– 0.413909 E–14 + i 0.670676 E–15

Table 5.4: Tensor components for scalar, vector, and rank $R = 2$ six-point functions; kinematics defined in Table 5.1 and Figure 5.1.

μ	ν	λ	$F^{\mu\nu\lambda}$
0	0	0	– 0.227754 E–11 – i 0.267244 E–12
0	0	1	0.140271 E–13 – i 0.119448 E–12
0	0	2	– 0.201270 E–13 + i 0.101968 E–12
0	0	3	0.102976 E–12 + i 0.624467 E–12
0	1	1	0.183904 E–12 + i 0.142429 E–12
0	1	2	– 0.131028 E–13 – i 0.610343 E–14
0	1	3	– 0.543316 E–13 – i 0.158809 E–13
0	2	2	0.181352 E–12 + i 0.141686 E–12
0	2	3	0.506408 E–13 + i 0.163568 E–13
0	3	3	0.600542 E–12 + i 0.130733 E–12
1	1	1	– 0.563539 E–13 + i 0.178403 E–13
1	1	2	0.210641 E–13 – i 0.584990 E–14
1	1	3	0.120482 E–12 – i 0.574688 E–13
1	2	2	– 0.201182 E–13 + i 0.620591 E–14
1	2	3	– 0.686164 E–14 + i 0.205457 E–14
1	3	3	– 0.447329 E–13 + i 0.193180 E–13
2	2	2	0.582201 E–13 – i 0.163889 E–13
2	2	3	0.119659 E–12 – i 0.570084 E–13
2	3	3	0.457464 E–13 – i 0.181141 E–13
3	3	3	0.557081 E–12 – i 0.374359 E–12

Table 5.5: Tensor components for a massive rank $R = 3$ six-point function; kinematics defined in Table 5.1 and Figure 5.1.

μ	ν	λ	ρ	$F^{\mu\nu\lambda\rho}$
0	0	0	0	0.666615 E–09 + i 0.247562 E–09
0	0	0	1	– 0.200049 E–10 + i 0.294036 E–10
0	0	0	2	0.200975 E–10 – i 0.237333 E–10
0	0	0	3	0.645477 E–10 – i 0.162236 E–09
0	0	1	1	– 0.116956 E–10 – i 0.516760 E–10
0	0	1	2	0.160357 E–11 + i 0.222284 E–11
0	0	1	3	0.792692 E–11 + i 0.729502 E–11
0	0	2	2	– 0.111838 E–10 – i 0.513133 E–10
0	0	2	3	– 0.681086 E–11 – i 0.708933 E–11
0	0	3	3	– 0.804454 E–10 – i 0.801909 E–10
0	1	1	1	0.100498 E–10 – i 0.151735 E–13
0	1	1	2	– 0.348984 E–11 – i 0.195436 E–12
0	1	1	3	– 0.211111 E–10 + i 0.295212 E–11
0	1	2	2	0.357455 E–11 + i 0.662809 E–14
0	1	2	3	0.121595 E–11 – i 0.807388 E–13
0	1	3	3	0.825803 E–11 – i 0.142086 E–11
0	2	2	2	– 0.958961 E–11 – i 0.585948 E–12
0	2	2	3	– 0.209232 E–10 + i 0.289031 E–11
0	2	3	3	– 0.802359 E–11 + i 0.994701 E–12
0	3	3	3	– 0.102576 E–09 + i 0.378476 E–10
1	1	1	1	– 0.246426 E–10 + i 0.276326 E–10
1	1	1	2	0.915670 E–12 – i 0.660629 E–12
1	1	1	3	0.303529 E–11 – i 0.287480 E–11
1	1	2	2	– 0.822697 E–11 + i 0.919635 E–11
1	1	2	3	– 0.116294 E–11 + i 0.100024 E–11
1	1	3	3	– 0.146918 E–10 + i 0.183799 E–10
1	2	2	2	0.908296 E–12 – i 0.654735 E–12
1	2	2	3	0.109510 E–11 – i 0.100875 E–11
1	2	3	3	0.717342 E–12 – i 0.557293 E–12
1	3	3	3	0.450661 E–11 – i 0.485065 E–11
2	2	2	2	– 0.245154 E–10 + i 0.274313 E–10
2	2	2	3	– 0.318500 E–11 + i 0.279750 E–11
2	2	3	3	– 0.146317 E–10 + i 0.182912 E–10
2	3	3	3	– 0.477335 E–11 + i 0.477368 E–11
3	3	3	3	– 0.730168 E–10 + i 0.112865 E–09

Table 5.6: Tensor components for a massive rank $R = 4$ six-point function; kinematics defined in Table 5.1 and Figure 5.1.

E_0	0.702503 E-14 + i 0.170006 E-14
E_1	3.56379 E-15 – i 5.58904 E-16
E_{12}	– 7.86411 E-16 + i 1.03994 E-15
E_{00}	– 8.18587 E-11 + i 1.80354 E-11
E_{123}	3.51267 E-16 + i 9.64413 E-17
E_{001}	9.38702 E-12 + i 2.18811 E-11

Table 5.7: Selected tensor coefficients of five-point tensor functions with massive internal particles; kinematics defined in Table 5.1.

	ϵ^0	$1/\epsilon$	$1/\epsilon^2$
F_0	– 57.8724994 – i 9248.84583	– 3167.69411 – i 2981.57728	– 1003.89197
F_3	– 867.761166 + i 859.212722	273.495904 + i 483.076108	153.767901
F_{22}	83.1234074 – i 271.20343	– 75.7263181 – i 95.1508846	– 30.2874673
F^{000}	– 185.635891 + i 1465.754753	487.259427 + i 525.6914058	174.2745041
F^{1111}	– 2.64116950 – i 4.28827971	– 0.8480346995 – i 0.4557274228	– 0.1450625441

Table 5.8: Tensor coefficients F_0, F_3, F_{22} and tensor components F^{000}, F^{1111} of six-point functions; all internal particles are massless, kinematics of Table 5.2.

	ϵ^0	$1/\epsilon$	$1/\epsilon^2$
E_0	202.168496 + i 3211.04072	1022.10601 + i 972.027061	309.405823 + i 0.0
E_2	264.996441 – i 303.068452	– 96.4696846 – i 149.228472	– 47.5008979 + i 0.0
E_{33}	1780.58042 + i 2914.50734	927.71650 + i 568.572069	180.982111 + i 0.0
E_{00}	9.56327810 + i 1.61648472E-13	4.70734562E-14 + i 2.48689958E-14	7.10542736E-15 + i 0.0
E_{444}	– 1035.29689 – i 1422.01085	– 452.640112 – i 254.226520	– 80.9228146 + i 0.0
E_{001}	– 0.81227772 – i 5.68434189 E–14	– 2.04281037 E–14 – i 2.84217094 E–14	– 7.10542736 E–15 + i 0.0

Table 5.9: Selected tensor coefficients of five-point tensor functions with massless internal particles; kinematics defined in Table 5.2.

	ϵ^0	$1/\epsilon$	$1/\epsilon^2$
E_0	– 0.289852933 E+04 + i 0.228935552 E+03	– 0.945038648 E+02 + i 0.454178453 E+02	0.7112330546 E+01 + i 0.0
E^3	0.168344624 E+03 – i 0.181758172 E+02	0.4242553725 E+01 – i 0.338838829 E+01	– 0.6442770877 E+00 + i 0.0
E^{23}	– 0.79409571852 E+01 + i 0.5445326927 E+00	– 0.3008645503 E+00 + i 0.9457613783 E–01	0.1027869989 E–01 + i 0.0
E^{012}	0.2472148936 E+01 – i 0.127011969 E+00	0.9699262574 E-01 – i 0.2560545796 E–01	– 0.2331885086 E–02 + i 0.0
E^{2130}	0.2733228280 E+02 – i 0.519106421 E+02	– 0.909476582 E+01 + i 0.1744459753 E–02	0.2112313083 E–03 + i 0.0

Table 5.10: Selected tensor components of five-point tensor functions with both massive and massless internal particles; kinematics defined in Table 5.3.

	ϵ^0	$1/\epsilon$	$1/\epsilon^2$
F_0	0.2403558675 E+04 – i 0.2058213187 E+03	0.7315208677 E+02 – i 0.4276718518 E+02	– 0.7543148872 E+01 + i 0.0
F^2	0.1112747404 E+03 – i 0.6809282900 E+01	0.4419243474 E+01 – i 0.1201033663 E+01	– 0.1044856909 E+00 + i 0.0
F^{13}	– 0.1014018623 E+02 + i 0.1797332619 E+01	– 0.5914958485 E–01 + i 0.3275539398 E+00	0.7678550480 E–01 + i 0.0
F^{123}	– 0.5007216712 E+00 + i 0.4194342396 E–01	– 0.1642316924 E–01 + i 0.7789453935 E–02	0.1225024390 E–02 + i 0.0
F^{3210}	0.1263455978 E+00 – i 0.6509987460 E–02	0.4610567958 E–02 – i 0.1506637282 E–02	– 0.1945123881 E–03 + i 0.0

Table 5.11: Selected tensor components of six-point tensor functions with both massive and massless internal particles; kinematics defined in Table 5.3.

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A. Gram determinants and algebra of signed minors

In this section relations are derived, which will turn out to be indispensable in our tensor reductions.

We begin with some notational remarks on Gram determinants G_{n-1},

$$G_{n-1} = |2q_j q_k|, \quad j,k = 1,\cdots,n-1. \tag{A.1}$$

The *modified Cayley determinant* of a diagram with n internal lines with chords q_j is:

$$\begin{aligned}()_n &= |C_{jk}|, \quad j,k = 0,\cdots,n, \\ &= \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & Y_{11} & Y_{12} & \dots & Y_{1n} \\ 1 & Y_{12} & Y_{22} & \dots & Y_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix},\end{aligned} \tag{A.2}$$

with

$$Y_{jk} = -(q_j - q_k)^2 + m_j^2 + m_k^2. \tag{A.3}$$

From our choice $q_n = 0$, it follows that both determinants are related:

$$()_n = -G_{n-1}, \tag{A.4}$$

and we will usually call $()_n$ the Gram determinant of the Feynman integral.

Signed minors [22] are determinants (with a sign convention) which are obtained by excluding rows and columns from the modified Cayley determinant $()_n$. They are denoted by the symbol

$$\begin{pmatrix} j_1 & j_2 & \cdots j_m \\ k_1 & k_2 & \cdots k_m \end{pmatrix}_n, \tag{A.5}$$

labelling the rows $j_1, j_2, \cdots, j_m$ and columns $k_1, k_2, \cdots, k_m$ which have been excluded from $()_n$. The sign of a signed minor is defined by

$$(-1)^{j_1+j_2+\cdots+j_m+k_1+k_2+\cdots+k_m} \times \texttt{Signature}[j_1, j_2, \cdots j_m] \times \texttt{Signature}[k_1, k_2, \cdots k_m] \tag{A.6}$$

where `Signature` gives the sign of permutations to place the indices in increasing order. This agrees e.g. with the definition of the operator `Signature[List]` in Mathematica. As an example may serve the quantity Δ_n:

$$\Delta_n = \begin{vmatrix} Y_{11} & Y_{12} & \dots & Y_{1n} \\ Y_{12} & Y_{22} & \dots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}_n. \tag{A.7}$$

We now will derive two relations between signed minors. Let us introduce

$$A_{ij}^s \equiv -\begin{pmatrix} 0 \\ j \end{pmatrix}_5 \begin{pmatrix} 0 & s \\ 0 & i \end{pmatrix}_5 \begin{pmatrix} s \\ s \end{pmatrix}_5 - \begin{pmatrix} s \\ j \end{pmatrix}_5 \begin{pmatrix} 0 & s \\ i & s \end{pmatrix}_5 \begin{pmatrix} 0 \\ 0 \end{pmatrix}_5 + \begin{pmatrix} 0 \\ s \end{pmatrix}_5 \begin{pmatrix} 0 & s \\ 0 & s \end{pmatrix}_5 \begin{pmatrix} i \\ j \end{pmatrix}_5. \tag{A.8}$$

We are going to show that this expression can be factorized as

$$A^s_{ij} = \begin{pmatrix} \end{pmatrix}_5 X^s_{ij}\,, \tag{A.9}$$

and provide an explicit expression forX^s_{ij}. To begin with, we show that A^s_{ij} is symmetric in the indices i and j for fixed s. Obviously the third term on the right hand side of (A.8) is symmetric since we consider a symmetric determinant. The symmetry of the first two terms means

$$\begin{pmatrix} s \\ s \end{pmatrix}_5 \left[\begin{pmatrix} 0 \\ i \end{pmatrix}_5 \begin{pmatrix} 0 & j \\ 0 & s \end{pmatrix}_5 - \begin{pmatrix} 0 \\ j \end{pmatrix}_5 \begin{pmatrix} 0 & i \\ 0 & s \end{pmatrix}_5 \right] + \begin{pmatrix} 0 \\ 0 \end{pmatrix}_5 \left[\begin{pmatrix} s \\ i \end{pmatrix}_5 \begin{pmatrix} 0 & s \\ j & s \end{pmatrix}_5 - \begin{pmatrix} s \\ j \end{pmatrix}_5 \begin{pmatrix} 0 & s \\ i & s \end{pmatrix}_5 \right] = 0. \tag{A.10}$$

The first square bracket of (A.10) can be evaluated using (A.13) of [22], i.e.

$$\begin{pmatrix} 0 \\ j \end{pmatrix}_5 \begin{pmatrix} 0 & i \\ 0 & s \end{pmatrix}_5 = -\begin{pmatrix} 0 \\ 0 \end{pmatrix}_5 \begin{pmatrix} 0 & s \\ i & j \end{pmatrix}_5 + \begin{pmatrix} 0 \\ i \end{pmatrix}_5 \begin{pmatrix} 0 & j \\ 0 & s \end{pmatrix}_5 \tag{A.11}$$

and (A.10) then results in

$$\begin{pmatrix} s \\ i \end{pmatrix}_5 \begin{pmatrix} 0 & s \\ j & s \end{pmatrix}_5 + \begin{pmatrix} s \\ j \end{pmatrix}_5 \begin{pmatrix} 0 & s \\ s & i \end{pmatrix}_5 + \begin{pmatrix} s \\ s \end{pmatrix}_5 \begin{pmatrix} 0 & s \\ i & j \end{pmatrix}_5 = 0. \tag{A.12}$$

This is proved by multiplication [5] with $()_5$ and using Eqn. (A.8) of [22] with $r = 2$, i.e.

$$\begin{pmatrix} i & l \\ j & k \end{pmatrix}_5 \begin{pmatrix} \end{pmatrix}_5 = \begin{pmatrix} i \\ j \end{pmatrix}_5 \begin{pmatrix} l \\ k \end{pmatrix}_5 - \begin{pmatrix} i \\ k \end{pmatrix}_5 \begin{pmatrix} l \\ j \end{pmatrix}_5 ; \;\; i,j,k,l = 0,\ldots,5. \tag{A.13}$$

Inserting this, products of three factors of the form $\begin{pmatrix} i \\ k \end{pmatrix}_5$ cancel by pairs, q.e.d. .

For the following, relations (A.11) and (A.12) of [22] become important, i.e.

$$\sum_{i=1}^{n} \begin{pmatrix} 0 \\ i \end{pmatrix}_5 = ()_5 \tag{A.14}$$

and

$$\sum_{i=1}^{n} \begin{pmatrix} j \\ i \end{pmatrix}_5 = 0, \;\; (j \neq 0). \tag{A.15}$$

Further, "extensionals" are needed, i.e. relations valid for $()_5$ can be extended to any minor of $()_5$; an extensional of (A.14) e.g. is

$$\sum_{i=1}^{n} \begin{pmatrix} j & 0 \\ k & i \end{pmatrix}_5 = \begin{pmatrix} j \\ k \end{pmatrix}_5 . \tag{A.16}$$

[5] Assuming here $()_5 \neq 0$ means no limitation since we are just looking for an algebraic relation.

As the simplest case we now immediately obtain from (A.8) $A^s_{ss} = 0$, i.e.

$$X^s_{ss} = 0. \tag{A.17}$$

Applying (A.14) and (A.15) to (A.8), we see

$$\sum_{j=1}^{5} A^s_{ij} = -\left(\ \right)_5 \begin{pmatrix} 0 & s \\ 0 & i \end{pmatrix}_5 \begin{pmatrix} s \\ s \end{pmatrix}_5 \tag{A.18}$$

and due to the symmetry in i and j we also have

$$\sum_{i=1}^{5} A^s_{ij} = -\left(\ \right)_5 \begin{pmatrix} 0 & s \\ 0 & j \end{pmatrix}_5 \begin{pmatrix} s \\ s \end{pmatrix}_5, \tag{A.19}$$

which gives us a hint of how X^s_{ij} might look, namely due to (A.18) it should contain a term $-\begin{pmatrix} 0 & s \\ 0 & i \end{pmatrix}_5 \begin{pmatrix} 0 & s \\ j & s \end{pmatrix}_5$. A further contribution must vanish after summing over i. Due to (A.17) it must contain a factor $\begin{pmatrix} 0 & j \\ s & i \end{pmatrix}_5$ [6]. The second factor of this contribution can only be depend on s and has been determined by explicit calculation to be $\begin{pmatrix} 0 & s \\ 0 & s \end{pmatrix}_5$. Thus we conclude:

$$X^s_{ji} = X^s_{ij} = -\begin{pmatrix} 0 & s \\ 0 & i \end{pmatrix}_5 \begin{pmatrix} 0 & s \\ j & s \end{pmatrix}_5 + \begin{pmatrix} 0 & j \\ s & i \end{pmatrix}_5 \begin{pmatrix} 0 & s \\ 0 & s \end{pmatrix}_5. \tag{A.20}$$

We come now to the second relation between signed minors. While (A.20) will be needed for the reduction of 4-point tensors to scalars I^s_4, for the reduction of 3-point tensors to scalars I^{st}_3 we also need

$$-\begin{pmatrix} 0 \\ j \end{pmatrix}_5 \begin{pmatrix} t & s \\ 0 & i \end{pmatrix}_5 \begin{pmatrix} s \\ s \end{pmatrix}_5 - \begin{pmatrix} s \\ j \end{pmatrix}_5 \begin{pmatrix} t & s \\ i & s \end{pmatrix}_5 \begin{pmatrix} 0 \\ 0 \end{pmatrix}_5 + \begin{pmatrix} 0 \\ s \end{pmatrix}_5 \begin{pmatrix} t & s \\ 0 & s \end{pmatrix}_5 \begin{pmatrix} i \\ j \end{pmatrix}_5 = \left(\ \right)_5 X^{st}_{ij}, \tag{A.21}$$

where again we have to show that indeed $()_5$ factorizes and we have to give an explicit expression for X^{st}_{ij}. The left-most term on the left hand side is an auxiliary term. It is antisymmetric in s and t after the cancellation of $\begin{pmatrix} s \\ s \end{pmatrix}_5$ and vanishes after summation over s and t because I^{st}_3 is symmetric in s and t. The cancellation of $\begin{pmatrix} s \\ s \end{pmatrix}_5$ has to be checked explicitly in every case where (A.21) is applied.

[6] Observe that $\sum_{j=1}^5 \begin{pmatrix} 0 & j \\ s & i \end{pmatrix}_5 = 0$ but $\sum_{i=1}^5 \begin{pmatrix} 0 & j \\ s & i \end{pmatrix}_5 = -\sum_{i=1}^5 \begin{pmatrix} 0 & j \\ i & s \end{pmatrix}_5 = -\begin{pmatrix} j \\ s \end{pmatrix}_5$.

We observe that the expressions for X^s_{ij} (A.8) and X^{st}_{ij} (A.21) differ only by replacing one 0 by t. Therefore the following ansatz is implied for X^{st}_{ij}.

$$X^{st}_{ij} = -\begin{pmatrix} 0 & s \\ 0 & j \end{pmatrix}_5 \begin{pmatrix} t & s \\ i & s \end{pmatrix}_5 + \begin{pmatrix} 0 & i \\ s & j \end{pmatrix}_5 \begin{pmatrix} t & s \\ 0 & s \end{pmatrix}_5 . \tag{A.22}$$

Now we directly evaluate $X^{st}_{ij}()_5$ using (A.13):

$$X^{st}_{ij}()_5 = -\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}_5 \begin{pmatrix} s \\ j \end{pmatrix}_5 - \begin{pmatrix} s \\ 0 \end{pmatrix}_5 \begin{pmatrix} 0 \\ j \end{pmatrix}_5\right] \begin{pmatrix} t & s \\ i & s \end{pmatrix} + \left[\begin{pmatrix} 0 \\ s \end{pmatrix}_5 \begin{pmatrix} i \\ j \end{pmatrix}_5 - \begin{pmatrix} i \\ s \end{pmatrix}_5 \begin{pmatrix} 0 \\ j \end{pmatrix}_5\right] \begin{pmatrix} t & s \\ 0 & s \end{pmatrix} \tag{A.23}$$

and the remaining equation to be verified is

$$\begin{pmatrix} s \\ 0 \end{pmatrix}_5 \begin{pmatrix} 0 \\ j \end{pmatrix}_5 \begin{pmatrix} t & s \\ i & s \end{pmatrix} - \begin{pmatrix} i \\ s \end{pmatrix}_5 \begin{pmatrix} 0 \\ j \end{pmatrix}_5 \begin{pmatrix} t & s \\ 0 & s \end{pmatrix} = -\begin{pmatrix} 0 \\ j \end{pmatrix}_5 \begin{pmatrix} t & s \\ 0 & i \end{pmatrix} \begin{pmatrix} s \\ s \end{pmatrix}_5 , \tag{A.24}$$

which is done by multiplying again with $()_5$ and again using (A.13). This gives us at the same time also a more general proof for X^s_{ij} (A.20), putting $t = 0$.

B. Reduction of dimensionally shifted five- and four-point integrals

In this appendix we provide explicitly the needed recursion relations for the reduction of the five- and four-point functions. In spite of the fact that here, essentially, only two different relations of [26] are applied for different indices and dimension, namely (I.30) and (I.31), we consider it helpful and sometimes even necessary, to provide them in detail. A special case of (I.31) is (B.4). The others are special cases of (I.30). For the six-point function relation (I.26) plays a major role and will be quoted when applied.

$$\nu_{ijk} I^{[d+]^3}_{5,ijk} = -\frac{\begin{pmatrix} 0 \\ k \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} I^{[d+]^2}_{5,ij} + \sum_{s=1, s\neq i,j}^{5} \frac{\begin{pmatrix} s \\ k \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} I^{[d+]^2,s}_{4,ij} + \frac{\begin{pmatrix} i \\ k \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} I^{[d+]^2}_{5,j} + \frac{\begin{pmatrix} j \\ k \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} I^{[d+]^2}_{5,i}, \tag{B.1}$$

$$\nu_{ij} I^{[d+]^2}_{5,ij} = -\frac{\begin{pmatrix} 0 \\ j \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} I^{[d+]}_{5,i} + \sum_{s=1, s\neq i}^{5} \frac{\begin{pmatrix} s \\ j \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} I^{[d+],s}_{4,i} + \frac{\begin{pmatrix} i \\ j \end{pmatrix}_5}{\begin{pmatrix} \\ \end{pmatrix}_5} I^{[d+]}_{5}. \tag{B.2}$$

The four-point function's shift is (I.44):

$$I^{[d+],s}_{4,i} = -\frac{\begin{pmatrix} 0 & s \\ i & s \end{pmatrix}_5}{\begin{pmatrix} s \\ s \end{pmatrix}_5} I^s_4 + \sum_{t=1, t\neq s}^{5} \frac{\begin{pmatrix} t & s \\ i & s \end{pmatrix}_5}{\begin{pmatrix} s \\ s \end{pmatrix}_5} I^{st}_3, \tag{B.3}$$

and the four-point integrals occurring in the reduction are (I.50):

$$I_4^{[d+],s} = \left[\frac{\binom{0\ s}{0\ s}_5}{\binom{s}{s}_5} I_4^s - \sum_{t=1,t\neq s}^{5} \frac{\binom{t\ s}{0\ s}_5}{\binom{s}{s}_5} I_3^{st} \right] \frac{1}{d-3} \tag{B.4}$$

In applications we can put $d = 4$ since $I_4^{[d+]}$ is UV- and IR-finite. Beyond that, as it is done frequently [8], $I_4^{[d+]}$ can be used as well as a "master integral" (see e.g. (3.14)) without reduction to the generic dimension.

References

[1] G. 't Hooft and M. Veltman, *Scalar one loop integrals*, *Nucl. Phys.* **B153** (1979) 365–401.

[2] G. Passarino and M. Veltman, *One loop corrections for e^+e^- annihilation into $\mu^+\mu^-$ in the Weinberg model*, *Nucl. Phys.* **B160** (1979) 151.

[3] G. van Oldenborgh, *FF: A package to evaluate one loop Feynman diagrams*, *Comput. Phys. Commun.* **66** (1991) 1–15.

[4] T. Hahn and M. Perez-Victoria, *Automatized one-loop calculations in four and d dimensions*, *Comput. Phys. Commun.* **118** (1999) 153, [hep-ph/9807565].

[5] A. Denner and S. Dittmaier, *Reduction of one-loop tensor 5-point integrals*, *Nucl. Phys.* **B658** (2003) 175–202, [hep-ph/0212259].

[6] A. Denner and S. Dittmaier, *Reduction schemes for one-loop tensor integrals*, *Nucl. Phys.* **B734** (2006) 62–115, [hep-ph/0509141].

[7] T. Binoth, J. Guillet, and G. Heinrich, *Reduction formalism for dimensionally regulated one-loop n-point integrals*, *Nucl. Phys.* **B572** (2000) 361–386, [hep-ph/9911342].

[8] T. Binoth, J. Guillet, G. Heinrich, E. Pilon, and C. Schubert, *An algebraic/numerical formalism for one-loop multi-leg amplitudes*, *JHEP* **10** (2005) 015, [hep-ph/0504267].

[9] A. Ferroglia, M. Passera, G. Passarino, and S. Uccirati, *All-purpose numerical evaluation of one-loop multi-leg Feynman diagrams*, *Nucl. Phys.* **B650** (2003) 162–228, [hep-ph/0209219].

[10] Y. Kurihara and T. Kaneko, *Numerical contour integration for loop integrals*, *Comput. Phys. Commun.* **174** (2006) 530–539, [hep-ph/0503003].

[11] G. Ossola, C. Papadopoulos, and R. Pittau, *Reducing full one-loop amplitudes to scalar integrals at the integrand level*, *Nucl. Phys.* **B763** (2007) 147–169, [hep-ph/0609007].

[12] G. Ossola, C. Papadopoulos, and R. Pittau, *CutTools: a program implementing the OPP reduction method to compute one-loop amplitudes*, *JHEP* **03** (2008) 042, [arXiv:0711.3596].

[13] R. K. Ellis, W. Giele, and G. Zanderighi, *Semi-numerical evaluation of one-loop corrections*, *Phys. Rev.* **D73** (2006) 014027, [hep-ph/0508308].

[14] Z. Bern, L. Dixon, and D. Kosower, *Progress in one-loop QCD computations*, *Ann. Rev. Nucl. Part. Sci.* **46** (1996) 109–148, [hep-ph/9602280].

[15] L. Dixon, *Calculating scattering amplitudes efficiently*, hep-ph/9601359.

[16] Z. Bern, L. Dixon, and D. Kosower, *On-Shell Methods in Perturbative QCD*, *Annals Phys.* **322** (2007) 1587–1634, [arXiv:0704.2798].

[17] W. T. Giele and G. Zanderighi, *On the Numerical Evaluation of One-Loop Amplitudes: the Gluonic Case*, hep-ph/0805.2152.

[18] S. Weinzierl, *Automated calculations for multi-leg processes*, *PoS* **ACAT** (2007) 005, [arXiv:0707.3342].

[19] Z. Bern *et al.*, *The NLO multileg working group: Summary report*, hep-ph/0803.0494.

[20] J. Blümlein, S. Moch and T. Riemann (eds.), *Loops and Legs in Quantum Field Theory, Proceedings of the 9th DESY Workshop on Elementary Particle Theory, 20-25 April 2008, Sondershausen, Germany*, *Nucl. Phys. Proc. Suppl.* **183** (2008).

[21] T. Binoth, *LHC phenomenology at next-to-leading order QCD: theoretical progress and new results*, *PoS* **ACAT** (2008). To appear.

[22] D. B. Melrose, *Reduction of Feynman diagrams*, *Nuovo Cim.* **40** (1965) 181–213.

[23] A. Davydychev, *A simple formula for reducing Feynman diagrams to scalar integrals*, *Phys. Lett.* **B263** (1991) 107–111.

[24] K. Chetyrkin and F. Tkachov, *Integration by parts: The algorithm to calculate β functions in four loops*, *Nucl. Phys.* **B192** (1981) 159–204.

[25] O. Tarasov, *Connection between Feynman integrals having different values of the space-time dimension*, *Phys. Rev.* **D54** (1996) 6479–6490, [hep-th/9606018].

[26] J. Fleischer, F. Jegerlehner, and O. Tarasov, *Algebraic reduction of one-loop Feynman graph amplitudes*, *Nucl. Phys.* **B566** (2000) 423–440, [hep-ph/9907327].

[27] F. Jegerlehner and O. Tarasov, *FIRCLA, one-loop correction to $e^+e^- \to \nu\bar{\nu}H$ and basis of Feynman integrals in higher dimensions*, *Nucl. Phys. Proc. Suppl.* **116** (2003) 83–87, [hep-ph/0212004].

[28] J. Campbell, E. W. N. Glover, and D. Miller, *One-loop tensor integrals in dimensional regularisation*, *Nucl. Phys.* **B498** (1997) 397–442, [hep-ph/9612413].

[29] G. Devaraj and R. G. Stuart, *Reduction of one-loop tensor form-factors to scalar integrals: A general scheme*, *Nucl. Phys.* **B519** (1998) 483–513, [hep-ph/9704308].

[30] J. Fleischer, J. Gluza, K. Kajda, and T. Riemann, *Pentagon diagrams of Bhabha scattering*, *Acta Phys. Polon.* **B38** (2007) 3529–3536, [arXiv:0710.5100].

[31] Silesian Univ., Katowice, webpage http://www.us.edu.pl/∼gluza/hexagon,
DESY, webpage http://www-zeuthen.desy.de/theory/research/CAS.html.

[32] T. Diakonidis, J. Fleischer, J. Gluza, K. Kajda, T. Riemann, and J. B. Tausk, *On the tensor reduction of one-loop pentagons and hexagons*, *Nucl. Phys. Proc. Suppl.* **183** (2008) 109–115, [arXiv:0807.2984].

[33] T. Hahn and M. Rauch, *News from FormCalc and LoopTools*, *Nucl. Phys. Proc. Suppl.* **157** (2006) 236–240, [hep-ph/0601248].

[34] A. van Hameren, J. Vollinga, and S. Weinzierl, *Automated computation of one-loop integrals in massless theories*, *Eur. Phys. J.* **C41** (2005) 361–375, [hep-ph/0502165].

[35] R. K. Ellis and G. Zanderighi, *Scalar one-loop integrals for QCD*, *JHEP* **02** (2008) 002, [arXiv:0712.1851].

[36] Z. Bern, L. Dixon, and D. Kosower, *Dimensionally regulated pentagon integrals*, *Nucl. Phys.* **B412** (1994) 751–816, [hep-ph/9306240].

[37] T. Binoth, J. P. Guillet, G. Heinrich, E. Pilon, and T. Reiter, *Golem95: a numerical program to calculate one-loop tensor integrals with up to six external legs*, arXiv:0810.0992.

[38] C. Bogner and S. Weinzierl, *Resolution of singularities for multi-loop integrals*, *Comput. Phys. Commun.* **178** (2008) 596–610, [arXiv:0709.4092].

[39] J. Gluza, K. Kajda, and T. Riemann, *AMBRE - a Mathematica package for the construction of Mellin-Barnes representations for Feynman integrals*, *Comput. Phys. Commun.* **177** (2007) 879–893, [arXiv:0704.2423].

[40] M. Czakon, *Automatized analytic continuation of Mellin-Barnes integrals*, *Comput. Phys. Commun.* **175** (2006) 559–571, [hep-ph/0511200].

[41] T. Stelzer and W. Long, *Automatic generation of tree level helicity amplitudes*, *Comput. Phys. Commun.* **81** (1994) 357–371, [hep-ph/9401258].

[42] F. Maltoni and T. Stelzer, *MadEvent: Automatic event generation with MadGraph*, *JHEP* **02** (2003) 027, [hep-ph/0208156].